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LIMIT OF VISCOUS DYNAMIC PROCESSES IN DELAMINATION AS THE VISCOSITY AND INERTIA VANISH

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Abstract. We introduce a model of dynamic evolution of a delaminated visco-elastic body with viscous adhesive. We prove the existence of solutions of the corresponding system of PDEs and then study the behavior of such solutions when the data of the problem vary slowly. We prove that a rescaled version of the dynamic evolutions converge to a "local" quasistatic evolution, which is an evolution satisfying an energy inequality and a momentum balance at all times. In the one-dimensional case we give a more detailed description of the limit evolution and we show that it behaves in a very similar way to the limit of the solutions of the dynamic model in [T. Roubicek, *SIAM J. Math. Anal.* **45** (2013) 101–126], where no viscosity in the adhesive is taken into account.

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1. Introduction

In the last years the field of contact mechanics has been becoming more and more studied, thanks also to the numerous engineering applications and simulations. The problem of delamination is an important part of these modelings. The setting consists of two elastic bodies glued by an adhesive on an interface. External forces and high stresses due to elastic deformations of the bodies may break the macromolecules of the adhesive, weakening its effect. Such process is irreversible, in the sense that the deteriorated adhesive cannot be restored. The state of the adhesive is described by the delamination coefficient z, that is a function defined on the interface which takes values in $[0,1]$ and is proportional to the efficacy of the glue. Until the glue is effective ($z > 0$) the movements of the bodies at the interface are constrained. Moreover some constraints at the interface are always considered due to the non-interpenetrability of the two bodies or to the pressure of the system. We then study the dynamics of the bodies taking into account the inertial and viscous effects. This model is very similar to the one considered in [30], with the difference that we also consider the viscosity related to the debonding of the adhesive. This makes the evolution of the coefficient z no more rate-independent. We are interested in considering all the dissipative effects due to viscosity, both in the bulk and in the adhesive, in order to develop a vanishing viscosity analysis. Instead we neglect every thermal effect. In [27,29] it is considered a system where also thermal

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effects are analyzed, while no viscosity of the delamination coefficient is considered. Viscosity in the flow rule has been studied in different settings where inertia is neglected (see, *e.g.*, [5, 26]).

The vanishing viscosity approach is an efficient tool to provide approximate solutions to rate-independent evolutions of mechanical systems. In [36] a gradient flow driven by a general potential energy in a finite-dimensional setting is considered, and the existence of a quasistatic evolution has been obtained by vanishing viscosity. A theoretic discussion of the vanishing viscosity technique can be found, *e.g.*, in [14, 24, 25], and references therein. In the infinite dimensional setting, let us quote some important contributions. In the specific framework of crack propagation in elastic media, see for instance [17, 20, 34]. Regularization of the flow rule *via* viscosity terms has been introduced, *e.g.*, [12, 18], in the framework of elasto-plasticity and damage model, respectively. Viscosity both in the bulk and in the flow rule has been taken into account in other works, as [4, 11]. In the context of delamination it is worth to cite [33], where the viscosity in the bulk is coupled with a rate-independent flow rule. In [5] also the viscosity for the internal variable is taken into account, but the inertial effects are neglected.

Our vanishing viscosity analysis is carried out in the second part of the paper. To be precise, we consider the solutions to the dynamic viscous system introduced in the first part of the paper and we study their asymptotic behavior when the external forces and boundary data vary in a still slower way (this is called, standing to the Roubicek terminology [30], the "slow loading" limit). Such analysis coincides to study the asymptotic behavior of the solutions when the viscosity and the inertia vanish in a specific ratio. If the viscosity goes to zero as ϵ, then the density of the body (inertia) must vanish as ϵ^2. In particular, such analysis is different from the standard vanishing viscosity approach, that is the asymptotic analysis as only the viscosity tends to zero. Let us remark that also a vanishing viscosity analysis, keeping fixed the density of the body, is possible in order to obtain a dynamic solution of the problem with neglected viscosity, even if this is not the aim of the present paper. This analysis is done in [30]. The slow loading limit have already been studied in different settings (see, *e.g.*, [1] for a general finite-dimensional framework, and [8] for a model of visco-plasticity).

In most the cases the solution corresponding to a dynamic system tends, as the viscosity and the inertial parameter goes to zero, to the quasistatic evolution of the same system. This means that the limiting function is, at every time, a stationary point of the energy functional of the system. In delamination, the quasistatic model is discussed in [19, 32]. However there are systems whose dynamic solutions do not converge to a quasistatic evolution. The same holds true for the vanishing viscosity limit when the limiting energy is not convex. In such a case it might happen that the limit evolution shows discontinuities in time. In some cases these jumps correspond to viscous transitions between stable states of the system. Indeed, using an ad-hoc rescaling of the time variable, one may prove that at each jump the limiting solution runs instantaneously between the initial stable state u^- and the final one u^+, following the trajectory of a solution of the dynamic-viscous problem which has u^- as limit at $-\infty$, and u^+ at $+\infty$. Results like this exist for different mechanical systems; a general framework in finite-dimension has been studied, *e.g.*, in [1, 25, 36], and, in the setting of Cam–Clay plasticity, in [12].

In [8] dynamic evolutions of a visco-elasto-plastic body are considered and a slow loading analysis is performed. When the parameter ϵ vanishes, it is proven that the solutions of the dynamic model approximate a quasistatic evolution in perfect plasticity. In particular, for a quasistatic evolution, the solution is characterized by coupling an energy balance and a momentum balance. In the present paper, the same asymptotic analysis gives rise to a limiting function that satisfies a momentum balance and only an energy inequality, which might be strict. In contrast with the system considered in [8], the energy driving the limiting rate-independent evolution is not convex, property which would avoid such discontinuities. The lack of convexity of the energy might indeed provide instantaneous discrete change of position of minimizers, thus forcing the driven quasistatic evolution to show discontinuities in time. In [8], the energy functional being convex and continuous in time, this cannot happen. In [30] some dynamic models for adhesive contact of visco-elastic bodies is considered (with no viscosity in the flow rule). A specific case of the models considered is the following: if $\Omega = \Omega_1 \cup \Omega_2$ represents the reference configuration of two visco-elastic bodies attached at an interface Γ, and u_μ represents the displacement of the bodies (we use the label μ since, in a second step, we will let μ go to zero), the momentum balance reads

$$\rho \ddot{u}_\mu - \operatorname{div}\big(\mathbb{C}^0 e(u_\mu) + \mu \mathbb{C}^1 e(\dot{u}_\mu)\big) = f, \tag{1.1a}$$

with $e(u_\mu)$ being the symmetric gradient of u_μ, $\mathbb{C}^0$ and $\mathbb{C}^1$ the elastic and visco-elastic tensors respectively, f the external load, and $\rho > 0$ and $\mu > 0$ the mass density of the body and the viscosity, respectively. The flow rule for the delamination coefficient $z_\mu \in L^1(\Gamma; [0,1])$ is

$$z_\mu \quad \text{is non-increasing}, \tag{1.1b}$$

$$\text{either} \;\; \frac{1}{2}\mathbb{K}[u_\mu]\cdot[u_\mu] - \alpha \leq 0 \quad \text{or} \quad z_\mu = 0, \tag{1.1c}$$

where $[u_\mu]$ is the jump of u_μ at Γ, $\alpha \in L^\infty(\Gamma)$ is a positive function, $\mathbb{K}$ the elasticity tensor for the adhesive. Equations (1.1a)−(1.1c) are supplemented with Dirichlet and Neumann conditions at the boundary, and with the contact condition

$$\sigma_\mu \nu = \mathbb{K}[u_\mu] z_\mu \qquad \text{on } \Gamma, \tag{1.1d}$$

with ν the unit normal to Γ and $\sigma_\mu := \mathbb{C}^0 e(u_\mu) + \mu \mathbb{C}^1 e(\dot{u}_\mu)$. Moreover, to avoid interpenetration of matter, it is required the following constraint on the jump of the displacement

$$[u_\mu]\cdot\nu = 0 \qquad \text{on } \Gamma. \tag{1.2}$$

In [30] it is shown that when the viscosity tends to zero the solutions of the equations above approximate a dynamic evolution (u,z) still satisfying (1.1b), (1.1d), (1.2), the boundary datum, and the following conditions:

(i) The momentum balance

$$\rho \ddot{u} - \operatorname{div}\big(\mathbb{C}^0 e(u)\big) = f. \tag{1.3}$$

(ii) The semi-stability condition (1.1c).

(iii) The energy balance (written for a homogeneous boundary datum and with $\langle\cdot,\cdot\rangle$ denoting the duality product in L^2) for all $t \in [0,T]$

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{u}(t)\|^2_{L^2} + \frac{1}{2}\langle \mathbb{C}^0 e(u(t)), e(u(t))\rangle + \frac{1}{2}\langle \mathbb{K}[u(t)], [u(t)]z(t)\rangle + \int_0^t \int_\Omega m \,\mathrm{d}x\mathrm{d}s - \langle \alpha, z(t)\rangle \\
&= \frac{\rho}{2}\|\dot{u}_0\|^2_{L^2} + \frac{1}{2}\langle \mathbb{C}^0 e(u_0), e(u_0)\rangle + \frac{1}{2}\langle \mathbb{K}[u_0], [u_0]z_0\rangle - \langle \alpha, z_0\rangle_\Gamma + \int_0^t \langle f, \dot{u}\rangle \mathrm{d}s.
\end{aligned} \tag{1.4}$$

Here m is a non-negative Borel measure on $[0,T]\times\Omega$ that arises as the limit of $\mu\int_0^t \langle \mathbb{C}^1 e(\dot{u}_\mu), e(\dot{u}_\mu)\rangle \mathrm{d}s$, which represents the viscosity dissipation of the evolution of (u_μ, z_μ). The asymptotic analysis for the slow loading limit, which is also discussed, gives rise to the same result, with the unique difference that condition (i) for the limit evolution (u,z) is replaced by

$$-\operatorname{div}\big(\mathbb{C}^0 e(u)\big) = f. \tag{1.5}$$

In the present paper we consider a model with equations (1.1a), (1.1b), (1.1d), (1.2), but with (1.1c) replaced by

$$\text{either} \;\; \frac{1}{2}\mathbb{K}[u]\cdot[u] + \mu \dot{z} - \alpha \leq 0 \quad \text{or} \quad z = 0, \tag{1.6}$$

where $\mu > 0$ represents the viscosity of the adhesive. We first prove the existence of solutions for this model (Thm. 3.9). This is obtained by a standard time-discretization argument and an Euler implicit scheme, which leads us to the approximate solutions (u_τ, z_τ), $\tau > 0$ being the time step. Let us emphasize that in order to let τ

go to zero, different arguments and proofs from those in [30] are needed. In particular, to prove (1.6), due to the presence of the viscosity in the adhesive, we need to show that the sequence z_τ is strongly convergent in $L^2(\Gamma)$ (see Lem. 3.6). This is a key step of the proof, relying to an application of the Fréchet−Kolmogorov criterion. In the second part of the paper we perform the slow loading limit of the obtained solutions. We show that the limit evolution (u, z) satisfies the condition (ii) above, (1.5), and an energy balance like (iii) above, but with the measure m replaced by the two measures μ_e and μ_z, on $[0, T] \times \Omega$ and $[0, T] \times \Gamma$ respectively, which arise from the terms $\epsilon \int_0^T \langle \mathbb{C}^1 e(\dot{u}_\epsilon), e(\dot{u}_\epsilon)\rangle \mathrm{d}s$ and $\epsilon \int_0^T \|\dot{z}_\epsilon\|^2 \mathrm{d}s$ (notice that here the parameter going to zero is ϵ). These two terms represent the dissipation due to the viscosity in the bulk and in the adhesive, respectively. In order to prove that the semi-stability condition (ii) holds for the limit we need some arguments of measure theory, again due to the presence of the viscosity term $\epsilon \dot{z}_\epsilon$ (see Lems. 4.7 and 4.8). Moreover some further regularity result, missing in [30], is given for the limit solution. It is proved in Lemma 4.9 that the set of discontinuities in time of u is at most countable and is a subset of the jump set of the delamination coefficient z. Theorem 4.11 shows instead that no time discontinuities of (u, z) can occur until the delamination variable z is still strictly positive a.e. on Γ. In particular, if this happens in a discrete time interval, the evolution is quasistatic and the measures μ_e and μ_z vanish in such interval.

The question whether the limit evolution obtained by the model in [30] and the one obtained by the ours are the same is still open. Indeed the non-uniqueness of solutions to (1.5), (ii), and (iii), does not guarantee that the two limits coincide. However, we focus on the one-dimensional example given in ([30], Sect. 4), where it is proven that under suitable external load, the limit obtained by vanishing viscosity shows a jump where the delamination coefficient switches instantaneously from 1 to 0. We then prove in Theorem 4.14 that our limit behaves in the same way, and coincides with the limit in [30] at least in the time interval before the jump occurs. This result is achieved after a finer analysis of the limit solutions, that, at least in the one-dimensional case, is easier thanks to some well-known geometric measure theory result on the description of the derivative of the composition of a smooth map with a function of bounded variation (see formula (4.41) below).

The final aim (and beyond the scope) of this work is to establish the basis for analyzing the nature of the jumps of the limiting solutions of mechanical systems obtained by vanishing viscosity (or slow loading limit). In particular, to find infinite-dimensional settings where it is possible to give a description of the instantaneous transitions of the limit in terms of the original dynamic evolutions on the whole real line (in the spirit of [1,25,36], stated in the finite-dimensional case).

The paper is organized as follows: in Section 2 we give some notation and preliminaries of our problem. In Section 3 we first show in Theorem 3.9 the existence of solutions to our model in the case that no constraint on the jump $[u]$ is prescribed. Theorem 3.11 states the existence of a solution satisfying also (1.2). In Section 4 we perform the slow loading limit of the viscous solutions, while in Section 4.1 we focus on the one-dimensional case.

2. Preliminaries

2.1. Reference configuration and notation

We consider a hyperelastic body that occupies a bounded open domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$, with Lipschitz boundary. We suppose that

$$\Omega = \Omega_1 \cup \Gamma \cup \Omega_2,$$

where Γ is a Lipschitz surface which is the common boundary of the two disjoint connected and open sets Ω_1 and Ω_2. The body is assumed perfectly elastic on $\Omega_1 \cup \Omega_2$ while the surface Γ represents the interface where Ω_1 and Ω_2 are glued and where delamination may occur. We denote by ν the unit normal to Γ pointing from Ω_1 into Ω_2. We suppose that the boundary $\partial\Omega$ writes as the union

$$\partial\Omega := \partial_D\Omega \cup \partial_N\Omega,$$

where $\partial_D \Omega$ and $\partial_N \Omega$ are the closures in $\partial\Omega$ of two disjoint open sets with a $(d-2)$-dimensional common boundary. We assume that $\partial_D \Omega$ has positive $(d-1)$-Hausdorff measure and that it has nonnegligible intersection with both $\partial\Omega_1$ and $\partial\Omega_2$. We also denote the outer unit normal to $\partial\Omega$ by ν.

Let us now introduce some notation. If X is a Banach space (usually, a space of functions defined on $\Omega_1 \cup \Omega_2$), we denote by $\|\cdot\|_X$ the norm in X. The duality product between elements of X' (the dual space of X) and elements of X is noted by $\langle\cdot,\cdot\rangle$. The same symbol is used for the duality pairing between general distributions and smooth functions. In order to distinguish the case when we deal with functions defined on $\Omega_1 \cup \Omega_2$ from the case when the functions are defined on Γ, we adopt the notation $\langle\cdot,\cdot\rangle_\Gamma$ in the latter case. In the sequel the symbol $\mathcal{M}_b(A;\mathbb{R}^d)$ denotes the space of Radon measures on the open set A with values in $\mathbb{R}^d$. The symbol $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ denotes the space of symmetric $d \times d$ real matrices.

2.2. The displacements

The class of admissible displacements for the delamination problem is the space $H^1(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$. It is convenient to define

$$H^1_D(\Omega_1 \cup \Omega_2;\mathbb{R}^d) := \{u \in H^1(\Omega_1 \cup \Omega_2;\mathbb{R}^d) : u = 0 \text{ on } \partial_D\Omega\}. \tag{2.1}$$

The corresponding dual space is denoted by $H^{-1}_D(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$. The jump on Γ of a function $u \in H^1(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$ is denoted by $[u] = u_2 - u_1$, where u_1 and u_2 are, respectively, the trace on Γ of $u \in H^1(\Omega_1;\mathbb{R}^d)$ and $u \in H^1(\Omega_2;\mathbb{R}^d)$. The continuity of the trace operator from $H^1(\Omega_i;\mathbb{R}^d)$ into $H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)$ gives

$$\|u\|_{H^{1/2}(\Gamma;\mathbb{R}^d)} \le \frac{\gamma}{2}\|u\|_{H^1(\Omega_i;\mathbb{R}^d)}, \quad i = 1,2, \tag{2.2}$$

for a positive constant γ, and then

$$\|[u]\|_{H^{1/2}(\Gamma;\mathbb{R}^d)} \le \gamma\|u\|_{H^1_D(\Omega_1 \cup \Omega_2;\mathbb{R}^d)}. \tag{2.3}$$

To the safe of readability, from now on, if no risk of confusion occurs, we will always omit the ambient spaces Γ and Ω appearing in the label of the norms. The symmetric gradient $e(u)$ of $u \in H^1(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$ is defined as

$$e(u) := \frac{1}{2}(\nabla u + (\nabla u)^T).$$

In $H^1_D(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$ the following Korn inequality holds

$$\|u\|_{H^1} \le \beta\|e(u)\|_{L^2} \quad \text{for every } u \in H^1_D(\Omega_1 \cup \Omega_2;\mathbb{R}^d), \tag{2.4}$$

for a positive constant β.

