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CONTROLLABILITY OF A 2×2 PARABOLIC SYSTEM BY ONE FORCE WITH SPACE-DEPENDENT COUPLING TERM OF ORDER ONE*

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Abstract. This paper is devoted to the controllability of linear systems of two coupled parabolic equations when the coupling involves a space dependent first order term. This system is set on an bounded interval $I \subset\subset \mathbb{R}$, and the first equation is controlled by a force supported in a subinterval of I or on the boundary. In the case where the intersection of the coupling and control domains is nonempty, we prove null controllability at any time. Otherwise, we provide a minimal time for null controllability. Finally we give a necessary and sufficient condition for the approximate controllability. The main technical tool for obtaining these results is the moment method.

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1. Introduction and main results

Let $T > 0$, $\omega := (a, b) \subseteq (0, \pi)$ and $Q_T := (0, \pi) \times (0, T)$. We consider in the present paper the following distributed control system

$$\begin{cases} \partial_t y_1 - \partial_{xx} y_1 = \mathbb{1}_\omega v & \text{in} \quad Q_T, \\ \partial_t y_2 - \partial_{xx} y_2 + p(x)\partial_x y_1 + q(x) y_1 = 0 & \text{in} \quad Q_T, \\ y_1(0,\cdot) = y_1(\pi,\cdot) = y_2(0,\cdot) = y_2(\pi,\cdot) = 0 & \text{on} \quad (0,T), \\ y_1(\cdot,0) = y_1^0, \ y_2(\cdot,0) = y_2^0 & \text{in} \quad (0,\pi) \end{cases} \tag{1.1}$$

and boundary control system

$$\begin{cases} \partial_t z_1 - \partial_{xx} z_1 = 0 & \text{in} \quad Q_T, \\ \partial_t z_2 - \partial_{xx} z_2 + p(x)\partial_x z_1 + q(x) z_1 = 0 & \text{in} \quad Q_T, \\ z_1(0,\cdot) = u, \ z_1(\pi,\cdot) = z_2(0,\cdot) = z_2(\pi,\cdot) = 0 & \text{on} \quad (0,T), \\ z_1(\cdot,0) = z_1^0, \ z_2(\cdot,0) = z_2^0 & \text{in} \quad (0,\pi), \end{cases} \tag{1.2}$$

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where $y^0 := (y_1^0, y_2^0) \in L^2(0,\pi)^2$ and $z^0 := (z_1^0, z_2^0) \in H^{-1}(0,\pi)^2$ are the initial conditions, $v \in L^2(Q_T)$ and $u \in L^2(0,T)$ are the controls, $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$.

It is known (see [20], p. 102 (resp. [16], Prop. 2.2)) that for given initial data $y^0 \in L^2(0,\pi)^2$ (resp. $z^0 \in H^{-1}(0,\pi)^2$) and a control $v \in L^2(Q_T)$ (resp. $u \in L^2(0,T)$) system (1.1) (resp. (1.2)) has a unique solution $y = (y_1, y_2)$ (resp. $z = (z_1, z_2)$) in

$$L^2(0,T; H_0^1(0,\pi)^2) \cap \mathcal{C}([0,T]; L^2(0,\pi)^2) (\text{resp. } L^2(Q_T)^2 \cap \mathcal{C}([0,T]; H^{-1}(0,\pi)^2)),$$

which depends continuously on the initial data and the control, that is

$$\|y\|_{L^2(0,T;H_0^1(0,\pi)^2)} + \|y\|_{\mathcal{C}([0,T];L^2(0,\pi)^2)} \leqslant C_T(\|y^0\|_{L^2(0,\pi)^2} + \|v\|_{L^2(Q_T)})$$

$$(\text{resp. } \|z\|_{L^2(Q_T)^2} + \|z\|_{\mathcal{C}([0,T];H^{-1}(0,\pi)^2)} \leqslant C_T(\|z^0\|_{H^{-1}(0,\pi)^2} + \|u\|_{L^2(0,T)})),$$

where C_T does not depend on y^0, v, z^0 and u.

Let us introduce the notion of null and approximate controllability for this kind of systems.

- System (1.1) (resp. system (1.2)) is *null controllable* at time T if for every initial condition $y^0 \in L^2(0,\pi)^2$ (resp. $z^0 \in H^{-1}(0,\pi)^2$) there exists a control $v \in L^2(Q_T)$ (resp. $u \in L^2(0,T)$) such that the solution to system (1.1) (resp. system (1.2)) satisfies

$$y(T) \equiv 0 \quad (\text{resp. } z(T) \equiv 0) \quad \text{in } (0,\pi).$$

- System (1.1) (resp. system (1.2)) is *approximately controllable* at time T if for all $\varepsilon > 0$ and all $y^0,\ y^1 \in L^2(0,\pi)^2$ (resp. $z^0,\ z^1 \in H^{-1}(0,\pi)^2$) there exists a control $v \in L^2(Q_T)$ (resp. $u \in L^2(0,T)$) such that the solution to system (1.1) (resp. system (1.2)) satisfies

$$\|y(T) - y^1\|_{L^2(0,\pi)^2} \leqslant \varepsilon \quad (\text{resp. } \|z(T) - z^1\|_{H^{-1}(0,\pi)^2} \leqslant \varepsilon).$$

We recall that null-controllability at some time T implies approximate controllability at the same time T for linear parabolic systems. This follows from the backward uniqueness result of ([17], Thm. 1.1) for first order perturbations and Propositions 2.5 and 2.6. Moreover the approximate controllability does not depend on the time of control T since we consider autonomous systems. It is a consequence of the analyticity in time of the adjoint semigroup.

The main goal of this article is to provide a complete answer to the null and approximate controllability issues for system (1.1) and (1.2). For a survey and some applications in physics, chemistry or biology concerning the controllability of this kind of systems, we refer to [6]. In the last decade, many papers studied this problem, however most of them are related to some parabolic systems with zero order coupling terms. Without first order coupling terms, some Kalman coupling conditions are made explicit in ([3], Thm. 1.4, [4], Thm. 1.1 and [16], Thm. 1.1) for distributed null controllability of systems of more than two equations with constant matrices and in higher space dimension and in the case of time dependent matrices, some Silverman−Meadows coupling conditions are given in ([3], Thm. 1.2).

Concerning the null and approximate controllability of systems (1.1) and (1.2) in the case $p \equiv 0$ and $q \not\equiv 0$ in $(0,\pi)$, a partial answer is given in [1,2,13,23] under the sign condition $q \leqslant 0$ or $q \geqslant 0$ in $(0,\pi)$. These results are obtained as a consequence of controllability results of a hyperbolic system using the transmutation method (see [21]). One can find a necessary and sufficient condition in [7] when $\int_0^\pi q(x)\mathrm{d}x \neq 0$. Finally, in [11], the authors gives a complete characterization of the approximate controllability and, in the recent work [8,9], we can find a complete study of the null controllability.

When $p \neq 0$, the approximate controllability of systems (1.1) and (1.2) in any dimension is studied in [22]. The author gives a sufficient condition for the approximate controllability on the boundary and, in the case of analytic coupling coefficients p and q, a necessary and sufficient condition for the internal approximate controllability.

Let us now remind known results concerning null controllability for systems of the following more general form. Let Ω be a bounded domain in $\mathbb{R}^N$ ($N \in \mathbb{N}^*$) of class $\mathcal{C}^2$ and ω_0 an arbitrary nonempty subset of Ω. We denote by $\partial\Omega$ the boundary of Ω. Consider the system of two coupled linear parabolic equations

$$\begin{cases} \partial_t y_1 = \Delta y_1 + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_{\omega_0} v & \text{in} \quad \Omega \times (0,T), \\ \partial_t y_2 = \Delta y_2 + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in} \quad \Omega \times (0,T), \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ y(\cdot, 0) = y^0 & \text{in} \quad \Omega, \end{cases} \tag{1.3}$$

where $y^0 \in L^2(\Omega)^2$, $g_{ij} \in L^\infty(\Omega \times (0,T))^N$ and $a_{ij} \in L^\infty(\Omega \times (0,T))$ for all $i, j \in \{1,2\}$.

As a particular case of the result in Section 4 of [18] (see also [5]), system (1.3) is null controllable whenever

$$g_{21} \equiv 0 \text{ in } \omega_1 \text{ and } (a_{21} > C \text{ in } \omega_1 \text{ or } a_{21} < -C \text{ in } \omega_1), \tag{1.4}$$

for a positive constant C and ω_1 a non-empty open subset of ω_0.

In ([19], Thm. 4), the author supposes that a_{11}, g_{11}, a_{22}, g_{22} are constant and the first order coupling operator $g_{21} \cdot \nabla + a_{21}$ can be written as

$$g_{21} \cdot \nabla + a_{21} = P_1 \circ \theta \text{ in } \Omega \times (0,T), \tag{1.5}$$

where $\theta \in \mathcal{C}^2(\overline{\Omega})$ satisfies $|\theta| > C$ in $\omega_1 \subseteq \omega_0$ for a positive constant C and P_1 is given by $P_1 := m_0 \cdot \nabla + m_1$, for some $m_0, m_1 \in \mathbb{R}$. Moreover the operator P_1 has to satisfy

$$\|u\|_{H^1(\Omega)} \leqslant C \|P_1^* u\|_{L^2(\Omega)} \quad \forall u \in H_0^1(\Omega).$$

Under these assumptions, the author proves the null controllability of system (1.3) at any time.

In ([10], Thm. 2.1), the authors prove that the same property holds true for system (1.3) if we assume that $a_{ij} \in \mathcal{C}^4(\overline{\Omega \times (0,T)})$, $g_{ij} \in \mathcal{C}^1(\overline{\Omega \times (0,T)})^N$ for all $i,j \in \{1,2\}$, $g_{21} \in \mathcal{C}^3(\overline{\Omega \times (0,T)})$ and the geometrical condition

$$\begin{cases} \partial\omega \cap \partial\Omega \text{ contains an open subset } \gamma \text{ for which the interior } \mathring{\gamma} \text{ is non-empty}, \\ \exists x_0 \in \gamma \text{ s.t. } g_{21}(t, x_0) \cdot \nu(x_0) \neq 0 \text{ for all } t \in [0,T], \end{cases} \tag{1.6}$$

where ν represents the exterior normal unit vector to the boundary $\partial\Omega$.

Lastly, for constant coefficients, it is proved in ([14], Thm. 1) that system (1.3) is null/approximately controllable at any time T if and only if

$$g_{21} \neq 0 \quad \text{or} \quad a_{21} \neq 0.$$

In ([14], Thm. 2), the authors give also a condition of null/approximate controllability in dimension one which can be written for system (1.1) as: $p \in \mathcal{C}^2(\omega_0)$, $q \in \mathcal{C}^3(\omega_0)$ and

$$\begin{aligned} -4\partial_x(q)\partial_x(p)p + \partial_{xx}(q)p^2 + 2q\partial_x(q)p - 3pq\partial_{xx}p + 6q(\partial_x p)^2 - 2q^2\partial_x p \\ -\partial_{xxx}(p)p^2 + 5\partial_x(p)\partial_{xx}(p)p - 4(\partial_x p)^3 \neq 0 \text{ in } \omega_0 \end{aligned}$$

for a subinterval ω_0 of ω.

Now let us go back to systems (1.1) and (1.2) for which we will provide a complete description of the null and approximate controllability. Our first and main result is the following

Theorem 1.1. *Let us suppose that $p \in W^1_\infty(0,\pi) \cap W^2_\infty(\omega)$, $q \in L^\infty(0,\pi) \cap W^1_\infty(\omega)$ and*

$$(\text{Supp } p \cup \text{Supp } q) \cap \omega \neq \varnothing. \tag{1.7}$$

Then system (1.1) is null controllable at any time T.

Let us compare this result with the previously described results to highlight our main contribution:

(1) Even though system (1.1) is considered in one space dimension, we remark first that our coupling operator has a more general form than the one in (1.5) assumed in [19]. Moreover unlike [14], its coefficients are non-constant with respect to the space variable.
(2) We do not have the geometrical restriction (1.6) assumed in [10]. More precisely we do not require the control support to be a neighbourhood of a part of the boundary.
(3) As said before, in ([22], Thm. 4.1), the author gives a necessary and sufficient condition for the approximate controllability of system (1.1) when p and q are analytic. We deduce that the condition of [22] is satisfied in dimension one as soon as p or q is not equal to zero.

For all $k \in \mathbb{N}^*$, we denote by $\varphi_k : x \mapsto \sqrt{\frac{2}{\pi}} \sin(kx)$ the normalized eigenvector of the Laplacian operator, with Dirichlet boundary condition, and consider the two following quantities

$$I_{a,k}(p,q) := \int_0^a \left(q - \frac{1}{2}\partial_x p\right) \varphi_k^2 \quad \text{and} \quad I_k(p,q) := \int_0^\pi \left(q - \frac{1}{2}\partial_x p\right) \varphi_k^2, \tag{1.8}$$

for all $k \in \mathbb{N}^*$. Combined with the criterion of Fattorini (see [15], Cor. 3.3 or Thm. 6.1 in the present paper), Theorem 1.1 leads to the following characterization:

Theorem 1.2. *Let us suppose that $p \in W^1_\infty(0,\pi) \cap W^2_\infty(\omega)$ and $q \in L^\infty(0,\pi) \cap W^1_\infty(\omega)$. System (1.1) is approximately controllable at any time $T > 0$ if and only if*

$$(\text{Supp } p \cup \text{Supp } q) \cap \omega \neq \varnothing \tag{1.9}$$

or

$$|I_k(p,q)| + |I_{a,k}(p,q)| \neq 0 \text{ for all } k \in \mathbb{N}^*. \tag{1.10}$$

This last result recovers the case $p \equiv 0$ studied in [11] for Supp $q \cap \omega = \varnothing$, where the authors also use the criterion of Fattorini.

