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APPROXIMATE CONTROLLABILITY OF LINEARIZED SHAPE-DEPENDENT OPERATORS FOR FLOW PROBLEMS

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Abstract. We study the controllability of linearized shape-dependent operators for flow problems. The first operator is a mapping from the shape of the computational domain to the tangential wall velocity of the potential flow problem and the second operator maps to the wall shear stress of the Stokes problem. We derive linearizations of these operators, provide their well-posedness and finally show approximate controllability. The controllability of the linearization shows in what directions the observable can be changed by applying infinitesimal shape deformations.

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1. Introduction

We study the controllability of linearized shape-dependent operators for flow problems. The first operator $\mathbf{S}_p$ is a mapping from the shape of the computational domain to the tangential wall velocity of the potential flow problem and the second operator $\mathbf{S}_s$ maps to the wall shear stress of the Stokes problem. On account of the shape dependence, both operators are highly nonlinear, despite of the underlying linear partial differential equations. We investigate linearizations $\mathrm{d}\mathbf{S}$ of these operators, *i.e.*, we study in which directions the observables can by changed by applying infinitesimal shape deformations. Our ultimate goal is to prove approximate controllability for these linearized shape-dependent operators. Approximate controllability means that we can find controls for the operator such that any element from the target space is approximated with arbitrary accuracy. In [11] we have utilized a conformal pull-back to study the operator $\mathbf{S}_p$ directly. However, the approach presented in the following is more general and can be extended to other flow problems, as we are going to show for the Stokes operator $\mathbf{S}_s$.

Our study of shape-dependent problems is motivated by the optimal shape design of polymer distributors used in the production process for filaments and nonwovens [9,10,12]. The goal is to design flow geometries with specific wall shear stress profiles, similar to the problems considered in [19, 20]. Numerically, we can solve the regularized inverse problem of finding a flow geometry which approximately realizes a given wall shear stress,

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using methods from shape optimization. However, here we address the theoretical question of what wall shear stress profiles are in fact attainable. Being able to establish some sort of controllability property, even though we can only do this for the linearization, suggests that the space of reachable profiles is rather large. For our application this means that we have a good chance to design polymer distributors, whose properties are close to our expectations. This agrees with our numerical results presented in [10], where we solve an optimization problem based on the Stokes operator $\mathbf{S}_s$.

The controllability of shape-dependent operators is rarely covered in the existing literature. Our approach is inspired by [4] where the authors study the controllability of a shape identification problem based on the Laplace problem. They show approximate controllability for the linearized operator using an adjoint argument (*cf.* [15,16]). The operator studied in [4] is comparable to our operator $\mathbf{S}_p$ because both are based on the Laplace problem. However, the operator in [4] maps to the trace evaluated on an interior curve whereas here $\mathbf{S}_p$ maps to the normal derivative evaluated on the variable wall boundary itself, which poses different technical challenges.

A good introduction to the general theory of shape optimization, the concept of shape derivatives and many examples can be found in [18,22]. The focus in [13] is more on the application of industrial airfoil design, but it can also be seen as an excellent access to the general topic. A rigorous treatment of shape derivatives and their existence theory is provided in [21]. A lot of theory on shape calculus and its application to shape optimization is given in [5]. Surveys on recent developments are found in [7,14]. While we mostly deal with flow problems, there are various other fields of application: for instance, see [1,8,17] for examples on structural optimization, [6,13] for airfoil design and [23] for applications in image processing.

We begin in Section 2 by introducing the geometric setup and give proper definitions for the space of admissible shapes and the linearized shape operator. In Section 3 we study the potential flow shape operator $\mathbf{S}_p$, derive its linearization, provide the well-posedness and finally show the approximate controllability. In Section 4 we follow the same path for the Stokes operator $\mathbf{S}_s$. Finally, we finish with a conclusion. In the Appendix, we state some basic facts about shape differentiability and the existence and uniqueness of solutions for partial differential equations (see Appendix A and B). The main results of this article are stated in Theorems 3.2 and 4.2.

2. Geometric setup

For $k \in \mathbb{N}_0$ let $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain of class $C^{k+1,1}$ (see [24]), where the boundary Γ_0 decomposes into the in- and outflow parts Γ_0^{in} and wall parts Γ_0^w, where Γ_0^{in} is closed and Γ_0^w is open. Let $\mathbf{n}$ be the outward pointing unit normal and let $\boldsymbol{\tau} := (-\mathbf{n}_2, \mathbf{n}_1)^\intercal$ be a tangential vector. We define

$$\Theta^k = \{\theta \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2); \|\theta\|_{C^{k,1}(\mathbb{R}^2,\mathbb{R}^2)} < 0.5\} \tag{2.1}$$

to be a ball around zero, where $C^{k,1}(\mathbb{R}^2, \mathbb{R}^2)$ denotes the space of k-times differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ with Lipschitz-continuous derivatives up to order k (see [24]). Let $\mathrm{Id} \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2)$ denote the identity mapping. For $\theta \in \Theta^k$ we consider the map

$$\mathrm{Id} + \theta : \mathbb{R}^2 \to \mathbb{R}^2, \tag{2.2}$$

i.e., $(\mathrm{Id} + \theta)(x) = x + \theta(x)$. If $\|\theta\|_{C^{k,1}(\mathbb{R}^2,\mathbb{R}^2)} < 0.5$, then θ is a strict contraction and a Neumann series argument implies that $\mathrm{Id} + \theta$ is invertable and that it is a $(k, 1)$-diffeomorphism (see [21]). In order to define the set of admissible shapes let the space of admissible deformation directions be

$$\mathcal{V}^k := \{\mathbf{V} \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2); \mathbf{V}|_{\Gamma_0^{in}} = 0; \mathbf{V}|_{\Gamma_0^w} = v_\mathbf{n}\mathbf{n}; v_\mathbf{n} \in C^{k,1}(\mathbb{R}^2)\}. \tag{2.3}$$

Note, that since Ω_0 is assumed to be of class $C^{k+1,1}$ we have $\mathbf{n} \in C^{k,1}(\Gamma_0, \mathbb{R}^2)$. Hence, this definition makes sense. We only consider normal shape deformations, because infinitesimal tangential deformations would shift the boundary along itself and are therefore no real shape deformations. Let the intersection with Θ^k be denoted by

$$\Theta^k_\mathcal{V} := \Theta^k \cap \mathcal{V}^k. \tag{2.4}$$

Then, the space of admissible shapes is given by

$$\mathcal{D}^k := \{\Omega_\theta = (\mathrm{Id} + \theta)(\Omega_0); \theta \in \Theta^k_{\mathcal{V}}\}. \tag{2.5}$$

Thus, $\mathcal{D}^k$ is a set of perturbations of the reference domain Ω_0 which leave Γ_0^{in} fixed and which are normal on Γ_0^w.

Definition 2.1. Let

$$\mathbf{S} : \mathcal{D}^k \to L^2(\Gamma_0^w). \tag{2.6}$$

be a given shape-dependent operator. Then the corresponding linearized shape operator is defined by

$$\begin{aligned} d\mathbf{S} : \mathcal{V}^k &\to L^2(\Gamma_0^w) \\ \mathbf{V} &\mapsto \lim_{s \searrow 0} \frac{\mathbf{S}(\Omega_{s\mathbf{V}}) - \mathbf{S}(\Omega_0)}{s}. \end{aligned} \tag{2.7}$$

Of course the important questions are whether the derivative does exist and how the operator can be evaluated. Both will be answered by applying the theory of material and shape derivatives which is provided in Appendix A.

Our goal is to show approximate controllability for two different linearized shape operators [4]:

Definition 2.2 (Approximate controllability)**.** Let $F : X \to Y$ be a linear operator. Then, F is approximately controllable if and only if $\operatorname{im} F$ lies dense in Y.

The definition immediately yields the following lemma which we use to show the property.

Lemma 2.3. *Let $F : X \to Y$ be a linear operator and let Y be a Hilbert space with scalar product $(\cdot,\cdot)_Y$. If and only if*

$$\operatorname{im} F^\perp := \{y \in Y; (F(x), y)_Y = 0 \textit{ for all } x \in X\} = \{0\} \tag{2.8}$$

then F is approximate controllable.

3. Potential flow

We begin our study with a potential flow shape operator which maps from the shape of the domain to the tangential wall velocity of the potential flow problem. We define the operator and derive its linearization. Then, we use the implicit function theorem to show the existence of the material derivative (see Def. A.1) which provides the well-definedness of the linearized shape operator. This also leads to the existence of the shape derivative (see Def. A.4), which can be computed as the solution of a boundary value problem. We can then write the linearized shape operator in terms of the shape derivative and use an adjoint argument to show that it is approximately controllable.

3.1. Definition of the shape operator and problem statement

Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain of class $C^{3,1}$ and let $g \in H^{\frac{5}{2}}(\Gamma_0)$ be given with $\partial_{\boldsymbol{\tau}} g|_{\Gamma_0^w} = 0$, where $\partial_{\boldsymbol{\tau}}$ denotes the derivative in tangential and $\partial_{\mathbf{n}}$ the derivative in normal direction. Note that g is constant on every connected component of the wall boundary by definition. This implies (see (3.3)) that the normal velocity vanishes at these boundary parts and thus that there is no flow through the wall. For $\theta \in \Theta^2$ let the stream function $\Psi(\theta) \in H^2(\Omega_\theta)$ be the solution of

$$\begin{aligned} \Delta\Psi(\theta) &= 0 && \text{in } \Omega_\theta \\ \Psi(\theta) &= g \circ (\mathrm{Id} + \theta)^{-1} && \text{on } \Gamma_\theta. \end{aligned} \tag{3.1}$$

Remark 3.1. The stream function $\Psi(\theta)$ is interpreted as the solution of a flow problem by defining the velocity $\mathbf{u}(\theta)$ through

$$\mathbf{u}(\theta) := \begin{pmatrix} \partial_2 \Psi(\theta) \\ -\partial_1 \Psi(\theta) \end{pmatrix} \qquad \text{in } \Omega_\theta. \tag{3.2}$$

In that case the normal wall velocity is

$$\mathbf{n} \cdot \mathbf{u}(\theta) = \partial_{\boldsymbol{\tau}} \Psi(\theta) = \partial_{\boldsymbol{\tau}} (g \circ (\mathrm{Id} + \theta)^{-1}) \qquad \text{on } \Gamma_\theta \tag{3.3}$$

and especially $\mathbf{n} \cdot \mathbf{u}(\theta)|_{\Gamma_\theta^w} = 0$ by definition of g. The tangential wall velocity is

$$\boldsymbol{\tau} \cdot \mathbf{u}(\theta) = -\partial_{\mathbf{n}} \Psi(\theta) \qquad \text{on } \Gamma_\theta \tag{3.4}$$

and we are going to define the operator $\mathbf{S}_p$ as a map to this tangential velocity.

We define the potential flow shape operator $\mathbf{S}_p$ by

$$\begin{aligned} \mathbf{S}_p : \mathcal{D}^2 &\to L^2(\Gamma_0^w) \\ \Omega_\theta &\mapsto -(\partial_{\mathbf{n}} \Psi(\theta)|_{\Gamma_\theta^w}) \circ (\mathrm{Id} + \theta). \end{aligned} \tag{3.5}$$

Note, that $\partial_{\mathbf{n}} \Psi(\theta)|_{\Gamma_\theta^w}$ is a function of $L^2(\Gamma_\theta^w)$. To get a well-defined operator where the target space is independent of Ω_θ we use the map $(\mathrm{Id} + \theta) : \Omega_0 \to \Omega_\theta$ to pull-back this function to the space $L^2(\Gamma_0^w)$.

