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INFINITE HORIZON JUMP-DIFFUSION FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS AND THEIR APPLICATION TO BACKWARD LINEAR-QUADRATIC PROBLEMS*

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Abstract. In this paper, we investigate infinite horizon jump-diffusion forward-backward stochastic differential equations under some monotonicity conditions. We establish an existence and uniqueness theorem, two stability results and a comparison theorem for solutions to such kind of equations. Then the theoretical results are applied to study a kind of infinite horizon backward stochastic linear-quadratic optimal control problems, and then differential game problems. The unique optimal controls for the control problems and the unique Nash equilibrium points for the game problems are obtained in closed forms.

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1. Introduction

Coupled forward-backward stochastic differential equations (SDEs) are encountered when one applies the classical stochastic maximum principle to optimal control or differential game problems (see [6, 20]). The existence and uniqueness of solutions to such kind of equations are closely linked to that of optimal controls or Nash equilibrium points. Forward-backward SDEs are also used to give a probabilistic interpretation for quasilinear second order partial differential equations (PDEs) of elliptic or parabolic type (see [10, 15]), which generalized the classical Feynman-Kac formula for linear PDEs. Moreover, in mathematical finance forward-backward SDEs are often adopted to describe the models involving large investors (see for example [3]).

Finite horizon forward-backward SDEs were first investigated by Antonelli [1] and a local existence and uniqueness result was obtained. He also constructed a counterexample showing that, a large time duration might lead to non-solvability just under the Lipschitz assumption. For the global solvability results, three fundamental methods are available. The first one is the *method of contraction mapping* used by Pardoux and Tang [10]. The second one concerns a kind of *four-step scheme* approach introduced by Ma, Protter and Yong [7] which can be regarded as a combination of the methods of PDEs and probability theory. This method requires the non-degeneracy of the forward diffusion, and is only effective in Markovian frameworks. The third one

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called *method of continuation* is probabilistic. This method gets rid of the restriction of non-degeneracy, and can deal with non-Markovian forward-backward SDEs. As a trade-off, a kind of monotonicity conditions on the coefficients is introduced to ensure the solvability, which is restrictive in a different way. This method is initiated by Hu and Peng [5], Peng and Wu [12]. Later, Yong [18,19] made improvements and made the method more systematic. For some recent developments on finite horizon forward-backward SDEs, one can refer to Ma *et al.* [8].

In 2000, Peng and Shi [11], for the first time, studied infinite horizon forward-backward SDEs driven by Brownian motions employing the method of continuation. Later, Wu [14] studied this problem in some different monotonicity framework from [11]. Yin [16,17] investigates the same issue by the method of contraction mapping. Some existence and uniqueness results and comparison theorems were obtained. In this paper, we consider a kind of generalized infinite horizon forward-backward SDEs driven by both Brownian motions and Poisson processes as follows:

$$\begin{cases} \mathrm{d}x(t) = b(t, x(t), y(t), z(t), r(t, \cdot))\mathrm{d}t + \sigma(t, x(t), y(t), z(t), r(t, \cdot))\mathrm{d}W(t) \\ \qquad\qquad + \displaystyle\int_{\mathcal{E}} \gamma(t, e, x(t-), y(t-), z(t), r(t, e))\tilde{N}(\mathrm{d}t, \mathrm{d}e), \\ -\mathrm{d}y(t) = g(t, x(t), y(t), z(t), r(t, \cdot))\mathrm{d}t - z(t)\mathrm{d}W(t) - \displaystyle\int_{\mathcal{E}} r(t, e)\tilde{N}(\mathrm{d}t, \mathrm{d}e), \\ x(0) = \Phi(y(0)), \end{cases} \tag{1.1}$$

where the notations and mappings will be given in Sections 2 and 3. We adopt the model with random jumps, since jump-diffusion processes characterize stochastic phenomena more often and accurate than just diffusion processes, which provide us with more realistic models in practice. For example, in finance, stock prices often exhibit some jump behaviors. Moreover, financial markets with jump stock prices provide a rich family of incomplete financial models. For more information about jump diffusion models, the interested readers may be referred to [2, 9, 13].

In this paper we study the solvability of forward-backward SDEs by virtue of the method of continuation. The idea is to introduce a family of infinite horizon forward-backward SDEs parameterized by $\alpha \in [0, 1]$ such that, when $\alpha = 1$ the forward-backward SDE coincides with (1.1) and when $\alpha = 0$ the forward-backward SDE is uniquely solvable. We will show that there exists a fixed step-length $\delta_0 > 0$, such that, if, for some $\alpha_0 \in [0, 1)$, the parameterized forward-backward SDE is uniquely solvable, then the same conclusion holds for α_0 being replaced by $\alpha_0 + \delta \leq 1$ with $\delta \in [0, \delta_0]$. Once this has been proved, we can increase the parameter α step by step and finally reach $\alpha = 1$.

It is worth noting that, we study a kind of more general coupled forward-backward SDEs in comparison with [11, 14, 16, 17]. Besides the coupling in b, σ, γ and g, in this paper the initial values are also in a coupled form: $x(0) = \Phi(y(0))$ (see (1.1)). The traditional technique treating the coupling in the initial values (or terminal values) when the horizon is finite (see for example [5,12,18]) is to parameterize and analyze the initial coefficient Φ as the same as other coefficients (b, σ, γ, g). When this traditional technique is used to the case of infinite horizon, we can solve two special cases: (i) The decoupled case: $\Phi(y(0)) = x_0$; (ii) The strong monotonicity case: there exists a constant $\nu > 0$ such that, for any $y_1, y_2 \in \mathbb{R}^n$, $\langle \Phi(y_1) - \Phi(y_2),\ y_1 - y_2\rangle \leq -\nu|y_1 - y_2|^2$. However, in many practical stochastic optimization problems, these two kinds of conditions are too strong to satisfy naturally. In the present paper, instead of the traditional parameterization technique, we employ the classical mean value theorem of continuous functions and some delicate techniques to handle the coupling between the two initial values. This technique was introduced for the first time by Wu and Yu [15] to analyze some algebraic equations. By virtue of the new technique, the conclusion is improved to a general monotonicity case: $\langle \Phi(y_1) - \Phi(y_2),\ y_1 - y_2\rangle \leq 0$ (see (H3.3)–(ii)) which is natural in the viewpoint of practical optimization problems. A potential application of the new technique is to deal with the associated finite horizon problems and hope to improve the corresponding results.

Since backward SDEs on an infinite time horizon are well defined dynamic systems (see Thm. 2.5), it is natural and appealing to study the corresponding optimal control and game problems arising from various fields. For example, in mathematical finance, the first process y of solution to some backward SDE is used to represent the price of some European contingent claim, and the other processes (z, r) of solution are used to characterize the corresponding portfolio. Then, in an incomplete security market, the minimum price of some contingent claim can be given by $\inf_{v(\cdot)\in\mathcal{V}} y^v(0)$, where (y^v, z^v, r^v) is the solution of some controlled backward SDEs and $v(\cdot) \in \mathcal{V}$ is the related control process.

As an application of theoretical results, we study a kind of backward stochastic linear-quadratic (LQ) optimal control problems, and then the general differential game problems. The LQ problems constitute an extremely important category of optimization problems, because many problems arising from practice can be modeled by them, and more importantly, many non-LQ problems can be approximated reasonably by LQ problems. On the other hand, LQ problems tend to have elegant and complete solutions due to their simple and nice structures, which also provide some understanding and preliminaries for the general nonlinear problems. By virtue of the unique solvability of forward-backward SDEs, we obtain unique optimal controls for control problems and unique Nash equilibrium points for game problems in closed forms. To our best knowledge, it is the first time to study this kind of infinite horizon backward LQ problems. The theoretical results of forward-backward SDEs can also be applied to nonlinear infinite horizon backward optimization problems. This subject will be detailed in our future works.

The present paper has the following improvements. (i) Compared with Peng and Shi [11], we clarify many ambiguous arguments, supplement some necessary details and improve some proofs. (ii) A general case in which the two initial values are in a coupled form is studied in this paper, and to deal with it we introduce a new technique which also can be applied to analyze other problems. (iii) We provide an important application of the theoretical results to infinite horizon backward LQ problems. (iv) In order to match practical problems more accurately, we adopt a wider jump-diffusion model.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and some preliminary results on infinite horizon (forward) SDEs and backward SDEs, especially an existence and uniqueness result of backward SDEs. In Section 3, we devote ourselves to investigating the infinite horizon jump-diffusion coupled forward-backward SDEs. We establish an existence and uniqueness theorem, two stability results and a comparison theorem for solutions to forward-backward SDEs. In Section 4, we apply the existence and uniqueness theorem to study a kind of infinite horizon backward stochastic LQ optimal control and differential game problems. We obtain the unique optimal control for the control problem, and the unique Nash equilibrium point for the game problem in closed forms.

2. Notations and preliminaries on SDEs and backward SDEs

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual Euclidean norm $|\cdot|$ and the usual Euclidean inner product $\langle\cdot,\ \cdot\rangle$. Let $\mathbb{R}^{n\times m}$ be the space consisting of all $(n \times m)$ matrices with the inner product:

$$\langle A,\ B\rangle = \mathrm{tr}\{AB^\top\}, \quad \text{for any } A, B \in \mathbb{R}^{n\times m},$$

where $\top$ denotes the transpose of matrices. Thus the norm $|A|$ of A induced by the inner product is given by $|A| = \sqrt{\mathrm{tr}AA^\top}$.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space. The filtration $\mathbb{F} = \{\mathcal{F}_t;\ 0 \le t < \infty\}$ is generated by two mutually independent stochastic sources augmented by all $\mathbb{P}$-null sets. One is a d-dimensional standard Brownian motion $W = (W_1, W_2, \ldots, W_d)^\top$, and the other one consists of l independent Poisson random measures $N = (N_1, N_2, \ldots, N_l)^\top$ defined on $\mathbb{R}_+ \times \mathcal{E}$, where $\mathcal{E} \subset \mathbb{R}^{\bar{l}} \setminus \{0\}$ is a nonempty Borel subset of some Euclidean space. The compensators of N are $\bar{N}(\mathrm{d}t, \mathrm{d}e) = (\pi_1(\mathrm{d}e)\mathrm{d}t, \pi_2(\mathrm{d}e)\mathrm{d}t, \ldots, \pi_l(\mathrm{d}e)\mathrm{d}t)$ which make $\{\tilde{N}((0,t] \times A) = (N - \bar{N})((0,t] \times A);\ 0 \le t < \infty\}$ a martingale for any A belonging to the Borel field $\mathcal{B}(\mathcal{E})$ with $\pi_i(A) < \infty$, $i = 1, 2, \ldots, l$. Here, for each $i = 1, 2, \ldots, l$, π_i is a given σ-finite measure on the measurable space $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ satisfying $\int_{\mathcal{E}}(1 \wedge |e|^2)\pi_i(\mathrm{d}e) < \infty$.

We introduce some spaces:

- $L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$ where $K\in\mathbb{R}$, the space of $\mathbb{R}^n$-valued $\mathbb{F}$-progressively measurable processes f defined on $[0,\infty)$ such that
$$\|f(\cdot)\|_{L^{2,K}_{\mathbb{F}}} := \left(\mathbb{E}\int_0^\infty |f(t)|^2 \mathrm{e}^{Kt}\mathrm{d}t\right)^{\frac{1}{2}} < \infty,$$
and for simplicity we denote $L^2_{\mathbb{F}}(0,\infty;\mathbb{R}^n) := L^{2,0}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$;
- $S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ where $T>0$, the space of $\mathbb{F}$-progressively measurable processes f which have right-continuous paths with left limits such that
$$\|f(\cdot)\|_{S^2_{\mathbb{F}}} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} |f(t)|^2\right]\right)^{\frac{1}{2}} < \infty,$$
and $S^{2,loc}_{\mathbb{F}}(0,\infty;\mathbb{R}^n) := \bigcap_{T>0} S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$;
- $\mathcal{X}^K(0,\infty;\mathbb{R}^n) := L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\cap S^{2,loc}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$; similarly, we denote $\mathcal{X}(0,\infty;\mathbb{R}^n) := \mathcal{X}^0(0,\infty;\mathbb{R}^n) = L^2_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\cap S^{2,loc}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$;
- $L^2(\mathcal{E},\mathcal{B}(\mathcal{E}),\pi;\mathbb{R}^{n\times l})$, the space of π-almost sure equivalence classes $r(\cdot) = (r_1(\cdot),\dots,r_l(\cdot))$ formed by the mappings from $\mathcal{E}$ to the space of $\mathbb{R}^{n\times l}$-valued matrices such that
$$\|r(\cdot)\| := \left(\int_{\mathcal{E}} \operatorname{tr}\left\{r(e)\operatorname{diag}(\pi(\mathrm{d}e))r(e)^\top\right\}\right)^{\frac{1}{2}} = \left(\sum_{i=1}^l \int_{\mathcal{E}} |r_i(e)|^2 \pi_i(\mathrm{d}e)\right)^{\frac{1}{2}} < \infty.$$
This space is equipped with the following inner product:
$$\langle r(\cdot),\ \bar{r}(\cdot)\rangle := \int_{\mathcal{E}} \operatorname{tr}\left\{r(e)\operatorname{diag}(\pi(\mathrm{d}e))\bar{r}(e)^\top\right\},\quad \forall\ r(\cdot),\bar{r}(\cdot)\in L^2(\mathcal{E},\mathcal{B}(\mathcal{E}),\pi;\mathbb{R}^{n\times l});$$
- $M^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})$ where $K\in\mathbb{R}$, the space of $\mathbb{R}^{n\times l}$-valued, $\mathcal{P}\otimes\mathcal{B}(\mathcal{E})$-measurable processes r such that
$$\begin{aligned}\|r(\cdot,\cdot)\|_{M^{2,K}_{\mathbb{F}}} &:= \left(\mathbb{E}\int_0^\infty \int_{\mathcal{E}} \operatorname{tr}\left\{r(t,e)\operatorname{diag}(\pi(\mathrm{d}e))r(t,e)^\top\right\}\mathrm{e}^{Kt}\mathrm{d}t\right)^{\frac{1}{2}}\\ &= \left(\mathbb{E}\int_0^\infty \|r(t,\cdot)\|^2\mathrm{e}^{Kt}\mathrm{d}t\right)^{\frac{1}{2}} < \infty,\end{aligned}$$
where $\mathcal{P}$ is the σ-algebra generated by the $\mathbb{F}$-progressively measurable processes on $[0,\infty)\times\Omega$, and we denote $M^2_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l}) := M^{2,0}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})$;

Clearly, for any $K_1<K_2$, $L^{2,K_2}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\subset L^{2,K_1}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$ and $M^{2,K_2}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})\subset M^{2,K_1}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})$, *i.e.* the sequences of spaces $\{L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\}_{K\in\mathbb{R}}$ and $\{M^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})\}_{K\in\mathbb{R}}$ are decreasing in K.

Further, we define the space $\mathcal{R} := \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\times L^2(\mathcal{E},\mathcal{B}(\mathcal{E}),\pi;\mathbb{R}^{n\times l})$. For any $\theta_1 = (x_1,y_1,z_1,r_1(\cdot))$, $\theta_2 = (x_2,y_2,z_2,r_2(\cdot))\in\mathcal{R}$, the inner product is defined by
$$\langle\theta_1,\ \theta_2\rangle := \langle x_1,\ x_2\rangle + \langle y_1,\ y_2\rangle + \langle z_1,\ z_2\rangle + \langle r_1(\cdot),\ r_2(\cdot)\rangle.$$
Then the norm of $\mathcal{R}$ is deduced by $|\theta| := \sqrt{\langle\theta,\ \theta\rangle}$. We also define
$$\mathcal{L}^{2,K}_{\mathbb{F}}(0,\infty) := L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\times L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\times L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times d})\times M^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})$$

with the norm

$$\|\theta(\cdot)\|_{\mathcal{L}_{\mathbb{F}}^{2,K}} = \left\{\mathbb{E}\int_0^\infty |\theta(t)|^2 \mathrm{e}^{Kt}\mathrm{d}t\right\}^{\frac{1}{2}} = \left\{\mathbb{E}\int_0^\infty \Big[|x(t)|^2 + |y(t)|^2 + |z(t)|^2 + \|r(t,\cdot)\|^2\Big]\mathrm{e}^{Kt}\mathrm{d}t\right\}^{\frac{1}{2}}.$$

Moreover, $\mathcal{L}_{\mathbb{F}}^2(0,\infty) := \mathcal{L}_{\mathbb{F}}^{2,0}(0,\infty)$.