2.3. Delamination parameter and energy stored by the adhesive

As in the modeling approach by Frémond (see [15, 16]), at a fixed time the state of the glue on the interface Γ is described by the variable $z : \Gamma \to [0,1]$. The class of admissible delamination parameters is denoted by

$$\mathcal{Z} := \{z \in L^2(\Gamma) : 0 \le z \le 1\}.$$

The state $z(x) = 1$ means that the adhesive is fully effective, while $z(x) = 0$ corresponds to the state when all the molecular links are broken and the interface is totally debonded at $x \in \Gamma$. The deterioration of the glue is considered irreversible, that is, the variable z is a non-increasing function of the time variable. This turns into the condition

$$\dot{z} \le 0.$$

During the evolution of the system the energy needed to delaminate is denoted by $\alpha \in L^\infty(\Gamma)$, and such energy is dissipated in two ways, by heat production, whose cost we denote by $a_1 = a_1(x) > 0$, $x \in \Gamma$, and by creation of new delaminated surfaces, whose cost we denote by $a - a_1 := a_0 = a_0(x) > 0$, $x \in \Gamma$. Hence the dissipation due to these effects in the time interval $[t_1, t_2]$ reads

$$\mathcal{D}_a(t_1, t_2) := -\int_{t_1}^{t_2} \langle a_0 + a_1, \dot{z}(s)\rangle_\Gamma \mathrm{d}s. \tag{2.5}$$

It is convenient then to define $\alpha \in L^\infty(\Gamma)$ by

$$\alpha := a_0 + a_1. \tag{2.6}$$

When the evolution is quite fast we also consider the dissipation due to the viscosity of the glue. We consider a parameter $\mu = \mu(x) > 0$, $x \in \Gamma$, for which the energy dissipated by viscosity effects during the delamination process in the time interval $[t_1, t_2]$ reads

$$\mathcal{D}_\mu(t_1, t_2) := \int_{t_1}^{t_2} \langle \mu \dot{z}(s), \dot{z}(s)\rangle_\Gamma \mathrm{d}s. \tag{2.7}$$

In the sequel we will adopt the simpler (but not restrictive) hypothesis that μ is constant on Γ.

The energy stored in Γ by the adhesive is modeled by the potential V defined as follows.

Definition 2.1. Let $V : \mathbb{R}^d \to \mathbb{R}$ be a smooth non-negative and convex map satisfying:

(i) $V(0) = 0$ and $V(x) > 0$ if $x \neq 0$. In particular $x = 0$ is the unique minimum of V.
(ii) $\nabla V : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz with constant $L > 0$.
(iii) There exists $1 \leq \delta \leq \delta^*$ and $C > 0$ such that $|V(x)| \leq C(|x| + 1)^\delta$ for all $x \in \mathbb{R}^d$. Here $\delta^* = +\infty$ for $d \leq 2$ and $\delta^* = \frac{d-1}{d-2}$ for $d > 2$. Since from (i) ∇V must vanish at the origin, property (ii) has the following consequence
(iv) For all $x \in \mathbb{R}^n$ it holds $|\nabla V(x)| \leq L|x|$.

The energy stored on Γ at a fixed time then reads:

$$\mathcal{E}_\Gamma(u, z) := \langle V([u]), z\rangle_\Gamma.$$

We remark that in dimension $d \leq 3$ we can take $V([u]) := \frac{1}{2}\mathbb{K}[u] \cdot [u]$ where $\mathbb{K}$ is called elastic coefficient of the adhesive. Such matrix is supposed positive definite and symmetric. With this choice we see that the growth of V in (iii) is $\delta = \delta^* = 2$.

2.4. Kinematic setting

The two elasticity tensors $\mathbb{C}^0$ and $\mathbb{C}^1$ in $\mathbb{R}^{d^4}$ are symmetric and positive definite, there exist positive constants α_i and β_i such that

$$\alpha_0|\eta|^2 \leq \langle \mathbb{C}^0\eta, \eta\rangle \leq \beta_0|\eta|^2, \tag{2.8a}$$
$$\alpha_1|\eta|^2 \leq \langle \mathbb{C}^1\eta, \eta\rangle \leq \beta_1|\eta|^2, \tag{2.8b}$$

for all $\eta \in \mathbb{R}^{d\times d}$. It is convenient to introduce the following notations

$$\mathcal{Q}_0(e) = \frac{1}{2}\langle \mathbb{C}^0 e, e\rangle, \tag{2.9}$$
$$\mathcal{Q}_1(e) = \langle \mathbb{C}^1 e, e\rangle, \tag{2.10}$$

for all $e \in L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}_{\mathrm{sym}})$.

Let $V : \mathbb{R}^d \to \mathbb{R}$ be the potential in Definition 2.1. For all $u \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ and $z \in L^\infty(\Gamma)$ we define $T(u, z) \in H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ as

$$\langle T(u,z), \varphi\rangle := \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma, \tag{2.11}$$

for every $\varphi \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$, so that, from (2.3) and from the hypothesis (ii) on V, one has

$$|\langle T(u,z),\varphi\rangle| \le \|\nabla V([u])\|_{L^2} \|[\varphi]\|_{L^2} \|z\|_{L^\infty} \le 2L\gamma \|u\|_{H^1_D} \|\varphi\|_{H^1_D} \|z\|_{L^\infty},$$

which implies that there exists a positive constant C such that

$$\|T(u,z)\|_{H^{-1}_D} \le C\|u\|_{H^1_D} \|z\|_{L^\infty}. \tag{2.12}$$

The stress σ satisfies the constitutive relation

$$\sigma = \mathbb{C}^0 e(u) + \mu \mathbb{C}^1 e(\dot u), \tag{2.13}$$

where $\mu > 0$ is the viscosity parameter in the bulk. Then the second principle of dynamics reads

$$\rho \ddot u(t) - \mathrm{div}\sigma(t) = f(t) \quad \text{ in } \Omega, \tag{2.14}$$

where we assume that the mass density of the elastic body is the constant $\rho > 0$. Together with (2.14) we require that the following boundary conditions are satisfied

$$u(t) = w(t) \quad \text{ on } \partial_D\Omega, \tag{2.15a}$$
$$\sigma(t)\nu = g(t) \quad \text{ on } \partial_N\Omega, \tag{2.15b}$$
$$\sigma(t)\nu = -\nabla V([u(t)])z(t) \quad \text{ on } \Gamma. \tag{2.15c}$$

We define the total external load of the system $\mathcal{L}(t) \in H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ by

$$\langle \mathcal{L}(t), \varphi\rangle := \langle f(t), \varphi\rangle + \int_{\partial_N\Omega} g(t)\cdot\varphi \mathrm{d}S, \tag{2.16}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$. To deal with (2.14) and (2.15), we define the continuous linear operator $\mathrm{div}_D : L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}) \to H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ by

$$\langle \mathrm{div}_D \sigma, \varphi\rangle := \langle \sigma, e(\varphi)\rangle, \tag{2.17}$$

for every $\sigma \in L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and every $\varphi \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$. Hence, if $f(t)$, $g(t)$, $\sigma(t)$, $u(t)$, $\partial_D\Omega$, and $\partial_N\Omega$ are sufficiently regular and $\mathcal{L}(t)$ is the total external load defined by (2.16), then (2.14), (2.15b), and (2.15c) are equivalent to

$$\rho\ddot u(t) - \mathrm{div}_D\sigma(t) = \mathcal{L}(t) - T(u,z), \tag{2.18}$$

where equality holds in $H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$, and where $T(u, z)$ is the linear operator defined in (2.11). In weak form (2.18) reads as

$$\langle \rho \ddot u(t), \varphi\rangle + \langle \sigma(t), e(\varphi)\rangle = \langle \mathcal{L}(t), \varphi\rangle - \langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma, \tag{2.19}$$

for every $\varphi \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$.

2.5. Mechanical constraints and delamination process

When delamination occurs on the interface $\varGamma$ it may happen that the two parts Ω_1 and Ω_2 of the body separate. In particular cavitation phenomena or shear movements may occur. Both these phenomena arise by the appearance of a non-zero jump of the displacement on $\varGamma$. Since interpenetration of Ω_1 and Ω_2 must be ruled out, such jump is constrained to have a non-negative normal component. Such condition is known in the literature as Signorini contact condition. A generalization of the Signorini condition is usually considered, in the following way. Let $D(x) \subset \mathbb{R}^d$ be a convex and closed cone, possibly depending on $x \in \varGamma$. This induces an ordering relation on the set of functions $v : \varGamma \to \mathbb{R}^d$, as follows,

$$v_1 \preceq v_2 \text{ if and only if } v_2(x) - v_1(x) \in D(x) \text{ for a.e. } x \in \varGamma.$$

The dual ordering $\preceq^*$ induced by the negative polar cone to D is given by

$$\zeta \preceq^* 0 \text{ if and only if } \zeta(x) \leq 0 \text{ for all } w \in D(x), \text{ for a.e. } x \in \varGamma.$$

Possible choices for the cone $D(x)$ are the following,

$$D(x) = \{v \in \mathbb{R}^d : v \cdot \nu(x) \geq 0\}, \tag{2.20a}$$
$$D(x) = \{v \in \mathbb{R}^d : v \cdot \nu(x) = 0\}, \tag{2.20b}$$

the first case being the classical unilateral Signorini contact condition, the latter being considered when cavitation cannot occur, for instance in systems under high pressure. The delamination modes (2.20a) and (2.20b) are usually referred to as *Mode I* and *Mode II* respectively. The constraint on the jump $[u]$ and the normal stress $t(\sigma) := \sigma\nu$ on $\varGamma$ reads

$$[u] \succeq 0, \tag{2.21a}$$
$$t(\sigma) + T(u, z) \succeq^* 0, \tag{2.21b}$$
$$(t(\sigma) + T(u, z)) \cdot [u] = 0. \tag{2.21c}$$

In this paper we will only treat the Mode II evolution, where D is actually a linear subspace of $\mathbb{R}^d$, which makes many proofs easier. The treatment of Mode I evolutions, requiring more sophisticated techniques is out of reach at the present stage.

The behavior of the variable z is strictly connected to the evolution of $[u]$. Whenever $[u]$ varies this has the effect of destroying molecular links on $\varGamma$, that turns into a decrease of the corresponding glue state z. When the bonding is completely broken, that is $z = 0$, any change of $[u]$ will not lead to energetic cost due to delamination. This is expressed by the constitutive equations

$$\dot{z} \leq 0, \tag{2.22a}$$
$$d \leq -\mu\dot{z}, \tag{2.22b}$$
$$\dot{z}(d + \mu\dot{z}) = 0, \tag{2.22c}$$
$$d \in \partial I_{[0,1]}(z) + V([u]) - \alpha, \tag{2.22d}$$

where $\partial I_{[0,1]}$ is the subdifferential of the function $I_{[0,1]}$, that is the function with equals 0 on $[0, 1]$ and $+\infty$ on $\mathbb{R} \setminus [0, 1]$. We assume that the parameter α defined in (2.6) is independent of time and there exists a constant $c > 0$ such that $\alpha \geq c$ a.e. on $\varGamma$. This quantity represents a threshold for the energy stored by the adhesive under which delamination cannot occur. The parameter $\mu > 0$ is the viscosity of the adhesive. Let us remark that as soon as $z = 0$ equations (2.22b)–(2.22d) reduce simply to the condition $\dot{z} = 0$, while the quantity $V([u]) - \alpha$ is no longer constrained. In particular no restriction to the evolution of $[u]$ is prescribed. At the same time, when $z > 0$ system (2.22) reads

$$\dot z \le 0, \tag{2.23a}$$
$$\dot z(V([u]) + \mu \dot z - \alpha) = 0, \tag{2.23b}$$
$$V([u]) + \mu \dot z - \alpha \le 0. \tag{2.23c}$$

Since z is a function defined on the interface Γ, equations (2.22) and (2.23) must be intended to hold almost everywhere on Γ.

3. Existence of unconstrained dynamic solutions

In this section we prove an existence result for solutions to the mechanical system introduced so far with no constraint on the jump $[u]$, *i.e.* with the cone D in (2.20) being $D \equiv \mathbb{R}^d$.

Theorem 3.1. *Let $\mathcal{L} \in L^2([0,T]; H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$, $u_0, v_0 \in H_D^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$, and $z_0 \in \mathcal{Z}$. Then there exists a triple (u, σ, z) with*

$$u \in H^1([0,T]; H_D^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.1a}$$
$$\dot u \in H^1([0,T]; H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^d)) \cap L^\infty([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.1b}$$
$$\sigma \in L^2([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}_{sym})), \tag{3.1c}$$
$$z \in H^1([0,T]; \mathcal{Z}), \tag{3.1d}$$

satisfying, for a.e. $t \in [0,T]$,

$$\rho \ddot u(t) - \operatorname{div} \sigma(t) = \mathcal{L}(t) - T(u(t), z(t)), \tag{3.2a}$$
$$\sigma(t) = \mathbb{C}^0 e(u(t)) + \mu \mathbb{C}^1 e(\dot u(t)), \tag{3.2b}$$

whereas

$$\dot z(t) \le 0, \tag{3.3a}$$
$$\dot z(t)(V([u(t)]) + \mu \dot z(t) - \alpha) = 0, \tag{3.3b}$$

on Γ,

$$V([u(t)]) + \mu \dot z(t) - \alpha \le 0, \tag{3.4}$$

on $\{z(t) > 0\} \subset \Gamma$, and with initial data

$$u(0) = u_0, \quad \dot u(0) = v_0, \quad z(0) = z_0. \tag{3.5}$$

Remark 3.2. Let us remark that, when $\mathcal{L}$ takes the form (2.16), in the regular case, (3.2a) is equivalent to (2.14), (2.15b), and (2.15c).

In fact, integrating by parts (2.19) and taking into account the definition (2.11) of T, we get

$$\langle \rho \ddot u, \varphi\rangle - \langle \operatorname{div}\sigma, \varphi\rangle - \langle \mathcal{L}, \varphi\rangle = -\langle \nabla V([u]) \cdot [\varphi], z\rangle_\Gamma - \langle \sigma\nu, [\varphi]\rangle_\Gamma - \langle \sigma\nu, \varphi\rangle_{\partial_N \Omega}. \tag{3.6}$$

If we set $[\varphi] = 0$ we obtain the strong form (2.14) and (2.15b), which together with (3.6) implies (2.15c).

To prove Theorem 3.1 we proceed by time discretization, and solve a minimal problem at every discrete time. For every integer $n > 0$ we divide the interval $[0,T]$ in n equal subintervals of length $\tau := T/n$. We set $t_i^n := i\tau$,

$$u_0^n = u_0, \quad u_{-1}^n := u_0 - \tau v_0, \quad z_0^n := z_0,$$

and define $\mathcal{L}_i^n := \frac{1}{\tau}\int_{t_i^n}^{t_{i+1}^n} \mathcal{L}(s)\mathrm{d}s$ for all $i > 0$. Then for $1 \le i \le n$ we recursively define $u_i^n \in H_D^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ as the minimizer of

$$U_i^n(u) := \frac{\rho}{2}\left\|\frac{u - u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}\right\|_{L^2}^2 + \mathcal{Q}_0(e(u)) + \langle V([u]), z_{i-1}^n\rangle_\Gamma + \frac{\mu}{2}\langle \mathbb{C}^1 e(u - u_{i-1}^n), e(u - u_{i-1}^n)\rangle - \langle \mathcal{L}_i^n, u\rangle, \tag{3.7}$$

and $z_i^n \in \mathcal{Z}$ as the minimizer of

$$W_i^n(z) := \frac{\mu}{2\tau}\|z - z_{i-1}^n\|_{L^2}^2 + \langle V([u_i^n]), z\rangle_\Gamma - \langle \alpha, z\rangle_\Gamma. \tag{3.8}$$

Computing variations in the variable u at the minimum u_i^n of (3.7) we get

$$\frac{\rho}{\tau}\left\langle \frac{u - u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}, \varphi\right\rangle + \langle \mathbb{C}^0 e(u_i^n), e(\varphi)\rangle + \frac{\mu}{\tau}\langle \mathbb{C}^1(e(u_i^n) - e(u_{i-1}^n)), e(\varphi)\rangle = \langle \mathcal{L}_i^n, \varphi\rangle - \langle \nabla V([u_i^n])\cdot[\varphi], z_{i-1}^n\rangle_\Gamma, \tag{3.9}$$

for every $\varphi \in H_D^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$. Instead taking variations η of the minimum z_i^n of (3.8), and taking into account that z_i must be non-negative, we get

$$\langle V([u_i^n]), \eta\chi_{\{z_i>0\}}\rangle_\Gamma + \frac{\mu}{\tau}\langle z_i^n - z_{i-1}^n, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma \ge 0, \tag{3.10}$$

for every $\eta \le 0$. Moreover, if the variation $\eta \le 0$ is such that $z_i \pm \epsilon\eta \in [0, z_{i-1}]$ for some $\epsilon > 0$, then we will have equality. Denoting by $\mathcal{V}(z_i)$ the set of such variations, we have

$$\langle V([u_i^n]), \eta)\rangle_\Gamma + \frac{\mu}{\tau}\langle z_i^n - z_{i-1}^n, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma = 0, \tag{3.11}$$

for all $\eta \in \mathcal{V}(z_i)$.