Remark 1.3. We will see in the proof of Theorems 1.1 and 1.2 that only the following regularity are needed for p and q

$$\begin{cases} p \in W^1_\infty(0,\pi) \cap W^2_\infty(\widetilde{\omega}), \\ q \in L^\infty(0,\pi) \cap W^1_\infty(\widetilde{\omega}), \end{cases}$$

for an open subinterval $\widetilde{\omega}$ of ω. These hypotheses are used in Definition (1.8) of $I_k(p,q)$ and $I_{a,k}(p,q)$ and the change of unknown described in Section 3.2. For more general coupling terms, these control problems are open.

When the supports of the control and the coupling terms are disjoint in system (1.1), following the ideas in ([9], Thm. 1.3) where the authors studied the case $p \equiv 0$, we obtain a minimal time of null controllability:

Theorem 1.4. *Let $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$. Suppose that condition (1.10) holds and*

$$(\text{Supp } p \cup \text{Supp } q) \cap \omega = \varnothing. \tag{1.11}$$

Let $T_0(p,q)$ be given by

$$T_0(p,q) := \limsup_{k\to\infty} \frac{\min(-\log|I_k(p,q)|, -\log|I_{a,k}(p,q)|)}{k^2}. \tag{1.12}$$

One has

(1) *If $T > T_0(p,q)$, then system (1.1) is null controllable at time T.*
(2) *If $T < T_0(p,q)$, then system (1.1) is not null controllable at time T.*

Concerning the boundary controllability, in ([22], Thm. 3.3), using the criterion of Fattorini, the author proves that system (1.2) is approximately controllable at time T if and only if

$$I_k(p,q) \neq 0 \text{ for all } k \in \mathbb{N}^*. \tag{1.13}$$

About null controllability of system (1.2), we can again generalize the results given in ([9], Thm. 1.1) to obtain a minimal time:

Theorem 1.5. *Let $p \in W^1_\infty(0,\pi)$, $q \in L^\infty(0,\pi)$ and suppose that condition (1.13) is satisfied. Let us define*

$$T_1(p,q) := \limsup_{k\to\infty} \frac{-\log|I_k(p,q)|}{k^2}\cdot \tag{1.14}$$

One has

(1) *If $T > T_1(p,q)$, then system (1.2) is null controllable at time T.*
(2) *If $T < T_1(p,q)$, then system (1.2) is not null controllable at time T.*

Remark 1.6. Using Riemann−Lebesgue Lemma, sequences $(I_k(p,q))_{k\in\mathbb{N}^*}$ and $(I_{a,k}(p,q))_{k\in\mathbb{N}^*}$ are convergent, more precisely

$$\lim_{k\to\infty} I_k(p,q) = I(p,q) := \frac{1}{\pi}\int_0^\pi \left(q - \frac{1}{2}\partial_x p\right) \quad \text{and} \quad \lim_{k\to\infty} I_{a,k}(p,q) = I_a(p,q) := \frac{1}{\pi}\int_0^a \left(q - \frac{1}{2}\partial_x p\right).$$

Thus, if one of the two limits $I(p,q)$ and $I_a(p,q)$ (resp. the first limit) is not equal to zero, then the minimal time $T_0(p,q)$ (resp. $T_1(p,q)$) is equal to zero.

This article is organized as follows. In the second section, we present some preliminary results useful to reduce the null controllability issues to the moment problem. In the third and fourth sections, we study the null controllability issue of system (1.1) in the two cases when the intersection of the coupling and control supports is empty or not. Then we give the proof of Theorems 1.2 and 1.5 in Sections 5 and 6, respectively. We finish with some comments and open problems in Section 7.

2. Preliminary results

Consider the differential operator

$$\begin{aligned} L : D(L) \subset L^2(0,\pi)^2 &\to L^2(0,\pi)^2 \\ f &\mapsto -\partial_{xx} f + A_0(p\partial_x f + qf), \end{aligned}$$

where the matrix A_0 is given by

$$A_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

the domain of L and its adjoint L^* is given by $D(L) = D(L^*) = H^2(0,\pi)^2 \cap H^1_0(0,\pi)^2$. In Section 2.1, we will first establish some properties of the operator L that will be useful for the moment method and, in Section 2.2, we will recall some characterizations of the approximate and null controllability of system (1.1).

2.1. Biorthogonal basis

Let us first analyze the spectrum of the operators L and L^*.

Proposition 2.1. *For all $k \in \mathbb{N}^*$ consider the two vectors*

$$\Phi^*_{1,k} := \begin{pmatrix} \psi^*_k \\ \varphi_k \end{pmatrix}, \Phi^*_{2,k} := \begin{pmatrix} \varphi_k \\ 0 \end{pmatrix},$$

where ψ^*_k is defined for all $x \in (0,\pi)$ by

$$\begin{cases} \psi^*_k(x) = \alpha^*_k \varphi_k(x) - \dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\xi))[I_k(p,q)\varphi_k(\xi) + \partial_x(p(\xi)\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]\mathrm{d}\xi, \\ \alpha^*_k = \dfrac{1}{k}\displaystyle\int_0^\pi \int_0^x \sin(k(x-\xi))[I_k(p,q)\varphi_k(\xi) + \partial_x(p(\xi)\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]\varphi_k(x)\mathrm{d}\xi\mathrm{d}x. \end{cases}$$

One has

1. *The spectrum of L^* is given by $\sigma(L^*) = \{k^2 : k \in \mathbb{N}^*\}$.*
2. *For $k \geqslant 1$, the eigenvalue k^2 of L^* is simple (algebraic multiplicity 1) if and only if $I_k(p,q) \neq 0$. In this case, $\Phi^*_{2,k}$ and $\Phi^*_{1,k}$ are respectively an eigenfunction and a generalized eigenfunction of the operator L^* associated with the eigenvalue k^2, more precisely*

$$\begin{cases} (L^* - k^2 Id)\Phi^*_{1,k} = I_k \Phi^*_{2,k}, \\ (L^* - k^2 Id)\Phi^*_{2,k} = 0. \end{cases} \tag{2.1}$$

3. *For $k \geqslant 1$, the eigenvalue k^2 of L^* is double (algebraic multiplicity 2) if and only if $I_k(p,q) = 0$. In this case, $\Phi^*_{1,k}$ and $\Phi^*_{2,k}$ are two eigenfunctions of the operator L^* associated with the eigenvalue k^2, that is for $i = 1,2$*

$$(L^* - k^2 Id)\Phi^*_{i,k} = 0.$$

Proof. The adjoint operator L^* of L is given by $D(L^*) = D(L)$ and $L^* f = -\partial_{xx} f + A_0^*(-\partial_x(pf) + qf)$. We can remark first that the resolvent of L^* is compact. Thus the spectrum of L^* reduces to its point spectrum. The eigenvalue problem associated with the operator L^* is

$$\begin{cases} -\partial_{xx}\psi - \partial_x(p(x)\varphi) + q(x)\varphi = \lambda\psi & \text{in} \quad (0,\pi), \\ -\partial_{xx}\varphi = \lambda\varphi & \text{in} \quad (0,\pi), \\ \varphi(0) = \psi(0) = \varphi(\pi) = \psi(\pi) = 0, \end{cases} \tag{2.2}$$

where $(\psi,\varphi) \in D(L^*)$ and $\lambda \in \mathbb{C}$. For $\varphi \equiv 0$ in $(0,\pi)$ and $\psi = \varphi_k$ in $(0,\pi)$, $\lambda = k^2$ is an eigenvalue of L^* and the vector $\Phi^*_{2,k} := (\varphi_k, 0)$ is an associated eigenfunction. If now $\varphi \not\equiv 0$ in $(0,\pi)$, then $\lambda = k^2$ is an eigenvalue and $\varphi = \kappa\varphi_k$ with $\kappa \in \mathbb{R}^*$. We remark that system (2.2) has a solution if and only if $I_k(p,q) = 0$. If $I_k(p,q) = 0$, $\Phi^*_{1,k} := (\psi^*_k, \varphi_k)$ is a second eigenfunction of L^* linearly independent of $\Phi^*_{2,k}$, where, applying the Fredholm alternative, ψ^*_k is the unique solution to the non-homogeneous Sturm−Liouville problem

$$\begin{cases} -\partial_{xx}\psi - k^2\psi = f \quad \text{in} \quad (0,\pi), \\ \psi(0) = \psi(\pi) = 0, \ \int_0^\pi \psi(x)\varphi_k(x)\,\mathrm{d}x = 0. \end{cases} \tag{2.3}$$

with $f := \partial_x(p(x)\varphi_k) - q(x)\varphi_k$. We recall that $\int_0^\pi f\varphi_k = 0$, since $I_k = 0$. Solving system (2.3) leads to the expression of ψ^*_k given in Proposition 2.1. The expression of α_k is given by the equality $\int_0^\pi \psi^*_k(x)\varphi_k(x)\,\mathrm{d}x = 0$ and the identity $\int_0^\pi f\varphi_k = 0$ leads to $\psi_k(\pi) = 0$. Thus, in the case $I_k(p,q) = 0$, $\lambda = k^2$ is a double eigenvalue of L^*. Items 1 and 3 are now proved.

Let us now suppose that $I_k(p,q) \neq 0$. The eigenvalue $\lambda = k^2$ is simple, $\Phi^*_{2,k} := (\varphi_k, 0)$ is an eigenfunction and a solution $\Phi^*_{1,k} := (\psi, \varphi)$ to $(L^* - k^2 Id)\Phi^*_{1,k} = I_k(p,q)\Phi^*_{2,k}$, that is

$$\begin{cases} -\partial_{xx}\psi - \partial_x(p(x)\varphi) + q(x)\varphi = k^2\psi + I_k(p,q)\varphi_k & \text{in} \quad (0,\pi), \\ -\partial_{xx}\varphi = k^2\varphi & \text{in} \quad (0,\pi), \\ \varphi(0) = \psi(0) = \varphi(\pi) = \psi(\pi) = 0, \end{cases} \tag{2.4}$$

is a generalized eigenfunction of L^*. We deduce that $\varphi = \varphi_k$ in $(0,\pi)$ and ψ is solution to the Sturm−Liouville problem (2.3) with $f = I_k(p,q)\varphi_k + \partial_x(p(x)\varphi_k) - q(x)\varphi_k$. We obtain the expression of ψ^*_k given in Proposition 2.1. □

The function ψ^*_k given in Proposition 2.1 will play an important role in this paper. Since φ_k, φ'_k and I_k are bounded, we have the following lemma

Lemma 2.2. *There exists a positive constant C such that*

$$|\alpha^*_k| \leqslant \frac{C}{k}, \qquad \|\psi^*_k\|_{L^\infty(0,\pi)} \leqslant \frac{C}{k}, \qquad \|\partial_x\psi^*_k\|_{L^\infty(0,\pi)} \leqslant C, \qquad \forall k \in \mathbb{N}^*. \tag{2.5}$$

Since the eigenvalues of the operator L^* are real, we deduce that L and L^* have the same spectrum and the associated eigenspaces have the same dimension. The eigenfunctions and the generalized eigenfunctions of L can be found as previously.

Proposition 2.3. *For all $k \in \mathbb{N}^*$ consider the two vectors*

$$\Phi_{1,k} := \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix}, \Phi_{2,k} := \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix},$$

where ψ_k is defined for all $x \in (0,\pi)$ by

$$\begin{cases} \psi_k(x) := \alpha_k\varphi_k(x) - \dfrac{1}{k}\displaystyle\int_0^x \sin(k(x-\xi))[I_k(p,q)\varphi_k(\xi) - p(\xi)\partial_x(\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]\mathrm{d}\xi, \\ \alpha_k := \dfrac{1}{k}\displaystyle\int_0^\pi\int_0^x \sin(k(x-\xi))[I_k(p,q)\varphi_k(\xi) - p(\xi)\partial_x(\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]\varphi_k(x)\mathrm{d}\xi\mathrm{d}x, \end{cases}$$

One has

(1) *The spectrum of L is given by $\sigma(L) = \sigma(L^*) = \{k^2 : k \in \mathbb{N}^*\}$.*
(2) *For $k \geqslant 1$, the eigenvalue k^2 of L is simple (algebraic multiplicity 1) if and only if $I_k(p,q) \neq 0$. In this case, $\Phi_{1,k}$ and $\Phi_{2,k}$ are an eigenfunction and a generalized eigenfunction of the operator L associated with the eigenvalue k^2, more precisely*

$$\begin{cases} (L - k^2 Id)\Phi_{1,k} = 0, \\ (L - k^2 Id)\Phi_{2,k} = I_k\Phi_{1,k}. \end{cases} \tag{2.6}$$

(3) *For $k \geqslant 1$, the eigenvalue k^2 of L is double (algebraic multiplicity 2) if and only if $I_k(p,q) = 0$. In this case, $\Phi_{1,k}$ and $\Phi_{2,k}$ are two eigenfunctions of the operator L associated with the eigenvalue k^2, that is for $i = 1,2$*

$$(L - k^2 Id)\Phi_{i,k} = 0.$$

Lemma 2.3 and Corollary 2.6 in [9] can be adapted easily to prove the following proposition.