Now, let us consider an infinite horizon (forward) stochastic differential equation (SDE):

$$\begin{aligned} x(t) = x_0 &+ \int_0^t b(s,x(s))\mathrm{d}s + \int_0^t \sigma(s,x(s))\mathrm{d}W(s) \\ &+ \int_0^t\int_{\mathcal{E}} \gamma(s,e,x(s-))\tilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t\in[0,\infty), \end{aligned}$$

which is also expressed in a differential form:

$$\begin{cases} \mathrm{d}x(t) = b(t,x(t))\mathrm{d}t + \sigma(t,x(t))\mathrm{d}W(t) + \displaystyle\int_{\mathcal{E}} \gamma(t,e,x(t-))\tilde{N}(\mathrm{d}t,\mathrm{d}e), \quad t\in[0,\infty), \\ x(0) = x_0, \end{cases} \tag{2.1}$$

where $x_0\in\mathbb{R}^n$, $b:\Omega\times[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$, $\sigma:\Omega\times[0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times d}$ and $\gamma:\Omega\times[0,\infty)\times\mathcal{E}\times\mathbb{R}^n\to\mathbb{R}^{n\times l}$. Moreover, we introduce the following assumptions:

(H1.1) For any $x\in\mathbb{R}^n$, $b(\cdot,x)$, $\sigma(\cdot,x)$ are $\mathbb{F}$-progressively measurable and $\gamma(\cdot,\cdot,x)$ is $\mathcal{P}\otimes\mathcal{B}(\mathcal{E})$-measurable. Moreover, there exists a constant $K\in\mathbb{R}$ such that $b(\cdot,0)\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$, $\sigma(\cdot,0)\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n\times d})$ and $\gamma(\cdot,\cdot,0)\in M_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n\times l})$.

(H1.2) b, σ and γ are Lipschitz continuous with respect to x, *i.e.* there exists a constant $C>0$ such that for any $t\in[0,\infty)$, any x_1, $x_2\in\mathbb{R}^n$,

$$|b(t,x_1)-b(t,x_2)| + |\sigma(t,x_1)-\sigma(t,x_2)| + \|\gamma(t,\cdot,x_1)-\gamma(t,\cdot,x_2)\| \le C|x_1-x_2|.$$

By the classical theory of SDEs, under assumptions (H1.1)–(H1.2), SDE (2.1) admits a unique strong solution. Precisely, for any $T\in[0,\infty)$,

$$\mathbb{E}\left[\sup_{x\in[0,T]}|x(t)|^2\right] < \infty,$$

i.e. $x\in S_{\mathbb{F}}^2(0,T;\mathbb{R}^n)$, and then $x\in S_{\mathbb{F}}^{2,loc}(0,\infty;\mathbb{R}^n)$.

Proposition 2.1. *Let assumptions* (H1.1)–(H1.2) *hold. We further assume the unique solution x of SDE* (2.1) *belongs to $L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$ where the constant K is given by* (H1.1). *Then, we have*

- (i) $x\in\mathcal{X}^K(0,\infty;\mathbb{R}^n)$;
- (ii) $\mathbb{E}[|x(t)|^2\mathrm{e}^{Kt}]$ *is bounded and continuous*;
- (iii) $\lim_{t\to\infty}\mathbb{E}[|x(t)|^2\mathrm{e}^{Kt}] = 0$.

Proof. The assertion (i) is obvious since x belongs to both $S_{\mathbb{F}}^{2,loc}(0,\infty;\mathbb{R}^n)$ and $L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$. For any $t\in[0,\infty)$, we apply Itô's formula to $|x(s)|^2\mathrm{e}^{Ks}$ on the interval $[0,t]$:

$$\begin{aligned} \mathbb{E}\left[|x(t)|^2\mathrm{e}^{Kt}\right] = |x_0|^2 &+ \mathbb{E}\int_0^t 2\Big\langle x(s),\ b(s,x(s))\Big\rangle \mathrm{e}^{Ks}\mathrm{d}s \\ &+ \mathbb{E}\int_0^t \Big[K|x(s)|^2 + |\sigma(s,x(s))|^2 + \|\gamma(s,\cdot,x(s))\|^2\Big]\mathrm{e}^{Ks}\mathrm{d}s. \end{aligned}$$

By the Lipschitz condition (H1.2), we have

$$\mathbb{E}\left[|x(t)|^2 \mathrm{e}^{Kt}\right] \le |x_0|^2 + (1+|K|+2C+4C^2)\mathbb{E}\int_0^t |x(s)|^2 \mathrm{e}^{Ks}\mathrm{d}s + \mathbb{E}\int_0^t \left[|b(s,0)|^2 + 2|\sigma(s,0)|^2 + 2\left\|\gamma(s,\cdot,0)\right\|^2\right]\mathrm{e}^{Ks}\mathrm{d}s,$$

where C is the Lipschitz constant. Due to the fact that $x(\cdot)$, $b(\cdot,0) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$, $\sigma(\cdot,0) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times d})$ and $\gamma(\cdot,\cdot,0) \in M^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})$ (see (H1.1)), the above inequality implies the deterministic process $\{\mathbb{E}[|x(t)|^2\mathrm{e}^{Kt}];\ t \ge 0\}$ is bounded. Moreover, in the same way, applying Itô's formula to $|x(s)|^2\mathrm{e}^{Ks}$ on the interval $[t_1,t_2]$ leads to

$$\left|\mathbb{E}\left[|x(t_2)|^2\mathrm{e}^{Kt_2}\right] - \mathbb{E}\left[|x(t_1)|^2\mathrm{e}^{Kt_1}\right]\right| \le (1+|K|+2C+4C^2)\mathbb{E}\int_{t_1}^{t_2} |x(s)|^2\mathrm{e}^{Ks}\mathrm{d}s + \mathbb{E}\int_{t_1}^{t_2}\left[|b(s,0)|^2 + 2|\sigma(s,0)|^2 + 2\left\|\gamma(s,\cdot,0)\right\|^2\right]\mathrm{e}^{Ks}\mathrm{d}s,$$

which implies that the process $\{\mathbb{E}[|x(t)|^2\mathrm{e}^{Kt}];\ t\ge 0\}$ is continuous. We have proved (ii). The above inequality also shows that, $\mathbb{E}\left[|x(t_2)|^2\mathrm{e}^{Kt_2}\right] - \mathbb{E}\left[|x(t_1)|^2\mathrm{e}^{Kt_1}\right] \to 0$ as $t_1,t_2\to\infty$, then $\lim_{t\to\infty}\mathbb{E}\left[|x(t)|^2\mathrm{e}^{Kt}\right]$ exists. Furthermore, due to

$$\int_0^\infty \mathbb{E}\left[|x(t)|^2\mathrm{e}^{Kt}\right]\mathrm{d}t < \infty,$$

we get the desired conclusion

$$\lim_{t\to\infty}\mathbb{E}\left[|x(t)|^2\mathrm{e}^{Kt}\right] = 0.$$

□

In the rest of this section, we shall consider an infinite horizon backward SDE as follows:

$$y(t) = \int_t^\infty \Big[G(s,y(s),z(s),r(s,\cdot)) + \varphi(s)\Big]\mathrm{d}s - \int_t^\infty z(s)\mathrm{d}W(s) - \int_t^\infty\int_{\mathcal{E}} r(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e),\quad t\in[0,\infty),$$

which is denoted also in the following differential form:

$$-\mathrm{d}y(t) = \Big[G(t,y(t),z(t),r(t,\cdot)) + \varphi(t)\Big]\mathrm{d}t - z(t)\mathrm{d}W(t) - \int_{\mathcal{E}} r(t,e)\tilde{N}(\mathrm{d}t,\mathrm{d}e),\quad t\in[0,\infty), \tag{2.2}$$

where $G : \Omega\times[0,\infty)\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(\mathcal{E},\mathcal{B}(\mathcal{E}),\pi;\mathbb{R}^{m\times l}) \to \mathbb{R}^m$ and $\varphi : \Omega\times[0,\infty)\to\mathbb{R}^m$. We assume the following assumptions on the coefficients (G,φ):

(H2.1) For any $(y,z,r(\cdot)) \in \mathbb{R}^m\times\mathbb{R}^{m\times d}\times L^2(\mathcal{E},\mathcal{B}(\mathcal{E}),\pi;\mathbb{R}^{m\times l})$, $G(\cdot,y,z,r(\cdot))$ is $\mathbb{F}$-progressively measurable, and satisfies $G(t,0,0,0) = 0$ for any $t\in[0,\infty)$. Moreover, there exists a constant $K\in\mathbb{R}$ such that $\varphi \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^m)$.

(H2.2) G is Lipschitz continuous with respect to $(y,z,r(\cdot))$, *i.e.* there exist constants $C_0\ge 0$, $C_1\ge 0$ and $C_2\ge 0$ such that for any $t\in[0,\infty)$, any $y_1,y_2\in\mathbb{R}^m$, any $z_1,z_2\in\mathbb{R}^{m\times d}$, any $r_1(\cdot),r_2(\cdot)\in L^2(\mathcal{E},\mathcal{B}(\mathcal{E}),\pi;\mathbb{R}^{m\times l})$,

$$|G(t,y_1,z_1,r_1(\cdot)) - G(t,y_2,z_2,r_2(\cdot))| \le C_0|y_1-y_2| + C_1|z_1-z_2| + C_2\left\|r_1(\cdot)-r_2(\cdot)\right\|.$$

(H2.3) G satisfies some 'weak monotonicity' conditions in the following sense: there exists a constant $\rho\in\mathbb{R}$ such that for any $t\in[0,\infty)$, any $y_1,y_2\in\mathbb{R}^m$, any $z\in\mathbb{R}^{m\times d}$, any $r(\cdot)\in L^2(\mathcal{E},\mathcal{B}(\mathcal{E}),\pi;\mathbb{R}^{m\times l})$,

$$\langle G(t,y_1,z,r(\cdot)) - G(t,y_2,z,r(\cdot)), y_1-y_2\rangle \le -\rho|y_1-y_2|^2.$$

(H2.4) $K + 2\rho - 2C_1^2 - 2C_2^2 > 0$.

Remark 2.2.

(i) In the classical form of backward SDEs, the coefficient is usually denoted by $g(t,y,z,r(\cdot))$. While in the present paper, for convenience, we set

$$G(t,y,z,r(\cdot))=g(t,y,z,r(\cdot))-g(t,0,0,0),\quad \varphi(t)=g(t,0,0,0),$$

and denote the coefficient g by $G+\varphi$.

(ii) From the density of real numbers, it is easy to see that (H2.4) is equivalent to the following statement: (H2.4') There exists a constant $\delta>0$ such that

$$K+2\rho-2C_1^2-2C_2^2-\delta>0. \tag{2.3}$$

Naturally, a triple of mappings $(y(\cdot),z(\cdot),r(\cdot,\cdot))$ is called an adapted solution to backward SDE (2.2) if and only if y is an $\mathbb{R}^m$-valued $\mathbb{F}$-progressively measurable process, z is an $\mathbb{R}^{m\times d}$-valued $\mathbb{F}$-progressively measurable process, r is an $\mathbb{R}^{m\times l}$-valued $\mathcal{P}\otimes\mathcal{B}(\mathcal{E})$-measurable process, and (y,z,r) satisfies (2.2). Similar to forward SDEs, we have the following

Corollary 2.3. *Let assumptions* (H2.1) and (H2.2) *hold. We further assume* $(y,z,r)\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^m)\times L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{m\times d})\times M_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{m\times l})$ *is a solution to backward SDE* (2.2) *where the constant* K *is given by* (H2.1)*. Then we have*

(i) $y\in\mathcal{X}^K(0,\infty;\mathbb{R}^m)$;
(ii) $\mathbb{E}[|y(t)|^2\mathrm{e}^{Kt}]$ *is bounded and continuous*;
(iii) $\lim_{t\to\infty}\mathbb{E}[|y(t)|^2\mathrm{e}^{Kt}]=0$.

Proof. Since (y,z,r) is a solution to backward SDE (2.2), then for any $t\in[0,\infty)$,

$$\begin{aligned}
y(t)&=\int_t^\infty\Big[G(s,y(s),z(s),r(s,\cdot))+\varphi(s)\Big]\mathrm{d}s-\int_t^\infty z(s)\mathrm{d}W(s)-\int_t^\infty\int_{\mathcal{E}}r(s,e)\tilde N(\mathrm{d}s,\mathrm{d}e)\\
&=\int_0^\infty\Big[G(s,y(s),z(s),r(s,\cdot))+\varphi(s)\Big]\mathrm{d}s-\int_0^\infty z(s)\mathrm{d}W(s)-\int_0^\infty\int_{\mathcal{E}}r(s,e)\tilde N(\mathrm{d}s,\mathrm{d}e)\\
&\quad-\int_0^t\Big[G(s,y(s),z(s),r(s,\cdot))+\varphi(s)\Big]\mathrm{d}s+\int_0^t z(s)\mathrm{d}W(s)+\int_0^t\int_{\mathcal{E}}r(s,e)\tilde N(\mathrm{d}s,\mathrm{d}e)\\
&=y(0)-\int_0^t\Big[G(s,y(s),z(s),r(s,\cdot))+\varphi(s)\Big]\mathrm{d}s+\int_0^t z(s)\mathrm{d}W(s)+\int_0^t\int_{\mathcal{E}}r(s,e)\tilde N(\mathrm{d}s,\mathrm{d}e).
\end{aligned}$$

So the process $y\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^m)$ can be regarded as an adapted solution to a forward SDE with

$$b(t,y)=-\Big[G(t,y,z(t),r(t,\cdot))+\varphi(t)\Big],\quad \sigma(t,y)=z(t),\quad \gamma(t,e,y)=r(t,e).$$

Under assumptions (H2.1) and (H2.2), it is easy to check that the above coefficients (b,σ,γ) satisfy assumptions (H1.1) and (H1.2). By Proposition 2.1, we get the conclusions. $\square$

In order to obtain an existence and uniqueness result for backward SDE (2.2), we first establish the following *a priori* estimate.

Lemma 2.4. *Let assumptions* (H2.1)–(H2.4) *hold. Let* (y_1, z_1, r_1) *and* $(y_2, z_2, r_2) \in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^m) \times L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{m\times d}) \times M^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{m\times l})$ *be solutions to backward SDEs* (2.2) *with* $\varphi = \varphi_1$ *and* $\varphi = \varphi_2$ *respectively. Then we have*

$$\begin{aligned}
&\mathbb{E}\int_0^\infty \Big[(K+2\rho-2C_1^2-2C_2^2-\delta)|y_1(t)-y_2(t)|^2\\
&\quad+\frac{1}{2}|z_1(t)-z_2(t)|^2+\frac{1}{2}\left\|r_1(t,\cdot)-r_2(t,\cdot)\right\|^2\Big]\mathrm{e}^{Kt}\mathrm{d}t\\
&\leq \frac{1}{\delta}\mathbb{E}\int_0^\infty |\varphi_1(t)-\varphi_2(t)|^2\mathrm{e}^{Kt}\mathrm{d}t,
\end{aligned}\tag{2.4}$$

where $\delta > 0$ *is defined in* (2.3).