Now we set $v_i^n := \frac{u_i^n - u_{i-1}^n}{\tau}$ and define the following piecewise affine (or constant) functions

$$\begin{aligned} &u_\tau(t) := u_i^n + (t - t_i^n)\frac{u_{i+1}^n - u_i^n}{\tau} && \text{for } t \in [t_i^n, t_{i+1}^n),\\ &z_\tau(t) := z_i^n + (t - t_i^n)\frac{z_{i+1}^n - z_i^n}{\tau} && \text{for } t \in [t_i^n, t_{i+1}^n),\\ &v_\tau(t) := v_i^n + (t - t_i^n)\frac{v_{i+1}^n - v_i^n}{\tau} && \text{for } t \in [t_i^n, t_{i+1}^n),\\ &\mathcal{L}_\tau(t) := \mathcal{L}_i^n && \text{for } t \in [t_i^n, t_{i+1}^n), \end{aligned} \tag{3.12}$$

for $i = 0, \dots, n-1$. The fact that

$$\mathcal{L}_\tau \to \mathcal{L} \quad \text{strongly in } L^2([0,T]; H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.13}$$

as $\tau \to 0$, is standard and will often be tacitly used in the sequel. The following statement holds.

Proposition 3.3. *There are functions $u \in H^1([0,T]; H_D^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ and $z \in L^\infty([0,T]; \mathcal{Z})$ such that, up to a subsequence,*

$$u_\tau \rightharpoonup u \quad \textit{weakly in } H^1([0,T]; H_D^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.14a}$$

$$u_\tau(t) \rightharpoonup u(t) \quad \textit{weakly in } H_D^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d), \textit{ for every } t \in [0,T], \tag{3.14b}$$

$$z_\tau \rightharpoonup z \quad \textit{weakly* in } L^\infty([0,T]; L^2(\Gamma)), \tag{3.14c}$$

$$z_\tau(t) \rightharpoonup z(t) \quad \textit{weakly* in } L^\infty(\Gamma) \textit{ for every } t \in [0,T], \tag{3.14d}$$

as $\tau \to 0$*. Moreover* $\dot{u} \in H^1([0,T]; H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$*,* $z \in H^1([0,T]; L^2(\Gamma))$*, and*

$$v_\tau \rightharpoonup \dot{u} \quad \textit{weakly* in } L^\infty([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.14e}$$
$$\dot{v}_\tau \rightharpoonup \ddot{u} \quad \textit{weakly in } L^2([0,T]; H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.14f}$$
$$\dot{z}_\tau \rightharpoonup \dot{z} \quad \textit{weakly in } L^2([0,T]; L^2(\Gamma)). \tag{3.14g}$$

Proof. Since z_i^n minimizes W_i^n in (3.8), we have $W_i^n(z_i^n) - W_i^n(z_{i-1}^n) \le 0$. Summing this expression with (3.9) with $\varphi = u_i^n - u_{i-1}^n$ we get

$$\begin{aligned}
&\frac{\rho}{2}\left\|\frac{u_i^n - u_{i-1}^n}{\tau}\right\|_{L^2}^2 - \frac{\rho}{2}\left\|\frac{u_{i-1}^n - u_{i-2}^n}{\tau}\right\|_{L^2}^2 + \frac{\rho}{2}\left\|\frac{u_i^n - u_{i-1}^n}{\tau} - \frac{u_{i-1}^n - u_{i-2}^n}{\tau}\right\|_{L^2}^2 \\
&+ \mathcal{Q}_0(e(u_i^n)) - \mathcal{Q}_0(e(u_{i-1}^n)) + \frac{1}{2}\langle \mathbb{C}^0(e(u_i^n) - e(u_{i-1}^n)), e(u_i^n) - e(u_{i-1}^n)\rangle \\
&+ \frac{\mu}{\tau}\langle \mathbb{C}^1(e(u_i^n) - e(u_{i-1}^n)), e(u_i^n) - e(u_{i-1}^n)\rangle - \langle \mathcal{L}_i^n, u_i^n - u_{i-1}^n\rangle \\
&- \langle \alpha, (z_i^n - z_{i-1}^n)\rangle_\Gamma + \langle \nabla V([u_i^n]) \cdot [u_i^n - u_{i-1}^n], z_{i-1}^n\rangle_\Gamma \\
&+ \langle V([u_i^n]), (z_i^n - z_{i-1}^n)\rangle_\Gamma + \frac{\mu}{2\tau}\|z_i^n - z_{i-1}^n\|_{L^2}^2 \le 0.
\end{aligned} \tag{3.15}$$

Using the notation introduced in (3.12) and taking into account the identities

$$\begin{aligned}
&\langle \nabla V([u_i^n]) \cdot [u_i^n - u_{i-1}^n], z_{i-1}^n\rangle_\Gamma + \langle V([u_i^n]), (z_i^n - z_{i-1}^n)\rangle_\Gamma = \langle V([u_i^n]), z_i^n\rangle_\Gamma \\
&- \langle V([u_{i-1}^n]), z_{i-1}^n\rangle_\Gamma - \langle \int_{t_{i-1}}^{t_i} \nabla V([u_\tau]) \cdot [\dot{u}_\tau] - \nabla V([u_i^n]) \cdot [\dot{u}_\tau] \mathrm{d}t, z_{i-1}^n\rangle_\Gamma,
\end{aligned}$$

we see that (3.15) implies

$$\begin{aligned}
&\frac{\rho}{2}\|v_\tau(t_i^n)\|_{L^2}^2 + \frac{\rho\tau}{2}\int_{t_{i-1}^n}^{t_i^n} \|\dot{v}_\tau\|_{L^2}^2 \mathrm{d}t + \mathcal{Q}_0(e(u_\tau(t_i^n))) - \mathcal{Q}_0(e(u_\tau(t_{i-1}^n))) \\
&+ \mu\int_{t_{i-1}^n}^{t_i^n} \mathcal{Q}_1(e(\dot{u}_\tau))\mathrm{d}t + \frac{\mu}{2}\int_{t_{i-1}^n}^{t_i^n} \|\dot{z}_\tau\|_{L^2}^2 \mathrm{d}t - \int_{t_{i-1}^n}^{t_i^n} \langle \alpha, \dot{z}_\tau\rangle \mathrm{d}t + \langle V([u_\tau(t_i^n)]), z_\tau(t_i^n)\rangle \\
&- \langle V([u_\tau(t_{i-1}^n)]), z_\tau(t_{i-1}^n)\rangle \\
&\le \int_{t_{i-1}^n}^{t_i^n} \langle \mathcal{L}_\tau, \dot{u}_\tau\rangle \mathrm{d}t + \int_{t_{i-1}^n}^{t_i^n} \langle \nabla V([u_\tau]) \cdot [\dot{u}_\tau] - \nabla V([u_i^n]) \cdot [\dot{u}_\tau], z_{i-1}^n\rangle_\Gamma \mathrm{d}t.
\end{aligned} \tag{3.16}$$

By the Lipschitz continuity of ∇V, the continuity of the trace operator (2.3), and the Korn inequality (2.4), we have

$$\begin{aligned}
&\left|\int_{t_{i-1}^n}^{t_i^n} \langle \nabla V([u_\tau]) \cdot [\dot{u}_\tau] - \nabla V([u_i^n]) \cdot [\dot{u}_\tau], z_{i-1}^n\rangle_\Gamma \mathrm{d}t\right| \le \tau k L \int_{t_{i-1}^n}^{t_i^n} \|[\dot{u}_\tau]\|_2^2 \mathrm{d}t \\
&\le \tau L\gamma^2 \int_{t_{i-1}^n}^{t_i^n} \|\dot{u}_\tau\|_{H^1}^2 \mathrm{d}t \le \tau L\gamma^2\beta^2 \int_{t_{i-1}^n}^{t_i^n} \|e(\dot{u}_\tau)\|_{L^2}^2 \mathrm{d}t.
\end{aligned} \tag{3.17}$$

Summing over $i = 1, \ldots, j$ expression (3.16) and using (2.8), one gets

$$\frac{\rho}{2}\|v_\tau(t_j^n)\|_{L^2}^2 + \frac{\alpha_0}{2}\|e(u_\tau(t_j^n))\|_{L^2}^2 + \alpha_1\mu\int_0^{t_j^n}\|e(\dot{u}_\tau)\|_{L^2}^2 \mathrm{d}t + \frac{\mu}{2}\int_0^{t_j^n}\|\dot{z}_\tau\|_{L^2}^2\mathrm{d}t$$
$$-\int_0^{t_j^n}\langle\alpha, \dot{z}_\tau\rangle \mathrm{d}t + \langle V([u_\tau(t_j^n)]), z_\tau(t_j^n)\rangle$$
$$\leq \int_0^{t_j^n}\langle\mathcal{L}_\tau, \dot{u}_\tau\rangle\mathrm{d}t + \tau L\gamma^2\beta^2\int_0^{t_j^n}\|e(\dot{u}_\tau)\|_{L^2}^2\mathrm{d}t + C, \tag{3.18}$$

for a constant $C > 0$ depending on u_0, v_0, z_0, μ, ρ, but independent of τ. Now

$$\int_0^{t_j^n}\langle\mathcal{L}_\tau, \dot{u}_\tau\rangle\mathrm{d}t \leq \frac{\lambda^{-1}}{2}\int_0^{t_j^n}\|\mathcal{L}_\tau\|_{H_D^{-1}}^2\mathrm{d}t + \frac{\lambda}{2}\int_0^{t_j^n}\|\dot{u}_\tau\|_{H^1}^2\mathrm{d}t \leq \frac{\lambda\beta^2}{2}\int_0^{t_j^n}\|e(\dot{u}_\tau)\|_{L^2}^2\mathrm{d}t + C, \tag{3.19}$$

by the Korn inequality (2.4), where $C > 0$ is a constant depending on the squared norm of $\mathcal{L} \in L^2([0,T]; H_D^{-1}(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ and on a fixed arbitrary positive number λ, but independent of τ. Then (3.18) implies

$$\frac{\rho}{2}\|v_\tau(t_j^n)\|_{L^2}^2 + \frac{\alpha_0}{2}\|e(u_\tau(t_j^n))\|_{L^2}^2 + \frac{\mu}{2}\int_0^{t_j^n}\|\dot{z}_\tau\|_{L^2}^2\mathrm{d}t$$
$$+\,\delta\int_0^{t_j^n}\|e(\dot{u}_\tau)\|_{L^2}^2\mathrm{d}t - \int_0^{t_j^n}\langle\alpha, \dot{z}_\tau\rangle\mathrm{d}t + \langle V([u_\tau(t_j^n)]), z_\tau(t_j^n)\rangle \leq C, \tag{3.20}$$

where $\delta := \alpha_1\mu - \frac{\lambda\beta^2}{2} - \tau L\gamma^2\beta^2$ and C is a positive constant. Since for λ sufficiently small and τ small enough all the terms in the left hand side are positive, we infer that all such terms are bounded. In particular there is a constant $L > 0$ such that

$$\|e(u_\tau)\|_{L^\infty([0,T];L^2(\Omega;\mathbb{R}^d))} \leq L, \tag{3.21}$$

for all n large enough and $\tau = \tau(n)$. Hence there are an increasing sequence n_k and a function $e \in L^\infty([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$ such that

$$e(u_{\tau(n_k)}) \rightharpoonup e \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{3.22a}$$

as $k \to \infty$. We will write $\tau \to 0$ for $k \to \infty$. Using the Korn inequality, from (3.21) we get

$$\|u_\tau\|_{L^\infty([0,T];H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))} \leq C, \tag{3.22b}$$

for some constant $C > 0$. This implies that, up to a subsequence, there is $u \in L^\infty([0,T]; H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ such that

$$u_\tau \rightharpoonup u \quad \text{weakly* in } L^\infty([0,T]; H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.22c}$$

as $\tau \to 0$. Convergence (3.22c) also implies that $e(u(t)) = e(t)$ for a.e. $t \in [0,T]$. Moreover (3.20) gives, up to passing to another subsequence,

$$e(\dot{u}_\tau) \rightharpoonup l \quad \text{weakly in } L^2([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{3.22d}$$
$$v_\tau \rightharpoonup v \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.22e}$$
$$z_\tau \rightharpoonup \hat{z} \quad \text{weakly* in } L^\infty([0,T]; L^2(\Gamma)), \tag{3.22f}$$
$$\dot{z}_\tau \rightharpoonup h \quad \text{weakly in } L^2([0,T]; L^2(\Gamma)), \tag{3.22g}$$

as $\tau \to 0$, for functions $l \in L^2([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$, $v \in L^\infty([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$, $\hat{z} \in L^\infty([0,T]; \mathcal{Z})$, and $h \in L^2([0,T]; L^2(\Gamma))$. Moreover z_τ are all functions with uniformly bounded variation on $[0,T]$. A generalization of Helly theorem (see Lem. 7.2 of [10]) then implies that

$$z_\tau(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma), \tag{3.22h}$$

for all $t \in [0,T]$ as $\tau \to 0$, for a function $z \in L^2([0,T]; \mathcal{Z})$.

Writing $z_\tau(t) = z_0 + \int_0^t \dot{z}_\tau(s)\mathrm{d}s$ and multiplying by a test function in $L^2(\Gamma)$ we obtain that we can identify $h = \dot{z}$. Multiplying z_τ by a test function in $L^1([0,T]; L^2(\Gamma))$ it is easily seen that it also must be $\hat{z} = z$. A similar argument shows that $l(t) = e(\dot{u}(t))$ for a.e. $t \in [0,T]$. The Korn inequality and (3.22d) implies that there is a function $\hat{u} \in L^2([0,T]; H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ such that

$$\dot{u}_\tau \rightharpoonup \hat{u} \quad \text{weakly in } L^2([0,T]; H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{3.22i}$$

and writing $u_\tau(t) = u_0 + \int_0^t \dot{u}_\tau(s)\mathrm{d}s$, arguing as before, we conclude that u in (3.22c) belongs to $H^1([0,T]; H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$, that $\hat{u} = \dot{u}$, and also that

$$u_\tau(t) \rightharpoonup u(t) \quad \text{weakly in } H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d), \tag{3.23}$$

for all $t \in [0,T]$.

From (3.9) it follows

$$\rho\dot{v}_\tau(t) = -\mathrm{div}_D(\mathbb{C}^0 e_\tau(t_i^n) + \mu\mathbb{C}^1 \dot{e}_\tau(t)) + \mathcal{L}_i^n - T(u_\tau(t_i^n), z_\tau(t_i^n)), \tag{3.24}$$

for all $t \in [t_i^n, t_{i+1}^n]$ and all i. From the continuity of the operators div_D and T, and from the convergences (3.22) we see that the right-hand side of the last expression is uniformly bounded in $L^2([0,T]; H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$, so that the same is true for $\dot{v}_\tau$ and, up to subsequences, there exists $\hat{v} \in L^2([0,T]; H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ such that

$$\dot{v}_\tau \rightharpoonup \hat{v} \quad \text{weakly in } L^2([0,T]; H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)). \tag{3.25}$$

Now, $v_\tau(t) - \dot{u}_\tau(t) = (\tau - (t - t_i^n))\dot{v}_\tau(t)$ when $t \in [t_i^n, t_{i+1}^n]$, for all i, so that $\int_0^T \|v_\tau - \dot{u}_\tau\|^2_{H^{-1}_D}\mathrm{d}s = \frac{\tau^2}{3}\int_0^T \|\dot{v}_\tau\|^2_{H^{-1}_D}\mathrm{d}s$, which, by the boundedness of $\dot{v}_\tau$, tends to zero. In particular, by (3.22i), since $\hat{u} = \dot{u}$, we find out that $\dot{u}(t) = v(t)$ for a.e. $t \in [0,T]$ and

$$v_\tau, \dot{u}_\tau \rightharpoonup \dot{u} \quad \text{weakly* in } L^\infty([0,T]; L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^d)). \tag{3.26}$$

Writing $v_\tau(t) = v_0 + \int_0^t \dot{v}_\tau(s)\mathrm{d}s$ and multiplying it by a test function in $L^2([0,T]; H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ we get $\dot{u}(t) = v_0 + \int_0^t \hat{v}(s)\mathrm{d}s$, and then $\dot{u}$ is differentiable in time and $\ddot{u} = \hat{v} \in L^2([0,T]; H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$. This concludes the proof. □

Corollary 3.4. *For the same subsequence of Theorem 3.1, it holds*

$$[u_\tau] \rightharpoonup [u] \quad \text{weakly in } H^1([0,T]; H^{\frac{1}{2}}(\Gamma)), \tag{3.27a}$$

$$[u_\tau(t)] \rightharpoonup [u(t)] \quad \text{weakly in } H^{\frac{1}{2}}(\Gamma), \text{ for every } t \in [0,T], \tag{3.27b}$$

$$[u_\tau(t)] \rightharpoonup [u(t)] \quad \text{strongly in } L^q(\Gamma), \text{ for every } t \in [0,T], \tag{3.27c}$$

for every $1 \le q < q^*$ *with* $\frac{1}{q^*} = \frac{d-2}{2(d-1)}$ *if* $d > 2$, $q^* = +\infty$ *otherwise. Moreover*

$$V([u_\tau(t)]) \rightharpoonup V([u(t)]) \quad \text{weakly in } L^{\frac{q^*}{2}}(\Gamma)), \text{ for every } t \in [0,T], \tag{3.28}$$

$$\dot{u}_\tau(t) \rightharpoonup \dot{u}(t) \quad \text{weakly in } L^2(\Omega; \mathbb{R}^d), \text{ for every } t \in [0,T]. \tag{3.29}$$

Proof. Convergences (3.27a) and (3.27b) are straightforward consequence of (3.14a), (3.14b), and the continuity of the trace operator, whereas (3.27c) follows from the compactness of the embedding $H^{\frac{1}{2}} \hookrightarrow L^q$ for all $q < q^*$. Convergence (3.28) follows by (3.27b) and the growth hypothesis on V. To prove (3.29) we first observe that such convergence holds with respect to the weak topology of $H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ since, by (3.14e) and (3.14f) we have

$$\dot{u}_\tau \rightharpoonup \dot{u} \text{ weakly in } H^1([0,T]; H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)). \tag{3.30}$$

Then (3.29) follows by (3.22b). □

Let us introduce the piecewise constant functions

$$\begin{aligned} \tilde{u}_\tau(t) &= u_\tau(t_i^n) && \text{for } t \in [t_i^n, t_{i+1}^n), \\ \tilde{z}_\tau(t) &= z_\tau(t_i^n) && \text{for } t \in [t_i^n, t_{i+1}^n), \end{aligned} \tag{3.31}$$

for all $i \leq (n-1)$. It is easy to show that convergences (3.14a), (3.14b), and (3.14d) holds true also for $\tilde{u}_\tau$ and $\tilde{z}_\tau$ in place of u_τ and z_τ. Now we are ready to prove the momentum balance.