We are going to show that the linearized shape operator $\mathrm{d}\mathbf{S}_p$ is well-defined and given by

$$\begin{aligned} \mathrm{d}\mathbf{S}_p : \mathcal{V}^2 &\to L^2(\Gamma_0^w) \\ \mathbf{V} &\mapsto -\partial_{\mathbf{n}} \Psi'(\Omega_0; \mathbf{V})|_{\Gamma_0^w} - \kappa \mathbf{S}_p(0)(\mathbf{n} \cdot \mathbf{V}), \end{aligned} \tag{3.6}$$

where $\Psi'(\Omega_0; \mathbf{V})$ is the solution of

$$\begin{aligned} \Delta \Psi'(\Omega_0; \mathbf{V}) &= 0 && \text{in } \Omega_0 \\ \Psi'(\Omega_0; \mathbf{V}) &= 0 && \text{on } \Gamma_0^{in} \\ \Psi'(\Omega_0; \mathbf{V}) &= -(\mathbf{n} \cdot \mathbf{V}) \partial_{\mathbf{n}} \Psi(0) && \text{on } \Gamma_0^w. \end{aligned} \tag{3.7}$$

In the rest of this section we establish the existence of $\mathrm{d}\mathbf{S}_p$ and prove the following result about the approximate controllability of the linearized shape operator:

Theorem 3.2. *Assume that $\mathbf{S}_p(0) \neq 0$ a.e. on Γ_0^w and suppose that the curvature $\kappa := \mathrm{div}_\Gamma\, \mathbf{n} \in C^0(\Gamma_0)$ is non negative $\kappa \geq 0$ on Γ_0^w. Then, $\mathrm{d}\mathbf{S}_p$ is approximately controllable.*

Remark 3.3. Especially the curvature condition is fulfilled if the wall boundaries are convex. Note, that the statement still holds for curvature $\kappa \geq -\delta$ for a sufficiently small constant $\delta \geq 0$. The constant δ must be small enough such that the bilinear form corresponding to (3.30) is still elliptic (see Def. B.1). Otherwise we can show that the bilinear form is coercive (see Def. B.2) and prove a result similar to the upcoming theorem 4.2.

3.2. Existence of the material derivative

One crucial point is to show the existence of the material derivative for the solution of (3.1), because it gives rise to the well-definedness of the linearized shape operator as well as the existence of the shape derivative. Let us define

$$z(s\mathbf{V}) := \partial_{\mathbf{n}} \Psi(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}} \tag{3.8}$$

for $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$ and $s \geq 0$ sufficiently small. Assume that the material derivative $\dot{z}(\Gamma_0; \mathbf{V})$ exists for $\mathbf{V} \in \mathcal{V}^2$ (see Def. A.2), then by Definition 2.1

$$\mathrm{d}\mathbf{S}_p(\mathbf{V}) = \left.\frac{\mathrm{d}(\mathbf{S}_p(\Omega_{s\mathbf{V}}))}{\mathrm{d}s}\right|_{s=0} = -\left.\frac{\mathrm{d}(z(s\mathbf{V}) \circ (\mathrm{Id} + s\mathbf{V}))}{\mathrm{d}s}\right|_{s=0} = -\dot{z}(\Gamma_0; \mathbf{V})|_{\Gamma_0^w}. \tag{3.9}$$

Therefore, to get a well-defined operator $\mathrm{d}\mathbf{S}_p$ we need to show that the material derivative $\dot{z}(\Gamma_0; \mathbf{V})$ exists. First we show the existence of the material derivative $\dot{\Psi}(\Omega_0; \mathbf{V})$ using the implicit function theorem. We need the following regularity result for (3.1):

Lemma 3.4. *Let Ω_0 be of class $C^{3,1}$ and assume that $g \in H^{\frac{5}{2}}(\Gamma_0)$. Then, there exists a unique $\Psi(\theta) \in H^2(\Omega_\theta)$ for every $\theta \in \Theta^2$. Furthermore, $\Psi(0) \in H^3(\Omega_0)$.*

Proof. For $\theta \in \Theta^2$, Ω_θ is of class $C^{2,1}$ and $g \circ (\mathrm{Id} + \theta)^{-1} \in H^{\frac{3}{2}}(\Gamma_\theta)$. Then, Lemma B.5 yields $\Psi(\theta) \in H^2(\Omega_\theta)$. Furthermore, Ω_0 is of class $C^{3,1}$ and $g \in H^{\frac{5}{2}}(\Gamma_0)$ which yields $\Psi(0) \in H^3(\Omega_0)$. □

To apply the implicit function theorem we require that the Laplace operator induces an isomorphism:

Lemma 3.5. *The Laplace operator $\Delta : H^2(\Omega_0) \cap H_0^1(\Omega_0) \to L^2(\Omega_0)$ is an isomorphism between the given spaces.*

Proof. The operator is clearly linear. Let $f \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$ then $\Delta f \in L^2(\Omega_0)$. On the other hand let $h \in L^2(\Omega_0)$, then there exists a unique solution $f \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$ of $\Delta f = h$ (see [24]). □

Now, we can show the existence of the material derivative $\dot{\Psi}(\Omega_0; \mathbf{V})$ of $\Psi(\theta)$. The proof relies on the implicit function theorem and the idea can be found in [21,22].

Lemma 3.6. *Suppose that for the solution of problem (3.1), $\Psi(\theta) \in H^2(\Omega_\theta)$ holds for $\theta \in \Theta^2$. Then, the material derivative $\dot{\Psi}(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ exists for all directions $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$.*

Proof. Let $\tilde{g} \in H^3(\Omega_0)$ be a continuation with $\tilde{g}|_{\Gamma_0} = g$ which exists due to ([24], Prop. 8.8). Let us define the function

$$\begin{aligned} F : \Theta^2 \times H^2(\Omega_0) \cap H_0^1(\Omega_0) &\to L^2(\Omega_0) \\ (\theta, u) &\mapsto \Delta_\theta u + \Delta_\theta \tilde{g}. \end{aligned} \tag{3.10}$$

See Lemma A.7 for the definition of the pulled-back Laplacian Δ_θ.

Let $\theta \in \Theta^2$. Then,

$$\Delta\Psi(\theta) = 0 \qquad \text{in } \Omega_\theta = (\mathrm{Id} + \theta)(\Omega_0) \tag{3.11}$$

and thus

$$(\Delta\Psi(\theta)) \circ (\mathrm{Id} + \theta) = 0 \qquad \text{in } \Omega_0. \tag{3.12}$$

Using Lemma A.7 this implies

$$\Delta_\theta(\Psi(\theta) \circ (\mathrm{Id} + \theta)) = 0 \qquad \text{in } \Omega_0, \tag{3.13}$$

where $\Psi(\theta) \circ (\mathrm{Id} + \theta) - \tilde{g} \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$ and thus

$$F(\theta, \Psi(\theta) \circ (\mathrm{Id} + \theta) - \tilde{g}) = 0. \tag{3.14}$$

Let $u_0 := \Psi(0) - \tilde{g} \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$. Then, $F(0, u_0) = 0$ and

$$D_2 F(0, u_0) = \Delta : H^2(\Omega_0) \cap H_0^1(\Omega_0) \to L^2(\Omega) \tag{3.15}$$

is an isomorphism by Lemma 3.5. Furthermore, from ([21], (1.3)) we know that the operator Δ_θ is differentiable with respect to θ at $\theta = 0$ and thus that F is differentiable at $\theta = 0$.

Then, because of the Implicit Function Theorem A.10 there exists a unique $\mathcal{G} : \Theta^2 \to H^2(\Omega_0) \cap H^1_0(\Omega_0)$ which is differentiable at $\theta = 0$. Then (3.14) implies

$$\mathcal{G}(\theta) = \Psi(\theta) \circ (\mathrm{Id} + \theta) - \tilde{g} \tag{3.16}$$

for $\theta \in \Theta^2$ and

$$\Psi(\theta) \circ (\mathrm{Id} + \theta) = \mathcal{G}(\theta) + \tilde{g} \tag{3.17}$$

is differentiable with respect to θ at $\theta = 0$ and the derivative lies in $H^2(\Omega_0)$. Thus the material derivative $\dot{\Psi}(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ exists for $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$. $\square$

This yields the well-definedness of the linearized shape operator:

Lemma 3.7. *The material derivative* $\dot{z}(\Gamma_0; \mathbf{V}) \in H^{\frac{1}{2}}(\Gamma_0)$ *exists for every* $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$. *Thus the operator* $\mathrm{d}\mathbf{S}_p$ *is well-defined.*

Proof. Let $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$. We know from Lemma 3.6 that $\dot{\Psi}(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ which implies the existence of $\dot{z}(\Gamma_0; \mathbf{V}) \in H^{\frac{1}{2}}(\Gamma_0)$ (see [10]). For $\mathbf{V} \in \mathcal{V}^2$ we have $\mathrm{d}\mathbf{S}_p(\mathbf{V}) = -\dot{z}(\Gamma_0; \mathbf{V}) \in L^2(\Gamma_0^w)$ and the operator is well-defined. $\square$

3.3. Existence of the shape derivative

Computing the operator $\mathrm{d}\mathbf{S}_p$ in an explicit way can be done *via* the shape derivative. The existence of the shape derivative can be derived from the existence of the material derivative and the following Lemma gives an explicit form for $\Psi'(\Omega_0; \mathbf{V})$.

Lemma 3.8. *For* $\theta \in \Theta^2$ *let* $\Psi(\theta) \in H^2(\Omega_\theta)$ *be the solution of* (3.1), *then for* $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$ *the shape derivative* $\Psi'(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ *exists and can be computed as the solution of*

$$\begin{aligned} \Delta\Psi'(\Omega_0; \mathbf{V}) &= 0 && \text{in } \Omega_0 \\ \Psi'(\Omega_0; \mathbf{V}) &= 0 && \text{on } \Gamma_0^{in} \\ \Psi'(\Omega_0; \mathbf{V}) &= -(\mathbf{n} \cdot \mathbf{V})\partial_{\mathbf{n}}\Psi(0) && \text{on } \Gamma_0^w. \end{aligned} \tag{3.18}$$

Proof. By Lemma 3.4, $\Psi(0) \in H^3(\Omega_0)$ and by Lemma 3.6 the material derivative $\dot{\Psi}(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ exists for $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$. Then, by Definition A.4 the shape derivative $\Psi'(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ exists. Furthermore, ([22], Prop. 3.1) yields that $\Psi'(\Omega_0; \mathbf{V})$ solves (3.18). $\square$

Lemma 3.9. *For* $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$ *and* $s \geq 0$ *sufficiently small it holds that* $z(s\mathbf{V}) = \partial_{\mathbf{n}}\Psi(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}} \in H^{\frac{1}{2}}(\Gamma_{s\mathbf{V}})$. *The shape derivative* $z'(\Gamma_0; \mathbf{V}) \in H^{\frac{1}{2}}(\Gamma_0)$ *exists and is given on the wall boundaries by*

$$z'(\Gamma_0; \mathbf{V}) = \partial_{\mathbf{n}}\Psi'(\Omega_0; \mathbf{V})|_{\Gamma_0^w} - \kappa z(0)(\mathbf{n} \cdot \mathbf{V}) \qquad \text{on } \Gamma_0^w. \tag{3.19}$$

Proof. Let $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$. We have shown in Lemma 3.7 that the material derivative $\dot{z}(\Gamma_0; \mathbf{V}) \in H^{\frac{1}{2}}(\Gamma_0)$ exists. Furthermore, we know that $\Psi(0) \in H^3(\Omega_0)$ and thus $z(0) \in H^{\frac{3}{2}}(\Gamma_0)$ by the trace theorem (see [24]). Then, Definition A.5 yields the existence of $z'(\Gamma_0; \mathbf{V}) \in H^{\frac{1}{2}}(\Gamma_0)$.