Proof. We denote

$$\begin{aligned}
&\hat{\varphi}(t) := \varphi_1(t)-\varphi_2(t), \quad \hat{y}(t) := y_1(t)-y_2(t),\\
&\hat{z}(t) := z_1(t)-z_2(t), \quad \hat{r}(t,e) := r_1(t,e)-r_2(t,e),
\end{aligned}$$

for any $(\omega,t,e)\in\Omega\times[0,\infty)\times\mathcal{E}$. For any given $T>0$, we apply Itô's formula to $|\hat{y}(t)|^2\mathrm{e}^{Kt}$ on the interval $[0,T]$:

$$\begin{aligned}
\mathbb{E}\Big[|\hat{y}(T)|^2\mathrm{e}^{KT}\Big]-|\hat{y}(0)|^2 &= \mathbb{E}\int_0^T \Big[K|\hat{y}(t)|^2+|\hat{z}(t)|^2+\|\hat{r}(t,\cdot)\|^2\Big]\,\mathrm{e}^{Kt}\mathrm{d}t\\
&\quad-2\mathbb{E}\int_0^T \Big\langle \hat{y}(t),\ G(t,y_1(t),z_1(t),r_1(t,\cdot))-G(t,y_2(t),z_2(t),r_2(t,\cdot))+\hat{\varphi}(t)\Big\rangle \mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}$$

Then

$$\begin{aligned}
&|\hat{y}(0)|^2+\mathbb{E}\int_0^T \Big[K|\hat{y}(t)|^2+|\hat{z}(t)|^2+\|\hat{r}(t,\cdot)\|^2\Big]\,\mathrm{e}^{Kt}\mathrm{d}t\\
&=\mathbb{E}\Big[|\hat{y}(T)|^2\mathrm{e}^{KT}\Big]+2\mathbb{E}\int_0^T \langle\hat{y}(t),\ \hat{\varphi}(t)\rangle\,\mathrm{e}^{Kt}\mathrm{d}t\\
&\quad+2\mathbb{E}\int_0^T \Big\langle\hat{y}(t),\ G(t,y_1(t),z_1(t),r_1(t,\cdot))-G(t,y_2(t),z_1(t),r_1(t,\cdot))\Big\rangle\mathrm{e}^{Kt}\mathrm{d}t\\
&\quad+2\mathbb{E}\int_0^T \Big\langle\hat{y}(t),\ G(t,y_2(t),z_1(t),r_1(t,\cdot))-G(t,y_2(t),z_2(t),r_2(t,\cdot))\Big\rangle\mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}$$

By assumptions (H2.2) and (H2.3), we have

$$\begin{aligned}
&|\hat{y}(0)|^2+\mathbb{E}\int_0^T \Big[K|\hat{y}(t)|^2+|\hat{z}(t)|^2+\|\hat{r}(t,\cdot)\|^2\Big]\,\mathrm{e}^{Kt}\mathrm{d}t\\
&\leq \mathbb{E}\Big[|\hat{y}(T)|^2\mathrm{e}^{KT}\Big]+\mathbb{E}\int_0^T\Big[2|\hat{y}(t)||\hat{\varphi}(t)|-2\rho|\hat{y}(t)|^2+2|\hat{y}(t)|(C_1|\hat{z}(t)|+C_2\left\|\hat{r}(t,\cdot)\right\|)\Big]\mathrm{e}^{Kt}\mathrm{d}t\\
&\leq \mathbb{E}\Big[|\hat{y}(T)|^2\mathrm{e}^{KT}\Big]+\mathbb{E}\int_0^T\Big[(\delta-2\rho+2C_1^2+2C_2^2)|\hat{y}(t)|^2+\frac{1}{2}|\hat{z}(t)|^2+\frac{1}{2}\left\|\hat{r}(t,\cdot)\right\|^2+\frac{1}{\delta}|\hat{\varphi}(t)|^2\Big]\mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\hat{y}(0)|^2+\mathbb{E}\int_0^T\Big[(K+2\rho-2C_1^2-2C_2^2-\delta)|\hat{y}(t)|^2+\frac{1}{2}|\hat{z}(t)|^2+\frac{1}{2}\left\|\hat{r}(t,\cdot)\right\|^2\Big]\mathrm{e}^{Kt}\mathrm{d}t\\
&\leq \mathbb{E}\Big[|\hat{y}(T)|^2\mathrm{e}^{KT}\Big]+\frac{1}{\delta}\mathbb{E}\int_0^T|\hat{\varphi}(t)|^2\mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}$$

Let $T \to \infty$. Thanks to Corollary 2.3, we have

$$|\hat{y}(0)|^2 + \mathbb{E}\int_0^\infty \left[(K + 2\rho - 2C_1^2 - 2C_2^2 - \delta)|\hat{y}(t)|^2 + \frac{1}{2}|\hat{z}(t)|^2 + \frac{1}{2}\|\hat{r}(t,\cdot)\|^2\right]\mathrm{e}^{Kt}\mathrm{d}t \le \frac{1}{\delta}\mathbb{E}\int_0^\infty |\hat{\varphi}(t)|^2\mathrm{e}^{Kt}\mathrm{d}t,$$

which implies (2.4). $\square$

Theorem 2.5. *Let assumptions* (H2.1)–(H2.4) *hold. When* $K > 0$*, the backward SDE* (2.2) *admits a unique solution* $(y, z, r) \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^m) \times L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{m\times d}) \times M_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{m\times l})$.

Proof. Clearly, the *a priori* estimate (2.4) implies the uniqueness. For the existence, we employ the method used in [11] to construct an adapted solution. In detail, for $n = 1, 2, \ldots$, we define

$$\varphi_n(t) = \mathbb{1}_{[0,n]}(t)\varphi(t), \quad t \in [0,\infty).$$

Obviously, the sequence $\{\varphi_n\}_{n=1}^\infty$ converges to φ in $L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^m)$. For each n, let $(\bar{y}_n, \bar{z}_n, \bar{r}_n)$ be the unique adapted solution of the following finite horizon backward SDE:

$$\begin{aligned}\bar{y}_n(t) = &\int_t^n \Big[G(s, \bar{y}_n(s), \bar{z}_n(s), \bar{r}_n(s,\cdot)) + \varphi_n(s)\Big]\mathrm{d}s - \int_t^n \bar{z}_n(s)\mathrm{d}W(s)\\ &- \int_t^n\int_{\mathcal{E}} \bar{r}_n(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t \in [0,n].\end{aligned}$$

Furthermore, we define

$$(y_n(t), z_n(t), r_n(t,\cdot)) := \begin{cases} (\bar{y}_n(t), \bar{z}_n(t), \bar{r}_n(t,\cdot)), & t \in [0,n],\\ (0,0,0), & t \in (n,\infty).\end{cases}$$

Obviously, $(y_n, z_n, r_n) \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^m) \times L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{m\times d}) \times M_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{m\times l})$. Since $G(s,0,0,0) = 0$ (see assumption (H2.1)), then (y_n, z_n, r_n) solves the following infinite horizon backward SDE:

$$\begin{aligned}y_n(t) = &\int_t^\infty \Big[G(s, y_n(s), z_n(s), r_n(s,\cdot)) + \varphi_n(s)\Big]\mathrm{d}s - \int_t^\infty z_n(s)\mathrm{d}W(s)\\ &- \int_t^\infty\int_{\mathcal{E}} r_n(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e), \quad t \in [0,\infty).\end{aligned}$$

Lemma 2.4 implies that $\{(y_n, z_n, r_n)\}_{n=1}^\infty$ is a Cauchy sequence in $L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^m) \times L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{m\times d}) \times M_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{m\times l})$. We denote by (y, z, r) the limit of $\{(y_n, z_n, r_n)\}_{n=1}^\infty$, and shall show that (y, z, r) solves the backward SDE (2.2).

First, when $K > 0$ we deduce that

$$\begin{aligned}\mathbb{E}\left[\int_t^\infty (z_n(s) - z(s))\mathrm{d}W(s)\right]^2 &= \mathbb{E}\int_t^\infty |z_n(s) - z(s)|^2\mathrm{d}s\\ &\le \mathbb{E}\int_0^\infty |z_n(s) - z(s)|^2\mathrm{e}^{Ks}\mathrm{d}s \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

i.e. the item $\int_t^\infty z_n(s)\mathrm{d}W(s)$ converges to $\int_t^\infty z(s)\mathrm{d}W(s)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^m)$ which is the space of $\mathcal{F}$-measurable square integrable random variables. The same argument also leads to a similar conclusion: the item

$\int_t^\infty \int_{\mathcal{E}} r_n(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e)$ converges to $\int_t^\infty \int_{\mathcal{E}} r(s,e)\tilde{N}(\mathrm{d}s,\mathrm{d}e)$ in $L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$. Second, for any $K>0$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^\infty \left(G(s,y_n(s),z_n(s),r_n(s,\cdot)) - G(s,y(s),z(s),r(s,\cdot))\right)\mathrm{d}s\right]^2\\
&\le \mathbb{E}\left[\int_0^\infty \Big|G(s,y_n(s),z_n(s),r_n(s,\cdot)) - G(s,y(s),z(s),r(s,\cdot))\Big|\mathrm{e}^{\frac{K}{2}s}\mathrm{e}^{-\frac{K}{2}s}\mathrm{d}s\right]^2\\
&\le \mathbb{E}\left[\left(\int_0^\infty \Big|G(s,y_n(s),z_n(s),r_n(s,\cdot)) - G(s,y(s),z(s),r(s,\cdot))\Big|^2\mathrm{e}^{Ks}\mathrm{d}s\right)\left(\int_0^\infty \mathrm{e}^{-Ks}\mathrm{d}s\right)\right]\\
&\le C\mathbb{E}\int_0^\infty \left[|y_n(s)-y(s)|^2 + |z_n(s)-z(s)|^2 + \|r_n(s,\cdot)-r(s,\cdot)\|^2\right]\mathrm{e}^{Ks}\mathrm{d}s\\
&\quad \to 0, \quad \text{as } n\to\infty,
\end{aligned}
$$

i.e. the item $\int_t^\infty G(s,y_n(s),z_n(s),r_n(s,\cdot))\mathrm{d}s$ converges to $\int_t^\infty G(s,y(s),z(s),r(s,\cdot))\mathrm{d}s$ in $L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$. We notice that, here in order to dominate the L^1-norm by the L^2-norm, we have to restrict $K>0$. This is different from the case of finite time intervals. The same argument also leads to $\int_t^\infty \varphi_n(s)\mathrm{d}s$ converges to $\int_t^\infty \varphi(s)\mathrm{d}s$ in $L^2(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^m)$. At last, since $\lim_{n\to\infty}\mathbb{E}\int_0^\infty |y_n(t)-y(t)|^2\mathrm{e}^{Kt}\mathrm{d}t = 0$, there exists a subsequence of $\{y_n\}$ such that

$$
\lim_{n\to\infty}\mathbb{E}\Big[|y_n(t)-y(t)|^2\Big] = 0, \quad \text{for almost everywhere } t\in[0,\infty).
$$

The proof is completed. □

3. Coupled forward-backward SDEs

In this section, we study the following kind of coupled forward-backward SDEs driven by both Brownian motions and Poisson processes on the infinite interval $[0,\infty)$:

$$
\begin{cases}
\mathrm{d}x(t) = b(t,x(t),y(t),z(t),r(t,\cdot))\mathrm{d}t + \sigma(t,x(t),y(t),z(t),r(t,\cdot))\mathrm{d}W(t)\\
\qquad\qquad + \displaystyle\int_{\mathcal{E}} \gamma(t,e,x(t-),y(t-),z(t),r(t,e))\tilde{N}(\mathrm{d}t,\mathrm{d}e),\\
-\mathrm{d}y(t) = g(t,x(t),y(t),z(t),r(t,\cdot))\mathrm{d}t - z(t)\mathrm{d}W(t) - \displaystyle\int_{\mathcal{E}} r(t,e)\tilde{N}(\mathrm{d}t,\mathrm{d}e),\\
x(0) = \Phi(y(0)),
\end{cases}
\tag{3.1}
$$

where $\Phi:\mathbb{R}^n\to\mathbb{R}^n$, $(b,\sigma,g):\Omega\times[0,\infty)\times\mathcal{R}\to\mathbb{R}^n\times\mathbb{R}^{n\times d}\times\mathbb{R}^n$ and $\gamma:\Omega\times[0,\infty)\times\mathcal{E}\times\mathcal{R}\to\mathbb{R}^{n\times l}$. Similar to Hu and Peng [5], for any $\theta=(x,y,z,r(\cdot))\in\mathcal{R}$, we use the notation $A(t,\theta) := (-g(t,\theta),b(t,\theta),\sigma(t,\theta),\gamma(t,\cdot,\theta))$. Now we give the following assumptions:

(H3.1) For any $\theta\in\mathcal{R}$, $b(\cdot,\theta)$, $\sigma(\cdot,\theta)$, $g(\cdot,\theta)$ are $\mathbb{F}$-progressively measurable and $\gamma(\cdot,\cdot,\theta)$ is $\mathcal{P}\otimes\mathcal{B}(\mathcal{E})$-measurable. Moreover, there exists a constant $K>0$ such that $A(\cdot,0)\in\mathcal{L}^{2,K}_{\mathbb{F}}(0,\infty)$.

(H3.2) A and Φ are Lipschitz continuous with respect to θ and y respectively, *i.e.* there exists a constant $C>0$ such that for any $t\in[0,\infty)$, any θ_1, $\theta_2\in\mathcal{R}$, any y_1, $y_2\in\mathbb{R}^n$,

$$
\begin{aligned}
&\text{(i)}\quad |A(t,\theta_1)-A(t,\theta_2)| \le C|\theta_1-\theta_2|,\\
&\text{(ii)}\quad |\Phi(y_1)-\Phi(y_2)| \le C|y_1-y_2|.
\end{aligned}
$$

(H3.3) A and Φ satisfy the monotonicity conditions in the sense: there exists a constant $\mu > 0$ such that for any $t \in [0, \infty)$, any $\theta_1, \theta_2 \in \mathcal{R}$, any $y_1, y_2 \in \mathbb{R}^n$,

$$\begin{aligned} &\text{(i)} \quad \langle A(t,\theta_1) - A(t,\theta_2),\ \theta_1 - \theta_2 \rangle \le -\mu|\theta_1 - \theta_2|^2, \\ &\text{(ii)} \quad \langle \Phi(y_1) - \Phi(y_2),\ y_1 - y_2 \rangle \le 0. \end{aligned}$$

(H3.4) $2\mu - K > 0$.

Remark 3.1. Assumption (H3.4) is artificial. In fact, if it does not hold true, then we can find a $\bar{K} \in (0, K)$ such that $2\mu - \bar{K} > 0$. Due to the decreasing property of $\{L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)\}_{K\in\mathbb{R}}$ and $\{M_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n\times l})\}_{K\in\mathbb{R}}$, assumption (H3.1) implies $A(\cdot, 0)$ also belongs to $\mathcal{L}_{\mathbb{F}}^{2,\bar{K}}(0,\infty)$. So we can deal with the corresponding problems in a larger space. However, for convenience, we would like to keep (H3.4) in this paper.

