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REFLECTED BSDES, OPTIMAL CONTROL AND STOPPING FOR INFINITE-DIMENSIONAL SYSTEMS *

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Abstract. We introduce the notion of mild supersolution for an obstacle problem in an infinite dimensional Hilbert space. The minimal supersolution of this problem is given in terms of a reflected BSDEs in an infinite dimensional Markovian framework. The results are applied to an optimal control and stopping problem.

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1. Introduction

The connection between backward stochastic differential equations (BSDEs) in $\mathbb{R}^n$ and semilinear parabolic PDEs is known since the seminal paper of Pardoux and Peng [18]. This result was then extended to the case of reflected BSDEs and of obstacle problem for PDEs (see [6]). It is also well known that BSDEs are related to optimal stochastic control problems and reflected BSDEs to optimal stopping or optimal control/stopping problems, see [20]. We notice that in the finite dimensional framework the above mentioned partial differential equations are intended either in classical sense, as in [18] or, more frequently, in viscosity sense, as, for instance, in [20].

On the other hand the relation among infinite dimensional BSDEs, optimal control of stochastic evolution equations with values in Hilbert spaces and parabolic equations in infinite dimensional spaces was investigated in [10] and in several successive papers where it appears that the concept of solution of the PDE has to be modified in the infinite dimensional case. Indeed classical solutions require too much regularity while the theory of viscosity solutions can be applied only in special cases with trace class noise and value function being continuous with respect to weak norms (see [9]). The definition that seems to best fit the infinite dimensional framework and the BSDE approach is the notion of *mild solution* (see [3]); namely if, on an Hilbert space H,

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we consider a semilinear parabolic PDE such as

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \mathcal{L}_t u\,(t,x) + \psi\,(t,x,u\,(t,x)\,,\nabla u\,(t,x))\,, & t \in [0,T]\,,\ x \in H \\ u(T,x) = \phi\,(x)\,, \end{cases}$$

and $(P_{s,t})_{0\leq s\leq t\leq T}$ is the transition semigroup related to the second order differential operators $(\mathcal{L}_t)_{t\in[0,T]}$ then a function $u : [0,T] \times H \to \mathbb{R}$ is called a mild solution of the above PDE whenever u admits a gradient (in a suitable sense) and it holds:

$$u(s,x) = P_{s,T}[\phi](x) + \int_s^T P_{s,\tau}\Big[\psi(\tau,\cdot,u(\tau,\cdot),\nabla u(\tau,\cdot))\Big](x)\ \mathrm{d}\tau.$$

Several works have then extended the BSDE approach in different directions both in the finite and in the infinite dimensional framework but, to our best knowledge, no results are available to connect reflected BSDEs in infinite dimensional spaces and obstacle problems for PDEs with infinitely many variables. The point is that it is not obvious how to include the reflection term (which is not absolutely continuous with respect to Lebesgue measure on $[0,T]$) into the definition of mild solution.

In this paper, to overcome such a difficulty and inspired by the techniques in A. Bensoussan [2], we propose the notion of *mild-supersolution*. To be more specific, our main result (see Thm. 3.5) will be to prove that:

if $(X^{s,x}, Y^{s,x}, Z^{s,x}, K^{s,x})$ is the solution of the following forward backward system with reflected BSDE:

$$\begin{cases} \mathrm{d}X_t^{s,x} = (AX_t^{s,x} + F(t,X_t^{s,x}))\,\mathrm{d}t + G(t,X_t^{s,x})\,\mathrm{d}W_t \quad & t \in [s,T] \\ X_s^{s,x} = x, \\ -\,\mathrm{d}Y_t^{s,x} = \psi(t,X_t^{s,x},Y_t^{s,x},Z_t^{s,x})\ \mathrm{d}t + \ \mathrm{d}K_t^{s,x} - Z_t^{s,x}\ \mathrm{d}W_t, & t \in [s,T], \\ Y_T^{s,x} = \phi(X_T^{s,x}), \\ Y_t^{s,x} \geq h(X_t^{s,x}), \\ \int_s^T (Y_t^{s,x} - h(X_t^{s,x}))\,\mathrm{d}K_t^{s,x} = 0 \end{cases}$$

and we set $u(t,x) := Y_t^{t,x}$, then u is the minimal mild supersolution of the obstacle problem

$$\begin{cases} \min\left(u(t,x) - h(x), -\dfrac{\partial u}{\partial t}(t,x) - \mathcal{L}_t u\,(t,x) - \psi\,(t,x,u\,(t,x)\,,\nabla u\,(t,x)\,G\,(t,x))\right) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T]\,,\ x \in H, \\ u(T,x) = \phi(x); \end{cases} \tag{1.1}$$

(see Def. 3.3 where the notion of supersolution is introduced). As we explain with more details in Remark 3.4, the notion of supersolution proposed in this paper, is not related to viscosity theory and, as a matter of fact, is inspired by the early work of A. Bensoussan, see [2]. Here we are able to characterize the value function as the minimal mild supersolution of the corresponding HJB obstacle problem. Such a definition, related to the minimality requirement included in the definition of solution to reflected BSDE (see [6]), seems natural in this context and has the advantage to bypass the question of uniqueness for the obstacle problem. Nevertheless a more explicit notion of uniqueness remains an open problem.

Another issue of this paper is that we do not assume any nondegeneracy on the coefficient G (and consequently any strong ellipticity on the second order differential operator in the PDE). Therefore (see the Example at the beginning of Sect. 3.1) we can not expect to have regular solutions of the obstacle problem nor regular value functions of the optimal stopping problem. In particular, the directional gradient ∇uG can not be understood according to its classical formulation. We employ the definition of generalized gradient (in probabilistic sense) introduced in [13]. It was proved in [13] that such generalized gradient exists for all locally Lipschitz functions. In Theorem 2.7 we show that our candidate solution $u(t,x) := Y_t^{t,x}$ is indeed locally Lipschitz.

Finally, let us remark that we work under general growth assumptions, with respect to x, on the nonlinear term ψ and on the final datum ϕ. This forces us to obtain L^p estimates on the solution of the reflected BSDE extending the ones proved in [6].

The structure of the paper is the following. In Section 2 we study reflected BSDEs obtaining, in the Markovian framework, L^p estimates and local Lipschitz continuity with respect to the initial datum. In Section 3 we introduce the notion of minimal mild supersolution of the obstacle problem in the sense of the generalized gradient and we show how it is related to the reflected BSDEs. Finally, in Section 4, we apply the above results to an optimal control and stopping problem.

2. Reflected BSDEs

In a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a cylindrical Wiener process $\{W_t, t \geq 0\}$ with values in a Hilbert space Ξ^* and we denote by $(\mathcal{F}_t)_{t\geq 0}$ its natural filtration, augmented in the usual way. We consider the following reflected backward stochastic differential equation (RBSDE in the following):

$$\begin{cases} -\,\mathrm{d}Y_t = f(t, Y_t, Z_t)\ \mathrm{d}t + \mathrm{d}K_t - Z_t\ \mathrm{d}W_t, \qquad t \in [0,T], \\ Y_T = \xi, \\ Y_t \geq S_t, \\ \int_0^T (Y_t - S_t)\,\mathrm{d}K_t = 0, \end{cases} \tag{2.1}$$

for the unknown adapted processes Y, Z and K. In the above equation Y and K are real processes while Z is a Ξ^*-valued process. Moreover Y admits a continuous modification and K is a continuous, adapted, non-decreasing process with $K_0 = 0$. The equation is understood in the usual integral way, namely:

$$Y_t + \int_t^T Z_r\ \mathrm{d}W_r = \xi + \int_t^T f(r, Y_r, Z_r)\ \mathrm{d}r + K_T - K_t, \qquad t \in [0,T],\ \mathbb{P}-\text{a.s.}. \tag{2.2}$$

In the following, if E is a separable Hilbert space, $0 < a < b$ and $p \geq 1$ by $\mathrm{L}^p_{\mathcal{P}}(\Omega \times [a,b], E)$ we denote the space of E-vauled $(\mathcal{F}_t)$-predictable processes ℓ such that:

$$\mathbb{E}\int_a^b |\ell(t)|^p\,\mathrm{d}t < \infty.$$

If $E = \mathbb{R}$ we write $\mathrm{L}^p_{\mathcal{P}}(\Omega \times [a,b])$ instead of $\mathrm{L}^p_{\mathcal{P}}(\Omega \times [a,b], \mathbb{R})$.

Moreover by $\mathrm{L}^p_{\mathcal{P}}(\Omega, \mathrm{L}^2([a,b], E))$, $p \geq 2$, we denote the subspace of $\mathrm{L}^2_{\mathcal{P}}(\Omega \times [a,b], E)$ given by processes verifying

$$\mathbb{E}\left[\int_a^b |\ell(t)|^2\,\mathrm{d}t\right]^{p/2} < \infty.$$

and by $\mathrm{L}^p_{\mathcal{P}}(\Omega, \mathrm{C}^0([a,b], E))$ the subspace of $\mathrm{L}^p_{\mathcal{P}}(\Omega \times [a,b], E)$ given by processes admitting a continuous version and verifying

$$\mathbb{E}\sup_{t\in[a,b]} |\ell(t)|^p < \infty.$$

An analogous definition is given for $\mathrm{L}^p_{\mathcal{P}}(\Omega, \mathrm{L}^2([a,b]))$ and for $\mathrm{L}^p_{\mathcal{P}}(\Omega, \mathrm{C}^0([a,b]))$, when $E = \mathbb{R}$.

Finally by $\mathcal{P}$ we denote the predictable σ-algebra on $\Omega \times [0,T]$, and by $\mathcal{B}(\Lambda)$ the Borel σ-algebra on any topological space Λ

The next result was proved in [6] (see Sect. 6 and Lem. 6.1) and it is reported here for the reader's convenience. We notice that while Y and K are real valued processes as in [6], here both $(W_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ take values in a general Hilbert space rather than in $\mathbb{R}^d$. However, all the arguments in [6] can be repeated just by replacing the norm of $\mathbb{R}^d$ by the natural Hilbertian norms, so we omit the proof.

We introduce the following class of penalized BSDEs:

$$\begin{cases} -\,\mathrm{d}Y_t^n = f(t, Y_t^n, Z_t^n)\ \mathrm{d}t + n(Y_t^n - S_t)^- \,\mathrm{d}t - Z_t^n\ \mathrm{d}W_t, \qquad t\in[0,T],\\ Y_T^n = \xi. \end{cases} \tag{2.3}$$

We denote by $(Y^n, Z^n) \in \mathrm{L}^2_{\mathcal{P}}(\Omega, \mathrm{C}^0([0,T])) \times \mathrm{L}^2_{\mathcal{P}}(\Omega\times[0,T], \Xi^*)$, their unique solution (see [10]) and by K^n the adapted, continuous, non-decreasing process:

$$K_t^n := n\int_0^t (Y_s^n - S_s)^-\,\mathrm{d}s. \tag{2.4}$$

Theorem 2.1. *Assume that $f : \Omega\times[0,T]\times\mathbb{R}\times\Xi\to\mathbb{R}$ is measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\Xi^*)$ and is Lipschitz with respect to y and z uniformly in t and ω. Moreover $f(\cdot,0,0)\in \mathrm{L}^2_{\mathcal{P}}(\Omega\times[0,T])$ and $\xi\in\mathrm{L}^2(\Omega,\mathcal{F}_T,\mathbb{P})$. Finally assume that the obstacle S is a continuous adapted real valued process satisfying*

$$\mathbb{E}\sup_{t\in[0,T]}|S_t|^2<\infty, \quad S_T\leq\xi.$$

Then equation (2.1) admits a unique solution (Y,Z,K) such that $Y\in\mathrm{L}^2_{\mathcal{P}}(\Omega,\mathrm{C}^0([0,T]))$, $Z\in\mathrm{L}^2_{\mathcal{P}}(\Omega\times[0,T],\Xi^*)$ and K is an adapted continuous non decreasing process with $K_0=0$ and $\mathbb{E}|K_T|^2<\infty$.

Moreover, for the pair of processes (Y^n, Z^n) solving (2.3) and for the process K^n given by (2.4), the following uniform estimate holds:

$$\mathbb{E}\sup_{s\in[0,T]}|Y_s^n|^2 + \mathbb{E}\int_0^T |Z_t^n|^2\ \mathrm{d}t + \mathbb{E}|K_T^n|^2 \leq c\mathbb{E}|\xi|^2 + c\mathbb{E}\int_0^T |f(t,0,0)|^2\,\mathrm{d}t, \tag{2.5}$$

for a constant c that does not depend on n.

Finally, for fixed $t\in[0,T]$, the sequence (Y_t^n) is $\mathbb{P}$-almost surely non decreasing and it holds:

$$\mathbb{E}\sup_{t\in[0,T]}|Y_t^n - Y_t|^2 + \mathbb{E}\int_0^T |Z_t^n - Z_t|^2\ \mathrm{d}t + \mathbb{E}|K_T^n - K_T|^2 \to 0. \tag{2.6}$$

Below we will need to show regular dependence of the solutions to reflected BSDEs with respect to some parameters (for instance the initial data of a forward stochastic differential equation). Due to the generality of our assumptions on the nonlinearity ψ this requires L^p estimates both on the solution of equation (2.1) and on the solutions of the penalized equations (2.3).

We assume the following:

Hypothesis 1. *The function $f : \Omega\times[0,T]\times\mathbb{R}\times\Xi^*\to\mathbb{R}$ is measurable with respect to $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\Xi^*)$ and Lipschitz continuous with respect to y and z uniformly in t and ω.*

Moreover there exists $p\geq 2$ such that:

$$\mathbb{E}\int_0^T |f(t,0,0)|^p\,\mathrm{d}t<\infty$$

and the final data ξ is in $L^p(\Omega,\mathcal{F}_T,\mathbb{P})$.

Finally the obstacle S is an adapted continuous real valued process satisfying

$$\mathbb{E}\sup_{t\in[0,T]}|S_t|^{2p-2}<\infty.$$

We notice that the integrability requests are not optimal (for instance we assume p-integrabi-lity jointly in $\Omega \times [0,T]$) for the generator f and $2(p-1)$ integrability for the obstacle S). Nevertheless such assumptions are verified in the Markovian framework (see Sect. 2.1) and will allow us to treat obstacle problems under general assumptions (see Sect. 3).