Proposition 2.4. *Consider the families*

$$\mathcal{B} := \{\Phi_{1,k}, \Phi_{2,k} : k \in \mathbb{N}^*\} \qquad \text{and} \qquad \mathcal{B}^* := \left\{\Phi^*_{1,k}, \Phi^*_{2,k} : k \in \mathbb{N}^*\right\}.$$

Then:

(1) *The sequences* $\mathcal{B}$ *and* $\mathcal{B}^*$ *are biorthogonal Riesz bases of* $L^2(0,\pi)^2$.
(2) *The sequence* $\mathcal{B}^*$ *is a Schauder basis of* $H_0^1(0,\pi)^2$ *and* $\mathcal{B}$ *is its biorthogonal basis in* $H^{-1}(0,\pi)$.

We recall that $\mathcal{B}$ and $\mathcal{B}^*$ are biorthogonal in $L^2(0,\pi)^2$ if $\langle \Phi_{i,k}, \Phi^*_{j,l}\rangle_{L^2(0,\pi)^2} = \delta_{i,j}\delta_{k,l}$ for all $k,l \in \mathbb{N}^*$ and $i,j \in \{1,2\}$.

2.2. Duality

As it is well known, the controllability has a dual concept called *observability* (see for instance ([6], Thm. 2.1, [12], Thm. 2.44, p. 5657)). Consider the dual system associated with system (1.1)

$$\begin{cases} -\partial_t\theta - \partial_{xx}\theta + A_0^*(-\partial_x(p(x)\theta) + q(x)\theta) = 0 & \text{in} \quad Q_T, \\ \theta(0,\cdot) = \theta(\pi,\cdot) = 0 & \text{on} \quad (0,T), \\ \theta(\cdot,T) = \theta^0 & \text{in} \quad (0,\pi), \end{cases} \tag{2.7}$$

where $\theta^0 \in L^2(0,\pi)^2$. Let B the matrix given by $B = (1\ 0)^*$. The approximate controllability is equivalent to a *unique continuation property*:

Proposition 2.5.

(1) *System* (1.1) *is approximately controllable at time* T *if and only if for all initial condition* $\theta^0 \in L^2(0,\pi)^2$ *the solution to system* (2.7) *satisfies the unique continuation property*

$$\mathbb{1}_\omega B^*\theta \equiv 0 \quad \text{in} \quad Q_T \ \Rightarrow\ \theta \equiv 0 \quad \text{in} \quad Q_T. \tag{2.8}$$

(2) *System* (1.2) *is approximately controllable at time* T *if and only if for all initial condition* $\theta^0 \in H_0^1(0,\pi)^2$ *the solution to system* (2.7) *satisfies the unique continuation property*

$$B^*\partial_x\theta(0,t) \equiv 0 \quad \text{in} \quad (0,T) \ \Rightarrow\ \theta \equiv 0 \quad \text{in} \quad Q_T. \tag{2.9}$$

The null controllability is characterized by an *observability inequality*:

Proposition 2.6.

(1) *System* (1.1) *is null controllable at time* T *if and only if there exists a constant* C_{obs} *such that for all initial condition* $\theta^0 \in L^2(0,\pi)^2$ *the solution to system* (2.7) *satisfies the observability inequality*

$$\|\theta(0)\|^2_{L^2(0,\pi)^2} \leqslant C_{\text{obs}} \iint_{Q_T} |\mathbb{1}_\omega(x)B^*\theta(x,t)|^2\mathrm{d}x\mathrm{d}t. \tag{2.10}$$

(2) *System* (1.1) *is null controllable at time* T *if and only if there exists a constant* C_{obs} *such that for all initial condition* $\theta^0 \in H_0^1(0,\pi)^2$ *the solution to system* (2.7) *satisfies the observability inequality*

$$\|\theta(0)\|^2_{H_0^1(0,\pi)^2} \leqslant C_{\text{obs}} \int_0^T |B^*\partial_x\theta(0,t)|^2\mathrm{d}t. \tag{2.11}$$

3. Resolution of the moment problem

In this section, we first establish the moment problem related to the null controllability for system (1.1) and then we will solve it in Section 3.2 (Thm. 1.1). The strategy involves finding an equivalent system (see Def. 3.1) to system (1.1), which has an associated quantity I_k satisfying "some good properties".

3.1. Reduction to a moment problem

Let $y^0 := (y_1^0, y_2^0) \in L^2(0,\pi)^2$. For $i \in \{1,2\}$ and $k \in \mathbb{N}^*$, if we consider $\theta^0 := \Phi^*_{i,k}$ in the dual system (2.7), we get after an integration by parts

$$\iint_{Q_T} v(x,t)\mathbb{1}_\omega(x) B^*\theta(x,t)\mathrm{d}x\mathrm{d}t = \langle y(T), \Phi^*_{i,k}\rangle_{L^2(0,\pi)^2} - \langle y^0, \theta(0)\rangle_{L^2(0,\pi)^2}.$$

Since $\mathcal{B}^*$ is a Riesz basis of $L^2(0,\pi)^2$, system (1.1) is null controllable if and only if for all $y^0 \in L^2(0,\pi)^2$, there exists a control $v \in L^2(Q_T)$ such that for all $k \in \mathbb{N}^*$ and $i \in \{1,2\}$ the solution y to system (1.1) satisfies the following equality

$$\iint_{Q_T} v(x,t)\mathbb{1}_\omega(x) B^*\theta_{i,k}(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\langle y^0, \theta_{i,k}(0)\rangle_{L^2(0,\pi)^2}, \tag{3.1}$$

where $\theta_{i,k}$ is the solution to the dual system (2.7) with the initial condition $\theta^0 := \Phi^*_{i,k}$.

In the moment problem (3.1), we will look for a control v of the form

$$v(x,t) := f^{(1)}(x)v^{(1)}(T-t) + f^{(2)}(x)v^{(2)}(T-t) \text{ for all } (x,t) \in Q_T, \tag{3.2}$$

with $v^{(1)}, v^{(2)} \in L^2(0,T)$ and $f^{(1)}, f^{(2)} \in L^2(0,\pi)$ satisfying $\operatorname{Supp} f^{(1)} \subseteq \omega$ and $\operatorname{Supp} f^{(2)} \subseteq \omega$.

The solutions $\theta_{1,k}$ and $\theta_{2,k}$ to the dual System (2.7) with the initial condition $\Phi^*_{1,k}$ and $\Phi^*_{2,k}$ are given for all $(x,t) \in Q_T$ by

$$\begin{cases} \theta_{1,k}(x,t) = \mathrm{e}^{-k^2(T-t)}\left(\Phi^*_{1,k}(x) - (T-t)I_k(p,q)\Phi^*_{2,k}(x)\right), \\ \theta_{2,k}(x,t) = \mathrm{e}^{-k^2(T-t)}\Phi^*_{2,k}(x). \end{cases} \tag{3.3}$$

Plugging (3.2) and (3.3) in the moment problem (3.1), we get for all $k \geqslant 1$

$$\begin{cases} \widetilde{f}_k^{(1)} \displaystyle\int_0^T v^{(1)}(t)\mathrm{e}^{-k^2 t}\,\mathrm{d}t + \widetilde{f}_k^{(2)} \int_0^T v^{(2)}(t)\mathrm{e}^{-k^2 t}\,\mathrm{d}t \\ \qquad\qquad -I_k(p,q) f_k^{(1)} \displaystyle\int_0^T v^{(1)}(t)t\mathrm{e}^{-k^2 t}\,\mathrm{d}t - I_k(p,q) f_k^{(2)} \int_0^T v^{(2)}(t)t\mathrm{e}^{-k^2 t}\,\mathrm{d}t \\ \qquad\qquad\qquad = -\mathrm{e}^{-k^2 T}\left\{y^0_{1,k} - T I_k(p,q) y^0_{2,k}\right\}, \\ f_k^{(1)} \displaystyle\int_0^T v^{(1)}(t)\mathrm{e}^{-k^2 t}\,\mathrm{d}t + f_k^{(2)} \int_0^T v^{(2)}(t)\mathrm{e}^{-k^2 t}\,\mathrm{d}t = -\mathrm{e}^{-k^2 T} y^0_{2,k}, \end{cases}$$

where $f_k^{(i)}$, $\widetilde{f}_k^{(i)}$ and $y^0_{i,k}$ are given for all $i \in \{1,2\}$ and $k \in \mathbb{N}^*$ by

$$f_k^{(i)} := \int_0^\pi f^{(i)}(x)\varphi_k(x)\mathrm{d}x, \quad \widetilde{f}_k^{(i)} := \int_0^\pi f^{(i)}(x)\psi_k^*(x)\mathrm{d}x, \tag{3.4}$$

and

$$y^0_{i,k} := \langle y^0, \Phi^*_{i,k}\rangle_{L^2(0,\pi)}. \tag{3.5}$$

In ([16], Prop. 4.1), the authors proved that the family $\{e_{1,k} := \mathrm{e}^{-k^2 t}, e_{2,k} := t\mathrm{e}^{-k^2 t}\}_{k\geq 1}$ admits a biorthogonal family $\{q_{1,k}, q_{2,k}\}_{k\geq 1}$ in the space $L^2(0,T)$, *i.e.* a family satisfying

$$\int_0^T e_{i,k} q_{j,l}(t)\,\mathrm{d}t = \delta_{ij}\delta_{kl}, \quad \forall k,l \geq 1, \quad 1 \leq i,j \leq 2. \tag{3.6}$$

Moreover for all $\varepsilon > 0$ there exists a constant $C_{\varepsilon,T} > 0$ such that

$$\|q_{i,k}\|_{L^2(0,T)} \leq C_{\varepsilon,T}\,\mathrm{e}^{\varepsilon k^2}, \quad \forall k \geq 1, \quad i = 1,2. \tag{3.7}$$

We will look for $v^{(1)}$ and $v^{(2)}$ of the form

$$v^{(i)}(t) = \sum_{k\geqslant 1} \left\{ v^{(i)}_{1,k} q_{1,k}(t) + v^{(i)}_{2,k} q_{2,k}(t) \right\}, \quad i = 1,2 \tag{3.8}$$

and prove that the series converges. The moment problem (3.1) can be written as

$$A_{1,k}V_{1,k} + A_{2,k}V_{2,k} = F_k, \quad \text{for all } k \geqslant 1, \tag{3.9}$$

with for all $k \in \mathbb{N}^*$

$$A_{1,k} = \begin{pmatrix} \widetilde{f}^{(1)}_k & \widetilde{f}^{(2)}_k \\ f^{(1)}_k & f^{(2)}_k \end{pmatrix}, \quad A_{2,k} = \begin{pmatrix} -I_k(p,q) f^{(1)}_k & -I_k(p,q) f^{(2)}_k \\ 0 & 0 \end{pmatrix}, \tag{3.10}$$

$$V_{1,k} := \begin{pmatrix} v^{(1)}_{1,k} \\ v^{(2)}_{1,k} \end{pmatrix}, \quad V_{2,k} := \begin{pmatrix} v^{(1)}_{2,k} \\ v^{(2)}_{2,k} \end{pmatrix} \tag{3.11}$$

and

$$F_k = \begin{pmatrix} -\mathrm{e}^{-k^2T}\left(y^0_{1,k} - T I_k(p,q) y^0_{2,k} \right) \\ -\mathrm{e}^{-k^2T} y^0_{2,k} \end{pmatrix}. \tag{3.12}$$

3.2. Solving the moment problem

In this section, we will prove the null controllability of system (1.1) at any time T when the supports of p or q intersects the control domain ω (Thm. 1.1). In [18], the authors obtain the null controllability of system (1.1) at any time under condition (1.4), so we will not consider this case and we will always suppose that Supp $p \cap \omega \neq \varnothing$. This implies that there exists $x_0 \in \omega$ such that $p(x_0) \neq 0$. By continuity of p, we deduce that $|p| > C$ in $\widetilde{\omega}$ for a positive constant C and an open subinterval $\widetilde{\omega}$ of ω.

Definition 3.1. Let $p_1,\ p_2 \in W^1_\infty(0,\pi)$ and $q_1,\ q_2 \in L^\infty(0,\pi)$. Consider the systems given for $i \in \{1,2\}$ by

$$\begin{cases} \text{For given } y^0 \in L^2(0,\pi)^2,\ v \in L^2(Q_T), \\ \text{Find } y := (y_1,y_2) \in L^2(0,T;H^1_0(0,\pi)^2) \cap \mathcal{C}([0,T];L^2(0,\pi)^2) \text{ such that:} \\ \partial_t y_1 - \partial_{xx} y_1 = \mathbb{1}_\omega v & \text{in} \quad Q_T, \\ \partial_t y_2 - \partial_{xx} y_2 + p_i(x)\partial_x y_1 + q_i(x) y_1 = 0 & \text{in} \quad Q_T, \\ y(0,\cdot) = y(\pi,\cdot) = 0 & \text{on} \quad (0,T), \\ y(\cdot,0) = y^0 & \text{in} \quad (0,\pi). \end{cases} \tag{$\mathcal{S}_i$}$$

We say that System $(\mathcal{S}_1)$ is *equivalent* to System $(\mathcal{S}_2)$ if System $(\mathcal{S}_1)$ is null controllable at time T if and only if System $(\mathcal{S}_2)$ is null controllable at time T.