Proposition 3.5. *Let u and z be the functions obtained in Proposition 3.3. Then it holds*

$$\langle \rho \ddot{u}, \varphi \rangle + \langle \mathbb{C}^0 e(u) + \mu \mathbb{C}^1 e(\dot{u}), e(\varphi) \rangle = \langle \mathcal{L}, \varphi \rangle - \langle \nabla V([u]) \cdot [\varphi], z \rangle_\Gamma, \tag{3.32}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ and for a.e. $t \in [0,T]$.

Proof. We start from (3.9), that, with the notation introduced so far, reads

$$\rho \langle \dot{v}_\tau, \varphi \rangle + \langle \mathbb{C}^0 e(\tilde{u}_\tau) + \mu \mathbb{C}^1 e(\dot{u}_\tau), e(\varphi) \rangle - \langle \mathcal{L}_\tau, \varphi \rangle + \langle \nabla V([\tilde{u}_\tau]) \cdot [\varphi], \tilde{z}_\tau \rangle_\Gamma = 0.$$

For $\psi \in C_c^\infty((0,T))$ we write

$$\int_0^T \big(\langle \mathbb{C}^0 e(\tilde{u}_\tau) + \mu \mathbb{C}^1 e(\dot{u}_\tau), e(\varphi) \rangle - \langle \mathcal{L}_\tau, \varphi \rangle + \langle \nabla V([\tilde{u}_\tau]) \cdot [\varphi], \tilde{z}_\tau \rangle_\Gamma \big) \psi \mathrm{d}t = - \int_0^T \rho \langle v_\tau, \varphi \rangle \dot{\psi} \mathrm{d}t, \tag{3.33}$$

and letting $\tau \to 0$, thanks to (3.22) we get

$$\int_0^T \big(\langle \mathbb{C}^0 e(u) + \mu \mathbb{C}^1 e(\dot{u}), e(\varphi) \rangle - \langle \mathcal{L}, \varphi \rangle + \langle \nabla V([u]) \cdot [\varphi], z \rangle_\Gamma \big) \psi \mathrm{d}t = - \int_0^T \rho \langle \dot{u}, \varphi \rangle \dot{\psi} \mathrm{d}t. \tag{3.34}$$

The arbitrariness of ψ then implies (3.32). □

The following Lemma plays a crucial role in the proof of Theorem 3.1. It shows that the convergence of z_τ to z also takes place with respect to the strong topology.

Lemma 3.6. *For all $q \geq 1$ and $t \in [0,T]$ we have*

$$z_\tau(t) \to z(t) \quad \textit{strongly in } L^q(\Gamma). \tag{3.35}$$

In order to prove Lemma 3.6 we recall the Fréchet−Kolmogorov theorem, whose proof can be found, *e.g.*, in [7]. For all $h \in \mathbb{R}^d$ we introduce the h-translation in $\mathbb{R}^d$, that is the function $s_h : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ defined by $s_h(f)(x) := f(x+h)$ for all $x \in \mathbb{R}^d$ and $f \in L^1(\mathbb{R}^d)$. Then the following Theorem holds true.

Theorem 3.7 (Fréchet–Kolmogorov). *Let B be a subset of $L^1(\mathbb{R}^d)$ such that for all $f \in B$ it holds $f = 0$ out of a bounded set $U \subset \mathbb{R}^d$. Then B is a relatively compact set in $L^1(\mathbb{R}^d)$ if and only if there exists a continuous non-negative function $\omega : \mathbb{R}^d \to \mathbb{R}$ such that $\omega(0) = 0$ and $\|f - s_h(f)\|_1 \leq \omega(h)$, for all $f \in B$ and for all $h \in \mathbb{R}^d$.*

Proof of Lemma 3.6. Since

$$z_i = \operatorname{argmin}_{0\le z\le z_{i-1}} \langle V([u_i]) - \alpha, z\rangle_\Gamma + \frac{\mu}{2\tau}\|z - z_{i-1}\|^2_{L^2(\Gamma)},$$

we see that the value of $z_i(x)$ at $x \in \Gamma$ is exactly the minimizer in $[0, z_{i-1}(x)]$ of

$$z \mapsto \langle V([u_i(x)]) - \alpha(x), z\rangle_\Gamma + \frac{\mu}{2\tau}|z - z_{i-1}(x)|^2, \tag{3.36}$$

so that, denoting $a(x) := V([u_i(x)]) - \alpha(x)$, we can explicitly compute the value of $z_i(x)$. If $\hat z(x) := -\frac{\tau}{\mu}a(x) + z_{i-1}(x)$ is the minimizer of (3.36) on $\mathbb{R}$, then we have (omitting the symbol x)

$$\begin{cases} \hat z > z_{i-1} \;\Leftrightarrow\; a < 0 \;\Rightarrow\; z_i = z_{i-1}, \\ 0 \le \hat z \le z_{i-1} \;\Leftrightarrow\; 0 \le a < \frac{\mu}{\tau} z_{i-1} \;\Rightarrow\; z_i = -\frac{\tau}{\mu}a + z_{i-1}, \\ \hat z < 0 \;\Leftrightarrow\; a > \frac{\mu}{\tau} z_{i-1} \;\Rightarrow\; z_i = 0, \end{cases} \tag{3.37}$$

from which it follows

$$z_i = z_{i-1} - \left(\frac{\tau}{\mu}a \wedge z_{i-1}\right)^+, \quad \text{and } \mu\dot z_\tau = -\left(a \wedge \frac{\mu}{\tau}z_{i-1}\right)^+. \tag{3.38}$$

From (3.27a) and the definition of V we see that $V([u_\tau])(t)$ is a converging sequence in $L^1(\Gamma)$. Therefore thanks to Theorem 3.7 there exists a function $\omega : \Gamma \cong \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\omega(0) = 0$ and

$$\|V([u_\tau](t)) - s_h(V([u_\tau](t)))\|_1 \le \omega(h), \tag{3.39}$$

for all $h \in \mathbb{R}^{d-1}$ and for all τ and $t \in [0, T]$. Without loss of generality we can also suppose that

$$\|a - s_h(a)\|_1 \le \omega(h), \tag{3.40}$$

since $\alpha \in L^\infty(\Gamma)$.

For fixed τ, let us prove by induction on i that $\|z_i - s_h(z_i)\|_1 \le \frac{i\tau}{\mu}\omega(h)$. Indeed, using the expression of z_i in (3.38), we have

$$\begin{aligned} \|z_i - s_h(z_i)\|_{L^1} =& \|z_{i-1} - \left(\frac{\tau}{\mu}a \wedge z_{i-1}\right)^+ - \left(s_h(z_{i-1}) - \left(\frac{\tau}{\mu}s_h(a) \wedge s_h(z_{i-1})\right)^+\right)\|_{L^1} \\ \le& \|z_{i-1} - s_h(z_{i-1})\|_{L^1} + \|\frac{\tau}{\mu}a - \frac{\tau}{\mu}s_h(a)\|_{L^1} \\ \le& \frac{(i-1)\tau}{\mu}\omega(h) + \frac{\tau}{\mu}\omega(h) = \frac{i\tau}{\mu}\omega(h), \end{aligned} \tag{3.41}$$

where the first inequality follows by the fact that the function $(x, y) \mapsto x - (x \wedge y)^+$ is 1-Lipschitz with respect to both its real variables, and the second inequality follows from the inductive hypothesis and (3.40). Now, recalling that $\tau = \frac{T}{N}$, (3.41) implies that for all τ and $t \in [0, T]$ it holds $\|z_\tau(t) - s_h(z_\tau(t))\|_1 \le \frac{T}{\mu}\omega(h)$. Since $z_\tau(t) \in [0, 1]$, we have $|z_\tau(t) - s_h(z_\tau(t))| \le 1$, and then also

$$\|z_\tau(t) - s_h(z_\tau(t))\|_q^q \le \frac{T}{\mu}\omega(h). \tag{3.42}$$

Using (3.22h), the last formula and Theorem 3.7 imply (3.35). □

We are now ready to prove the conditions governing the flow rule.

Proposition 3.8. *Let $u \in L^\infty([0,T]; H^1(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ and $z \in L^2([0,T]; \mathcal{Z})$ be the functions defined in (3.22c) and (3.22h). Then for a.e. $t \in [0,T]$ it holds*

$$\langle V([u(t)]), \dot{z}(t)\rangle_\Gamma + \mu \|\dot{z}(t)\|^2_{L^2} - \langle \alpha, \dot{z}(t)\rangle_\Gamma = 0, \tag{3.43}$$

and

$$\langle V([u(t)]), \eta\chi_{\{z(t)>0\}}\rangle_\Gamma + \mu\langle \dot{z}(t), \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma \geq 0, \tag{3.44}$$

for all $\eta \in L^\infty(\Gamma)$, $\eta \leq 0$.

Proof. Let us fix $t \in [0,T]$, and for all τ we decompose the interface Γ as the union of the three sets $\Gamma = A^t_\tau \cup B^t_\tau \cup C^t_\tau$ where, if $t \in [t_i - 1, t_i)$, then $A^t_\tau := \{z_i = 0 < z_{i-1}\}$, $B^t_\tau := \{z_i = z_{i-1}\}$, $C^t_\tau := \{0 < z_i < z_{i-1}\}$. We recognize these three cases as the three options of (3.37), so that it is readily seen that

$$V([u_\tau])\dot{z}_\tau + \mu|\dot{z}_\tau|^2 - \alpha\dot{z}_\tau = 0, \tag{3.45}$$

on B^t_τ and C^t_τ, while on A^t_τ

$$V([u_\tau]) + \mu\dot{z}_\tau - \alpha \geq 0. \tag{3.46}$$

The latter being true for all $t \in [0,T]$. In particular recalling that $\dot{z}_\tau \leq 0$, we have

$$\int_0^T \langle V([u_\tau]), \dot{z}_\tau\rangle_\Gamma + \mu\|\dot{z}_\tau\|_2^2 - \langle \alpha, \dot{z}_\tau\rangle_\Gamma \mathrm{d}t \leq 0. \tag{3.47}$$

We would like to pass to the limit in (3.47). To this aim, from (3.27c) and the definition of V we first observe that actually $V([u_\tau])(t)$ is converging in $L^2(\Gamma, \mathbb{R})$ when $d < 3$ or the growth condition for V in (iii) is $\delta < \delta^*$. Thus, in this case, we have $V([u_\tau]) \to V([u])$ strongly in $L^2([0,T], L^2(\Gamma))$. This, together with (3.14g), implies

$$\int_0^T \langle V([u]), \dot{z}\rangle_\Gamma + \mu\|\dot{z}\|_2^2 - \langle \alpha, \dot{z}\rangle_\Gamma \mathrm{d}t \leq 0. \tag{3.48}$$

To treat the critical case $d \geq 3$ and $\delta = \delta^*$ we need a semicontinuity argument. We claim that

$$\int_0^T \langle V([u]), \dot{z}\rangle_\Gamma \mathrm{d}t \leq \liminf_{\tau\to 0} \int_0^T \langle V([u_\tau]), \dot{z}_\tau\rangle_\Gamma \mathrm{d}t. \tag{3.49}$$

This, together with the semicontinuity of the $L^2([0,T]; L^2(\Gamma))$ norm of $\dot{z}_\tau$, allows us to pass to the limit in (3.47) getting (3.48). Integrating by parts in time the term

$$\int_0^T \langle V([u_\tau]), \dot{z}_\tau\rangle_\Gamma \mathrm{d}t = \langle V([u_\tau(T)]), z_\tau(T)\rangle_\Gamma - \langle V([u_0]), z_0\rangle_\Gamma - \int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t} V([u_\tau]), z_\tau \right\rangle_\Gamma \mathrm{d}t,$$

we observe that, thanks to (3.28) and (3.35), the claim is equivalent to the inequality

$$\liminf_{\tau\to 0} \left(-\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t} V([u_\tau]), z_\tau \right\rangle_\Gamma \mathrm{d}t \right) \geq -\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t} V([u]), z \right\rangle_\Gamma \mathrm{d}t. \tag{3.50}$$

Let us prove this. We put $\varphi = u^n_i - u^n_{i-1}$ in (3.9), and then sum this expression on $i = 1, \ldots, n$. Arguing as in the proof of Proposition 3.3, easy computations leads us to

$$\begin{aligned} -\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t} V([u_\tau]), z_\tau \right\rangle_\Gamma \mathrm{d}t &= \frac{\rho}{2}\|\dot{u}_\tau\|^2_{L^2} - \frac{\rho}{2}\|v_0\|^2_{L^2} + \frac{\rho\tau}{2}\int_0^T \|\dot{v}_\tau\|^2_{L^2}\mathrm{d}t + \mathcal{Q}_0(e(u_\tau(T))) \\ &- \mathcal{Q}_0(e(u_0)) + \tau\int_0^T \mathcal{Q}_0(e(u_\tau))\mathrm{d}t + \mu\int_0^T \mathcal{Q}_1(e(\dot{u}_\tau))\mathrm{d}t - \int_0^T \langle \mathcal{L}_\tau, \dot{u}_\tau\rangle \mathrm{d}t + \int_0^T D_\tau \mathrm{d}t, \end{aligned} \tag{3.51}$$

where

$$D_\tau(t) := \tau\langle \nabla V([u_i])[\dot{u}_\tau(t)], z_{i-1}\rangle - \langle \nabla V([u_\tau(t)])[\dot{u}_\tau(t)], z_\tau(t)\rangle \qquad \text{if } t \in [t_{i-1}, t_i).$$

We now take the lim inf in (3.51). The third and sixth term in the right-hand side vanish as $\tau \to 0$ thanks to (3.14a) and (3.14e), and it is easy to estimate the last one, and to prove that is vanishes as well thanks to (ii) in Definition 2.1 and convergences (3.14a) and (3.14g). The eighth term passes to the limit by (3.13), whereas the first, fourth, and seventh one are lower semicontinuous, thanks to (3.14b) and (3.29). Therefore the liminf of (3.51) is greater or equal to

$$\frac{\rho}{2}\|\dot{u}\|^2_{L^2} - \frac{\rho}{2}\|v_0\|^2_{L^2} + \mathcal{Q}_0(e(u(T))) - \mathcal{Q}_0(e(u_0)) + \mu\int_0^T \mathcal{Q}_1(e(\dot{u}))\mathrm{d}t - \int_0^T \langle \mathcal{L}, \dot{u}\rangle\,\mathrm{d}t = -\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t}V([u]), z\right\rangle_\Gamma \mathrm{d}t,$$

where the equality follows from (3.32). Hence the claim (3.50) is proved.

Now formula (3.10) provides

$$\int_0^T \langle V([\tilde{u}_\tau]), \eta\chi_{\{\tilde{z}_\tau>0\}}\rangle_\Gamma + \mu\langle \dot{z}_\tau, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma \mathrm{d}t \geq 0. \tag{3.52}$$

for all $\eta \leq 0$. We note that, by definitions of z_τ and $\tilde{z}_\tau$ it holds $\chi_{\{\tilde{z}_\tau>0\}} = \chi_{\{z_\tau>0\}}$. From Lemma 3.6 we know that $z_\tau \to z$ strongly in $L^1(\Gamma\times[0,T])$, so that we can suppose it converges almost everywhere in $\Gamma\times[0,T]$. As a consequence we get

$$\liminf_{\tau\to 0} \chi_{\{z_\tau>0\}} \geq \chi_{\{z>0\}} \ \text{ a.e. on } \Gamma.$$

Then, from (3.52), the Fatou Lemma, and taking into account that $\eta \leq 0$ and that $V([u_\tau]) \to V([u])$ strongly in $L^1(\Gamma\times[0,T])$, we obtain

$$\begin{aligned} &\int_0^T \langle V([u]), \eta\chi_{\{z>0\}}\rangle_\Gamma \mathrm{d}t \geq \limsup_{\tau\to 0}\int_0^T \langle V([u_\tau]), \eta\chi_{\{z_\tau>0\}}\rangle_\Gamma \mathrm{d}t \\ &\geq -\liminf_{\tau\to 0}\int_0^T \mu\langle \dot{z}_\tau, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma \mathrm{d}t = \int_0^T \mu\langle \dot{z}, \eta\rangle_\Gamma - \langle \alpha, \eta\rangle_\Gamma \mathrm{d}t, \end{aligned} \tag{3.53}$$

for all $\eta \leq 0$. Thus (3.44) follows. Now, plugging $\eta = \dot{z}$ we recover the opposite inequality of (3.48) provided $\dot{z} = 0$ almost everywhere on the set $\{z = 0\}$. But this is a straightforward consequence of the fact that z is non-negative, and also (3.43) is proved.

