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SMALL-TIME LOCAL ATTAINABILITY FOR A CLASS OF CONTROL SYSTEMS WITH STATE CONSTRAINTS

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Abstract. In this paper we consider the problem of *small time local attainability* (STLA) for nonlinear finite-dimensional time-continuous control systems in presence of state constraints. More precisely, given a nonlinear control system subjected to state constraints and a closed set S, we provide sufficient conditions to steer to S every point of a suitable neighborhood of S along admissible trajectories of the system, respecting the constraints, and giving also an upper estimate of the minimum time needed for each point to reach the target. Methods of nonsmooth analysis are used.

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1. Introduction

Consider a finite-dimensional control system of the form

$$\begin{cases} \dot{y}(t) = f(y(t), u(t)), \\ y(0) = x. \end{cases}$$

where U is a given compact subset of $\mathbb{R}^m$, $x \in \mathbb{R}^d$,

$$u(\cdot) \in \mathscr{U} := \{v : [0, +\infty[\to U \text{ such that } v \text{ is measurable}\},$$

and $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ satisfies some standard smoothness assumptions in order to guarantee the existence and uniqueness of the solution of the above Cauchy problem (see Sect. 2) for every choice of $u \in \mathscr{U}$. Such solutions are called admissible trajectories of the system starting from x.

Given a closed subset S of $\mathbb{R}^d$, called the target set, an important problem studied in control theory is to provide sufficient conditions on f and U ensuring that, for every horizon time $T > 0$, all the points sufficiently near to S can be steered to S by admissible trajectories of the system in time less than T.

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We refer to this property as *small-time local attainability* (STLA), see Definition 2.4. We recall that a *(small-time) fully controllable* system is a system enjoying the following property: given $\bar{x} \in \mathbb{R}^d$ and $T > 0$ there exists $\delta = \delta_{T,\bar{x}} > 0$ such that every couple of points $x, y \in B(\bar{x}, \delta_{T,\bar{x}})$ can be joined by an admissible trajectory in time less than T. Clearly, any (small-time) fully controllable system enjoy also STLA, however the class of STLA systems is much wider.

When the target reduces to a point x_0, STLA is equivalent to the well-known *small time local contollability* from x_0 for the system obtained reversing the dynamics: namely, for all $T > 0$ the reachable set at time less or equal than T from x_0 for the reversed dynamics system contains a neighborhood of x_0. Small time local controllability property from a point has been extensively studied under various degree of generality from the beginning of the history of control theory. In the early 60's, Kalman proved the following result. Assume that f is *linear*, *i.e.*, $f(x, u) = Ax + Bu$ where $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, $B \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ are constant matrices, $S = \{0\}$. Then the following are equivalent:

(1) the system is controllable to the equilibrium point 0, *i.e.*, every point can be steered to the origin in finite time;
(2) the matrix $(B|AB|A^2B|\dots|A^{n-1}B)$ has full rank (equals n).

The second condition above is the celebrated *Kalman rank condition*, and implies the Hölder continuity of T, with exponent depending on the smallest $0 \le k \le n-1$ such that the matrix

$$(B|AB|A^2B|\dots|A^kB)$$

has full rank.

Later, in the 70's, several generalizations, mainly concerned the case when target set S is an equilibrium point for the system, of this condition to nonlinear systems were proved by several authors among which we recall Hermes, Sussmann, Hörmander and many others (we refer to [1, 9] for more details). All these results involve a suitable expansion around the equilibrium point, and STLA is achieved by imposing some conditions on the Lie algebra generated by the vector fields. The *Agrachev–Gamkrelidze formalism* (AGF), also known as *chronological calculus*, introduced by Agrachev and Gamkrelidze in 1978 (see *e.g.* [1]) turned out to be a powerful tool in such kind of analysis, allowing to formulate many problems in a purely algebraic way on abstract Lie algebras (see [1, 12] for further details).

However, as pointed out in [11], where also is provided a comparison between STLA and classical conditions for small time local controllability, the problem of local attainability of a closed set with respect to the trajectories of a control system can not be reduced to the problem of small-time local attainability at every point of its boundary (or to small-time controllability from every point of the boundary of the target for the reversed dynamics). The reason is that the small-time local attainability depends not only on the dynamics of the control system, but also on the geometry of the considered closed set. So, it needs a specific study.

One of the most common conditions ensuring STLA is Petrov's condition, which can be stated as follows in the case of compact target S: there exist $\delta, \mu > 0$, such that for every $x \in \mathbb{R}^d \setminus S$ whose distance $d_S(x)$ from S is less than δ there exist $u \in U$ and a point $\bar{x} \in S$ with $\|x - \bar{x}\| = d_S(x)$ and

$$\langle x - \bar{x}, f(x, u)\rangle \le -\mu d_S(x).$$

When the function $d_S(\cdot)$ is differentiable at $x \notin S$, the above condition can be written as $\langle \nabla d_S(x), f(x, u)\rangle \le -\mu$. Since given $x \in \mathbb{R}^d$ we have that $F(x) := \{f(x, u) : u \in U\}$ ranges among all the possible instantaneous velocities of admissible trajectories passing through x, and $-\nabla d_S(x)$ is the direction along which we have the fastest decrease of the distance from the target, at least in a smooth setting, we can interpret Petrov condition in the following way: for each point sufficiently near to x there exists an admissible trajectory which at the first order points *sufficiently forward* to the target.

Petrov's condition is very strong, and it was proved in [16] that it is equivalent to the local Lipschitz continuity of the minimum time function $T(\cdot)$ in a neighborhood of the target (see Def. 2.2), *i.e.*, in the compact case it is equivalent for any given x near to the target to the existence of an admissible trajectory starting from x and reaching the target in time less than $Cd_S(x)$ for a suitable $C > 0$ depending on μ but not on x. Generalization to noncompact nonsmooth target S of Petrov's condition are also well known.

A natural formulation of a naive extension of Petrov's condition would be of the following type: assume that for any point x near to S there exists an admissible trajectory starting from x along which the distance decreases at a sufficiently high rate, then we can expect STLA. Indeed, in classical smooth Petrov condition we obtain that we have at least one admissible trajectory along which the distance decreases linearly in time, with decreasing rate proportional to $1/\mu$.

A generalization of the above problem could be the following one: instead of searching for admissible trajectories along which the distance is decreasing, given x as above we look for a curve $\Delta t \mapsto y_{\Delta t,x}$ such that $y_{\Delta t,x}$ can be reached from x in time Δt and such that $d_S(y_{\Delta t,x}) - d_S(x) \leq \Delta r$, *without paying any attention* to the behavior of the distance along the admissible trajectory joining x and $y_{\Delta t}$, and considering only the relationship between Δt and Δr. The map $t \mapsto y_{t,x}$ is called an $\mathscr{A}$-trajectory, see Definition 3.1. Clearly, every admissible trajectory is an $\mathscr{A}$-trajectory, but the converse is not true in general.

A first step in this direction was taken by Krastanov-Quincampoix in [11], in which it is assumed that there exists $\mu > 0$ such that for every x near to S and t sufficiently small we can find an $\mathscr{A}$-trajectory $y(t,x)$ such that $y(0,x) = x$ and

$$y(t,x) = x + a(t;x) + t^\alpha A(x) + o(t^\alpha;x),$$

where

(1) $\alpha > 0$, $A(\cdot)$ is a locally Lipschitz continuous function,
(2) the reminder satisfies a uniform estimate $\|o(t^\alpha;x)\| \leq Kt^{\alpha+\beta}$ with K, β suitable positive constants independent of x,
(3) $\|a(\cdot)\|$ is bounded from above by $Mt^s d_S(x)$ where $M, s > 0$ are suitable constants,
(4) there exists a point $\bar{x} \in S$ with $\|x - \bar{x}\| = d_S(x)$ and

$$\langle x - \bar{x}, A(x)\rangle \leq -\mu d_S(x).$$

Roughly speaking, Petrov's condition amounts to require an infinitesimal decreasing condition for the distance involving the first order term of at least one admissible trajectory, *i.e.*, on admissible velocity. Now, we require it for the *essential leading term* of at least an $\mathscr{A}$-trajectory which now is a term of order $\alpha \geq 1$. The name "essential leading term" is motivated by the fact that as long as x is taken near to S, we have that $\|a(\cdot)\|$ vanishes. By the equivalency between Petrov's condition and local Lipschitz continuity of $T(\cdot)$ we can not expect any more an estimate like $T(x) \leq Cd_S(x)$ in the case $\alpha > 1$, however it turns out that a similar estimate holds true, yielding $T(x) \leq Cd_S^{1/\alpha}(x)$.

Later, in [13] was treated the case in which the constant μ appearing in Petrov's condition is a function $\mu = \mu(d_S(x))$ allowed to slowly vanish as $d_S(x) \to 0$. This was not covered by [11], since they assumed μ to be always constant. In [13] Petrov-like sufficient conditions were provided for STLA at first and second order, linking the exponent α of the final estimate $T(x) \leq Cd_S^\alpha(x)$ also with the dependency of $\mu(\cdot)$ on $d_S(\cdot)$. Many of the results of this paper were obtained under additional geometrical assumptions on the target, which were removed in a later paper [10] by Krastanov, where the results of [11,13] are subsumed in a unique formulation, but still under strong smoothness hypothesis on the terms appearing in the expression of $y_{t,x}$ and taking into account a decay of $r \mapsto \mu(r)$ only as suitable powers of r.

The recent paper [15] weakened some smoothness assumptions required in [10, 11] of the terms appearing in the expression of the $\mathscr{A}$-trajectory $t \mapsto y_{t,x}$, but instead of them, the authors assumed more regularity on the target set than in [10]. With even more regularity, in [15] is also defined a *generalized curvature* by means of suitable generalized gradients of higher order of the distance function. This allows us to consider not only first-order expansion of the distance along an $\mathscr{A}$-trajectory, but also second-order, improving further STLA sufficient conditions.

The higher order controllability conditions studied in [13, 15] were recently extended to the discrete case in [7]. In particular, the original system is approximated by a higher order one-step method whose order has to be compatible with the order of controllability conditions. In [7], the discrete minimum time to reach S, which converges to the true one as time discretization step tends to 0, is also bounded by an amount proportional to a fractional power of the distance to S of the initial point. Furthermore, the robustness of controllability conditions with respect to a shrinking of S under natural assumptions can be found in [7].

In general it is difficult to describe the set of $\mathscr{A}$-trajectories on which the conditions for STLA must be checked. But for *control-affine systems*, *i.e.*, special systems where the dynamics is given by $\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{N} u_i(t)f_i(x(t))$, $\|u_i\|_{L^\infty} \leq 1$, see (2.2), additional information can be provided by studying the Lie algebra generated by the vector fields $\{f_0 \pm f_i\}_{i=1,\dots,N}$. However, the presence of a nontrivial drift term f breaks the time-reversal symmetry (which would hold if $f_0 = 0$), which was an essential ingredient for many results obtained by means of Lie algebraic methods in the 60's and in the 70's by Kalman, Hermes, Sussmann, Hörmander and many other authors. We refer to [11] for a brief history of these results.