Let us present the main technique used all along this section. Suppose that system (1.1) is null controllable at time T. Let v a control such that the solution y to system (1.1) verifies $y(T) = 0$ in $(0,\pi)$ and $\omega_0 := (\alpha,\beta)$ a subinterval of $\omega = (a,b)$. Consider a function $\theta \in W^2_\infty(0,\pi)$ satisfying

$$\begin{cases} \theta \equiv \kappa_1 & \text{in} \quad (0,\alpha), \\ \theta \equiv \kappa_2 & \text{in} \quad (\beta,\pi), \\ \theta > \kappa_3 & \text{in} \quad (0,\pi), \end{cases} \tag{3.13}$$

with $\kappa_1,\ \kappa_2,\ \kappa_3 \in \mathbb{R}_+^*$. Thus if we consider the change of unknown

$$\widehat{y} := (\widehat{y}_1, y_2) \text{ with } \widehat{y}_1 := \theta^{-1} y_1, \tag{3.14}$$

then $\widehat{y}$ is solution in $L^2(0,T;H_0^1(0,\pi)^2) \cap \mathcal{C}([0,T];L^2(0,\pi)^2)$ to the system

$$\begin{cases} \partial_t \widehat{y}_1 - \partial_{xx}\widehat{y}_1 = \mathbb{1}_\omega \widehat{v} & \text{in} \quad Q_T, \\ \partial_t y_2 - \partial_{xx} y_2 + \widehat{p}\partial_x \widehat{y}_1 + \widehat{q}\widehat{y}_1 = 0 & \text{in} \quad Q_T, \\ \widehat{y}(0,\cdot) = \widehat{y}(\pi,\cdot) = 0 & \text{on} \quad (0,T), \\ \widehat{y}(\cdot,0) = \widehat{y}^0 & \text{in} \quad (0,\pi), \end{cases} \tag{3.15}$$

where the initial condition is $\widehat{y}^0 := (\theta^{-1}y_1^0, y_2^0) \in L^2(0,\pi)^2$, the control is $\widehat{v} := -\partial_{xx}(\theta^{-1})y_1 - 2\partial_x(\theta^{-1})\partial_x y_1 + \theta^{-1}\mathbb{1}_\omega v \in L^2(Q_T)$ and the coupling terms are given by $\widehat{p} := p\theta$ and $\widehat{q} := p\partial_x\theta + q\theta$. Since θ is constant in $(0,\pi)\backslash\omega_0$, we have Supp $\widehat{v} \subseteq \omega \times (0,T)$. Since y is controlled, then $\widehat{y}$ also. The converse is clearly true: starting from the controlled system (3.15) the same process leads to the construction of a controlled solution of system (1.1). Thus through the change of unknown (3.14), following Definition 3.1, systems (1.1) and (3.15) are equivalent.

The next main result of this section is Proposition 3.6 that will be introduced after some lemmas. The first of them is the following.

Lemma 3.2. *Let $p \in W_\infty^1(0,\pi) \cap W_\infty^2(\omega)$ and $q \in L^\infty(0,\pi) \cap W_\infty^1(\omega)$ with $|p| > C$ in an open subinterval $\widetilde{\omega}$ of ω for a positive constant C. There exists a subinterval $\omega_1 := (\alpha,\beta) \subset \widetilde{\omega}$ and a function $\theta \in W_\infty^2(0,\pi)$ satisfying (3.13) such that system (1.1) is equivalent to system (3.15) with $\widehat{q} \equiv 0$ in ω_1. Moreover, for all $\epsilon > 0$, the interval ω_1 can be chosen in order to obtain for all $k \in \mathbb{N}^*$*

$$|I_k(p,q) - I_k(\widehat{p},\widehat{q})| \leqslant \varepsilon. \tag{3.16}$$

Consequently, taking the limit, we deduce that $|I(p,q) - I(\widehat{p},\widehat{q})| \leqslant \varepsilon$.

Proof. Let $\omega_1 := (\alpha,\beta)$ be an interval strictly included in $\widetilde{\omega} := (\widetilde{a},\widetilde{b})$ and $\theta \in W_\infty^2(0,\pi)$ satisfying

$$\begin{cases} p\partial_x\theta + q\theta = 0 & \text{in} \quad \omega_1, \\ \theta \equiv 1 & \text{in} \quad (0,\pi)\backslash\widetilde{\omega}, \\ |\theta| > C & \text{in} \quad (0,\pi), \end{cases} \tag{3.17}$$

for a positive constant C. In the intervals $(\widetilde{a},\alpha]$ and $[\beta,\widetilde{b})$, we can take θ of class $\mathcal{C}^\infty$ in order to have $\theta \in W_\infty^2(0,\pi)$. Thus the function θ verifies (3.13) and, following the change of unknown described in (3.14), system (1.1) is equivalent to system (3.15) with $\widehat{q} \equiv 0$ in ω_1. The estimates in (3.16) are obtained taking the interval ω_1 small enough, so θ will be close to 1. □

Let us first study system (1.1) in a particular case.

Lemma 3.3. *Consider $p \in W_\infty^1(0,\pi) \cap W_\infty^2(\omega)$ and $q \in L^\infty(0,\pi) \cap W_\infty^1(\omega)$. Let us suppose that $p \equiv C \in \mathbb{R}^*$ and $q \equiv 0$ in an open subinterval $\widetilde{\omega}$ of ω. Then system (1.1) is equivalent to a system of the form (3.15) with coupling terms $\widehat{p},\ \widehat{q}$ satisfying*

$$|I_k(\widehat{p},\widehat{q})| > C/k^6,\ \ \forall k \in \mathbb{N}^*.$$

To prove this result we will need this lemma:

Lemma 3.4. *Let $(u_k)_{k\in\mathbb{N}^*}$ be a real sequence. Then there exists $\kappa \in \mathbb{R}^*_+$ such that for all $k \in \mathbb{N}^*$*

$$|u_k + \kappa| \geqslant 1/k^2.$$

Proof of Lemma 3.4. By contradiction let us suppose that for all $\kappa \in \mathbb{R}^*_+$ there exists $k \in \mathbb{N}^*$ such that $|u_k+\kappa| < 1/k^2$. Then

$$\mathbb{R}^*_+ \subseteq \bigcup_{k\in\mathbb{N}^*} (-u_k - 1/k^2, -u_k + 1/k^2). \tag{3.18}$$

The convergence of the series $\sum_{k\in\mathbb{N}^*} 1/k^2$ implies that the measure of the set in the right-hand side in (3.18) is finite and leads to the conclusion. $\square$

Proof of Lemma 3.3. Let (α, β) an open subinterval of $\widetilde{\omega}$ with α and β to be determined later, $\kappa \in \mathbb{R}^*_+$ and $\theta \in W^2_\infty(0,\pi)$ satisfying

$$\begin{cases} \theta \equiv 1 & \text{in} \quad (0,\pi)\backslash(\alpha,\beta), \\ \theta \equiv 1 + \kappa\xi & \text{in} \quad (\alpha,\beta), \end{cases} \tag{3.19}$$

where

$$\xi(x) := \left[\sin\left(\frac{\pi(x-\alpha)}{\beta-\alpha}\right)\right]^2 \text{ for all } x \in (\alpha,\beta). \tag{3.20}$$

In particular, we have $\theta \geqslant 1$ in $(0,\pi)$. Let $k \in \mathbb{N}^*$, $\widehat{y}_1 := \theta^{-1} y_1$ and $\widehat{y} := (\widehat{y}_1, y_2)$ the solution to system (3.15). For system (3.15) the quantity I_k defined in the introduction is given by

$$\begin{aligned} I_k(\widehat{p},\widehat{q}) &= \int_0^\pi \left\{\widehat{q} - \frac{1}{2}\partial_x \widehat{p}\right\} \varphi_k^2 \\ &= I_k(p,q) + \kappa J_k, \end{aligned}$$

with $\widehat{p}$, $\widehat{q}$ given by $\widehat{p} := p\theta$ and $\widehat{q} := p\partial_x\theta + q\theta$ and J_k defined by

$$J_k := \frac{1}{2}\int_\alpha^\beta \partial_x(\xi)\varphi_k^2.$$

Then, after a simple calculation, we obtain

$$J_k = \frac{\overline{\frac{2\pi}{(\beta-\alpha)^2}}}{\left(2k + \frac{2\pi}{\beta-\alpha}\right)\left(2k - \frac{2\pi}{\beta-\alpha}\right)} \sin(k(\beta+\alpha))\sin(k(\beta-\alpha)). \tag{3.21}$$

Let $n \in \mathbb{N}^*$ large enough such that $\frac{a}{n} < \frac{b}{n+1}$. There exists ℓ an algebraic number of order two, *i.e.* a root of a polynomial of degree 2 with integer coefficients, satisfying

$$\frac{a}{n} < \ell < \frac{b}{n+1} \quad \text{ and } \quad \ell \neq \frac{\pi}{j} \quad \text{ for all } j \in \mathbb{N}^*,$$

since the set of such numbers is dense in $\mathbb{R}$. Let us take $\alpha := n\ell$ and $\beta := (n+1)\ell$. Thus $\alpha,\ \beta \in (a,b)$,

$$k(\beta+\alpha) = k(2n+1)\ell \quad \text{ and } \quad k(\beta-\alpha) = k\ell. \tag{3.22}$$

Moreover

$$\left|2k + \frac{2\pi}{\beta-\alpha}\right| \times \left|2k - \frac{2\pi}{\beta-\alpha}\right| < Rk^2,$$

with $R > 0$. Since ℓ is an algebraic number of order two, using diophantine approximations it can be proved that

$$\inf_{j\geqslant 1}(j|\sin(j\ell)|) \geqslant \gamma, \tag{3.23}$$

for a positive constant γ (see [9], Eq. (5.13)). The expressions (3.21)–(3.23) give

$$|J_k| \geqslant \frac{2\pi}{(\beta-\alpha)^2}\frac{\gamma^2}{R(2n+1)k^4} \tag{3.24}$$

for all $k \in \mathbb{N}^*$. Using Lemma 3.4, there exists $\kappa \in \mathbb{R}_+^*$ satisfying

$$\left|\frac{I_k(p,q)}{J_k} + \kappa\right| \geqslant 1/k^2.$$

Combining the last inequality with estimate (3.24),

$$|I_k(\widehat{p},\widehat{q})| = |I_k(p,q) + \kappa J_k| \geqslant |J_k|/k^2 \geqslant C/k^6. \qquad \square$$

The next lemma is proved in ([9], Lem. 5.1).

Lemma 3.5. *There exist functions $f^{(1)}, f^{(2)} \in L^2(0,\pi)$ satisfying* Supp $f^{(1)}$, Supp $f^{(2)} \subseteq \omega$ *and such that for all $k \in \mathbb{N}^*$*

$$\min\left\{\left|f_k^{(1)}|, |f_k^{(2)}\right|\right\} \geqslant \frac{C}{k^3} \quad \text{and} \quad |B_k| := \left|\widehat{f}_k^{(1)} f_k^{(2)} - \widehat{f}_k^{(2)} f_k^{(1)}\right| \geqslant \frac{C}{k^5}, \tag{3.25}$$

where for $i \in \{1,2\}$ the terms $f_k^{(i)}$ and $\widehat{f}_k^{(i)}$ are given by

$$f_k^{(i)} := \int_0^\pi f^{(i)}(x)\varphi_k(x)\mathrm{d}x \quad \text{and} \quad \widehat{f}_k^{(i)} := \int_0^\pi f^{(i)}(x)\cos(kx)\mathrm{d}x. \tag{3.26}$$

With the help of Lemma 3.5, we deduce the following proposition:

Proposition 3.6. *Consider $p \in W^1_\infty(0,\pi) \cap W^2_\infty(\omega)$ and $q \in L^\infty(0,\pi) \cap W^1_\infty(\omega)$. Let us suppose that $|p| > C$ in an open subinterval $\widetilde{\omega}$ of ω for a positive constant C. Then system* (1.1) *is equivalent to a system of the form* (3.15) *with coupling terms $\widehat{p}$, $\widehat{q}$ satisfying condition* (1.10)*, $T_0(\widehat{p},\widehat{q}) = 0$ and*

$$|\det A_{1,k}| \geq \frac{C_1}{k^7}\,|I_{a,k}(\widehat{p},\widehat{q})| - \frac{C_2}{k}\,|I_k(\widehat{p},\widehat{q})| \ \ \forall k \in \mathbb{N}^*, \tag{3.27}$$

where C_1 and C_2 are two positive constants independent on k (the notion of equivalent systems is defined at the beginning of Sect. 3.2*).*

Proof. Using Lemma 3.2, without loss of generality, we can suppose that $q \equiv 0$ and $|p| > C$ in a subinterval $\widehat{\omega}$ of $\widetilde{\omega}$ for a positive constant C. If $\partial_x p \equiv 0$ in $\widehat{\omega}$, Lemma 3.3 leads to

$$|I_k(p,q)| \geqslant C/k^6, \ \forall k \in \mathbb{N}^*,$$

which implies that condition (1.10) is satisfied and the right-hand side of inequality (3.27) is negative for some appropriate constants C_1 and C_2. Otherwise, let $(\alpha,\beta) \subseteq \widehat{\omega}$ such that $\partial_x p > C$ in (α,β) or $\partial_x p < -C$ in (α,β) for a positive constant C. The rest of the proof is divided into three steps:

Step 1. If $I(p,q) := \int_0^\pi \{q - \frac{1}{2}\partial_x p\} = 0$, we will prove in this step that system (1.1) is equivalent to a system with coupling terms $\widehat{p}$, $\widehat{q}$ satisfying $I(\widehat{p},\widehat{q}) \neq 0$. Assume that $I(p,q) = 0$ and consider $\theta \in W^2_\infty(0,\pi)$ defined in (3.19), with $\kappa := 1$. We remark that $|\theta| \geqslant 1$. If we consider the change of unknown described in (3.14), then for all $k \in \mathbb{N}^*$, using the definition of I_k, we obtain

$$\begin{aligned} I_k(\widehat{p},\widehat{q}) &= I_k(p,q) + \int_\alpha^\beta \left\{\frac{1}{2}\partial_x(\xi)p - \frac{1}{2}\xi\partial_x(p)\right\}\varphi_k^2 \mathrm{d}x \\ &= I_k(p,q) + J_k(p,q), \end{aligned}$$

where

$$\begin{aligned} J_k(p,q) &= \frac{1}{2\pi}\int_\alpha^\beta \{\partial_x(\xi)p - \xi\partial_x(p)\}\{1-\cos(2kx)\}\mathrm{d}x \\ &\underset{k\to\infty}{\longrightarrow} -\frac{1}{\pi}\int_\alpha^\beta \xi\partial_x(p)\mathrm{d}x =: J(p,q). \end{aligned}$$

Using the definition of ξ given in (3.20), we get

$$\begin{aligned} |J(p,q)| &\geqslant \frac{1}{\pi}\inf_{(\alpha,\beta)}|\partial_x p|\int_\alpha^\beta \sin^2\left(\frac{\pi(x-\alpha)}{\beta-\alpha}\right)\mathrm{d}x \\ &= \frac{1}{2\pi}\inf_{(\alpha,\beta)}|\partial_x p|\int_\alpha^\beta \left\{1-\cos\left(\frac{2\pi(x-\alpha)}{\beta-\alpha}\right)\right\}\mathrm{d}x \\ &= \frac{(\beta-\alpha)}{2\pi}\inf_{(\alpha,\beta)}|\partial_x p| \neq 0. \end{aligned}$$

We recall that $I_k(p,q)\to I(p,q)=0$. Thus, we obtain $I_k(\widehat{p},\widehat{q})\to I(\widehat{p},\widehat{q}) = J(p,q)\neq 0$.

Step 2. We will show in this second step that system (1.1) is equivalent to a system with coupling terms $\widehat{p}$, $\widehat{q}$ such that $|I_k(p,q)| > C > 0$ for all $k\in\mathbb{N}^*$ satisfying $p\varphi_k$ non-constant. In view of Step 1, we can assume that $I(p,q)\neq 0$. Using Lemma 3.2, up to the change of unknown (3.14) we can also suppose that $q\equiv 0$ in an open subinterval $\widehat{\omega}$ of $\widetilde{\omega}$. Moreover, by (3.16), the function θ and $\widehat{\omega}$ can be chosen in order to keep the quantity I different of zero. Let $(\alpha,\beta)\subseteq\widehat{\omega}$ such that $|p|>C>0$ in (α,β). Since $I(p,q)\neq 0$ and $I_k(p,q)\to I(p,q)$, there exists $k_0\in\mathbb{N}^*$ such that $|I_k(p,q)|>C$ for a constant $C>0$ and all $k\geqslant k_0$. Let us define the set

$$S_0 := \{k\in\mathbb{N}^* : I_k(p,q)=0 \quad \text{and} \quad p\varphi_k \text{ non}-\text{constant} \ \text{ in } (\alpha,\beta)\}$$

and $M := \#S_0 < \infty$. Let $\theta\in W^2_\infty(0,\pi)$ satisfying

$$\begin{cases} \theta = 1+\sum_{m=1}^M \xi_m,\ |\theta|>C>0, \\ \xi_m\in W^2_\infty(0,\pi),\ \operatorname{Supp}\xi_m\subseteq(\alpha,\beta), \text{ for all } m\in\{1,\ldots,M\}, \end{cases}$$

where $\xi_1,\ldots,\xi_M$ are to be determined. Again, if we consider the change of unknown (3.14), then for all $k\in\mathbb{N}^*$, using the definition of I_k, we obtain

$$\begin{aligned} I_k(\widehat{p},\widehat{q}) &= I_k(p,q) + \sum_{m=1}^M \int_\alpha^\beta \left\{\frac{1}{2}\partial_x(\xi_m)p - \frac{1}{2}\xi_m\partial_x(p)\right\}\varphi_k^2\mathrm{d}x \\ &=: I_k(p,q) + \sum_{m=1}^M J_{m,k}(p,q). \end{aligned}$$

The goal is to choose the functions $\xi_1,\ldots,\xi_M$ such that for a constant $C>0$ we have $|I_k(\widehat{p},\widehat{q})|>C$ for all $k\in\mathbb{N}^*$ satisfying $p\varphi_k$ non-constant in (α,β). We will construct $\xi_1,\ldots,\xi_M$ from ξ_1 until ξ_M.

Let $k\in S_0$ and consider $(f_1,\xi_1)\in W^1_\infty(\alpha,\beta)\times W^2_\infty(\alpha,\beta)$ a solution to

$$\begin{cases} \frac{1}{2}\partial_x(\xi_1)p - \frac{1}{2}\xi_1\partial_x(p) = f_1 & \text{in } (\alpha,\beta), \\ \xi_1(\alpha)=\xi_1(\beta)=\partial_x\xi_1(\alpha)=\partial_x\xi_1(\beta)=0. \end{cases}$$

This system is equivalent to

$$\begin{cases} \xi_1(x) = p(x) \displaystyle\int_\alpha^x \frac{2f_1(s)}{p^2(s)} \mathrm{d}s, \quad \text{for all } x \in (\alpha, \beta), \\ \displaystyle\int_\alpha^\beta \frac{2f_1(s)}{p^2(s)} \mathrm{d}s = 0, f_1(\alpha) = f_1(\beta) = 0. \end{cases}$$

We remark that we need that $p \in W^2_\infty(\alpha, \beta)$. Finding a function f_1 satisfying

$$f_1(\alpha) = f_1(\beta) = 0, \int_\alpha^\beta \frac{2f_1(s)}{p^2(s)} \mathrm{d}s = 0 \quad \text{and} \quad J_{1,k}(p,q) = \int_\alpha^\beta f_1(s)\varphi_k^2(s)\mathrm{d}s \neq 0, \tag{3.28}$$

is equivalent to finding a function $g := 2f_1/p^2$ satisfying

$$g_1(\alpha) = g_1(\beta) = 0, \int_\alpha^\beta g_1(s)\mathrm{d}s = 0 \quad \text{and} \quad \int_\alpha^\beta g_1(s)p^2(s)\varphi_k^2(s)\mathrm{d}s \neq 0.$$

Let $\kappa_1 \in \mathbb{R}$ and define for all $j \in \mathbb{N}^*$ and all $x \in (\alpha, \beta)$

$$g_{1,j}(x) := \kappa_1 \sin\left(\frac{2\pi j(x-\alpha)}{\beta - \alpha}\right).$$

Using the fact that $p\varphi_k$ is non-constant in (α, β), without loss of generality, we can suppose that

$$\varphi_k\left(\alpha + \frac{\beta-\alpha}{4}\right) p\left(\alpha + \frac{\beta-\alpha}{4}\right) \neq \varphi_k\left(\alpha + \frac{3(\beta-\alpha)}{4}\right) p\left(\alpha + \frac{3(\beta-\alpha)}{4}\right),$$

otherwise we adapt the interval (α, β) at the beginning of Step 2. We deduce that the function h_k of $L^2(\alpha, \alpha + (\beta-\alpha)/2)$ defined by

$$h_k(s) := p^2(s)\varphi_k^2(s) - p^2(\beta+\alpha-s)\varphi_k^2(\beta+\alpha-s)$$

is not equal to zero in $(\alpha, \alpha + (\beta-\alpha)/2)$. Since $(g_{1,j})_{j\in\mathbb{N}^*}$ is a Riesz basis of $L^2(\alpha, \alpha + (\beta-\alpha)/2)$, there exists $j_1 \in \mathbb{N}^*$ such that

$$\int_\alpha^{\alpha+(\beta-\alpha)/2} g_{1,j_1}(s) \left[p^2(s)\varphi_k^2(s) - p^2(\beta+\alpha-s)\varphi_k^2(\beta+\alpha-s)\right] \mathrm{d}s \neq 0.$$

Moreover, using the fact that $g_{1,j_1}(s) = g_{1,j_1}(\beta+\alpha-s)$ $\forall s \in (\alpha, \alpha + (\beta-\alpha)/2)$, we have

$$\int_\alpha^{\alpha+(\beta-\alpha)/2} g_{1,j_1}(s)p^2(s)\varphi_k^2(s)\mathrm{d}s \neq -\int_{\alpha+(\beta-\alpha)/2}^\beta g_{1,j_1}(s)p^2(s)\varphi_k^2(s)\mathrm{d}s.$$

Thus

$$\int_\alpha^\beta g_{1,j_1}(s)p^2(s)\varphi_k^2(s)\mathrm{d}s \neq 0.$$

Plugging $g_1 := g_{1,j_1}$ and $f_1 := \dfrac{g_{1,j_1}p^2}{2}$ in (3.28), we obtain

$$J_{1,k}(p,q) = \frac{\kappa_1}{2} \int_\alpha^\beta \sin\left(\frac{2\pi j_1(s-\alpha)}{\beta-\alpha}\right) p(s)^2\varphi_k(s)^2\mathrm{d}s \neq 0.$$

We have also for all $j \in \mathbb{N}^*$

$$J_{1,j}(p,q) = \frac{\kappa_1}{2} \int_\alpha^\beta \sin\left(\frac{2\pi j_1(s-\alpha)}{\beta-\alpha}\right) p(s)^2\varphi_j(s)^2\mathrm{d}s.$$

We fix κ_1 in order to have

$$\sup_{i\in\mathbb{N}^*}|J_{1,i}(p,q)| \leqslant \frac{1}{2}\inf_{i\in\mathbb{N}^*\setminus S_0}|I_i(p,q)|.$$

Let $m\in\{2,\dots,M\}$ and let us assume that $\xi_1,\dots,\xi_{m-1}$ are already constructed. Consider the set

$$S_{m-1} := \left\{k\in\mathbb{N}^* : I_k(p,q)+\sum_{j=1}^{m-1}J_{j,k}(p,q)=0 \text{ and } p\varphi_k \text{ non}-\text{constant in } (\alpha,\beta)\right\}.$$

If $S_{m-1}=\varnothing$, then we take $\xi_m=0$ in $(0,\pi)$. Otherwise, let $k\in S_{m-1}$ and consider $(f_m,\xi_m)\in W^1_\infty(\alpha,\beta)\times W^2_\infty(\alpha,\beta)$ a solution to

$$\begin{cases} \frac{1}{2}\partial_x(\xi_m)p-\frac{1}{2}\xi_m\partial_x(p)=f_m & \text{in } (\alpha,\beta),\\ \xi_m(\alpha)=\xi_m(\beta)=\partial_x\xi_m(\alpha)=\partial_x\xi_m(\beta)=0. \end{cases}$$

This system is equivalent to

$$\begin{cases} \xi_m(x)=p(x)\int_\alpha^x\frac{2f_m(s)}{p^2(s)}\mathrm{d}s, & \text{for all } x\in(\alpha,\beta),\\ \int_\alpha^\beta\frac{2f_m(s)}{p^2(s)}\mathrm{d}s=0, f_m(\alpha)=f_m(\beta)=0. \end{cases}$$

Let $\kappa_m>0$. Again, there exists $j_m\in\mathbb{N}^*$ such that the function f_m given for all $x\in(\alpha,\beta)$ by

$$f_m(x):=\frac{\kappa_m}{2}\sin\left(\frac{2\pi j_m(x-\alpha)}{\beta-\alpha}\right)p(x)^2$$

is solution to this system. Then, we obtain

$$J_{m,j}(p,q)=\frac{\kappa_m}{2}\int_\alpha^\beta\sin\left(\frac{2\pi j_m(s-\alpha)}{\beta-\alpha}\right)p(s)^2\varphi_j(s)^2\mathrm{d}s.$$

The last quantity is different of zero for $j=k$. Let us fix κ_m in order to have

$$\sup_{i\in\mathbb{N}^*}|J_{m,i}(p,q)|\leqslant\frac{1}{2}\inf_{i\in\mathbb{N}^*\setminus S_{m-1}}\left|I_i(p,q)+\sum_{j=1}^{m-1}J_{j,i}(p,q)\right|.$$

Thus, after constructing the functions $\xi_1,\dots,\xi_M$, the obtained functions $\widehat{p}$ and $\widehat{q}$ are such that

$$|I_k(\widehat{p},\widehat{q})|>C \text{ for all } k\in\mathbb{N}^* \text{ satisfying } \widehat{p}\varphi_k \text{ non-constant in } (\alpha,\beta),$$

where C is a positive constant which does not depend on k.