Next we employ the method of continuation originally introduced by Hu and Peng [5] to obtain the existence and uniqueness of the forward-backward SDE (3.1). For this purpose, we introduce a family of infinite horizon forward-backward SDEs parametrized by $\alpha \in [0, 1]$:

$$\begin{cases} \mathrm{d}x^\alpha(t) = & [\alpha b(t,\theta^\alpha(t)) - \mu(1-\alpha)y^\alpha(t) + \phi(t)]\,\mathrm{d}t \\ & + [\alpha\sigma(t,\theta^\alpha(t)) - \mu(1-\alpha)z^\alpha(t) + \psi(t)]\,\mathrm{d}W(t) \\ & + \displaystyle\int_{\mathcal{E}} [\alpha\gamma(t,e,\theta^\alpha(t-)) - \mu(1-\alpha)r^\alpha(t,e) + \xi(t,e)]\,\tilde{N}(\mathrm{d}t,\mathrm{d}e), \\ -\mathrm{d}y^\alpha(t) = & [\alpha g(t,\theta^\alpha(t)) + \mu(1-\alpha)x^\alpha(t) + \eta(t)]\,\mathrm{d}t - z^\alpha(t)\mathrm{d}W(t) \\ & - \displaystyle\int_{\mathcal{E}} r^\alpha(t,e)\tilde{N}(\mathrm{d}t,\mathrm{d}e), \\ x^\alpha(0) = \Phi(y^\alpha(0)), & \end{cases} \tag{3.2}$$

where $(\eta,\phi,\psi,\xi) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$ and we denote $\theta^\alpha(t) := (x^\alpha(t), y^\alpha(t), z^\alpha(t), r^\alpha(t,\cdot))$, $\theta^\alpha(t-) := (x^\alpha(t-), y^\alpha(t-), z^\alpha(t), r^\alpha(t,e))$. We notice that, the coefficient Φ is not parameterized as the same as other coefficients (b,σ,γ,g). This is a difference from the traditional parameterization technique used in [5,11,12,18,19].

When $\alpha = 0$, the forward-backward SDE (3.2) is reduced to

$$\begin{cases} \mathrm{d}x^0(t) = & \left[-\mu y^0(t) + \phi(t)\right]\mathrm{d}t + \left[-\mu z^0(t) + \psi(t)\right]\mathrm{d}W(t) \\ & + \displaystyle\int_{\mathcal{E}} \left[-\mu r^0(t,e) + \xi(t,e)\right]\tilde{N}(\mathrm{d}t,\mathrm{d}e), \\ -\mathrm{d}y^0(t) = & \left[\mu x^0(t) + \eta(t)\right]\mathrm{d}t - z^0(t)\mathrm{d}W(t) - \displaystyle\int_{\mathcal{E}} r^0(t,e)\tilde{N}(\mathrm{d}t,\mathrm{d}e), \\ x^0(0) = \Phi(y^0(0)). & \end{cases} \tag{3.3}$$

Before proving the unique solvability result for (3.3), we need to consider an algebraic equation related to the coupling of initial conditions. The technique dealing with the algebraic equation is similar to the proof of Lemma 3.4 in [15].

Lemma 3.2. *Let assumptions* (H3.1)–(H3.4) *hold. For any $p \in \mathbb{R}^n$, the following algebraic equation*

$$x = \Phi(x + p) \tag{3.4}$$

admits a unique solution $x \in \mathbb{R}^n$.

Proof. Firstly, we show the uniqueness. If both x_1 and x_2 satisfy the algebraic equation (3.4), then

$$x_1 - x_2 = \Phi(x_1 + p) - \Phi(x_2 + p).$$

Making inner product with $x_1 - x_2$, by the monotonicity condition on Φ (see (H3.3)), we get

$$|x_1 - x_2|^2 = \langle \Phi(x_1 + p) - \Phi(x_2 + p),\ x_1 - x_2 \rangle \leq 0.$$

We proved the uniqueness.

Secondly, we prove the existence. We define a new function

$$\lambda(x,p) = \Phi(x+p) - x, \quad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{3.5}$$

Making inner product of λ and x, by the monotonicity condition of Φ, we get

$$\begin{aligned} \langle \lambda(x,p),\ x \rangle &= \langle \Phi(x+p),\ x \rangle - |x|^2 \\ &= \langle \Phi(x+p) - \Phi(p),\ x \rangle + \langle \Phi(p),\ x \rangle - |x|^2 \\ &\leq\ -|x|^2 + \langle \Phi(p),\ x \rangle. \end{aligned}$$

From the inequality: $\langle a,\ b \rangle \leq (1/2)(|a|^2 + |b|^2)$, we have

$$\langle \lambda(x,p),\ x \rangle \leq -\frac{1}{2}|x|^2 + \frac{1}{2}|\Phi(p)|^2. \tag{3.6}$$

We assert that the above inequality implies that, for any $p \in \mathbb{R}^n$, there exists an $x(p)$ such that $\lambda(x(p),p) = 0$. This is equivalent to the existence of the algebraic equation (3.4). In order to highlight the idea of the proof, here we only prove this conclusion in a simple case where $n = 1$. For the general case $n > 1$, the proof is a bit complicated and technical, we would like to omit it. The interested readers can be referred to Appendix in [15]. When $n = 1$, the inequality (3.6) is rewritten as

$$\lambda(x,p)x \leq -\frac{1}{2}x^2 + \frac{\Phi^2(p)}{2}\cdot \tag{3.7}$$

(i) When $x < 0$, dividing x on both sides of the inequality (3.7), we have

$$\lambda(x,p) \geq -\frac{1}{2}x + \frac{\Phi^2(p)}{2x}\cdot$$

Letting $x \to -\infty$, we have $\lambda(x,p) \to +\infty$.

(ii) When $x > 0$, dividing x on both sides of the inequality (3.7), we have

$$\lambda(x,p) \leq -\frac{1}{2}x + \frac{\Phi^2(p)}{2x}\cdot$$

Letting $x \to +\infty$, we have $\lambda(x,p) \to -\infty$.

Obviously λ is a continuous function. From the classical mean value theorem of continuous functions, we know that, for any p, there exists a real number $x(p)$ such that $\lambda(x(p),p) = 0$. We finish the proof of existence. □

Remark 3.3. One conventional approach to prove the solvability of some algebraic equations is by virtue of a monotone operator with coerciveness (see for example [22], Thm. 26.A). Due to assumption (H3.3), Φ is a monotone operator. However, there is no coercive condition imposed. So this conventional method cannot be applied for our problem.

Lemma 3.4. *Let assumptions* (H3.1)–(H3.4) *hold. For any* $(\eta,\phi,\psi,\xi)\in\mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$*, the forward-backward SDE* (3.3) *admits a unique solution in* $\mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$*.*

Proof. Let us consider a linear infinite horizon backward SDE:

$$\begin{aligned}-\mathrm{d}p(t) =\ & [-\mu p(t)+\phi(t)+\eta(t)]\,\mathrm{d}t-[(1+\mu)q(t)-\psi(t)]\,\mathrm{d}W(t)\\ &-\int_{\mathcal{E}}[(1+\mu)k(t,e)-\xi(t,e)]\,\tilde{N}(\mathrm{d}t,\mathrm{d}e),\end{aligned}\tag{3.8}$$

and a linear infinite horizon (forward) SDE combined with an algebraic equation:

$$\begin{cases}\mathrm{d}x(t) =\ [-\mu(x(t)+p(t))+\phi(t)]\,\mathrm{d}t+[-\mu q(t)+\psi(t)]\,\mathrm{d}W(t)\\ \qquad\qquad +\displaystyle\int_{\mathcal{E}}[-\mu k(t,e)+\xi(t,e)]\,\tilde{N}(\mathrm{d}t,\mathrm{d}e),\\ x(0)=\varPhi(x(0)+p(0)).\end{cases}\tag{3.9}$$

Due to Theorem 2.5 with $C_1=C_2=0$ and $\rho=\mu$, the backward SDE (3.8) admits a unique solution $(p,q,k)\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)\times L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n\times d})\times M_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n\times l})$. Once (p,q,k) is solved, by Lemma 3.2, we can uniquely solve $x(0)$ from the initial condition $x(0)=\varPhi(x(0)+p(0))$. then we solve SDE (3.9). It admits a unique solution x. Next we shall show that $x\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$. For any constant $T>0$, we apply Itô's formula to $|x(t)|^2\mathrm{e}^{Kt}$ on the finite interval $[0,T]$:

$$\begin{aligned}&\mathbb{E}\left[|x(T)|^2\mathrm{e}^{KT}\right]+(2\mu-K)\mathbb{E}\int_0^T|x(t)|^2\mathrm{e}^{Kt}\mathrm{d}t\\ &=|x(0)|^2+\mathbb{E}\int_0^T\Big[2\langle x(t),\ \phi(t)-\mu p(t)\rangle+|\psi(t)-\mu q(t)|^2+\|\xi(t,\cdot)-\mu k(t,\cdot)\|^2\Big]\mathrm{e}^{Kt}\mathrm{d}t.\end{aligned}$$

Since $K<2\mu$, then there exists a constant $\varepsilon>0$ such that $2\mu-K-\varepsilon>0$. By the inequality $2\langle a,\ b\rangle\le\varepsilon|a|^2+(1/\varepsilon)|b|^2$, we have

$$\begin{aligned}&\mathbb{E}\left[|x(T)|^2\mathrm{e}^{KT}\right]+(2\mu-K-\varepsilon)\mathbb{E}\int_0^T|x(t)|^2\mathrm{e}^{Kt}\mathrm{d}t\\ &\le|x(0)|^2+\mathbb{E}\int_0^T\left[\frac{1}{\varepsilon}|\phi(t)-\mu p(t)|^2+|\psi(t)-\mu q(t)|^2+\|\xi(t,\cdot)-\mu k(t,\cdot)\|^2\right]\mathrm{e}^{Kt}\mathrm{d}t.\end{aligned}$$

Letting $T\to\infty$, we have

$$\begin{aligned}&(2\mu-K-\varepsilon)\mathbb{E}\int_0^\infty|x(t)|^2\mathrm{e}^{Kt}\mathrm{d}t\\ &\le|x(0)|^2+\mathbb{E}\int_0^\infty\left[\frac{1}{\varepsilon}|\phi(t)-\mu p(t)|^2+|\psi(t)-\mu q(t)|^2+\|\xi(t,\cdot)-\mu k(t,\cdot)\|^2\right]\mathrm{e}^{Kt}\mathrm{d}t.\end{aligned}$$

We have proved $x\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$.

It is easy to verify that $(x^0,y^0,z^0,r^0)=(x,x+p,q,k)$ is a solution to the forward-backward SDE (3.3). We proved the existence.

We would like to prove the uniqueness in a bigger space: $\mathcal{L}_{\mathbb{F}}^{2}(0,\infty)$. Let $\theta_1(\cdot)=(x_1(\cdot),y_1(\cdot),z_1(\cdot),r_1(\cdot,\cdot))$ and $\theta_2(\cdot)=(x_2(\cdot),y_2(\cdot),z_2(\cdot),r_2(\cdot,\cdot))$ belonging to $\mathcal{L}_{\mathbb{F}}^{2}(0,\infty)$ be two solutions to the forward-backward SDE (3.3). We denote $\hat{\theta}(\cdot)=(\hat{x}(\cdot),\hat{y}(\cdot),\hat{z}(\cdot),\hat{r}(\cdot,\cdot))=(x_1(\cdot)-x_2(\cdot),y_1(\cdot)-y_2(\cdot),z_1(\cdot)-z_2(\cdot),r_1(\cdot,\cdot)-r_2(\cdot,\cdot))$ and apply Itô's formula to $\langle\hat{x}(t),\ \hat{y}(t)\rangle$ on the interval $[0,T]$ to get

$$\mathbb{E}\Big[\langle\hat{x}(T),\ \hat{y}(T)\rangle\Big]+\mu\mathbb{E}\int_0^T|\hat{\theta}(t)|^2\mathrm{d}t=\Big\langle\varPhi(y_1(0))-\varPhi(y_2(0)),\hat{y}(0)\Big\rangle.$$

By the monotonicity condition of $\varPhi$, we get

$$\mathbb{E}\Big[\langle \hat{x}(T),\ \hat{y}(T)\rangle\Big] + \mu\mathbb{E}\int_0^T |\hat{\theta}(t)|^2 \mathrm{d}t \leq 0.$$

Letting $T \to \infty$, thanks to Proposition 2.1 and Corollary 2.3, we have

$$\mu\mathbb{E}\int_0^\infty |\hat{\theta}(t)|^2 \mathrm{d}t \leq 0.$$

The uniqueness is proved. □

The above Lemma 3.4 shows that, when $\alpha = 0$, the forward-backward SDE (3.2) is in a simple form and then is uniquely solvable. It is clear that, when $\alpha = 1$ and (η, ϕ, ψ, ξ) vanish, the forward-backward SDE (3.2) coincides with (3.1). We will show that there exists a fixed step-length $\delta_0 > 0$, such that, if, for some $\alpha_0 \in [0, 1)$, (3.2) is uniquely solvable for any $(\eta, \phi, \psi, \xi) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0, \infty)$, then the same conclusion holds for α_0 being replaced by $\alpha_0 + \delta \leq 1$ with $\delta \in [0, \delta_0]$. Once this has been proved, we can increase the parameter α step by step and finally reach $\alpha = 1$, which gives the unique solvability of the forward-backward SDE (3.1). This idea is adopted from [5, 11, 12, 18, 19], and this method is called the method of continuation.

Now, we prove the following continuation lemma.

Lemma 3.5. *Under assumptions* (H3.1)–(H3.4)*, there exists an absolute constant $\delta_0 > 0$ such that, if, for some $\alpha_0 \in [0, 1)$, the forward-backward SDE* (3.2) *is uniquely solvable in $\mathcal{L}_{\mathbb{F}}^{2,K}(0, \infty)$ for any $(\eta, \phi, \psi, \xi) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0, \infty)$, then the same is true for $\alpha = \alpha_0 + \delta$ with $\delta \in [0, \delta_0]$ and $\alpha_0 + \delta \leq 1$.*

Proof. Let δ_0 be determined as follows. Let $\delta \in [0, \delta_0]$. For each $\theta(\cdot) = (x(\cdot), y(\cdot), z(\cdot), r(\cdot, \cdot)) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0, \infty)$, we consider the following forward-backward SDE (compared to (3.2) with $\alpha = \alpha_0 + \delta$):

$$\begin{cases} dX(t) = \Big[\alpha_0 b(t, \varTheta(t)) - \mu(1-\alpha_0)Y(t) + \delta\big(b(t, \theta(t)) + \mu y(t)\big) + \phi(t)\Big]\mathrm{d}t \\ \qquad\qquad + \Big[\alpha_0 \sigma(t, \varTheta(t)) - \mu(1-\alpha_0)Z(t) + \delta\big(\sigma(t, \theta(t)) + \mu z(t)\big) + \psi(t)\Big]\mathrm{d}W(t) \\ \qquad\qquad + \displaystyle\int_{\mathcal{E}} \Big[\alpha_0 \gamma(t, e, \varTheta(t-)) - \mu(1-\alpha_0)R(t, e) \\ \qquad\qquad + \delta\big(\gamma(t, e, \theta(t-)) + \mu r(t, e)\big) + \xi(t, e)\Big]\tilde{N}(\mathrm{d}t, \mathrm{d}e), \\ -dY(t) = \Big[\alpha_0 g(t, \varTheta(t)) + \mu(1-\alpha_0)X(t) + \delta\big(g(t, \theta(t)) - \mu x(t)\big) + \eta(t)\Big]\mathrm{d}t \\ \qquad\qquad - Z(t)\mathrm{d}W(t) - \displaystyle\int_{\mathcal{E}} R(t, e)\tilde{N}(\mathrm{d}t, \mathrm{d}e), \\ X(0) = \varPhi(Y(0)). \end{cases} \tag{3.10}$$

It is easy to check that $\big(\delta\big(g(\cdot, \theta(\cdot)) - \mu x(\cdot)\big) + \eta(\cdot),\ \ \delta\big(b(\cdot, \theta(\cdot)) + \mu y(\cdot)\big) + \phi(\cdot),\ \ \delta\big(\sigma(\cdot, \theta(\cdot)) + \mu z(\cdot)\big) + \psi(\cdot),\ \ \delta\big(\gamma(\cdot, \cdot, \theta(\cdot-)) + \mu r(\cdot, \cdot)\big) + \xi(\cdot, \cdot)\big) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0, \infty)$. Then, by our assumptions, the above forward-backward SDE is uniquely solvable in the space $\mathcal{L}_{\mathbb{F}}^{2,K}(0, \infty)$. We denote the unique solution by $\varTheta(\cdot) = (X(\cdot), Y(\cdot), Z(\cdot), R(\cdot, \cdot))$. We have established a mapping

$$\varTheta = I_{\alpha_0+\delta}(\theta) : \mathcal{L}_{\mathbb{F}}^{2,K}(0, \infty) \to \mathcal{L}_{\mathbb{F}}^{2,K}(0, \infty).$$

Next we shall prove it is a contraction.