Next, we show that the shape derivative has the given form on the wall boundaries. Therefore, let $\mathbf{V} \in C^{2,1}(\mathbb{R}^2, \mathbb{R}^2)$ be given. Let the smooth test function

$$\phi \in \{\phi \in C^\infty(\mathbb{R}^2); \partial_{\mathbf{n}}\phi = 0 \text{ on } \Gamma_0;\ \phi = 0 \text{ on } \Gamma_0^{in}\} \tag{3.20}$$

be arbitrary. For $s \geq 0$ and sufficiently small, integration by parts yields

$$\begin{aligned} 0 &= \int_{\Omega_{s\mathbf{V}}} \Delta\Psi(s\mathbf{V})\phi \,\mathrm{d}x \\ &= -\int_{\Omega_{s\mathbf{V}}} \nabla\Psi(s\mathbf{V}) \cdot \nabla\phi \,\mathrm{d}x + \int_{\Gamma_{s\mathbf{V}}} z(s\mathbf{V})\phi \,\mathrm{d}s. \end{aligned} \tag{3.21}$$

Using Lemmas A.8 and A.9 to differentiate with respect to s yields

$$-\int_{\Omega_0} \nabla\Psi'(\Omega_0;\mathbf{V})\cdot\nabla\phi \,\mathrm{d}x - \int_{\Gamma_0} \nabla\Psi(0)\cdot\nabla\phi \,(\mathbf{n}\cdot\mathbf{V}) \,\mathrm{d}s + \int_{\Gamma_0} (z'(\Gamma_0;\mathbf{V})\phi + (z(0)\underbrace{\partial_{\mathbf{n}}\phi}_{=0} + \kappa z(0)\phi)\,(\mathbf{n}\cdot\mathbf{V}))\,\mathrm{d}s = 0. \tag{3.22}$$

By Lemma 3.8 we have $\Delta\Psi'(\Omega_0;\mathbf{V}) = 0$ in Ω_0 and integration by parts yields

$$\int_{\Omega_0} \nabla\Psi'(\Omega_0;\mathbf{V}) \cdot \nabla\phi \,\mathrm{d}x = \int_{\Gamma_0^w} \partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V})\phi \,\mathrm{d}s. \tag{3.23}$$

On the other hand, it holds

$$\begin{aligned} \int_{\Gamma_0} \nabla\Psi(0) \cdot \nabla\phi \,(\mathbf{n}\cdot\mathbf{V}) \,\mathrm{d}s &= \int_{\Gamma_0^w} \partial_{\mathbf{n}}\Psi(0)\,\partial_{\mathbf{n}}\phi\,(\mathbf{n}\cdot\mathbf{V})\,\mathrm{d}s + \int_{\Gamma_0^w} \partial_{\boldsymbol{\tau}}\Psi(0)\,\partial_{\boldsymbol{\tau}}\phi\,(\mathbf{n}\cdot\mathbf{V})\,\mathrm{d}s \\ &= 0, \end{aligned} \tag{3.24}$$

where we have used $\mathbf{n}\cdot\mathbf{V} = 0$ on Γ_0^{in} and $\partial_{\mathbf{n}}\phi = 0$, $\partial_{\boldsymbol{\tau}}\Psi(0) = \partial_{\boldsymbol{\tau}} g = 0$ on Γ_0^w. Then, plugging (3.23) and (3.24) into (3.22) and using that ϕ vanishes on Γ_0^{in} yields

$$\int_{\Gamma_0^w} (-\partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V}) + z'(\Gamma_0;\mathbf{V}) + \kappa z(0)(\mathbf{n}\cdot\mathbf{V}))\phi \,\mathrm{d}s = 0. \tag{3.25}$$

Finally since $\phi|_{\Gamma_0^w}$ is arbitrary in $C_0^\infty(\Gamma_0^w)$ and since $C_0^\infty(\Gamma_0^w)$ is dense in $L^2(\Gamma_0^w)$ we conclude

$$z'(\Gamma_0;\mathbf{V}) = \partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V})|_{\Gamma_0^w} - \kappa z(0)(\mathbf{n}\cdot\mathbf{V}) \qquad \text{on } \Gamma_0^w. \tag{3.26}$$

□

Lemma 3.10. *The linearized shape operator* $\mathrm{d}\mathbf{S}_p$ *is well-defined and given by*

$$\begin{aligned} \mathrm{d}\mathbf{S}_p : \mathcal{V}^2 &\to L^2(\Gamma_0^w) \\ \mathbf{V} &\mapsto -\partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V})|_{\Gamma_0^w} + \kappa z(0)(\mathbf{n}\cdot\mathbf{V}), \end{aligned} \tag{3.27}$$

where $\Psi'(\Omega_0;\mathbf{V})$ *is the solution of*

$$\begin{aligned} \Delta\Psi'(\Omega_0;\mathbf{V}) &= 0 && \text{in } \Omega_0 \\ \Psi'(\Omega_0;\mathbf{V}) &= 0 && \text{on } \Gamma_0^{in} \\ \Psi'(\Omega_0;\mathbf{V}) &= -(\mathbf{n}\cdot\mathbf{V})\partial_{\mathbf{n}}\Psi(0) && \text{on } \Gamma_0^w. \end{aligned} \tag{3.28}$$

Proof. We have shown in Lemma 3.7 that the material derivative of z exists and thus that the operator is well-defined. Let $\mathbf{V} \in \mathcal{V}^2$. Remember that by definition $\mathbf{V}$ is normal on Γ_0. We conclude using Definition A.5 and Lemma 3.9

$$\begin{aligned} \mathrm{d}\mathbf{S}_p(\mathbf{V}) &= -\dot{z}(\Gamma_0;\mathbf{V}) = -z'(\Gamma_0;\mathbf{V}) + \partial_{\boldsymbol{\tau}} z(0)(\boldsymbol{\tau}\cdot\mathbf{V}) \\ &= -\partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V})|_{\Gamma_0^w} + \kappa z(0)(\mathbf{n}\cdot\mathbf{V}). \end{aligned} \tag{3.29}$$

□

3.4. Approximate controllability

We have derived the linearized potential flow shape operator and use it to show our approximate controllability result. To do this we need the following uniqueness lemma:

Lemma 3.11. *Assume that the curvature $\kappa \in C^0(\Gamma_0)$ is nonnegative, i.e., $\kappa \geq 0$ on Γ_0^w. If $\phi \in H^2(\Omega_0)$ solves*

$$\begin{aligned} \Delta\phi &= 0 && \text{in } \Omega_0 \\ \phi &= 0 && \text{on } \Gamma_0^{in} \\ \partial_{\mathbf{n}}\phi + \kappa\phi &= 0 && \text{on } \Gamma_0^w \end{aligned} \tag{3.30}$$

then $\phi = 0$.

Proof. Define the space $V := \{y \in H^1(\Omega_0); y|_{\Gamma_0^{in}} = 0\}$. Let $\phi \in H^2(\Omega_0)$ solve (3.30). Testing the equation with ϕ yields after integration by parts

$$\begin{aligned} 0 &= -\int_{\Omega_0} \Delta\phi\,\phi\,\mathrm{d}x \\ &= \int_{\Omega_0} \|\nabla\phi\|^2\,\mathrm{d}x + \int_{\Gamma_0^w} \kappa\phi^2\,\mathrm{d}s. \end{aligned} \tag{3.31}$$

Due to $\kappa \geq 0$ this implies $\phi \equiv$ *const* a.e. in Ω_0 and the Dirichlet condition yields $\phi \equiv 0$ a.e. in Ω_0. □

Finally, we have everything at hand to show the approximate controllability for $\mathrm{d}\mathbf{S}_p$ using an adjoint argument.

Proof of Theorem 3.2. Define

$$H^{\frac{3}{2}}_{i=0}(\Gamma_0) = \{\mu \in H^{\frac{3}{2}}(\Gamma_0); \mu = 0 \text{ on } \Gamma_0^{in}\} \tag{3.32}$$

and for $\mu \in H^{\frac{3}{2}}_{i=0}(\Gamma_0)$ let $\phi(\mu) \in H^2(\Omega_0)$ be the unique solution of the adjoint problem

$$\begin{aligned} \Delta\phi(\mu) &= 0 && \text{in } \Omega_0 \\ \phi(\mu) &= \mu && \text{on } \Gamma_0, \end{aligned} \tag{3.33}$$

where uniqueness and regularity follow from Lemma B.5. For $(\mathbf{V}, \mu) \in \mathcal{V}^2 \times H^{\frac{3}{2}}_{i=0}(\Gamma_0)$ integration by parts yields

$$\begin{aligned} 0 &= \int_{\Omega_0} \Delta\Psi'(\Omega_0;\mathbf{V})\phi(\mu)\,\mathrm{d}x \\ &= \int_{\Omega_0} \Psi'(\Omega_0;\mathbf{V})\Delta\phi(\mu)\,\mathrm{d}x + \int_{\Gamma_0^w} \partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V})\phi(\mu)\,\mathrm{d}s - \int_{\Gamma_0^w} \Psi'(\Omega_0;\mathbf{V})\partial_{\mathbf{n}}\phi(\mu)\,\mathrm{d}s \end{aligned} \tag{3.34}$$

and therefore

$$\int_{\Gamma_0^w} \partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V})\mu\,\mathrm{d}s = -\int_{\Gamma_0^w} (\mathbf{n}\cdot\mathbf{V})\partial_{\mathbf{n}}\Psi(0)\partial_{\mathbf{n}}\phi(\mu)\,\mathrm{d}s. \tag{3.35}$$

Now, assume that $\mu \in H^{\frac{3}{2}}_{i=0}(\Gamma_0)$ is arbitrary where

$$\int_{\Gamma_0^w} \mathrm{d}\mathbf{S}_p(\mathbf{V})\mu\,\mathrm{d}s = 0 \qquad \text{for all } \mathbf{V} \in \mathcal{V}^2 \tag{3.36}$$

holds. Then, we conclude using (3.29) and (3.35)

$$
\begin{aligned}
0 &= \int_{\Gamma_0^w} \mathrm{d}\mathbf{S}_p(\mathbf{V})\mu \,\mathrm{d}s \\
&= -\int_{\Gamma_0^w} \partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V})\mu \,\mathrm{d}s + \int_{\Gamma_0^w} \kappa\partial_{\mathbf{n}}\Psi(0)(\mathbf{n}\cdot\mathbf{V})\mu \,\mathrm{d}s \\
&= \int_{\Gamma_0^w} (\mathbf{n}\cdot\mathbf{V})\partial_{\mathbf{n}}\Psi(0)\,\partial_{\mathbf{n}}\phi(\mu)\,\mathrm{d}s + \int_{\Gamma_0^w} (\mathbf{n}\cdot\mathbf{V})\partial_{\mathbf{n}}\Psi(0)\,\kappa\phi(\mu)\,\mathrm{d}s \\
&= \int_{\Gamma_0^w} (\mathbf{n}\cdot\mathbf{V})\partial_{\mathbf{n}}\Psi(0)\,\big(\partial_{\mathbf{n}}\phi(\mu) + \kappa\phi(\mu)\big)\,\mathrm{d}s.
\end{aligned} \tag{3.37}
$$