Step 3. Finally, in this third step, we will prove that system (1.1) is equivalent to a system satisfying $T_0(\widehat{p},\widehat{q})=0$ and conditions (1.10) and (3.27). In view of Step 2, we can assume that

$$|I_k(p,q)|>C \text{ for all } k\in\mathbb{N}^* \text{ satisfying } p\varphi_k \text{ non-constant in } (\alpha,\beta),$$

where C is a positive constant which does not depend on k. If $|I_k(p,q)|>C_0$ for all $k\in\mathbb{N}^*$ and a constant $C_0>0$, then condition (1.10) is satisfied and the right-hand side of inequality (3.27) is negative for some appropriate constants C_1 and C_2. Let us now suppose that, for a $m\in\mathbb{N}^*$, we have

$$I_m(p,q)=0 \text{ and } p\varphi_m \text{ constant in } (\alpha,\beta). \tag{3.29}$$

Again, using Lemma 3.2, up to the change of unknown (3.14) described at the beginning of the section we can also suppose that $q \equiv 0$ in a subinterval (α,β) of $\widetilde{\omega}$. Moreover, using (3.16), this change of unknown can be chosen in order to keep the property: $|I_k(p,q)| > C > 0$ for all $k \in \mathbb{N}^*\backslash\{m\}$. Let $m \in \mathbb{N}^*$ such that $I_m(p,q) = 0$ and $p\varphi_m$ is constant in (α,β), otherwise we argue as in Step 2. Let $\theta \in W^2_\infty(0,\pi)$ satisfying

$$\begin{cases} \theta = 1+\xi \quad \text{in} \quad (0,\pi), \\ \xi \in W^2_\infty(0,\pi), |\theta| > C > 0, \\ \xi \equiv \xi_\alpha \in \mathbb{R}^*_+ \quad \text{in} \quad (0,\alpha), \\ \xi \equiv 0 \quad \text{in} \quad (\beta,\pi), \end{cases}$$

Again, if we consider the change of unknown described in (3.14), then for all $k \in \mathbb{N}^*$

$$\begin{aligned} I_k(\widehat{p},\widehat{q}) &= I_k(p,q) + \int_0^\beta \left\{ \frac{1}{2}\partial_x(\xi)p + \xi q - \frac{1}{2}\xi\partial_x(p) \right\} \varphi_k^2 \mathrm{d}x \\ &=: I_k(p,q) + J_k(p,q). \end{aligned}$$

We will distinguish the cases $I_{\alpha,m}(p,q) = 0$ and $I_{\alpha,m}(p,q) \neq 0$ (see (1.8) for the definition of this quantity) for the new control domain $\omega := (\alpha,\beta)$.

Case 1. Assume that $I_{\alpha,m}(p,q) = 0$. Let $(\xi,h) \in W^2_\infty(\alpha,\beta) \times W^1_\infty(\alpha,\beta)$ be a solution to the system

$$\begin{cases} \frac{1}{2}\partial_x(\xi)p - \frac{1}{2}\xi\partial_x(p) = h & \text{in } (\alpha,\beta), \\ \xi(\beta) = \partial_x\xi(\alpha) = \partial_x\xi(\beta) = 0, \ \xi(\alpha) = \xi_\alpha \in \mathbb{R}^*. \end{cases}$$

This system is equivalent to

$$\begin{cases} \xi(x) = -p(x)\int_x^\beta \frac{2h(s)}{p^2(s)}\mathrm{d}s, \quad \text{for all } x \in (\alpha,\beta), \\ \int_\alpha^\beta \frac{2h(s)}{p^2(s)}\mathrm{d}s = \frac{-\xi_\alpha}{p(\alpha)}, \ h(\alpha) = \frac{-\xi_\alpha \partial_x p(\alpha)}{2}, h(\beta) = 0. \end{cases}$$

Taking into account that $I_{\alpha,m}(p,q) = 0$, $q \equiv 0$ in (α,β) and $p\varphi_m \equiv \gamma$ in (α,β) for a $\gamma \in \mathbb{R}^*$, one gets

$$J_m(p,q) = \xi_\alpha \int_0^\alpha \left(q - \frac{1}{2}\partial_x(p) \right) \varphi_m^2 \mathrm{d}x + \frac{\gamma^2}{2}\int_\alpha^\beta \partial_x\left(\frac{\xi}{p}\right)\mathrm{d}x = -\frac{\gamma^2\xi_\alpha}{2p(\alpha)} \neq 0.$$

Let ξ_α and h be such that

$$\sup_{k\in\mathbb{N}^*} |J_k(p,q)| \leqslant \frac{1}{2}\inf_{k\in\mathbb{N}^*\backslash\{m\}} |I_k(p,q)|.$$

Then $|I_k(\widehat{p},\widehat{q})| > C$ for all $k \in \mathbb{N}^*$ and a positive constant C. Thus condition (1.10) is satisfied and the right-hand side of inequality (3.27) is negative for some appropriate constants C_1 and C_2.

Case 2. Let us now assume that $I_{\alpha,m}(p,q) \neq 0$. Then condition (1.10) is verified. We recall that, in the moment problem described in the last section, we have

$$\det A_{1,m} = \widetilde{f}_m^{(1)} f_m^{(2)} - \widetilde{f}_m^{(2)} f_m^{(1)},$$

where $f_m^{(1)}$, $f_m^{(2)}$, $\widetilde{f}_m^{(1)}$ and $\widetilde{f}_m^{(2)}$ are given in (3.4). Since $p\varphi_m$ is constant in (α,β), the function ψ_m^* of Proposition 2.1 reads for all $x\in(\alpha,\beta)$

$$\begin{cases} \psi_m^*(x) = \alpha_m^*\varphi_m - \dfrac{1}{m}\displaystyle\int_0^\alpha \sin(m(x-\xi))[\partial_x(p(\xi)\varphi_m(\xi)) - q(\xi)\varphi_m(\xi)]\mathrm{d}\xi \\ \qquad\quad = \tau_m\varphi_m(x) - \sqrt{\dfrac{\pi}{2}}\dfrac{1}{m}I_{\alpha,m}(p,q)\cos(mx), \\ \tau_m := \alpha_m^* - \sqrt{\dfrac{\pi}{2}}\dfrac{1}{m}\displaystyle\int_0^\alpha \cos(m\xi)[\partial_x(p(\xi)\varphi_m(\xi)) - q(\xi)\varphi_m(\xi)]\mathrm{d}\xi. \end{cases}$$

We deduce that

$$\det A_{1,m} = -\sqrt{\frac{\pi}{2}}\frac{1}{m}I_{\alpha,m}(p,q)\left(\widehat{f}_m^{(1)}f_m^{(2)} - \widehat{f}_m^{(2)}f_m^{(1)}\right),$$

where $\widehat{f}_m^{(1)}$ and $\widehat{f}_m^{(2)}$ are given in Lemma 3.5. Using Lemma 3.5, we obtain $\det A_{1,m}\neq 0$. Thus, for C_1 small enough (3.27) is true for $k=m$ and, for all $k\neq m$, the right-hand side of (3.27) is negative for C_2 be enough.

We conclude this proof remarking that, in each case, there exists $C>0$ and $k_0\in\mathbb{N}^*$ such that, for all $k\geqslant k_0$, we have $|I_k(\widehat{p},\widehat{q})|\geqslant C/k^6$, which implies that $T_0(\widehat{p},\widehat{q})=0$. □

We recall that $T_0(p,q)$ is given by (1.12). Before proving Theorem 1.1, we will establish the following proposition which is true even in the case where the coupling region and the control domain are disjoint.

Proposition 3.7. *Assume that conditions* (1.10) *and* (3.27) *hold and* $T>T_0(p,q)$.
Then system (1.1) *is null controllable at time* T.

Proof. We will use the same strategy than [9]. Let $\varepsilon>0$. Using the definition of the minimal time $T_0(p,q)$ in (1.12), there exists a positive integer k_ε for which

$$\min\left\{\log|I_{a,k}(p,q)|^{-1}, \log|I_k(p,q)|^{-1}\right\} < k^2(T_0(p,q)+\varepsilon), \quad \forall k>k_\varepsilon. \tag{3.30}$$

The goal is to solve the moment problem described in Section 3.1. We recall that we look for a control v of the form (3.2) and (3.8) with $f^{(1)}$ and $f^{(2)}$ defined in Lemma 3.5. We will solve the moment problem (3.9) depending on whether k belongs to Λ_1, Λ_2 or Λ_3, where

$$\begin{cases} \Lambda_1 := \{k\in\mathbb{N}^* : I_k(p,q)\neq 0,\ I_{a,k}(p,q)\neq 0\}, \\ \Lambda_2 := \{k\in\mathbb{N}^* : I_k(p,q)\neq 0,\ I_{a,k}(p,q)= 0\}, \\ \Lambda_3 := \{k\in\mathbb{N}^* : I_k(p,q)= 0,\ I_{a,k}(p,q)\neq 0\}. \end{cases}$$

Case 1. Consider the case $k\in\Lambda_1$ with $k\leq k_\varepsilon$.
Let us take $v_{1,k}^{(2)}=v_{2,k}^{(2)}=0$. The moment problem (3.9) becomes

$$\begin{cases} \widetilde{f}_k^{(1)}v_{1,k}^{(1)} - I_k(p,q)f_k^{(1)}v_{2,k}^{(1)} = -\mathrm{e}^{-k^2T}\left(y_{1,k}^0 - TI_k(p,q)y_{2,k}^0\right), \\ f_k^{(1)}v_{1,k}^{(1)} = -\mathrm{e}^{-k^2T}y_{2,k}^0. \end{cases}$$

Since $I_k(p,q)\neq 0$ and using the estimate of $f_k^{(1)}$ and $f_k^{(2)}$ in Lemma 3.5, the last system has a unique solution

$$\begin{cases} v_{1,k}^{(1)} = -\mathrm{e}^{-k^2T}\dfrac{y_{2,k}^0}{f_k^{(1)}}, \\ v_{2,k}^{(1)} = \dfrac{\mathrm{e}^{-k^2T}}{I_k(p,q)f_k^{(1)}}\left(y_{1,k}^0 - TI_k(p,q)y_{2,k}^0 - \widetilde{f}_k^{(1)}\dfrac{y_{2,k}^0}{f_k^{(1)}}\right). \end{cases} \tag{3.31}$$

Moreover, since the set of the k considered in this case is finite, we get the inequality

$$\left|v_{j,k}^{(i)}\right| \leq C_\varepsilon \mathrm{e}^{-k^2 T} \|y^0\|_{L^2(0,\pi)^2}, \quad i,j = 1,2. \tag{3.32}$$

Case 2. Let $k \in \Lambda_1$ such that $k > k_\varepsilon$ and $|I_k(p,q)|^{-1} \leq \mathrm{e}^{k^2(T_0(p,q)+2\varepsilon)}$.
As in the previous case, we take $v_{1,k}^{(2)} = v_{2,k}^{(2)} = 0$ and the moment problem (3.9) has a unique solution, given by (3.31). Thanks to the property of ψ_k^* (see (2.5)) and Lem. 3.5, we get for $i = 1,2$ the following estimates

$$|f_k^{(1)}| \geqslant C/k^3, \quad |\widetilde{f}_k^{(i)}| \leqslant \frac{C}{k}, \; |y_{i,k}^0| \leqslant C\|y^0\|_{L^2(0,\pi)^2}, \; \forall k \in \mathbb{N}^*. \tag{3.33}$$

Thus, using the assumptions on k, we obtain

$$\begin{cases} |v_{1,k}^{(1)}| \leq Ck^3 \mathrm{e}^{-k^2 T} \|y^0\|_{L^2(0,\pi)^2} \leq C_\varepsilon \mathrm{e}^{-(T-\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \\ |v_{1,k}^{(2)}| \leq \dfrac{C_\varepsilon \mathrm{e}^{-(T-\varepsilon)k^2}}{|I_k(p,q)|} \|y^0\|_{L^2(0,\pi)^2} \leq C_\varepsilon \mathrm{e}^{-(T-T_0-3\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \end{cases}$$

where C_ε is a constant which is independent on k and y^0.