Let $\theta_1 = (x_1, y_1, z_1, r_1)$, $\theta_2 = (x_2, y_2, z_2, r_2) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$ and $\Theta_1 = (X_1, Y_1, Z_1, R_1) = I_{\alpha_0+\delta}(\theta_1)$, $\Theta_2 = (X_2, Y_2, Z_2, R_2) = I_{\alpha_0+\delta}(\theta_2)$. Let

$$\hat{\theta} = (\hat{x}, \hat{y}, \hat{z}, \hat{r}) = (x_1 - x_2, y_1 - y_2, z_1 - z_2, r_1 - r_2),$$
$$\hat{\Theta} = (\hat{X}, \hat{Y}, \hat{Z}, \hat{R}) = (X_1 - X_2, Y_2 - Y_2, Z_1 - Z_2, R_1 - R_2).$$

For any $T > 0$, applying Itô's formula to $\langle \hat{X}(t),\ \hat{Y}(t)\rangle \mathrm{e}^{Kt}$ on the interval $[0,T]$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big\langle \hat{X}(T),\ \hat{Y}(T)\Big\rangle \mathrm{e}^{KT}\Big] - \Big\langle \Phi(Y_1(0)) - \Phi(Y_2(0)),\ \hat{Y}(0)\Big\rangle \\
&= \alpha_0 \mathbb{E}\int_0^T \Big\langle A(t,\Theta_1(t)) - A(t,\Theta_2(t)),\ \hat{\Theta}(t)\Big\rangle \mathrm{e}^{Kt}\mathrm{d}t - \mu(1-\alpha_0)\mathbb{E}\int_0^T |\hat{\Theta}(t)|^2 \mathrm{e}^{Kt}\mathrm{d}t \\
&\quad + \delta\mathbb{E}\int_0^T \Big\langle A(t,\theta_1(t)) - A(t,\theta_2(t)),\ \hat{\Theta}(t)\Big\rangle \mathrm{e}^{Kt}\mathrm{d}t + \delta\mu\mathbb{E}\int_0^T \Big\langle \hat{\theta}(t),\ \hat{\Theta}(t)\Big\rangle \mathrm{e}^{Kt}\mathrm{d}t \\
&\quad + K\mathbb{E}\int_0^T \Big\langle \hat{X}(t),\ \hat{Y}(t)\Big\rangle \mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}$$

By assumptions (H3.2) and (H3.3), we deduce

$$\begin{aligned}
&\mathbb{E}\Big[\Big\langle \hat{X}(T),\ \hat{Y}(T)\Big\rangle \mathrm{e}^{KT}\Big] + (\mu - \frac{1}{2}K)\mathbb{E}\int_0^T |\hat{\Theta}(t)|^2 \mathrm{e}^{Kt}\mathrm{d}t \\
&\leq \delta(C+\mu)\mathbb{E}\int_0^T |\hat{\theta}(t)||\hat{\Theta}(t)|\mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}$$

Since $\frac{1}{2}K < \mu$ (see assumption (H3.4)), then there exists a constant $\varepsilon > 0$ such that $\mu - \frac{1}{2}K - \varepsilon > 0$. Then we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big\langle \hat{X}(T),\ \hat{Y}(T)\Big\rangle \mathrm{e}^{KT}\Big] + (\mu - \frac{1}{2}K - \varepsilon)\mathbb{E}\int_0^T |\hat{\Theta}(t)|^2 \mathrm{e}^{Kt}\mathrm{d}t \\
&\leq \delta^2 \cdot \frac{(C+\mu)^2}{4\varepsilon}\mathbb{E}\int_0^T |\hat{\theta}(t)|^2 \mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}$$

Letting $T \to \infty$, we have

$$\mathbb{E}\int_0^\infty |\hat{\Theta}(t)|^2 \mathrm{e}^{Kt}\mathrm{d}t \leq \delta^2 \cdot \frac{(C+\mu)^2}{2\varepsilon(2\mu - K - 2\varepsilon)}\mathbb{E}\int_0^T |\hat{\theta}(t)|^2 \mathrm{e}^{Kt}\mathrm{d}t.$$

Now we choose $\delta_0^2 = \frac{\varepsilon(2\mu - K - 2\varepsilon)}{2(C+\mu)^2}$, then for any $\delta \in [0, \delta_0]$, we have the following estimate:

$$\Big\|\hat{\Theta}(\cdot)\Big\|_{\mathcal{L}_{\mathbb{F}}^{2,K}} \leq \frac{1}{2}\Big\|\hat{\theta}(\cdot)\Big\|_{\mathcal{L}_{\mathbb{F}}^{2,K}}.$$

This implies that the mapping $I_{\alpha_0+\delta}$ is a contraction. Hence, it has a unique fixed point, which is the unique solution of (3.2) for $\alpha = \alpha_0 + \delta$. We complete the proof. $\square$

Now, we give an existence and uniqueness result for the forward-backward SDE (3.1).

Theorem 3.6. *Under assumptions* (H3.1)–(H3.4)*, the forward-backward SDE* (3.1) *admits a unique solution* $(x(\cdot), y(\cdot), z(\cdot), r(\cdot,\cdot)) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$.

Proof. By Lemmas 3.4 and 3.5, we can solve the forward-backward SDE (3.2) uniquely for any $\alpha \in [0,1]$ and $(\eta, \phi, \psi, \xi) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$. Particularly, (3.2) with $\alpha = 1$ and $(\eta, \phi, \psi, \xi) = 0$, which is (3.1), admits a unique solution. We finish the proof. $\square$

Remark 3.7.

(i) By a similar proof as that of Lemma 3.4, the uniqueness holds true in the bigger space $\mathcal{L}^2_{\mathbb{F}}(0,\infty)$.
(ii) By Proposition 2.1 and Corollary 2.3, the unique solution (x,y,z,r) of the forward-backward SDE (3.1) belongs to $\mathcal{X}^K(0,\infty;\mathbb{R}^n)\times\mathcal{X}^K(0,\infty;\mathbb{R}^n)\times L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times d})\times M^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})$ exactly.

In the rest of this section, we would like to establish some properties of the solutions to forward-backward SDEs including two stability results and a comparison theorem. First we establish the stability results.

Proposition 3.8. *Let $(b_1,\sigma_1,\gamma_1,g_1,\Phi_1)$ and $(b_2,\sigma_2,\gamma_2,g_2,\Phi_2)$ be two sets of coefficients of forward-backward SDEs satisfying assumptions* (H3.1)–(H3.4). *Let $\theta_1=(x_1,y_1,z_1,r_1)$ and $\theta_2=(x_2,y_2,z_2,r_2)$ be the corresponding solutions.*

(i) *If we assume that $\Phi_1=\Phi_2$, then*

$$\mathbb{E}\int_0^\infty|\theta_1(t)-\theta_2(t)|^2\mathrm{e}^{Kt}\mathrm{d}t\le C\mathbb{E}\int_0^\infty|A_1(t,\theta_2(t))-A_2(t,\theta_2(t))|^2\mathrm{e}^{Kt}\mathrm{d}t,\tag{3.11}$$

where C is a constant depending on μ and K.

(ii) *If we strengthen the monotonicity condition on Φ_1 as follows: there exists a constant $\nu>0$ such that for any $y_1,\ y_2\in\mathbb{R}^n$,*

$$\langle\Phi_1(y_1)-\Phi_1(y_2),\ y_1-y_2\rangle\le-\nu|y_1-y_2|^2,\tag{3.12}$$

then

$$\begin{aligned}&|y_1(0)-y_2(0)|^2+\mathbb{E}\int_0^\infty|\theta_1(t)-\theta_2(t)|^2\mathrm{e}^{Kt}\mathrm{d}t\\&\le C\left\{|\Phi_1(y_2(0))-\Phi_2(y_2(0))|^2+\mathbb{E}\int_0^\infty|A_1(t,\theta_2(t))-A_2(t,\theta_2(t))|^2\mathrm{e}^{Kt}\mathrm{d}t\right\},\end{aligned}\tag{3.13}$$

where C is a constant depending on μ, ν and K.

Proof. We apply Itô's formula to $\langle\hat{x}(t),\ \hat{y}(t)\rangle\mathrm{e}^{Kt}$ on the interval $[0,T]$:

$$\begin{aligned}&\mathbb{E}\Big[\langle\hat{x}(T),\ \hat{y}(T)\rangle\mathrm{e}^{KT}\Big]-\langle\Phi_1(y_1(0))-\Phi_1(y_2(0)),\ \hat{y}(0)\rangle\\&\quad-\mathbb{E}\int_0^T\Big\langle A_1(t,\theta_1(t))-A_1(t,\theta_2(t)),\ \hat{\theta}(t)\Big\rangle\mathrm{e}^{Kt}\mathrm{d}t\\&=\langle\Phi_1(y_2(0))-\Phi_2(y_2(0)),\ \hat{y}(0)\rangle\\&\quad+\mathbb{E}\int_0^T\Big\{\Big\langle A_1(t,\theta_2(t))-A_2(t,\theta_2(t)),\hat{\theta}(t)\Big\rangle+K\Big\langle\hat{x}(t),\ \hat{y}(t)\Big\rangle\Big\}\mathrm{e}^{Kt}\mathrm{d}t,\end{aligned}$$

where the notations $\hat{x}:=x_1-x_2$, *etc.* By the monotonicity condition on A_1, we have

$$\begin{aligned}&\mathbb{E}\Big[\langle\hat{x}(T),\ \hat{y}(T)\rangle\mathrm{e}^{KT}\Big]-\langle\Phi_1(y_1(0))-\Phi_1(y_2(0)),\ \hat{y}(0)\rangle+\left(\mu-\frac{K}{2}\right)\mathbb{E}\int_0^T|\hat{\theta}(t)|^2\mathrm{e}^{Kt}\mathrm{d}t\\&\le\langle\Phi_1(y_2(0))-\Phi_2(y_2(0)),\ \hat{y}(0)\rangle+\mathbb{E}\int_0^T\Big\langle A_1(t,\theta_2(t))-A_2(t,\theta_2(t)),\hat{\theta}(t)\Big\rangle\mathrm{e}^{Kt}\mathrm{d}t.\end{aligned}\tag{3.14}$$

(i) When $\Phi_1=\Phi_2$, considering the monotonicity condition on Φ_1 (see assumption (H3.3)), (3.14) is reduced to

$$\mathbb{E}\Big[\langle\hat{x}(T),\ \hat{y}(T)\rangle\mathrm{e}^{KT}\Big]+\left(\mu-\frac{K}{2}\right)\mathbb{E}\int_0^T|\hat{\theta}(t)|^2\mathrm{e}^{Kt}\mathrm{d}t\le\mathbb{E}\int_0^T\Big\langle A_1(t,\theta_2(t))-A_2(t,\theta_2(t)),\hat{\theta}(t)\Big\rangle\mathrm{e}^{Kt}\mathrm{d}t.$$

By a similar technique as that in the proof of Lemma 3.5, we obtain the estimate (3.11).

(ii) When Φ_1 satisfies the strong monotonicity condition (3.12), the inequality (3.14) is reduced to

$$\begin{aligned}&\mathbb{E}\Big[\langle\hat{x}(T),\ \hat{y}(T)\rangle \mathrm{e}^{KT}\Big]+\nu|\hat{y}(0)|^2+\left(\mu-\frac{K}{2}\right)\mathbb{E}\int_0^T|\hat{\theta}(t)|^2\mathrm{e}^{Kt}\mathrm{d}t\\ &\leq \langle \Phi_1(y_2(0))-\Phi_2(y_2(0)),\ \hat{y}(0)\rangle+\mathbb{E}\int_0^T\Big\langle A_1(t,\theta_2(t))-A_2(t,\theta_2(t)),\hat{\theta}(t)\Big\rangle\mathrm{e}^{Kt}\mathrm{d}t.\end{aligned}$$

Similar to the proof of Lemma 3.5, we get the estimate (3.13). □

Remark 3.9. In the proof of the above proposition, it is easy to see, even if the Lipschitz condition and the monotonicity conditions are not satisfied by $(b_2,\sigma_2,\gamma_2,g,\Phi)$, the estimates (3.11) and (3.13) still hold.

We would like to point out that the L^2-estimate (see (3.11) and (3.13)) for solutions of forward-backward SDEs will play a key role in studying Pontryagin's maximum principle and Bellman's dynamic programming principle for stochastic optimal control and stochastic differential game problems of infinite horizon forward-backward SDEs.

Next we prove a comparison theorem. As same as before, let $\theta_1=(x_1,y_1,z_1,r_1)$ and $\theta_2=(x_2,y_2,z_2,r_2)$ be the solutions of (3.1) with coefficients $(b,\sigma,\gamma,g,\Phi_1)$ and $(b,\sigma,\gamma,g,\Phi_2)$ respectively. Denote

$$\hat{\theta}:=\theta_1-\theta_2=(x_1-x_2,y_1-y_2,z_1-z_2,r_1-z_2)=:(\hat{x},\hat{y},\hat{z},\hat{r}).$$

Similar to Lemma 7 in [11], one can easily prove the following lemma.

Lemma 3.10. *Let assumptions* (H3.1)–(H3.4) *holds for* $(b,\sigma,\gamma,g,\Phi_1)$ *and* $(b,\sigma,\gamma,g,\Phi_2)$.

(i) *For any* $t\in[0,\infty)$*, we have*

$$\langle\hat{x}(t),\ \hat{y}(t)\rangle\geq 0.$$

(ii) *If we define an* $\mathbb{F}$*-stopping time:* $\tau=\inf\{t\geq 0;\ \langle\hat{x}(t),\ \hat{y}(t)\rangle=0\}$*, then we further have*

$$\hat{\theta}(t)\mathbb{1}_{[\tau,\infty)}(t)=(\hat{x}(t),\hat{y}(t),\hat{z}(t),\hat{r}(t,\cdot))\mathbb{1}_{[\tau,\infty)}(t)=0.$$

Proof.