We claim that under Hypotheses 1 we can estimate the L^p norms of the solution (Y, Z, K) of equation (2.1). Namely:

Theorem 2.2. *Let Hypothesis 1 hold and let (Y, Z, K) be the unique solution to equation (2.1). Then the following holds:*

$$\begin{aligned} &\mathbb{E} \sup_{t\in[0,T]} |Y_t|^p + \mathbb{E}\left(\int_0^T |Z_t|^2 \,\mathrm{d}t\right)^{p/2} + \mathbb{E}|K_T|^p \\ &\quad \leq C\mathbb{E}|\xi|^p + C\mathbb{E}\int_0^T |f(t,0,0)|^p \,\mathrm{d}t + C\left(\mathbb{E}\sup_{t\in[0,T]} |S_t|^{2p-2}\right)^{p/(2p-2)}, \end{aligned} \tag{2.7}$$

(*we notice that C only depends on p, T and on the Lipschitz constant of f*).

The above result will be a consequence of the corresponding estimates for the penalized approximations.

Proposition 2.3. *Let Hypothesis 1 hold and, for all $n \in \mathbb{N}$, let (Y^n, Z^n) be the unique solution of equation (2.3) and let K^n be the process defines in (2.4). Then the following holds:*

$$\begin{aligned} &\mathbb{E} \sup_{t\in[0,T]} |Y^n_t|^p + \mathbb{E}\left(\int_0^T |Z^n_t|^2 \,\mathrm{d}t\right)^{p/2} + \mathbb{E}|K^n_T|^p \\ &\quad \leq C\mathbb{E}|\xi|^p + C\mathbb{E}\int_0^T |f(t,0,0)|^p \,\mathrm{d}t + C\left(\mathbb{E}\sup_{t\in[0,T]} |S_t|^{2p-2}\right)^{p/(2p-2)}, \end{aligned} \tag{2.8}$$

where C only depends on p, T and on the Lipschitz constant of f.

Proof. First of all we notice that we can always reduce ourselves to the case in which

$$\frac{y}{|y|} f(t,y,z) \leq |f(t,0,0)| + \mu|y| + \lambda|z| \qquad \text{with } \mu + \lambda^2 \leq -1. \tag{2.9}$$

Indeed, setting $\tilde{Y}^n_t = \mathrm{e}^{at} Y^n_t$, $\tilde{Z}^n_t = \mathrm{e}^{at} Z^n_t$, $\tilde{S}_t = \mathrm{e}^{at} S_t$ we get that $\left(\tilde{Y}^n, \tilde{Z}^n\right)$ satisfies

$$\begin{cases} -\,\mathrm{d}\tilde{Y}^n_t = \mathrm{e}^{at} f\left(t, \mathrm{e}^{-at}\tilde{Y}^n_t, \mathrm{e}^{-at}\tilde{Z}^n_t\right) \,\mathrm{d}t - a\tilde{Y}^n_t \,\mathrm{d}t + n\left(\tilde{Y}^n_t - \tilde{S}_t\right)^- \,\mathrm{d}t \\ \qquad\qquad -\tilde{Z}^n_t \,\mathrm{d}W_t, \qquad t \in [0,T], \\ \tilde{Y}^n_T = \mathrm{e}^{aT}\xi. \end{cases}$$

So the generator is given by

$$\tilde{f}(t,y,z) := \mathrm{e}^{at} f(t, \mathrm{e}^{-at}y, \mathrm{e}^{-at}z) - ay,$$

and, by choosing a sufficiently large (depending only on the Lipschitz constant of f), we have $\mu + \lambda^2 \leq -1$. From now on we assume that (2.9) holds true and, for simplicity, we omit the superscript $\sim$.

Moreover by c we shall denote a suitable constant that depends only on the Lipschitz constant of f, T and p and by $c(\delta)$ a constant that depends, besides the above parameters, on an auxiliary constant $\delta > 0$. The values of c and $c(\delta)$ may change from line to line.

To start with, we apply Itô formula to $|Y_t^n|^p$, $s \le t \le T$ and obtain:

$$\begin{aligned}-\,\mathrm{d}|Y_t^n|^p = & \, p|Y_t^n|^{p-1}\hat{Y}_t^n f(t,Y_t^n,Z_t^n)\,\mathrm{d}t + np|Y_t^n|^{p-1}\hat{Y}_t^n (Y_t^n - S_t)^-\,\mathrm{d}t \\ & - p|Y_t^n|^{p-1}\hat{Y}_t^n Z_t^n\,\mathrm{d}W_t - \frac{p(p-1)}{2}|Y_t^n|^{p-2}|Z_t^n|^2\,\mathrm{d}t,\end{aligned}$$

where $\hat{Y}_t^n := \dfrac{Y_t^n}{|Y_t^n|}$. Integrating between s and T, $0 \le s \le t \le T$, we get:

$$\begin{aligned}& |Y_s^n|^p + \frac{p(p-1)}{2}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2\;\mathrm{d}t \\ & = |\xi|^p + p\int_s^T |Y_t^n|^{p-1}\hat{Y}_t^n f(t,Y_t^n,Z_t^n)\,\mathrm{d}t + np\int_s^T |Y_t^n|^{p-1}\hat{Y}_t^n (Y_t^n - S_t)^-\;\mathrm{d}t \\ & \quad - p\int_s^T |Y_t^n|^{p-1}\hat{Y}_t^n Z_t^n\,\mathrm{d}W_t \\ & \le |\xi|^p + p\int_s^T |Y_t^n|^{p-1}|f(t,0,0)|\,\mathrm{d}t + p\mu\int_s^T |Y_t^n|^p\,\mathrm{d}t + p\lambda\int_s^T |Y_t^n|^{p-1}|Z_t^n|\,\mathrm{d}t \\ & \quad + np\int_s^T |S_t|^{p-1}(Y_t^n - S_t)^-\;\mathrm{d}t - p\int_s^T |Y_r^n|^{p-1}\hat{Y}_t^n Z_t^n\,\mathrm{d}W_t \\ & \le |\xi|^p + c\int_t^T |f(t,0,0)|^p\,\mathrm{d}t + p\int_s^T |Y_t^n|^p\,\mathrm{d}t + p\mu\int_s^T |Y_t^n|^p\,\mathrm{d}t + \frac{p\lambda^2}{(p-1)}\int_s^T |Y_t^n|^p\,\mathrm{d}t \\ & \quad + \frac{p(p-1)}{4}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2\,\mathrm{d}t + \sup_{t\in[s,T]}|S_t|^{p-1} np\int_s^T (Y_t^n - S_t)^-\;\mathrm{d}t \\ & \quad - p\int_s^T |Y_r^n|^{p-1}\hat{Y}_t^n Z_t^n\,\mathrm{d}W_t,\end{aligned}$$

where we have applied Young inequality and we have noticed that

$$|y|^{p-1}\hat{y}(y-s)^- \le |s|^{p-1}(y-s)^-,\ \forall y,s\,\in\mathbb{R},\ \text{where } \hat{y} := \frac{y}{|y|},\ \text{if } y \neq 0.$$

The above inequality is trivial if $y \le 0$ and if $s \le y$; as far as the remaining case $0 < y < s$ is concerned it follows again by a straightforward computation.

Recalling that, by (2.9), $\mu + \lambda^2 + 1 \le 0$ (and consequently $p + p\mu + p(p-1)^{-1}\lambda^2 \le 0$, since $p \ge 2$) we get:

$$\begin{aligned}& |Y_s^n|^p + \frac{p(p-1)}{4}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2\;\mathrm{d}t \\ & \qquad \le |\xi|^p + c\int_t^T |f(t,0,0)|^p\,\mathrm{d}t + p\sup_{t\in[s,T]}|S_t|^{p-1} n\int_s^T (Y_t^n - S_t)^-\;\mathrm{d}t - c\int_s^T |Y_r^n|^{p-1}\hat{Y}_t^n Z_t^n\,\mathrm{d}W_t.\end{aligned}$$

By the penalized BSDE (2.3) in integral form we deduce that

$$\int_s^T n(Y_t^n - S_t)^-\,\mathrm{d}t = -\xi + Y_t^n - \int_s^T f(t,Y_t^n,Z_t^n)\;\mathrm{d}t + \int_s^T Z_t^n\;\mathrm{d}W_t, \tag{2.10}$$

therefore

$$
\begin{aligned}
&|Y_s^n|^p + \frac{p(p-1)}{4}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2 \, \mathrm{d}t \\
&\le |\xi|^p + c\int_s^T |f(t,0,0)|^p \, \mathrm{d}t - p\int_s^T |Y_t^n|^{p-1}\hat{Y}_t^n Z_t^n \, \mathrm{d}W_t \\
&\quad + p\left[\sup_{t\in[s,T]} |S_t|^{p-1}\left(-\xi + Y_s^n - \int_s^T f(t,Y_t^n,Z_t^n)\ \mathrm{d}t + \int_s^T Z_t^n\ \mathrm{d}W_t\right)\right] \\
&\le |\xi|^p + c\int_s^T |f(t,0,0)|^p \, \mathrm{d}t - p\int_s^T |Y_t^n|^{p-1}\hat{Y}_t^n Z_t^n \, \mathrm{d}W_t + c\sup_{t\in[s,T]} \left|S_t|^p + \frac{1}{2}\right| Y_s^n|^p \\
&\quad + p\sup_{t\in[s,T]} |S_t|^{p-1}\left[|\xi| + \left|\int_s^T f(t,Y_t^n,Z_t^n)\ \mathrm{d}t\right| + \left|\int_s^T Z_t^n\ \mathrm{d}W_t\right|\right],
\end{aligned}
$$

and moving the term $1/2|Y_s^n|^p$ to the left hand side:

$$
\begin{aligned}
\frac{1}{2}|Y_s^n|^p + \frac{p(p-1)}{4}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2\ \mathrm{d}t \le |\xi|^p + c\int_s^T |f(t,0,0)|^p \, \mathrm{d}t - p\int_s^T |Y_t^n|^{p-1}\hat{Y}_t^n Z_t^n \, \mathrm{d}W_t + c\sup_{t\in[s,T]} |S_t|^p \\
+ p\sup_{t\in[s,T]} |S_t|^{p-1}\left[|\xi| + \left|\int_s^T f(t,Y_t^n,Z_t^n)\ \mathrm{d}t\right| + \left|\int_s^T Z_t^n\ \mathrm{d}W_t\right|\right].
\end{aligned}
\tag{2.11}
$$

Since $(Y^n, Z^n) \in \mathrm{L}^p_{\mathcal{P}}(\Omega, \mathrm{C}^0([0,T])) \times \mathrm{L}^p_{\mathcal{P}}(\Omega, \mathrm{L}^2([0,T], \Xi))$ (see [10]), the Itô integral $\int_s^T |Y_r^n|^{p-1}\hat{Y}_t^n Z_t^n \, \mathrm{d}W_t$ has null expectation. Consequently, computing the mean value and recalling that, by Hölder inequality:

$$
\mathbb{E}\sup_{t\in[s,t]} |S_t|^p \le \left(\mathbb{E}\sup_{t\in[s,t]} |S_t|^{2p-2}\right)^{p/(2p-2)},
$$

we deduce that:

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}|Y_s^n|^p + \frac{p(p-1)}{4}\mathbb{E}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2\ \mathrm{d}t \le c\,\Theta \\
+ p\mathbb{E}\sup_{t\in[s,T]} |S_t|^{p-1}\left[|\xi| + \left|\int_s^T f(t,Y_t^n,Z_t^n)\ \mathrm{d}t\right| + \left|\int_s^T Z_t^n\ \mathrm{d}W_t\right|\right] \\
\le c\,\Theta + p\left(\mathbb{E}\sup_{t\in[s,T]} |S_t|^{2(p-1)}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|\xi| + \int_s^T (|f(t,0,0)| + c|Y_t^n| + c|Z_t^n|)\ \mathrm{d}t + \left|\int_s^T Z_t^n\ \mathrm{d}W_t\right|\right]^2\right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.12}
$$

To shorten notation, we have set:

$$
\Theta := \mathbb{E}|\xi|^p + \mathbb{E}\int_0^T |f(t,0,0)|^p\ \mathrm{d}t + \left(\mathbb{E}\sup_{t\in[0,T]} |S_t|^{2p-2}\right)^{p/(2p-2)}.
$$

Plugging (2.5) in (2.12) we get, by the BDG inequality:

$$\frac{1}{2}\mathbb{E}|Y_s^n|^p + \frac{p(p-1)}{4}\mathbb{E}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2 \,\mathrm{d}t \le c\,\Theta$$
$$+ c\left(\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}}\mathbb{E}\left[|\xi|^2 + \int_0^T |f(t,0,0)|^2 \,\mathrm{d}t + \int_s^T |Y_t^n|^2 \,\mathrm{d}t + \int_s^T |Z_t^n|^2 \,\mathrm{d}t\right]^{\frac{1}{2}}$$
$$\le c\,\Theta + c\left(\mathbb{E}\sup_{t\in[0,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}} * \left(\mathbb{E}|\xi|^2 + c\mathbb{E}\int_0^T |f(t,0,0)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}. \tag{2.13}$$

In particular

$$\frac{p(p-1)}{4}\mathbb{E}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2 \,\mathrm{d}t \le c\,\Theta + c\left(\mathbb{E}\sup_{t\in[0,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}} * \left(\mathbb{E}|\xi|^2 + C\mathbb{E}\int_0^T |f(t,0,0)|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \tag{2.14}$$

By (2.11) with r instead of s, such that $0 \le s \le r \le T$ we get (after multiplication by 2)

$$|Y_r^n|^p \le 2|\xi|^p + c\int_0^T |f(t,0,0)|^p \,\mathrm{d}t - 2p\int_r^T |Y_r^n|^{p-1}\hat{Y}_t^n Z_t^n \,\mathrm{d}W_t + c\sup_{r\in[s,T]}|S_t|^p$$
$$+ 2p\sup_{r\in[s,T]}|S_r|^{p-1}\left[|\xi| + \left|\int_r^T f(t,Y_t^n,Z_t^n)\,\mathrm{d}t\right| + \left|\int_r^T Z_t^n \,\mathrm{d}W_t\right|\right].$$

Computing the supremum over the time r we arrive at:

$$\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p \le c\,\Theta + 2p\mathbb{E}\sup_{r\in[s,T]}\left|\int_r^T\right| Y_t^n|^{p-1}\hat{Y}_t^n Z_t^n \,\mathrm{d}W_t|$$
$$+ 2p\left(\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|\xi|^2 + \int_s^T |f(t,Y_t^n,Z_t^n)|^2 \,\mathrm{d}t + \sup_{t\in[s,T]}\left|\int_t^T Z_t^n \,\mathrm{d}W_t\right|^2\right]\right)^{\frac{1}{2}}$$

