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ADAPTIVE STABILIZATION FOR A CLASS OF PDE-ODE CASCADE SYSTEMS WITH UNCERTAIN HARMONIC DISTURBANCES *

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Abstract. Adaptive boundary stabilization is investigated for a class of PDE-ODE cascade systems with general uncertain harmonic disturbances. The essential difference between this paper and the existing related literature is the presence of the uncertain disturbances belonging to an unknown interval, which makes the problem unsolved so far. Motivated by the existing related literature, the paper develops the adaptive boundary stabilization for the PDE-ODE cascade system in question. First, an adaptive boundary feedback controller is constructed in two steps by adaptive and Lyapunov techniques. Then, it is shown that the resulting closed-loop system is well-posed and asymptotically stable, by the semigroup approach and LaSalle's invariance principle, respectively. Moreover, the parameter estimates involved in the designed controller are shown to ultimately converge to their own real values. Finally, the effectiveness of the proposed method is illustrated by a simulation example.

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1. Introduction and problem formulation

In this paper, we consider the adaptive boundary stabilization for the following PDE-ODE cascade system with general uncertain harmonic disturbance:

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0, \\ w_x(0,t) = qw_t(0,t), \\ w(L,t) = X_1(t), \\ \dot{X}_1(t) = X_2(t), \\ \dot{X}_2(t) = \lambda a(L)w_x(L,t) + \frac{1}{M}u(t) + d(t), \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), \end{cases} \tag{1.1}$$

where $w : [0,\, L] \times \mathbb{R}^+ \to \mathbb{R}$ and $X = [X_1,\, X_2]^{\mathrm{T}} : \mathbb{R}^+ \to \mathbb{R}^2$ are the states of the PDE and ODE subsystems, respectively; x takes values in $[0,\, L]$ on which the PDE subsystem evolves, called the spatial variable; $u : \mathbb{R}^+ \to \mathbb{R}$

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is the control input of the entire system; w_x (resp. w_{xx}) and w_t (resp. w_{tt}) denote the (resp. second) partial derivatives of w with respect to x and t, respectively; q, λ and M are positive constants; $a : [0, L] \to \mathbb{R}^+$ is defined as follows:

$$a(x) = a_1 x + a_2 \tag{1.2}$$

for positive constants a_1 and a_2; $d : \mathbb{R}^+ \to \mathbb{R}$ is the general uncertain harmonic disturbance given by:

$$d(t) = \sum_{i=1}^{n} (\theta_i \sin \omega_i t + \varphi_i \cos \omega_i t) \tag{1.3}$$

with ω_i's and θ_i's, φ_i's being known and unknown nonnegative constants, called frequencies and amplitudes of the harmonic disturbance, respectively.

System (1.1) can describe the motion of a crane with flexible cable [2,4]. The boundary stabilization of such systems has received much effort in the past decades (see, *e.g.*, [2–4, 8, 13–15, 20, 28, 29] and references therein), since cranes are widely used in new buildings, assembly plants, nuclear waste-handling facilities, shipyards and so on. It is necessary to point out that, systems considered in [2–4, 8, 20, 28, 29] don't allow any disturbance. Although systems with disturbances have been investigated in [13–15], the involved disturbances are required to belong to a known interval. However, in practice, due to the uncontrollable conditions, such as wind, rain and complex terrains for cranes, disturbances cannot be ignored and the bounds are usually difficult to be obtained. Consequently, how to stabilize system (1.1) with disturbance $d(t)$ which doesn't belong to a known interval is of much interest from both practical and theoretical viewpoints, and deserves intensive investigation.

Much attention has been made on the control of PDE-ODE cascade systems, see, *e.g.*, [1–4, 6, 8–15, 20, 21, 23, 24, 26, 28, 29, 31] and references therein, where the uncertain disturbances are limited in a known interval or aren't involved at all. Quite differently, system (1.1) allows the general uncertain harmonic disturbance to belong to an unknown interval, which makes the problem under discussion unsolved so far. It is worth pointing out, the case with the uncertain harmonic disturbance satisfying (1.3) has been studied in [16, 17], but the proposed approaches are to PDE systems, rather than to PDE-ODE cascade systems. Motivated by [2, 17], the paper develops the adaptive boundary stabilization for PDE-ODE cascade system (1.1). First, an adaptive boundary feedback controller is constructed in two steps by adaptive and Lyapunov techniques. Then, it is shown that the resulting closed-loop system is well-posed and asymptotically stable, by the semigroup approach and LaSalle's invariance principle, respectively. Moreover, the parameter estimates involved in the designed controller are shown to ultimately converge to their own real values. It is worth mentioning that, as [2, 4], the designed adaptive controller depends merely on the measures at the end $x = L$ of system (1.1), which makes the controller much easier to implement.

The remainder of the paper proceeds as follows. Section 2 provides the procedure of the adaptive control design. Section 3 shows the well-posedness of the closed-loop system. Section 4 presents the main results of the paper. Section 5 gives a numerical example to illustrate the effectiveness of the theoretical results. Section 6 addresses some concluding remarks. The paper ends with an appendix which gives the proof of a proposition and several useful inequalities.