Now, by assumption $\partial_{\mathbf{n}}\Psi(0) = -\mathbf{S}_p(0) \neq 0$ a.e. on Γ_0^w, therefore,

$$
\{(\mathbf{n}\cdot\mathbf{V})(\partial_{\mathbf{n}}\Psi(0))|_{\Gamma_0^w};\mathbf{V}\in\mathcal{V}^2\} \tag{3.38}
$$

is dense in $L^2(\Gamma_0^w)$ and we conclude

$$
\partial_{\mathbf{n}}\phi(\mu) + \kappa\phi(\mu) = 0 \qquad \text{on } \Gamma_0^w. \tag{3.39}
$$

This leads to a problem independent of μ:

$$
\begin{aligned}
\Delta\phi(\mu) &= 0 && \text{in } \Omega_0 \\
\phi(\mu) &= 0 && \text{on } \Gamma_0^{in} \\
\partial_{\mathbf{n}}\phi(\mu) + \kappa\phi(\mu) &= 0 && \text{on } \Gamma_0^w.
\end{aligned} \tag{3.40}
$$

Lemma 3.11 yields that $\phi(\mu) = 0$ is the only solution which implies $\mu = \phi(\mu)|_{\Gamma_0} = 0$. Thus we have shown that

$$
\mathcal{H}_p := \left\{\mu \in H^{\frac{3}{2}}_{i=0}(\Gamma_0^w);\int_{\Gamma_0^w} \mathrm{d}\mathbf{S}_p(\mathbf{V})\mu\,\mathrm{d}s = 0 \text{ for all } \mathbf{V}\in\mathcal{V}^2\right\} \subset \{0\} \tag{3.41}
$$

with $H^{\frac{3}{2}}_{i=0}(\Gamma_0^w) := H^{\frac{3}{2}}_{i=0}(\Gamma_0)|_{\Gamma_0^w}$.

Now, by definiton

$$
\mathrm{im}(\mathrm{d}\mathbf{S}_p)^\perp = \left\{\mu \in L^2(\Gamma_0^w);\int_{\Gamma_0^w} \mathrm{d}\mathbf{S}_p(\mathbf{V})\mu\,\mathrm{d}s = 0 \text{ for all } \mathbf{V}\in\mathcal{V}^2\right\} \tag{3.42}
$$

holds. We know that $C_0^\infty(\Gamma_0^w) \subset H^{\frac{3}{2}}_{i=0}(\Gamma_0^w) \subset L^2(\Gamma_0^w)$ and that $C_0^\infty(\Gamma_0^w)$ is dense in $L^2(\Gamma_0^w)$ (see [24]). Thus $H^{\frac{3}{2}}_{i=0}(\Gamma_0^w)$ is also dense in $L^2(\Gamma_0^w)$. Therefore, if $\mu \in \mathrm{im}(\mathrm{d}\mathbf{S}_p)^\perp$ there exists a sequence $\mu_i \in \mathcal{H}_p$ which converges to μ and because $\mathcal{H}_p$ is closed $\mu \in \mathcal{H}_p$. Thus

$$
\mathrm{im}(\mathrm{d}\mathbf{S}_p)^\perp \subset \mathcal{H}_p \subset \{0\} \tag{3.43}
$$

and hence $\mathrm{im}(\mathrm{d}\mathbf{S}_p)^\perp = \{0\}$. Finally Lemma 2.3 yields that $\mathrm{d}\mathbf{S}_p$ is approximately controllable. □

Thus, we have shown that the linearized shape operator of this potential flow problem is approximately controllable.

4. Stokes flow

We want to continue with an operator based on the Stokes equation, which maps to the wall shear stress. This operator is motivated by our application of designing optimal distributor geometries for polymer spin packs. We want to generate a better understanding on the inverse problem of finding a flow geometry with a certain wall shear stress profile. Especially, we want to explore whether the space of reachable profiles is rather large or small. We show that the operator $d\mathbf{S}_s$ is approximate controllable in the sense of Theorem 4.2. This backs our expectations on the numerics and we can hope to design distributor geometries with a wall shear stress close to the desired target stress.

4.1. Definition of the shape operator and problem statement

Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded domain of class $C^{6,1}$ and let $g \in H^{5+\frac{1}{2}}(\Gamma_0)$ be given with $\partial_{\boldsymbol{\tau}} g|_{\Gamma_0^w} = 0$. See Remark 4.3 for a justification of the high regularity requirement. We define the Stokes flow shape operator $\mathbf{S}_s$ by

$$\begin{aligned} \mathbf{S}_s : \mathcal{D}^4 &\to L^2(\Gamma_0^w) \\ \Omega_\theta &\mapsto (\omega(\theta)|_{\Gamma_\theta^w}) \circ (\mathrm{Id} + \theta). \end{aligned} \tag{4.1}$$

For Θ^4 the stream function $\Psi(\theta)$ and vorticity $\omega(\theta)$ are the solutions of

$$\begin{aligned} \Delta\Psi(\theta) &= -\omega(\theta) && \text{in } \Omega_\theta \\ \Delta\omega(\theta) &= 0 && \text{in } \Omega_\theta \\ \Psi(\theta) &= g \circ (\mathrm{Id} + \theta)^{-1} && \text{on } \Gamma_\theta \\ \partial_{\mathbf{n}}\Psi(\theta) &= 0 && \text{on } \Gamma_\theta \end{aligned} \tag{4.2}$$

where the existence and regularity is shown in Lemma 4.4.

Remark 4.1. The flow velocity is given by

$$\mathbf{u}(\theta) = \begin{pmatrix} \partial_2\Psi(\theta) \\ -\partial_1\Psi(\theta) \end{pmatrix} \tag{4.3}$$

and $\mathbf{u}(\theta)$ solves Stokes equation (see [3])

$$\begin{aligned} -\Delta\mathbf{u}(\theta) + \nabla p &= 0 && \text{in } \Omega_\theta \\ \operatorname{div} \mathbf{u}(\theta) &= 0 && \text{in } \Omega_\theta \end{aligned} \tag{4.4}$$

with boundary conditions

$$\begin{aligned} \boldsymbol{\tau} \cdot \mathbf{u}(\theta) &= -\partial_{\mathbf{n}}\Psi(\theta) = 0 && \text{on } \Gamma_\theta \\ \mathbf{n} \cdot \mathbf{u}(\theta) &= \partial_{\boldsymbol{\tau}}\Psi(\theta) = \partial_{\boldsymbol{\tau}}(g \circ (\mathrm{Id} + \theta)^{-1}) && \text{on } \Gamma_\theta \end{aligned} \tag{4.5}$$

and especially $\mathbf{n} \cdot \mathbf{u}(\theta)|_{\Gamma_\theta^w} = 0$ by definition of g. Furthermore, $\mathbf{S}_s$ maps to the wall shear stress $\sigma(\theta) = \omega(\theta)|_{\Gamma_\theta^w}$.

We show that the linearized shape operator $\mathrm{d}\mathbf{S}_s$ is well-defined and given by

$$\begin{aligned} \mathrm{d}\mathbf{S}_s : \mathcal{V}^4 &\to L^2(\Gamma_0^w) \\ \mathbf{V} &\mapsto \omega'(\Omega_0; \mathbf{V})|_{\Gamma_0^w} + \partial_{\mathbf{n}}\omega(0)(\mathbf{n} \cdot \mathbf{V}), \end{aligned} \tag{4.6}$$

where $\Psi'(\Omega_0;\mathbf{V})$ and $\omega'(\Omega_0;\mathbf{V})$ are the solution (see Lem. 4.8) of

$$\begin{aligned} \Delta\Psi'(\Omega_0;\mathbf{V}) &= -\omega'(\Omega_0;\mathbf{V}) && \text{in } \Omega_0 \\ \Delta\omega'(\Omega_0;\mathbf{V}) &= 0 && \text{in } \Omega_0 \\ \Psi'(\Omega_0;\mathbf{V}) &= 0 && \text{on } \Gamma_0 \\ \partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V}) &= 0 && \text{on } \Gamma_0^{in} \\ \partial_{\mathbf{n}}\Psi'(\Omega_0;\mathbf{V}) &= (\mathbf{n}\cdot\mathbf{V})\omega(0) && \text{on } \Gamma_0^{w}. \end{aligned} \tag{4.7}$$

In the rest of this section we establish the existence of $\mathrm{d}\mathbf{S}_s$ and prove the following result about the approximate controllability of the linearized shape operator:

Theorem 4.2. *Let Ω_0 be bounded and of class $C^{6,1}$ and assume that $\mathbf{S}_s(0) \neq 0$ on Γ_0^w. Then, the operator $\mathrm{d}\mathbf{S}_s : \mathcal{V}^4 \to L^2(\Gamma_0^w)/\mathcal{Z}_{\partial_\mathbf{n}}$ is approximately controllable. Here $\mathcal{Z}_{\partial_\mathbf{n}} = \{\partial_\mathbf{n}\phi|_{\Gamma_0^w} \in L^2(\Gamma_0^w); \phi \in H^4(\Omega_0)$ solution of (4.28)$\}$ is a finite dimensional subspace of $L^2(\Gamma_0^w)$.*

Remark 4.3. The assumptions of this section include a very high regularity requirement of $C^{6,1}$ for the reference domain Ω_0. For the well-definedness of the operator $\mathbf{S}_s$ itself, $C^{4,1}$ would suffice, because this would provide the existence of the trace of $\omega(\theta)$. It is also true that in many parts of this section the regularity assumptions can be relaxed by applying arguments for the weak formulation. However, a key part for the final proof is the coercivity (see Def. B.2) of the bilinear form (4.32), which due to [24] does require $c_{11} \in C^1(\bar{\Omega}_0)$ for the coefficient of the boundary form. And by definition of that coefficient this requires $C^{6,1}$ for Ω_0 (*cf.* Lem. 4.12).