Proceeding as in the proof of (2.13) we get:

$$\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p \le c\,\Theta + c\mathbb{E}\sup_{r\in[s,T]|}\int_r^T |Y_t^n|^{p-1}\hat{Y}_t^n Z_t^n \,\mathrm{d}W_t|$$
$$+ c\left(\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|\xi|^2 + \int_s^T |f(t,0,0)|^2 \,\mathrm{d}t\right]\right)^{\frac{1}{2}},$$

and, again by BDG inequality,

$$\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p \le c\mathbb{E}|\xi|^p + c\int_s^T |f(t,0,0)|^p \,\mathrm{d}t + c\mathbb{E}\left(\int_r^T |Y_t^n|^{2(p-1)}|Z_t^n|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}$$
$$+ c\left(\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|\xi|^2 + \int_s^T |f(t,0,0)|^2 \,\mathrm{d}t\right]\right)^{\frac{1}{2}}.$$

Therefore

$$\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p \le c\,\Theta + c\mathbb{E}\left(\sup_{t\in[s,T]}|Y_t^n|^p\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2\,\mathrm{d}t\right)^{\frac{1}{2}} + c\left(\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|\xi|^2+\int_s^T |f(t,0,0)|^2\ \mathrm{d}t\right]\right)^{\frac{1}{2}},$$

and finally:

$$\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p \le c\,\Theta + \frac{1}{2}\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p + c\mathbb{E}\int_s^T |Y_t^n|^{p-2}|Z_t^n|^2\,\mathrm{d}t + c\left(\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|\xi|^2+\int_s^T |f(t,0,0)|^2\ \mathrm{d}t|\right]\right)^{\frac{1}{2}}.$$

Applying estimate (2.14) we can conclude:

$$\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p \le c\,\Theta + \left(\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|\xi|^2+\int_s^T |f(t,0,0)|^2\ \mathrm{d}t\right]\right)^{\frac{1}{2}}. \tag{2.15}$$

Next we estimate $\mathbb{E}(\int_s^T |Z_t^n|^2\ \mathrm{d}t)^{\frac{p}{2}}$. We apply Itô formula to $|Y_t^n|^2$, $s\le t\le T$ and get:

$$\begin{aligned}\mathrm{d}|Y_t^n|^2 &= -2Y_t^n f(t,Y_t^n,Z_t^n)\,\mathrm{d}t - nY_t^n(Y_t^n-S_t)^-\,\mathrm{d}t + 2Y_t^nZ_t^n\,\mathrm{d}W_t + |Z_t^n|^2\,\mathrm{d}t\\ &\le -2Y_t^n f(t,Y_t^n,Z_t^n)\,\mathrm{d}t - nS_t(Y_t^n-S_t)^-\,\mathrm{d}t + 2Y_t^nZ_t^n\,\mathrm{d}W_t + |Z_t^n|^2\,\mathrm{d}t.\end{aligned} \tag{2.16}$$

since $(y-s)^- \le s(y-s)^-$ for all $y,s\in\mathbb{R}$ (indeed $xx^-\le 0$ for all $x\in\mathbb{R}$).

We integrate on $[s,T]$ and we raise to the power $\frac{p}{2}$:

$$\begin{aligned}|Y_s^n|^p + \left(\int_s^T |Z_t^n|^2\ \mathrm{d}t\right)^{\frac{p}{2}} &\le c|\xi|^p + c\left|\int_s^T Y_t^n f(t,Y_t^n,Z_t^n)\,\mathrm{d}t\right|^{\frac{p}{2}} + c\left|n\int_s^T S_t(Y_t^n-S_t)^-\ \mathrm{d}t\right|^{\frac{p}{2}}\\ &\quad + c\left|\int_s^T Y_r^nZ_t^n\,\mathrm{d}W_t\right|^{\frac{p}{2}}\\ &\le c|\xi|^p + c\left|\int_s^T Y_t^n f(t,Y_t^n,Z_t^n)\,\mathrm{d}t\right|^{\frac{p}{2}} + c\left(\sup_{t\in[s,T]} n|S_t|\left|\int_s^T (Y_t^n-h(X_t))^-\ \mathrm{d}t\right|\right)^{\frac{p}{2}}\\ &\quad + c\left|\int_s^T Y_r^nZ_t^n\,\mathrm{d}W_t\right|^{\frac{p}{2}}.\end{aligned}$$

Using the expression (2.10) for $n\int_s^T (Y_t^n - S_t)^-\,\mathrm{d}t$ we get:

$$\left(\int_s^T |Z_t^n|^2\,\mathrm{d}t\right)^{\frac{p}{2}} \le c|\xi|^p + c\left|\int_s^T Y_t^n f(t,Y_t^n,Z_t^n)\,\mathrm{d}t\right|^{\frac{p}{2}} + c\left|\int_s^T Y_t^n Z_t^n\,\mathrm{d}W_t\right|^{\frac{p}{2}}$$
$$+ c\sup_{t\in[s,T]}|S_t|^{\frac{p}{2}}\left|-\xi + Y_t^n - \int_s^T f(t,Y_t^n,Z_t^n)\,\mathrm{d}t + \int_s^T Z_t^n\,\mathrm{d}W_t\right|^{\frac{p}{2}}$$
$$\le c|\xi|^p + c\left(\int_s^T \left(|Y_t^n|\,|f(t,0,0)| + \mu|Y_t^n|^2 + \lambda|Y_t^n||Z_t^n|\right)\mathrm{d}t\right)^{\frac{p}{2}} + c\left|\int_s^T Y_t^n Z_t^n\,\mathrm{d}W_t\right|^{\frac{p}{2}}$$
$$+ c\sup_{t\in[s,T]}|S_t|^{\frac{p}{2}}\left(|\xi| + |Y_t^n| + \int_s^T \left(|f(t,0,0)| + \mu|Y_t^n| + \lambda|Z_t^n|\right)\mathrm{d}t + \left|\int_s^T Z_t^n\,\mathrm{d}W_t\right|\right)^{\frac{p}{2}}.$$

Computing expectation, applying BDG and Young inequalities and estimate (2.15), we have:

$$\mathbb{E}\left(\int_s^T |Z_t^n|^2\,\mathrm{d}t\right)^{\frac{p}{2}} \le c\mathbb{E}|\xi|^p + c\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p + c\mathbb{E}\left(\int_s^T |f(t,0,0)|^2\,\mathrm{d}t\right)^{\frac{p}{2}}$$
$$+\frac{1}{4}\left(\mathbb{E}\int_s^T |Z_t^n|^2\,\mathrm{d}t\right)^{\frac{p}{2}} + c\mathbb{E}\left(\int_s^T |Y_t^n Z_t^n|^2\,\mathrm{d}t\right)^{\frac{p}{4}} + c\mathbb{E}\sup_{t\in[s,T]}|S_t|^p$$
$$\le c\,\Theta + \frac{1}{2}\mathbb{E}\left(\int_s^T |Z_t^n|^2\,\mathrm{d}t\right)^{\frac{p}{2}} + c\mathbb{E}\sup_{r\in[s,T]}|Y_r^n|^p.$$

Finally by estimate (2.15), we obtain:

$$\mathbb{E}\left(\int_s^T |Z_t^n|^2\,\mathrm{d}t\right)^{\frac{p}{2}} \le c\,\Theta + c\left(\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)}\right)^{\frac{1}{2}}\left(\mathbb{E}\left[|\xi|^2 + \left|\int_s^T\right| f(t,0,0)|^2\,\mathrm{d}t|\right]\right)^{\frac{1}{2}}$$

and this concludes the proof of the estimate of $\mathbb{E}(\int_s^T |Z_t^n|^2\,\mathrm{d}t)^{\frac{p}{2}}$.

The estimate of $\mathbb{E}|K_T^n|^p$ is now a straightforward consequence of the previous ones and of relation (2.10). □

We are now ready to prove Theorem 2.2.

Theorem 2.2. We already know (see Thm. 2.1) that $Y_t^n \uparrow Y_t$ and $\mathbb{E}\sup_{t\in[0,T]}|Y_t - Y_t^n|^2 \to 0$.

Choosing a suitable subsequence we can assume that $\sup_{t\in[0,T]}|Y_t - Y_t^n|^2 \to 0$, $\mathbb{P}$-almost surely. Therefore, Fatou Lemma and (2.8) yield:

$$\mathbb{E}\sup_{t\in[0,T]}|Y_t|^p \le C\mathbb{E}|\xi|^p + C\mathbb{E}\int_0^T |f(t,0,0)|^p\,\mathrm{d}t + C\left(\mathbb{E}\sup_{t\in[0,T]}|S_t|^{2p-2}\right)^{p/(2p-2)}.$$

As far as the convergence of Z^n is concerned, we already know that $Z^n \to Z$ in $L^2_{\mathcal{P}}(\Omega\times[0,T])$, see (2.6). By Proposition 2.3 we know that Z^n is bounded in $\mathrm{L}^p_{\mathcal{P}}(\Omega, \mathrm{L}^2([0,T],\Xi^*))$, so, extracting, if needed, a subsequence, we can assume that (Z^n) converges weakly in $\mathrm{L}^p_{\mathcal{P}}(\Omega, L^2([0,T],\Xi^*))$ and, consequently, weakly in $\mathrm{L}^2_{\mathcal{P}}(\Omega\times[0,T],\Xi^*)$.

Therefore the weak limit of (Z^n) in $\mathrm{L}^p_{\mathcal{P}}(\Omega, \mathrm{L}^2([0,T], \Xi^*))$ must coincide with the strong limit Z in $\mathrm{L}^2_{\mathcal{P}}(\Omega \times [0,T], \Xi^*)$ topology. Consequently again by (2.8) we have that Z satisfies

$$\mathbb{E}\left(\int_0^T |Z_t|^2 \, \mathrm{d}t\right)^{p/2} \leq C\mathbb{E}|\xi|^p + C\mathbb{E}\int_0^T |f(t,0,0)|^p \, \mathrm{d}t + C\left(\mathbb{E}\sup_{t\in[0,T]} |S_t|^{2p-2}\right)^{p/(2p-2)}.$$

To prove the convergence of $(K^n)_n$ we argue in a similar way. By (2.6) we know that $\mathbb{E}|K^n_T - K_T|^2 \to 0$. The claim follows as before by Fatou Lemma and estimate (2.8). □

2.1. Reflected BSDEs in a Markovian framework

Now we consider a RBSDE depending on a forward equation with values in another real and separable Hilbert space H. Namely, we consider the forward backward system

$$\begin{cases} \mathrm{d}X^{s,x}_t = AX^{s,x}_t \, \mathrm{d}t + F(t, X^{s,x}_t) \, \mathrm{d}t + G(t, X^{s,x}_t) \, \mathrm{d}W_t, \qquad t \in [s,T], \\ X^{s,x}_s = x, \\ -\mathrm{d}Y^{s,x}_t = \psi(t, X^{s,x}_t, Y^{s,x}_t, Z^{s,x}_t) \; \mathrm{d}t + \; \mathrm{d}K^{s,x}_t - Z^{s,x}_t \; \mathrm{d}W_t, \qquad t \in [s,T], \\ Y^{s,x}_T = \phi(X^{s,x}_T), \\ Y^{s,x}_t \geq h(X^{s,x}_t), \\ \int_s^T (Y^{s,x}_t - h(X^{s,x}_t)) \, \mathrm{d}K^{s,x}_t = 0. \end{cases} \tag{2.17}$$

When we need to stress the dependence on the initial conditions of the forward equation we shall denote the solution of the above RBSDE by $(Y^{s,x}, Z^{s,x}, K^{s,x})$ otherwise, when no confusion is possible, we shall denote it just by (Y, Z, K).

Below and in the rest of the paper we will use the following notation:

- if K_1 and K_2 are Hilbert spaces by $\mathrm{L}_2(K_1, K_2)$ we denote the Hilbert space of Hilbert−Schmidt operators $K_1 \to K_2$ endowed with the Hilbert−Schmidt norm;
- by $\mathcal{G}(K_1, K_2)$ we denote the space of all continuously Gateaux differentiable mappings from K_1 to K_2. That is of all maps f such that the directional derivative $\nabla f(x)h$ exists in every point $x \in K_1$ and for every direction $h \in K_1$, moreover for all points $x \in K_1$ the map $h \to \nabla f(x)h$ is continuous and for all directions $h \in K_1$ the map $x \to \nabla f(x)h$ is continuous. See [10] for details.

We will work under the following assumptions on the coefficients of the forward equation:

Hypothesis 2.

(1) *$A : H \supset \mathcal{D}(A) \to H$ is the generator of a strongly continuous semigroup of linear operators $(\mathrm{e}^{tA})_{t\geq 0}$.*
(2) *The mapping $F : [0,T] \times H \to H$ is measurable and satisfies, for some constant $C > 0$ and $0 \leq \gamma < 1$,*

$$\begin{aligned} &\left|\mathrm{e}^{sA}F(t,x)\right| \leq Cs^{-\gamma}(1+|x|), \qquad t\in[0,T], \\ &\left|\mathrm{e}^{sA}F(t,x) - \mathrm{e}^{sA}F(t,y)\right| \leq Cs^{-\gamma}|x-y|, \qquad s>0,\ t\in[0,T], \quad x,y\in H. \end{aligned} \tag{2.18}$$

(3) *G is a mapping $[0,T]\times H \to L(\Xi, H)$ such that for every $v \in \Xi$, the mapping $Gv : [0,T]\times H \to H$ is measurable and, if $s > 0$, $t \in [0,T]$ and $x \in H$ we have $\mathrm{e}^{sA}G(t,x) \in L_2(\Xi, H)$. Moreover there exists $0 < \theta < \frac{1}{2}$ such that*

$$\begin{aligned} [c]l \, &\left|\mathrm{e}^{sA}G(t,x)\right|_{L_2(\Xi,H)} \leq Ls^{-\theta}(1+|x|), \\ &\left|\mathrm{e}^{sA}G(t,x) - \mathrm{e}^{sA}G(t,y)\right|_{L_2(\Xi,H)} \leq Ls^{-\theta}|x-y|, \qquad s>0,\ t\in[0,T], \quad x,y\in H. \end{aligned} \tag{2.19}$$

(4) *For every $s > 0$ and $t \in [0,T]$, $F(t,\cdot) \in \mathcal{G}(H,H)$ and $\mathrm{e}^{sA}G(t,\cdot) \in \mathcal{G}(H,L_2(\Xi,H))$.*

The following Proposition is proved in [10].

Proposition 2.4. *Under Hypothesis 2, the forward equation*

$$\begin{cases} \mathrm{d}X_t^{s,x} = AX_t^{s,x}\,\mathrm{d}t + F(t,X_t^{s,x})\,\mathrm{d}t + G(t,X_t^{s,x})\,\mathrm{d}W_t \qquad t \in [s,T] \\ X_s^{s,x} = x, \end{cases} \tag{2.20}$$

admits a unique continuous mild solution (for the well established definition of mild solution of stochastic evolution equations see [3]). Moreover $\mathbb{E}\sup_{t\in[s,T]}|X_t^{s,x}|^p \le C_p(1+|x|)^p$, for every $p \in (2,\infty)$, and some constant $C_p > 0$.