Notations. Throughout the paper, $\mathbb{L}^2(0, L)$ denotes the space of all measurable functions on $(0, L)$ with the property that $\int_0^L |f(x)|^2 \, \mathrm{d}x < +\infty$; $\mathbb{L}^\infty(0, +\infty)$ denotes the space of all bounded measurable functions on $(0, +\infty)$ with the property that $\operatorname{ess\,sup}_{t\in(0,+\infty)}|f(t)| < +\infty$; $\mathbb{C}_0^\infty(0, L)$ denotes the space of all real-valued functions on $[0, L]$ having continuous derivatives of all orders and having compact support contained in $(0, L)$; $\mathbb{H}^i(0, L)$ denotes the usual Sobolev space of functions in $\mathbb{L}^2(0, L)$ with derivatives up to ith order also in $\mathbb{L}^2(0, L)$; $\mathbb{L}^\infty(0, +\infty; \mathbb{H}^i(0, L))$ denotes the space of all bounded measurable functions $f(x,t) : (0, +\infty) \to \mathbb{H}^i(0, L)$ with the property that $|f(x,t)|_{\mathbb{H}^i(0, L)} \in \mathbb{L}^\infty(0, +\infty)$; $\mathbb{W}^{1,\infty}(0, +\infty; \mathbb{H}^1(0, L))$ denotes the space of all functions $t \to f(x,t)$ in $\mathbb{L}^\infty(0, +\infty; \mathbb{H}^1(0, L))$ with first derivative with respect to t also in $\mathbb{L}^\infty(0, +\infty; \mathbb{H}^1(0, L))$. Moreover, for simplicity of expression, we sometimes drop the arguments of a function if no confusion is caused.

2. Adaptive boundary control design

This section is to design an adaptive boundary feedback controller for system (1.1) to achieve the desired stability of the resulting closed-loop system, which is presented in two steps. Specifically, in Step 1, as in the famous backstepping technique for ODEs in the finite dimensional framework [22], we first design the controller $X_2^*(t)$ for the following subsystem peeled from (1.1):

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0, \\ w_x(0,t) = qw_t(0,t), \\ w(L,t) = X_1(t), \\ \dot{X}_1(t) = X_2^*(t), \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \end{cases} \tag{2.1}$$

to ensure the asymptotic stability of the resulting closed-loop system of (2.1). Then, in Step 2, an adaptive controller is successfully constructed for the original system, which guarantees that the error $X_2(t) - X_2^*(t)$ converges to zero and the resulting closed-loop system of (1.1) is asymptotically stable, and furthermore the estimates of the parameters ultimately converge to their own real values.

Step 1. Motivated by [2], we introduce the following energy function for subsystem (2.1):

$$V_1(t) = \frac{1}{2}\int_0^L \left(w_t^2(x,t) + a(x)w_x^2(x,t)\right) \mathrm{d}x + \frac{1}{2}X_1^2(t),$$

which satisfies

$$\begin{aligned} \dot{V}_1(t) &= \int_0^L \left(w_t(x,t)w_{tt}(x,t) + a(x)w_x(x,t)w_{xt}(x,t)\right) \mathrm{d}x + X_1(t)\dot{X}_1(t) \\ &= \int_0^L \left(w_t(x,t)\left(a(x)w_x(x,t)\right)_x + a(x)w_x(x,t)w_{xt}(x,t)\right) \mathrm{d}x + X_1(t)\dot{X}_1(t) \\ &= \left(a(x)w_x(x,t)w_t(x,t)\right)\big|_0^L + X_1(t)\dot{X}_1(t) \\ &= \dot{X}_1(t)\left(a(L)w_x(L,t) + X_1(t)\right) - qa(0)w_t^2(0,t). \end{aligned} \tag{2.2}$$

Then, we can choose

$$X_2^*(t) = -K\left(a(L)w_x(L,t) + X_1(t)\right), \tag{2.3}$$

where K is a positive constant, which makes (2.2) become

$$\dot{V}_1(t) = -K\left(a(L)w_x(L,t) + X_1(t)\right)^2 - qa(0)w_t^2(0,t) \le 0, \tag{2.4}$$

noting $q > 0$ and $a(0) = a_2 > 0$.

Based on (2.4), it will be shown that the following closed-loop system deriving from (2.1) and (2.3) is asymptotically stable:

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0, \\ w_x(0,t) = qw_t(0,t), \\ w(L,t) = X_1(t), \\ \dot{X}_1(t) = -K\left(a(L)w_x(L,t) + X_1(t)\right), \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \end{cases} \tag{2.5}$$

in the sense

$$\lim_{t\to\infty}\left(\int_0^L \left(w_t^2(x,t) + a(x)w_x^2(x,t)\right) \mathrm{d}x + X_1^2(t)\right) = 0.$$

To see this, we introduce for system (2.5) the Hilbert space $\mathbb{F}_1 = \left\{(\beta,\gamma,\nu)\in\mathbb{H}^1(0,L)\times\mathbb{L}^2(0,L)\times\mathbb{R}\,\middle|\,\nu=\beta(L)\right\}$ with the following inner product:

$$\left\langle(\beta,\gamma,\nu),(\beta',\gamma',\nu')\right\rangle_{\mathbb{F}_1} = \int_0^L (a\beta_x\beta_x' + \gamma\gamma')\,\mathrm{d}x + \nu\nu', \tag{2.6}$$

and define the linear operator $\mathcal{A}_1 : \mathbb{D}_1 \to \mathbb{F}_1$ as

$$\begin{cases}\mathcal{A}_1(\beta,\gamma,\nu) \;=\; (\gamma,(a\beta_x)_x, -K(a(L)\beta_x(L)+\nu)), \quad \forall(\beta,\gamma,\nu)\in\mathbb{D}_1,\\ \mathbb{D}_1 \;=\; \left\{(\beta,\gamma,\nu)\in\mathbb{H}^2(0,L)\times\mathbb{H}^1(0,L)\times\mathbb{R}\,\middle|\,\nu=\beta(L),\ \beta_x(0)=q\gamma(0)\right\},\end{cases} \tag{2.7}$$

where a is defined the same as (1.2), and q and K have been specified in (1.1) and (2.3), respectively.

Thus, by (2.7), system (2.5) can be changed as the following evolution equation:

$$\begin{cases}Y_t(x,t) \;=\; \mathcal{A}_1 Y(x,t),\\ Y(x,0) \;=\; Y_0(x),\end{cases}$$

where $Y(x,t) = (w(x,t), w_t(x,t), X_1(t))$ and $Y_0(x) = (w_0(x), w_1(x), w_0(L)) \in \mathbb{F}_1$. Then, for the weak solution (see Def. 3.1.6, p. 105 of [7]) of system (2.5), there holds the following theorem, whose proof is so similar to that of Theorem 2 in [4] and hence is omitted here.