(i) Clearly, for any $t\in[0,\infty)$, there exists a sequence of times $\{T_i\}_{i=1}^{\infty}$, which increases and diverges as $i\to\infty$, such that

$$\lim_{i\to\infty}\mathbb{E}\Big[\langle\hat{x}(T_i),\ \hat{y}(T_i)\rangle\ \Big|\ \mathcal{F}_t\Big]=0.$$

We apply Itô's formula to $\langle\hat{x}(s),\ \hat{y}(s)\rangle$ on the interval $[t,T_i]$ to have

$$\begin{aligned}\mathbb{E}\Big[\langle\hat{x}(T_i),\ \hat{y}(T_i)\rangle\ \Big|\ \mathcal{F}_t\Big]-\langle\hat{x}(t),\ \hat{y}(t)\rangle&=\mathbb{E}\bigg[\int_t^{T_i}\Big\langle A(s,\theta_1(s))-A(s,\theta_2(s)),\ \hat{\theta}(s)\Big\rangle\mathrm{d}s\ \bigg|\ \mathcal{F}_t\bigg]\\ &\leq\ -\mu\mathbb{E}\bigg[\int_t^{T_i}|\hat{\theta}(s)|^2\mathrm{d}s\ \bigg|\ \mathcal{F}_t\bigg].\end{aligned}$$

Then, letting $i\to\infty$, we have

$$\langle\hat{x}(t),\ \hat{y}(t)\rangle\geq\lim_{i\to\infty}\mu\mathbb{E}\bigg[\int_t^{T_i}|\hat{\theta}(s)|^2\mathrm{d}s\ \bigg|\ \mathcal{F}_t\bigg]\geq 0.$$

(ii) For the given $\mathbb{F}$-stopping time τ, it is easy to see, for any $T \in [0, \infty)$,

$$\langle \hat{x}(T),\ \hat{y}(T)\rangle - \langle \hat{x}(\tau \wedge T),\ \hat{y}(\tau \wedge T)\rangle \geq 0.$$

On the other hand, from Itô's formula,

$$\begin{aligned}\mathbb{E}\Big[\langle \hat{x}(T),\ \hat{y}(T)\rangle - \langle \hat{x}(\tau \wedge T),\ \hat{y}(\tau \wedge T)\rangle\Big] &= \mathbb{E}\int_{\tau\wedge T}^{T} \Big\langle A(s,\theta_1(s)) - A(s,\theta_2(s)),\ \hat{\theta}(s)\Big\rangle \mathrm{d}s \\ &\leq\ -\mu\mathbb{E}\int_{\tau\wedge T}^{T} |\hat{\theta}(s)|^2 \mathrm{d}s \leq 0.\end{aligned}$$

Then, by the above two inequalities, we obtain

$$\hat{\theta}(s)\mathbb{1}_{[\tau\wedge T, T]}(t) = 0.$$

Due to the arbitrariness of T, we get the desired conclusion $\hat{\theta}(s)\mathbb{1}_{[\tau,\infty)}(t) = 0$. □

Theorem 3.11. *Let $n = 1$. Let assumptions* (H3.1)–(H3.4) *hold for* $(b, \sigma, \gamma, g, \Phi_1)$ *and* $(b, \sigma, \gamma, g, \Phi_2)$.

(i) *If $\Phi_1(y_2(0)) > \Phi_2(y_2(0))$, then $\hat{y}(0) > 0$.*
(ii) *If $\Phi_1(y_2(0)) = \Phi_2(y_2(0))$, then $\hat{y}(0) = 0$.*

Proof. When $n = 1$, Lemma 3.10–(i) is read as $\hat{x}(t)\hat{y}(t) \geq 0$ for any $t \in [0, \infty)$. Especially, taking $t = 0$, we have $\hat{x}(0)\hat{y}(0) \geq 0$. Then from the monotonicity condition on Φ_1,

$$\begin{aligned}0 &\leq \Big(\Phi_1(y_1(0)) - \Phi_2(y_2(0))\Big)\hat{y}(0) \\ &= \Big[\Big(\Phi_1(y_1(0)) - \Phi_1(y_2(0))\Big) + \Big(\Phi_1(y_2(0)) - \Phi_2(y_2(0))\Big)\Big]\hat{y}(0) \\ &\leq \Big(\Phi_1(y_2(0)) - \Phi_2(y_2(0))\Big)\hat{y}(0).\end{aligned} \tag{3.15}$$

(i) If $\Phi_1(y_2(0)) > \Phi_2(y_2(0))$, then (3.15) implies $\hat{y}(0) \geq 0$. Moreover, if $\hat{y}(0) = 0$, then the stopping time τ defined in Lemma 3.10–(ii) is equal to 0. By Lemma 3.10–(ii), we have $\hat{x}(0) = 0$. Since $y_1(0) = y_2(0)$, then

$$\Phi_1(y_2(0)) - \Phi_2(y_2(0)) = \Phi_1(y_1(0)) - \Phi_2(y_2(0)) = \hat{x}(0) = 0.$$

This is a contradiction. Therefore, in this case we must have $\hat{y}(0) > 0$.

(ii) When $\Phi_1(y_2(0)) = \Phi_2(y_2(0))$, we also use a framework of reduction to absurdity to show $\hat{y}(0) = 0$. We assume that $\hat{y}(0) \neq 0$. From (3.15), we deduce that

$$\hat{x}(0) = \Phi_1(y_1(0)) - \Phi_2(y_2(0)) = 0.$$

By Lemma 3.10–(ii) once again, we have $\hat{y}(0) = 0$. We obtain a contradiction, and then finish the proof. □

4. Application to backward stochastic LQ problems

In this section, we apply the solvability result of forward-backward SDEs studied in the above section to deal with two kinds of backward stochastic linear-quadratic (LQ) problems with jumps, including an LQ stochastic optimal control (SOC) problem and an LQ nonzero-sum stochastic differential game (NZSSDG) problem.

Firstly, for the control problem, the system is given by the following controlled linear backward SDE on the infinite interval $[0, \infty)$:

$$
\begin{aligned}
-\mathrm{d}y(t) = &\left[A(t)y(t) + \sum_{i=1}^{d} B_i(t)z_i(t) + \sum_{i=1}^{l} \int_{\mathcal{E}} C_i(t,e)r_i(t,e)\pi_i(\mathrm{d}e) + D(t)v(t) + \alpha(t)\right]\mathrm{d}t \\
&- \sum_{i=1}^{d} z_i(t)\mathrm{d}W_i(t) - \sum_{i=1}^{l} \int_{\mathcal{E}} r_i(t,e)\tilde{N}_i(\mathrm{d}t, \mathrm{d}e),
\end{aligned}
\tag{4.1}
$$

where A, B_i $(i = 1, 2, \ldots, d)$, D are $\mathbb{F}$-progressively measurable, matrix-valued, bounded processes with appropriate dimensions; C_i $(i = 1, 2, \ldots, l)$ is a $\mathcal{P} \otimes \mathcal{B}(\mathcal{E})$-measurable, $(n \times n)$ matrix-valued process such that $\int_{\mathcal{E}} |C_i(t,e)|^2 \pi_i(\mathrm{d}e)$ is uniformly bounded for any $(\omega, t) \in \Omega \times [0, \infty)$; and the nonhomogeneous term $\alpha \in L_{\mathbb{F}}^{2,K}(0, \infty; \mathbb{R}^n)$ where $K > 0$ is a constant. The admissible control set is defined by

$$
\begin{aligned}
\mathcal{V} := \Big\{ v \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^k) \;\Big|\; &\text{with respect to } v, \text{ (4.1) admits a unique solution} \\
&(y^v, z^v, r^v) \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n) \times L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n\times d}) \times M_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^{n\times l}) \Big\},
\end{aligned}
$$

in which each element v is called an admissible control, and (y^v, z^v, r^v) is called the state trajectory corresponding to v. In what follows, we will show the admissible control set $\mathcal{V}$ is nonempty under some suitable conditions. In addition, we are given a cost functional associated with v in a quadratic form:

$$
\begin{aligned}
J(v(\cdot)) = &\frac{1}{2}\langle Qy(0),\ y(0)\rangle + \frac{1}{2}\mathbb{E}\int_0^\infty \Bigg[\langle L(t)y(t),\ y(t)\rangle + \sum_{i=1}^{d}\langle M_i(t)z_i(t),\ z_i(t)\rangle \\
&+ \sum_{i=1}^{l}\int_{\mathcal{E}}\langle S_i(t,e)r_i(t,e),\ r_i(t,e)\rangle\pi_i(\mathrm{d}e) + \langle R(t)v(t),\ v(t)\rangle\Bigg]\mathrm{d}t,
\end{aligned}
\tag{4.2}
$$

where Q is an $(n \times n)$ symmetric and positive semi-definite matrix; L, M_i $(i = 1, 2, \ldots, d)$ are $\mathbb{F}$-progressively measurable, $(n \times n)$ symmetric and positive semi-definite matrix-valued, bounded processes; S_i $(i = 1, 2, \ldots, l)$ is a $\mathcal{P} \otimes \mathcal{B}(\mathcal{E})$-measurable, $(n \times n)$ symmetric and positive semi-definite matrix-valued, bounded process; R is an $\mathbb{F}$-progressively measurable, $(k \times k)$ symmetric and positive definite matrix-valued, bounded process. Moreover, R^{-1} is also bounded.

Problem (SOC). The problem is to find an admissible control $u \in \mathcal{V}$ such that

$$
J(u(\cdot)) = \inf_{v(\cdot)\in\mathcal{V}} J(v(\cdot)). \tag{4.3}
$$

Such an admissible control u is called an optimal control, and $(y, z, r) := (y^u, z^u, r^u)$ is called the corresponding optimal state trajectory.

The following result links Problem (SOC) to a forward-backward SDE.

Lemma 4.1. *If the following forward-backward SDE:*

$$\begin{cases} \mathrm{d}x(t) = \Big[A^\top(t)x(t) - L(t)y(t)\Big]\mathrm{d}t + \displaystyle\sum_{i=1}^{d}\Big[B_i^\top(t)x(t) - M_i(t)z_i(t)\Big]\mathrm{d}W_i(t) \\ \qquad\qquad + \displaystyle\sum_{i=1}^{l}\int_{\mathcal{E}}\Big[C_i^\top(t,e)x(t-) - S_i(t,e)r_i(t,e)\Big]\tilde{N}_i(\mathrm{d}t,\mathrm{d}e), \\ -\mathrm{d}y(t) = \Big[D(t)R^{-1}(t)D^\top(t)x(t) + A(t)y(t) + \displaystyle\sum_{i=1}^{d}B_i(t)z_i(t) \\ \qquad\qquad + \displaystyle\sum_{i=1}^{l}\int_{\mathcal{E}}C_i(t,e)r_i(t,e)\pi_i(\mathrm{d}e) + \alpha(t)\Big]\mathrm{d}t - \displaystyle\sum_{i=1}^{d}z_i(t)\mathrm{d}W_i(t) \\ \qquad\qquad - \displaystyle\sum_{i=1}^{l}\int_{\mathcal{E}}r_i(t,e)\tilde{N}_i(\mathrm{d}t,\mathrm{d}e), \\ x(0) = -Qy(0) \end{cases} \tag{4.4}$$

admits a solution $(x,y,z,r) \in \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$, *then*

$$u(t) = R^{-1}(t)D^\top(t)x(t), \quad t \in [0,\infty), \tag{4.5}$$

provides an optimal control of Problem (SOC). *Moreover the optimal control is unique.*

Proof. First, we prove that u defined by (4.5) is an optimal control for Problem (SOC). For each $v \in \mathcal{V}$, the corresponding state trajectory is denoted by (y^v, z^v, r^v). Let us consider the difference of $J(v(\cdot))$ and $J(u(\cdot))$ (the argument (t,e) is suppressed):

$$\begin{aligned} J(v(\cdot)) - J(u(\cdot)) &= \frac{1}{2}\Big[\langle Qy^v(0),\ y^v(0)\rangle - \langle Qy(0),\ y(0)\rangle\Big] \\ &\quad + \frac{1}{2}\mathbb{E}\int_0^\infty\bigg\{\Big[\langle Ly^v,\ y^v\rangle - \langle Ly,\ y\rangle\Big] + \sum_{i=1}^{d}\Big[\langle M_i z_i^v,\ z_i^v\rangle - \langle M_i z_i,\ z_i\rangle\Big] \\ &\quad + \sum_{i=1}^{l}\int_{\mathcal{E}}\Big[\langle S_i r_i^v,\ r_i^v\rangle - \langle S_i r_i,\ r_i\rangle\Big]\pi_i(\mathrm{d}e) + \Big[\langle Rv,\ v\rangle - \langle Ru,\ u\rangle\Big]\bigg\}\mathrm{d}t \\ &= \frac{1}{2}\langle Q(y^v(0) - y(0)),\ y^v(0) - y(0)\rangle \\ &\quad + \frac{1}{2}\mathbb{E}\int_0^\infty\bigg\{\langle L(y^v - y),\ y^v - y\rangle + \sum_{i=1}^{d}\langle M_i(z_i^v - z_i),\ z_i^v - z_i\rangle \\ &\quad + \sum_{i=1}^{l}\int_{\mathcal{E}}\langle S_i(r_i^v - r_i),\ r_i^v - r_i\rangle\pi_i(\mathrm{d}e) + \langle R(v-u),\ v-u\rangle\bigg\}\mathrm{d}t + \varLambda, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \varLambda = \langle Qy(0),\ y^v(0) - y(0)\rangle + \mathbb{E}\int_0^\infty\bigg\{\langle Ly,\ y^v - y\rangle + \sum_{i=1}^{d}\langle M_i z_i,\ z_i^v - z_i\rangle \\ + \sum_{i=1}^{l}\int_{\mathcal{E}}\langle S_i r_i,\ r_i^v - r_i\rangle\pi_i(\mathrm{d}e) + \langle Ru,\ v-u\rangle\bigg\}\mathrm{d}t. \end{aligned}$$

Applying Itô's formula to $\langle x(t),\ y^v(t)-y(t)\rangle$ on the interval $[0,T]$, by the initial condition of x (see (4.4)) and the definition of u (see (4.5)), we have

$$\begin{aligned}&\mathbb{E}\Big[\langle x(T),\ y^v(T)-y(T)\rangle\Big]+\langle Qy(0),\ y^v(0)-y(0)\rangle\\ =&\ -\mathbb{E}\int_0^T\Big\{\langle Ly,\ y^v-y\rangle+\sum_{i=1}^d\langle M_iz_i,\ z_i^v-z_i\rangle+\sum_{i=1}^l\int_{\mathcal{E}}\langle S_ir_i,\ r_i^v-r_i\rangle\pi_i(\mathrm{d}e)+\langle Ru,\ v-u\rangle\Big\}\mathrm{d}t.\end{aligned}$$

Letting $T\to\infty$, we get $\Lambda=0$. Then, since Q, L, M_i $(i=1,2,\dots,d)$, S_i $(i=1,2,\dots,l)$ are positive semi-definite, and R is positive definite, we have $J(v(\cdot))-J(u(\cdot))\geq 0$. Due to the arbitrariness of v, we prove that u defined by (4.5) is an optimal control.

For the uniqueness, besides u given by (4.5), let $\bar u\in\mathcal{V}$ be another optimal control, and denote by $(y^{\bar u},z^{\bar u},r^{\bar u})$ the corresponding optimal state trajectory. Obviously $J(\bar u(\cdot))=J(u(\cdot))$. Coming back to (4.6), we have

$$\begin{aligned}0=&\ \frac12\langle Q(y^{\bar u}(0)-y(0)),\ y^{\bar u}(0)-y(0)\rangle\\ &+\frac12\mathbb{E}\int_0^\infty\Big\{\langle L(y^{\bar u}-y),\ y^{\bar u}-y\rangle+\sum_{i=1}^d\langle M_i(z_i^{\bar u}-z_i),\ z_i^{\bar u}-z_i\rangle\\ &+\sum_{i=1}^l\int_{\mathcal{E}}\langle S_i(r_i^{\bar u}-r_i),\ r_i^{\bar u}-r_i\rangle\pi_i(\mathrm{d}e)+\langle R(\bar u-u),\ \bar u-u\rangle\Big\}\mathrm{d}t\\ \geq&\ \frac12\mathbb{E}\int_0^\infty\langle R(\bar u-u),\ \bar u-u\rangle\mathrm{d}t.\end{aligned}$$

Because R is positive definite, we get $\bar u(\cdot)=u(\cdot)$. We have proved the uniqueness of the optimal control. □

In order to obtain the solvability of (4.4), we assume the following assumptions:

(A1.1) There exists a constant $\mu>0$ such that for any $(\omega,t,e)\in\Omega\times[0,\infty)\times\mathcal{E}$, any $i=1,2,\dots,d$, any $j=1,2,\dots,l$,

$$D(t)R^{-1}(t)D^\top(t)\geq\mu I,\quad L(t)\geq\mu I,\quad M_i(t)\geq\mu I,\quad S_j(t,e)\geq\mu I.$$

(A1.2) $2\mu-K\geq 0$.

Here, I denotes the $(n\times n)$ identity matrix and the expression $A\geq B$ means $A-B$ is positive semi-definite as usual. We notice that, in the viewpoint of Remark 3.1, assumption (A1.2) is artificial. If it does not hold true, then we can consider Problem (SOC) in some larger space. However, for the convenience of presentation, we keep it here.