All the results of [10, 11, 13, 15] are obtained for systems in which the state space is the whole of $\mathbb{R}^d$. The first paper investigating the problem of STLA in presence of restriction on the state space was [4], where the authors extended first-order Petrov's condition to such kind of systems. To this aim, they assume on the boundary of the constraint an *inward pointing condition*. In its simplest form (*i.e.*, the system is autonomous and the constraint is given by $\overline{\Omega}$, where Ω is an open bounded subset of $\mathbb{R}^d$), this condition amounts to ask that at every point of $\partial\Omega$ there are admissible velocities belonging to the interior of the tangent cone to Ω (see Rem. 3.3 in [4]). It turns out that even in its full general version this condition implies that $\partial\Omega$ is locally Lipschitz continuous (see Rem. 3.2 in [4]).

The first aim of this paper is to remove the smoothness assumptions on the terms appearing in the expression of the $\mathscr{A}$-trajectory $t \mapsto y_{t,x}$, as in [15], but without any additional regularity hypothesis on the target set used in [15], thus fully generalizing the results of [10] also in presence of the additional state constraint $y(t) \in \overline{\Omega}$, where Ω is an open subset of $\mathbb{R}^d$ with $\Omega \setminus S \neq \emptyset$. This first result, contained in Theorem 2.8, relies on the assumption of the existence of certain $\mathscr{A}$-trajectories fulfilling suitable properties and respecting the state constraints. The presence of state constraints would affect only the existence of such curves, which is assumed to be granted. So, in particular, state constraints do not play explicitly any role in the proof of this first result.

After proving such general STLA results, we turn our attention to control-affine systems in presence of state constraints, providing an approximate representation formula for the elements of the set of $\mathscr{A}$-trajectory and thus proving a better sufficient condition for STLA in this case. The presence of state constraints is taken into account by means of a condition different from the inward pointing condition of [4], assuming some smoothness of the distance function near to the boundary of the constraint, which in particular allows us to treat systems with a class of constraints whose boundaries are not necessarily Lipschitz continuous. The main ingredient of this part is a suitable approximate representation formula for the expansion of the distance function along the trajectories of control-affine systems with nontrivial drift. The comparison between the inward pointing condition of [4] and ours is postponed to the end of Section 3.

The paper is structured as follows: in Section 2 we state and prove our general results on STLA with state constraints, outlining a comparison between the results of [10, 11, 13, 15] and ours. In Section 3 we turn our attention to control-affine systems, providing some explicit sufficient conditions for STLA. Finally, in Section 4 we give an example illustrating our approach.

2. A general result for small time local attainability

Definition 2.1. Let K be a closed subset of $\mathbb{R}^d$, $S \subset \mathbb{R}^d$, $x = (x_1, \dots, x_d) \in K$, $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, $r > 0$, X be a vector space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a function.

We denote by:

$\langle x,y\rangle$	$:=\sum_{i=1}^{d} x_i y_i$	the *scalar product* in $\mathbb{R}^d$;
$\|x\|$	$:=\sqrt{\langle x,x\rangle}$	the *Euclidean norm* in $\mathbb{R}^d$;
	∂S, $\mathrm{int}(S)$, $\overline{S}$	the *topological boundary*, *interior* and *closure* of S;
$B(y,r)$	$:=\{z\in\mathbb{R}^d:\ \|z-y\|<r\}$	the *open ball* centered at y of radius r;
$d_K(y)$	$:=\mathrm{dist}(y,K)=\min\{\|z-y\|:\ z\in K\}$	the *distance* of y from K;
$\pi_K(y)$	$:=\{z\in K:\ \|z-y\|=d_K(y)\}$	the *set of projections* of y onto K;
S^c	$:=\mathbb{R}^d\setminus S$	the *complement* of S;
S_δ	$=B(S,\delta):=\{y\in\mathbb{R}^d:\ d_S(y)<\delta\}$	the δ*-neighborhood* of S;
	$\mathrm{Id}_{\mathbb{R}^d}$	the identity function in $\mathbb{R}^d$.

If $\pi_K(y)=\{\xi\}$, *i.e.*, it is a singleton, we will identify it with its unique element and write $\pi_K(y)=\xi$.

Given an open subset $\Omega\subseteq\mathbb{R}^d$ and a compact set $U\subseteq\mathbb{R}^m$, we will consider the following *state constrained control system*:

$$\begin{cases}\dot y(t)=f(y(t),u(t)), & \text{for a.e. } t>0,\\ y(0)=x_0\in\overline{\Omega}, & \\ y(t)\in\overline{\Omega}, & t\geq 0.\end{cases}\tag{2.1}$$

where for every compact K there exists $L=L_K>0$ such that $f:\mathbb{R}^d\times\mathbb{R}^m\to\mathbb{R}^d$ satisfies

$$\|f(x,u)-f(y,u)\|\leq L_K\|x-y\|,\ \text{for all } x,y\in K,\ u\in U,$$

$u(\cdot)\in\mathscr{U}:=\{v\colon [0,+\infty[\to U:\ v \text{ is measurable}\}$, and a closed subset $S\subseteq\mathbb{R}^d$ (called the *target set*) with $S\cap\overline{\Omega}\neq\emptyset$ is given.

A special case of the above system is given by the following *state constrained control-affine system*:

$$\begin{cases}\dot y(t)=f_0(y(t))+\displaystyle\sum_{i=1}^{N}u_i(t)f_i(y(t)), & \text{for a.e. } t>0,\\ y(0)=x_0\in\overline{\Omega}, & \\ y(t)\in\overline{\Omega}, & t\geq 0,\end{cases}\tag{2.2}$$

where $f_0,f_i\in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$, $u_i\in\mathscr{U}$, with $i=1,\dots,N$ and

$$\mathscr{U}:=\{v\colon [0,+\infty[\to[-1,1]:\ v \text{ is measurable}\}.$$

Definition 2.2. Given the system (2.1), we define the *state constrained reachable set from* $x_0\in\overline{\Omega}$ *at time* $\tau\geq 0$:

$$\mathscr{R}^{\Omega}_{x_0}(\tau):=\Big\{y(\tau):\ y(\cdot) \text{ is a solution of (2.1) defined on } [0,\tau]\Big\}.$$

The *state constrained minimum time function* from $x_0\in\overline{\Omega}$ is

$$T_\Omega(x_0):=\begin{cases}+\infty, & \text{if } \mathscr{R}^{\Omega}_{x_0}(\tau)\cap S=\emptyset \text{ for all } \tau\geq 0,\\ \inf\{\tau\geq 0:\ \mathscr{R}^{\Omega}_{x_0}(\tau)\cap S\neq\emptyset\}, & \text{otherwise.}\end{cases}$$

If $\Omega=\mathbb{R}^d$, in both cases we will omit the adjective *constrained* and we will write simply $\mathscr{R}_x(t)$ and $T(x)$.

Remark 2.3 (Estimates on trajectories)**.** Consider the system 2.1. Let $\bar{x} \in \mathbb{R}^d$, $\delta_{\bar{x}} > 0$. Choose

$$M_{\bar{x}} \geq \max\{\|f(z,u)\| : z \in \overline{B(\bar{x},\delta_{\bar{x}})},\, u \in U\}.$$

Then for any $0 < \delta' < \delta_{\bar{x}}$ we have $\mathscr{R}_x(t) \subseteq \overline{B(\bar{x},\delta_{\bar{x}})}$ for all $x \in B(\bar{x},\delta')$ and $0 \leq t \leq \dfrac{\delta_{\bar{x}} - \delta'}{M_{\bar{x}}}$. The proof is classical, and is based on Schauder's fixed point theorem and Gronwall's inequality. See *e.g.* Section 5 in Chapter III of [2].

The property in which we are interested is the following (see also [10, 11, 15]).

Definition 2.4 (STLA)**.** We say that S is *small-time local attainable* for the system (2.1) if for any $T > 0$ there exists an open neighborhood $U_T \subseteq \mathbb{R}^d$ of S such that $T_\Omega(x) \leq T$ for all $x \in U_T \cap \overline{\Omega}$.

Remark 2.5. A sufficient condition for STLA is to be able to bound from above the minimum time function $T_\Omega(\cdot)$ in a relative neighborhood of the target by a continuous increasing function of the distance from the target itself, and vanishing exactly on the target.

Lemma 2.6 (Localization)**.** *Consider system (2.1) with closed target $S \subseteq \mathbb{R}^d$. Assume that for every $\bar{x} \in \partial S \cap \overline{\Omega}$ there exists a continuous increasing function $\omega_{\bar{x}} : [0,+\infty[\to [0,+\infty[$ and $0 < \delta_{\bar{x}} < +\infty$ such that*

(1) $\omega_{\bar{x}}(p) = 0$ *if and only if* $p = 0$,
(2) $T_\Omega(x) \leq \omega_{\bar{x}}(d_S(x))$ *for all* $x \in B(\bar{x},\delta_{\bar{x}}) \cap \overline{\Omega}$.

Then STLA holds. If moreover $\partial S \cap \overline{\Omega}$ is compact, then $\omega_{\bar{x}}(\cdot)$ and $\delta_{\bar{x}}$ can be chosen independently on $\bar{x}$.

Proof. Let $T > 0$ be fixed. For any $\bar{x} \in \partial S \cap \overline{\Omega}$ there exists $r_{\bar{x}} > 0$ such that $\omega_{\bar{x}}(s) \leq T$ for all $s \in [0, r_{\bar{x}}]$. Set $U_{\bar{x}} := B(\bar{x},\delta_{\bar{x}}) \cap S_{r_{\bar{x}}}$, and notice that, in particular, if $x \in U_{\bar{x}} \cap \overline{\Omega}$ we have $T_\Omega(x) \leq \omega_{\bar{x}}(d_S(\bar{x})) \leq T$. Moreover, trivially, we have $T_\Omega(x) \leq T$ for all $x \in S$. Defined

$$U_T := \left(\bigcup_{\bar{x} \in \partial S \cap \overline{\Omega}} U_{\bar{x}} \cup S \right),$$

we have that $U_T \cap \overline{\Omega}$ is an open neighborhood of S in the topology of $\overline{\Omega}$ and, by construction, we have $T_\Omega(x) \leq T$ for all $x \in U_T \cap \overline{\Omega}$. So STLA holds.

In the compact case, we can cover $\partial S \cap \overline{\Omega}$ by finitely many balls $\{B(\bar{x}_i, \delta_{\bar{x}_i}) : i = 1\ldots, N\}$, where $\bar{x}_i \in \partial S \cap \overline{\Omega}$ and $\delta_{\bar{x}_i} > 0$, thus we can set $\omega(p) = \max\limits_{i=1,\ldots,N} \omega_{\bar{x}_i}(p)$ and $\delta = \min\limits_{i=1,\ldots,N} \delta_{\bar{x}_i}$. □

The following simple result will be extensively used in our analysis.

Lemma 2.7. *Let $\delta > 0$ be a constant, $\lambda : [0,+\infty[\times[0,+\infty[\to \mathbb{R}$, $\theta : [0,+\infty[\to [0,+\infty[$ be continuous functions such that*

(1) $r \mapsto \dfrac{\theta(r)}{\lambda(\theta(r), r)}$ *is bounded from above by a nonincreasing nonnegative function* $\beta(\cdot) \in L^1(]0,\delta[)$;
(2) $\lambda(\theta(r), r) > 0$ *for* $0 < r < \delta$, *and* $\lambda(0, r) = 0$ *for* $r > 0$.

Consider any sequence $\{r_i\}_{i\in\mathbb{N}}$ in $[0,\delta]$ satisfying for all $i \in \mathbb{N}$:

(S_1) $r_{i+1} - r_i \leq -\lambda(\theta(r_i), r_i)$.
(S_2) $\theta(r_i) \neq 0$ *implies* $r_i \neq 0$.