Case 3. Consider now $k \in \Lambda_1$ such that $k > k_\varepsilon$ and $|I_k(p,q)|^{-1} > \mathrm{e}^{k^2(T_0(p,q)+2\varepsilon)}$.
This implies with (3.30) that

$$|I_{a,k}(p,q)|^{-1} < \mathrm{e}^{k^2(T_0(p,q)+\varepsilon)}. \tag{3.34}$$

The two last inequalities lead to

$$|I_k(p,q)| < \mathrm{e}^{-\varepsilon k^2} \left|I_{a,k}(p,q)\right|.$$

Combined with inequality (3.27), taking k_ε large enough, we get

$$|\det A_{1,k}| > C_\varepsilon \mathrm{e}^{-\varepsilon k^2} \left|I_{a,k}(p,q)\right|, \tag{3.35}$$

with C_ε independent on k. To solve the moment problem (3.9), we take here $v_{2,k}^{(1)} = v_{2,k}^{(2)} = 0$. Then the moment problem (3.9) reads $A_{1,k}V_{1,k} = F_k$. Since $\det A_{1,k} \neq 0$ and using (3.35), the inverse of $A_{1,k}$ is given by

$$(A_{1,k})^{-1} = (\det\, A_{1,k})^{-1} \begin{pmatrix} f_k^{(2)} & -\widetilde{f}_k^{(2)} \\ -f_k^{(1)} & \widetilde{f}_k^{(1)} \end{pmatrix}.$$

We deduce that the solution to the moment problem (3.9) is

$$\begin{cases} v_{1,k}^{(1)} = \dfrac{\mathrm{e}^{-k^2 T}}{\det\, A_{1,k}} \left\{ -f_k^{(2)} y_{1,k}^0 + (TI_k(p,q) f_k^{(2)} + \widetilde{f}_k^{(2)}) y_{2,k}^0 \right\}, \\ v_{1,k}^{(2)} = \dfrac{\mathrm{e}^{-k^2 T}}{\det\, A_{1,k}} \left\{ f_k^{(1)} y_{1,k}^0 - (TI_k(p,q) f_k^{(1)} + \widetilde{f}_k^{(1)}) y_{2,k}^0 \right\}. \end{cases}$$

The last expression together with (3.34) and (3.35) gives

$$|v_{1,k}^{(i)}| \leq C_\varepsilon \mathrm{e}^{-(T-T_0-2\varepsilon)k^2} \|y^0\|_{L^2(0,\pi)^2}, \quad i = 1,2. \tag{3.36}$$

Case 4. Let us consider $k \in \Lambda_2$.
If $k \leq k_\varepsilon$, we can argue as in Case 1. Let us suppose that $k > k_\varepsilon$. In this case, $I_{a,k}(p,q) = 0$, $I_k(p,q) \neq 0$ and

inequality (3.30) reads $|I_k(p,q)|^{-1} < \mathrm{e}^{k^2(T_0(p,q)+\varepsilon)}$. We take here $v_{1,k}^{(2)} = v_{2,k}^{(2)} = 0$ and the solution of moment problem (3.9) is given by (3.31). We get

$$\left|v_{j,k}^{(i)}\right| \leq C_\varepsilon \mathrm{e}^{-k^2(T-T_0(p,q)-2\varepsilon)}\|y^0\|_{L^2(0,\pi)^2}, \quad i,j=1,2.$$

Case 5. Let us now deal with the case $k \in \Lambda_3$.
We recall that $I_k(p,q)=0$, $I_{1,k}(p,q)\neq 0$ and inequality (3.30) reads

$$|I_{a,k}(p,q)|^{-1} < \mathrm{e}^{k^2(T_0(p,q)+\varepsilon)}. \tag{3.37}$$

The moment problem (3.9) is now $A_{1,k}V_{1,k} = F_k$ with $A_{1,k}$ and F_k given in (3.10) and (3.12), respectively. From (3.27), the matrix $A_{1,k}$ is invertible and

$$\begin{cases} v_{1,k}^{(1)} = \dfrac{\mathrm{e}^{-k^2T}}{\det\, A_{1,k}}\left\{-f_k^{(2)}y_{1,k}^0 + \widetilde{f}_k^{(2)}y_{2,k}^0\right\}, \\ v_{1,k}^{(2)} = \dfrac{\mathrm{e}^{-k^2T}}{\det\, A_{1,k}}\left\{f_k^{(1)}y_{1,k}^0 - \widetilde{f}_k^{(1)}y_{2,k}^0\right\}. \end{cases}$$

Using inequalities (3.27) and (3.37), we obtain estimate (3.36).

4. Conclusion

We have constructed a control v of the form (3.2) and (3.8), which satisfies

$$\left|v_{j,k}^{(i)}\right| \leq C_\varepsilon \mathrm{e}^{-k^2(T-T_0(p,q)-3\varepsilon)}\|y^0\|_{L^2(0,\pi)^2}, \quad i,j=1,2,\ k\in\mathbb{N}^*.$$

The last inequality, the estimate (3.7) of $q_{i,k}$ and the expression (3.8) of $v^{(i)}$ $(i=1,2)$ lead to

$$\left\|v^{(i)}\right\|_{L^2(0,T)} \leq C_{\varepsilon,T}\mathrm{e}^{-k^2(T-T_0(p,q)-4\varepsilon)},\ i=1,2.$$

Thus, taking $\varepsilon \in (0,(T-T_0(p,q))/4)$, we have the absolute convergence of the series defining $v^{(1)}$ and $v^{(2)}$ in $L^2(0,T)$. This ends the proof. □

Proof of Theorem 1.1. Using Proposition 3.6, system (1.1) is equivalent to a system with coupling terms $\widehat{p}$ and $\widehat{q}$ satisfying condition (1.10) and (3.27). Proposition 3.7 leads to the null controllability of system (1.1) when $T > T_0(\widehat{p},\widehat{q})$. We end the proof of Theorems 1.1 remarking that $T_0(\widehat{p},\widehat{q})=0$. □

5. Proof of Theorem 1.4

5.1. Positive null controllability result

Before studying the case where the intersection of the coupling and control domains is empty, we will first rewrite the function ψ_k^* given in Proposition 2.1.

Lemma 5.1. *Let* $k\in\mathbb{N}^*$. *Consider the function* ψ_k^* *defined in Proposition* 2.1. *If we suppose that condition* (1.11) *holds, then for all* $x\in\omega$

$$\psi_k^*(x) = \tau_k\varphi_k(x) + g_k(x) \text{ for all } x\in\omega,$$

where

$$\begin{cases} \tau_k := \alpha_k^* - \sqrt{\dfrac{\pi}{2}}\dfrac{1}{k}\displaystyle\int_0^a \cos(k\xi)[\partial_x(p(\xi)\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]\mathrm{d}\xi, \\ g_k(x) := -\dfrac{I_k(p,q)}{k}\displaystyle\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\mathrm{d}\xi - \sqrt{\dfrac{\pi}{2}}\dfrac{1}{k}I_{a,k}(p,q)\cos(kx). \end{cases}$$

Proof. Since $p = q \equiv 0$ in ω, we get for all $x \in \omega$,

$$\psi_k^*(x) = \alpha_k^* \varphi_k(x) - \frac{I_k(p,q)}{k} \int_0^x \sin(k(x-\xi))\varphi_k(\xi)\,\mathrm{d}\xi - \frac{1}{k}\int_0^a \sin(k(x-\xi))[\partial_x(p(\xi)\varphi_k(\xi)) - q(\xi)\varphi_k(\xi)]\mathrm{d}\xi. \quad \square$$

Proof of Theorem 1.4. We will follow the strategy of [9]. More precisely, we will prove Theorem 1.3 with the help of Proposition 3.7. Assume that conditions (1.10) and (1.11) hold. Consider the functions $f^{(1)}$ and $f^{(2)}$ defined in Lemma 3.5 and the matrix $A_{1,k}$ given in (3.10). Let $k \in \mathbb{N}^*$. We recall that

$$\det A_{1,k} = \widetilde{f}_k^{(1)} f_k^{(2)} - \widetilde{f}_k^{(2)} f_k^{(1)},$$

where, for $i = 1, 2$, $f_k^{(i)}$ and $\widetilde{f}_k^{(i)}$ are defined in (3.4). Since Supp $f^{(i)} \subseteq \omega$, using the expression of ψ_k^* given in Lemma 5.1, we obtain

$$\widetilde{f}_k^{(i)} = \tau_k f_k^{(i)} + \int_0^\pi f^{(i)}(x)\, g_k(x)\,\mathrm{d}x,$$

where for all $x \in \omega$

$$g_k(x) = -\frac{I_k(p,q)}{k}\int_0^x \sin(k(x-\xi))\varphi_k(\xi)\mathrm{d}\xi - \sqrt{\frac{\pi}{2}}\frac{1}{k} I_{a,k}(p,q)\cos(kx).$$

We deduce that

$$\begin{aligned}
\det A_{1,k} &= f_k^{(2)} \int_0^\pi f^{(1)}(x)\, g_k(x)\mathrm{d}x - f_k^{(1)} \int_0^\pi f^{(2)}(x)\, g_k(x)\,\mathrm{d}x \\
&= -\frac{I_k(p,q)}{k}\left(f_k^{(2)} \int_0^\pi \int_0^x f^{(1)}(x) \sin(k(x-\xi))\varphi_k(\xi)\mathrm{d}\xi\,\mathrm{d}x \right. \\
&\quad \left. - f_k^{(1)} \int_0^\pi \int_0^x f^{(2)}(x)\sin(k(x-\xi))\varphi_k(\xi)\mathrm{d}\xi\,\mathrm{d}x \right) - \sqrt{\frac{\pi}{2}}\frac{1}{k} I_{a,k}(p,q)\left(\widehat{f}_k^{(1)} f_k^{(2)} - \widehat{f}_k^{(2)} f_k^{(1)}\right),
\end{aligned}$$

where $\widehat{f}_k^{(i)}$ are defined in (3.26). Since the integrals

$$\int_0^\pi \int_0^x f^{(i)}(x)\sin(k(x-\xi))\varphi_k(\xi)\,\mathrm{d}\xi\,\mathrm{d}x$$

and the sequence $(f_k^{(i)})_{k\in\mathbb{N}^*, i\in\{1,2\}}$ are uniformly bounded with respect to k and i, we conclude with the help of Lemma 3.5.

We deduce that condition (3.27) holds. Thus, using Proposition 3.7, system (1.1) is null controllable at time T. $\square$

5.2. Negative null controllability result

Let us now prove the negative part of Theorem 1.3 with the strategy used in [9]. Let $T < T_0(p,q)$. We will argue by contradiction: assume that system (1.1) is null controllable at time T. Using Proposition 2.6, there exists a constant $C_{\mathrm{obs}} > 0$ such that for all $\theta^0 \in L^2(0,\pi)^2$, the solution to system (2.7) satisfies the observability inequality

$$\|\theta(0)\|^2_{L^2(0,\pi)^2} \leqslant C_{\mathrm{obs}} \iint_{Q_T} |\mathbb{1}_\omega(x) B^*\theta(x,t)|^2 \mathrm{d}x\mathrm{d}t. \tag{5.1}$$

Using the Definition of $T_0(p,q)$ (see (1.12)) there exists a strictly increasing sequence $(k_n)_{n\in\mathbb{N}^*} \subseteq \mathbb{N}$ satisfying:

$$T_0(p,q) = \lim_{n\to\infty} \frac{\min\left(\log\left|I_{a,k_n}(p,q)^{-1}\right|, \log\left|I_{k_n}(p,q)^{-1}\right|\right)}{k_n^2}. \tag{5.2}$$

Let us fix $n \geq 1$ and $\theta^{0n} := a_n \Phi^*_{1,k_n} + b_n \Phi^*_{2,k_n}$ with $(a_n, b_n) \in \mathbb{R}^2$ to be determined later and Φ^*_{2,k_n}, Φ^*_{1,k_n} the eigenfunction and generalized eigenfunction associated with k_n^2 given in Proposition 2.1. If we denote by θ^n the solution to the dual system (2.7) for initial data θ^{0n}, then

$$\theta^n(x,t) = \mathrm{e}^{-k_n^2(T-t)} \left\{ a_n \Phi^*_{1,k_n} + (b_n - (T-t) I_{k_n}(p,q) a_n) \Phi^*_{2,k_n} \right\},$$

thus, using the orthogonality $\langle \psi^*_{k_n}, \varphi_{k_n} \rangle_{L^2(0,\pi)} = 0$, we have

$$\begin{cases} D_{1,n} := \|\theta^n(0)\|^2_{L^2(0,\pi)^2} = \mathrm{e}^{-2k_n^2 T} \left\{ |a_n|^2 |\psi^*_{k_n}|^2 + (b_n - T I_{k_n}(p,q) a_n)^2 + |a_n|^2 \right\} \\ D_{2,n} := \iint_{Q_T} |\mathbb{1}_\omega(x) B^* \theta^n(x,t)|^2 \mathrm{d}x\mathrm{d}t = \displaystyle\int_0^T \int_\omega \mathrm{e}^{-2k_n^2 t} \left| a_n \psi^*_{k_n}(x) + (b_n - t I_{k_n}(p,q) a_n) \varphi_{k_n}(x) \right|^2 \mathrm{d}x\,\mathrm{d}t. \end{cases}$$

The observability inequality (5.1) reads

$$D_{1,n} \leqslant C_{\mathrm{obs}} D_{2,n}. \tag{5.3}$$

By choosing $a_n := 1$ and $b_n := -\tau_{k_n}$, we get

$$D_{1,n} \geqslant \mathrm{e}^{-2k_n^2 T} \tag{5.4}$$

and the expression of $\psi^*_{k_n}(x)$ given in Lemma 5.1 leads to

$$\begin{aligned} D_{2,n} = &\int_0^T \int_\omega \mathrm{e}^{-2k_n^2 t} \Bigg| -\sqrt{\frac{\pi}{2}} \frac{1}{k_n} I_{a,k_n}(p,q) \cos(k_n x) \\ &-I_{k_n}(p,q) \frac{1}{k_n} \int_0^x \sin\left(k_n(x-\xi)\right) \varphi_{k_n}(\xi)\,\mathrm{d}\xi - t I_{k_n}(p,q) \varphi_{k_n}(x) \Bigg|^2 \mathrm{d}x\,\mathrm{d}t \\ \leqslant &\, C(I_{a,k_n}(p,q)^2 + I_{k_n}(p,q)^2). \end{aligned}$$

Let $\varepsilon > 0$. Equality (5.2) implies that there is $k_\varepsilon \in \mathbb{N}^*$ such that for all $k_n \geqslant k_\varepsilon$

$$\max\left(|I_{a,k_n}(p,q)|^2, |I_{k_n}(p,q)|^2 \right) \leqslant \mathrm{e}^{-2k_n^2 (T_0(p,q) - \varepsilon)}.$$

We deduce that for $\varepsilon := (T_0(p,q) - T)/2$, we get

$$D_{2,n} \leqslant C \mathrm{e}^{-2k_n^2 (T+\varepsilon)}. \tag{5.5}$$

Thus, since k_n goes to ∞, estimates (5.4) and (5.5) are in contradiction with inequality (5.3) for n large enough.