Theorem 2.1. *For any initial value $Y_0(x)\in\mathbb{F}_1$, the weak solution $Y(x,t) = S(t)Y_0(x)$ of system (2.5) is asymptotically stable in the following sense:*

$$\lim_{t\to\infty}\|Y(x,t)\|^2_{\mathbb{F}_1} = \lim_{t\to\infty}\left(\int_0^L \left(w_t^2(x,t)+a(x)w_x^2(x,t)\right)\mathrm{d}x + X_1^2(t)\right) = 0,$$

where $S(t)$ is the contraction semigroup on $\mathbb{F}_1$ generated by $\mathcal{A}_1$ and $\|Y(x,t)\|_{\mathbb{F}_1}$ is the inner product induced norm given by (2.6).

Step 2. The estimates of unknown parameters θ_i's and φ_i's are denoted by $\hat{\theta}_i(t)$'s and $\hat{\varphi}_i(t)$'s, respectively, whose updating laws will be determined later. Accordingly, the parameter estimate errors are defined as $\tilde{\theta}_i(t) = \theta_i - \hat{\theta}_i(t)$ and $\tilde{\varphi}_i(t) = \varphi_i - \hat{\varphi}_i(t)$, and for late use $\Theta(t) \triangleq \left(\tilde{\theta}_1(t),\ldots,\tilde{\theta}_n(t),\tilde{\varphi}_1(t),\ldots,\tilde{\varphi}_n(t)\right)^{\mathrm{T}}$.

To derive the suitable form of $u(t)$ in this step, we define

$$V(t) = V_1(t) + \frac{1}{2}\tilde{X}_2^2(t) + \frac{1}{2}\Theta^{\mathrm{T}}(t)\Theta(t), \tag{2.8}$$

as the energy function for the entire system:

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0,\\ w_x(0,t) = q w_t(0,t),\\ w(L,t) = X_1(t),\\ \dot{X}_1(t) = X_2^*(t) + \tilde{X}_2(t),\\ \dot{\tilde{X}}_2(t) = U(t) + d(t) - \dot{X}_2^*(t),\end{cases} \tag{2.9}$$

where $\tilde{X}_2 = X_2 - X_2^*$, and $U(t)$ simply denotes $\lambda a(L)w_x(L,t) + \frac{1}{M}u(t)$.

By (1.1), (2.3) and similar to the derivation process of (2.2), there holds

$$
\begin{aligned}
\dot V(t) &= \dot V_1(t) + \tilde X_2(t)\left(U(t)+d(t)-\dot X_2^*(t)\right) - \sum_{i=1}^n \left(\tilde\theta_i(t)\dot{\hat\theta}_i(t) + \tilde\varphi_i(t)\dot{\hat\varphi}_i(t)\right)\\
&= \left(X_2^*(t)+\tilde X_2(t)\right)\left(a(L)w_x(L,t)+X_1(t)\right) - qa(0){w_t}^2(0,t)\\
&\quad + \tilde X_2(t)\left(U(t)+d(t)-\dot X_2^*(t)\right) - \sum_{i=1}^n \left(\tilde\theta_i(t)\dot{\hat\theta}_i(t) + \tilde\varphi_i(t)\dot{\hat\varphi}_i(t)\right)\\
&= -K\left(a(L)w_x(L,t)+X_1(t)\right)^2 + \tilde X_2(t)\left(a(L)w_x(L,t)+X_1(t)\right) - qa(0){w_t}^2(0,t)\\
&\quad + \tilde X_2(t)\left(U(t)-\dot X_2^*(t)+\sum_{i=1}^n\left(\hat\theta_i(t)\sin\omega_i t + \hat\varphi_i(t)\cos\omega_i t\right)\right)\\
&\quad + \tilde X_2(t)\sum_{i=1}^n\left(\tilde\theta_i(t)\sin\omega_i t + \tilde\varphi_i(t)\cos\omega_i t\right) - \sum_{i=1}^n \left(\tilde\theta_i(t)\dot{\hat\theta}_i(t) + \tilde\varphi_i(t)\dot{\hat\varphi}_i(t)\right). \qquad (2.10)
\end{aligned}
$$

Noting $q>0$, $a(0)=a_2>0$ and by the method of completing square, we have

$$
\begin{aligned}
\dot V(t) &\le -K\left(a(L)w_x(L,t)+X_1(t)\right)^2 + \frac{K}{2}\left(a(L)w_x(L,t)+X_1(t)\right)^2 + \frac{1}{2K}\tilde X_2^2(t)\\
&\quad + \tilde X_2(t)\left(U(t)-\dot X_2^*(t)+\sum_{i=1}^n\left(\hat\theta_i(t)\sin\omega_i t + \hat\varphi_i(t)\cos\omega_i t\right)\right)\\
&\quad + \sum_{i=1}^n\left(\tilde\theta_i(t)\left(\tilde X_2(t)\sin\omega_i t - \dot{\hat\theta}_i(t)\right) + \tilde\varphi_i(t)\left(\tilde X_2(t)\cos\omega_i t - \dot{\hat\varphi}_i(t)\right)\right)\\
&\le -\frac{K}{2}\left(a(L)w_x(L,t)+X_1(t)\right)^2 + \frac{1}{2K}\tilde X_2^2(t) + \tilde X_2(t)\big(U(t)-\dot X_2^*(t)\\
&\quad + \sum_{i=1}^n\left(\hat\theta_i(t)\sin\omega_i t + \hat\varphi_i(t)\cos\omega_i t\right)\big) + \sum_{i=1}^n\Big(\tilde\theta_i(t)\left(\tilde X_2(t)\sin\omega_i t - \dot{\hat\theta}_i(t)\right)\\
&\quad + \tilde\varphi_i(t)\left(\tilde X_2(t)\cos\omega_i t - \dot{\hat\varphi}_i(t)\right)\Big). \qquad (2.11)
\end{aligned}
$$