4.2. Existence of the material derivative

To prove the well-posedness of the linearized shape operator $d\mathbf{S}_s$ let us define

$$z(s\mathbf{V}) := \omega(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}} = \Delta\Psi(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}} \tag{4.8}$$

for $\mathbf{V} \in C^{4,1}(\mathbb{R}^2,\mathbb{R}^2)$ and $s \geq 0$ sufficiently small. Again, our first task is to show the existence of the material derivative $\dot{\Psi}(\Omega_0;\mathbf{V})$ of the stream function as the solution of the biharmonic problem

$$\begin{aligned} \Delta\Delta\Psi(\theta) &= 0 && \text{in } \Omega_\theta \\ \Psi(\theta) &= g\circ(\mathrm{Id}+\theta)^{-1} && \text{on } \Gamma_\theta \\ \partial_\mathbf{n}\Psi(\theta) &= 0 && \text{on } \Gamma_\theta. \end{aligned} \tag{4.9}$$

We start by stating the standard regularity result:

Lemma 4.4. *For $\theta \in \Theta^4$ let $\Psi(\theta)$ be the solution of problem (4.9), then $\Psi(\theta) \in H^4(\Omega_\theta)$. Furthermore, $\Psi(0) \in H^6(\Omega_0)$.*

Proof. Let $\theta \in \Theta^4$, then $\Omega_\theta \in C^{4,1}$ and $g\circ(\mathrm{Id}+\theta)^{-1} \in H^{5+\frac{1}{2}}(\Gamma_\theta)$. Then, Lemma B.6 implies $\Psi(\theta) \in H^4(\Omega_\theta)$. Furthermore, since Ω_0 is of class $C^{6,1}$ and $g \in H^{5+\frac{1}{2}}(\Gamma_0)$ Lemma B.6 implies $\Psi(0) \in H^6(\Omega_0)$. □

In the same way as for the Laplace operator (*cf.* Lem. 3.5) the elliptic existence and regularity theory yields:

Lemma 4.5. *The biharmonic operator $\Delta\Delta : H^4(\Omega_0)\cap H_0^2(\Omega_0) \to L^2(\Omega_0)$ is an isomorphism between the given spaces.*

Again, we use the implicit function theorem to show the existence of the material derivative (*cf.* [21,22]).

Lemma 4.6. *Suppose that the solution of (4.2) fulfills $\Psi(\theta) \in H^4(\Omega_\theta)$ for $\theta \in \Theta^4$. Then, the material derivative $\dot{\Psi}(\Omega_0;\mathbf{V}) \in H^4(\Omega_0)$ exists for all directions $\mathbf{V} \in C^{4,1}(\mathbb{R}^2,\mathbb{R}^2)$.*

Proof. Let $\tilde{g} \in H^6(\Omega_0)$ be a continuation with $\tilde{g}|_{\Gamma_0} = g$ and $\partial_{\mathbf{n}} \tilde{g}|_{\Gamma_0} = 0$ which exists due to ([24], Prop. 8.8). Define the function

$$\begin{aligned} F : \Theta^4 \times H^4(\Omega_0) \cap H_0^2(\Omega_0) &\to L^2(\Omega_0) \\ (\theta, u) &\mapsto \Delta_\theta \Delta_\theta u + \Delta_\theta \Delta_\theta \tilde{g}. \end{aligned} \tag{4.10}$$

Let $\theta \in \Theta^4$. Then, it holds

$$\Delta\Delta\Psi(\theta) = 0 \qquad \text{in } \Omega_\theta = (\mathrm{Id} + \theta)(\Omega_0) \tag{4.11}$$

and thus

$$(\Delta\Delta\Psi(\theta)) \circ (\mathrm{Id} + \theta) = 0 \qquad \text{in } \Omega_0. \tag{4.12}$$

Using Lemma A.7 this implies

$$\Delta_\theta \Delta_\theta (\Psi(\theta) \circ (\mathrm{Id} + \theta)) = 0 \qquad \text{in } \Omega_0, \tag{4.13}$$

where $\Psi(\theta) \circ (\mathrm{Id} + \theta) - \tilde{g} \in H^4(\Omega_0) \cap H_0^2(\Omega_0)$ and thus

$$F(\theta, \Psi(\theta) \circ (\mathrm{Id} + \theta) - \tilde{g}) = 0. \tag{4.14}$$

Let $u_0 := \Psi(0) - \tilde{g} \in H^4(\Omega_0) \cap H_0^2(\Omega_0)$. Then, $F(0, u_0) = 0$ and

$$D_2 F(0, u_0) = \Delta\Delta : H^4(\Omega_0) \cap H_0^2(\Omega_0) \to L^2(\Omega) \tag{4.15}$$

is an isomorphism by Lemma 4.5. Furthermore, from ([21], (1.3)) we conclude that the operator F is differentiable at $\theta = 0$.

Then, because of the Implicit Function Theorem A.10 there exists a unique $\mathcal{G} : \Theta^4 \to H^4(\Omega_0) \cap H_0^2(\Omega_0)$ which is also differentiable at $\theta = 0$. Then equation (4.14) implies

$$\mathcal{G}(\theta) = \Psi(\theta) \circ (\mathrm{Id} + \theta) - \tilde{g} \tag{4.16}$$

for $\theta \in \Theta^4$ and

$$\Psi(\theta) \circ (\mathrm{Id} + \theta) = \mathcal{G}(\theta) + \tilde{g} \tag{4.17}$$

is differentiable with respect to θ at $\theta = 0$ where the derivative lies in $H^4(\Omega_0)$. Thus, the material derivative $\dot{\Psi}(\Omega_0; \mathbf{V}) \in H^4(\Omega_0)$ exists for $\mathbf{V} \in C^{4,1}(\mathbb{R}^2, \mathbb{R}^2)$. □

Since we have established the existence of $\dot{\Psi}(\Omega_0; \mathbf{V})$, the existence of $\dot{\omega}(\Omega_0; \mathbf{V})$ and $\dot{z}(\Gamma_0; \mathbf{V})$ with $z(s\mathbf{V}) = \omega(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}}$ follow directly:

Lemma 4.7. *The material derivative $\dot{\omega}(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ exists for every $\mathbf{V} \in C^{4,1}(\mathbb{R}^2, \mathbb{R}^2)$. Let $z(s\mathbf{V}) = \omega(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}}$ for all $s \geq 0$ sufficiently small. Then, the material derivative $\dot{z}(\Gamma_0; \mathbf{V}) \in H^{\frac{3}{2}}(\Gamma_0)$ exists for $\mathbf{V} \in C^{4,1}(\mathbb{R}^2, \mathbb{R}^2)$. Thus the operator $d\mathbf{S}_s$ is well-defined.*

Proof. Let $\mathbf{V} \in C^{4,1}(\mathbb{R}^2, \mathbb{R}^2)$. By Lemma 4.6, $\dot{\Psi}(\Omega_0; \mathbf{V}) \in H^4(\Omega_0)$ which implies $\dot{\omega}(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ and thus $\dot{z}(\Gamma_0; \mathbf{V}) \in H^{\frac{3}{2}}(\Gamma_0)$ by Lemma A.3. □

4.3. Existence of the shape derivative

Since we have shown the existence of the material derivatives we get the following result for the shape derivatives.

Lemma 4.8. *For $\mathbf{V} \in C^{4,1}(\mathbb{R}^2, \mathbb{R}^2)$ the shape derivatives $\Psi'(\Omega_0; \mathbf{V}) \in H^4(\Omega_0)$ and $\omega'(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ exist. Furthermore, for $\mathbf{V} \in \mathcal{V}^4$ it is given as the solution of*

$$\begin{aligned} \Delta\Psi'(\Omega_0; \mathbf{V}) &= -\omega'(\Omega_0; \mathbf{V}) && \text{in } \Omega_0 \\ \Delta\omega'(\Omega_0; \mathbf{V}) &= 0 && \text{in } \Omega_0 \\ \Psi'(\Omega_0; \mathbf{V}) &= 0 && \text{on } \Gamma_0 \\ \partial_{\mathbf{n}}\Psi'(\Omega_0; \mathbf{V}) &= 0 && \text{on } \Gamma_0^{in} \\ \partial_{\mathbf{n}}\Psi'(\Omega_0; \mathbf{V}) &= (\mathbf{n} \cdot \mathbf{V})\omega(0) && \text{on } \Gamma_0^{w}. \end{aligned} \tag{4.18}$$

Proof. Let $\mathbf{V} \in C^{4,1}(\mathbb{R}^2, \mathbb{R}^2)$. We have shown that $(\dot{\Psi}(\Omega_0; \mathbf{V}), \dot{\omega}(\Omega_0; \mathbf{V})) \in H^4(\Omega_0) \times H^2(\Omega_0)$ exists and that $(\Psi(0), \omega(0)) \in H^6(\Omega_0) \times H^4(\Omega_0)$ by Lemma 4.4. Therefore, by Definition A.5 the shape derivative $(\Psi'(\Omega_0; \mathbf{V}), \omega'(\Omega_0; \mathbf{V})) \in H^4(\Omega_0) \times H^2(\Omega_0)$ exists.

Now, let $\mathbf{V} \in \mathcal{V}^4$. Then from ([22], Prop. 3.1) we conclude

$$\Delta\Psi'(\Omega_0; \mathbf{V}) = -\omega'(\Omega_0; \mathbf{V}) \qquad \text{in } \Omega_0 \tag{4.19}$$

and

$$\Delta\omega'(\Omega_0; \mathbf{V}) = 0 \qquad \text{in } \Omega_0. \tag{4.20}$$

For $\theta \in \Theta^4$ we have $\Psi(\theta) \circ (\mathrm{Id} + \theta) = g$ on Γ_0 and thus by definition of the material derivative

$$\dot{\Psi}(\Omega_0; \mathbf{V})|_{\Gamma_0} = 0. \tag{4.21}$$

Then,

$$\begin{aligned} \Psi'(\Omega_0; \mathbf{V})|_{\Gamma_0} &= \dot{\Psi}(\Omega_0; \mathbf{V})|_{\Gamma_0} - (\nabla\Psi(0) \cdot \mathbf{V})|_{\Gamma_0} \\ &= -(\partial_{\mathbf{n}}\Psi(0)(\mathbf{n} \cdot \mathbf{V}))|_{\Gamma_0} \\ &= 0, \end{aligned} \tag{4.22}$$

because $\mathbf{V} \in \mathcal{V}^4$ is normal and $\partial_{\mathbf{n}}\Psi(0) = 0$ on Γ_0. Finally, we deduce from ([22], (3.12))

$$\partial_{\mathbf{n}}\Psi'(\Omega_0; \mathbf{V}) = \partial_{\boldsymbol{\tau}}((\mathbf{n} \cdot \mathbf{V})\partial_{\boldsymbol{\tau}}\Psi(0)) + (\mathbf{n} \cdot \mathbf{V})\omega(0). \tag{4.23}$$

Then, $\partial_{\boldsymbol{\tau}}((\mathbf{n} \cdot \mathbf{V})\partial_{\boldsymbol{\tau}}\Psi(0))$ vanishes because $\mathbf{V} = 0$ on Γ_0^{in} and $\partial_{\boldsymbol{\tau}}\Psi(0) = 0$ on Γ_0^w. We get

$$\begin{aligned} \partial_{\mathbf{n}}\Psi'(\Omega_0; \mathbf{V}) &= 0 && \text{on } \Gamma_0^{in} \\ \partial_{\mathbf{n}}\Psi'(\Omega_0; \mathbf{V}) &= (\mathbf{n} \cdot \mathbf{V})\omega(0) && \text{on } \Gamma_0^{w}. \end{aligned} \tag{4.24}$$

□

Lemma 4.9. *For $\mathbf{V} \in C^4(\mathbb{R}^2, \mathbb{R}^2)$ let $z(s\mathbf{V}) = \omega(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}}$ for all $s \geq 0$ sufficiently small. Then, the shape derivative $z'(\Gamma_0; \mathbf{V}) \in H^{\frac{1}{2}}(\Gamma_0)$ exists and is given by*

$$z'(\Gamma_0; \mathbf{V}) = \omega'(\Omega_0; \mathbf{V})|_{\Gamma_0} + \partial_{\mathbf{n}}\omega(0)(\mathbf{n} \cdot \mathbf{V}). \tag{4.25}$$