Theorem 4.2. *Under assumptions* (A1.1)–(A1.2)*, the forward-backward SDE* (4.4) *admits a unique solution* $(x,y,z,r)\in\mathcal{L}^{2,K}_{\mathbb{F}}(0,\infty)$. *Moreover, u defined by* (4.5) *is the unique optimal control for Problem* (SOC).

Proof. It is easy to check that assumptions (A1.1)–(A1.2) imply assumptions (H3.1)–(H3.4). Then by Theorem 3.6, the forward-backward SDE (4.4) is uniquely solvable. Moreover, thanks to Lemma 4.1, we can finish the proof. □

Next we extend the LQ SOC problem to an LQ nonzero-sum stochastic differential game (NZSSDG) problem. Without loss of generality, we only consider the case of two players in this paper. The case of n ($\geq$3) players can be treated in the same way. In detail, the game system is described by the following controlled linear backward SDE on $[0,\infty)$:

$$
\begin{aligned}
-\mathrm{d}y(t) = \Big[&A(t)y(t) + \sum_{i=1}^{d} B_i(t)z_i(t) + \sum_{i=1}^{l}\int_{\mathcal{E}} C_i(t,e)r_i(t,e)\pi_i(\mathrm{d}e) + D_1(t)v_1(t) \\
&+ D_2(t)v_2(t) + \alpha(t)\Big]\mathrm{d}t - \sum_{i=1}^{d} z_i(t)\mathrm{d}W_i(t) - \sum_{i=1}^{l}\int_{\mathcal{E}} r_i(t,e)\tilde{N}_i(\mathrm{d}t,\mathrm{d}e),
\end{aligned}
\tag{4.7}
$$

where A, B_i $(i=1,2,\dots,d)$, D_1, D_2 are $\mathbb{F}$-progressively measurable, matrix-valued, bounded processes with appropriate dimensions; C_i $(i=1,2,\dots,l)$ is a $\mathcal{P}\otimes\mathcal{B}(\mathcal{E})$-measurable, $(n\times n)$ matrix-valued process such that $\int_{\mathcal{E}}|C_i(t,e)|^2\pi_i(\mathrm{d}e)$ is uniformly bounded for any $(\omega,t)\in\Omega\times[0,\infty)$; and the nonhomogeneous term $\alpha\in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)$ where $K>0$ is a constant. v_1 and v_2 are the control processes of Player 1 and Player 2, respectively. We introduce the admissible control set for the two players:

$$
\begin{aligned}
\mathcal{V} := \Big\{&(v_1,v_2)\in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{k_1})\times L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{k_2}) \;\Big|\; \text{with respect to } (v_1,v_2), \text{ (4.7) admits} \\
&\text{a unique solution } (y^{v_1,v_2},z^{v_1,v_2},r^{v_1,v_2})\in L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^n)\times L^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times d}) \\
&\times M^{2,K}_{\mathbb{F}}(0,\infty;\mathbb{R}^{n\times l})\Big\},
\end{aligned}
$$

in which each element (v_1,v_2) is called an admissible control pair, and $(y^{v_1,v_2},z^{v_1,v_2},r^{v_1,v_2})$ is called the state trajectory corresponding to (v_1,v_2). As same as Problem (SOC), we will show the admissible control set $\mathcal{V}$ is nonempty under suitable conditions. For any $(v_1,v_2)\in\mathcal{V}$, let

$$
\begin{aligned}
\mathcal{V}_1(v_2) &= \{\bar{v}_1 \mid (\bar{v}_1,v_2)\in\mathcal{V}\},\\
\mathcal{V}_2(v_1) &= \{\bar{v}_2 \mid (v_1,\bar{v}_2)\in\mathcal{V}\}.
\end{aligned}
$$

Additionally, the cost functionals of the two players are given as follows: for $i=1,2$,

$$
\begin{aligned}
J_i(v_1(\cdot),v_2(\cdot)) = \frac{1}{2}\langle Q_i y(0),\ y(0)\rangle + \frac{1}{2}\mathbb{E}\int_0^\infty \Big[&\langle L_i(t)y(t),\ y(t)\rangle + \sum_{j=1}^{d}\langle M_{ij}(t)z_j(t),\ z_j(t)\rangle \\
&+ \sum_{j=1}^{l}\int_{\mathcal{E}}\langle S_{ij}(t,e)r_j(t,e),\ r_j(t,e)\rangle\pi_j(\mathrm{d}e) + \langle R_i(t)v_i(t),\ v_i(t)\rangle\Big]\mathrm{d}t,
\end{aligned}
\tag{4.8}
$$

where Q_i is an $(n\times n)$ symmetric and positive semi-definite matrix; L_i, M_{ij} $(j=1,2,\dots,d)$ are $\mathbb{F}$-progressively measurable, $(n\times n)$ symmetric and positive semi-definite matrix-valued, bounded processes; S_{ij} $(j=1,2,\dots,l)$ is a $\mathcal{P}\otimes\mathcal{B}(\mathcal{E})$-measurable, $(n\times n)$ symmetric and positive semi-definite matrix-valued, bounded process; R_i is an $\mathbb{F}$-progressively measurable, $(k\times k)$ symmetric and positive definite matrix-valued, bounded process. Moreover, R_i^{-1} is also bounded.

Suppose each player hopes to minimize his/her cost functional $J_i(v_1(\cdot),v_2(\cdot))$ by selecting an appropriate admissilbe control v_i $(i=1,2)$, then the game problem is formulated as follows.

Problem (NZSSDG). The problem is to find a pair of admissible controls $(u_1,u_2)\in\mathcal{V}$ such that

$$
\begin{aligned}
J_1(u_1(\cdot),u_2(\cdot)) &= \inf_{v_1(\cdot)\in\mathcal{V}_1(u_2)} J_1(v_1(\cdot),u_2(\cdot)),\\
J_2(u_1(\cdot),u_2(\cdot)) &= \inf_{v_2(\cdot)\in\mathcal{V}_2(u_1)} J_2(u_1(\cdot),v_2(\cdot)).
\end{aligned}
\tag{4.9}
$$

Such a pair of admissible controls (u_1,u_2) is called a Nash equilibrium point. For the sake of notations, we denote the state trajectory corresponding to (u_1,u_2) by $(y,z,r):=(y^{u_1,u_2},z^{u_1,u_2},r^{u_1,u_2})$.

Lemma 4.3. *If the following forward-backward SDE:*

$$
\begin{cases}
\mathrm{d}x_1(t) = \Big[A^\top(t)x_1(t) - L_1(t)y(t)\Big]\mathrm{d}t + \displaystyle\sum_{j=1}^{d}\Big[B_j^\top(t)x_1(t) - M_{1j}(t)z_j(t)\Big]\mathrm{d}W_j(t) \\
\qquad\qquad + \displaystyle\sum_{j=1}^{l}\int_{\mathcal{E}}\Big[C_j^\top(t,e)x_1(t-) - S_{1j}(t,e)r_j(t,e)\Big]\tilde{N}_j(\mathrm{d}t,\mathrm{d}e), \\
\mathrm{d}x_2(t) = \Big[A^\top(t)x_2(t) - L_2(t)y(t)\Big]\mathrm{d}t + \displaystyle\sum_{j=1}^{d}\Big[B_j^\top(t)x_2(t) - M_{2j}(t)z_j(t)\Big]\mathrm{d}W_j(t) \\
\qquad\qquad + \displaystyle\sum_{j=1}^{l}\int_{\mathcal{E}}\Big[C_j^\top(t,e)x_2(t-) - S_{2j}(t,e)r_j(t,e)\Big]\tilde{N}_j(\mathrm{d}t,\mathrm{d}e), \\
-\mathrm{d}y(t) = \Big[D_1(t)R_1^{-1}(t)D_1^\top(t)x_1(t) + D_2(t)R_2^{-1}(t)D_2^\top(t)x_2(t) + A(t)y(t) \\
\qquad\qquad + \displaystyle\sum_{j=1}^{d}B_j(t)z_j(t) + \sum_{j=1}^{l}\int_{\mathcal{E}}C_j(t,e)r_j(t,e)\pi_j(\mathrm{d}e) + \alpha(t)\Big]\mathrm{d}t \\
\qquad\qquad - \displaystyle\sum_{j=1}^{d}z_j(t)\mathrm{d}W_j(t) - \sum_{j=1}^{l}\int_{\mathcal{E}}r_j(t,e)\tilde{N}_j(\mathrm{d}t,\mathrm{d}e), \\
x_1(0) = -Q_1y(0), \quad x_2(0) = -Q_2y(0)
\end{cases}
\tag{4.10}
$$

admits a solution $(x_1, x_2, y, z, r) \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n) \times \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$*, then*

$$
\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} R_1^{-1}(t)D_1^\top(t)x_1(t) \\ R_2^{-1}(t)D_2^\top(t)x_2(t) \end{pmatrix}, \quad t \in [0,\infty),
\tag{4.11}
$$

provides a Nash equilibrium point for Problem (NZSSDG).

Proof. We shall link Problem (NZSSDG) with two LQ SOC problems. Precisely, for $i = 1, 2$, we fix $u_{3-i}(\cdot)$ which is defined in (4.11). To minimize (the argument (t,e) is suppressed)

$$
\begin{aligned}
J_i(v_i(\cdot), u_{3-i}(\cdot)) = \frac{1}{2}\langle Q_i y^{v_i}(0),\ y^{v_i}(0)\rangle + \frac{1}{2}\mathbb{E}\int_0^\infty \Big[\langle L_i y^{v_i},\ y^{v_i}\rangle + \sum_{j=1}^{d}\langle M_{ij}z_j^{v_i},\ z_j^{v_i}\rangle \\
+ \sum_{j=1}^{l}\int_{\mathcal{E}}\langle S_{ij}r_j^{v_i},\ r_j^{v_i}\rangle\pi_j(\mathrm{d}e) + \langle R_i v_i,\ v_i\rangle\Big]\mathrm{d}t
\end{aligned}
\tag{4.12}
$$

subject to

$$
\begin{aligned}
-\mathrm{d}y^{v_i} = \Big[Ay^{v_i} + \sum_{j=1}^{d}B_j z_j^{v_i} + \sum_{j=1}^{l}\int_{\mathcal{E}}C_j r_j^{v_i}\pi_j(\mathrm{d}e) + D_i v_i + \Big(D_{3-i}u_{3-i} + \alpha\Big)\Big]\mathrm{d}t \\
- \sum_{j=1}^{d}z_j^{v_i}\mathrm{d}W_j - \sum_{j=1}^{l}\int_{\mathcal{E}}r_j^{v_i}\tilde{N}_j(\mathrm{d}t,\mathrm{d}e)
\end{aligned}
\tag{4.13}
$$

over $\mathcal{V}_i(u_{3-i})$ is an LQ SOC problem. Since (4.10) admits a solution (x_1, x_2, y, z, r), then (x_i, y, z, r) solves the following forward-backward SDE:

$$\begin{cases} \mathrm{d}x_i = \Big[A^\top x_i - L_i y\Big]\mathrm{d}t + \sum_{j=1}^{d}\Big[B_j^\top x_i - M_{ij}z_j\Big]\mathrm{d}W_j \\ \qquad + \sum_{j=1}^{l}\int_{\mathcal{E}}\Big[C_j^\top x_i - S_{ij}r_j\Big]\tilde{N}_j(\mathrm{d}t, \mathrm{d}e), \\ -\mathrm{d}y = \Big[D_i R_i^{-1} D_i^\top x_i + Ay + \sum_{j=1}^{d} B_j z_j + \sum_{j=1}^{l}\int_{\mathcal{E}} c_j r_j \pi_j(\mathrm{d}e) + \Big(D_{3-i}u_{3-i} + \alpha\Big)\Big]\mathrm{d}t \\ \qquad - \sum_{j=1}^{d} z_j \mathrm{d}W_j - \sum_{j=1}^{l}\int_{\mathcal{E}} r_j \tilde{N}_j(\mathrm{d}t, \mathrm{d}e), \\ x_i(0) = -Q_i y(0). \end{cases} \tag{4.14}$$

By Lemma 4.1, the LQ SOC problem (4.12)–(4.13) admits a unique optimal control with the form

$$u_i(t) = R_i^{-1}(t)D_i^\top(t)x_i(t), \quad t \in [0, \infty),$$

which is coincided with (4.11). In other words, the following equation holds:

$$J_i(u_i(\cdot), u_{3-i}(\cdot)) = \inf_{v_i(\cdot)\in\mathcal{V}_i(u_{3-i})} J_i(v_i(\cdot), u_{3-i}(\cdot)).$$

Since $i = 1, 2$, from the definition of the Nash equilibrium point (see (4.9)), (u_1, u_2) defined by (4.11) provides a Nash equilibrium point for Problem (NZSSDG). $\square$

The forward-backward SDE (4.10) is more complicated. In order to obtain the solvability of (4.10), we would like to employ a linear transform which is originally introduced by Hamadène [4] (see also Yu [21]). For this transform, we need to introduce the following assumptions:

(A2.1) The matrix-valued processes $D_i R_i^{-1} D_i^\top$, $i = 1, 2$, are independent of t.
(A2.2) The following commutation relations among matrices hold true:

$$D_i(t)R_i^{-1}(t)D_i^\top(t)H(t) = H(t)D_i(t)R_i^{-1}(t)D_i^\top(t), \quad t \in [0, \infty), \quad i = 1, 2,$$

where $H(t) = A^\top(t)$, $B_j^\top(t)$ $(j = 1, 2, \dots, d)$, $C_j^\top(t)$ $(j = 1, 2, \dots, l)$.
(A2.3) There exists a constant $\delta > 0$ such that, for any $(\omega, t) \in \Omega \times [0, \infty)$,

$$2A(t) + \sum_{j=1}^{d} B_j(t)B_j^\top(t) + \sum_{j=1}^{l}\int_{\mathcal{E}} C_j(t, e)C_j^\top(t, e)\pi_j(\mathrm{d}e) + KI \le -\delta I.$$

We notice that, assumption (A2.3) is not necessary when the corresponding finite horizon game problems were studied (see for example [4, 21]). Here we assume it due to the infinite time horizon.

Now we introduce another forward-backward SDE (the argument (t,e) is suppressed):

$$\left\{\begin{aligned} d\bar{x} =& \Big[A^\top \bar{x} - \big(D_1R_1^{-1}D_1^\top L_1 + D_2R_2^{-1}D_2^\top L_2\big)\,\bar{y}\Big]\mathrm{d}t \\ &+ \sum_{j=1}^{d}\Big[B_j^\top \bar{x} - \big(D_1R_1^{-1}D_1^\top M_{1j} + D_2R_2^{-1}D_2^\top M_{2j}\big)\,\bar{z}_j\Big]\mathrm{d}W_j \\ &+ \sum_{j=1}^{l}\int_{\mathcal{E}}\Big[C_j^\top \bar{x} - \big(D_1R_1^{-1}D_1^\top S_{1j} + D_2R_2^{-1}D_2^\top S_{2j}\big)\,\bar{r}_j\Big]\tilde{N}_j(\mathrm{d}t,\mathrm{d}e), \\ -d\bar{y} =& \Big[\bar{x} + A\bar{y} + \sum_{j=1}^{d}B_j\bar{z}_j + \sum_{j=1}^{l}\int_{\mathcal{E}}C_j\bar{r}_j\pi_j(\mathrm{d}e) + \alpha\Big]\mathrm{d}t \\ &- \sum_{j=1}^{d}\bar{z}_j\mathrm{d}W_j - \sum_{j=1}^{l}\bar{r}_j\tilde{N}_j(\mathrm{d}t,\mathrm{d}e), \\ \bar{x}(0) =& \,-\big(D_1(0)R_1^{-1}(0)D_1^\top(0)Q_1 + D_2(0)R_2^{-1}(0)D_2^\top(0)Q_2\big)\,\bar{y}(0), \end{aligned}\right. \tag{4.15}$$

and give the following result.