□

Proof of Theorem 3.1. Equations (3.2) and (3.3) follows by definition of σ and from (3.32). Due to the arbitrariness of η, equation (3.44) readily implies

$$V([u(t)]) + \mu\dot{z}(t) - \alpha \leq 0 \qquad \text{a.e. on } \Gamma\cap\{z(t)>0\},$$

that is (3.4), while (3.43) implies (3.3a) and (3.3b), keeping into account that z is non-negative and non-increasing. To prove (3.5), we use (3.14b), (3.14d), and the fact that $u_\tau(0) = u_0$ and $z_\tau(0) = z_0$ for all τ. It remains to show that $\dot{u}(0) = v_0$. By (3.30) we have $v_\tau(t) \rightharpoonup \dot{u}(t)$ weakly in $H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)$ for all $t\in[0,T]$. The thesis follows since by definition $v_\tau(0) = v_0$ for all τ. □

When we deal with a nonhomogeneous boundary datum for u the existence theorem is stated as follows:

Theorem 3.9. *Let $\mathcal{L} \in L^2([0,T]; H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))$, $u_0, v_0 \in H^1(\Omega_1\cup\Omega_2;\mathbb{R}^d)$, $z_0 \in \mathcal{Z}$, and let $w \in H^1([0,T]; H^1_D(\Omega,\mathbb{R}^d))$ with $\dot{w} \in H^1([0,T]; H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))$ be such that $w(0) = u_0$ and $\dot{w}(0) = v_0$ on $\partial_D\Omega$.*

Then there exists a triple (u,σ,z) *with*

$$u \in H^1([0,T];H^1(\Omega_1\cup\Omega_2;\mathbb{R}^d)), \tag{3.54a}$$

$$\dot u \in H^1([0,T];H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))\cap L^\infty([0,T];L^2(\Omega_1\cup\Omega_2;\mathbb{R}^d)), \tag{3.54b}$$

$$\sigma \in L^2([0,T];L^2(\Omega_1\cup\Omega_2;\mathbb{R}^{d\times d}_{sym}), \tag{3.54c}$$

$$z \in H^1([0,T];\mathcal{Z}), \tag{3.54d}$$

satisfying (3.2), (3.3), *and* (3.4), *the initial data*

$$u(0)=u_0, \qquad \dot u(0)=v_0, \qquad z(0)=z_0, \tag{3.55}$$

and the Dirichlet condition

$$u(t)=w(t) \ \textit{on}\ \partial_D\Omega \qquad \textit{for a.e. } t\in[0,T]. \tag{3.56}$$

The proof is essentially the same of Theorem 3.1, that can be easily arranged.

Proof. We set $w^n_{-1}:=w(0)-\tau\dot w(0)$, $w^n_i=w(t^n_i)$, $\omega^n_i:=\frac{w^n_i-w^n_{i-1}}{\tau}$ for $i=0,\dots,n$, then we define the piecewise affine functions

$$w_\tau = w^n_i+(t-t^n_i)\frac{w^n_{i+1}-w^n_i}{\tau} \qquad \text{for } t\in[t^n_i,t^n_{i+1}), \tag{3.57a}$$

$$\omega_\tau = v^n_i+(t-t^n_i)\frac{\omega^n_{i+1}-\omega^n_i}{\tau} \qquad \text{for } t\in[t^n_i,t^n_{i+1}), \tag{3.57b}$$

for $i=0,\dots,n-1$. The fact that

$$w_\tau\to w \quad \text{strongly in } H^1([0,T];H^1(\Omega;\mathbb{R}^d)), \tag{3.58a}$$

$$\omega_\tau\to \dot w \quad \text{strongly in } H^1([0,T];H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)), \tag{3.58b}$$

is standard and easily checked. We also define the piecewise affine function $l_\tau:[0,T]\to H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)$ by setting

$$l_\tau:=\rho\dot\omega_\tau-\mathrm{div}_D(\mathbb{C}^0e(w_\tau)+\mu\mathbb{C}^1e(\dot w_\tau)), \tag{3.59}$$

so that property (2.8), the continuity of div_D, and (3.58) imply that

$$l_\tau\to l \quad \text{strongly in } L^2([0,T];H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)), \tag{3.60}$$

where $l:=\rho\ddot w-\mathrm{div}_D(\mathbb{C}^0e(w)+\mu\mathbb{C}^1e(\dot w))$. Arguing as in the proof of Theorem 3.1 we solve the minimum problems (3.7) and (3.8) with $\mathcal{L}^n_i-l(t^n_i)$ in place of $\mathcal{L}^n_i$ and denote by u^n_i and z^n_i their minimizers. Standard arguments taking into account relation (3.60) ensure that the same estimates (3.20) hold for the functions u^0_τ, z_τ, v^0_τ defined as in (3.12). So that we found functions $u^0\in H^1([0,T];H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))$ with $\dot u^0\in H^1([0,T];H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))$ and $z\in L^\infty([0,T];L^2(\Gamma))\cap H^1([0,T];\mathcal{Z})$ such that

$$u^0_\tau\rightharpoonup u^0 \quad \text{weakly in } H^1([0,T];H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)), \tag{3.61a}$$

$$u^0_\tau(t)\rightharpoonup u^0(t) \quad \text{weakly in } H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d),\ \text{for every } t\in[0,T], \tag{3.61b}$$

$$z_\tau\rightharpoonup z \quad \text{weakly* in } L^\infty([0,T];L^2(\Gamma)), \tag{3.61c}$$

$$z_\tau(t)\rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for every } t\in[0,T],, \tag{3.61d}$$

$$v^0_\tau\rightharpoonup \dot u^0 \quad \text{weakly* in } L^\infty([0,T];L^2(\Omega_1\cup\Omega_2;\mathbb{R}^d)), \tag{3.61e}$$

$$\dot v^0_\tau\rightharpoonup \ddot u^0 \quad \text{weakly in } L^2([0,T];H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)), \tag{3.61f}$$

$$\dot z_\tau\rightharpoonup \dot z \quad \text{weakly in } L^2([0,T];L^2(\Gamma)). \tag{3.61g}$$

Moreover we also get (3.10), (3.11), while (3.9) is replaced by the following

$$\rho\langle \dot v^0_\tau, \varphi\rangle + \langle \mathbb{C}^0 e(\tilde u^0_\tau) + \mu\mathbb{C}^1 e(\dot u^0_\tau), e(\varphi)\rangle + \langle \nabla V([\tilde u^0_\tau])\cdot[\varphi], \tilde z_\tau\rangle_\Gamma = \langle \tilde{\mathcal{L}}_\tau - \tilde l_\tau, \varphi\rangle,$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ and for a.e. $t \in [0,T]$. Arguing as in Proposition 3.5 we see that this expression passes to the limit as $\tau \to 0$ and leads one to

$$\rho\langle \ddot u^0, \varphi\rangle + \langle \mathbb{C}^0 e(\tilde u^0_\tau) + \mu\mathbb{C}^1 e(\dot u^0_\tau), e(\varphi)\rangle + \langle \nabla V([u^0])\cdot[\varphi], z\rangle_\Gamma = \langle \mathcal{L} - l, \varphi\rangle, \tag{3.62}$$

for all $\varphi \in H^1_D(\Omega_1\cup\Omega_2; \mathbb{R}^d)$ and for a.e. $t \in [0,T]$. If we define $u := u^0 + w$, observing that, since $w \in H^1(\Omega, \mathbb{R}^d)$, $[w] = 0$ on Γ, then (3.62) reads

$$\rho\langle \ddot u, \varphi\rangle + \langle \mathbb{C}^0 e(u) + \mu\mathbb{C}^1 e(\dot u), e(\varphi)\rangle + \langle \nabla V([u])\cdot[\varphi], z\rangle_\Gamma = \langle \mathcal{L}, \varphi\rangle.$$

At the same time (3.10) and (3.11) pass to the limit like in the case of a homogeneous boundary datum, and give rise to the same equations (3.43) and (3.44). The conclusion easily follows. □

The following Proposition provides the energy balance of the system.

Proposition 3.10. *Let u be the solution of Theorem 3.9. Then for all $0 \le t_1 < t_2 \le T$, the following energy balance holds*

$$\begin{aligned}
&\frac{\rho}{2}\|\dot u(t_2) - \dot w(t_2)\|^2_{L^2} + \mathcal{Q}_0(e(u(t_2))) + \langle V([u(t_2)]), z(t_2)\rangle_\Gamma + \mu\int_{t_1}^{t_2}\mathcal{Q}_1(e(\dot u))\mathrm{d}s\\
&+\mu\int_{t_1}^{t_2}\|\dot z\|^2_{L^2}\mathrm{d}s - \langle\alpha, z(t_2)\rangle_\Gamma = \frac{\rho}{2}\|\dot u(t_1) - \dot w(t_1)\|^2_{L^2} + \mathcal{Q}_0(e(u(t_1))) - \langle \alpha, z(t_1)\rangle_\Gamma\\
&+\langle V([u(t_1)]), z(t_1)\rangle_\Gamma + \int_{t_1}^{t_2}\langle\sigma, e(\dot w)\rangle\mathrm{d}s + \int_{t_1}^{t_2}\langle \mathcal{L} - \rho\ddot w, \dot u - \dot w\rangle\mathrm{d}s,
\end{aligned}\tag{3.63}$$

where $\sigma = \mathbb{C}^0 e(u) + \mu\mathbb{C}^1 e(\dot u)$.

Proof. We put $\varphi = \dot u - \dot w$ in (3.62) and sum this expression with (3.43). Integrating in time on $[t_1, t_2]$ we get (3.63). □

3.1. Processes in Mode II

In this section we discuss how to obtain a solution of the problem in Theorem 3.9 which also satisfy a constraint as in (2.21).

Let $D \subset \mathbb{R}^d$ be a linear subspace. For instance we can take the convex and closed cone defined in (2.20b). Let us define

$$\mathcal{N}_D := \{u \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d) \text{ such that } [u] \in D \text{ a.e. on } \Gamma\}.$$

Since $\sigma(t)$ is not in general an element of $L^1(\Gamma; \mathbb{R}^d)$, we prove a theorem where the solutions satisfy (2.21) in a weak form.

Theorem 3.11. *Let D be a linear subspace of $\mathbb{R}^d$ and let $\mathcal{L}$, u_0, v_0, z_0, and w be as in Theorem 3.9. Then there exists a couple (u,z) satisfying (3.54), (3.56), (3.55), and such that, for a.e. $t \in [0,T]$, it satisfies conditions (3.43), (3.44), and*

$$[u(t)] \in D \qquad \text{a.e. on } \Gamma, \tag{3.64a}$$

$$\langle \rho\ddot u, \varphi\rangle + \langle \mu\mathbb{C}^1 e(\dot u) + \mathbb{C}^0 e(u), e(\varphi)\rangle + \langle \nabla V([u])\cdot[\varphi], z\rangle_\Gamma = \langle \mathcal{L}, \varphi\rangle, \tag{3.64b}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ with $[\varphi] \in D$ a.e. on Γ.

We will give a sketch of the proof, being it very similar to the one of Theorem 3.1. For simplicity, let us treat the case with a homogeneous boundary datum.

Proof. We set u_i^n be the minimizer of U_i^n in (3.7) among all the functions $u \in \mathcal{N}_D$ and let z_i^n be the minimum of (3.8). Since $\mathcal{N}_D$ is a linear subspace of $H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ the discrete Euler condition (3.9) follows for all $\varphi \in \mathcal{N}_D$. Moreover we can follows the lines of the proof of Theorem 3.1 only replacing $H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ by $\mathcal{N}_D$. In particular since $\mathcal{N}_D$ is closed in $H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$ we get

$$[u(t)] \in D. \tag{3.65}$$

Now it is easy to pass to the limit as $h \to +\infty$ in (3.43) and (3.44), arguing as in as in the proof of Proposition 3.8. Instead (3.9) easily passes to the limit in the case that $\varphi \in \mathcal{N}_D$ providing condition (3.64b). $\square$

Corollary 3.12. *Let (u, z) be a solution of (3.1), (3.43), and (3.64). Then the energy balance (3.63) holds.*

Proof. The proof is the same as Proposition 3.10, since $\dot{u}$ satisfies the constraint $[\dot{u}] \in D$ and we can employ (3.64b) with $\varphi = \dot{u} - \dot{w}$. $\square$

4. Limit of solutions in rescaled time

In this section we study the asymptotic behavior of dynamic evolutions when the rate of the external loads and the boundary conditions becomes slower and slower. This is done by mean of a suitable rescaling of the data. If we start with an external load $\mathcal{L}$ and a datum w on $[0, T]$, we set $\mathcal{L}^\epsilon(t) := \mathcal{L}(t/\epsilon)$ and $w^\epsilon(t) := w(t/\epsilon)$ so that $\mathcal{L}^\epsilon$ and w^ϵ are defined on $[0, T/\epsilon]$. If (u^ϵ, z^ϵ) is the solution given by Theorem 3.9 with these data, we are interested in studying its behavior as $\epsilon \to 0$. To handle with this, another rescaling is required. We define $(u_\epsilon(t), z_\epsilon(t)) := (u^\epsilon(\epsilon t), z^\epsilon(\epsilon t))$, in such a way that the functions (u_ϵ, z_ϵ) are now defined on the same interval $[0, T]$. A straightforward change of variables shows that (u_ϵ, z_ϵ) solves the same equations of (u, z), with a scalar ϵ appearing besides all the terms with one time derivative, and ϵ^2 appearing beside the second derivative. In other words (u_ϵ, z_ϵ) are the solutions of the beginning delamination problem with a density mass equal to $\rho\epsilon^2$ and a viscosity parameter equal to $\mu\epsilon$. For simplicity in what follows we set $\rho = \mu = 1$, so that $\rho\epsilon^2$ reads ϵ^2 and $\mu\epsilon$ reads ϵ.

Now we are ready to perform the analysis of (u_ϵ, z_ϵ) as the parameter ϵ vanishes. We will restrict the attention to the dimension case $d \le 3$, and we will assume that the potential $V([u])$ has the form

$$V([u]) := \frac{1}{2}\mathbb{K}[u] \cdot [u],$$

where $\mathbb{K}$ is the elastic coefficient of the adhesive, constant on Γ. We assume also that $\mathbb{K}$ is positive definite, so that $\langle \mathbb{K}[u] \cdot [u] \rangle_\Gamma$ is a an equivalent norm on $L^2(\Gamma; \mathbb{R}^d)$. Such hypothesis is classical in the literature. Moreover we will need to assume more regularity on the data. In particular we suppose that $w \in H^2([0, T]; H^1_D(\Omega; \mathbb{R}^d))$ and $\mathcal{L} \in H^1([0, T]; H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$.

We first state the Theorem in the case of a homogeneous boundary datum.

Theorem 4.1. *Let $\mathcal{L} \in H^1([0, T]; H^{-1}_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ and u_0, v_0, z_0 as in Theorem 3.1. Let $(u_\epsilon, z_\epsilon)_\epsilon$ be a family of solutions to the problem of Theorem 3.1, then there exist $u \in L^\infty([0, T]; H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d))$ and $z \in L^2([0, T]; \mathcal{Z})$ such that, up to a subsequence,*

$$u_\epsilon \to u \text{ strongly in } L^2([0, T]; H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)), \tag{4.1a}$$

$$z_\epsilon \rightharpoonup z \quad \text{weakly* in } L^\infty([0, T]; \mathcal{Z}), \tag{4.1b}$$

$$z_\epsilon(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma) \text{ for all } t \in [0, T], \tag{4.1c}$$

as $\epsilon \to 0$. There also exist two non-negative Borel measures $\mu_z \in \mathcal{M}_b([0,T]\times\Gamma)$ and $\mu_e \in \mathcal{M}_b([0,T]\times\Omega)$ such that, for the same subsequence,

$$\epsilon \dot{z}_\epsilon^2 \rightharpoonup \mu_z \text{ weakly* in } \mathcal{M}_b([0,T]\times\Gamma), \tag{4.1d}$$
$$\epsilon \mathbb{C}^1 e(\dot{u}_\epsilon)\cdot e(\dot{u}_\epsilon) \rightharpoonup \mu_e \text{ weakly* in } \mathcal{M}_b([0,T]\times\Omega). \tag{4.1e}$$

as $\epsilon \to 0$. Moreover (u,z) satisfies for a.e. $t\in[0,T]$ the momentum balance

$$\langle \mathbb{C}^0 e(u(t)), e(\varphi)\rangle + \langle \mathbb{K}[u(t)]\cdot[\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}(t), \varphi\rangle, \tag{4.2}$$

for all $\varphi \in H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)$, and the energy equality

$$\begin{aligned}
&\mathcal{Q}_0(e(u)(t_2)) + \left\langle \frac{1}{2}\mathbb{K}[u(t_2)]\cdot[u(t_2)], z(t_2)\right\rangle_\Gamma - \langle \alpha, z(t_2)\rangle_\Gamma - \langle \mathcal{L}(t_2), u(t_2)\rangle \\
&+ \mu_z(]t_1,t_2]\times\Gamma) + \mu_b(]t_1,t_2]\times\Omega) = \mathcal{Q}_0(e(u(t_1))) + \langle\frac{1}{2}\mathbb{K}[u(t_1)]\cdot[u(t_1)], z(t_1)\rangle_\Gamma \\
&- \langle \alpha, z(t_1)\rangle_\Gamma - \langle \mathcal{L}(t_1), u(t_1)\rangle - \int_{t_1}^{t_2}\langle \dot{\mathcal{L}}, u\rangle \mathrm{d}s,
\end{aligned} \tag{4.3}$$

for a.e. $0\le t_1 < t_2 \le T$.