Then we have

(a) $r_i \to 0$,

(b) $\displaystyle\sum_{i=0}^{\infty}\theta(r_i) \le \int_0^{r_0} \beta(r)\,\mathrm{d}r$.

Proof. According to (S_1), the sequence $\{r_i\}_{i\in\mathbb{N}}$ is monotonically decreasing and bounded from below, thus it admits a limit. Let $r_\infty = \lim_{i\to+\infty} r_i$, it is evident that $0 \le r_\infty < \delta$. Now we would like to show that $r_\infty = 0$. Assume, by contradiction, $r_\infty > 0$. By passing to the limit for $i \to +\infty$ in (S_1) and recalling that $\lambda(\cdot,\cdot)$ is a continuous function and $\lambda(\theta(r_i), r_i) \ge 0$, we obtain that $0 = \lambda(\theta(r_\infty), r_\infty)$ and this contradicts the fact that $\lambda(\theta(r), r) > 0$ for any $0 < r < \delta$, thus $r_\infty = 0$. Since if $\theta(r_i) \ne 0$ we have $r_i \ne 0$ and $\dfrac{r_i - r_{i+1}}{\lambda(\theta(r_i), r_i)} \ge 1$, we obtain

$$\sum_{i=0}^{\infty}\theta(r_i) = \sum_{\substack{i=0\\ \theta(r_i)\ne 0}}^{\infty} \theta(r_i) \le \sum_{\substack{i=0\\ \theta(r_i)\ne 0}}^{\infty} \frac{\theta(r_i)}{\lambda(\theta(r_i), r_i)}(r_i - r_{i+1})$$

$$\le \sum_{\substack{i=0\\ \theta(r_i)\ne 0}}^{\infty} \beta(r_i)(r_i - r_{i+1}) \le \int_0^{r_0} \beta(r)\mathrm{d}r,$$

recalling the monotonicity property of $r \mapsto \beta(r)$. □

We will apply this Lemma to the following situation: take a sequence of points $\{x_i\}_{i\in\mathbb{N}}$, define $r_i = d_S^2(x_i)$ for all $i \in \mathbb{N}$, and assume that the time needed to reach x_{i+1} from x_i is $\theta(r_i)$. Then we can bound from above $\sum_{i=0}^{\infty}\theta(r_i)$, and thus the time needed to steer x_0 to S, provided that we are able to construct $\lambda(\cdot)$ fulfilling the assumptions of Lemma 2.7. If the bound is locally uniform in a neighborhood of S, STLA follows from Remark 2.5.

The map $\lambda(t, r)$ will measure the *decreasing of the squared distance from the target* starting from a point that is at distance r from the target after having followed a particular admissible trajectory for time t. Roughly speaking, if the decreasing of the distance is *too slow*, we will not be able to reach the target in finite time, so we need a quantitive bound of the ratio between the time passed and the amount of the decreasing of the distance.

It is clear that, in the above discussion, we can replace the squared distance $d_S^2(\cdot)$ with a general locally Lipschitz continuous function $\Phi_S : \mathbb{R}^d \to \mathbb{R}$, satisfying some extra regularity assumptions, and such that $S = \{x \in \mathbb{R}^d : \Phi_S(x) \le 0\}$. More precisely, we will assume that $\Phi_S(\cdot)$ is a *semiconcave function* of constant K: *i.e.* for every $p \in \partial^P v(x)$ we have

$$\Phi_S(y) - \Phi_S(x) \le \langle p, y - x\rangle + K\|y - x\|^2,$$

for any x, y and $p \in \partial^P \Phi_S(x)$ (the superdifferential of Φ_S at x). For further details on semiconcave functions and the explanation of the notation, we refer to the monograph [3]. A function f is called semiconvex if $-f$ is semiconcave. We recall here that in general $d_S^2(\cdot)$ is semiconcave on the whole of $\mathbb{R}^d$ with constant 2, while to have semiconcavity of $d_S(\cdot)$ further regularity assumptions are needed (see Prop. 2.2.1 p. 36 in [3]).

We state and prove now a first general result on STLA in the spirit of Remark 2.5.

Theorem 2.8 (General attainability)**.** *Consider the system* (2.1)*. Let $\bar{x} \in \partial S \cap \overline{\Omega}$, $\delta_{\bar{x}} > 0$ and assume that*

$$S \cap \overline{B(\bar{x}, \delta_{\bar{x}})} := \{x \in \overline{B(\bar{x}, \delta_{\bar{x}})} : \Phi_S(x) \le 0\},$$

where $\Phi_S : \mathbb{R}^d \to \mathbb{R}$ is a locally Lipschitz function. Set $\Phi_{\bar{x}} = \max_{x\in\overline{B(\bar{x},\delta_{\bar{x}})}} \{\Phi_S(x)\}$, denote by $L(r) > 0$ a Lipschitz constant of $\Phi_S(\cdot)$ on $\overline{B(\bar{x}, \delta_{\bar{x}})} \cap \{x : \Phi_S(x) \le r\}$, and let $M_{\bar{x}} = \max_{\substack{v\in U\\ z\in\overline{B(\bar{x},\delta_{\bar{x}})}}} \{\|f(z, v)\|\}$. Let

$\sigma, \delta, \mu, \chi : [0, +\infty[\times[0, +\infty[\to [0, +\infty[$, *and* $\tau, \theta : [0, +\infty[\to [0, +\infty[$ *be continuous function. We assume that:*

(1) $\tau(r) = 0$ *iff* $r = 0$, $0 < \theta(r) \le \tau(r)$ *for every* $0 < r < \Phi_{\bar{x}}$*;*
(2) *for any* $x \in (\overline{B(\bar{x}, \delta_{\bar{x}})} \cap \overline{\Omega}) \setminus S$ *and* $0 < t \le \tau(\Phi_S(x))$ *the following holds*
  (2a) $[\mathscr{R}_x^{\Omega}(t)]_{\delta(t,\Phi_S(x))} \cap S_{2\delta_{\bar{x}}} \neq \{x\}$,
  (2b) *if* $\mathscr{R}_x^{\Omega}(t) \cap S = \emptyset$, *there exists* $y_{t,x} \in [\mathscr{R}_x^{\Omega}(t)]_{\delta(t,\Phi_S(x))} \cap \overline{B(x, \chi(t,r))}$ *with*

$$\min_{\zeta \in \partial^P \Phi_S(x)} \langle \zeta, y_{t,x} - x\rangle + K\|y_{t,x} - x\|^2 \le -\mu(t, \Phi_S(x)) + \sigma(t, \Phi_S(x));$$

  (2c) $\Phi_S(\cdot)$ *is semiconcave on* $\overline{B(\bar{x}, \delta_{\bar{x}})}$ *with semiconcavity constant* $K = K_{\bar{x}} > 0$.
(3) *the continuous function* $\lambda : [0, +\infty[\times[0, +\infty[\to \mathbb{R}$, *defined as*

$$\lambda(t, r) := \mu(t, r) - \sigma(t, r) - (L(r) + K\delta(t, r) + 2K\chi(t, r))\,\delta(t, r),$$

satisfies the following properties:
  (3a) $0 < \lambda(\theta(r), r) < r$, $\lambda(0, r) = 0$ *for all* $0 < r < \Phi_{\bar{x}}$*;*
  (3b) $r \mapsto \dfrac{\theta(r)}{\lambda(\theta(r), r)}$ *is bounded from above by a nonincreasing nonnegative function* $\beta(\cdot) \in L^1(]0, \Phi_{\bar{x}}[)$.

Then, if we set

$$\omega(r_0) := \int_0^{r_0} \beta(r)\, \mathrm{d}r,$$

we have that there exists $\delta'_{\bar{x}} > 0$ *such that* $T_{\Omega}(x) \le \omega(\Phi_S(x))$ *for any* $x \in B(\bar{x}, \delta'_{\bar{x}}) \cap \overline{\Omega} \setminus S$.

Before the proof of Theorem 2.8, we make some comment on the assumptions.

(i) Assumption (1) is just technical, and fix an upper bound $\tau(\Phi_S(x))$ on time sampling, depending only on the level set of $\Phi_S(\cdot)$ to which the considered starting point x belongs.
(ii) Assumption (2a) states that sufficiently near to the target there are no points where the unique admissible trajectory is the constant one. This is quite reasonable since if $\bar{x}$ would be one of such points, we would have $T(\bar{x}) = +\infty$, so STLA could not hold.
(iii) Assumption (2b) provides a quantitative estimate of the variation of the Φ_S between two sampling times in the case that we are not able to reach the target in the sampling time $\tau(\Phi_S(x))$.
(iv) Assumption (2c) gives the technical assumptions on $\Phi_S(\cdot)$ (see also Rem. 2.9).
(v) Assumption (3) ensures that between two sampling times the function Φ_S actually decreases with a decreasing rate *fast enough* to reach the target in finite time, thanks to Lemma 2.7.

Proof. If $\Phi_{\bar{x}} = 0$ then $B(\bar{x}, \delta_{\bar{x}}) \subseteq S$, and so $T_{\Omega}(x) = 0$ for all $x \in B(\bar{x}, \delta_{\bar{x}}) \cap \overline{\Omega}$, and there is nothing to prove. We suppose now $\Phi_{\bar{x}} > 0$. Since f is bounded on $\overline{B(\bar{x}, \delta_{\bar{x}})}$, by Remark 2.3 we can choose $0 < \delta'_{\bar{x}} < \frac{\delta_{\bar{x}}}{2}$ such that, if we set

$$T_{\delta'_{\bar{x}}} = \max_{x \in \overline{B(\bar{x}, \delta'_{\bar{x}})} \setminus S} \int_0^{\Phi_S(x)} \beta(s)\, \mathrm{d}s,$$

we have $\mathscr{R}_x(t) \subseteq \overline{B(\bar{x}, \delta_{\bar{x}})}$ for all $0 < t \le T_{\delta'_{\bar{x}}}$ and $x \in \overline{B(\bar{x}, \delta'_{\bar{x}})}$. Recalling that, by continuity of $\Phi_S(\cdot)$, by the definition of S, and by the fact that $\beta \in L^1$, we have $T_{\delta'_{\bar{x}}} \to 0^+$ as $\delta'_{\bar{x}} \to 0^+$. Moreover, we have also $\mathscr{R}_x(t) \subseteq \overline{B(x, M_{\bar{x}} t)}$ for all $0 < t \le T_{\delta'_{\bar{x}}}$. We define a sequence of points and times $\{(x_i, t_i, r_i)\}_{i \in \mathbb{N}}$ by induction as follows. We choose $x_0 \in (\overline{B(\bar{x}, \delta'_{\bar{x}})} \cap \overline{\Omega}) \setminus S$, and set $r_0 = \Phi_S(x_0)$, $t_0 = \theta(r_0)$. Suppose to have defined x_i, t_i, r_i. We distinguish the following cases:

(1) if $x_i \in S$, we define $x_{i+1} = x_i$, $t_{i+1} = 0$, $r_{i+1} = 0$.
(2) if $x_i \notin S$

if $\mathscr{R}^{\Omega}_{x_i}(t_i) \cap S \neq \emptyset$, take $x_{i+1} \in \mathscr{R}^{\Omega}_{x_i}(t_i) \cap S$ and define $r_{i+1} = 0$, $t_{i+1} = 0$.
if $\mathscr{R}^{\Omega}_{x_i}(t_i) \cap S = \emptyset$, we choose $y_i \in [\mathscr{R}^{\Omega}_{x_i}(t_i)]_{\delta(t_i,r_i)} \cap \overline{B(x_i, \chi(x_i, r_i))}$ such that

$$\min_{\zeta_{x_i} \in \partial^P \Phi_S(x_i)} \langle \zeta_{x_i}, y_i - x_i \rangle + \|y_i - x_i\|^2 \leq -\mu(t_i, r_i) + \sigma(t_i, r_i).$$