Thus, we choose the adaptive boundary feedback controller as follows:

$$
\begin{aligned}
u(t) &= M\left(\dot X_2^*(t) - \alpha\tilde X_2(t) - \sum_{i=1}^n\left(\hat\theta_i(t)\sin\omega_i t + \hat\varphi_i(t)\cos\omega_i t\right) - \lambda a(L)w_x(L,t)\right)\\
&= -M\left(Ka(L)w_{xt}(L,t) - K^2\left(a(L)w_x(L,t)+X_1(t)\right) + (K+\alpha)\tilde X_2(t)\right)\\
&\quad - M\left(\sum_{i=1}^n\left(\hat\theta_i(t)\sin\omega_i t + \hat\varphi_i(t)\cos\omega_i t\right) + \lambda a(L)w_x(L,t)\right), \qquad (2.12)
\end{aligned}
$$

where α is a design parameter satisfying $\alpha > \frac{1}{2K}$; the corresponding updating laws for $\hat\theta_i(t)$'s and $\hat\varphi_i(t)$'s are given as

$$
\begin{cases}
\dot{\hat\theta}_i(t) = \tilde X_2(t)\sin\omega_i t, & i=1,\ldots,n,\\
\dot{\hat\varphi}_i(t) = \tilde X_2(t)\cos\omega_i t, & i=1,\ldots,n.
\end{cases} \qquad (2.13)
$$

By (2.12) and noting that $U(t) = \lambda a(L)w_x(L,t) + \frac{1}{M}u(t)$, there is

$$U(t) = \dot{X}_2^*(t) - \alpha\tilde{X}_2(t) - \sum_{i=1}^{n}\left(\hat{\theta}_i(t)\sin\omega_i t + \hat{\varphi}_i(t)\cos\omega_i t\right).$$

By substituting this and (2.13) into (2.10) (its first equation) and (2.11) (noting $\alpha > \frac{1}{2K}$), we have

$$\begin{cases} \dot{V}(t) & = \dot{V}_1(t) - \alpha\tilde{X}_2^2(t), \\ \dot{V}(t) & \leq -\frac{K}{2}\left(a(L)w_x(L,t) + X_1(t)\right)^2 - \left(\alpha - \frac{1}{2K}\right)\tilde{X}_2^2(t) \leq 0. \end{cases} \tag{2.14}$$

Next two sections are devoted to respectively address the well-posedness and the asymptotical stability of the entire system (2.9) with the controller (2.12) and (2.13) in the loop.

3. Well-posedness of the closed-loop system

Let's analyze the well-posedness of the following closed-loop system which consists of (2.9), (2.12) and (2.13):

$$\begin{cases} w_{tt}(x,t) - \left(a(x)w_x(x,t)\right)_x = 0, \\ w_x(0,t) = qw_t(0,t), \\ w(L,t) = X_1(t), \\ \dot{X}_1(t) = -K\left(a(L)w_x(L,t) + X_1(t)\right) + \tilde{X}_2(t), \\ \dot{\tilde{X}}_2(t) = -\alpha\tilde{X}_2(t) + \sum_{i=1}^{n}\left(\tilde{\theta}_i(t)\sin\omega_i t + \tilde{\varphi}_i(t)\cos\omega_i t\right), \\ \dot{\tilde{\theta}}_i(t) = -\tilde{X}_2(t)\sin\omega_i t, \quad i = 1,\dots,n, \\ \dot{\tilde{\varphi}}_i(t) = -\tilde{X}_2(t)\cos\omega_i t, \quad i = 1,\dots,n, \\ \tilde{\theta}_i(0) = \tilde{\theta}_{i0},\ \tilde{\varphi}_i(0) = \tilde{\varphi}_{i0}, \quad i = 1,\dots,n, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x). \end{cases} \tag{3.1}$$

For this system, inspired by [2,18], we introduce Hilbert space $\mathbb{F}_2 = \left\{(f,g,h,k,\Phi) \in \mathbb{H}^1(0,L)\times\mathbb{L}^2(0,L)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{2n}\,\middle|\, h = f(L)\right\}$ with the inner product:

$$\left\langle (f,g,h,k,\Phi),(f',g',h',k',\Phi')\right\rangle_{\mathbb{F}_2} = \int_0^L \left(af_xf_x' + gg'\right)\,\mathrm{d}x + hh' + kk' + \Phi^{\mathrm{T}}\Phi', \tag{3.2}$$

where a is defined the same as (1.2). Define the linear operator $\mathcal{A}_2 : \mathbb{D}_2 \to \mathbb{F}_2$ as follows:

$$\begin{cases} \mathcal{A}_2(f,g,h,k,\Phi) = \left(g,(af_x)_x,-K\left(a(L)f_x(L)+h\right)+k,-\alpha k+\Phi^{\mathrm{T}}\Omega,-k\Omega\right),\forall(f,g,h,k,\Phi)\in\mathbb{D}_2, \\ \mathbb{D}_2 = \left\{(f,g,h,k,\Phi)\in\mathbb{H}^2(0,L)\times\mathbb{H}^1(0,L)\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^{2n}\,\middle|\, h = f(L),\ f_x(0) = qg(0),\right. \\ \qquad\qquad \left. g(L) + K\left(a(L)f_x(L)+h\right) = k\right\}, \end{cases} \tag{3.3}$$

where $\Omega = (\sin\omega_1 t,\dots,\sin\omega_n t,\cos\omega_1 t,\dots,\cos\omega_n t)^{\mathrm{T}}$; q, K and α have been specified in (1.1), (2.3) and (2.12), respectively. Then system (3.1) can be formulated as the following abstract evolution equation:

$$\begin{cases} Z_t(x,t) = \mathcal{A}_2 Z(x,t), \\ Z(x,0) = Z_0(x), \end{cases} \tag{3.4}$$

where $Z(x,t) = \left(w(x,t),w_t(x,t),X_1(t),\tilde{X}_2(t),\Theta(t)\right)$ and $Z_0(x) = \left(w_0(x),w_1(x),w_0(L),\tilde{X}_2(0),\Theta(0)\right) \in \mathbb{F}_2$.