Lemma 4.4. *Under assumptions* (A2.1)–(A2.3)*, the existence and uniqueness of* (4.10) *are equivalent to that of* (4.15).

Proof. On the one hand, by assumptions (A2.1) and (A2.2), if $(x_1,x_2,y,z,r)\in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)\times\mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$ is a solution of (4.10), then

$$\left\{\begin{aligned} &\bar{x}(t) = D_1(t)R_1^{-1}(t)D_1^\top(t)x_1(t) + D_2(t)R_2^{-1}(t)D_2^\top(t)x_2(t), \\ &\bar{y}(t) = y(t), \quad \bar{z}_j(t) = z_j(t)\ (j=1,2,\dots,d), \quad \bar{r}_j(t,e) = r_j(t,e)\ (j=1,2,\dots,l), \end{aligned}\right. \qquad t\in[0,\infty)$$

belonging to $\mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$ solves (4.15).

On the other hand, if $(\bar{x},\bar{y},\bar{z},\bar{r})\in\mathcal{L}_{\mathcal{F}}^{2,K}(0,\infty)$ is a solution of (4.15), we let $y=\bar{y}$, $z=\bar{z}$, $r=\bar{r}$ and (x_1,x_2) be the unique solution of the following SDE:

$$\left\{\begin{aligned} \mathrm{d}x_1 =& \Big[A^\top x_1 - L_1y\Big]\mathrm{d}t + \sum_{j=1}^{d}\Big[B_j^\top x_1 - M_{1j}z_j\Big]\mathrm{d}W_j + \sum_{j=1}^{l}\int_{\mathcal{E}}\Big[C_j^\top x_1 - S_{1j}r_j\Big]\tilde{N}_j(\mathrm{d}t,\mathrm{d}e), \\ \mathrm{d}x_2 =& \Big[A^\top x_2 - L_2y\Big]\mathrm{d}t + \sum_{j=1}^{d}\Big[B_j^\top x_2 - M_{2j}z_j\Big]\mathrm{d}W_j + \sum_{j=1}^{l}\int_{\mathcal{E}}\Big[C_j^\top x_2 - S_{2j}r_j\Big]\tilde{N}_j(\mathrm{d}t,\mathrm{d}e), \\ x_1(0) =& \,-Q_1y(0), \quad x_1(0) = -Q_2y(0). \end{aligned}\right.$$

Obviously $(x_1,x_2)\in S_{\mathbb{F}}^{2,loc}(0,\infty;\mathbb{R}^n)\times S_{\mathbb{F}}^{2,loc}(0,\infty;\mathbb{R}^n)$. Moreover, assumption (A2.3) ensures it also belongs to $L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)\times L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$. In fact, for $i=1,2$, we apply Itô's formula to $|x_i(t)|^2\mathrm{e}^{Kt}$ on the interval $[0,T]$

to have

$$
\begin{aligned}
&\mathbb{E}\Big[|x_i(T)|^2 \mathrm{e}^{KT}\Big] - |x_i(0)|^2 \\
&= \mathbb{E}\int_0^T \Big[\Big\langle\Big(2A + \sum_{j=1}^d B_j B_j^\top + \sum_{j=1}^l \int_{\mathcal{E}} C_j C_j^\top \pi_j(\mathrm{de}) + KI\Big)x_i,\ x_i\Big\rangle \\
&\quad + 2\Big\langle\Big(- L_i y + \sum_{j=1}^d B_j M_{ij} z_j + \sum_{j=1}^l \int_{\mathcal{E}} C_j S_{ij} r_j \pi_j(\mathrm{de})\Big),\ x_i\Big\rangle \\
&\quad + \sum_{j=1}^d |M_{ij} z_j|^2 + \sum_{j=1}^l \int_{\mathcal{E}} |S_{ij} r_j|^2 \pi_j(\mathrm{de})\Big]\mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}
$$

By assumption (A2.3) and the inequality: $2\langle y,\ x\rangle \le (2/\delta)|y|^2 + (\delta/2)|x|^2$,

$$
\begin{aligned}
\frac{\delta}{2}\mathbb{E}\int_0^T |x_i|^2 \mathrm{e}^{Kt}\mathrm{d}t \le |x_i(0)|^2 + \mathbb{E}\int_0^T \Big[\frac{2}{\delta}\Big| - L_i y + \sum_{j=1}^d B_j M_{ij} z_j + \sum_{j=1}^l \int_{\mathcal{E}} C_j S_{ij} r_j \pi_j(\mathrm{de})\Big|^2 \\
+ \sum_{j=1}^d |M_{ij} z_j|^2 + \sum_{j=1}^l \int_{\mathcal{E}} |S_{ij} r_j|^2 \pi_j(\mathrm{de})\Big]\mathrm{e}^{Kt}\mathrm{d}t.
\end{aligned}
$$

Letting $T \to \infty$, we get $x_i \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n)$. In what follows, we shall show that $(x_1, x_2, y, z, r) \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n) \times \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$ defined above is a solution of the forward-backward SDE (4.10). Actually, the remaining thing is to show (x_1, x_2, y, z, r) satisfies the backward equation in (4.10). Compared with the backward equation in (4.15), we only need to show $D_1 R_1^{-1} D_1^\top x_1 + D_2 R_2^{-1} D_2^\top x_2 = \bar{x}$. For the convenience, we let

$$
\tilde{x}(t) = D_1(t) R_1^{-1}(t) D_1^\top(t) x_1(t) + D_2(t) R_2^{-1}(t) D_2^\top(t) x_2(t).
$$

By assumptions (A2.1) and (A2.2), we know $\tilde{x}$ satisfies

$$
\begin{cases}
d\tilde{x} = \Big[A^\top \tilde{x} - \big(D_1 R_1^{-1} D_1^\top L_1 + D_2 R_2^{-1} D_2^\top L_2\big)\, y\Big]\mathrm{d}t \\
\qquad + \displaystyle\sum_{j=1}^d \Big[B_j^\top \tilde{x} - \big(D_1 R_1^{-1} D_1^\top M_{1j} + D_2 R_2^{-1} D_2^\top M_{2j}\big)\, z_j\Big]\mathrm{d}W_j \\
\qquad + \displaystyle\sum_{j=1}^l \int_{\mathcal{E}} \Big[C_j^\top \tilde{x} - \big(D_1 R_1^{-1} D_1^\top S_{1j} + D_2 R_2^{-1} D_2^\top S_{2j}\big)\, r_j\Big]\tilde{N}_j(\mathrm{d}t, \mathrm{de}), \\
\tilde{x}(0) = -\big(D_1(0) R_1^{-1}(0) D_1^\top(0) Q_1 + D_2(0) R_2^{-1}(0) D_2^\top(0) Q_2\big)\, y(0),
\end{cases}
$$

which coincides with the forward equation in (4.15). Regarding (y, z, r) as fixed processes, from the uniqueness of SDE, we have $\bar{x} = \tilde{x}$ and we proved that (x_1, x_2, y, z, r) solves (4.10).

In a similar way, one can prove that the uniqueness of (4.10) is equivalent to that of (4.15). □

For the solvability of (4.15), and then (4.10), we impose the following assumptions:

(A3.1) The matrix $D_1(0) R_1^{-1}(0) D_1^\top(0) Q_1 + D_2(0) R_2^{-1}(0) D_2^\top(0) Q_2$ is positive semi-definite.

(A3.2) There exists a constant $\mu > 0$ such that

$$D_1(t)R_1^{-1}(t)D_1^\top(t)H_1(t) + D_2(t)R_2^{-1}(t)D_2^\top(t)H_2(t) \geq \mu I, \quad t \in [0,\infty),$$

where $H_i(t) = L_i,\ M_{ij}\ (j = 1,2,\ldots,d),\ S_{ij}\ (j = 1,2,\ldots,l),\ i = 1,2$.

(A3.3) $2\min\{\mu, 1\} - K \geq 0$.

Once again, in the viewpoint of Remark 3.1, assumption (A3.3) is artificial. We keep it here for the convenience of presentation.

Remark 4.5.

(i) For the symmetric matrices $D_i(0)R_i^{-1}(0)D_i^\top(0)$ and Q_i $(i = 1,2)$, if they are commutative, then there exists an orthogonal matrix P such that both $P^{-1}D_i(0)R_i^{-1}(0)D_i^\top(0)P$ and $P^{-1}Q_iP$ are diagonal matrices. Moreover, due to the semi-definiteness of $D_i(0)R_i^{-1}(0)D_i^\top(0)$ and Q_i, we have the following statement: if $D_i(0)R_i^{-1}(0)D_i^\top(0)Q_i = Q_iD_i(0)R_i^{-1}(0)D_i^\top(0)$ $(i = 1,2)$, then (A3.1) holds true.

(ii) Similarly, if there exist two constants $\beta_1 \geq 0$ and $\beta_2 \geq 0$ satisfying $\beta_1 + \beta_2 > 0$ such that

$$\begin{aligned}&H_i(t) \geq \beta_i I, \quad D_i(t)R_i^{-1}(t)D_i^\top(t) \geq \beta_i I,\\ &D_i(t)R_i^{-1}(t)D_i^\top(t)H_i(t) = H_i(t)D_i(t)R_i^{-1}(t)D_i^\top(t),\end{aligned}$$

where $H_i(t) = L_i,\ M_{ij}\ (j = 1,2,\ldots,d),\ S_{ij}\ (j = 1,2,\ldots,l),\ i = 1,2$, then (A3.2) holds true.

Theorem 4.6. *Let assumptions* (A2.1)–(A2.3) *and* (A3.1)–(A3.3) *hold.*

(i) *The forward-backward SDE* (4.10) *admits a unique solution* $(x_1,x_2,y,z,r) \in L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n) \times \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$. *Moreover,* (u_1,u_2) *defined by* (4.11) *is a Nash equilibrium point for Problem* (NZSSDG).

(ii) *If we further assume that, for any* $(\omega,t,e) \in \Omega \times [0,\infty) \times \mathcal{E}$, *any* $i = 1,2,\ j = 1,2,\ldots,d,\ k = 1,2,\ldots,l$

$$D_i(t)R_i^{-1}(t)D_i^\top(t) \geq \mu I, \quad L_i(t) \geq \mu I, \quad M_{ij}(t) \geq \mu I, \quad S_{ik}(t,e) \geq \mu I, \tag{4.16}$$

then (u_1,u_2) *defined by* (4.11) *is the unique Nash equilibrium point for Problem* (NZSSDG).

Proof.

(i) Under assumptions (A3.1)–(A3.3), by Theorem 3.6, the forward-backward SDE (4.15) admits a unique solution in $\mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$. With the help of assumptions (A2.1)–(A2.3) and Lemma 4.4, the forward-backward SDE (4.10) admits also a unique solution in the space $L_{\mathbb{F}}^{2,K}(0,\infty;\mathbb{R}^n) \times \mathcal{L}_{\mathbb{F}}^{2,K}(0,\infty)$. Moreover, by Lemma 4.3, (u_1,u_2) defined by (4.11) provides a Nash equilibrium point for Problem (NZSSDG).

(ii) Let $(\bar{u}_1,\bar{u}_2)$ be another Nash equilibrium point for Problem (NZSSDG). From the viewpoint of Lemma 4.3, for any $i = 1,2$, fix $\bar{u}_{3-i}$, then $\bar{u}_i$ is an optimal control of the LQ SOC problem (4.12)–(4.13). Thanks to (4.16) and Theorem 4.2, $\bar{u}_i$ must have the form:

$$\bar{u}_i(t) = R_i^{-1}(t)D_i^\top(t)x_i(t), \quad t \in [0,\infty),$$

where (x_i,y,z) satisfies (4.14). Combining the two cases: $i = 1$ and $i = 2$, we get the conclusion:

$$\begin{pmatrix}\bar{u}_1(t)\\ \bar{u}_2(t)\end{pmatrix} = \begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix} = \begin{pmatrix}R_1^{-1}(t)D_1^\top(t)x_1(t)\\ R_2^{-1}(t)D_2^\top(t)x_2(t)\end{pmatrix}, \quad t \in [0,\infty),$$

where (x_1,x_2,y,z,r) is the unique solution of (4.10). We complete the proof. ∎

Example 4.7. In this example, we would like to focus ourselves on a special case to illustrate the results obtained in this section. Let the dimension of the state process y, the dimension of Brownian motion and the number of Poisson random measures are 1. Under this '1 dimension' setting, all the involving matrix-valued processes are real-valued indeed. Moreover, let A, B, C, D, Q, L, M, S, R in Problem (SOC) and A, B, C, D_i, Q_i, L_i, M_i, S_i, R_i $(i=1,2)$ in Problem (NZSSDG) are independent of the time variable t (and then they are independent of ω also due to the $\mathbb{F}$-adaptedness of processes). Furthermore, let the nonhomogeneous term α in (4.1) and (4.7) vanish.

(i). Let $D \neq 0$, $Q \geq 0$, $L > 0$, $M > 0$, $R > 0$, and there exists a constant $\kappa > 0$ such that $S(e) \geq \kappa$ for all $e \in \mathcal{E}$. Define

$$\mu = \min\left\{\frac{D^2}{R}, L, M, \kappa\right\}.$$

For any $K \in (0, 2\mu)$, assumptions (A1.1)–(A1.2) are satisfied. By Theorem 4.2, the forward-backward SDE (4.4) admits a unique solution, and Problem (SOC) has a unique optimal control which is given by (4.5).

(ii). For $i = 1, 2$, let $D_i \neq 0$, $Q_i \geq 0$, $L_i > 0$, $M_i > 0$, $R_i > 0$, and there exists a constant $\kappa > 0$ such that $S(e) \geq \kappa$ for all $e \in \mathcal{E}$. Moreover, we assume

$$A < -\frac{1}{2}B^2 - \frac{1}{2}\int_{\mathcal{E}} |C(e)|^2 \pi(\mathrm{d}e). \tag{4.17}$$

Define

$$\begin{aligned}
\mu &= \min\left\{\frac{D_1^2}{R_1}, \frac{D_2^2}{R_2}, L_1, L_2, M_1, M_2, \kappa, \frac{D_1^2}{R_1}L_1 + \frac{D_2^2}{R_2}L_2, \frac{D_1^2}{R_1}M_1 + \frac{D_2^2}{R_2}M_2, \left(\frac{D_1^2}{R_1} + \frac{D_2^2}{R_2}\right)\kappa\right\},\\
\rho &= \min\left\{-\frac{1}{2}B^2 - \frac{1}{2}\int_{\mathcal{E}} |C(e)|^2 \pi(\mathrm{d}e) - A, \mu, 1\right\}.
\end{aligned}$$

For any $K \in (0, 2\rho)$, it is easy to check assumptions (A2.1)–(A2.3), (A3.1)–(A3.3), and (4.16) are satisfied. By Theorem 4.6, the forward-backward SDE (4.10) admits a unique solution, and Problem (NZSSDG) has a unique Nash equilibrium point which is given by (4.11).

5. Conclusion

In this paper, we investigate a kind of forward-backward stochastic differential equations (SDEs) driven by both Brownian motions and Poisson processes on an infinite horizon. In our setting, besides the coupling of mappings b, σ, γ and g, the two initial values are also coupled. We employ a new technique to treat the coupling between the initial values. For this kind of forward-backward SDEs, we establish an existence and uniqueness theorem by virtue of the method of continuation under some monotonicity conditions. Some important properties including stability and comparison of solutions are also addressed. These results generalize that of Peng and Shi [11].

The theoretical results are applied to solve an infinite horizon linear-quadratic (LQ) backward stochastic optimal control problem and an LQ nonzero-sum backward stochastic differential game. Under suitable conditions, we get the solvability of the Hamiltonian systems related to the LQ control problem and LQ game problem, which are linear forward-backward SDEs of the type studied previously. Then the unique optimal control and the unique Nash equilibrium point are obtained in closed forms, respectively.

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