Proof. This is a direct consequence of Lemma A.6. □

Lemma 4.10. *The linearized shape operator* $\mathrm{d}\mathbf{S}_s$ *is well-defined and given by*

$$\begin{aligned} \mathrm{d}\mathbf{S}_s : \mathcal{V}^4 &\to L^2(\Gamma_0^w) \\ \mathbf{V} &\mapsto \omega'(\Omega_0;\mathbf{V})|_{\Gamma_0^w} + \partial_{\mathbf{n}}\omega(0)(\mathbf{n}\cdot\mathbf{V}). \end{aligned} \tag{4.26}$$

Proof. We have shown in Lemma 4.7 that the operator is well-defined. Let $\mathbf{V} \in \mathcal{V}^4$. Remember that by definition $\mathbf{V}$ is normal on Γ_0. We conclude using Definition A.5 and Lemma 4.9

$$d\mathbf{S}_s(\mathbf{V}) = \dot{z}(\Gamma_0;\mathbf{V}) = z'(\Gamma_0;\mathbf{V}) + \partial_{\boldsymbol{\tau}} z(0)(\boldsymbol{\tau}\cdot\mathbf{V}) = \omega'(\Omega_0;\mathbf{V})|_{\Gamma_0^w} + \partial_{\mathbf{n}}\omega(0)(\mathbf{n}\cdot\mathbf{V}). \tag{4.27}$$

□

4.4. Approximate controllability

The approximate controllability of the operator $\mathrm{d}\mathbf{S}_s$ depends on the uniqueness question addressed in the following lemma. However, we can only show that the corresponding bilinear form is coercive but not that it is elliptic. Therefore, we have to rely on the weaker argument of Theorem B.4, which states that the homogeneous solutions form a finite dimensional subspace. In the case that zero is no eigenvalue of the corresponding representation operator, this subspace is trivial. There is no way to tell whether zero is an eigenvalue or not. We know that there are only countably many eigenvalues which do not accumulate in a finite region (see [24]).

Lemma 4.11. *Assume that* Ω_0 *is bounded and of class* $C^{4,1}$ *and* $c_{11} \in C^1(\bar{\Omega}_0)$*. We consider*

$$\begin{aligned} \Delta\Delta\phi &= 0 && \text{in } \Omega_0 \\ \phi &= 0 && \text{on } \Gamma_0 \\ \partial_{\mathbf{n}}\phi &= 0 && \text{on } \Gamma_0^{in} \\ \Delta\phi + c_{11}\partial_{\mathbf{n}}\phi &= 0 && \text{on } \Gamma_0^w \end{aligned} \tag{4.28}$$

and define $\mathcal{Z} := \{\phi \in H^4(\Omega_0); \phi \text{ solves } (4.28)\}$*. Then,* $\mathcal{Z}$ *is a finite dimensional subspace of* $H^4(\Omega_0)$*.*

Proof. Define $V := \{u \in H^2(\Omega_0); u|_{\Gamma_0} = 0; \partial_{\mathbf{n}} u|_{\Gamma_0^{in}} = 0\}$. Let $\phi \in \mathcal{Z}$ and let $\eta \in V$ be a test function. Then,

$$\begin{aligned} 0 &= \int_{\Omega_0} \Delta\Delta\phi\eta \,\mathrm{d}x \\ &= \int_{\Omega_0} \Delta\phi\Delta\eta \,\mathrm{d}x + \int_{\Gamma_0} \partial_{\mathbf{n}}\Delta\phi\eta \,\mathrm{d}s - \int_{\Gamma_0} \Delta\phi\partial_{\mathbf{n}}\eta \,\mathrm{d}s \\ &= \int_{\Omega_0} \Delta\phi\Delta\eta \,\mathrm{d}x + \int_{\Gamma_0^w} c_{11}\partial_{\mathbf{n}}\phi\partial_{\mathbf{n}}\eta \,\mathrm{d}s. \end{aligned} \tag{4.29}$$

We define the bilinear form

$$a(\varphi,\eta) := \int_{\Omega_0} \Delta\varphi\Delta\eta \,\mathrm{d}x \tag{4.30}$$

and the boundary form

$$c(\varphi,\eta) := \int_{\Gamma_0^w} c_{11}\partial_{\mathbf{n}}\varphi\partial_{\mathbf{n}}\eta \,\mathrm{d}s. \tag{4.31}$$

The space V is a closed subspace of $H^2(\Omega_0)$ with $H_0^2(\Omega_0) \subset V \subset H^2(\Omega)$ and $a(\varphi,\eta)$ is V-coercive (*cf.* Def. B.2 and [24]). Because of $c_{11} \in C^1(\bar{\Omega}_0)$ the bilinear form $a(\varphi,\eta)+c(\varphi,\eta)$ is also V-coercive (see [24]). The embedding

$V \hookrightarrow L^2(\Omega_0) \hookrightarrow V'$ is a Gelfand triple and $V \hookrightarrow L^2(\Omega_0)$ is compact (see [24]). Thus, the assumptions of Theorem B.4 hold for the weak formulation:

Find $\varphi \in V$ such that

$$a(\varphi, \eta) + c(\varphi, \eta) = 0 \qquad \text{for all } \eta \in V. \tag{4.32}$$

From Theorem B.4 we conclude that $\tilde{\mathcal{Z}} := \{\varphi \in V; \varphi \text{ solves } (4.32)\}$ is finite dimensional. Because of (4.29) we know that every $\phi \in \mathcal{Z}$ solves (4.32) and thus we conclude

$$\mathcal{Z} = \tilde{\mathcal{Z}} \cap H^4(\Omega_0) \tag{4.33}$$

which yields the result. $\square$

The next lemma shows the regularity of the coefficient appearing in the approximate controllability proof.

Lemma 4.12. *Assume that $\omega(0) \neq 0$ on Γ_0^w. Then,*

$$c_{11} := -\frac{\partial_{\mathbf{n}}\omega(0)}{\omega(0)} \in C^1(\Gamma_0^w). \tag{4.34}$$

Proof. We have shown that $\Psi(0) \in H^6(\Omega_0)$ and thus $\omega(0) \in H^4(\Omega_0)$. Then, $\omega|_{\Gamma_0} \in H^{5+\frac{1}{2}}(\Gamma_0)$ and $\partial_{\mathbf{n}}\omega|_{\Gamma_0} \in H^{\frac{5}{2}}(\Gamma_0)$. By the Lemma of Sobolev (see [24]) we have $\omega|_{\Gamma_0} \in C^1(\Gamma_0)$ and $\partial_{\mathbf{n}}\omega|_{\Gamma_0} \in C^1(\Gamma_0)$ and since ω is non-zero on Γ_0^w, $c_{11} \in C^1(\Gamma_0^w)$ holds. $\square$

Finally, we are prepared to show the main result for the operator $\mathrm{d}\mathbf{S}_s$.

Proof of Theorem 4.2. Define

$$H^{\frac{5}{2}}_{i=0}(\Gamma_0) = \{\mu \in H^{\frac{5}{2}}(\Gamma_0); \mu = 0 \text{ on } \Gamma_0^{in}\} \tag{4.35}$$

and for $\mu \in H^{\frac{5}{2}}_{i=0}(\Gamma_0)$ let $\phi(\mu) \in H^4(\Omega_0)$ be the solution of the adjoint problem

$$\begin{aligned} \Delta\Delta\phi(\mu) &= 0 && \text{in } \Omega_0 \\ \phi(\mu) &= 0 && \text{on } \Gamma_0 \\ \partial_{\mathbf{n}}\phi(\mu) &= \mu && \text{on } \Gamma_0 \end{aligned} \tag{4.36}$$

where the existence and regularity follows from Lemma B.6. For $(\mathbf{V}, \mu) \in \mathcal{V}^4 \times H^{\frac{5}{2}}_{i=0}(\Gamma_0)$ integration by parts yields

$$\begin{aligned} 0 &= \int_{\Omega_0} \Delta\Delta\Psi'(\Omega_0; \mathbf{V})\phi(\mu)\,\mathrm{d}x \\ &= \int_{\Omega_0} \Delta\Psi'(\Omega_0; \mathbf{V})\Delta\phi(\mu)\,\mathrm{d}x - \int_{\Gamma_0} \Delta\Psi'(\Omega_0; \mathbf{V})\partial_{\mathbf{n}}\phi(\mu)\,\mathrm{d}s \\ &= \int_{\Omega_0} \Psi'(\Omega_0; \mathbf{V})\underbrace{\Delta\Delta\phi(\mu)}_{=0}\,\mathrm{d}x - \int_{\Gamma_0} \underbrace{\Delta\Psi'(\Omega_0; \mathbf{V})}_{=-\omega'(\Omega_0;\mathbf{V})}\underbrace{\partial_{\mathbf{n}}\phi(\mu)}_{=\mu}\,\mathrm{d}s \\ &\quad + \int_{\Gamma_0} \underbrace{\partial_{\mathbf{n}}\Psi'(\Omega_0; \mathbf{V})}_{(\mathbf{n}\cdot\mathbf{V})\omega(0)}\Delta\phi(\mu)\,\mathrm{d}s \end{aligned} \tag{4.37}$$

and we get the identity

$$-\int_{\Gamma_0^w} \omega'(\Omega_0;\mathbf{V})\mu\,\mathrm{d}s = \int_{\Gamma_0^w} (\mathbf{n}\cdot\mathbf{V})\omega(0)\Delta\phi(\mu)\,\mathrm{d}s. \tag{4.38}$$

Now, assume that $\mu \in H^{\frac{5}{2}}_{i=0}(\Gamma_0)$ is arbitrary with

$$\int_{\Gamma_0^w} \mathrm{d}\mathbf{S}_s(\mathbf{V})\mu\,\mathrm{d}s = 0 \qquad \text{for all } \mathbf{V}\in\mathcal{V}^4. \tag{4.39}$$

We conclude

$$\begin{aligned}0 &= \int_{\Gamma_0^w} \mathrm{d}\mathbf{S}_s(\mathbf{V})\mu\,\mathrm{d}s\\ &= \int_{\Gamma_0^w} \omega'(\Omega_0;\mathbf{V})\mu\,\mathrm{d}s + \int_{\Gamma_0^w} \partial_{\mathbf{n}}\omega(0)(\mathbf{n}\cdot\mathbf{V})\mu\,\mathrm{d}s\\ &= \int_{\Gamma_0^w} (\mathbf{n}\cdot\mathbf{V})(-\omega(0)\Delta\phi(\mu) + \partial_{\mathbf{n}}\omega(0)\partial_{\mathbf{n}}\phi(\mu))\,\mathrm{d}s.\end{aligned} \tag{4.40}$$

Since $\{\mathbf{n}\cdot\mathbf{V};\mathbf{V}\in\mathcal{V}^4\}$ is dense in $L^2(\Gamma_0^w)$ we derive

$$-\omega(0)\Delta\phi(\mu) + \partial_{\mathbf{n}}\omega(0)\partial_{\mathbf{n}}\phi(\mu) = 0 \qquad \text{on } \Gamma_0^w. \tag{4.41}$$