Next, we show that linear operator $\mathcal{A}_2$ defined by (3.3) generates a contraction semigroup on Hilbert space $\mathbb{F}_2$. By Lumer–Phillips theorem (see Thm. 4.3, p. 14 of [27]), we need to prove that $\mathcal{A}_2$ is dissipative and there exits a $\lambda_0 > 0$ such that the range of $\lambda_0 I - \mathcal{A}_2$ is $\mathbb{F}_2$, seeing Lemmas 3.1 and 3.3 below, respectively.

Lemma 3.1. *Linea operator $\mathcal{A}_2$ defined by (3.3) is dissipative.*

Proof. For any $(f, g, h, k, \varPhi) \in \mathbb{D}_2$, by (3.2) and integration by parts, we have

$$\begin{aligned}\left\langle \mathcal{A}_2(f,g,h,k,\varPhi),(f,g,h,k,\varPhi)\right\rangle_{\mathbb{F}_2} &= \int_0^L (af_x g_x + (af_x)_x g)\,\mathrm{d}x + h\left(-K\left(a(L)f_x(L)+h\right)+k\right)\\ &\quad + k(-\alpha k + \varPhi^{\mathrm{T}}\varOmega) - k\varOmega^{\mathrm{T}}\varPhi\\ &= g(L)\left(a(L)f_x(L)+h\right) - qa(0)g^2(0) - \alpha k^2,\\ &= -K\left(a(L)f_x(L)+h\right)^2 + k\left(a(L)f_x(L)+h\right) - qa(0)g^2(0) - \alpha k^2.\end{aligned}$$

Then, align to the derivation process of (2.11) and noting $\alpha > \frac{1}{2K}$, there holds

$$\left\langle \mathcal{A}_2(f,g,h,k,\varPhi),(f,g,h,k,\varPhi)\right\rangle_{\mathbb{F}_2} \le -\frac{K}{2}\left(a(L)f_x(L)+h\right)^2 - \left(\alpha - \frac{1}{2K}\right)k^2 \le 0. \tag{3.5}$$

Hence, $\mathcal{A}_2$ is dissipative in $\mathbb{F}_2$. □

Motivated by [19], we obtain the following proposition.

Proposition 3.2. *For any $(p^*, \eta^*, r^*, v^*, \varPsi^*) \in \mathbb{F}_2$, there exists a $p \in \mathbb{H}^2(0, L)$ such that*

$$\begin{cases} (ap_x)_x - p \;=\; -p^* - \eta^*,\\ p_x(0) \;=\; q(p(0) - p^*(0)),\\ Ka(L)p_x(L) \;=\; -(K+1)p(L) + r^* + \frac{v^* + \varPsi^{*\mathrm{T}}\varOmega}{1+\alpha+\varOmega^{\mathrm{T}}\varOmega},\end{cases} \tag{3.6}$$

where a is the same as (1.2), q, K, α and $\varOmega$ have been specified in (1.1), (2.3), (2.12) and (3.3), respectively.

Proof. See Appendix of the paper. □

Based on Proposition (3.2), we establish the following lemma.

Lemma 3.3. *Let $\mathcal{A}_2$ be defined by (3.3). Then the range of $I - \mathcal{A}_2$ is $\mathbb{F}_2$.*

Proof. To prove that $I - \mathcal{A}_2$ is surjective, it suffices to prove that for any $(p^*, \eta^*, r^*, v^*, \varPsi^*) \in \mathbb{F}_2$, there exists a $(p, \eta, r, v, \varPsi) \in \mathbb{D}_2$ such that

$$(I - \mathcal{A}_2)(p, \eta, r, v, \varPsi) = (p^*, \eta^*, r^*, v^*, \varPsi^*),$$

which together with the definition of $\mathcal{A}_2$ given by (3.3) yields that

$$\begin{cases} p - \eta \;=\; p^*,\\ \eta - (ap_x)_x \;=\; \eta^*,\\ r - v + K\left(a(L)p_x(L) + r\right) \;=\; r^*,\\ v + \alpha v - \varPsi^{\mathrm{T}}\varOmega \;=\; v^*,\\ \varPsi + v\varOmega \;=\; \varPsi^*,\\ r \;=\; p(L),\\ p_x(0) \;=\; q\eta(0),\\ \eta(L) + K\left(a(L)p_x(L) + r\right) \;=\; v.\end{cases} \tag{3.7}$$

For simplicity of presentation, the first, second, ..., and eighth equations of (3.7) are denoted by ①, ②, ..., and ⑧, respectively.

From ④ and ⑤, it follows that

$$v + \alpha v + v\Omega^{\mathrm{T}}\Omega = v^* + {\Psi^*}^{\mathrm{T}}\Omega,$$

by which and noting that $1 + \alpha + \Omega^{\mathrm{T}}\Omega > 0$, we arrive at

$$v = \frac{v^* + {\Psi^*}^{\mathrm{T}}\Omega}{1 + \alpha + \Omega^{\mathrm{T}}\Omega}. \tag{3.8}$$

Substituting (3.8) into ⑤ results in

$$\Psi = \Psi^* - v\Omega = \Psi^* - \frac{v^* + {\Psi^*}^{\mathrm{T}}\Omega}{1 + \alpha + \Omega^{\mathrm{T}}\Omega}\Omega. \tag{3.9}$$

From ①, we can directly conclude

$$\eta = p - p^*,$$

by which and $p^* \in \mathbb{H}^1(0, L)$, we have $\eta \in \mathbb{H}^1(0, L)$ if there exists a $p \in \mathbb{H}^2(0, L)$ satisfying (3.7). In addition, by ⑥, (3.8) and (3.9), we can directly conclude that $(r, v, \Psi) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n}$. Thus, the remainder of the proof is to prove that there exists a $p \in \mathbb{H}^2(0, L)$ satisfying (3.7).