The proof of the theorem is essentially the same of ([30], Prop. 3.2), with the only difference that we have the addition of the viscosity of the adhesive. We summarize some important steps and emphasize some differences, and then refer to [30] for the detail.

Proof.
Step 1: *A priori* bounds. We start from the energy balance for the solution (u_ϵ, z_ϵ), that is

$$\begin{aligned}
&\frac{\epsilon^2}{2}\|\dot{u}_\epsilon(t)\|^2_{L^2} + \mathcal{Q}_0(e(u_\epsilon)(t)) + \left\langle \frac{1}{2}\mathbb{K}[u_\epsilon(t)]\cdot[u_\epsilon(t)], z_\epsilon(t)\right\rangle_\Gamma + \epsilon\int_0^t \mathcal{Q}_1(e(\dot{u}_\epsilon))\mathrm{d}s \\
&+ \epsilon\int_0^t \|\dot{z}_\epsilon\|^2_{L^2}\mathrm{d}s - \int_0^t \langle \alpha, \dot{z}_\epsilon\rangle_\Gamma \\
&= \frac{\epsilon^2}{2}\|u_0\|^2_{L^2} + \mathcal{Q}_0(e(u_0)) + \left\langle \frac{1}{2}\mathbb{K}[u_0]\cdot[u_0], z_0\right\rangle_\Gamma + \int_0^t \langle \mathcal{L}, \dot{u}_\epsilon\rangle\,\mathrm{d}s.
\end{aligned} \tag{4.4}$$

Integrating by parts in time the term $\int_0^t \langle \mathcal{L}, \dot{u}_\epsilon\rangle \mathrm{d}s$, and then using the Cauchy and the Korn inequalities, we see that the right-hand side of (4.4) is bounded by

$$\frac{C_0}{\lambda} + \frac{\beta\lambda}{2}\|e(u_\epsilon)(t)\|^2_2 + C_1\int_0^t \|e(u_\epsilon)\|^2_2\mathrm{d}s,$$

for some constants $C_0,\ C_1 > 0$ depending on the data of the problem but independent of ϵ, and for an arbitrary constant $\lambda > 0$. Setting $\lambda = \frac{\alpha_0}{2\beta}$, from (4.4) we obtain

$$\begin{aligned}
&\frac{\epsilon^2}{2}\|\dot{u}_\epsilon(t)\|^2_{L^2} + \frac{\alpha_0}{4}\|e(u_\epsilon)(t)\|^2_2 + \left\langle \frac{1}{2}\mathbb{K}[u_\epsilon(t)]\cdot[u_\epsilon(t)], z_\epsilon(t)\right\rangle_\Gamma + \epsilon\alpha_1\int_0^t \|e(\dot{u_\epsilon})\|^2_2\mathrm{d}s \\
&+ \epsilon\int_0^t \|\dot{z}_\epsilon\|^2_{L^2}\mathrm{d}s - \int_0^t \langle \alpha, \dot{z}_\epsilon\rangle_\Gamma \le \frac{2\beta C_0}{\alpha_0} + C_1\int_0^t \|e(u_\epsilon)\|^2_2\mathrm{d}s,
\end{aligned} \tag{4.5}$$

and in particular, since all the terms in the left-hand side are non-negative, we infer

$$\|e(u_\epsilon)(t)\|^2_2 \le C + C\int_0^t \|e(u_\epsilon)\|^2_2\mathrm{d}s, \tag{4.6}$$

for some constant $C > 0$ independent of ϵ. The Gronwall Lemma then implies that the right-hand side of (4.5) is bounded by a constant. This provides the following estimates: there exists a constant $C > 0$ such that

$$\|e(u_\epsilon)(t)\|^2_{L^\infty([0,T];L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \le C, \tag{4.7a}$$
$$\epsilon\|\dot{u}_\epsilon(t)\|_{L^\infty([0,T];L^2(\Omega;\mathbb{R}^d))} \le C, \tag{4.7b}$$
$$\epsilon\|e(\dot{u}_\epsilon)\|^2_{L^2([0,T];L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \le C, \tag{4.7c}$$
$$\epsilon\|\dot{z}_\epsilon\|^2_{L^2([0,T];L^2(\Gamma))} \le C. \tag{4.7d}$$

and arguing as in ([30], Prop.3.2) we find $z \in L^\infty([0,T];\mathcal{Z})$ such that

$$z_\epsilon(t) \rightharpoonup z(t) \quad \text{weakly* in } L^\infty(\Gamma), \tag{4.8}$$

for all $t \in [0,T]$. The boundedness (4.7a) and the Korn inequality imply that there exists $u \in L^\infty([0,T];H^1_D(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$ such that, up to a subsequence,

$$u_\epsilon \rightharpoonup u \text{ weakly* in } L^\infty([0,T];H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)), \tag{4.9a}$$
$$[u]_\epsilon \rightharpoonup [u] \text{ weakly* in } L^\infty([0,T];H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)), \tag{4.9b}$$

as $\epsilon \to 0$. Finally, the estimates (4.7c) and (4.7d) show that the functions $\epsilon\dot{z}^2_\epsilon$ and $\epsilon\mathbb{C}^1 e(\dot{u}_\epsilon)\cdot e(\dot{u}_\epsilon)$ are uniformly bounded in $L^1([0,T]\times\Gamma)$ and $L^1([0,T]\times\Omega)$ respectively, so that there exist two non-negative Borel measures μ_z and μ_e such that, up to a subsequence,

$$\epsilon\dot{z}^2_\epsilon \rightharpoonup \mu_z \text{ weakly* in } \mathcal{M}_b([0,T]\times\Gamma), \tag{4.9c}$$
$$\epsilon\mathbb{C}^1 e(\dot{u}_\epsilon)\cdot e(\dot{u}_\epsilon) \rightharpoonup \mu_e \text{ weakly* in } \mathcal{M}_b([0,T]\times\Omega). \tag{4.9d}$$

Step 2. The two following key lemmas are proved in ([30], Prop. 3.2).

Lemma 4.2. *For all $\varphi \in H^1(\Omega_1\cup\Omega_2;\mathbb{R}^d)$ and all ψ compactly supported real smooth function on $[0,T]$, it holds*

$$\lim_{\epsilon\to 0}\int_0^T \langle \mathbb{K}[u_\epsilon(s)]\psi(s)\cdot[\varphi], z_\epsilon(s)\rangle_\Gamma \mathrm{d}s = \int_0^T \langle \mathbb{K}[u(s)]\psi(s)\cdot[\varphi], z(s)\rangle_\Gamma \mathrm{d}s. \tag{4.10}$$

Lemma 4.3. *It holds*

$$\int_0^t \langle \mathbb{K}[u(s)]\cdot[u(s)], z(s)\rangle_\Gamma \mathrm{d}s \le \liminf_{\epsilon\to 0}\int_0^t \langle \mathbb{K}[u_\epsilon(s)]\cdot[u_\epsilon(s)], z_\epsilon(s)\rangle_\Gamma \mathrm{d}s. \tag{4.11}$$

Step 3. Let ψ be a smooth and compactly supported positive function on $[0,T]$. Multiplying equation (3.32) for (u_ϵ, z_ϵ) by ψ, and then integrating in time on $[0,T]$, we obtain

$$\begin{aligned}&\int_0^T \big(\langle \mathbb{C}^0 e(u_\epsilon) + \epsilon\mathbb{C}^1 e(\dot{u}_\epsilon), e(\varphi)\rangle + \langle \mathbb{K}[u_\epsilon]\cdot[\varphi], z_\epsilon\rangle_\Gamma\big)\psi \mathrm{d}s \\ &= \int_0^T \langle \epsilon^2\dot{u}_\epsilon, \varphi\rangle\dot{\psi} + \langle \mathcal{L},\varphi\rangle\psi \mathrm{d}s.\end{aligned} \tag{4.12}$$

Lemma 4.2 allows us to pass to the limit obtaining, thanks to (4.7b)−(4.7d), (4.8), (4.9), and the arbitrariness of ψ, so that

$$\langle \mathbb{C}^0 e(u(t)), e(\varphi)\rangle + \langle \mathbb{K}[u(t)]\cdot[\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}(t),\varphi\rangle, \tag{4.13}$$

for all $\varphi \in H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)$ and a.e. $t \in [0,T]$.

Putting $\varphi = u_\epsilon$ in (3.32) for (u_ϵ, z_ϵ) and then integrating in time on $[0, t]$ we obtain

$$\epsilon^2\langle \dot{u}_\epsilon(t), u_\epsilon(t)\rangle + \frac{\epsilon}{2}\mathcal{Q}_1(e(u_\epsilon(t))) - \int_0^t \epsilon^2\|\dot{u}_\epsilon\|^2 + \mathcal{Q}_0(e(u_\epsilon))\mathrm{d}s$$
$$= \epsilon^2\langle v_0, u_0\rangle + \frac{\epsilon}{2}\mathcal{Q}_1(e(u_0))) - \int_0^t \langle \mathbb{K}[u_\epsilon]\cdot[u_\epsilon], z_\epsilon\rangle_\Gamma + \langle \mathcal{L}, u_\epsilon\rangle \mathrm{d}s, \tag{4.14}$$

and taking into account the estimates (4.7b)–(4.7d), we obtain

$$\lim_{\epsilon\to 0}\int_0^t \mathcal{Q}_0(e(u_\epsilon)) + \langle \mathbb{K}[u_\epsilon]\cdot[u_\epsilon], z_\epsilon\rangle_\Gamma \mathrm{d}s = \int_0^t \langle \mathcal{L}, u\rangle \mathrm{d}s. \tag{4.15}$$

Setting $\varphi = u$ in (4.13), the right-hand side equals $\int_0^t \mathcal{Q}_0(e(u)) + \langle \mathbb{K}[u]\cdot[u], z\rangle_\Gamma \mathrm{d}s$. Now

$$\int_0^t \mathcal{Q}_0(e(u)) \le \liminf_{\epsilon\to 0}\int_0^t \mathcal{Q}_0(e(u_\epsilon))\mathrm{d}s,$$

and, from Lemma 4.3,

$$\int_0^t \langle \mathbb{K}[u]\cdot[u], z\rangle_\Gamma \mathrm{d}s \le \liminf_{\epsilon\to 0}\int_0^t \langle \mathbb{K}[u_\epsilon]\cdot[u_\epsilon], z_\epsilon\rangle_\Gamma \mathrm{d}s,$$

so that, by (4.13) and (4.15), we infer that equalities hold, and hence

$$u_\epsilon \to u \text{ strongly in } L^2([0,T]; H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)). \tag{4.16}$$

In particular for a.e. $t\in[0,T]$ one has

$$u_\epsilon(t) \to u(t) \text{ strongly in } H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d), \tag{4.17}$$
$$[u_\epsilon](t) \to [u](t) \text{ strongly in } H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d), \tag{4.18}$$

and, thanks to (4.8), we have

$$\langle \mathbb{K}[u_\epsilon(t)]\cdot[u_\epsilon(t)], z_\epsilon(t)\rangle_\Gamma \to \langle \mathbb{K}[u(t)]\cdot[u(t)], z(t)\rangle_\Gamma, \tag{4.19}$$

for a.e. $t\in[0,T]$. This allows us to pass to the limit as $\epsilon\to 0$ in the energy balance for (u_ϵ, z_ϵ) in order to obtain (4.3), provided

$$\epsilon\int_{t_1}^{t_2} \mathcal{Q}_1(e(\dot{u}_\epsilon))\mathrm{d}s \to \mu_b(]t_1,t_2]\times\Omega), \tag{4.20a}$$
$$\epsilon\int_{t_1}^{t_2} \|\dot{z}_\epsilon\|^2\mathrm{d}s \to \mu_z(]t_1,t_2]\times\Gamma), \tag{4.20b}$$

for a.e. $0\le t_1 < t_2 \le T$.

Step 4. To prove (4.20) we follow the lines of the proof of ([30], Prop. 3.2), which straightforwardly applies to this case. This concludes the proof. □

Theorem 4.1 easily generalizes to the case of a non-homogeneous boundary datum. We state here the result without the proof since it is a simple arrangement of the one above.

Theorem 4.4. *Let* $\mathcal{L}\in H^1([0,T]; H^{-1}_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))$, $w\in H^2([0,T]; H^1_D(\Omega;\mathbb{R}^d))$, *and* u_0, v_0, z_0 *as in Theorem 3.9. Let* (u_ϵ, z_ϵ) *be the solution given by Theorem 3.1, then there exist* $u\in L^\infty([0,T]; H^1(\Omega_1\cup\Omega_2;\mathbb{R}^d))$

with $u(t) = w(t)$ on $\partial_D \Omega$, and $z \in L^2([0,T];\mathcal{Z})$ such that, up to subsequence, (4.1) holds as $\epsilon \to 0$, and the momentum balance holds for a.e. $t \in [0,T]$

$$\langle \mathbb{C}^0 e(u(t)), e(\varphi)\rangle + \langle \mathbb{K}[u(t)]\cdot[\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}(t), \varphi\rangle, \tag{4.21}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$. Moreover the energy equality

$$\begin{aligned}
&\mathcal{Q}_0(e(u)(t_2)) + \left\langle \frac{1}{2}\mathbb{K}[u(t_2)]\cdot[u(t_2)], z(t_2)\right\rangle_\Gamma - \langle \alpha, z(t_2)\rangle_\Gamma - \langle \mathcal{L}(t_2), u(t_2) - w(t_2)\rangle \\
&= \mathcal{Q}_0(e(u(t_1))) + \left\langle \frac{1}{2}\mathbb{K}[u(t_1)]\cdot[u(t_1)], z(t_1)\right\rangle_\Gamma - \langle \mathcal{L}(t_1), u(t_1) - w(t_1)\rangle - \langle \alpha, z(t_1)\rangle_\Gamma \\
&\quad + \mu_z(]t_1,t_2]\times\Gamma) + \mu_b(]t_1,t_2]\times\Omega) - \int_{t_1}^{t_2}\langle \dot{\mathcal{L}}, u - w\rangle \mathrm{d}s + \int_{t_1}^{t_2}\langle \sigma, e(\dot{w})\rangle \mathrm{d}s,
\end{aligned} \tag{4.22}$$

holds true for a.e. $0 \le t_1 < t_2 \le T$, where $\sigma := \mathbb{C}^0 e(u)$.

An immediate consequence of (4.22) is the following:

Corollary 4.5. *Let (u,z) be the evolution obtained in the previous theorem. Then*

$$\begin{aligned}
&\mathcal{Q}_0(e(u)(t_2)) + \left\langle \frac{1}{2}\mathbb{K}[u(t_2)]\cdot[u(t_2)], z(t_2)\right\rangle_\Gamma - \langle \alpha, z(t_2)\rangle_\Gamma - \langle \mathcal{L}(t_2), u(t_2) - w(t_2)\rangle \\
&\le \mathcal{Q}_0(e(u(t_1))) + \left\langle \frac{1}{2}\mathbb{K}[u(t_1)]\cdot[u(t_1)], z(t_1)\right\rangle_\Gamma - \langle \mathcal{L}(t_1), u(t_1) - w(t_1)\rangle \\
&\quad - \langle \alpha, z(t_1)\rangle_\Gamma - \int_{t_1}^{t_2}\langle \dot{\mathcal{L}}, u - w\rangle \mathrm{d}s + \int_{t_1}^{t_2}\langle \sigma, e(\dot{w})\rangle \mathrm{d}s,
\end{aligned} \tag{4.23}$$

for a.e. $0 \le t_1 < t_2 \le T$.

Remark 4.6 (Limit of processes in mode II)**.** The limit of evolution with constraints as provided by Theorem 3.11 is straightforwardly arranged. The limit (u,z) will satisfy for all $t \in [0,T]$ the property

$$u(t) \in D, \tag{4.24}$$

while the momentum balance (4.21) is replaced by

$$\langle \mathbb{C}^0 e(u(t)), e(\varphi)\rangle + \langle \mathbb{K}[u(t)]\cdot[\varphi], z(t)\rangle_\Gamma = \langle \mathcal{L}, \varphi\rangle, \tag{4.25}$$

for all $\varphi \in H^1_D(\Omega_1 \cup \Omega_2;\mathbb{R}^d)$ with $[\varphi] \in D$ a.e. on Γ.

We are now in position to discuss the flow rule of the limit evolution (u,z). The presence of the viscosity term $\dot{z}$ in the flow rules (3.43) and (3.44), in contrast to [30] where the flow rule is rate-independent, makes the following analysis necessary.