Because of $\omega(0) = \mathbf{S}_s(0) \neq 0$ on Γ_0^w, we can define

$$c_{11} := -\frac{\partial_{\mathbf{n}}\omega(0)}{\omega(0)} \in C^1(\Gamma_0^w), \tag{4.42}$$

where the regularity follows from Lemma 4.12. This yields the uniqueness problem

$$\begin{aligned}\Delta\Delta\phi(\mu) &= 0 && \text{in } \Omega_0\\ \phi(\mu) &= 0 && \text{on } \Gamma_0\\ \partial_{\mathbf{n}}\phi(\mu) &= 0 && \text{on } \Gamma_0^{in}\\ \Delta\phi(\mu) + c_{11}\partial_{\mathbf{n}}\phi(\mu) &= 0 && \text{on } \Gamma_0^w.\end{aligned} \tag{4.43}$$

Define

$$\mathcal{Z} := \{\phi(\mu)\in H^4(\Omega_0);\phi(\mu) \text{ is solution of } (4.43)\} \tag{4.44}$$

and

$$\mathcal{Z}_{\partial_{\mathbf{n}}} := \{\mu = \partial_{\mathbf{n}}\phi|_{\Gamma_0^w};\phi\in\mathcal{Z}\}. \tag{4.45}$$

By Lemma 4.11 we know that $\mathcal{Z}$ is a finite dimensional subspace of $H^4(\Omega_0)$. Then, $\mathcal{Z}_{\partial_{\mathbf{n}}}$ is a finite dimensional subspace of $H^{\frac{5}{2}}_{i=0}(\Gamma_0^w) := H^{\frac{5}{2}}_{i=0}(\Gamma_0)|_{\Gamma_0^w}$ and thus of $L^2(\Gamma_0^w)$. Thus we have shown that

$$\mathcal{H}_s := \left\{\mu\in H^{\frac{5}{2}}_{i=0}(\Gamma_0^w);\int_{\Gamma_0^w}\mathrm{d}\mathbf{S}_s(\mathbf{V})\mu\,\mathrm{d}s = 0 \text{ for all } \mathbf{V}\in\mathcal{V}^4\right\} \subset \mathcal{Z}_{\partial_{\mathbf{n}}}. \tag{4.46}$$

Now, by definiton

$$\mathrm{im}(\mathrm{d}\mathbf{S}_s)^{\perp} = \left\{\mu\in L^2(\Gamma_0^w)/\mathcal{Z}_{\partial_{\mathbf{n}}};\int_{\Gamma_0^w}\mathrm{d}\mathbf{S}_s(\mathbf{V})\mu\,\mathrm{d}s = 0 \text{ for all } \mathbf{V}\in\mathcal{V}^4\right\} \tag{4.47}$$

holds. We know that $C_0^\infty(\Gamma_0^w) \subset H^{\frac{5}{2}}_{i=0}(\Gamma_0^w) \subset L^2(\Gamma_0^w)$ and that $C_0^\infty(\Gamma_0^w)$ is dense in $L^2(\Gamma_0^w)$ (see [24]). Thus $H^{\frac{5}{2}}_{i=0}(\Gamma_0^w)$ is also dense in $L^2(\Gamma_0^w)$. Therefore, if $\mu \in \operatorname{im}(\mathrm{d}\mathbf{S}_s)^\perp$, there exists a sequence $\mu_i \in \mathcal{H}_s/\mathcal{Z}_{\partial_\mathbf{n}}$ which converges to μ and because $\mathcal{H}_s/\mathcal{Z}_{\partial_\mathbf{n}}$ is closed $\mu \in \mathcal{H}_s/\mathcal{Z}_{\partial_\mathbf{n}}$. Thus

$$\operatorname{im}(\mathrm{d}\mathbf{S}_s)^\perp \subset \mathcal{H}_s/\mathcal{Z}_{\partial_\mathbf{n}} \subset \mathcal{Z}_{\partial_\mathbf{n}}/\mathcal{Z}_{\partial_\mathbf{n}} = \{0\} \tag{4.48}$$

and hence $\operatorname{im}(\mathrm{d}\mathbf{S}_s)^\perp = \{0\}$. Finally Lemma 2.3 yields that $\mathrm{d}\mathbf{S}_s$ is approximately controllable.

Using Lemma 2.3 we conclude that $\mathrm{d}\mathbf{S}_s$ is approximately controllable as a mapping to $L^2(\Gamma_0^w)/\mathcal{Z}_{\partial_\mathbf{n}}$. Furthermore we have shown that $\mathcal{Z}_{\partial_\mathbf{n}}$ is a finite dimensional subspace of $L^2(\Gamma_0^w)$. □

5. Conclusion

We have studied the controllability of two shape-dependent operators based on flow problems. We were able to prove approximate controllability for linearizations of these operators using an adjoint argument. For the Stokes operator we have to note that a small subspace remains which is not reachable, but this subspace is finite dimensional. Even though we have studied linearizations, we can draw conclusions for the actual operators. Having the approximate controllability property for the linearization means that we can change the observable into almost every direction by applying infinitesimal shape perturbations. Our application in view is the design of polymer distributors with specific wall shear stress profiles. Theorem 4.2 does suggest that the space of reachable wall shear stress profiles is rather large. Therefore, we can expect a good performance of the shape optimization algorithm, meaning that the optimal stress profiles lie close to the desired target stress in the L^∞-sense, even though we are only using L^2 shape optimization. This statement does agree with our numerical experience form [10], where we have solved an optimization problem based on the Stokes operator.

On the other hand it remains an open challenge to proof a kind of controllability property for the nonlinear operator $\mathbf{S}$ in a mathematical rigorous way. One could certainly make use of our results for the linearization $d\mathbf{S}$. However, one drawback of the shape deformation approach considered here is that the perturbation decreases the regularity of the shape. Therefore, the regularity requirements which were valid for Ω_0 are not fulfilled for $\Omega_\theta = (\mathrm{Id} + \theta)\Omega_0$. One could either try to increase the regularity of the perturbations even further or one could use a different characterization of the shape variations: A promising approach would certainly be the characterization through distance functions, which is described in [5].

Appendix A. Shape differentiation

We provide the concepts of material and shape derivatives and cite the essential theory on the differentiation of shape-dependent integrals. Further details can be found in [22].

Definition A.1 (Material derivative)**.** For $\mathbf{V} \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2)$ let $y(s\mathbf{V}) \in H^m(\Omega_{s\mathbf{V}})$ hold for all $s \geq 0$ sufficiently small. Then, $\dot{y}(\Omega_0; \mathbf{V})$ is called material derivative of y in direction $\mathbf{V}$ if and only if the limit

$$\dot{y}(\Omega_0; \mathbf{V}) = \lim_{s \searrow 0} \frac{1}{s}\big(y(s\mathbf{V}) \circ (\mathrm{Id} + s\mathbf{V}) - y(0)\big) \in H^m(\Omega_0) \tag{A.1}$$

exists.

The material derivative of a boundary function is defined in a similar way:

Definition A.2 (Boundary material derivative)**.** For $\mathbf{V} \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2)$ let $z(s\mathbf{V}) \in H^r(\Gamma_{s\mathbf{V}})$ hold for all $s \geq 0$ sufficiently small. Then, $\dot{z}(\Gamma_0; \mathbf{V})$ is called material derivative in direction of $\mathbf{V}$ if and only if the limit

$$\dot{z}(\Gamma_0; \mathbf{V}) = \lim_{s \searrow 0} \frac{1}{s}\big(z(s\mathbf{V}) \circ (\mathrm{Id} + s\mathbf{V}) - z(0)\big) \in H^r(\Gamma_0) \tag{A.2}$$

exists.

The following relation holds between the material derivatives and the boundary material derivative:

Lemma A.3 (from [22])**.** *Let $k \geq m \geq 1$. For $\mathbf{V} \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2)$ let $y(s\mathbf{V}) \in H^m(\Omega_{s\mathbf{V}})$ and let $z(s\mathbf{V}) = y(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}} \in H^{m-\frac{1}{2}}(\Gamma_{s\mathbf{V}})$ for all $s \geq 0$ sufficiently small. Suppose that the material derivative $\dot{y}(\Omega_0; \mathbf{V}) \in H^m(\Omega_0)$ exists. Then, the material derivative of the boundary function exists and is given by $\dot{z}(\Gamma_0; \mathbf{V}) = \dot{y}(\Omega_0; \mathbf{V})|_{\Gamma_0} \in H^{m-\frac{1}{2}}(\Gamma_0)$*

Next, we define the shape derivative. The difference between material and shape derivative is that the first is the derivative of $y(\theta) \circ (\mathrm{Id} + \theta)$ and the second the derivative of just $y(\theta)$ without the pull-back. It is convenient to derive the definition of the shape derivative from the material derivative by just subtracting the part originating from differentiating the map $(\mathrm{Id} + \theta)$. This way, we can directly derive the existence from the existence of the material derivative.

Definition A.4 (Shape derivative)**.** Let $y(0) \in H^m(\Omega_0)$ and assume that the material derivative $\dot{y}(\Omega_0; \mathbf{V}) \in H^m(\Omega_0)$ exists for $\mathbf{V} \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2)$. Then, the shape derivative in direction $\mathbf{V}$ is defined by

$$y'(\Omega_0; \mathbf{V}) := \dot{y}(\Omega_0; \mathbf{V}) - \nabla y(0) \cdot \mathbf{V} \in H^{m-1}(\Omega_0). \tag{A.3}$$

Furthermore, we can see directly from the definition that $y(0) \in H^{m+1}(\Omega_0)$ implies $y'(\Omega_0; \mathbf{V}) \in H^m(\Omega_0)$.

On the boundary we define the shape derivative in the following way:

Definition A.5 (Boundary shape derivative)**.** For a $\mathbf{V} \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2)$ let $z(s\mathbf{V}) \in H^r(\Gamma_{s\mathbf{V}})$ for all $s \geq 0$ sufficiently small and assume that the material derivative $\dot{z}(\Gamma_0; \mathbf{V}) \in H^r(\Gamma_0)$ exists. Then, the shape derivative in direction $\mathbf{V}$ is defined by

$$z'(\Gamma_0; \mathbf{V}) := \dot{z}(\Gamma_0; \mathbf{V}) - \partial_{\boldsymbol{\tau}} z(0)\, \boldsymbol{\tau} \cdot \mathbf{V} \in H^{r-1}(\Gamma_0). \tag{A.4}$$

Furthermore, if $z(0) \in H^{r+1}(\Gamma_0)$, then $z'(\Gamma_0; \mathbf{V}) \in H^r(\Gamma_0)$.

The following lemma draws a connection between shape derivatives on the domain and the boundary:

Lemma A.6 (from [22])**.** *Let $k \geq m \geq 1$. For $\mathbf{V} \in C^{k,1}(\mathbb{R}^2, \mathbb{R}^2)$ and $s \geq 0$ sufficiently small let $y(s\mathbf{V}) \in H^m(\Omega_{s\mathbf{V}})$ and $z(s\mathbf{V}) = y(s\mathbf{V})|_{\Gamma_{s\mathbf{V}}} \in H^{m-\frac{1}{2}}(\Gamma_{s\mathbf{V}})$. Suppose that $y(0) \in H^{m+1}(\Omega_0)$ and that $y'(\Omega_0; \mathbf{V}) \in H^m(\Omega_0)$ exists. Then,*

$$z'(\Gamma_0; \mathbf{V}) = y'(\Omega_0; \mathbf{V})|_{\Gamma_0} + \partial_{\mathbf{n}} y(0)(\mathbf{V} \cdot \mathbf{n}) \in H^{m-\frac{1}{2}}(\Gamma_0). \tag{A.5}$$

For the pull-back of the Laplacian the following holds:

Lemma A.7 (from [21])**.** *For $k \geq m \geq 2$ let $\theta \in \Theta^k$. Then*

$$(\Delta f) \circ (\mathrm{Id} + \theta) = \Delta_\theta (f \circ (\mathrm{Id} + \theta)) \tag{A.6}$$

for all $f \in H^m(\Omega_\theta)$, where $\Delta_\theta : H^m(\Omega_0) \to H^{m-2}(\Omega_0)$ is defined by

$$\Delta_\theta f := \sum_{i,j,l=1}^{d} M_{ij}(\theta) \frac{\partial}{\partial x_j} \left(M_{il}(\theta) \frac{\partial f}{\partial x_l} \right) \tag{A.7}$$

with $M(\theta) := \left[(D(\mathrm{Id} + \theta))^{-1} \right]^T$.