First, from ①, ② and ⑦, it follows that

$$\begin{cases} p - (ap_x)_x \;=\; p^* + \eta^*, \\ p_x(0) \;=\; q(p(0) - p^*(0)). \end{cases} \tag{3.10}$$

Then, by ③, ⑥ and (3.8), there holds

$$Ka(L)p_x(L) + (K+1)p(L) = r^* + \frac{v^* + {\Psi^*}^{\mathrm{T}}\Omega}{1 + \alpha + \Omega^{\mathrm{T}}\Omega},$$

which together with (3.10) and Proposition 3.2 concludes that there exists a $p \in \mathbb{H}^2(0, L)$ satisfying (3.7). Thus, $(p, \eta, r, v, \Psi) \in \mathbb{D}_2$ and hence completes the proof. □

Now, similar to Theorem 1 in [2], we obtain the following theorem which mainly states the well-posedness of the closed-loop system.

Theorem 3.4. *Let $\mathcal{A}_2$ be a linear operator defined by (3.3). Then $\mathcal{A}_2$ generates a contraction semigroup $T(t)$ on $\mathbb{F}_2$. Therefore, for system (3.4), there hold*

(i) *For any $Z_0(x) \in \mathbb{F}_2$, system (3.4) has a unique mild solution $Z(x, t) = T(t)Z_0(x)$.*
(ii) *For any $Z_0(x) \in \mathbb{D}_2$, system (3.4) has a unique strong solution $Z(x, t) = T(t)Z_0(x)$, and furthermore $w(x, t) \in \mathbb{W}^{1,\infty}(0, \infty; \mathbb{H}^1(0, L)) \cap \mathbb{L}^\infty(0, \infty; \mathbb{H}^2(0, L))$.*
(iii) *For any $Z_0(x) \in \mathbb{D}_2$, $\|Z(x, t)\|_{\mathbb{F}_2}$ and $\|\mathcal{A}_2 Z(x, t)\|_{\mathbb{F}_2}$ are nonincreasing with respect to t, where $\|Z(x, t)\|_{\mathbb{F}_2}$ and $\|\mathcal{A}_2 Z(x, t)\|_{\mathbb{F}_2}$ are the inner product induced norm given by (3.2) in $\mathbb{F}_2$.*

Proof. We first prove that $\mathcal{A}_2$ generates a contraction semigroup $T(t)$ on $\mathbb{F}_2$. By Lemmas 3.1, 3.3 and Lumer–Phillips theorem, it suffices to show that the closure of $\mathbb{D}_2$ denoted by $\overline{\mathbb{D}}_2$ is $\mathbb{F}_2$. Assume $\overline{\mathbb{D}}_2 \neq \mathbb{F}_2$, then there exists $Z^*(x, t) \in \mathbb{F}_2$ such that, for all $Z(x, t) \in \mathbb{D}_2$,

$$\left\langle Z^*(x, t), Z(x, t) \right\rangle_{\mathbb{F}_2} = 0. \tag{3.11}$$

Noting $I - \mathcal{A}_2$ is surjective, we have that for $Z^*(x, t)$, there exists a $Z'(x, t) \in \mathbb{D}_2$ such that

$$(I - \mathcal{A}_2)Z'(x, t) = Z^*(x, t), \tag{3.12}$$

which together with (3.11) results in

$$\left\langle (I-\mathcal{A}_2)Z'(x,t), Z(x,t)\right\rangle_{\mathbb{F}_2} = 0.$$

Since $Z(x,t)$ is arbitrary in $\mathbb{D}_2$, we take $Z(x,t) = Z'(x,t)$. Then by (3.5), it can be established that

$$\left\langle Z'(x,t), Z'(x,t)\right\rangle_{\mathbb{F}_2} \le 0,$$

which implies $Z'(x,t) = 0$. By (3.12), we have $Z^*(x,t) = 0$. Thus, $\overline{\mathbb{D}}_2 = \mathbb{F}_2$ is proved. By Lumer–Phillips theorem, $\mathcal{A}_2$ generates a contraction semigroup $T(t)$ on $\mathbb{F}_2$, and hence claim (i) can be directly concluded.

We next turn to prove claim (ii). By Lemma 3.3, it follows that $-\mathcal{A}_2$ is m-accretive. Hence, by Proposition 3.3 on page 102 of [5], we have

$$\mathbb{R}_1 = \mathbb{F}_2,\ \text{for all}\, \lambda > 0,$$

where $\mathbb{R}_1$ denotes the range of $I - \lambda\mathcal{A}_2$, which implies that

$$\mathbb{R}_1 \supset \overline{\mathbb{D}}_2,\ \text{for all small}\, \lambda > 0.$$

Moreover, by Corollary 2.5 on page 5 of [27] and using the fact that any a contraction semigroup is a C_0-semigroup, we have that $\mathcal{A}_2$ is closed. Thus, noting $\mathbb{F}_2$ is a reflexive space and by Theorem 1.5 on page 216 of [5], for any $Z_0(x) \in \mathbb{D}_2$, system (3.4) has a unique strong solution $Z(x,t) \in \mathbb{W}^{1,\infty}(0,\infty;\mathbb{F}_2)$ which implies that $w(x,t) \in \mathbb{W}^{1,\infty}(0,\infty;\mathbb{H}^1(0,L))$.