Lemma 4.7. *For a.e. $(x,t) \in \Gamma \times [0,T]$ it holds*

$$\textit{either}\ \ \frac{1}{2}\mathbb{K}[u(x,t)]\cdot[u(x,t)] - \alpha(x) \le 0 \qquad \textit{or}\ z(x,t) = 0. \tag{4.26}$$

Proof. By (3.4), for all $\epsilon > 0$ it holds

$$\left(\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon] + \epsilon\dot{z}_\epsilon - \alpha\right)\chi_{\{z_\epsilon > 0\}} \le 0.$$

Up to a subsequence we have that $\chi_{\{z_\epsilon>0\}} \rightharpoonup \zeta$ weakly* in $L^\infty([0,T]\times\Gamma)$ for some $\zeta \in L^\infty([0,T]\times\Gamma)$. Thanks to (4.7d) we know that $\epsilon\dot{z}_\epsilon \to 0$ strongly in $L^2([0,T];L^2(\Gamma))$, while thanks to (4.17) and (4.18) we know that $\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon] \to \frac{1}{2}\mathbb{K}[u]\cdot[u]$ strongly in $L^1([0,T];L^1(\Gamma))$, so that at the limit as $\epsilon\to 0$ the previous relation gives rise to

$$\left(\frac{1}{2}\mathbb{K}[u]\cdot[u]-\alpha\right)\zeta \leq 0, \tag{4.27}$$

almost everywhere on $[0,T]\times\Gamma$. Now the thesis follows if we prove that $\zeta > 0$ on the set $\{z > 0\}$. Let $A := \{(t,x)\in[0,T]\times\Gamma : 0 = \zeta(t,x) < z(x,t)\}$, and let us prove that $|A| = 0$. Then suppose $|A| > 0$. From the fact that $z_\epsilon(t) \rightharpoonup z(t)$ weakly* in $L^\infty(\Gamma)$ for all $t\in[0,T]$, the Fubini theorem and the Dominated Convergence Theorem imply

$$0 < \int_A z = \lim_{\epsilon\to 0}\int_A z_\epsilon.$$

On the other side we see that the right-hand side must be zero since $z_\epsilon \to 0$ strongly in $L^1(A)$. Indeed, by the fact that $z_\epsilon \leq 1$, we have that $\|z_\epsilon\|_{L^1(A)} \leq |\{z_\epsilon > 0\}\cap A| = \int_A \chi_{\{z_\epsilon>0\}} \to \int_A \zeta = 0$ by hypothesis, and the lemma is proved.

□

Now we prove that there is a representative $\bar{z} : [0,T]\times\Gamma \to [0,1]$ in the class of $z \in L^1([0,T]\times\Gamma)$ that is non-increasing in the time variable, and thus for a.e. $(t,x)\in[0,T]\times\Gamma$ there exists the time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\bar{z}(t,x)\in\mathbb{R}$. Let us define

$$\bar{z}(t,x) := \liminf_{\delta\to 0}\int_{B_{x,\delta}} z(t,y)\mathrm{d}y, \tag{4.28}$$

where $B_{x,\delta}$ is the ball in Γ centered at x and with radius $\delta > 0$. It turns out that such limit always exists and coincides with $z(t,x)$ for a.e. $(t,x)\in[0,T]\times\Gamma$. Moreover for all x and all $0 \leq t_1 < t_2 \leq T$ it holds $\bar{z}(t_1,x) \leq \bar{z}(t_2,x)$, since this inequality holds for z_ϵ and we have $\int_{B_{x,\delta}} z(t,y)dy = \lim_{\epsilon\to 0}\int_{B_{x,\delta}} z_\epsilon(t,y)dy$ for all $\delta > 0$ by (4.8). In particular for all fixed $x\in\Gamma$ the function $t \to \bar{z}(t,x)$ is non-increasing, and then differentiable almost everywhere on $[0,T]$.

For $\bar{z}$ the following is true.

Lemma 4.8. *For a.e.* $(t,x)\in[0,T]\times\Gamma$ *it holds*

$$\left(\frac{1}{2}\mathbb{K}[u(t,x)]\cdot[u(t,x)]-\alpha(x)\right)\dot{\bar{z}}(t,x) = 0. \tag{4.29}$$

Proof. For all real numbers $0 \leq a < b \leq T$ and all open sets $A\subset\Gamma$ we can define the total variation of z_ϵ on $[a,b]\times A$ as

$$\mathrm{Var}(z_\epsilon,[a,b]\times A) := \langle\chi_A, z_\epsilon(a)-z_\epsilon(b)\rangle_\Gamma, \tag{4.30}$$

that defines a non-negative measure on the Borel subsets of $[0,T]\times\Gamma$. Defining similarly the total variation of z we see that $\mathrm{Var}(z_\epsilon,\cdot) \rightharpoonup \mathrm{Var}(z,\cdot)$ weakly* in the space of non-negative Radon measures $\mathcal{M}_b([0,T]\times\Gamma)$. Writing $z_\epsilon(a)-z_\epsilon(b) = -\int_a^b \dot{z}_\epsilon(s)\mathrm{d}s$ and similarly $z(a)-z(b) = -\int_a^b D_t\bar{z}(s)\mathrm{d}s$ (where D_t is the distributional derivative in time), we also get

$$-\int_B \dot{\bar{z}} \leq \mathrm{Var}(\bar{z},B) \leq \liminf_{\epsilon\to 0}\mathrm{Var}(z_\epsilon,B) = \liminf_{\epsilon\to 0}\left(-\int_B \dot{z}_\epsilon\right), \tag{4.31}$$

for all Borel sets $B \subset [0,T] \times \Gamma$, where the first inequality is due to the fact that $-\dot{\bar{z}}$ is only the part of $-D_t\bar{z}$ that is absolutely continuous with respect to the Lebesgue measure, while the second one follows by the lower semicontinuity of the mass.

From the fact that $\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon] \to \frac{1}{2}\mathbb{K}[u]\cdot[u]$ strongly in $L^1([0,T];L^1(\Gamma))$ we have that $\frac{1}{2}\mathbb{K}[u_\epsilon(t,x)]\cdot[u_\epsilon(t,x)] \to \frac{1}{2}\mathbb{K}[u(t,x)]\cdot[u(t,x)]$ for a.e. $(t,x) \in [0,T]\times\Gamma$. Let us define $C := \{(t,x) \in [0,T]\times\Gamma : \dot{\bar{z}}(t,x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t,x)]\cdot[u(t,x)] - \alpha(x) \neq 0\}$. Since $\bar{z}$ is non-negative and non-increasing it is straightforward that $\dot{\bar{z}} = 0$ on the set $\bar{z} = 0$, so that condition (4.26) tells us that $|(C \setminus C') \cup (C' \setminus C)| = 0$, with $C' := \{(t,x) \in [0,T]\times\Gamma : \dot{\bar{z}}(t,x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t,x)]\cdot[u(t,x)] - \alpha(x) < 0\}$. To prove the Lemma it then suffices to show that $|C'| = 0$. Suppose it is not the case and for some $n > 0$ it holds $|C_n| > 0$, with $C_n := \{(t,x) \in [0,T]\times\Gamma : \dot{\bar{z}}(t,x) \neq 0,\ \frac{1}{2}\mathbb{K}[u(t,x)]\cdot[u(t,x)] - \alpha(x) < -\frac{1}{n}\}$. Thanks to the pointwise convergence of $\frac{1}{2}\mathbb{K}[u_\epsilon]\cdot[u_\epsilon]$ to $\frac{1}{2}\mathbb{K}[u]\cdot[u]$ we can find a subset $B \subset C_n$ with positive measure and a number $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and all $(t,x) \in B$ it holds $\mathbb{K}[u_\epsilon(t,x)]\cdot[u_\epsilon(t,x)] - \alpha(x) < 0$. This means that, thanks to (3.3b), $\dot{z}_\epsilon(t,x) = 0$ for all $\epsilon < \epsilon_0$ and all $(t,x) \in B$. Hence

$$0 = -\lim_{\epsilon\to 0}\int_B \dot{z}_\epsilon \geq -\int_B \dot{\bar{z}},$$

where we have used (4.31). But since $-\dot{\bar{z}}$ is non-negative we find $\dot{\bar{z}} = 0$ almost everywhere on B, contradicting the fact that $B \subset C_n$. $\square$

To simplify the notation let us now confine our discussion to the case of a homogeneous boundary datum. We remark that the same following facts can be stated for the case $w \neq 0$. Let $\mathcal{E} : [0,T] \to \mathbb{R}$ be the energy of the limit evolution (u,z) obtained in Theorem 4.1 defined as

$$\mathcal{E}(t) := \mathcal{Q}_0(e(u)(t)) + \left\langle \frac{1}{2}\mathbb{K}[u(t)]\cdot[u(t)], z(t)\right\rangle_\Gamma - \langle\alpha, z(t)\rangle_\Gamma - \langle\mathcal{L}(t), u(t)\rangle + \int_0^t \langle\dot{\mathcal{L}}, u\rangle \mathrm{d}s, \tag{4.32}$$

for all $t \in [0,T]$. Inequality (4.23) says exactly that $\mathcal{E}$ is an essentially non-increasing function. Essentially means that there exists a negligible set $N \subset [0,T]$ such that $\mathcal{E}$ is non-increasing on $[0,T]\setminus N$. We can then always extend it to a (unique) left-continuous non-increasing function on the whole $[0,T]$, denoted by the same symbol $\mathcal{E}$. As a consequence the new $\mathcal{E}$ is discontinuous on an at most countable set $J_E \subset [0,T]$, and this set does not depend on the value of $\mathcal{E}$ on N. We will also denote by J_z the subset of $[0,T]$ where the function z is discontinuous with respect to the strong topology of $L^1(\Gamma)$. Since z is a non-increasing function with values in $[0,1]$, we see that J_z is at most countable as well.

Theorem 4.1 shows that the evolution (u,z), limit of (u_ϵ, z_ϵ), satisfies the momentum balance almost everywhere on $[0,T]$. The next Lemma gives a more precise description of the set of times where the momentum balance holds, and at the same time tells us that we can change the map $u \in L^\infty([0,T];L^2(\Omega;\mathbb{R}^d))$ on the negligible set $N \subset [0,T]$ in such a way that the energy $\mathcal{E}$ is globally non-increasing.

Lemma 4.9. *Suppose $\bar{t} \in [0,T]\setminus(J_E \cup N)$ is such that z is continuous at $\bar{t}$ with respect to the strong topology of $L^1(\Gamma)$,* i.e. *$\bar{t} \notin J_z$. Then the momentum balance (4.2) holds at such $\bar{t}$.*

Moreover there exists a representative of $u \in L^\infty([0,T];H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d))$, still denoted by u, such that the momentum balance (4.2) holds at all $t \in [0,T]\setminus J_z$ and the corresponding energy (4.32) is non-increasing and continuous at all $t \in [0,T]\setminus J_z$.

Proof. Condition (4.2) tells us that $u(t)$ is the (unique) minimizer in $H^1_D(\Omega_1\cup\Omega_2;\mathbb{R}^d)$ of the potential

$$W_t(u) := \mathcal{Q}_0(e(u)) + \langle\frac{1}{2}\mathbb{K}[u]\cdot[u], z(t)\rangle_\Gamma - \langle\mathcal{L}(t), u\rangle. \tag{4.33}$$

Let us denote by $M(t) := \min W_t$. The fact that z is continuous at $\bar{t}$ entails that also M is continuous at $\bar{t}$. Let us choose a sequence t_n with $t_n \to \bar{t}$ such that $t_n \notin N$ and $u(t_n)$ satisfies the momentum balance (4.2) for all

$n > 0$. Then we have

$$\lim_{n\to\infty} \mathcal{E}(t_n) = \lim_{n\to\infty} \left(M(t_n) + \langle \alpha, z(t_n) \rangle - \int_0^{t_n} \langle \dot{\mathcal{L}}, u \rangle \mathrm{d}s \right)$$
$$= M(\bar{t}) + \langle \alpha, z(\bar{t}) \rangle - \int_0^{\bar{t}} \langle \dot{\mathcal{L}}, u \rangle \mathrm{d}s = \mathcal{E}(\bar{t}), \tag{4.34}$$

where the last equality follows from the continuity of $\mathcal{E}$. This says that $W_{\bar{t}}(u(\bar{t})) = M(\bar{t})$, which, thanks to the uniqueness of the minimizer of $W_{\bar{t}}$, entails that $u(\bar{t})$ is such minimizer, so that it also satisfies (4.2), and the first part of the statement is proved.

Let us now fix $t \in [0,T] \setminus J_z$, if we choose $t_n \in [0,T] \setminus J_z$ such that $t_n \to t$ and $u(t_n)$ satisfies the momentum balance (4.2), formula (4.34) still holds with $\bar{t}$ replaced by t thanks to the continuity of z, therefore proving that we can redefine u at all points $t \in N \setminus J_z$ as the minimizer of W_t. We see that the new u coincides with the old one almost everywhere and satisfies (4.2) at all $t \in N \setminus J_z$ by definition. This concludes the proof, noting that the new $\mathcal{E}$ corresponding to the new u is continuous on $[0,T] \setminus J_z$. □

Remark 4.10. A consequence of Lemma 4.9 is that the set of times $t \in [0,T]$ such that the new $u(t)$ does not satisfy the momentum balance (4.2) is an at most countable set, being J_z at most countable. Let us denote it by S_u. Lemma 4.9 then reads

$$(S_u \cup J_E) \subset J_z.$$

Another consequence of Lemma 4.9 is that at every time where z is continuous (*i.e.*, outside J_z), also u is continuous with respect to the strong topology of $H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$. This follows from the continuity of $M(t)$, and from the fact that $\mathcal{Q}_0(e(u)) + \langle \frac{1}{2}\mathbb{K}[u] \cdot [u], z \rangle_\Gamma$ is an equivalent norm in $H^1_D(\Omega_1 \cup \Omega_2; \mathbb{R}^d)$, for every fixed $z \in \mathcal{Z}$. Then, if we denote by J_u the set of times where u is discontinuous, J_u is at most countable and $J_u \subset J_z$.

Thanks to the strong continuity of u and z we also infer that for all $t \in [0,T] \setminus J_z$ relation (4.26) holds true for $\mathcal{H}^{d-1}$-a.e. $x \in \Gamma$.

Let us finally remark that, with the new definition of $\mathcal{E}$, the energy inequality (4.23) holds for all $t_1, t_2 \in [0,T] \setminus J_z$.

Theorem 4.11. *Suppose that there exists $0 < s \leq T$, $s \notin J_z$, such that $z(t,x) > 0$ at a.e. $x \in \Gamma$ for all $0 \leq t \leq s$. Then the energy $\mathcal{E}$ is constant on $[0,s] \setminus J_z$,* i.e. *$\mathcal{E}(t) = \mathcal{E}(0)$ for all $t \in [0,s] \setminus J_z$. In particular $\mu_z = 0$ on $[0,s] \times \Gamma$ and $\mu_e = 0$ on $[0,s] \times \Omega$.*

Proof. Taking into account (4.23), it suffices to show that $\mathcal{E}(0) \leq \mathcal{E}(s)$. To prove this, for all integers $n > 0$ let us choose a sequence of times $0 = t_0 < t_1 < \cdots < t_n = s$ such that $t_i \in [0,T] \setminus S_u$ for all $i \leq n$ and such that $\max_{i<n} |t_{i+1} - t_i| \to 0$ as $n \to \infty$. The minimality of W_{t_i} at $u(t_i)$ implies $W_{t_i}(u(t_i)) \leq W_{t_i}(u(t_{i+1}))$ for all $0 \leq i < n$. This implies

$$\begin{aligned}
&\mathcal{Q}_0(e(u(t_i))) - \mathcal{Q}_0(e(u(t_{i+1}))) - \langle \mathcal{L}(t_i), u(t_i) \rangle + \langle \mathcal{L}(t_{i+1}), u(t_{i+1}) \rangle \\
&+ \left\langle \frac{1}{2}\mathbb{K}[u(t_i)] \cdot [u(t_i)], z(t_i) \right\rangle_\Gamma - \langle \frac{1}{2}\mathbb{K}[u(t_{i+1})] \cdot [u(t_{i+1})], z(t_{i+1}) \rangle_\Gamma \\
&\leq \left\langle \frac{1}{2}\mathbb{K}[u(t_{i+1})] \cdot [u(t_{i+1})], z(t_i) - z(t_{i+1}) \right\rangle_\Gamma + \langle \mathcal{L}(t_{i+1}) - \mathcal{L}(t_i), u(t_{i+1}) \rangle \\
&\leq \langle \alpha, z(t_i) - z(t_{i+1}) \rangle_\Gamma + \langle \mathcal{L}(t_{i+1}) - \mathcal{L}(t_i), u(t_{i+1}) \rangle,
\end{aligned} \tag{4.35}$$

where in the last inequality we have used (4.26) with Remark 4.10. Summing this expression on $i = 0, \dots, n-1$ we obtain

$$\mathcal{Q}_0(e(u(0))) - \mathcal{Q}_0(e(u(s))) - \langle \mathcal{L}(0), u(0)\rangle + \langle \mathcal{L}(s), u(s)\rangle + \left\langle \frac{1}{2}\mathbb{K}[u(0)]\cdot[u(0)], z(0)\right\rangle_{\Gamma} - \left\langle \frac{1}{2}\mathbb{K}[u(s)]\cdot[u(s)], z(s)\right\rangle_{\Gamma}$$
$$\leq \langle \alpha, z(0)\rangle_{\Gamma} - \langle \alpha, z(s)\rangle_{\Gamma} + \sum_{i=0}^{n-1}\langle \mathcal{L}(t_{i+1}) - \mathcal{L}(t_i), u(t_{i+1})\rangle.$$

The last term tends to $\int_0^s \langle \dot{\mathcal{L}}, u\rangle \mathrm{d}s$ as $n \to \infty$ thanks to the regularity of $\mathcal{L}$ and the fact that J_u is at most countable. Hence the inequality above implies exactly $\mathcal{E}(0) \leq \mathcal{E}(s)$, and the thesis follows. □

Remark 4.12. If we do not redefine the functions $\mathcal{E}$ and u as in Lemma 4.9, Theorem 4.11 still holds, with the only difference that the equality $\mathcal{E}(t) = \mathcal{E}(0)$ holds only for a.e. $t \in [0, s] \setminus (N \cup J_z)$. To see this it suffices to apply the same proof with the only difference that we have to choose the times t_i in the set where (4.21) holds for the original u.