We select $x_{i+1} \in \mathscr{R}^{\Omega}_{x_i}(t_i)$ such that $\|y_i - x_{i+1}\| \leq \delta(t_i, r_i)$, and define $r_{i+1} = \Phi_S(x_{i+1})$, $t_{i+1} = \theta(r_{i+1})$. According to the semiconcavity of $\Phi_S(\cdot)$ (with semiconcavity constant K), we have that there exists $\zeta_{x_i} \in \partial \Phi_S(x_i)$ such that

$$\begin{aligned} r_{i+1} - r_i &\leq \langle \zeta_{x_i}, x_{i+1} - x_i \rangle + K\|x_{i+1} - x_i\|^2 \\ &\leq \langle \zeta_{x_i}, x_{i+1} - y_i + y_i - x_i \rangle + K\|(x_{i+1} - y_i) + (y_i - x_i)\|^2 \\ &\leq \langle \zeta_{x_i}, x_{i+1} - y_i + y_i - x_i \rangle + K\|x_{i+1} - y_i\|^2 \\ &\quad + 2K\|x_{i+1} - y_i\|\|y_i - x_i\| + K\|y_i - x_i\|^2. \end{aligned}$$

By recalling that by assumption (2b) and the selection of x_{i+1}, we have $\|y_i - x_i\| \leq \chi(t_i, x_i)$ and $\|x_{i+1} - y_i\| \leq \delta(t_i, x_i)$. Therefore

$$\begin{aligned} r_{i+1} - r_i &\leq \langle \zeta_{x_i}, x_{i+1} - y_i \rangle + \langle \zeta_{x_i}, y_i - x_i \rangle + K\delta^2(t_i, x_i) \\ &+ 2K\delta(t_i, x_i)\chi(t_i, x_i) + K\|y_i - x_i\|^2 \\ &\leq L(r_i)\delta(t_i, r_i) + \left(\langle \zeta_{x_i}, y_i - x_i \rangle + K\|y_i - x_i\|^2\right) + K\delta^2(t_i, x_i) + 2K\delta(t_i, x_i)\chi(t_i, x_i) \\ &\leq L(r_i)\delta(t_i, r_i) - \mu(t_i, r_i) + \sigma(t_i, r_i) + K\delta^2(t_i, x_i) + 2K\delta(t_i, x_i)\chi(t_i, x_i) \\ &\leq \left(L(r_i) + K\delta(t_i, r_i) + 2K\chi(t_i, r_i)\right)\delta(t_i, r_i) - \mu(t_i, r_i) + \sigma(t_i, r_i) = -\lambda(t_i, r_i), \end{aligned} \tag{2.3}$$

recalling that $\|\zeta_{x_i}\| \leq L(r_i)$ by definition of $L(\cdot)$. We notice that in this case $x_{i+1} \notin S$ since $x_{i+1} \in \mathscr{R}^{\Omega}_{x_i}(t_i)$ and $\mathscr{R}^{\Omega}_{x_i}(t_i) \cap S = \emptyset$, thus $t_{i+1} > 0$ and $r_{i+1} > 0$.

The assumptions of Lemma 2.7 are satisfied:

(1) $r_{i+1} - r_i \leq -\lambda(\theta(r_i), r_i)$,
(2) it is obvious that $\theta(r_i) \neq 0$ implies $r_i \neq 0$. Indeed, assume that $r_i = 0$. Since $0 \leq \theta(r) \leq \tau(r)$, and $\tau(r) = 0$ iff $r = 0$, we have $\theta(0) = 0$.
(3) By assumption, there exists $\beta \in L^1(]0, \delta_0[)$ such that $\dfrac{\theta(s)}{\lambda(\theta(s), s)} \leq \beta(r)$.

Applying Lemma 2.7, we have that

(a) $r_i \to 0$,
(b) $\displaystyle\sum_{i=0}^{\infty} \theta(r_i) \leq \int_0^{r_0} \beta(r)\, \mathrm{d}r$.

Since $\displaystyle\sum_{i=0}^{\infty} t_i \leq \sum_{i=0}^{\infty} \theta(s_i)$, we have $\displaystyle\sum_{i=0}^{\infty} t_i \leq \int_0^{r_0} \beta(r)\, \mathrm{d}r \leq T_{\delta'_x}$.

Noticing that $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathscr{R}_x\left(\displaystyle\sum_{i=0}^{\infty} t_i\right)$ and $\displaystyle\sum_{i=0}^{\infty} t_i \leq T_{\delta'}$, we have that $\{x_i\}_{i\in\mathbb{N}} \subseteq \overline{B(\bar{x}, \delta_{\bar{x}})}$, thus is bounded. Up to subsequence, still denoted by $\{x_i\}_{i\in\mathbb{N}}$, we have that there exists $x_\infty \in \mathbb{R}^d$ such that $x_i \to x_\infty$. Since $\Phi_S(x_i) \to 0$, we have $x_\infty \in S$ and so $T_\Omega(x_0) \leq \displaystyle\sum_{i=0}^{\infty} t_i \leq \omega(\Phi_S(x_0))$. $\square$

Remark 2.9. Recalling the properties of semiconcave functions, for every general closed set S we can take $\Phi_S(\cdot) = d_S^2(\cdot)$ with $K = 2$ and $L(r) = 2\sqrt{r}$. In this case, given $x \notin S$, we have $\zeta \in \partial^P d_S^2(x)$ if and only if $\zeta = 2d_S(x)\xi$ with $\xi \in \partial^P d_S(x)$. If S satisfies the ρ-internal sphere condition, we can take $\Phi_S(\cdot) = d_S(\cdot)$ with $K = 1/\rho$, $L(r) \equiv 1$.

Remark 2.10. In order to compare our result with Theorem 3.1 of [10], which generalized the classical Petrov's Condition, the Theorem 3.1 of [11], and the results of [13], it is enough to take $\Phi_S(\cdot) = d_S^2(\cdot)$ and define the functions appearing in the statement of Theorem 2.8 as suitable ones of (sum of) powers of r, t.

In Example 5.21 of [15] is presented a situation where Theorem 3.1 of [10] cannot be used, since the requirement of Lipschitz continuity of the essential leading term $A(\cdot)$ of the $\mathcal{A}$-trajectory (see Def. 3.1) prevents the choice of λ. This requirement was essential in the proof of Theorem 3.1 in [10]. The first main result of [15], *i.e.*, Theorem 5.10 in [15], used a different argument which does not require that Lipschitz condition. It works only under the additional assumption which is the internal sphere condition of the target S (see Example 5.21 in [15]). Observe that Theorem 2.8 requires neither Lipschitz continuity on the essential leading term $A(x)$, nor the internal sphere condition (which will lead to the semiconcavity of $d_S(\cdot)$ in $\mathbb{R}^d \setminus \operatorname{int} S$), thus generalizing also Theorem 5.10 of [15].

However, the second main result of [15] which exploits also generalized curvature of the target S – which is supposed to be sufficiently smooth such that d_S is at least of class $C^{1,1}$ in a neighborhood of S – is still not covered by Theorem 2.8, as shown by Example 5.22 in [15], where Theorem 5.10 of [15] fails even if the target S is smooth, since the exploitation of its curvature properties is also of essence in order to obtain STLA.

3. Attainability conditions for control-affine systems

We turn now our attention to *control-affine systems* described in (2.2). For such kind of systems it turns out that it is possible to construct *explicit approximations* of $\mathscr{R}_x^{\Omega}(t)$, on which we are going to check the conditions of Theorem 2.8.

We first recall this following definition.

Definition 3.1 ($\mathscr{A}^{\Omega}$-trajectory)**.** Let $\bar{x} \in \mathbb{R}^d$, $T > 0$. We say that a continuous curve $y_{\bar{x}} : [0, T] \to \mathbb{R}^d$ is an *$\mathscr{A}^{\Omega}$-trajectory* starting from $\bar{x}$ if we have $y_{\bar{x}}(0) = \bar{x}$ and $y_{\bar{x}}(t) \in \mathscr{R}_{\bar{x}}^{\Omega}(t)$ for any $t \in [0, T]$ (see also Sect. 3.1 in [11]). If $\Omega = \mathbb{R}^d$ we will omit it.

Our aim is to provide for these system conditions on the data of problem (*i.e.*, on vector fields f_j, on S and on Ω as appear in (2.2)) ensuring the applicability of Theorem 2.8 for a *given* system.

The problem can be split in two parts:

(1) construct suitable approximated $\mathscr{A}$-trajectories of the systems approaching the target sufficiently fast;
(2) among the previous trajectories, select the ones along which it is possible to provide a suitable lower bound of the distance from Ω^c, thus granting the fulfillment of the state constraints.

The first issue is strictly related to the possibility of providing a description at least of some suitable subsets of $\mathscr{R}_x(t)$ for any $x \in \mathbb{R}^d$ near to the target and $t > 0$ sufficiently small.

The second issue amounts to provide a quantitative estimate of the variation of the (squared) distance function from Ω^c along the $\mathscr{A}$-trajectories, in a very similar way as it was done with the (squared) distance function from the target S, or, more generally, of $\Phi_S(\cdot)$. While in the latter case we provided an *upper estimate* by means of a *semiconcavity inequality* satisfied by Φ_S, in the first case we will need the *reversed inequality*, *i.e.*, a *semiconvexity inequality* to bound the (squared) distance function from below. Without any extra smoothness hypothesis, the (squared) distance from a set is not semiconvex, thus, while for the upper bound we do not put any smoothness assumption on S, for the lower bound we will need some regularity hypothesis on Ω^c.

We recall the following definition, referring the reader to [1] for further details.