To prove $w(x,t) \in \mathbb{L}^\infty(0,\infty;\mathbb{H}^2(0,L))$, by $Z_0(x) \in \mathbb{D}_2$ and Theorem 2.1.10.a on page 21 of [7], we have $Z(x,t) \in \mathbb{D}_2$ which implies that $w(x,t) :\ [0,+\infty] \to \mathbb{H}^2(0,L)$, so it remains to show $|w(x,t)|_{\mathbb{H}^2(0,L)} \in \mathbb{L}^\infty(0,+\infty)$. For this, by (2.8) and (2.14), we have

$$\sup_{t\ge 0}\left(\int_0^L \left(w_t^2(x,t) + a(x)w_x^2(x,t)\right)\,\mathrm{d}x + X_1^2(t) + \tilde{X}_2^2(t) + \sum_{i=1}^n \left(\tilde{\theta}_i^2(t) + \tilde{\varphi}_i^2(t)\right)\right) < \infty,$$

which implies that

$$\begin{cases} \sup_{t\ge 0}\left(\int_0^L w_x^2(x,t)\,\mathrm{d}x\right) < \infty, \\ \sup_{t\ge 0} w(L,t) < \infty. \end{cases} \tag{3.13}$$

From (3.13) and Poincaré's inequality (see Lem. A.1 of Appendix A in the paper), it follows that

$$\sup_{t\ge 0}\left(\int_0^L w^2(x,t)\,\mathrm{d}x\right) < \infty. \tag{3.14}$$

Thus, by (3.13), (3.14) and Agmon's inequality (see Lem. A.2 of Appendix A in the paper), there holds

$$\max_{x\in[0,\,L]} |w(x,t)|^2 < \infty,$$

and hence $w(x,t) \in \mathbb{L}^\infty(0,\infty;\mathbb{H}^2(0,L))$ is proved.

We then prove claim (iii). Noting $\|Z(x,t)\|_{\mathbb{F}_2}^2 = \left\langle (Z(x,t), Z(x,t)\right\rangle_{\mathbb{F}_2}$ and by (3.4), we can directly obtain

$$\frac{\mathrm{d}\|Z(x,t)\|_{\mathbb{F}_2}}{\mathrm{d}t} = \frac{\left\langle \mathcal{A}_2 Z(x,t), Z(x,t)\right\rangle_{\mathbb{F}_2}}{\|Z(x,t)\|_{\mathbb{F}_2}},$$

which together with (3.5) yields

$$\frac{\mathrm{d}\|Z(x,t)\|_{\mathbb{F}_2}}{\mathrm{d}t} \le 0. \tag{3.15}$$

Thus, $t \to \|Z(x,t)\|_{\mathbb{F}_2}$ is nonincreasing.

Noting $Z(x,t) \in \mathbb{D}_2$ and by Theorem 2.1.10.b on page 21 of [7], we have $\mathcal{A}_2 Z(x,t) \in \mathbb{D}_2$ whose proof is similar to that of Theorem 2.1.10.a on page 21 of [7]. Then, again by Theorem 2.1.10.b, there holds

$$\frac{\mathrm{d}\mathcal{A}_2 Z(x,t)}{\mathrm{d}t} = \frac{\mathrm{d}^2 Z(x,t)}{\mathrm{d}t^2} = \mathcal{A}_2^2 Z(x,t),$$

by which and a similar argument as in deriving (3.15), it follows that

$$\frac{\mathrm{d}\|\mathcal{A}_2 Z(x,t)\|_{\mathbb{F}_2}}{\mathrm{d}t} \le 0,$$

and hence claim (iii) is proved. This completes the proof of the theorem. □

4. Main results

In this section, we adopt LaSalle's invariance principle (see Thm. 3.64, p. 161 of [25]) to study the asymptotic stability of closed-loop system (3.1). For this, the following lemma is required which will play a key role in the later analysis.

Lemma 4.1. *Let $\mathcal{A}_2$ be defined by (3.3). Then $0 \in \mathbb{R}_2$ and $(I-\mathcal{A}_2)^{-1}$ is compact, where $\mathbb{R}_2$ denotes the range of $\mathcal{A}_2$.*

Proof. From (3.3), it is easy to verify that

$$\mathcal{A}_2(0) = 0,$$

and hence $0 \in \mathbb{R}_2$ is proved.

We next turn to prove that $(I-\mathcal{A}_2)^{-1}$ is compact. For this, let $\left\{\bar{Z}_{n^*}(x,t) \,\middle|\, \|\bar{Z}_{n^*}(x,t)\|_{\mathbb{F}_2} \le c,\, n^* \in \mathbb{N}\right\}$ be a bounded sequence in $\mathbb{F}_2$. On the one hand, from Lemma 3.3 we know that there exists a sequence $\left\{Z_{n^*}(x,t) \,\middle|\, Z_{n^*}(x,t) \in \mathbb{D}_2\right\}$ satisfying

$$(I-\mathcal{A}_2)Z_{n^*}(x,t) = \bar{Z}_{n^*}(x,t). \tag{4.1}$$

Therefore, we have

$$\begin{aligned}\left\langle \bar{Z}_{n^*}(x,t), Z_{n^*}(x,t)\right\rangle_{\mathbb{F}_2} &= \left\langle (I-\mathcal{A}_2)Z_{n^*}(x,t), Z_{n^*}(x,t)\right\rangle_{\mathbb{F}_2} \\ &= \left\langle Z_{n^*}(x,t), Z_{n^*}(x,t)\right\rangle_{\mathbb{F}_2} - \left\langle \mathcal{A}_2 Z_{n^*}(x,t), Z_{n^*}(x,t)\right\rangle_{\mathbb{F}_2},\end{aligned}$$

which together with (3.5) results in

$$\left\langle \bar{Z}_{n^*}(x,t), Z_{n^*}(x,t)\right\rangle_{\mathbb{F}_2} \ge \left\langle Z_{n^*}(x,t), Z_{n^*}(x,t)\right\rangle_{\mathbb{F}_2}. \tag{4.2}$$