We will work under the following assumptions on ψ:

Hypothesis 3. *The function $\psi : [0,T]\times H \times \mathbb{R} \times \Xi^* \to \mathbb{R}$ is Borel measurable and satisfies the following:*

(1) *there exists a constant $L > 0$ such that:*

$$|\psi(t,x,y_1,z_1) - \psi(t,x,y_2,z_2)| \le L\left(|y_1-y_2| + |z_1-z_2|_{\Xi^*}\right), \quad |\psi(t,0,0,0)| \le L,$$

for every $t \in [0,T]$, $x \in H$, $y_1,y_2 \in \mathbb{R}$, $z_1,z_2 \in \Xi^*$;

(2) *for every $t \in [0,T]$, $\psi(t,\cdot,\cdot,\cdot)$ is continuous $H \times \mathbb{R} \times \Xi^* \to \mathbb{R}$;*

(3) *there exists $L' > 0$ and $m \ge 0$ such that:*

$$|\psi(t,x_1,y,z) - \psi(t,x_2,y,z)| \le L'|x_1-x_2|\left(1+|x_1|^m+|x_2|^m+|y|^m\right)\left(1+|z|_{\Xi^*}\right),$$

for every $t \in [0,T]$, $x_1,x_2 \in H$, $y \in \mathbb{R}$, $z \in \Xi$;

(4) *$h,\phi : H \to \mathbb{R}$ verify $h \le \phi$, moreover there exists $L > 0$ such that:*

$$|\phi(x_1)-\phi(x_2)| \le L|x_1-x_2|\left(1+|x_1|^m+|x_2|^m\right),$$

$$|h(x_1)-h(x_2)| \le L|x_1-x_2|\left(1+|x_1|^m+|x_2|^m\right),$$

for all $x_1,x_2 \in H$.

We notice that Hypothesis 3 implies that:

$$|\psi(t,x,y,z)| \le L\left(1+|x|^{m+1}+|y|+|z|_{\Xi^*}\right), \quad |\phi(x)| \le L(1+|x|^{m+1}), \quad |h(x)| \le L(1+|x|^{m+1}), \tag{2.21}$$

for all $t \in [0,T]$, $x \in H$, $y \in \mathbb{R}$, $z \in \Xi$.

Proposition 2.5. *Let Hypotheses 2 and 3 hold true and fix $s \in [0,T]$ and $x \in H$. Then the RBSDE in (2.17) admits a unique solution $(Y^{s,x},Z^{s,x},K^{s,x})$ on $[s,T]$ in the sense of equation (2.2). Moreover $Y^{s,x}$ admits a continuous version, $(K^{s,x})$ is continuous and non-decreasing with $K_s^{s,x} = 0$ and, for all $p \ge 2$ there exists $C_p > 0$ such that*

$$\mathbb{E}\sup_{t\in[s,T]}|Y_t^{s,x}|^p + \mathbb{E}\left(\int_s^T |Z_t^{s,x}|^2\,\mathrm{d}t\right)^{p/2} + \mathbb{E}|K_T^{s,x}|^p \le C\left(1+|x|^{p(m+1)}\right). \tag{2.22}$$

Moreover if we consider the penalized version of the RBSDE in (2.17):

$$\begin{cases} -\mathrm{d}Y_t^{n,s,x} = \psi(t,X_t^{s,x},Y_t^{n,s,x},Z_t^{n,s,x})\,\mathrm{d}t + n(Y_t^{n,s,x} - h(X_t^{s,x}))^-\,\mathrm{d}t - Z_t^{n,s,x}\,\mathrm{d}W_t, \quad t \in [s,T], \\ Y_T^{n,s,x} = \phi(X_T^{s,x}). \end{cases} \tag{2.23}$$

then (2.22) *holds for equation* (2.23) *as well* (*with constant* C *independent of* n). *Finally*

$$\mathbb{E}\sup_{t\in[0,T]}|Y_t^n - Y_t|^2 + \mathbb{E}\int_0^T |Z_t^n - Z_t|^2 \,\mathrm{d}t \to 0. \tag{2.24}$$

Proof. It suffices to notice that if we set:

$$f(t,y,z) := \psi(t, X_t^{s,x}, y, z),\ S_t := h(X_t^{s,x}),\ \xi := \phi(X_T^{s,x})$$

where $t\in[0,T]$, $y\in\mathbb{R}$, $z\in\Xi^*$ and $X^{s,x}$ is the solution of equation (2.20), then, by (2.21), f, h and S satisfy Hypothesis 1. In particular:

$$\mathbb{E}\int_s^T |f(t,0,0)|^p \,\mathrm{d}t = \mathbb{E}\int_s^T \left|\psi\left(t, X_t^{s,x}, 0, 0\right)\right|^p \,\mathrm{d}t \le c\left(1+|x|^{p(m+1)}\right), \tag{2.25}$$

$$\mathbb{E}\sup_{t\in[s,T]}|S_t|^{2(p-1)} = \mathbb{E}\sup_{t\in[s,T]}|h(X_t^{s,x})|^{2(p-1)} \le c\left(1+|x|^{2(p-1)(m+1)}\right), \tag{2.26}$$

$$\mathbb{E}|\xi|^p = \mathbb{E}|\phi(X_t^{s,x})|^p \le (1+|x|^{p(m+1)}).$$

The claim follows by Proposition 2.3 and Theorem 2.2. □

Remark 2.6. We note that the processes $(X_t^{s,x}, Y_t^{s,x}, Z_t^{s,x})_{t\ge s}$ are independent of $\mathcal{F}_s$.

For further use we also note some useful identities. The first one is the well-known flow property (see for instance [3]): for fixed $0\le r\le s\le T$ we have, $\mathbb{P}$-a.s.,

$$X_t^{r,x} = X_t^{s,X_s^{r,x}}, \qquad t\in[s,T].$$

Thus the triples $(Y_t^{s,X_s^{r,x}}, Z_t^{s,X_s^{r,x}}, K_t^{s,X_s^{r,x}})_{t\in[s,T]}$ and $(Y_t^{r,x}, Z_t^{r,x}, K_t^{r,x})_{t\in[s,T]}$ are solutions to the same backward equation on $[s,T]$. According to the uniqueness statement in Theorem 2.1, the first two components coincide as elements of the space $\mathrm{L}^2_{\mathcal{P}}(\Omega, C([s,T]))\times \mathrm{L}^2_{\mathcal{P}}(\Omega\times[s,T],\Xi^*)$ and we conclude that for fixed $0\le r\le s\le T$ we have, $\mathbb{P}$-a.s.,

$$Y_t^{s,X_s^{r,x}} = Y_t^{r,x}, \qquad t\in[s,T], \tag{2.27}$$

$$Z_t^{s,X_s^{r,x}} = Z_t^{r,x}, \qquad \text{for a.e. } t\in[s,T]. \tag{2.28}$$

The next theorem is devoted to the local Lipschitz continuity of $Y^{s,x}$ with respect to x.

Theorem 2.7. *Let Hypotheses* 2 *and* 3 *hold true and let* $(Y^{s,x}, Z^{s,x}, K^{s,x})$ *be the unique solution of the RBSDE in* (2.17). *Then there exists a constant* $C>0$ *such that,* $\forall x_1, x_2\in H$,

$$|Y_s^{s,x_1} - Y_s^{s,x_2}| \le C\left(1+|x_1|^{m(m+1)}+|x_2|^{m(m+1)}\right)|x_1-x_2|. \tag{2.29}$$

Proof. We initially assume that the generator ψ, the final datum ϕ and the obstacle h are smooth, namely for every $t\in[0,T]$ we assume that $\psi\,(t,\cdot,\cdot,\cdot)\in\mathcal{G}(H\times\mathbb{R}\times\Xi^*,\mathbb{R})$, $h\,(t,\cdot)\in\mathcal{G}(H,\mathbb{R})$ and $\phi\in\mathcal{G}(H,\mathbb{R})$. The idea is to prove that, in the case of smooth (differentiable) coefficients, the solution of the penalized equation (2.23) is differentiable with respect to x, and the derivative is bounded uniformly with respect to n.

In order to work in a "smooth framework", in the penalized BSDE (2.23) we have to replace the non-differentiable penalization $n(y-h)^-$ by the smooth one $n\gamma(y-h)$ where $\gamma:\mathbb{R}\to\mathbb{R}^+$ is a function in $\mathrm{C}^1_b(\mathbb{R})$ verifying:

$$\gamma(y)=0 \text{ for } y\ge 0, \qquad \gamma(y)>0 \text{ for } y<0,$$
$$\gamma(y)=-y \text{ for } y\le -1, \qquad \dot\gamma(y)<0 \text{ for } y<0.$$

Notice that to construct γ it is enough to set $\gamma(y) = \displaystyle\int_0^{-y} \ell(r)\,\mathrm{d}r$ with

$$\ell(r) = 0 \text{ for } r \leq 0, \qquad \ell(r) > 0 \text{ for } r > 0, \qquad \ell(r) = 1 \text{ for } r \geq 1, \qquad \int_0^1 \ell(r)\,\mathrm{d}r = 1.$$

So we consider the following smooth penalized BSDE

$$\begin{cases} -\,\mathrm{d}Y_t^{n,s,x} = \psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x})\ \mathrm{d}t + n\gamma(Y_t^{n,s,x} - h(X_t^{s,x}))\,\mathrm{d}t - Z_t^{n,s,x}\ \mathrm{d}W_t, \quad t \in [0,T], \\ Y_T^{n,s,x} = \phi(X_T^{s,x}), \end{cases} \tag{2.30}$$

and we notice that estimates obtained in Proposition 2.5 are still true for the pair of processes $(Y^{n,s,x}, Z^{n,s,x})$ solution of equation (2.30). Indeed, it is still true that $|y|^{p-1}\hat{y}\gamma(y-s) \leq |s|^{p-1}\gamma(y-s)$ for all $y, s \in \mathbb{R}$, where $\hat{y} : \frac{y}{|y|}$ when $y \neq 0$.

By [10] we know that we can differentiate $(Y^{n,s,x}, Z^{n,s,x})$ with respect to x, and that $(\nabla_x Y^{n,s,x},\ \nabla_x Z^{n,s,x})$ is the solution of the following linear BSDE:

$$\begin{cases} -\,\mathrm{d}\nabla_x dY_t^{n,s,x} = \nabla_x\psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x})\nabla_x X_t^{s,x}\ \mathrm{d}t + \nabla_y\psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x})\nabla_x Y_t^{n,s,x}\ \mathrm{d}t \\ \qquad\qquad + n\dot{\gamma}(Y_t^{n,s,x} - h(X_t^{s,x}))(\nabla_x Y_t^{n,s,x} - \nabla h(X_t^{s,x})\nabla_x X_t^{s,x})\,\mathrm{d}t \\ \qquad\qquad + \nabla_z\psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x})\nabla_x Z_t^{n,s,x}\ \mathrm{d}t - \nabla_x Z_t^{n,s,x}\ \mathrm{d}W_t, \qquad t \in [s,T], \\ \nabla_x Y_T^{n,s,x} = \nabla\phi(X_T^{s,x})\nabla_x X_T^{s,x}. \end{cases}$$

where (see again [10]) $\nabla_x X^{s,x}$ is the mild solution to the following forward equation (to be intended in mild form):

$$\begin{cases} \mathrm{d}\nabla_x X_t^{s,x} = A\nabla_x X_t^{s,x}\ \mathrm{d}t + \nabla_x F(t, X_t^{s,x})\nabla_x X_t^{s,x}\ \mathrm{d}t + \nabla_x G(t, X_t^{s,x})\nabla_x X_t^{s,x}\ \mathrm{d}W_t, \quad t \in [s,T], \\ \nabla_x X_s^{s,x} = I \end{cases}$$

and $I : H \to H$ is the identity operator in H.

We set $\tilde{\mathbb{P}} := \mathcal{E}_T\mathbb{P}$, with

$$\mathcal{E}_T = \exp\left(\int_s^T \nabla_z\psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x})\ \mathrm{d}W_t - \frac{1}{2}\int_s^T |\nabla_z\psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x})|^2\,\mathrm{d}t\right). \tag{2.31}$$

Since $\nabla_z\psi$ is bounded, Girsanov Theorem yields that $\tilde{\mathbb{P}}$ is a probability measure equivalent to $\mathbb{P}$ and

$$\tilde{W}_\tau = -\int_s^\tau \nabla_z\psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x})\,\mathrm{d}t + W_\tau, \quad s \leq \tau \leq T$$

is a $\tilde{\mathbb{P}}$-cylindrical Wiener process.