The following results provide the derivatives of integral expressions:

Lemma A.8 (Differentiation of domain integrals, see [22])**.** *Let $k \geq 1$. For $\mathbf{V} \in C^k(\mathbb{R}^2, \mathbb{R}^2)$ let $y(t\mathbf{V}) \in H^1(\Omega_{t\mathbf{V}})$ for all $t \geq 0$ sufficiently small. Let $f \in H^1(\mathbb{R}^2)$, assume that the shape derivative $y'(\Omega_0; \mathbf{V}) \in H^1(\Omega_0)$ exists and let*

$$J(t\mathbf{V}) := \int_{\Omega_{t\mathbf{V}}} y(t\mathbf{V}) f \, \mathrm{d}x. \tag{A.8}$$

Then, the derivative of J in direction $\mathbf{V}$ is given by

$$\mathrm{d}J(\mathbf{V}) := \left.\frac{\mathrm{d}J(t\mathbf{V})}{\mathrm{d}t}\right|_{t=0} = \int_{\Omega_0} y'(\Omega_0; \mathbf{V}) f \, \mathrm{d}x + \int_{\Gamma_0} y(0) f (\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}s. \tag{A.9}$$

Lemma A.9 (Differentiation of boundary integrals, see [22])**.** *Let $k \geq 2$. For $\mathbf{V} \in C^k(\mathbb{R}^2, \mathbb{R}^2)$ let $z(t\mathbf{V}) \in H^{\frac{3}{2}}(\Gamma_{t\mathbf{V}})$ for all $t \geq 0$ sufficiently small. Assume that the shape derivative $z'(\Gamma_0; \mathbf{V}) \in H^{\frac{3}{2}}(\Gamma_0)$ exits and let $f \in H^2(\mathbb{R}^2)$. Define*

$$J(t\mathbf{V}) = \int_{\Gamma_{t\mathbf{V}}} z(t\mathbf{V}) f \, \mathrm{d}s. \tag{A.10}$$

Then, the derivative of J in direction $\mathbf{V}$ is given by

$$\mathrm{d}J(\mathbf{V}) := \left.\frac{\mathrm{d}J(t\mathbf{V})}{\mathrm{d}t}\right|_{t=0} = \int_{\Gamma_0} z'(\Gamma_0; \mathbf{V}) f + (z(0) \partial_{\mathbf{n}} f + \kappa z(0) f)(\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}s. \tag{A.11}$$

In particular if $z(t\mathbf{V}) = y(t\mathbf{V})|_{\Gamma_{t\mathbf{V}}}$ with $y'(\Omega_0; \mathbf{V}) \in H^2(\Omega_0)$ we have

$$\mathrm{d}J(\mathbf{V}) = \int_{\Gamma_0} y'(\Omega_0; \mathbf{V}) f + (\partial_{\mathbf{n}} y(0) f + z(0) \partial_{\mathbf{n}} f + \kappa z(0) f)(\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}s. \tag{A.12}$$

Proof. From [22] we know that

$$\mathrm{d}J(\mathbf{V}) = \int_{\Gamma_0} (zf)'(\Gamma_0; \mathbf{V}) + \kappa z(0) f (\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}s \tag{A.13}$$

$$= \int_{\Gamma_0} z'(\Gamma_0; \mathbf{V}) f + z(0) f'(\Gamma_0; \mathbf{V}) + \kappa z(0) f (\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}s. \tag{A.14}$$

Since $f \in H^2(\mathbb{R}^2)$ is independent of the shape its derivative simplifies to $f'(\Gamma_0; \mathbf{V}) = \partial_{\mathbf{n}} f \, (\mathbf{V} \cdot \mathbf{n})$. Plugging this in yields (A.11) and making use of Lemma A.6 yields (A.12). □

The existence proofs for the material derivatives rely on the implicit function theorem:

Theorem A.10 (Implicit function Theorem, from [2])**.** *Let E_1, E_2, F be Banach spaces, let W be open in $E_1 \times E_2$ and let $f \in C^q(W, F)$. Suppose that $(x_0, y_0) \in W$ such that $f(x_0, y_0) = 0$ and*

$$D_2 f(x_0, y_0) : E_2 \to F \tag{A.15}$$

is an isomorphism. Then, there are open neighborhoods $U \subset W$ of (x_0, y_0) and $V \subset E_1$ of x_0 and a unique $\mathcal{G} \in C^q(V, E_2)$ such that

$$((x, y) \in U \text{ and } f(x, y) = 0) \Leftrightarrow (x \in V \text{ and } y = \mathcal{G}(x)). \tag{A.16}$$

Appendix B. Existence and uniqueness of solutions for PDE

Based on the notation of [24] we introduce elliptic and coercive bilinear forms which give rise to usual existence existence results for partial differential equations.

Definition B.1 (V-Elliptic)**.** Let $m \geq 1$ and let V be a closed subspace equipped with the $H^m(\Omega)$-norm between $H_0^m(\Omega) \subset V \subset H^m(\Omega)$. We call a bilinear form $a : H^m(\Omega) \times H^m(\Omega) \to \mathbb{R}$ V-elliptic if and only if

(1) $|a(\psi, \phi)| \leq c_1 \|\psi\|_{H^m(\Omega)} \|\phi\|_{H^m(\Omega)}$, for all $\psi, \phi \in H^m(\Omega)$;

(2) $a(\psi, \psi) \geq c_2 \|\psi\|^2_{H^m(\Omega)}$, for all $\psi \in V$;
where $c_1, c_2 > 0$ are independent of ψ and ϕ.

Definition B.2 (V-Coercive)**.** Let $m \geq 1$ and let V be a closed subspace equipped with the $H^m(\Omega)$-norm between $H_0^m(\Omega) \subset V \subset H^m(\Omega)$. We call a bilinear form $a : H^m(\Omega) \times H^m(\Omega) \to \mathbb{R}$ V-coercive if and only if

(1) $|a(\psi, \phi)| \leq c_1 \|\psi\|_{H^m(\Omega)} \|\phi\|_{H^m(\Omega)}$, for all $\psi, \phi \in H^m(\Omega)$;

(2) $a(\psi, \psi) + k \|\psi\|^2_{L^2(\Omega)} \geq c_2 \|\psi\|^2_{H^m(\Omega)}$, for all $\psi \in V$;
where $c_1, c_2 > 0$ and $k \in \mathbb{R}$ are constants independent of ψ and ϕ.

For V-elliptic problems we can apply the Lax−Milgram lemma to provide the existence of a unique solution.

Theorem B.3 (Lax−Milgram, from [24])**.** *Let $a(\psi, \phi)$ be V-elliptic and let $f \in V'$. Then there exists a unique $\psi \in V$ such that*

$$a(\psi, \phi) = \langle f, \phi \rangle_{V', V} \tag{B.1}$$

for all $\phi \in V$.

A key part in our line of proof is the uniqueness question addressed in Lemma 4.11. However, since the corresponding bilinear form is only V-coercive we rely on the following theorem, which does not provide uniqueness, but states that the space of homogeneous solutions is finite dimensional.

Theorem B.4 (from [24])**.** *Let $V \hookrightarrow L^2(\Omega) \hookrightarrow V'$ be a Gelfand triple and let the embedding $V \hookrightarrow L^2(\Omega)$ be compact. Let $a(\psi, \phi)$ be V-coercive, then*

$$\mathcal{Z} = \{\psi \in V; a(\psi, \phi) = 0 \textit{ for all } \phi \in V\} \tag{B.2}$$

is a finite dimensional subspace of V. Furthermore, if 0 is no eigenvalue of the corresponding representation operator then $\mathcal{Z} = \{0\}$ holds.

Lemma B.5. *For $l \in \mathbb{N}$ let Ω be a domain of class $C^{2+l,1}$. Let $g_0 \in H^{\frac{3}{2}+l}(\Gamma_0)$. Then, for*

$$\begin{aligned} \Delta\Psi &= 0 \qquad && \textit{in } \Omega \\ \Psi &= g_0 && \textit{on } \Gamma \end{aligned} \tag{B.3}$$

there exists a unique solution $\Psi \in H^{2+l}(\Omega)$.

Proof. Making use of the inverse trace theorem from ([24], Prop. 8.8) there exist a continuation $g \in H^{2+l}(\Omega)$ with $g|_\Gamma = g_0$. Using $\Phi = \Psi - g$ we can reformulate and derive the following weak problem: find $\Phi \in H_0^1(\Omega)$ with

$$\int_\Omega \nabla\Phi \cdot \nabla\xi \mathrm{d}x = \int_\Omega \Delta g \, \xi \mathrm{d}x \qquad \text{for all } \xi \in H_0^1(\Omega). \tag{B.4}$$

Finally, Theorem B.3 yields the existence of a unique solution $\Phi \in H_0^1(\Omega)$. And ([24], Prop. 20.4) yields the regularity $\Phi \in H^{2+l}(\Omega)$ and thus $\Psi \in H^{2+l}(\Omega)$. □

Lemma B.6. *For $l \in \mathbb{N}$ let Ω be a domain of class $C^{4+l,1}$. Let $g_0 \in H^{2+\frac{3}{2}+l}(\Gamma_0)$ and $g_1 \in H^{2+\frac{1}{2}+l}(\Gamma_0)$. Then, for*

$$\begin{aligned} \Delta\Delta\Psi &= 0 && \text{in } \Omega \\ \Psi &= g_0 && \text{on } \Gamma \\ \partial_{\mathbf{n}}\Psi &= g_1 && \text{on } \Gamma \end{aligned} \tag{B.5}$$

there exists a unique solution $\Psi \in H^{4+l}(\Omega)$.

Proof. Making use of the inverse trace theorem from ([24], Prop. 8.8) there exist a continuation $g \in H^{4+l}(\Omega)$ with $g|_\Gamma = g_0$ and $\partial_{\mathbf{n}} g|_\Gamma = g_1$. Using $\Phi = \Psi - g$ we can reformulate and derive the following weak problem: find $\Phi \in H_0^2(\Omega)$ with

$$\int_\Omega \Delta\Phi\, \Delta\xi \mathrm{d}x = -\int_\Omega \Delta\Delta g\, \xi \mathrm{d}x \qquad \text{for all } \xi \in H_0^2(\Omega). \tag{B.6}$$

Finally, Theorem B.3 yields the existence of a unique solution $\Phi \in H_0^2(\Omega)$. And ([24], Prop. 20.4) yields the regularity $\Phi \in H^{4+l}(\Omega)$ and thus $\Psi \in H^{4+l}(\Omega)$. □

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