4.1. The one-dimensional case

In this section we consider the case $d = 1$. Without loss of generality we set $\Omega_1 :=]0, 1[$, $\Omega_2 :=]-1, 0[$, $\Gamma := \{0\}$, $\partial_D\Omega := \{-1, 1\}$, and assume that $\mathbb{C}^0 = 1$ and $\mathbb{K} = 1$. We denote by u the displacement, and we want to study an evolution with Dirichlet conditions $u(t, 1) = a_1(t)$, $u(t, -1) = a_{-1}(t)$ for all $t \in [0, T]$, and external forces $\mathcal{L}(t, x)$. This arises imposing $w(t, x) := a_{-1}(t) + \frac{x+1}{2}(a_1(t) - a_{-1}(t))$. We assume that at the initial time we have $z_0 = 1$.

Let us first state the following preliminary fact:

Lemma 4.13. *$\mathcal{L} \in H_D^{-1}(]-1, 0[\cup]0, 1[)$ if and only if there exists $F \in L^2(]-1, 0[\cup]0, 1[)$ such that $\langle \mathcal{L}, \varphi\rangle = -\langle F, \varphi_x\rangle$, for all $\varphi \in H_D^1(]-1, 0[\cup]0, 1[)$.*

Proof. We can write

$$\langle \mathcal{L}, \varphi\rangle \leq C_1\|\varphi\|_{H^1} \leq C_2\|\varphi_x\|_2,$$

thanks to the Poincaré inequality. In particular, since the linear map $A : H_D^{-1}(]-1, 0[\cup]0, 1[) \to L^2(]-1, 0[\cup]0, 1[)$ given by $A(\varphi) = \varphi_x$ is bijective, we see that $\mathcal{L} \circ A^{-1}$ belongs to the dual of $L^2(]-1, 0[\cup]0, 1[)$, and then there exists $F \in L^2(]-1, 0[\cup]0, 1[)$ such that $\mathcal{L} \circ A^{-1}(\psi) = -\langle F, \psi\rangle$ for all $\psi \in L^2(]-1, 0[\cup]0, 1[)$. The claim follows by writing $\psi = \varphi_x$. □

Lemma 4.9 guarantees that (u, z) satisfies (4.21) and (4.23) everywhere on $[0, T] \setminus J_z$. Now we prove that, up to suitably change the function $t \to (u(t), z(t))$ on a negligible set, we can assume that such conditions are satisfied for all $t \in [0, T[$. In the one-dimensional case $z(t)$ is just a real number, and convergence (4.1c) ensures that z is non-increasing, and then coincides with $\bar{z}$ defined in (4.28). We define

$$\tilde{z}(t) := \lim_{s \to t^-} z(s).$$

In particular $\tilde{z}$ is left-continuous. Let $S_u \subset [0, T]$ be the set of all t at which (4.21) does not hold. Then for all $t \in S_u$ we define $\hat{u}(t)$ as the (unique) solution of problem (4.21) with $z(t)$ replaced by $\tilde{z}(t)$ and boundary datum $w(t)$. Then we set

$$\tilde{u}(t) := \begin{cases} \hat{u}(t) & \text{if } t \in S_u \\ u(t) & \text{otherwise.} \end{cases}$$

Not to overburden the notation since now on we will still denote $(\tilde{u}, \tilde{z})$ by (u, z). Let us remark that, thanks to Lemma 4.9 and the fact that z is left-continuous at all $t \in [0, T]$, it is easily seen that the energy (4.32) turns out to be globally non-increasing, *i.e.* it is a non-increasing function on the whole interval $[0, T]$.

In other words we have first redefined z in order that it is left-continuous, and then we have redefined u as in Lemma 4.9. Thanks to the left-continuity of z the proof of Lemma 4.9 shows that the new u satisfies (4.21) on the whole $[0,T]$.

When (t,z) are fixed, (4.21) is equivalent to the fact that u is the minimizer of the functional

$$u \to \frac{1}{2}\langle u_x, u_x\rangle + \frac{1}{2}[u]^2 z - \langle \mathcal{L}, u\rangle,$$

among all the functions $u \in H^1(]-1,0[\cup]0,1[)$ with $u(1) = w(t,-1)$ and $u(-1) = w(t,1)$. Equivalently, this is expressed by the following system of equations

$$\begin{cases} -u_{xx}(t,x) = \mathcal{L}(t,x) \quad \text{on} \quad]-1,0[\ \cup\ ]0,1[, \\ u_x(t,0) = [u(t,0)]z(t) \\ u(1) = w(t,-1) \\ u(-1) = w(t,1). \end{cases} \tag{4.36}$$

It is easy to compute explicitly the solutions of such system. Let $F \in H^1([0,T];L^2(]-1,0[\cup]0,1[))$ be the function, provided by Lemma 4.13, such that $\langle \mathcal{L}(t),\varphi\rangle = -\langle F(t),\varphi_x\rangle$ and set $G(t,x) := \int_0^x F(t)(y)dy$ for all $x \in]-1,0[\cup]0,1[$, the solution $u = u(t,x)$ of (4.36) takes the form

$$u(t,x) = \begin{cases} G(t,1) - G(t,x) + \xi(t)\frac{z(t)}{1+2z(t)}(x-1) + w(t,1) & \text{if } x > 0 \\ G(t,-1) - G(t,x) + \xi(t)\frac{z(t)}{1+2z(t)}(x+1) + w(t,-1) & \text{if } x < 0, \end{cases} \tag{4.37}$$

where $\xi(t) := G(t,1) - G(t,-1) + w(t,1) - w(t,-1)$. We also find

$$[u(t)] := \frac{\xi(t)}{1+2z(t)}\cdot \tag{4.38}$$

Let us set

$$\eta(t) := \frac{1}{2}\frac{\xi(t)^2}{(1+2z(t))^2} - \alpha,$$

and define

$$\begin{aligned} t_0 &:= \inf_t \left\{ \frac{1}{2}[u(t)]^2 - \alpha = \eta(t) \geq 0 \right\}, \\ t_1 &:= \inf_t \left\{ \frac{1}{2}[u(t)]^2 - \alpha = \eta(t) > 0 \right\}, \end{aligned} \tag{4.39}$$

with these values being T if the corresponding infima are computed on empty sets. Obviously we have $t_0 \leq t_1$. The times t_0 and t_1 depend only on ξ and the value of z, and in particular

$$\begin{aligned} t_0 &:= \inf_t \left\{ z(t) \leq \frac{\xi(t) - \sqrt{2\alpha}}{2\sqrt{2\alpha}} \right\}, \\ t_1 &:= \inf_t \left\{ z(t) < \frac{\xi(t) - \sqrt{2\alpha}}{2\sqrt{2\alpha}} \right\}. \end{aligned} \tag{4.40}$$

Moreover the energy (4.32) reads

$$\begin{aligned} \mathcal{E}(t) =& \frac{1}{2}\langle u_x(t,x), u_x(t,x)\rangle + \frac{1}{2}[u(t)]^2 z(t) - \alpha z(t) + \langle F(t), u_x(t) - w_x(t)\rangle \\ &- \int_0^t \langle \dot{F}(s), u_x(s) - w_x(s)\rangle \mathrm{d}s - \int_0^t \langle u_x(s), \dot{w}_x(s)\rangle \mathrm{d}s, \end{aligned}$$

and plugging the formulas found above in this expression we obtain

$$\mathcal{E}(t)=\frac{1}{2}\frac{\xi(t)^2z(t)}{1+2z(t)}-\alpha z(t)-\frac{(G(0,1)-G(0,-1))(w(0,1)-w(0,-1))}{2}-\int_0^t\frac{\xi(s)\dot{\xi}(s)}{1+2z(s)}z(s).$$

We will now employ a standard formula providing the expression of the distributional derivative of the composition of a smooth function with a function of bounded variation (see, *e.g.*, [35], or [2]). If $z:[0,T]\to\mathbb{R}$ is a function of bounded variation and $\Phi:\mathbb{R}^2\to\mathbb{R}$ is smooth, such formula applied to the function $t\to\Phi(t,z(t))$ reads

$$D_t\Phi(\cdot,z(\cdot))=\Phi_1(\cdot,z(\cdot))\mathcal{L}^1+\Phi_2(\cdot,\bar{z}(\cdot))D_tz\llcorner_{C_z}+\sum_{s\in\mathbb{R}^+}[\Phi(s,z(s^+))-\Phi(s,z(s^-))]\delta_s, \tag{4.41}$$

where Φ_i is the derivative of Φ with respect to the i-th variable, $\mathcal{L}^1$ is the Lebesgue measure on $\mathbb{R}$, $\bar{z}$ is the continuous representative of z on the set C_z, the set where z is continuous, $z(s^+)$ (resp. $z(s^-)$) is the limit from the right (resp. left) of z at $s\in\mathbb{R}$, and δ_s is the Dirac delta at $s\in\mathbb{R}$. We use this formula to compute the distributional derivative of $\mathcal{E}$. Let us recall that the function z itself is continuous at every t except at the jump times. Therefore we find

$$D_t\mathcal{E}(t)=\left(\frac{1}{2}\frac{\xi(t)^2}{(1+2z(t))^2}-\alpha\right)(\dot{z}+\dot{z}^c)+\sum_{s\in[0,T]}\left(\frac{1}{2}\frac{\xi(s^+)z(s^+)}{(1+2z(s^+))}-\frac{1}{2}\frac{\xi(s^-)z(s^-)}{(1+2z(s^-))}-\alpha z(s^+)+\alpha z(s^-)\right)\delta_s, \tag{4.42}$$

where $\dot{z}$ and $\dot{z}^c$ are the absolutely continuous part of $D_tz\llcorner_{C_z}$ with respect to $\mathcal{L}^1$ and the Cantor part respectively. We can write the jumps of (4.42) in the following equivalent way

$$-\sum_{s\in[0,T]}\left(\int_{z(t^+)}^{z(t^-)}\frac{1}{2}\frac{\xi(s)^2}{(1+2r)^2}-\alpha\mathrm{d}r\right)\delta_s. \tag{4.43}$$

From the energy inequality we know that the energy is a non-increasing function, so its total derivative (4.42) must be a non-positive measure on $[0,T]$. Since the absolutely continuous, the Cantor, and the jump parts of this measure are mutually singular, they must all be non-positive. This applied to the jumps implies that the integrals appearing in the sum (4.43) are all non-negative. On the other hand

$$\int_{z(t^+)}^{z(t^-)}\frac{1}{2}\frac{\xi(s)^2}{(1+2r)^2}-\alpha\mathrm{d}r\le\int_{z(t^+)}^{z(t^-)}\frac{1}{2}\frac{\xi(s)^2}{(1+2z(t^+))^2}-\alpha\mathrm{d}s\le 0,$$

where the first inequality follows from the fact that $r\to\frac{1}{2}\frac{\xi(s)^2}{(1+2r)^2}-\alpha$ is non-increasing, and the second inequality follows until $t\in[0,t_1[$. Moreover, the first inequality is strict if $\xi(s)\neq 0$, since $r\to\frac{1}{2}\frac{\xi(s)^2}{(1+2r)^2}-\alpha$ is strictly decreasing in this case, while if $\xi(s)=0$ the second inequality is strict since $\alpha>0$. In particular we find out that no jump can occur in the interval $[0,t_1[$.

We claim that, if there is a jump of z, than such jump is unique and takes place at $t=t_1$. Moreover $z(t)=0$ for $t>t_1$. Without loss of generality suppose $t_1<T$. Since z is left-continuous, the function $\eta(t)=\frac{1}{2}\frac{\xi(t)^2}{(1+2z(t))^2}-\alpha$ is left-continuous, so that by definition of t_1 there is a sequence $t_k\searrow t_1$ such that $\eta(t_k)>0$ for all k. Again, since η is left-continuous we obtain that for all $\delta>0$ the set of all t such that $\eta(t)>0$ has positive Lebesgue measure on $[t_1,t_1+\delta]$. This, thanks to (4.26), implies that $z(t)=0$ for $t>t_1$, getting the claim.

Let us now consider the Cantor and absolutely continuous part of (4.42). We see that $\dot{z}$ and $\dot{z}^c$ might concentrate only on the set $A:=\{t\in[0,t_1]:\frac{1}{2}\frac{\xi(t)^2}{(1+2z(t))^2}-\alpha=0\}=\{t\in[0,t_1]:z(t)=\frac{\xi(t)-\sqrt{2\alpha}}{2\sqrt{2\alpha}}\}$. This is the set where the continuous function $\xi(t)$ coincides with $f(t):=\sqrt{2\alpha}(1+2z(t))$. We claim that the distributional derivatives of the BV functions ξ and f coincide on A. Indeed this is a particular case of a more general result provided by Theorem A.1 of [9]. As a consequence we get

$$\dot{\xi} = 2\sqrt{2\alpha}(\dot{z} + \dot{z}^c),$$

which implies that $\dot{z}^c = 0$ since the right-hand side is absolutely continuous with respect to the Lebesgue measure. Moreover we find out that $\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{\xi}$. We can summarize the discussion carried out so far with the following result, which holds in the 1-dimensional case:

Theorem 4.14 (1-dimensional case)**.** *Let (u, z) be the limit obtained in Theorem 4.4. Then there is a representative of z that is left-continuous. Let t_0, t_1 be as in (4.40). Then there is a representative of u such that $u(t)$ is the solution of (4.36) for all $t \in [0, t_1]$. For these representatives, still denoted by (u, z), it holds that z is constant on the interval $[0, t_0]$ and it is such that $z(t) \equiv 0$ for $t > t_1$. Moreover z can jump only at $t = t_1$, $\dot{z}^c \equiv 0$ on $[0, T]$, and $\dot{z}$ is concentrated on the set*

$$A := \left\{ t \in [t_0, t_1] : z(t) = \frac{\xi(t) - \sqrt{2\alpha}}{2\sqrt{2\alpha}} \right\}, \tag{4.44}$$

where it also holds $\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{\xi}$, with $\xi(t) := G(t, 1) - G(t, -1) + w(t, 1) - w(t, -1)$. In formula

$$\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{\xi}\chi_A.$$

In terms of the data of the problem we can state the following:

Theorem 4.15. *Let (u, z) be the limit obtained in Theorem 4.4 with initial condition $z(0) = z_0 > 0$ and suppose z is left-continuous. Let*

$$\tilde{t}_0 := \inf_{t\in[0,T]} \left\{ \xi(t) \geq (1 + 2z_0)\sqrt{2\alpha} \right\}, \qquad \tilde{t}_1 := \inf_{t\in[0,T]} \left\{ \xi(t) > (1 + 2z_0)\sqrt{2\alpha} \right\},$$

then it holds $z(t) = z_0$ if $t \leq \tilde{t}_0$, $z(t) = 0$ if $t > \tilde{t}_1$, $\dot{z} = \frac{1}{2\sqrt{2\alpha}}\dot{\xi}\chi_A$, and z can jump only at $t = \tilde{t}_1$.

Corollary 4.16. *If ξ is strictly increasing and is such that $\xi(0) < (1 + 2z_0)\sqrt{2\alpha}$, then there is only one time $\bar{t} > 0$ such that $A = \{\bar{t}\}$ and $z(t) = z_0$ for $t \leq \bar{t}$, while $z(t) = 0$ for $t > \bar{t}$.*

Proof. In such a case $t_0 = t_1 = \bar{t}$. Note that hypothesis $\xi(0) < (1 + 2z_0)\sqrt{2\alpha}$ prevents that $\bar{t} = 0$. $\square$

The last statement proves that the function (u, z) given by an external load and a boundary condition as in the example of ([30], Sect. 4) coincides with the couple of such example. We emphasize that Theorem 4.14 refers to a couple (u, z) which evolves without constrains on the jump of the displacement $[u]$. However, if the jump remains positive during the evolution, *i.e.* $[u] \geq 0$ on $[0, T]$, as in the example of ([30], Sect. 4), the evolution itself satisfies the constraint of mode I.

We conclude the section with the following remark, that show that the conditions we have obtained by the analysis of the limit (u, z) are not sufficient to conclude whether jumps of z (and then of u) occur or not.

Remark 4.17. Suppose that the function $\xi \in \mathcal{C}^\infty(\mathbb{R})$ is such that $\xi(0) = 0$, $\xi(1) = 3\sqrt{2\alpha}$, $\xi(2) = \sqrt{2\alpha}$, and ξ is strictly monotone in the intervals $[0, 1]$ and $[1, 2]$. Let then $z = 1$ on $[0, 1]$, $z(t) = \frac{\xi(t)-\sqrt{2\alpha}}{2\sqrt{2\alpha}}$ for $t \in [1, 2]$, and $z(t) = 0$ for $t > 2$. Then let $u(t)$ be the solution of (4.36), *i.e.* the function in (4.37). For such (u, z) we see that (4.21) holds by definition while (4.42) shows that (4.22) holds true with $\mu_e = \mu_z = 0$. This is an example of an evolution satisfying the conditions of the limit of dynamic processes with initial condition $z_0 = 1$, and which does not show any jump, actually being smooth in time. However it is still not clear if there exists some dynamic process whose limit is such function. In particular it is not clear if the measures μ_e and μ_z must be strictly positive, as in the case of Corollary 4.16, or may vanish.

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