The pair $(\nabla_x Y^{n,s,x}, \nabla_x Z^{n,s,x})$ solves the following BSDE for $t \in [s,T]$:

$$\begin{cases} -\,\mathrm{d}\nabla_x Y_t^{n,s,x} = \nabla_x \psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x}) \nabla_x X_t^{s,x}\ \mathrm{d}t \\ \qquad\qquad\qquad +\nabla_y \psi(t, X_t^{s,x}, Y_t^{n,s,x}, Z_t^{n,s,x}) \nabla_x Y_t^{n,s,x}\ \mathrm{d}t \\ \qquad\qquad\qquad +n\dot{\gamma}(Y_t^{n,s,x} - h(X_t^{s,x}))(\nabla_x Y_t^{n,s,x} - \nabla h(X_t^{s,x})\nabla_x X_t^{s,x})\,\mathrm{d}t - \nabla_x Z_t^{n,s,x}\ \mathrm{d}\tilde{W}_t, \\ \nabla_x Y_T^{n,s,x} = \nabla \phi(X_T^{s,x}) \nabla_x X_T^{s,x}. \end{cases} \tag{2.32}$$

Multiplying $\nabla_x Y_t^{n,s,x}$ by $\exp\{\int_s^t (\nabla_y \psi(t, X_\sigma^{s,x}, Y_\sigma^{n,s,x}, Z_\sigma^{n,s,x}) + n\dot{\gamma}(Y_\sigma^{n,s,x} - h(X_\sigma^{s,x})))\ \mathrm{d}\sigma\}$ and choosing $t = s$ we get:

$$\begin{aligned} \nabla_x Y_s^{n,s,x} =\, & \mathbb{E}\Bigg[\mathcal{E}_T \int_s^T \exp\left\{\int_s^\tau (\nabla_y \psi(t, X_\sigma^{s,x}, Y_\sigma^{n,s,x}, Z_\sigma^{n,s,x}) + n\dot{\gamma}(Y_\sigma^{n,s,x} - h(X_\sigma^{s,x}))\ \mathrm{d}\sigma\right\} \\ & \times \Big(\nabla_x \psi(\tau, X_\tau^{s,x}, Y_\tau^{n,s,x}, Z_\tau^{n,s,x}) - n\dot{\gamma}(Y_\tau^{n,s,x} - h(X_\tau^{s,x})\nabla h(X_\tau^{s,x}))\nabla_x X_\tau^{s,x}\Big)\ \mathrm{d}\tau\Bigg] \\ & + \mathbb{E}\left[\mathcal{E}_T \exp\left\{\int_s^T \nabla_y \psi(t, X_\sigma^{s,x}, Y_\sigma^{n,s,x}, Z_\sigma^{n,s,x}) + n\dot{\gamma}(Y_\sigma^{n,s,x} - h(X_\sigma^{s,x}))\ \mathrm{d}\sigma\right\} \nabla\phi(X_T^{s,x})\nabla_x X_T^{s,x}\right]. \end{aligned} \tag{2.33}$$

Taking into account that $\nabla_y \psi$ is bounded by Hypothesis 3 and that $\dot{\gamma} \leq 0$:

$$\begin{aligned} |\nabla_x Y_s^{n,s,x}| \leq\, & c\mathbb{E}\left[\mathcal{E}_T \left(|\nabla\phi(X_T^{s,x})\nabla_x X_T^{s,x}| + |\int_s^T \nabla_x \psi(\tau, X_\tau^{s,x}, Y_\tau^{n,s,x}, Z_\tau^{n,s,x}) \nabla_x X_\tau^{s,x}\ \mathrm{d}\tau|\right)\right] \\ & + cn\mathbb{E}\left[\mathcal{E}_T |\int_s^T \exp\left\{\int_s^\tau n\dot{\gamma}(Y_\sigma^{n,s,x} - h(X_\sigma^{s,x}))\ \mathrm{d}\sigma\right\} \dot{\gamma}(Y_\tau^{n,s,x} - h(X_\tau^{s,x}))\nabla h(X_\tau^{s,x})\nabla_x X_\tau^{s,x}\ \mathrm{d}\tau|\right] \\ =\, & I + II. \end{aligned}$$

We start by estimating I. Here and in the following we again denote by c a constant that can vary from line to line but does not depend neither on n nor on x.

$$I \leq c\mathbb{E}\left[\mathcal{E}_T|\nabla\phi(X_T^{s,x})\nabla_x X_T^{s,x}|\right] + c\mathbb{E}\left[\mathcal{E}_T \int_s^T |\nabla_x \psi(\tau, X_\tau^{s,x}, Y_\tau^{n,s,x}, Z_\tau^{n,s,x}) \nabla_x X_\tau^{s,x}|\ \mathrm{d}\tau\right].$$

Taking into account that $\mathbb{E}\mathcal{E}_T^p \leq c$, by Hölder inequality, with p, q, r conjugate exponents $p > 1$, $1 < q < 2$, $qm > 2$, (m being as in Hypothesis 3) we get:

$$\mathbb{E}\Big[\mathcal{E}_T|\nabla\phi(X_T^{s,x})\nabla_x X_T^{s,x}|\Big] \leq c\Big(\mathbb{E}\Big[|\nabla\phi(X_T^{s,x})|^q\Big]\Big)^{1/q}\Big(\mathbb{E}\Big[|\nabla_x X_T^{s,x}|^r\Big]\Big)^{1/r} \leq c(1+|x|^m),$$

where we have used the estimate on $\nabla_x X_T^{s,x}$ stated in Proposition 3.3 of [10].

In a similar way we can estimate (for $q > 2$):

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{E}_T \int_s^T |\nabla_x \psi(\tau, X_\tau^{s,x}, Y_\tau^{n,s,x}, Z_\tau^{n,s,x}) \nabla_x X_\tau^{s,x}| \, \mathrm{d}\tau\right] \\
&\leq c\mathbb{E}\left[\mathcal{E}_T \int_s^T (1 + |X_\tau^{s,x}|^m + |Y_\tau^{n,s,x}|^m)\,(1 + |Z_\tau^{n,s,x}|)\, |\nabla_x X_\tau^{s,x}| \, \mathrm{d}\tau\right] \\
&\leq c\left(\mathbb{E}\left[\left(1 + \sup_{\tau\in[s,T]} |X_\tau^{s,x}|^{mq} + \sup_{\tau\in[s,T]} |Y_\tau^{n,s,x}|^{mq}\right) \sup_{\tau\in[s,T]} |\nabla_x X_\tau^{s,x}|^q \left(\int_s^T (1 + |Z_\tau^{n,s,x}|) \, \mathrm{d}\tau\right)^q\right]\right)^{1/q} \\
&\leq c\left(\mathbb{E}\left[1 + \sup_{\tau\in[s,T]} |X_\tau^{s,x}|^{3mq} + \sup_{\tau\in[s,T]} |Y_\tau^{n,s,x}|^{3mq}\right]\right)^{1/3q} \left(\mathbb{E} \sup_{\tau\in[s,T]} |\nabla_x X_\tau^{s,x}|^{3q}\right)^{1/3q} \\
&\qquad \times \left(\mathbb{E}\left[\int_s^T \left(1 + |Z_\tau^{n,s,x}|^2\right)^{\frac{3}{2}q} \, \mathrm{d}\tau\right]^{3q}\right)^{1/3q} \\
&\leq c\left(1 + |x|^{m(m+1)}\right).
\end{aligned}
$$

where we have used estimates (2.22) and Proposition 2.4.

As far as II is concerned, let p, q and $\bar{p}$, $\bar{q}$ be two pairs of conjugate exponents, and define:

$$
l(\tau) := -n\dot{\gamma}(Y_\tau^{n,s,x} - h(X_\tau^{s,x})) \geq 0, \ \tau \in [s, T].
$$

Then:

$$
\begin{aligned}
&\mathbb{E}\Big[\mathcal{E}_T \Big| \int_s^T \exp\Big(-\int_s^\tau l_\sigma \, \mathrm{d}\sigma\Big) l_\tau \nabla h(X_\tau^{s,x}) \nabla_x X_\tau^{s,x} \, \mathrm{d}\tau\Big|\Big] \\
&\leq c\left(\mathbb{E}\Big[\Big(1 + \sup_{\tau\in[s,T]} |X_\tau^{s,x}|^m\Big) \sup_{\tau\in[s,T]} |\nabla_x X_\tau^{s,x}| \int_s^T \exp\Big(-\int_s^\tau l_\sigma \, \mathrm{d}\sigma\Big) l_\tau \, \mathrm{d}\tau\Big]^q\right)^{1/q} \\
&\leq c\left(\mathbb{E}\Big[\left(1 + \sup_{\tau\in[s,T]} |X_\tau^{s,x}|^{m\bar{p}q}\right) \sup_{\tau\in[s,T]} |\nabla_x X_\tau^{s,x}|^{\bar{p}q}\Big]\right)^{1/(\bar{p}q)} \left(\mathbb{E}\Big[\int_s^T \exp\Big(-\int_s^\tau l_\sigma \, \mathrm{d}\sigma\Big) l_\tau \, \mathrm{d}\tau\Big]^{q\bar{q}}\right)^{1/(q\bar{q})} \\
&\leq c\,(1 + |x|^m) \left(\mathbb{E}\Big[1 - \exp\Big(-\int_s^\tau l_\sigma \, \mathrm{d}\sigma\Big)\Big]^{q\bar{q}}\right)^{1/(q\bar{q})} \leq c\,(1 + |x|^m),
\end{aligned}
$$

where, in the last inequality, we have used that:

$$
\int_s^T \exp\left\{-\int_s^\tau l(\sigma) \, \mathrm{d}\sigma\right\} l(\tau) \, \mathrm{d}\tau = 1 - \exp\left\{-\int_s^T l(\sigma) \, \mathrm{d}\sigma\right\}.
$$

So

$$
|\nabla_x Y_s^{n,s,x}| \leq c\left(1 + |x|^{m(m+1)}\right), \tag{2.34}
$$

where c may depend on T, on the coefficients A, F, G, ψ, h, ϕ, but not on n. By (2.34) we get that, for any $x, y \in H$:

$$
|Y_s^{n,s,x} - Y_s^{n,s,y}| \leq c|x - y| \left(1 + |x|^{m(m+1)} + |y|^{m(m+1)}\right). \tag{2.35}
$$

Letting $n \to \infty$ (see (2.6)) we obtain the desired Lipschitz continuity of $Y_s^{s,x}$, namely:

$$
|Y_s^{s,x} - Y_s^{s,y}| \leq c|x - y| \left(1 + |x|^{m(m+1)} + |y|^{m(m+1)}\right), \qquad \forall\, x,\, y \in H. \tag{2.36}
$$

We notice that in the equations (2.35) and (2.36) the constant c depends on ϕ, h and ψ only through the constants L and L'.

We have now to remove the assumption of differentiability on the coefficients ψ, h, ϕ in the reflected BSDE. Since ψ is Lipschitz continuous with respect to y and z, and $\forall\, t\,, y\,, z\, \in [0,T]\times\mathbb{R}\times\varXi$, $\psi(t,\cdot,y,z)\,, h\,, \phi$ are locally Lipschitz continuous with respect to x, then by taking their inf-sup convolution $(\psi_k,\, \phi_k,\, h_k)_{k\geq 1}$ (see *e.g.* [4] for the notion of inf-sup convolution, and [15,16] for the use of inf-sup convolutions in the Lipschitz and locally Lipschitz case) we obtain a sequence of differentiable functions that verify Hypothesis 3 with constants L and L' independent on k and that converge pointwise to $(\psi,\, \phi,\, h)$.

We denote by $(Y^{n,k,s,x}, Z^{n,k,s,x}, K^{n,k,s,x})$ the solution of the penalized RBSDEs with regularized coefficients, namely

$$\begin{cases} -\,\mathrm{d}Y_t^{n,k,s,x} = \psi_k(t, X_t^{s,x}, Y_t^{n,k,s,x}, Z_t^{n,k,s,x})\ \mathrm{d}t + n\gamma(Y_t^{n,k,s,x} - h_k(X_t^{s,x}))\,\mathrm{d}t - Z_t^{n,k,s,x}\ \mathrm{d}W_t, \\ \hfill t\in[0,T], \\ Y_T^{n,k,s,x} = \phi_k(X_T^{s,x}). \end{cases} \tag{2.37}$$

By (2.35) and (2.36) we get that for any $x, y \in H$:

$$|Y_s^{n,k,s,x} - Y_s^{n,k,s,y}| \leq c|x-y|\left(1 + |x|^{m(m+1)} + |y|^{m(m+1)}\right), \tag{2.38}$$

where c does not depend nor on n neither on k. By standard results on BSDEs (see [11]) we know that:

$$(Y^{n,k,s,x}, Z^{n,k,s,x}) \to (Y^{n,s,x}, Z^{n,s,x}) \text{ in } \mathrm{L}^p_{\mathcal{P}}(\varOmega, \mathrm{C}^0([0,T])) \times \mathrm{L}^p_{\mathcal{P}}(\varOmega\times[0,T], \varXi),$$

where $(Y^{n,s,x}, Z^{n,s,x})$ is the solution of equation (2.30). In particular $Y_s^{n,k,s,x} \to Y_s^{n,s,x}$. Finally by (2.6) we have $Y^{n,s,x} \uparrow Y^{s,x}$ (the fact that here the smooth and strictly monotone function γ replaces the negative part $(\,\cdot\,)^-$ is inessential). The claim then follows by (2.38). □

Remark 2.8. Notice that if h and ϕ are bounded and Lipschitz, if for every $s\in[0,T]$, $\sup_{x\in H}|\psi(s,x,0,0)| < \infty$ and, as a function of x, ψ is Lipschitz uniformly with respect to the other variables (that is Hypothesis 3, point 3 holds true with $m = 0$) then, repeating the arguments in Proposition 2.5, we deduce that the processes $(Y^{s,x},\, Z^{s,x})$ are bounded with respect to x, that is:

$$\mathbb{E}\sup_{t\in[s,T]}|Y_t^{s,x}|^p + \mathbb{E}\left(\int_0^T |Z_t^{s,x}|^2\ \mathrm{d}t\right)^{p/2} < C. \tag{2.39}$$

3. Obstacle problem for a semilinear parabolic PDE: Solution *via* RBSDEs

In this Section we consider an obstacle problem for a semilinear PDE in an infinite dimensional Hilbert space H and we solve it in a suitable sense by means of reflected BSDEs. We will deal with the following obstacle problem, formally written:

$$\begin{cases} \min\left(u(t,x) - h(x), -\dfrac{\partial u}{\partial t}(t,x) - \mathcal{L}_t u\,(t,x) - \psi\,(t,x,u\,(t,x)\,, \nabla u\,(t,x)\,G\,(t,x))\right) = 0 \\ \hfill t\in[0,T]\,,\ x\in H \\ u(T,x) = \phi\,(x)\,, \end{cases} \tag{3.1}$$

where the directional generalized gradient $\nabla u\,(t,x)\,G(t,x)$ will be defined below (see also [13]). Moreover $(\mathcal{L}_t)_{t\in[0,T]}$ is the infinitesimal generator of the Markov process generated by equation (2.20), namely, for a sufficiently smooth function $f : H \to \mathbb{R}$, $\mathcal{L}_t$ is defined by:

$$\mathcal{L}_t f\,(x) = \frac{1}{2}\mathrm{Trace}\left(G\,(t,x)\,G^*\,(t,x)\,\nabla^2 f\,(x)\right) + \langle Ax, \nabla f\,(x)\rangle_H + \langle F\,(t,x)\,, \nabla f\,(x)\rangle_H.$$

It is worth noticing that, if $X^{s,x}$, $0 \le s \le t \le T$ is the mild solution to equation (2.20) and we denote by $P_{s,t}$ the operator

$$P_{s,t}[f](x) = \mathbb{E} f(X_t^{s,x}), \quad f : H \to \mathbb{R} \text{ continuous and bounded.}$$

then $\mathcal{L}_t$ is (at least formally) the generator of the transition semigroup $(P_{s,t})_{0\le s\le t\le T}$.

The above leads us to consider solutions of the obstacle problem (3.1) in mild sense, as we are going to state in the next sections.