Definition 3.2 (Vector fields on manifolds)**.** Given a real smooth d-dimensional manifold M, it is well known that the set $\mathrm{Vec}(M)$ of vector fields on M can be defined as the set of all maps $V : M \to \mathcal{L}(C^1(M;\mathbb{R}), C^0(M;\mathbb{R}))$, where $\mathcal{L}(C^1(M;\mathbb{R}), C^0(M;\mathbb{R}))$ is the vector space of the linear maps from $C^1(M;\mathbb{R})$ to $C^0(M;\mathbb{R})$, such that Leibniz's Rule holds:

$$V(q)(fg) = V(q)(f)g(q) + f(q)V(q)(g)$$

for all $f, g \in C^1(M)$, $q \in M$. If we fix a coordinate system $\psi : M \to \mathbb{R}^d$ around $q \in M$, such that $\psi(q') = x \in \mathbb{R}^d$ for every q' sufficiently near to q, given $V \in \mathrm{Vec}(M)$ we have that there exists a unique map $\tilde{V} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\tilde{V}(x) = (V_1(x), \dots, V_d(x))$ and

$$V(q')\varphi = \langle \nabla(\varphi \circ \psi^{-1})(x), \tilde{V}(x) \rangle = \sum_{i=1}^{d} \frac{\partial(\varphi \circ \psi^{-1})}{\partial x_i}(x)\tilde{V}_i(x), \text{ for all } \varphi \in C^1(\mathbb{R}^d;\mathbb{R}),$$

conversely, given a map $\tilde{V} : \mathbb{R}^d \to \mathbb{R}^d$, $\tilde{V}(x) = (\tilde{V}_1(x), \dots, \tilde{V}_d(x))$, we can define an element $V \in \mathrm{Vec}(M)$ by mean of the above formula. In this case, we will say that $V(q')$ is represented in the chart ψ by $\sum_{i=1}^{d} \tilde{V}_i(x)\dfrac{\partial}{\partial x_i}$. The set $T_qM := \{V(q) : V \in \mathrm{Vec}(M)\}$ is the *tangent space* to M at q. The disjoint union of all T_qM for $q \in M$ is the tangent bundle TM to M. With some abuse of notation, we will write $(V\varphi)(q)$ in place of $V(q)\varphi$. A vector field V is of class $C^{k,\alpha}_{\mathrm{loc}}$, *i.e.* k-times differentiable with Hölder continuous kth differential of exponent α, if the corresponding representation in local charts $\tilde{V} : \mathbb{R}^d \to \mathbb{R}^d$ is of class $C^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)$. When $V, W \in \mathrm{Vec}(M)$, $W \in C^1(M)$ we can define $VW : C^2(M) \to C^0(M)$ by setting $(VW)\varphi = V(W\varphi)$, for all $\varphi \in C^2$, since $W\varphi \in C^1(M)$. More generally, if $V_1, \dots, V_m$ are of class C^{m-1} and $\varphi \in C^m$, we can define $(V_1 \dots V_m)\varphi$ by induction setting $(V_1 \dots V_m)\varphi = (V_1 \dots V_{m-1})(V_m\varphi)$. If $M = \mathbb{R}^d$, we can take ψ to be equal to the identity $\mathrm{Id}_{\mathbb{R}^d}$ and $\varphi = \pi_h \in C^\infty$ to be the hth coordinate function $\pi_h \circ \psi(q) = x_h$ if $\psi(q) = (x_1, \dots, x_d)$, thus identifying V and $\tilde{V}$. In this case we write also

$$\begin{aligned} V_1V_2(x) &= ((V_1V_2)\pi_1(x), \dots, (V_1V_2)\pi_d(x),) = (\nabla(\nabla\pi_i(x) \cdot V_2(x)) \cdot V_1(x))_{i=1,\dots,d} \\ &= \left(\nabla V_2^{(i)}(x) \cdot V_1(x)\right)_{i=1,\dots,d} = \nabla V_2(x) \cdot V_1(x), \end{aligned}$$

where $\nabla V_2(x)$ is the Jacobian matrix of V_2 at x, moreover this construction can be iterated by induction to define $V_1V_2 \dots V_m(x)$. Finally, we define the *first order Lie bracket* of the vector fields g_1, g_2 to be the differential operator

$$[g_1, g_2]\varphi(q) := (g_1g_2\varphi)(q) - (g_2g_1\varphi)(q), \text{ for all } \varphi \in C^\infty(M).$$

When $M = \mathbb{R}^d$, we have

$$[g_1, g_2](x) = (g_1g_2 - g_2g_1)(x) = \nabla g_2(x) \cdot g_1(x) - \nabla g_1(x) \cdot g_2(x).$$

Definition 3.3 (Characters, alphabet and words)**.** We consider a nonempty finite set of symbols $\boldsymbol{X} := \{x_0, \dots, x_N\}$, called the *alphabet*. The elements of $\boldsymbol{X}$ will be called *characters*. A *word* on $\boldsymbol{X}$ is any *finite* sequence of characters $w = x_{i_1}x_{i_2} \dots x_{i_M}$, where $i_j \in \{0, \dots, N\}$ for all $j \in \{0, \dots, M\}$. In this case, $|w| = M$ is called the *length* of the word. The *empty word* is the unique word of length 0 and will be denoted by Λ. If $w = x_{i_1}x_{i_2} \dots x_{i_M} \neq \Lambda$, we will define $I = (i_1, \dots, i_M) \in \mathbb{N}^M$ and write $w = x_I$. Given two words $x_I = x_{i_1} \dots x_{i_M}$ and $x_J = x_{j_1} \dots x_{j_H}$, we define their *concatenation* $x_{IJ} = x_Ix_J = x_{i_1} \dots x_{i_M}x_{j_1} \dots x_{j_H}$. We have clearly $x_I\Lambda = \Lambda x_I = x_I$ for all words x_I. The set $\Sigma(\boldsymbol{X})$ of all words on $\boldsymbol{X}$ together with the operation of concatenation is a monoid, since this operation is associative (but in general noncommutative) with Λ as the identity element. Given $k \in \mathbb{N}$, we will denote by $\Sigma_k(\boldsymbol{X})$ the subset of $\Sigma(\boldsymbol{X})$ made of words of length less or equal than k.

Definition 3.4 (Free Lie algebras)**.** Given an alphabet $\boldsymbol{X}$ and the set of words $\Sigma(\boldsymbol{X})$, we can consider the free module on $\mathbb{R}$ generated by $\Sigma(\boldsymbol{X})$, *i.e.*, the set of all formal finite linear combinations of words $\sum_{h=1}^{P} c_h w_h$, where $w_h \in \Sigma(\boldsymbol{X})$ and $c_h \in \mathbb{R}$, with the usual identifications: *i.e.*, if $c = 1$ then $cw = w$ for all $w \in \Sigma(\boldsymbol{X})$, and for all $w_1, \ldots, w_P \in \Sigma(\boldsymbol{X})$, $c_1, \ldots, c_P \in \mathbb{R}$, we have $\sum_{h=1}^{P} c_h w_h = \sum_{\substack{h=1\\h\neq j}}^{P} c_h w_h$ if $c_j = 0$. The free module on $\mathbb{R}$ generated by $\Sigma(\boldsymbol{X})$ together with the operation of concatenation is the free algebra $A(\boldsymbol{X})$ generated by $\Sigma(\boldsymbol{X})$, namely, the product of two words is given by $(c_1 x_I)(c_2 x_J) = c_1 c_2 x_{IJ}$ for all $c_1, c_2 \in \mathbb{R}$ and $x_I, x_J \in \Sigma(\boldsymbol{X})$. On $A(\boldsymbol{X}) \times A(\boldsymbol{X})$, we define the *Lie bracket* (or *commutator*) by setting $[w, z] = wz - zw \in A(\boldsymbol{X})$ for every $w, z \in \Sigma(\boldsymbol{X})$, where wz is the concatenation of w and z; and then extending it on the whole of $A(\boldsymbol{X})$ by linearity. The Lie bracket operation gives to $A(\boldsymbol{X})$ the structure of a Lie algebra. Given $k \in \mathbb{N}$, we will denote by $A_k(\boldsymbol{X})$ the subset of $A(\boldsymbol{X})$ made of all finite linear combinations of words in $\Sigma_k(\boldsymbol{X})$ with real coefficients.

Definition 3.5 (Chen−Fliess series)**.** Given the alphabet $\boldsymbol{X} := \{x_0, \ldots, x_N\}$, consider now the following Cauchy problem in $A(\boldsymbol{X})$

$$\begin{cases} \dot{S}(t) = S(t) \cdot \left(\sum_{j=0}^{N} u_j(t) x_j \right), \quad \text{t¿0}, \\ S(0) = \Lambda, \end{cases} \tag{3.1}$$

where the maps $u_j \in \mathscr{U}$ for any $j = 0, \ldots, N$ and $\cdot$ denotes here the concatenation operation. Given $u(\cdot) = (u_0(\cdot), \ldots, u_N(\cdot)) \in \mathscr{U}^{N+1}$, $t > 0$, we set

$$\begin{cases} \Upsilon_\Lambda(t, u) = 1, \\ \Upsilon_{x_j}(t, u) = \int_0^t u_j(s)\,\mathrm{d}s, & \text{for } j = 0, \ldots, N, \\ \Upsilon_{wx_j}(t, u) = \int_0^t \Upsilon_w(s, u) u_j(s)\,\mathrm{d}s, & \text{for } w \in \Sigma(\boldsymbol{X}),\ j = 0, \ldots, N. \end{cases}$$

This defines by recurrence a map $\Upsilon : \Sigma(\boldsymbol{X}) \times [0, +\infty[\times \mathscr{U}^{N+1} \to \mathbb{R}$, which can be extended by linearity to a map $\Upsilon : A(\boldsymbol{X}) \times [0, +\infty[\times \mathscr{U}^{N+1} \to \mathbb{R}$. With this definition, the explicit solution of (3.1) is given by *Chen−Fliess* series

$$S(t) = \sum_{n \in \mathbb{N}} \sum_{\substack{w \in \Sigma(\boldsymbol{X})\\|w|=n}} \Upsilon_w(t, u) w.$$

Remark 3.6. The number of terms appearing in $\sum_{\substack{w \in \Sigma(\boldsymbol{X})\\|w|=n}} \Upsilon_w(t, u) w$ increases very rapidly with n. Since many terms turn out to be repeated or can collected into terms involving possibly nested commutators of lower-length words w', w'', this motivates the need of finding *alternative description* for the Chen−Fliess series, exploiting as much as possible the symmetries in the iterated products and factorizing the words w appearing in the sum with respect to suitable Lie algebra basis (*e.g.* Hall–Viennot basis) for which the terms can be computed efficiently.

We want to link now the above abstract setting to the original control-affine system (2.2). We will give an idea of this connection, referring the reader to [12] for the details.

Definition 3.7. Consider the system (2.2), and let $\boldsymbol{X} = \{x_0, \dots, x_N\}$. Assume that $f_0, \dots, f_N$ are of class $C^{k,1}$ for some $k \geq 1$. Define a map ψ on $\Sigma_k(\boldsymbol{X})$ by setting for all $j = 0, \dots, N$, $w \in \Sigma_{k-1}(\boldsymbol{X})$ and $\varphi \in C^\infty(M)$

$$\begin{cases} \psi(\Lambda)\varphi = \varphi, \\ \psi(x_j)\varphi = f_j\varphi, \\ \psi(wx_j)\varphi = \psi(w)(f_j\varphi), \end{cases}$$

where $f_j\varphi$ is the usual action of the vector field f_j on M as a differential operator on the function φ. By linearity, we can extend ψ by linearity on the whole of $A_k(\boldsymbol{X})$.

Remark 3.8. Fixed a coordinate system around x_0 on the d-dimensional manifold M, and choosen φ_h as the hth coordinate function, it has been proved by Sussmann that, when all the vector fields are analytic, the vector-valued series

$$(\psi(S(t))\varphi)(x_0) := ((\psi(S(t))\varphi_1)(x_0), \dots, (\psi(S(t))\varphi_d)(x_0))$$

converges exactly to the solution of (2.2) (we set $u_0 \equiv 1$). When the vector fields are not analytic, we cannot expect convergence of this series in any sense, not even if they are C^∞, however its truncation yields an approximation of the solution.

When we consider $M = \mathbb{R}^d$, we can take the coordinate functions $\varphi = (\varphi_1, \dots, \varphi_d)$ to be the identity function. In this case, we can identify a differential operator acting on φ with a map from $\mathbb{R}^d$ to $\mathbb{R}^d$. We will use systematically this identification for systems in $\mathbb{R}^d$.