6. Proof of Theorem 1.2

We will proved Theorem 1.2 using the criterion of Fattorini, as in the pioneer work [22].

Theorem 6.1 (See [15], Cor. 3.3)**.** *System* (1.1) *is approximatively controllable at time T if and only if for any $s \in \mathbb{C}$ and for any $u \in \mathcal{D}(L^*)$ we have*

$$\left.\begin{array}{lll} L^* u = su & \text{in} & (0,\pi) \\ B^* u = 0 & \text{in} & \omega \end{array}\right\} \Rightarrow u = 0.$$

Proof of Theorem 1.2.

Necessary condition: Let us suppose that conditions (1.9)–(1.10) do not hold *i.e.* there exists $k_0 \in \mathbb{N}^*$ such that

$$I_{k_0}(p,q) = I_{a,k_0}(p,q) = 0 \text{ and } (\mathrm{Supp}\ p \cup \mathrm{Supp}\ q) \cap \omega = \varnothing.$$

We remark that the function $\psi_{k_0}^*$ of Lemma 5.1 satisfy $\psi_{k_0}^* = \tau_{k_0}\varphi_{k_0}$ in ω, then

$$\Phi_{1,k_0}^* - \tau_{k_0}\Phi_{2,k_0}^* = \begin{pmatrix} 0 \\ \varphi_k \end{pmatrix} \text{ in } \omega.$$

We deduce that $\Phi_{1,k_0}^* - \tau_{k_0}\Phi_{2,k_0}^*$ is an non-trivial eigenfunction associated with the eigenvalue k_0^2 of the operator L^* satisfying

$$B^*(\Phi_{1,k_0}^* - \tau_{k_0}\Phi_{2,k_0}^*) \equiv 0 \qquad \text{in} \quad \omega.$$

Thus, using Theorem 6.1, system (1.1) is not approximately controllable at time T.

Sufficient condition: Let us suppose that conditions (1.9)−(1.10) hold. If $(\operatorname{Supp} p \cup \operatorname{Supp} q) \cap \omega \neq \varnothing$, then we conclude using Theorem 1.1. Let us now suppose that

$$(\operatorname{Supp} p \cup \operatorname{Supp} q) \cap \omega = \varnothing \qquad \text{and} \qquad |I_k(p,q)| + |I_{a,k}(p,q)| \neq 0 \text{ for all } k \in \mathbb{N}^*.$$

If $I_k(p,q) \neq 0$, the set of the eigenvectors associated with the eigenvalue k^2 of L^* is generated by $\Phi_{2,k}^*$ (see Prop. 2.1). In this case, we remark that for all $k \in \mathbb{N}^*$

$$B^*\Phi_{2,k}^* = \varphi_k \not\equiv 0 \qquad \text{in} \quad \omega. \tag{6.1}$$

If $I_k(p,q) = 0$, the eigenvectors associated with the eigenvalue k^2 of L^* are linear combinations of $\Phi_{1,k}^*$ and $\Phi_{2,k}^*$. Let $\alpha, \beta \in \mathbb{R}$ and $\Phi^* := \alpha\Phi_{1,k}^* + \beta\Phi_{2,k}^*$ satisfying

$$B^*\Phi^* \equiv 0 \qquad \text{in} \quad \omega. \tag{6.2}$$

Using Lemma 5.1, it is equivalent to

$$(\alpha + \beta\tau_k)\varphi_k(x) - \beta\sqrt{\frac{\pi}{2}}\frac{1}{k}I_{a,k}(p,q)\cos(kx) = 0 \text{ for all } x \in \omega.$$

Since $I_{a,k}(p,q) \neq 0$, we deduce that $\beta = 0$. Then $\alpha = 0$. We conclude with the help of Theorem 6.1. □

7. Proof of Theorem 1.5

As in Section 3.1, system (1.2) is null controllable at time T if and only if for all $y^0 \in H^{-1}(0,\pi)^2$, $k \in \mathbb{N}^*$ and $i \in \{1,2\}$ the solution $\theta_{i,k}$ to the dual system (2.7) for the initial data $\Phi_{i,k}^*$ satisfies

$$\int_0^T u(t)B^*\partial_x\theta_{i,k}(0,t)\mathrm{d}t = -\langle y^0, \theta_{i,k}(\cdot,0)\rangle_{H^{-1},H_0^1}. \tag{7.1}$$

We recall that, for all $k \in \mathbb{N}^*$, $\theta_{1,k}$ and $\theta_{2,k}$ are given for all $(x,t) \in Q_T$ by

$$\theta_{1,k}(x,t) = \mathrm{e}^{-k^2(T-t)}\left(\Phi_{1,k}^*(x) - (T-t)I_k(p,q)\Phi_{2,k}^*(x)\right) \text{ and } \theta_{2,k}(x,t) = \mathrm{e}^{-k^2(T-t)}\Phi_{2,k}^*(x).$$

Proof of Theorem 1.5. Again, we will follow the strategy used in [9]. Assume that $T > T_1$ and $I_k(p,q) \neq 0$ for all $k \in \mathbb{N}^*$. We will look for the control u under the form

$$u(t) := \sum_{k\in\mathbb{N}^*}\{u_{1,k}q_{1,k}(T-t) + u_{2,k}q_{2,k}(T-t)\}, \tag{7.2}$$

for all $t \in (0,T)$, where $q_{1,k}$ and $q_{2,k}$ are defined in Section 3.1. Plugging the expressions of u, $\theta_{1,k}$ and $\theta_{2,k}$ in equality (7.1), we obtain the *moment problem*

$$\begin{cases} u_{1,k} = -\mathrm{e}^{-k^2T}\dfrac{\langle y_1^0, \varphi_k\rangle_{H^{-1},H_0^1}}{\partial_x\varphi_k(0)}, \\ u_{2,k} = \dfrac{\mathrm{e}^{-k^2T}}{I_k\partial_x\varphi_k(0)}\left\{\langle y_1^0, \psi_k^*\rangle_{H^{-1},H_0^1} + \langle y_2^0, \varphi_k\rangle_{H^{-1},H_0^1} - \left(I_kT + \dfrac{\partial_x\psi_k^*(0)}{\partial_x\varphi_k(0)}\right)\langle y_1^0, \varphi_k\rangle_{H^{-1},H_0^1}\right\}. \end{cases}$$

Let $\varepsilon > 0$. Using the definition of T_1 (see (1.14)), we have $I_k(p,q) > C_\varepsilon \mathrm{e}^{-k^2(T_1+\varepsilon)}$ for all $k \in \mathbb{N}^*$. Then, using the estimates (2.5) and (3.33), we get

$$|u_{1,k}| + |u_{2,k}| \leqslant C\mathrm{e}^{-k^2(T-T_1-2\varepsilon)}\|y^0\|_{H^{-1}(0,\pi)^2}.$$

Thus for $\varepsilon < (T-T_1)/2$, the control u defined in (7.2) is an element of $L^2(0,T)$.

Assume now that $T < T_1$ and $I_k(p,q) \neq 0$ for all $k \in \mathbb{N}^*$. By contradiction let us suppose that there exists a constant C_{obs} such that for all $\theta^0 \in H_0^1(0,\pi)^2$ the solution to the dual system (2.7) satisfies

$$\|\theta(0)\|^2_{H_0^1(0,\pi)^2} \leqslant C_{\mathrm{obs}} \int_0^T |B^*\partial_x\theta(0,t)|^2 \mathrm{d}t. \tag{7.3}$$

Let $\varepsilon = (T_1 - T)/2$. Using the definition of T_1, there exists a sequence $(k_n)_{n\in\mathbb{N}^*}$ such that

$$I_{k_n}(p,q) < \mathrm{e}^{-k_n^2(T+\varepsilon)}. \tag{7.4}$$

Let $\theta_n^0 := a_n\Phi^*_{1,k_n} + b_n\Phi^*_{2,k_n}$ with $(a_n,b_n) \in \mathbb{R}^2$. We recall that

$$\theta^n(x,t) = \mathrm{e}^{-k_n^2(T-t)}\{a_n\Phi^*_{1,k_n} + (b_n - (T-t)I_k(p,q)a_n)\Phi^*_{2,k_n}\}.$$

Then, after calculation, we get

$$\|\theta(0)\|^2_{H_0^1(0,\pi)^2} = \mathrm{e}^{-2k_n^2T}(a_n^2\|\psi_{k_n}\|^2_{H_0^1} + a_n^2k_n^2 + (b_n - TI_k(p,q)a_n)^2k_n^2)$$

and

$$\int_0^T |B^*\partial_x\theta(0,t)|^2\mathrm{d}t = \int_0^T \mathrm{e}^{-2k_n^2(T-t)}|a_n\partial_x\psi_{k_n}(0) + \sqrt{\frac{2}{\pi}}(b_n - (T-t)I_{k_n}(p,q)a_n)k_n|^2\mathrm{d}t.$$

For $a_n := 1$ and $b_n := -\sqrt{\frac{\pi}{2}}\partial_x\psi_{k_n}(0)/k_n$, taking into account inequality (7.4) and using the estimate (2.5), we obtain

$$\|\theta(0)\|^2_{H_0^1(0,\pi)^2} \geqslant k_n^2\mathrm{e}^{-2k_n^2T} \quad \text{and} \quad \int_0^T |B^*\partial_x\theta(0,t)|^2\mathrm{d}t \leqslant Ck_n^2\mathrm{e}^{-2k_n^2(T+\varepsilon)}.$$

Thus for n large enough we get a contradiction with observability inequality (7.3). $\square$

8. Comments and open problems

When the control domain and the support of the coupling coefficients p and q is disjoint in the system

$$\begin{cases} \partial_t y_1 - \partial_{xx}y_1 = \mathbb{1}_\omega v & \text{in} \quad Q_T, \\ \partial_t y_2 - \partial_{xx}y_2 + p(x)\partial_x y_1 + q(x)y_1 = 0 & \text{in} \quad Q_T, \\ y_1(0,\cdot) = y_1(\pi,\cdot) = y_2(0,\cdot) = y_2(\pi,\cdot) = 0 & \text{on} \quad (0,T), \\ y_1(\cdot,0) = y_1^0,\ y_2(\cdot,0) = y_2^0 & \text{in} \quad (0,\pi) \end{cases} \tag{8.1}$$

(resp. system (1.2)), it is legitimate to ask if the minimal time T_1 (resp. T_0) given in Theorem 1.3 (resp. Thm. 1.4) can be different of zero and finite. For $p \equiv 0$ in $(0,\pi)$, it is proved in ([9], Lem. 7.1) that for any $\tau_0 \in [0,\infty]$ there exists a function $q \in L^\infty(0,\pi)$ such that the minimal time of null controllability $T_0(p,q)$ associated with system (1.1) is given by $T_0(p,q) = \tau_0$. The authors give explicit functions and one can easily adapt them to the case $p \not\equiv 0$ in $(0,\pi)$. In the other hand, the null controllability in the cases $T = T_0$ in Theorem 1.4 and $T = T_1$ in Theorem 1.5 are open problems.

In higher space dimension, even for this simplified system (8.1) (resp. system (1.2)), distributed and boundary controllability are also open problems. Considering the different results described in the introduction of the present paper, we can conjecture that the system of two coupled linear parabolic equations

$$\begin{cases} \partial_t y_1 = \Delta y_1 + g_{11} \cdot \nabla y_1 + g_{12} \cdot \nabla y_2 + a_{11} y_1 + a_{12} y_2 + \mathbb{1}_\omega v & \text{in} \quad \Omega \times (0,T), \\ \partial_t y_2 = \Delta y_2 + g_{21} \cdot \nabla y_1 + g_{22} \cdot \nabla y_2 + a_{21} y_1 + a_{22} y_2 & \text{in} \quad \Omega \times (0,T), \\ y = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ y(\cdot,0) = y^0 & \text{in} \quad \Omega, \end{cases} \tag{8.2}$$

is null controllable at time $T > 0$ if there exists an open nonempty subset ω_0 of ω such that

$$|a_{21}| > C \text{ in } \omega_0 \times (0,T) \quad \text{or} \quad |g_{21}^k| > C \quad \text{in} \quad \omega_0 \times (0,T), \tag{8.3}$$

for a $k \in \{1, \dots, N\}$.

It seems that the main difficulty is to prove a Carleman estimate for the adjoint problem of system (8.2) under condition (8.3) when the coupling term is a differential operator (see for instance [10, 19] and also [14] for a different approach). In the one-dimensional case, we were not able to adapt the strategy developed in this paper in this general setting.

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