3.1. The generalized directional gradient

We observe that, under our assumptions we can not expect that the value function of an optimal stopping problem is differentiable. For instance, consider the following trivial deterministic state equation without control:

$$\begin{cases} \mathrm{d}X_t^{s,x} = -1 \\ X_s^{s,x} = x \in \mathbb{R}, \end{cases}$$

and the following stopping problem:

$$J(s,x,\tau) = \phi(X_T^{s,x})\,\chi_{\{\tau=1\}} + h(X_\tau^{s,x})\,\chi_{\{\tau<1\}},$$

where χ denotes the indicator function, $h \le \phi$ and we choose a non decreasing h. It is clear that the value function is given by:

$$v(x) = \max(\phi(x-1), h(x)),$$

and that, even if h and ϕ are differentiable, v may fail to be differentiable (for instance take $h(x) = \arctan(x)$ and $\phi(x) = \arctan(x) + \pi/8$).

Notice that in the above example we rely on the degeneracy of the noise. The issue of differentiability of u when noise is non degenerate is very interesting but falls out of the scope of the present work.

To handle the lack of regularity of u the classical derivative ∇u will be replaced by the notion of generalized gradient, whose definition is given below.

Theorem 3.1. *Assume that Hypothesis 2 holds and that $u : [0,T] \times H \to \mathbb{R}$ is a Borel measurable function satisfying, for some $r > 0$:*

$$|u(t,x) - u(t,x')| \le c(1 + |x|^r + |x'|^r)|x - x'|. \tag{3.2}$$

Then there exists a Borel measurable function $\zeta : [0,T] \times H \to \Xi^*$ with the following properties.

(i) *For every $s \in [0,T]$, $x \in H$ and $p \in [2,\infty)$,*

$$\mathbb{E}\int_s^T |\zeta(\tau, X_\tau^{t,x})|^p \;\mathrm{d}\tau < +\infty. \tag{3.3}$$

(ii) *For any $\xi \in \Xi$ denote by $W^\xi = (W_t^\xi)_{t\ge 0}$ the projection $W_t^\xi := \langle W_t, \xi\rangle$. Then, for any $x \in H$ and $0 \le s \le T' < T$, the processes $\{u(t, X_t^{s,x}), t \in [s,T]\}$ and W^ξ admit a joint quadratic variation, in the sense of [19], on the interval $[s,T']$ and*

$$\langle u(\cdot, X_\cdot^{s,x}), W^\xi\rangle_{[s,T']} = \int_s^{T'} \zeta(t, X_t^{s,x})\xi \;\mathrm{d}t, \qquad \mathbb{P}-a.s..$$

(iii) *There exists a Borel measurable function $\rho : [0,T] \times H \to H^*$ such that for all $t \in [s,T]$ and all $x \in H$*

$$\zeta(t, X_t^{s,x}) = \rho(t, X_t^{s,x})G(t, X_t^{s,x}) \quad \mathbb{P}\text{-a.s. for a.e. } t \in [s,T].$$

Proof. The proof is given in [13], Section 4. In that paper it is also noticed, see remark 3.1, that uniqueness can be stated in the following sense: if $\hat{\zeta}$ is another function with the stated properties then for $0 \le s \le t \le T$ and $x \in H$ we have $\zeta(t, X_t^{s,x}) = \hat{\zeta}(t, X_t^{s,x})$, $\mathbb{P}-$a.s. for a.e. $t \in [s,T]$. □

Definition 3.2. Let $u : [0,T] \times H \to \mathbb{R}$ be a Borel measurable function satisfying (3.2). The family of all measurable functions $\zeta : [0,T] \times H \to \varXi^*$ satisfying properties (i) and (ii) in Theorem 3.1 will be called the *generalized directional gradient of* u and denoted by $\widetilde{\nabla}^G u$.

3.2. Mild solutions of the obstacle problem in the sense of the generalized directional gradient

We are now in a position to give the precise definition of supersolution for the problem (3.1).

Definition 3.3. We say that a Borel measurable function $\bar{u} : [0,T] \times H \to \mathbb{R}$ is a mild supersolution of the obstacle problem (3.1) in the sense of the generalized directional gradient if the following holds:

(1) for some $C > 0, r \ge 0$ and for every $s \in [0,T]$, $x, x' \in H$

$$|\bar{u}(s,x) - \bar{u}(s,x')| \le C|x - x'|(1 + |x| + |x'|)^r, \qquad |u(s,0)| \le C;$$

(2) for every $s \in [0,T]$, $x \in H$,

$$\bar{u}(s,x) \ge h(x);$$

(3) for all $0 \le s \le t \le T$ and $x \in H$

$$\bar{u}(s,x) \ge P_{s,t}[u(t,\cdot)](x) + \int_s^t P_{s,\tau}\Big[\psi(\tau,\cdot,\bar{u}(\tau,\cdot),\zeta(\tau,\cdot))\Big](x)\ \mathrm{d}\tau, \tag{3.4}$$

where ζ is an arbitrary element of the generalized gradient $\widetilde{\nabla}^G \bar{u}$;

(4) $\bar{u}(T,\cdot) = \phi$.

Remark 3.4. We notice that the notion of supersolution stated above is related to the inequality in formula (3.4) and does not refer to the definition of supersolution in viscosity sense (see, for instance [9]). In this paper the issue of uniqueness of the solution to the obstacle problem is bypassed by constructing a minimal supersolution but it remains essentially open. In this context viscosity theory (at least in its classical formulation) will probably not be helpful since it requires strong assumptions to be applied to infinite dimensional HJB equations even when there is no obstacle.

We are now ready to state the main result of this paper.

Theorem 3.5. *Assume that Hypotheses 2 and 3 hold true. Let us define*

$$u(s,x) = Y_s^{s,x}, \tag{3.5}$$

where $(Y^{s,x}, Z^{s,x})$ *is the solution to the reflected BSDE in* (2.17). *Then* u *is a mild supersolution in the sense of the generalized directional gradient for the obstacle problem* (3.1).

Moreover u *is minimal in the sense that given any* $\bar{u}$, *supersolution of* (3.1) *in the sense of Definition 3.3, it holds* $u(s,x) \le \bar{u}(s,x)$, *and* $s \in [0,T]$, $x \in H$

Finally, if in addition $\sup_{s\in[0,T],x\in H} |\psi(s,x,0,0)| < \infty$ *and* ϕ *and* h *are bounded then* u *is also bounded.*

Proof. If we set $u(s,x) := Y_s^{s,x}$ then, by Theorem 2.7, u has the regularity required in Definition 3.3, point 1, moreover points 2 and 4 immediately follow since Y the is solution to the RBSDE in (2.17).

As far as point 3 of definition 3.3 is concerned, by (2.17) we get:

$$u(s,x) = Y_t^{s,x} + \int_s^t \psi(\tau, X_\tau^{s,x}, Y_\tau^{s,x}, Z_\tau^{s,x})\ \mathrm{d}\tau + K_t^{s,x} - K_s^{s,x} - \int_s^t Z_\tau^{s,x}\ \mathrm{d}W_\tau, \tag{3.6}$$

Fixed $\xi \in \Xi$, we now compute joint quadratic variation of both sides of (3.6) with W^ξ. Proposition 2.1 in [13] and Theorem 3.1 yield that $\widetilde{\nabla}^G u$ exists. Moreover, if $\zeta \in \widetilde{\nabla}^G u$, we have:

$$\langle u(\cdot, X_\cdot^{s,x}), W_\cdot^\xi\rangle_{[s,t]} = \int_s^t \zeta(\sigma, X_\sigma^{s,x})\xi\ \mathrm{d}\sigma.$$

On the other hand by the Markov property stated in Remark 2.6:

$$u(t, X_t^{s,x}) = Y_t^{t, X_t^{s,x}} = Y_t^{s,x}$$

and by (2.17) we deduce:

$$\langle Y_\cdot^{s,x}, W_\cdot\rangle_{[s,t]} = \int_s^t Z_\sigma^{s,x}\xi\ \mathrm{d}\sigma.$$

Comparing these two expressions we get

$$\int_s^t \zeta(\sigma, X_\sigma^{s,x})\xi\ \mathrm{d}\sigma = \int_s^t Z_\sigma^{s,x}\xi\ \mathrm{d}\sigma, \quad \mathbb{P}-\text{a.s.}\ . \tag{3.7}$$

Since both sides of (3.7) are continuous with respect to t, it follows that, $\mathbb{P}$-a.s., they coincide for all $t \in [s,T]$. This implies that $\zeta(\sigma, X_\sigma^{s,x}) = Z_\sigma^{s,x}$, $\mathbb{P}$-a.s. for a.e. $\sigma \in [s,T]$. Equation (3.6) can be rewritten as

$$u(s,x) = Y_t^{s,x} + \int_s^t \psi(\tau, X_\tau^{s,x}, u(\tau, X_\tau^{s,x}), \zeta(\tau, X_\tau^{s,x}))\ \mathrm{d}\tau + K_t^{s,x} - K_s^{s,x} - \int_s^t Z_\tau^{s,x}\ \mathrm{d}W_\tau. \tag{3.8}$$

Computing the conditional expectation $\mathbb{E}^{\mathcal{F}_s}$, taking into account that K is nondecreasing we get:

$$u(s,x) \geq Y_t^{s,x} + \int_s^t P_{s,\tau}\Big[\psi(\tau, \cdot, u(\tau,\cdot), \zeta(\tau,\cdot))\Big](x)\ \mathrm{d}\tau. \tag{3.9}$$

We have shown that u is a mild supersolution along Definition 3.3.

Now we have to prove that u is the minimal supersolution. Let $\bar{u}$ be any supersolution and set $\bar{Y}_t^{s,x} = \bar{u}(t, X_t^{s,x})$. Then for every $\sigma \in [s,t]$, with $0 \leq s \leq t$, by point 3 of Definition 3.3, having replaced x by $X_\sigma^{s,x}$ which is $\mathcal{F}_\sigma$-measurable,

$$\bar{u}(\sigma, X_\sigma^{s,x}) \geq \mathbb{E}^{\mathcal{F}_\sigma}\bar{u}\left(t, X_t^{\sigma, X_\sigma^{s,x}}\right) + \mathbb{E}^{\mathcal{F}_\sigma}\int_\sigma^t \psi\left(\tau, X_\tau^{\sigma, X_\sigma^{s,x}}, \bar{Y}_\tau^{\sigma, X_\sigma^{s,x}}, \bar{\zeta}\left(\tau, X_\tau^{\sigma, X_\sigma^{s,x}}\right)\right)\ \mathrm{d}\tau. \tag{3.10}$$

So it turns out that

$$(L_\sigma^{s,x})_{\sigma\in[s,T]} := \left(-\bar{u}(\sigma, X_\sigma^{s,x}) - \int_s^\sigma \psi\left(\tau, X_\tau^{s,x}, \bar{Y}_\tau^{s,x}, \bar{\zeta}(\tau, X_\tau^{s,x})\right)\ \mathrm{d}\tau\right)_{\sigma\in[s,T]}$$

is a submartingale. Hypothesis 3 on ψ; the growth property of u as required in Definition 3.3, point 1; relation (3.3) and finally Proposition 2.4 imply that $L^{s,x}$ is a uniformly integrable submartingale, so it is of class (D) and the Doob−Meyer decomposition can be applied, see *e.g.* Definition 4.8 and Theorem 4.10 in Chapter 1 of [8]. Namely $L^{s,x}$ can be decomposed into:

$$L_\sigma^{s,x} = \bar{M}_\sigma^{s,x} + \bar{K}_\sigma^{s,x},$$

where $\bar{K}^{s,x}$ is an integrable nondecreasing process such that $\bar{K}^{s,x}_s = 0$, and $\bar{M}^{s,x}$ is a uniformly integrable martingale. Moreover (see [5], Chap. VII relation (15.1)) since

$$\mathbb{E} \sup_{\sigma\in[s,T]} |L^{s,x}_\sigma|^2 < \infty$$

we also have

$$\bar{K}^{s,x}_T \in \mathrm{L}^2(\Omega).$$

Notice that $(\bar{M}^{s,x}_t)_{t\in[s,T]}$ is a martingale with respect to the complete filtration generated by the Wiener process W, thus, by the martingale representation theorem, see again [3, 8], there exists a process $\bar{Z} \in \mathrm{L}^2_{\mathcal{P}}(\Omega\times[s,T];\mathrm{L}^2(\Xi,\mathbb{R}))$ such that

$$\bar{M}^{s,x}_\sigma = -\left[u(s,x) + \int_s^\sigma \bar{Z}^{s,x}_\tau \,\mathrm{d}W_\tau\right].$$

The above, together with the definition of $L^{s,x}$, imply that, for all $\sigma \in [s,T]$:

$$u(s,x) = \bar{u}(\sigma, X^{s,x}_\sigma) + \int_s^t \psi(\tau, X^{s,x}_\tau, \bar{Y}^{s,x}_\tau, \bar{\zeta}(\tau, X^{s,x}_\tau) \,\mathrm{d}\tau + \bar{K}^{s,x}_\sigma - \bar{K}^{s,x}_s - \int_s^t \bar{Z}^{s,x}_\tau \,\mathrm{d}W_\tau, \tag{3.11}$$

that is, $\forall\, 0 \le s \le t \le T$:

$$\bar{Y}^{s,x}_t = \bar{Y}^{s,x}_T + \int_t^T \psi(\tau, X^{s,x}_\tau, \bar{Y}^{s,x}_\tau, \bar{\zeta}(\tau, X^{s,x}_\tau)) \,\mathrm{d}\tau + \bar{K}^{s,x}_t - \bar{K}^{s,x}_s - \int_t^T \bar{Z}^{s,x}_\tau \,\mathrm{d}W_\tau. \tag{3.12}$$

Finally we have to identify $\zeta(\tau, X^{s,x}_\tau)$ and $\bar{Z}^{s,x}_\tau$, $\mathbb{P}$-a.s. for a.e. $\tau \in [s,T]$. To this aim, proceeding as before, for $\xi \in \Xi$, let us compute the joint quadratic variation of both sides of (3.11) with W^ξ. Notice that the finite variation term K does not give any contribution. So Proposition 2.1 in [13] and Theorem 3.1 yield, for $s \le \sigma < T$ and $\zeta \in \widetilde{\nabla}^G u$,

$$\int_s^\sigma \zeta(\tau, X^{s,x}_\tau)\xi \,\mathrm{d}\tau = \int_s^\sigma \bar{Z}^{s,x}_\tau \xi \,\mathrm{d}\tau, \tag{3.13}$$

$\mathbb{P}$-a.s.. Again both sides of (3.13) are continuous with respect to σ and, $\mathbb{P}$-a.s., they coincide for all $\sigma \in [s,T]$. This implies that $\zeta(\sigma, X^{s,x}_\sigma) = \bar{Z}^{s,x}_\sigma$, $\mathbb{P}$-a.s. for a.e. $\sigma \in [s,T]$. So we get that, by defining: $\bar{Y}^{s,x}_t := \bar{u}(t, X^{s,x}_t)$, $t \in [s,T]$, the couple of processes $(\bar{Y}^{s,x}, \bar{Z}^{s,x})$ verifies:

$$\begin{cases} -\,\mathrm{d}\bar{Y}^{s,x}_t = \psi(t, X^{s,x}_t, \bar{Y}^{s,x}_t, \bar{Z}^{s,x}_t) \,\mathrm{d}t + \mathrm{d}\bar{K}^{s,x}_t - \bar{Z}^{s,x}_t \,\mathrm{d}W_t, \qquad t \in [s,T], \\ \bar{Y}_T = \phi(X^{s,x}_T), \\ \bar{Y}^{s,x}_t \ge h(X^{s,x}_t). \end{cases} \tag{3.14}$$

Now we have to compare $\bar{Y}^{s,x}$ with $Y^{s,x}$. To this aim, extending an argument used in [2], we compare $\bar{Y}^{s,x}$ with the penalized solution $Y^{n,s,x}$ of equation (2.23).