Lemma 3.9. *Consider the system (2.2) in $\mathbb{R}^d$, and let $\boldsymbol{X} = \{x_0, \dots, x_N\}$. Assume that $f_0, \dots, f_N$ are of class $C^{k,1}_{\mathrm{loc}}(\mathbb{R}^d)$ for some $k \geq 1$. Consider the mth partial sum with $0 \leq m \leq k$*

$$S_m(t) = \sum_{w \in \Sigma_m(\boldsymbol{X})} \Upsilon_w(t,u)w.$$

Then for each compact neighborhood K of x, there exists $t_K > 0$ and $C_K > 0$ such that

$$\|y_x(t) - \psi(S_m(t))(x)\| \leq C_K t^{m+1}, \text{ for any } 0 < t < t_K,$$

where $y_x(\cdot)$ is the solution of (2.2) satisfying $y_x(0) = x$. Thus, in particular, we have that

$$\left\{ \sum_{w \in \Sigma_m(\boldsymbol{X})} \Upsilon_w(t,u)\psi(w)(x) : 0 < t < t_K, u(\cdot) = (1, u_1(\cdot), \dots, u_N(\cdot)) \in \mathscr{U}^{N+1} \right\} \subseteq [\mathscr{R}_x(t)]_{C_K t^{m+1}}.$$

Proof. The result is a special case in $\mathbb{R}^d$ of equation (2.13) of Section 2.4.4 in [1] obtained by choosing $\varphi = \mathrm{Id}_{\mathbb{R}^d}$ and the identification for any $w \in \Sigma_m(\boldsymbol{X})$ of the differential operator $\psi(w)$ acting on φ with a map $\psi(w)(\cdot)$ from $\mathbb{R}^d$ to $\mathbb{R}^d$. □

Remark 3.10. Of course, in Lemma 3.9 we can choose convenient subsets $\mathscr{C}$ of $\mathscr{U}^{N+1}$ to obtain similar inclusions. For instance, we can restrict ourselves to use only piecewise constant controls in order to simplify the computation of the iterated integrals appearing in $\Upsilon_w(t,u)$. This will give a tool to check in practice condition (2b) of Theorem 2.8.

Lemma 3.11. *Fix $t > 0$, a partition $0 = t_0 < \ldots < t_N = t$ of $[0,t]$, $\{\lambda_i\}_{i=1,\dots,N} \in \mathbb{R}$. Define $I_i = [t_{i-1}, t_i]$, $\ell_i = t_i - t_{i-1}$ for $i = 1, \dots, N$ and $u(\cdot) = (u_1(\cdot), \dots, u_N(\cdot))$ with $u_i(\tau) = \lambda_i \chi_{I_i}(\tau)$ (where $\chi_{I_i}(\cdot)$ is the map assuming value 1 on I_i and 0 otherwise) for all $\tau \in [0,t]$, $i = 1, \dots, N$. Let $\boldsymbol{X} = \{x_1, \dots, x_N\}$ and consider $w \in \Sigma(\boldsymbol{X}) \setminus \{\Lambda\}$. Then*

(1) *if $\Upsilon_w(\tau, u) \neq 0$ with $w = x_{j_1} \dots x_{j_n}$, the sequence $\{j_h\}_{h=1}^n$ must be nondecreasing;*

(2) *if* $w = x_{h_1}^{\alpha_1} x_{h_2}^{\alpha_2} \dots x_{h_m}^{\alpha_m}$, *with* $\alpha_i \in \mathbb{N} \setminus \{0\}$ *and* $h_1 < \dots < h_m$, *then*

$$\Upsilon_w(t,u) = \prod_{h=1}^{m} \frac{(\lambda_{j_h}\ell_{j_h})^{\alpha_h}}{\alpha_h!}.$$

Proof. With the above choice of $u(\cdot)$, given $w = x_{j_1} \dots x_{j_n} \in \Sigma(\boldsymbol{X})$, and according to the definition of $\Upsilon_w(\tau,u)$, we have

$$\Upsilon_w(\tau,u) = \lambda_{j_1} \cdots \lambda_{j_n} \int \dots \int_{0 \le s_n \le \dots \le s_1 \le \tau} \chi_{I_{j_1}}(s_n) \dots \chi_{I_{j_n}}(s_1)\, \mathrm{d}s_n \dots \mathrm{d}s_1$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Upsilon_{wx_j}(\tau,u) = \lambda_j \chi_{I_j}(\tau)\Upsilon_w(\tau,u), \text{ for all } \tau \in [0,t] \setminus \{t_{j-1}, t_j\}.$$

In particular, $\Upsilon_{wx_j}(\tau,u) = \Upsilon_{wx_j}(t_j,u)$ for all $t_j \le \tau \le t$. Similarly, if $0 \le \tau \le t_{j_n-1}$ we have $\Upsilon_w(\tau,u) = \Upsilon_w(0,u) = 0$.

(1) Consider $\Upsilon_{vx_ix_j}(\tau,u)$ with a given $v \in \Sigma(\boldsymbol{X})$. If $j < i$, $\Upsilon_{vx_i}(\tau,u) = 0$ for $\tau \in]t_{j-1}, t_j[$ since $t_j \le t_{i-1}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Upsilon_{vx_ix_j}(\tau,u) = \lambda_j \chi_{I_j}(\tau)\Upsilon_{vx_i}(\tau,u) = 0, \qquad \text{for all } \tau \in]t_{j-1}, t_j[.$$

Due to absolute continuity of $\Upsilon_{vx_ix_j}(\cdot,u)$, $\Upsilon_{vx_ix_j}(\cdot,u)$ is constant in $]t_{j-1},t_j[$ and hence on the whole of $[0,t]$. Thus $\Upsilon_{vx_ix_j}(\tau,u) = \Upsilon_{vx_ix_j}(0,u) = 0$. If we consider now $\Upsilon_{vx_ix_jx_k}(\tau,u)$, we have that its derivative w.r.t. τ vanishes in $[0,t] \setminus \{t_{k-1}, t_k\}$, and so $\Upsilon_{vx_ix_jx_k}(\tau,u) = \Upsilon_{vx_ix_jx_k}(0,u) = 0$ in $[0,t]$. Iterating this argument adding new letters, this implies that if $j < i$ we have $\Upsilon_{vx_ix_jv'}(\tau,u) = 0$ for all $\tau \in [0,t]$ and $v' \in \Sigma(\boldsymbol{X})$, and thus if $\Upsilon_w(\tau,u) \neq 0$ then the sequence $\{j_h\}_{h=1}^n$ must be nondecreasing.

(2) Assume now $w = x_{h_1}^{\alpha_1} x_{h_2}^{\alpha_2} \dots x_{h_m}^{\alpha_m}$, with $\alpha_i \in \mathbb{N} \setminus \{0\}$ and $0 < h_1 < \dots < h_m$. Recall that $\Upsilon_w(\tau,u) = 0$ for $0 \le \tau \le t_{h_m-1}$ and $\Upsilon_w(\tau,u) = \Upsilon_w(t_{h_m},u)$ for $t_{h_m} \le \tau \le t$. Given $\tau \in]t_{h_m-1}, t_{h_m}[$, we have for all $0 < \alpha \le \alpha_m$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}\tau^\alpha}\Upsilon_w(\tau,u) = \lambda_j^\alpha \Upsilon_{w'x_{h_m}^{\alpha_m-\alpha}}(\tau,u),$$

where $w' = x_{h_1}^{\alpha_1} x_{h_2}^{\alpha_2} \dots x_{h_{m-1}}^{\alpha_{m-1}}$. In particular, recalling the smoothness of $\Upsilon_w(\cdot,u)$, we have

$$\lim_{\tau \to t_{h_{m-1}}^+} \frac{\mathrm{d}^\alpha}{\mathrm{d}\tau^\alpha}\Upsilon_w(\tau,u) = \begin{cases} 0, \text{ for } 0 < \alpha < \alpha_m, \\ \lambda_j^{\alpha_m}\Upsilon_{w'}(t_{h_{m-1}},u), \text{ for } \alpha = \alpha_m, \\ 0, \text{ for } \alpha > \alpha_m. \end{cases}$$

The third case is obtained since $\tau \in]t_{h_m-1}, t_{h_m}[$, which has an empty intersection with $I_{h_{m-1}}$ by the assumption $h_m > h_{m-1}$, and $\Upsilon_{w'}(\tau,u)$ is constant if $\tau \notin I_{h_{m-1}}$. This implies

$$\Upsilon_w(\tau,u) = \begin{cases} 0, & \text{if } 0 \le \tau \le t_{j_m-1}, \\ \dfrac{(\lambda(\tau - t_{j_m-1}))^{\alpha_m}}{\alpha_m!}\Upsilon_{w'}(\tau,u), & \text{if } t_{j_m-1} \le \tau \le t_{j_m}, \\ \dfrac{(\lambda_{j_m}\ell_{j_m})^{\alpha_m}}{\alpha_m!}\Upsilon_{w'}(\tau,u), & \text{if } t_{j_m} \le \tau \le t. \end{cases}$$

The assertion now follows by repeating the argument on w' (which is a product of $m-1$ powers of letters) and choosing $\tau = t$, recalling that $\Upsilon_\Lambda(t,u) = 1$. □

The following result allows us to construct the desired approximation of $\mathscr{R}_x(t)$ using Lemma 3.11.

Lemma 3.12. *Consider the system* (2.2) *in* $\mathbb{R}^d$ *with* $f_0, f_i \in C^{k,1}_{\rm loc}(\mathbb{R}^d)$, $i = 1, \ldots, N$, $0 < m \le k$.

Let $M > 0$, $\boldsymbol{X} := \{x_1, \ldots, x_M\}$, $\sigma = (\sigma_1, \ldots, \sigma_M) \in \{1, \ldots, N\}^M$, $\ell = (\ell_1, \ldots, \ell_M) \in]0, 1]^M$ *such that* $\sum_{j=1}^M \ell_j = 1$, $\lambda = (\lambda_1, \ldots, \lambda_M) \in [-1, 1]^M$. *Define* $g = (g_1, \ldots, g_M)$ *by setting* $g_i = f_0 + \lambda_i f_{\sigma_i}$, $i = 1, \ldots, M$.

Given a word $v \in \Sigma_k(\boldsymbol{X})$, *define*

$$\tilde{\psi}_v(t) = \begin{cases} t^{|\alpha|} \dfrac{(\ell_i)^\alpha}{\alpha!} (g_i)^\alpha, & \text{if } i_1 < i_2 < \ldots < i_h, \\ \mathrm{Id}_{\mathbb{R}^d}, & \text{if } v = \Lambda, \\ 0, & \text{otherwise,} \end{cases}$$

where if $v \neq \Lambda$ *we wrote it in a unique way as* $v = x_{i_1}^{\alpha_1} \ldots x_{i_h}^{\alpha_h}$ *with* $i_j \in \{1, \ldots, M\}$, $\alpha_j \in \mathbb{N} \setminus \{0\}$ *and* $i_j \neq i_{j+1}$ *for all* $j = 1, \ldots, h-1$, *and we have denoted* $(\ell_i)^\alpha = {\ell_{i_1}}^{\alpha_1} \ldots {\ell_{i_h}}^{\alpha_h}$, *and* $(g_i)^\alpha = g_{i_1}^{\alpha_1} \cdot \ldots \cdot g_{i_h}^{\alpha_h}$.

Then, if we set $P_x^{m,M,\ell,\lambda,\sigma}(t) := x + \sum_{w \in \Sigma_m \setminus \{\Lambda\}} \tilde{\psi}_w(t)(x)$, *for any compact neighborhood* K *of* x *there exists* $t_K > 0$ *and* $C_K > 0$ *such that* $P_x^{m,M,\ell,\lambda,\sigma}(t) \in [\mathscr{R}_x(t)]_{C_K t^{m+1}}$ *for all* $0 < t < t_K$.