Applying Itô formula to the process $\mathrm{e}^{n(T-t)}Y^{n,s,x}_t$ and taking into account that $y + (y-h)^- = y \vee h$, we get:

$$\begin{cases} -\,\mathrm{d}\mathrm{e}^{n(T-t)}Y^{n,s,x}_t = \mathrm{e}^{n(T-t)}\psi(t, X^{s,x}_t, Y^{n,s,x}_t, Z^{n,s,x}_t) \,\mathrm{d}t + n\mathrm{e}^{n(T-t)}\,[Y^{n,s,x}_t \vee h(X^{s,x}_t)] \,\mathrm{d}t \\ \qquad\qquad -\mathrm{e}^{n(T-t)} Z^{n,s,x}_t \,\mathrm{d}W_t, \qquad t \in [s,T], \\ Y^{n,s,x}_T = \phi(X^{s,x}_T). \end{cases} \tag{3.15}$$

Applying Itô formula to the process $\mathrm{e}^{n(T-t)}\bar{Y}^{s,x}_t$ we get

$$\begin{cases} -\,\mathrm{d}\mathrm{e}^{n(T-t)}\bar{Y}^{s,x}_t = n\mathrm{e}^{n(T-t)}\bar{Y}^{s,x}_t \,\mathrm{d}t + \mathrm{e}^{n(T-t)}\psi(t, X^{s,x}_t, \bar{Y}^{s,x}_t, \bar{Z}^{s,x}_t) \,\mathrm{d}t + \mathrm{e}^{n(T-t)} \,\mathrm{d}\bar{K}^{s,x}_t \\ \qquad\qquad -\mathrm{e}^{n(T-t)}\bar{Z}^{s,x}_t \,\mathrm{d}W_t, \qquad t \in [s,T], \\ \bar{Y}^{s,x}_T = \phi(X^{s,x}_T). \end{cases} \tag{3.16}$$

Notice that in (3.16) we can replace $\bar{Y}_t^{s,x}$ by $\bar{Y}_t^{s,x} \vee h(X_t^{s,x})$ (since $\bar{u} \geq h$ by the definition of supersolution to problem (3.1)). Assume for a moment the following Lemma.

Lemma 3.6 (Comparison)**.** *Let $f^i : \Omega \times [0,T] \times \mathbb{R} \times \Xi \to \mathbb{R}$, $i = 1, 2$ satisfy Hypothesis 1 with $p = 2$, fix $\xi \in \mathrm{L}^2_{\mathcal{F}_T}(\Omega)$ and let K be a progressively measurable nondecreasing processes with $\mathbb{E}\, K_T^2 < \infty$. If (Y^1, Z^1) and (Y^2, Z^2) with $Y^i \in \mathrm{L}^2_{\mathcal{P}}(\Omega, \mathrm{C}^0([0,T]))$ and $Z^i \in \mathrm{L}^2_{\mathcal{P}}(\Omega \times [0,T], \Xi)$, $i = 1, 2$, are the solutions to the following equations of backward type:*

$$\begin{cases} -\,\mathrm{d}Y_t^1 = f^1(t, Y_t^1, Z_t^1)\ \mathrm{d}t + \mathrm{d}K_t - Z_t^1\ \mathrm{d}W_t, \qquad t \in [0,T], \\ Y_T^1 = \xi, \end{cases} \tag{3.17}$$

$$\begin{cases} -\,\mathrm{d}Y_t^2 = f^2(t, Y_t^2, Z_t^2)\ \mathrm{d}t - Z_t^2\ \mathrm{d}W_t, \qquad t \in [0,T], \\ Y_T^2 = \xi. \end{cases} \tag{3.18}$$

and

$$(\delta f)_t := f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2) \geq 0, \ \ \mathrm{d}\mathbb{P} \times \mathrm{d}t \quad a.s, \tag{3.19}$$

then we have that $Y_t^1 \geq Y_t^2$, $\mathbb{P}$-almost surely for any $t \in [0,T]$.

By applying Lemma 3.6 to the BSDEs (3.15) and (3.16) we get a comparison for the processes $\left(\mathrm{e}^{n(T-t)} Y_t^{n,s,x}\right)_{t \in [s,T]}$ and $\left(\mathrm{e}^{n(T-t)} \bar{Y}_t^{s,x}\right)_{t \in [s,T]}$, namely:

$$\mathrm{e}^{n(T-t)} \bar{Y}_t^{s,x} \geq \mathrm{e}^{n(T-t)} Y_t^{n,s,x} \tag{3.20}$$

$\mathbb{P}$-almost surely and for any time t. Consequently:

$$\bar{Y}_t^{s,x} \geq Y_t^{n,s,x}. \tag{3.21}$$

Now we let $n \to \infty$. By [6], Section 6, $Y_t^{n,s,x} \uparrow Y_t^{s,x}$ for any $s \leq t \leq T$ and $\mathbb{P}$-a.s.. So taking $s = t$ in (3.21) we finally get

$$\bar{u}(s,x) \geq u(s,x). \tag{3.22}$$

So the minimality of u is proved.

The other properties required by Definition 3.3 follow by estimates (2.22), which pass to the limit as $n \to \infty$. $\square$

In order to complete the proof of Theorem 3.5, we have to prove Lemma 3.6.

Proof. **Lemma 3.6**. We adapt the proof of the classical comparison theorem for BSDEs given in [7], Theorem 2.2, to the equations (3.17) and (3.18). Let:

$$\Delta_y f_t^1 = \frac{f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^1)}{Y_t^1 - Y_t^2} \qquad \text{if } Y_t^1 - Y_t^2 \neq 0, \qquad \Delta_y f_t^1 = 0 \text{ otherwise},$$

$$\Delta_z f_t^1 = \frac{f^1(t, Y_t^2, Z_t^1) - f^1(t, Y_t^2, Z_t^2)}{|Z_t^1 - Z_t^2|^2}(Z_t^1 - Z_t^2) \qquad \text{if } Z_t^1 - Z_t^2 \neq 0, \qquad \Delta_z f_t^1 = 0 \text{ otherwise}.$$

Moreover let (δf) be defined as in (3.19), and $(\delta Y)_t := Y_t^1 - Y_t^2$, $(\delta Z)_t := Z_t^1 - Z_t^2$.

It holds

$$\begin{cases} -\,\mathrm{d}(\delta Y)_t = \Delta_y f_t^1\,(\delta Y)_t\ \mathrm{d}t + (\Delta_z f_t^1)^*\,(\delta Z)_t\ \mathrm{d}t + (\delta f)_t\ \mathrm{d}t + \mathrm{d}K_t - (\delta Z)_t\ \mathrm{d}W_t, \qquad t \in [0,T], \\ (\delta Y)_T = 0. \end{cases} \tag{3.23}$$

We notice that $\Delta_y f_t^1$ and $\Delta_z f_t^1$ are bounded and that $(\delta f) \in \mathrm{L}^2_{\mathcal{P}}(\Omega \times [0,T], \mathbb{R})$.

Multiplying $(\delta Y)_t$ by $\exp(\int_0^t \varDelta_y f^1_\tau d\tau)$ and then applying Girsanov theorem we obtain:

$$(\delta Y)_t = \mathbb{E}\left(\rho_{t,T}\left[\int_t^T \exp\left(\int_t^s \varDelta_y f^1_\tau\,\mathrm{d}\tau\right)\mathrm{d}K_s + \int_t^T \exp\left(\int_t^s \varDelta_y f^1_\tau\,\mathrm{d}\tau\right)(\delta f)_s\,\mathrm{d}s\right]\middle|\mathcal{F}_t\right) \tag{3.24}$$

where $\rho_{t,T}$ is the Girsanov density:

$$\rho_{t,T} = \exp\left(\int_t^T (\varDelta_z f^1_s)^*\,\mathrm{d}W_s - \frac{1}{2}\int_t^T |\varDelta_z f^1_s|^2\,\mathrm{d}s\right).$$

The claim follows from (3.24) since (K_t) is a non decreasing process and $(\delta f)_t \geq 0$, $\mathbb{P}$ a.s. for a.e. $t \in [0,T]$. □

4. The optimal control-stopping problem

We apply our results to an optimal control and stopping problem. We start by specifying the setting. An *admissible system* will be a set

$$\mathcal{S} = \left(\varOmega^{\mathcal{S}}, \mathcal{F}^{\mathcal{S}}, (\mathcal{F}^{\mathcal{S}}_t)_{t\geq 0}, \mathbb{P}^{\mathcal{S}}, (W^{\mathcal{S}}_t)_{t\geq 0}\right)$$

where $(\varOmega^{\mathcal{S}}, \mathcal{F}^{\mathcal{S}}, (\mathcal{F}^{\mathcal{S}}_t)_{t\geq 0}, \mathbb{P}^{\mathcal{S}})$, is a complete probability space endowed with a filtration satisfying the usual assumptions and $(W^{\mathcal{S}}_t)_{t\geq 0}$ is a $(\mathcal{F}^{\mathcal{S}})$-cylindrical Wiener process in $\varXi$. Fixed a complete separable metric space U, an *admissible control* in the setting $\mathcal{S}$ is any $\left(\mathcal{F}^{\mathcal{S}}_t\right)$-predictable process $\alpha : \varOmega^{\mathcal{S}} \times [0,T] \to U$. The set of all admissible controls will be denoted by $\mathcal{U}^{\mathcal{S}}$.

We fix a function $R : H \times U \to \varXi$ bounded, continuous such that:

$$|R(\alpha,x) - R(\alpha,x')| \leq |x - x'| \qquad \forall \alpha \in U,\ x,x' \in H \tag{4.1}$$

Given an admissible setting $\mathcal{S}$, an admissible control $\alpha \in \mathcal{U}^{\mathcal{S}}$, an initial time $s \in [0,T]$ and an initial datum $x \in H$, by $X^{\alpha,s,x}$ we denote the solution to the controlled stochastic differential equation in H (notice that $X^{\alpha,s,x}$ depends on $\mathcal{S}$ but for notational simplicity we do not indicate it)

$$\begin{cases} \mathrm{d}X^{\alpha,s,x}_t = AX^{\alpha,s,x}_t\,\mathrm{d}t + F(t,X^{\alpha,s,x}_t)\,\mathrm{d}t + G(t,X^{\alpha,s,x}_t)(R(X^{\alpha,s,x}_t,\alpha_t)\,\mathrm{d}t + \mathrm{d}W^{\mathcal{S}}_t), \qquad t\in[s,T]\,, \\ X^{\alpha,s,x}_s = x \in H. \end{cases} \tag{4.2}$$

Moreover given $l : [0,T] \times H \times U \to \mathbb{R}$ we introduce the cost functional:

$$J\left(s,x,\tau,\alpha\right) = \mathbb{E}\int_s^\tau l\left(r, X^{\alpha,s,x}_r, \alpha_r\right)\,\mathrm{d}r + \mathbb{E}[\phi\left(X^{\alpha,s,x}_T\right)\chi_{\{\tau=T\}}] + \mathbb{E}\left[h\left(\tau, X^{\alpha,s,x}_\tau\right)\chi_{\{\tau<T\}}\right], \tag{4.3}$$

that we wish to maximize over all admissible control $\alpha \in \mathcal{U}^{\mathcal{S}}$ and over all $(\{\mathcal{F}^{\mathcal{S}}_t\})_{t\geq 0}$-stopping times τ satisfying $t \leq \tau \leq T$ (the mean value is computed with respect to $\mathbb{P}^{\mathcal{S}}$).

For $s \in [0,T]$, $x \in H$, $z \in \varXi^*$ we define the Hamiltonian function in the usual way as:

$$\psi(s,x,z) = \sup_{\alpha\in U}\{zR(x,\alpha) + l(s,x,\alpha)\}. \tag{4.4}$$

We notice that since R is bounded ψ is Lipschitz with respect to z uniformly in s and α. We will assume throughout this section that A, F and G verify Hypothesis 2 and that ϕ, ψ and h verify Hypothesis 3. Moreover we assume that $|l(s,x,\alpha)| \leq c(1+|x|^r)$ for some $c,r>0$.

Under the above assumptions, fixed $s \in [0,T]$ and $x \in H$, for all $\alpha \in \mathcal{U}^{\mathcal{S}}$ there exists a unique mild solution $X^{\alpha,s,x}$ to equation (4.2). Moreover $X^{\alpha,s,x} \in \mathrm{L}^p_{\mathcal{P}}(\varOmega, \mathrm{C}^0([s,T],H))$ for all $p \geq 1$, see [10]. Consequently $J(s,x,\tau,\alpha)$ is a well defined real number for all $\alpha \in \mathcal{U}^{\mathcal{S}}$ and all $(\{\mathcal{F}^{\mathcal{S}}_t\})_{t\geq 0}$-stopping time $\tau \leq T$.

By the Girsanov theorem, there exists a probability measure $\mathbb{P}^{\mathcal{S},\alpha}$ such that the process

$$W_t^{\mathcal{S},\alpha} := W_t^{\mathcal{S}} + \int_s^t R(X_r^{\alpha,s,x}, \alpha_r)\,\mathrm{d}r \quad t \geq s$$

is a cylindrical $\mathbb{P}^{\mathcal{S},\alpha}$-Wiener process in Ξ. We denote by $(\mathcal{F}_t^{\mathcal{S},\alpha})_{t\geq 0}$ its natural filtration, augmented in the usual way. Then $X^{\alpha,s,x}$ satisfies the following equation:

$$\begin{cases} \mathrm{d}X_t^{\alpha,s,x} = AX_t^{\alpha,s,x}\,\mathrm{d}t + F(t, X_t^{\alpha,s,x})\,\mathrm{d}t + G(t, X_t^{\alpha,s,x})\,\mathrm{d}W_t^{\mathcal{S},\alpha}, \qquad t \in [s,T] \\ X_s^{\alpha,s,x} = x. \end{cases} \tag{4.5}$$

In particular (notice that the above equation enjoys strong existence, in probabilistic sense, and pathwise uniqueness) $X_t^{\alpha,s,x}$ turns out to be adapted to $(\mathcal{F}_t^{\mathcal{S},\alpha})_{t\geq 0}$.