Proof. Indeed, for each $t > 0$ we consider the partition $t_0 = 0 < t_1 < \ldots < t_M = t$, where $t_i - t_{i-1} = \ell_i t$. We set $I_i = [t_{i-1}, t_i]$, and define $u_{\sigma_i}(s) = \lambda_i \chi_{I_i}(s)$ for $i = 1, \ldots, M$. This is equivalent to consider the system $\dot{x}(s) = \sum_{i=1}^M \mu_i(s) g_i(x(s))$, where $\mu_{\sigma_i}(s) = \chi_{I_i}(s)$ for $i = 1, \ldots, M$. According to Lemma 3.11, in this case the mth order truncation of the Chen−Fliess series is $P_x^{m,M,\ell,\lambda,\sigma}(t)$ and, together with the error estimate of Lemma 3.9, this concludes the proof. □

Remark 3.13. $P_x^{m,M,\ell,\lambda,\sigma}(t)$ is the approximation at mth order of the point reached at time t by an admissible trajectory using piecewise constant controls activating at each time only one controlled vector field and with total amount of switchings equal to M. Some variants are possible, for example we may consider $\lambda \in \mathrm{Mat}_{M\times N}([-1, 1])$, where $\mathrm{Mat}_{M\times N}([-1, 1])$ is the set of $M \times N$ matrices with entries in $[-1, 1]$, $\sigma \in \mathrm{Mat}_{M\times N}(\{1, \ldots, N\})$ and define accordingly $g_i = f + \sum_{j=1}^N \lambda_{ij} f_{\sigma_{ij}}$, $i = 1, \ldots, M$, keeping inalterate the definitions of $\tilde{\psi}_v$ and of $P_x^{m,M,\ell,\lambda,\sigma}(t)$. The result still holds exactly with the same proof, and in this way we drop the restriction to use only one controlled vector field at each time.

Definition 3.14. Consider the system (2.2) in $\mathbb{R}^d$ with $f_0, f_i \in C^{k,1}_{\rm loc}(\mathbb{R}^d)$, $i = 1, \ldots, N$, $0 < m \le k$. Define

$$\mathscr{T}_x^m := \left\{ P_x^{m,M,\ell,\lambda,\sigma}(\cdot) - x : M > 0, \sigma \in \{1, \ldots, N\}^M, \ell \in]0, 1]^M \text{ with } \sum_{j=1}^M \ell_j = 1, \lambda \in [-1, 1]^M \right\}.$$

According to Lemma 3.12, given $P_x(\cdot) \in \mathscr{T}_x^m$, for any compact neighborhood K of x there exists $t_K > 0$ and $C_K > 0$ such that $x + P_x(t) \in [\mathscr{R}_x(t)]_{C_K t^{m+1}}$ for all $0 < t < t_K$.

Now we state the second main result of the paper, concerning sufficient conditions for STLA in the control-affine case (2.2).

Theorem 3.15 (Local STLA for constrained control-affine systems)**.** *Consider the system* (2.2) *with* $f_0, f_1, \ldots, f_N \in C^{k,1}_{\rm loc}(\mathbb{R}^d; \mathbb{R}^d)$. *Fix* $\bar{x} \in \partial S \cap \overline{\Omega}$, $\delta_{\bar{x}} > 0$, *and assume that*

$$S \cap \overline{B(\bar{x}, \delta_{\bar{x}})} \cap \overline{\Omega} := \{x : \Phi_S(x) \le 0\},$$
$$\overline{B(\bar{x}, \delta_{\bar{x}})} \cap \Omega^c := \{x : \Psi_{\Omega^c}(x) \le 0\},$$

for suitable locally Lipschitz functions $\Phi_S, \Psi_{\Omega^c} : \mathbb{R}^d \to \mathbb{R}$. *Let* $C_{\bar{x}} > 0$, $\tau_{\bar{x}} > 0$ *be the constants appearing in Definition* 3.14 *by taking* $\overline{B(\bar{x}, \delta_{\bar{x}})}$ *as a compact neighborhood of* $\bar{x}$.

Assume that $\Phi_S(\cdot)$ *is semiconcave on* $\overline{B(\bar{x}, \delta_{\bar{x}})}$ *with semiconcavity constant* $K_{\bar{x}} > 0$. *Define* $\Phi_{\bar{x}}$, $L(\cdot) \ge 0$, $M_{\bar{x}}$, $K = K_{\bar{x}}$, $\sigma, \mu, \chi, \tau, \theta, \beta$ *as in Theorem* 2.8. *If* $\bar{x} \in \partial S \cap \partial \Omega$, *suppose that* Ψ_{Ω^c} *is semiconvex on* $\overline{B(\bar{x}, \delta_{\bar{x}})}$.

Let $\varepsilon :]0,+\infty[\times]0,+\infty[\to\mathbb{R}$ *be a continuous function such that* $\displaystyle\lim_{t\to 0^+}\frac{\varepsilon(t,r)}{t^2}=+\infty$*, uniformly w.r.t.* $r\in]0, \max_{z\in\overline{B(\bar{x},\delta_{\bar{x}})}}\{\Psi_{\Omega^c}(z)\}]$.

Assume that for every $x\in\overline{\Omega}\cap B(\bar{x},\delta_{\bar{x}})\setminus S$ *there exist* $0<k_x\le k$, $P_x(\cdot)\in\mathscr{T}_x^{k_x}$, *and* $\zeta_x\in\partial^P\Phi_S(x)$, $\theta_x\in\partial_P\Psi_{\Omega^c}$ *satisfying for all* $0\le t\le\tau(\Phi_S(x))$:

(App) *approaching condition:*

$$\langle\zeta_x,P_x(t)\rangle+K\|P_x(t)\|^2\le-\mu(t,\Phi_S(x))+\sigma(t,\Phi_S(x)),$$

(Con) *constraint condition: if* $\bar{x}\in\partial\Omega$ *we require also that*

$$\langle\theta_x,P_x(t)\rangle>\varepsilon(t,\Psi_{\Omega^c}(x)),\ \textit{for all } x\in\Omega\cap\overline{B(\bar{x},\delta_{\bar{x}})}\setminus S.$$

Moreover, set $\delta(t,r)=C_{\bar{x}}t^k$ *and suppose that* (1), (3) *in Theorem* 2.8 *are satisfied. Then there exists* $0<\delta''_{\bar{x}}<\frac{\delta_{\bar{x}}}{2}$ *and a continuous increasing function* $\omega_{\bar{x}}:[0,+\infty[\to[0,+\infty[$ *such that* $\omega_{\bar{x}}(0)=0$ *and* $T_\Omega(x)\le\omega_{\bar{x}}(\Phi_S(x))$ *for every* $x\in B(\bar{x},\delta''_{\bar{x}})\cap\overline{\Omega}$.

Before proving the theorem, we make some comments on the assumptions.

(1) We are considering the approximation of $\mathscr{R}_x(t)$ provided by all the truncation of Chen−Fliess series obtained by using piecewise constant controls, this gives a family of $\mathscr{A}$-trajectories, among which we assume to be able to apply Theorem 2.8, ignoring for the moment any state constraint.
(2) The semiconvexity assumption on Ψ_{Ω^c} together with Assumption **(Con)** yields a quantitative estimate of the variation of $\Psi_c(\cdot)$ along the $\mathscr{A}$-trajectory.
(3) The assumptions on $\varepsilon(\cdot)$ will prevent the vanishing of $\Psi_c(\cdot)$, thus implying that the unconstrained $\mathscr{A}$-trajectory will satisfy also the state constraint, and so that it is actually an $\mathscr{A}^\Omega$-trajectory.

Proof. Without loss of generality, we may assume that if $\bar{x}\in\partial S\cap\Omega$ we have $\overline{B(\bar{x},\delta_{\bar{x}})}\cap\Omega^c=\emptyset$ and $\int_0^{\Phi_{\bar{x}}}\beta(r)\,\mathrm{d}r<1$. As in the proof of Theorem 2.8, we can choose $0<\delta'_{\bar{x}}<\dfrac{\delta_{\bar{x}}}{2}$ such that $\mathscr{R}_x(t)\subseteq\overline{B(\bar{x},\delta_{\bar{x}})}$ for all $x\in B(\bar{x},\delta'_{\bar{x}})$ and $0\le t\le\displaystyle\max_{x\in\overline{B(\bar{x},\delta'_{\bar{x}})\setminus S}}\int_0^{\Phi_S(x)}\beta(r)\,\mathrm{d}r$, where $\beta(\cdot)\in L^1$ is a function as in Theorem 2.8.

Given $P_x(\cdot)\in\mathscr{T}_x^{k_x}$ as in the statement, we can find $P'_x(\cdot)\in\mathscr{T}_x^k$ such that $\|P'_x(t)-P_x(t)\|=o(t^{k_x})$. We set $y_{t,x}=x+P'_x(t)$. Then we apply Theorem 2.8 ignoring the state constaint to obtain the upper bound $T(x)\le\displaystyle\int_0^{\Phi_S(x)}\beta(r)\,\mathrm{d}r$.

If $\bar{x}\in\partial S\cap\Omega$ the proof is concluded, recalling that in this case $\mathscr{R}_x(t)=\mathscr{R}_x^\Omega(t)$ for all $0\le t\le T(x)$, and so $T(x)=T_\Omega(x)$.

Assume that $\bar{x}\in\partial\Omega\cap\partial S$, and take $x\in B(\bar{x},\delta'_{\bar{x}})\cap\Omega\setminus S$. Let $C,\tau_C>0$ and $y_x(\cdot)$ be an admissible trajectory for the unconstrained system such that $\|y_x(t)-(x+P_x(t))\|\le Ct^2$ for $0<t\le\tau_C$. By taking $0<\delta''_{\bar{x}}<\delta'_{\bar{x}}$ such that $\displaystyle\int_0^{\Phi_S(x)}\beta(r)\,\mathrm{d}r<\tau_C$ for all $x\in\overline{B(\bar{x},\delta''_{\bar{x}})}\cap\Omega$, we will show that $y_x(t)\in\overline{\Omega}$ for all $0\le t\le T(x)$.

We denote by $\Psi_{\bar{x}}$ the semiconvexity constant of Φ_{Ω^c} on $\overline{B(\bar{x},\delta_{\bar{x}})}$. For $0<t<T(x)$, there exists $\theta_x\in\partial_P\Psi_{\Omega^c}(x)$ such that

$$\begin{aligned}\Psi_{\Omega^c}(y_x(t))&\ge\Psi_{\Omega^c}(x)+\langle\theta_x,y_x(t)-x\rangle-\Psi_{\bar{x}}\|y_x(t)-x\|^2\\&\ge\Psi_{\Omega^c}(x)+\langle\theta_x,P_x(t)\rangle-\|\theta_x\|Ct^2-\Psi_{\bar{x}}(\|P_x(t)\|^2+2Ct^2\|P_x(t)\|+C^2t^4).\end{aligned}$$

Since $\Psi_{\Omega^c}(\cdot)$ is locally Lipschitz continuous, we have that $\|\theta_{\bar{x}}\|$ is uniformly bounded in $\overline{B(\bar{x},\delta''_{\bar{x}})}$, furthermore, by the smoothness of the vector fields, we have that $\dfrac{\|P_x(t)\|}{t}$ is uniformly bounded for all $x \in \overline{B(\bar{x},\delta''_{\bar{x}})}$ and $t \in]0, T(x)]$. In particular, there exists $D > 0$ and $\tau_D > 0$ such that for all $0 < t < \tau_D$ we have

$$\Psi_{\Omega^c}(y_x(t)) \geq \Psi_{\Omega^c}(x) + \varepsilon(t, \Psi_{\Omega^c}(x)) - Dt^2 \geq \Psi_{\Omega^c}(x) > 0,$$

recalling the fact that given $D > 0$ there exists $\tau_D > 0$ such that $\varepsilon(t, \Phi_S(x)) - Dt^2 \geq 0$ for all $x \in B(\bar{x},\delta''_{\bar{x}})$ and $0 < t < \tau_D$, due to the assumptions of $\varepsilon(\cdot)$. Thus we take $0 < \delta''_{\bar{x}} < \delta'_{\bar{x}}$ such that $\displaystyle\int_0^{\Phi_S(x)} \beta(r)\,\mathrm{d}r < \min\{\tau_C,\, \tau_D\}$ for all $x \in \overline{B(\bar{x},\delta''_{\bar{x}})} \cap \overline{\Omega}$.