In $\left(\Omega^{\mathcal{S}}, \mathcal{F}^{\mathcal{S}}, (\mathcal{F}_t^{\mathcal{S},\alpha})_{t\geq 0}, \mathbb{P}^{\mathcal{S},\alpha}\right)$ we consider the solution $(\widetilde{Y}^{s,x}, \widetilde{Z}^{s,x}, \widetilde{K}^{s,x})$ of the following reflected backward stochastic differential equation:

$$\begin{cases} -\,\mathrm{d}\widetilde{Y}_t^{s,x} = \psi(s, X_t^{\alpha,s,x}, \widetilde{Z}_t^{s,x})\ \mathrm{d}t + \mathrm{d}\widetilde{K}_t^{s,x} - \widetilde{Z}_t^{s,x}\ \mathrm{d}W_t^{\mathcal{S},\alpha}, \qquad t \in [0,T], \\ \widetilde{Y}_T^{s,x} = \phi(X_T^{\alpha,s,x}), \\ \widetilde{Y}_t^{s,x} \geq h(X_t^{\alpha,s,x}), \\ \int_0^T (\widetilde{Y}_t^{s,x} - h(X_t^{\alpha,s,x}))\,\mathrm{d}\widetilde{K}_t^{s,x} = 0. \end{cases} \tag{4.6}$$

By its construction we know that the law of the solution to equation (2.17) does not depend on the choice of the specific probability space $(\Omega, \mathcal{E}, \mathbb{P})$ and of the Wiener process (W_t). Therefore the law of the processes $\widetilde{Y}^{s,x}$, $\widetilde{Z}^{s,x}$ and $\widetilde{K}^{s,x}$ defined above do not depend neither on the admissible setting $\mathcal{S}$ nor on the admissible control α. In particular $\widetilde{Y}_s^{s,x}$ is a well defined real number (regardless of the choice of $\mathcal{S}$ and α).

We argue as in [6], Proposition 2.3. Rewriting (4.6) in terms of the original noise $(W^{\mathcal{S}})$ and integrating it between s and any $(\mathcal{F}_s^{\mathcal{S}})$-stopping time τ, we get that $\mathbb{P}$-a.s.:

$$\widetilde{Y}_s^{s,x} = \widetilde{Y}_\tau^{s,x} + \int_s^\tau \psi(r, X_r^{\alpha,s,x}, \widetilde{Z}_r^{s,x})\ \mathrm{d}r + \widetilde{K}_\tau^{s,x} - \widetilde{K}_t^{s,x} - \int_s^\tau \widetilde{Z}_r^{s,x}\ \mathrm{d}W_r^{\mathcal{S}} - \int_s^\tau \widetilde{Z}_r^{s,x}\ R(X_r^{\alpha,s,x}, \alpha_r)\,\mathrm{d}r.$$

Noticing that $(\int_0^t \widetilde{Z}_r^{\alpha,s,x}\ \mathrm{d}W_r^{\mathcal{S}})_{t\,\geq\,0}$ is a $\mathbb{P}^{\mathcal{S}}$-martingale and that $\widetilde{Y}_r^{\alpha,s,x} \geq h(r, X_r^{\alpha,s,x})$; by computing expectation with respect to $\mathbb{P}^{\mathcal{S}}$ we get:

$$\begin{aligned} \widetilde{Y}_s^{s,x} =& \,\mathbb{E}\int_s^\tau \psi(r, X_r^{\alpha,s,x}, \widetilde{Z}_r^{\alpha,s,x})\ \mathrm{d}r - \mathbb{E}\int_s^\tau \widetilde{Z}_r^{\alpha,s,x}\alpha_r\ \mathrm{d}r \\ &+ \mathbb{E}[\widetilde{K}_\tau^{s,x} - \widetilde{K}_t^{s,x}] + \mathbb{E}\widetilde{Y}_\tau^{s,x}\chi_{\{\tau<T\}} + \mathbb{E}\phi(X_T^{\alpha,s,x})\chi_{\{\tau=T\}}], \end{aligned}$$

Finally adding and subtracting the running cost we have:

$$\begin{aligned} \widetilde{Y}_s^{s,x} =& \,J(s,x,\tau,\alpha) + \mathbb{E}\left[\widetilde{Y}_s^{s,x} - h(\tau, X_\tau^{\alpha,s,x})\right] + \mathbb{E}[\widetilde{K}_\tau^{s,x} - \widetilde{K}_s^{s,x}] \\ &+ \mathbb{E}\int_s^\tau \left[\psi(r, X_r^{\alpha,s,x}, \widetilde{Z}_r^{\alpha,s,x}) - l(r, X_r^{\alpha,s,x}, \alpha_r) - Z_r^{s,x}\alpha_r\right]\ \mathrm{d}r. \end{aligned} \tag{4.7}$$

We have therefore proved the following result

Theorem 4.1. *For every admissible setting $\mathcal{S}$, for every $(\{\mathcal{F}_t^{\mathcal{S}}\})_{t\geq 0}$-stopping time τ and for every admissible control $u \in \mathcal{U}^{\mathcal{S}}$ we have:*

$$J(s,x,\tau,\alpha) \leq \widetilde{Y}_s^{s,x},$$

moreover the equality holds if and only if the three statements below hold true:

$$\psi(r, X_r^{\alpha,s,x}, \widetilde{Z}_r^{s,x}) - l(r, X_r^{\alpha,s,x}, \alpha_r) - Z_r^{s,x}\alpha_r = 0, \quad \mathbb{P}^{\mathcal{S}} - a.s.\ for\ a.e.\ r \in [s,\tau] \tag{4.8}$$

$$\widetilde{K}_\tau^{s,x} - \widetilde{K}_s^{s,x} = 0, \quad \mathbb{P}^{\mathcal{S}} - a.s. \tag{4.9}$$

$$\widetilde{Y}_\tau^{s,x} I_{\{\tau<T\}} = h(\tau, X_\tau^{\alpha,s,x}) I_{\{\tau<T\}}, \quad \mathbb{P}^{\mathcal{S}} - a.s. \tag{4.10}$$

Remark 4.2. Fixed an admissible setting $\mathcal{S}$ and an admissible control $\alpha \in \mathcal{U}^{\mathcal{S}}$ let $\bar{\tau}$ be defined as follows:

$$\overline{\tau} = \inf\{t \leq r \leq T : \widetilde{Y}_r^{s,x} = h(r, X_r^{\alpha,s,x})\} \wedge T. \tag{4.11}$$

Condition $\displaystyle\int_0^T (\widetilde{Y}_t^{s,x} - h(t, X_t^{\alpha,s,x}))\, \mathrm{d}\widetilde{K}_t^{s,x} = 0$ together with continuity and monotonicity of $\widetilde{K}$ implies that:

$$\widetilde{K}_{\overline{\tau}}^{s,x} - \widetilde{K}_s^{s,x} = 0.$$

Moreover (4.10) follows by definition. Consequently we have:

$$\widetilde{Y}_s^{s,x} = J(s,x,\bar{\tau},\alpha) + \mathbb{E}\int_s^{\bar{\tau}} \left[\psi(r, X_r^{\alpha,s,x}, \widetilde{Z}_r^{\alpha,s,x}) - l(r, X_r^{\alpha,s,x}, \alpha_r) - Z_r^{\alpha,s,x}\alpha_r\right]\ \mathrm{d}r. \tag{4.12}$$

Taking into account equations (4.5), (4.6) and Proposition 3.5 the above results can be refomulated as follows.

Corollary 4.3. *Let u be the minimal mild supersolution to the obstacle problem and let ζ be any element of its generalized gradient. Given any admissible setting $\mathcal{S}$ and any admissible control $\alpha \in \mathcal{U}^{\mathcal{S}}$ we have:*

$$J(s,x,\tau,\alpha) \leq u(s,x) \tag{4.13}$$

moreover the equality holds if and only if the following holds $\mathbb{P}^{\mathcal{S}} - a.s.$ for a.e.$r \in [s,\tau]$

$$\psi(r, X_r^{\alpha,s,x}, \zeta(r, X_r^{\alpha,s,x})) - l(r, X_r^{\alpha,s,x}, \alpha_r) - \zeta(r, X_r^{\alpha,s,x})\alpha_r = 0, \tag{4.14}$$

$$K_\tau^{s,x} - \widetilde{K}_s^{s,x} = 0, \quad \mathbb{P} - a.s.,$$

$$u(\tau, X_\tau^{\alpha,s,x}) I_{\{\tau<T\}} = h(\tau, X_\tau^{\alpha,s,x}) I_{\{\tau<T\}}, \quad \mathbb{P} - a.s..$$

Finally if

$$\overline{\tau} = \inf\{t \leq r \leq T : u(r, X_r^{\alpha,s,x}) = h(r, X_r^{\alpha,s,x})\} \wedge T. \tag{4.15}$$

then the equality in (4.13) holds if and only if (4.14) holds.

We come now to the existence of optimal controls. We shall exploit the weak formulation of the control problem and select a suitable admissible setting $\bar{\mathcal{S}}$. We assume the following

Hypothesis 4. *The maximum in the definition (4.4) is attained for all $t \in [s,T]$, $x \in H$ and $z \in \Xi^*$ e.g. if we define*

$$\Gamma(s,x,z) = \{\alpha \in \mathcal{U} : zR(x,\alpha) + l(s,x,\alpha) = \psi(s,x,z)\} \tag{4.16}$$

then $\Gamma(s,x,z) \neq \emptyset$ for every $s \in [0,T]$, every $x \in H$ and every $z \in \Xi^*$.

Remark 4.4. By [1], see Theorems 8.2.10 and 8.2.11, under Hypothesis 4, Γ always admits a measurable selection, *i.e.* there exists a measurable function $\gamma : [0,T] \times H \times \Xi^* \to U$ with $\gamma(s,x,z) \in \Gamma(s,x,z)$ for every $s \in [0,T]$, every $x \in H$ and every $z \in \Xi^*$.

Moreover, if U is compact, then Hypothesis 4 always holds.

Theorem 4.5. *Assume Hypothesis 4 and fix $s \in [0,T]$, $x \in H$, a measurable selection γ of Γ and an element ζ of the generalized gradient of the minimal supersolution u of the obstacle problem (3.1). There exists at least an admissible setting $\bar{\mathcal{S}}$ in which the closed loop equation*

$$\begin{cases} \mathrm{d}\bar{X}_t = A\bar{X}_t\,\mathrm{d}t + F(t,\bar{X}_t)\,\mathrm{d}t + G(t,\bar{X}_t)[R(\bar{X}_t,\gamma(t,\bar{X}_t,\zeta(t,\bar{X}_t)))\,\mathrm{d}t + \mathrm{d}W_t^{\mathcal{S}}], & t \in [s,T] \\ \bar{X}_s = x, \end{cases} \tag{4.17}$$

admits a mild solution.

Proof. We fix any admissible setting:

$$\mathcal{S} = \left(\Omega^{\mathcal{S}}, \mathcal{F}^{\mathcal{S}}, (\mathcal{F}_t^{\mathcal{S}})_{t\geq 0}, \mathbb{P}^{\mathcal{S}}, (W_t^{\bar{\mathcal{S}}})_{t\geq 0}\right)$$

and consider the uncontrolled forward SDE

$$\begin{cases} \mathrm{d}X_t = AX_t\,\mathrm{d}t + F(t,X_t)\,\mathrm{d}t + G(t,X_t)\,\mathrm{d}W_t^{\mathcal{S}}, & t \in [s,T] \\ X_s = x. \end{cases} \tag{4.18}$$

By the Girsanov theorem, there exists a probability measure $\hat{\mathbb{P}}$ such that the process

$$\hat{W}_t := W_t^{\mathcal{S}} - \int_s^t R(X_r^{s,x},\gamma(r,X_r^{s,x},\zeta(r,X_r^{s,x}))\,\mathrm{d}r \quad t \geq s$$

is a cylindrical $\hat{\mathbb{P}}$-Wiener process in Ξ. We denote by $(\hat{\mathcal{F}}_t)_{t\geq s}$ its natural filtration, augmented in the usual way. Clearly X solves

$$\begin{cases} \mathrm{d}X_t = AX_t\,\mathrm{d}t + F(t,X_t)\,\mathrm{d}t + G(t,X_t)[R(X_t,\gamma(t,X_t,\zeta(t,X_t)))\,\mathrm{d}t + \mathrm{d}\hat{W}_t], & t \in [s,T] \\ \hat{X}_s = x. \end{cases} \tag{4.19}$$

and $(\Omega^{\mathcal{S}}, \mathcal{F}^{\mathcal{S}}, (\hat{\mathcal{F}}_t)_{t\geq 0}, \hat{\mathbb{P}}, (\hat{W}_t)_{t\geq 0})$ is the desired admissible system. $\square$

We finally get the following

Theorem 4.6. *Assume Hypothesis 4 and fix $s \in [0,T]$, $x \in H$, a measurable selection γ of Γ and an element ζ of the generalized gradient of u. Moreover let $\bar{\mathcal{S}}$ be an admissible setting in which the closed loop equation (4.17) admits a mild solution. Under the above conditions there exists $\bar{\alpha} \in \mathcal{U}^{\bar{\mathcal{S}}}$ and an $(\mathcal{F}_t^{\bar{\mathcal{S}}})_{t\geq 0}$ stopping time $\bar{\tau}$ for which*

$$\widetilde{Y}_s^{s,x} = u(s,x) = J(s,x,\bar{\tau},\bar{\alpha}).$$

Consequently $\bar{\alpha}$ and $\bar{\tau}$ are optimal.

Proof. Just let $\bar{X}$ be the mild solution of equation (4.17) and define $\bar{\alpha} = \gamma(t,\bar{X}_t,\zeta(t,\bar{X}_t))$. Clearly $\bar{X}_t = X^{\bar{\alpha},s,x}$ and relation (4.14) holds. Thus by Corollary 4.3 it is enough to choose

$$\overline{\tau} = \inf\{t \leq r \leq T : u(r,\bar{X}_r) = h(r,\bar{X}_r)\} \wedge T.$$

$\square$

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