This implies that x_i and x_{i+1}, constructed as in the proof of Theorem 2.8 for the unconstrained system starting from $x \in \overline{B(\bar{x},\delta''_{\bar{x}})} \cap \Omega$, are actually connected by an admissible trajectory also for the constrained system for every $i \in \mathbb{N}$, since Ψ_{Ω^c} is nondecreasing, and so in particular it remains strictly positive. Thus also in this case we have

$$T_\Omega(x) \leq \int_0^{\Phi_S(x)} \beta(r)\,\mathrm{d}r,$$

for all $x \in \overline{B(\bar{x},\delta''_{\bar{x}})} \cap \Omega$.

Finally, assume that $\bar{x} \in \partial\Omega \cap \partial S$, and take $x \in B(\bar{x},\delta''_{\bar{x}}) \cap \partial\Omega \setminus S$. We can find a sequence of points $\{z_i\}_{i\in\mathbb{N}} \subseteq B(\bar{x},\delta''_{\bar{x}}) \cap \Omega \setminus S$, a sequence of admissible trajectories $\{y_{z_i}(\cdot)\}_{i\in\mathbb{N}}$ and a sequence of times$\{T_i\}_{i\in\mathbb{N}}$, such that $y_{z_i}(0) = z_i$, $y_{z_i}(t) \in \overline{\Omega} \cap \overline{B(\bar{x},\delta_{\bar{x}})}$ for all $0 \leq t \leq T_i$, $y_{z_i}(T_i) \in S$, and $T_i \leq \displaystyle\int_0^{\Phi_\Omega(z_i)} \beta(s)\,\mathrm{d}s$.

It is well known that up to passing to a subsequence, we have that $T_i \to T_\infty$ and $\{y_{z_i}(\cdot)\}_{i\in\mathbb{N}}$ uniformly converges to an admissible trajectory $y_x(\cdot)$ satisfying $y_x(0) = x$. Since the constraint and the target set are closed, we have also $y_x(t) \in \overline{\Omega} \cap \overline{B(\bar{x},\delta''_{\bar{x}})}$ for all $0 \leq t \leq T_\infty$ and $y_x(T_\infty) \in S$. Thus

$$T_\Omega(x) \leq T_\infty \leq \lim_{i\to\infty} \int_0^{\Phi_S(z_i)} \beta(r)\,\mathrm{d}r = \int_0^{\Phi_S(x)} \beta(r)\,\mathrm{d}r. \qquad \square$$

Applying Lemma 2.6, we can give a global STLA estimate.

Corollary 3.16 (Global STLA for constrained control-affine systems)**.** *Consider the system* (2.2)*. Assume that at every $\bar{x} \in \partial S \cap \overline{\Omega}$ the assumptions of Theorem 3.15 are satisfied, then STLA holds. Moreover, if $\partial S \cap \overline{\Omega}$ is compact we have that there exists $\delta_S > 0$ and a continuous function $\omega : [0, +\infty[\to [0, +\infty[$ such that $\omega(0) = 0$ and $T_\Omega(x) \leq \omega(d_S(x))$ for every $x \in S_{\delta_S} \cap \overline{\Omega}$.*

Proof. It is a straightforward application of Lemma 2.6. $\square$

As already said, if we want to take $\Psi_{\Omega^c} = d_{\Omega^c}$, we have to assume some smoothness property on the constraint $\overline{\Omega}$. In particular, if Ω^c belongs to the class of *locally positive reach sets*, there exists a neighborhood V of Ω^c where $d_{\Omega^c} \in C^{1,1}(V \setminus \Omega^c)$, thus it is both semiconcave and semiconvex. This class of sets was introduced in [8], and has been extensively studied by many authors both in finite and infinite dimensions. We refer the reader to [5, 6, 14] for further details and extension of such kind of results.

We end this section by comparing the inward pointing condition of [4] and ours. Indeed, [4] deals with constraints satisfying the so-called *wedgedness property*, *i.e.* the nonemptiness of the Clarke's tangent cone (see Rem. 3.2 of [4] for further details). However it is easy to show that the classes of wedged sets and of sets whose complement has locally positive reach are distinct (even if smooth $C^{1,1}$ sets belong to both of them), thus the results are not directly comparable.

4. An example

In this section we present an example illustrating our approach.

Example 4.1. In $\mathbb{R}^3$ we consider the control-affine system (2.2) with $N = 2$, and set $S := \overline{B(0,1/2)}$, $f_0(x_1,x_2,x_3) = \frac{1}{8}(-x_2,x_1,0)$, $f_1(x_1,x_2,x_3) = (x_1x_3,x_2x_3,0)$, $f_2(x_1,x_2,x_3) = (0,0,1)$, $\Omega := \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_1^2+x_2^2>\frac{1}{16}\right\}$.

We take $\Phi_S(x) = \|x\|-1/2$. This map agrees with $d_S(\cdot)$ on $\overline{\mathbb{R}^d\setminus S}$ and is smooth on $\mathbb{R}^d\setminus\{0\}$. We have $\nabla\Phi_S(x) = \dfrac{x}{\|x\|}$ for all $x\neq 0$. Moreover, $\partial^P\Phi_S(x) = \{\dfrac{x}{\|x\|}\}$ for all $x\neq 0$, and $\Phi_S(\cdot)$ is semiconcave of constant $K = K_{\bar{x}} = 2$ in every ball $\overline{B(\bar{x},\delta_{\bar{x}})}$ with $\bar{x}\in\partial S$ and $\delta_{\bar{x}}\le 1/4$. Finally, we have $L(r) = 1$ by 1-Lipschitz continuity of $\Phi_S(\cdot)$. Notice that the constraint has positive reach, thus we take $\Psi_{\Omega^c}(x) = d_{\Omega^c}(x) = \sqrt{x_1^2+x_2^2}-\frac{1}{4}$.

We consider now the unconstrained problem.

We notice that at every point $(x_1,x_2,x_3)\in\partial S$ we have

$$\langle\nabla\Phi_S(x),(f_0(x)+u_1f_1(x)+u_2f_2(x)\rangle = 2x_3((x_1^2+x_2^2)u_1+u_2) \tag{4.1}$$

For any $\bar{x}\in D := \partial S\cap\Omega\setminus\{x_3=0\}$, there exists $\delta_{\bar{x}}>0$ such that Petrov's condition for any $x\in B(\bar{x},\delta_{\bar{x}})$: indeed, by choosing $u_2(s) = -\mathrm{sign}(x_3)$, $-1\le u_1\le 1$, we have that 4.1 is continuous and strictly negative on D thus every point of D possesses a neighborhood where the above expression remains bounded away from 0.

Given $t>0$, we consider the following choice of controls:

$$u_1(s) = \begin{cases} 1, & \text{if } 8s/t\in[0,1]\cup[6,7],\\ -1, & \text{if } 8s/t\in[2,3]\cup[4,5],\\ 0, & \text{elsewhere,}\end{cases} \qquad u_2(s) = u_1(s-t/8).$$

Given $x\in\mathbb{R}^3$, in this case Chen−Fliess series yields an $\mathscr{A}$-trajectory of the form

$$\tilde{y}_x(t) = x+tf_0(x)+\frac{t^2}{32}(16f_0f_0+[f_1,f_2])(x)+o(t^2)$$

and, by the smoothness of the vector fields, there exists $L>0$ such that $\|o(t^2)\|\le Lt^3$ for every $x\in B(0,1)\supset S$. Set $P_x(t) = tf_0(x)+\dfrac{t^2}{32}(16f_0f_0+[f_1,f_2])(x)$. We notice that, by the smoothness of the vector fields, the map $x\mapsto P_x(t)$ is continuous. Given $x=(x_1,x_2,0)\in\partial S$, we have

$$\langle\nabla\Phi_S(x),P_x(t)\rangle+2\|P_x(t)\|^2 = \frac{t^2(-128+64(x_1+x_2))}{16384}+o(t^2)\le\frac{-t^2}{256}+o(t^2).$$

In particular, there exist $\tau,C>0$ such that for $0\le t\le\tau$ and every $x=(x_1,x_2,0)\in\partial S$

$$\langle\nabla\Phi_S(x),P_x(t)\rangle+2\|P_x(t)\|^2<-2Ct^2.$$

Thus every point $\bar{x}\in\partial S\cap\{x_3=0\}$ possesses a neighborhood $V_{\bar{x}}$ such that $\langle\nabla\Phi_S(x),P_x(t)\rangle+2\|P_x(t)\|^2<-Ct^2$, and so we can define $k_x=2$, $\mu(t,r)=Ct^2$, $\delta(t,r)=Lt^3$, $\sigma(t,r)=0$, $\chi(t,r)=1$, $\tau(r)=\theta(r)=\min\{\tau,\dfrac{1}{L},\dfrac{1}{2C}\sqrt{r}\}$. Condition **(App)** thus holds at these points, and by Theorem 3.15 and Corollary 3.16, we obtain that there exists $C'>0$ such that $T(x)\le C'd_S^{1/2}(x)$ on a suitable neighborhood of S.

Now we pass to consider the constraints. Since $\partial S\cap\overline{\Omega}\cap\{x_3=0\}=\emptyset$, for any $\bar{x}\in\partial S\cap\overline{\Omega}$ there exists $\delta_{\bar{x}}>0$ such that Petrov's condition holds at every $x\in\overline{B(\bar{x},\delta_{\bar{x}})}\setminus S$ by taking $u_2=-\mathrm{sign}(x_3)$ and $-1\le u_1\le 1$. Moreover, for all $x\in\Omega\cap\overline{B(\bar{x},\delta_{\bar{x}})}\setminus S$ we have

$$\langle\nabla\Psi_{\Omega^c}(x_1,x_2,x_3),f_0(x_1,x_2,x_3)+u_1f_1(x_1,x_2,x_3)-\mathrm{sign}(x_3)f_2(x_1,x_2,x_3)\rangle = u_1\sqrt{x_1^2+x_2^2}\,x_3\ge\frac{u_1x_3}{4}\cdot$$

By taking $u_1 = \mathrm{sign}(x_3)$ we have that the above expression is strictly positive at any point of $\Omega \cap \overline{B(\bar{x}, \delta_{\bar{x}})} \setminus S$, thus both (App) and the constraint condition are fulfilled, so $T_\Omega(x) \le C' d_S^{1/2}(x)$ on a suitable neighborhood of S in $\overline{\Omega}$ by Theorem 3.15 and Corollary 3.16.

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