On the other hand, by (4.1), it follows that

$$\begin{aligned}\left\langle \bar{Z}_{n^*}(x,t), \bar{Z}_{n^*}(x,t)\right\rangle_{\mathbb{F}_2} &= \left\langle Z_{n^*}(x,t), \bar{Z}_{n^*}(x,t)\right\rangle_{\mathbb{F}_2} - \left\langle \mathcal{A}_2 Z_{n^*}(x,t), Z_{n^*}(x,t)\right\rangle_{\mathbb{F}_2} \\ &\quad + \left\langle \mathcal{A}_2 Z_{n^*}(x,t), \mathcal{A}_2 Z_{n^*}(x,t)\right\rangle_{\mathbb{F}_2},\end{aligned}$$

by which and (3.5), we arrive at

$$\left\langle Z_{n^*}(x,t), \bar{Z}_{n^*}(x,t)\right\rangle_{\mathbb{F}_2} \le \left\langle \bar{Z}_{n^*}(x,t), \bar{Z}_{n^*}(x,t)\right\rangle_{\mathbb{F}_2}. \tag{4.3}$$

Thus, by (4.2) and (4.3), there holds

$$\|Z_{n^*}(x,t)\|_{\mathbb{F}_2} \le \|\bar{Z}_{n^*}(x,t)\|_{\mathbb{F}_2} \le c,$$

which together with Sobolev Imbedding Theorem implies that there exist a subsequence $\left\{Z_{n^*_{k^*}}(x,t)\,\middle|\,k^*\in\mathbb{N}\right\}$ of $\left\{Z_{n^*}(x,t)\right\}$ and $Z_0(x)\in\mathbb{F}_2$, such that

$$Z_{n^*_{k^*}}(x,t)\;\to\;Z_0(x).$$

Hence, $(I-\mathcal{A}_2)^{-1}$ is compact and the proof is complete. □

We are now in a position to address the main result of the paper, which is summarized in the following theorem.

Theorem 4.2. *For any initial value $Z_0(x)\in\mathbb{F}_2$, if design parameters α and K satisfying $\alpha>\frac{1}{2K}$, then the weak solution $Z(x,t)=T(t)Z_0(x)$ of closed-loop system (3.1) is asymptotically stable in the following sense:*

$$\lim_{t\to\infty}\left(\int_0^L\left(w_t^2(x,t)+a(x)w_x^2(x,t)\right)\,\mathrm{d}x+X_1^2(t)\right)=0, \tag{4.4}$$

$$\lim_{t\to\infty}\tilde{X}_2^2(t)=0, \tag{4.5}$$

and

$$\begin{cases}\lim_{t\to\infty}\hat{\theta}_i(t)=\theta_i, & i=1,\ldots,n,\\ \lim_{t\to\infty}\hat{\varphi}_i(t)=\varphi_i, & i=1,\ldots,n.\end{cases} \tag{4.6}$$

Proof. We first prove that (4.4), (4.5) and (4.6) hold for any initial value $Z_0(x)\in\mathbb{D}_2$. For this, by Theorem 3.65 on page 162 of [25] and Lemma 4.1, we can directly conclude that the trajectory $\gamma\left(Z_0(x)\right)=\left\{Z(x,t)\,\middle|\,t\ge 0\right\}$ of system (3.4) is precompact in $\mathbb{F}_2$. Then by using LaSalle's invariance principle, any solution of system (3.4) tends to the largest invariant set of the following:

$$S=\left\{\left(w(x,t),w_t(x,t),X_1(t),\tilde{X}_2(t),\Theta(t)\right)\,\middle|\,\dot{V}(t)\;=\;0\right\}.$$

For any $\left(\breve{w}(x,t),\breve{w}_t(x,t),\breve{X}_1(t),\breve{\tilde{X}}_2(t),\breve{\Theta}(t)\right)\in S$, by the inequality of (2.14), we have

$$\begin{cases}\breve{\tilde{X}}_2(t)\;=\;0,\\ a(L)\breve{w}_x(L,t)+\breve{X}_1(t)\;=\;0,\end{cases} \tag{4.7}$$

by which, the forth equation of (2.2), (2.3), (2.13) and (2.14), there holds

$$\begin{cases}\breve{X}_2(t)\;=\;0,\\ \breve{w}_t(0,t)\;=\;0,\\ \breve{\tilde{\theta}}_i(t)\;\equiv\;\breve{\tilde{\theta}}_i(0),\;\;i=1,\ldots,n,\\ \breve{\tilde{\varphi}}_i(t)\;\equiv\;\breve{\tilde{\varphi}}_i(0),\;\;i=1,\ldots,n.\end{cases} \tag{4.8}$$

Hence, by (3.1), (4.7) and (4.8), $\breve{w}(x,t)$ should satisfy

$$\begin{cases}\breve{w}_{tt}(x,t)-(a(x)\breve{w}_x(x,t))_x\;=\;0,\\ \breve{w}_x(0,t)\;=\;0,\\ a(L)\breve{w}_x(L,t)+\breve{w}(L,t)\;=\;0,\\ \breve{w}_t(L,t)\;=\;0.\end{cases} \tag{4.9}$$

Noting that $Z_0(x)\in\mathbb{D}_2$ and by claim (ii) of Theorem 3.4, there holds

$$\breve{w}(x,t)\;\in\;\mathbb{W}^{1,\infty}(0,\infty;\mathbb{H}^1(0,L))\cap\mathbb{L}^\infty(0,\infty;\mathbb{H}^2(0,L)),$$

which together with Lemma 3 in [4] yields that, for system (4.9)

$$\breve{w}(x,t) \equiv 0.$$

Moreover, by (3.1) and (4.8), we obtain that for any $t \geq 0$

$$\sum_{i=1}^{n} \left(\breve{\tilde{\theta}}_i(t) \sin \omega_i t + \breve{\tilde{\varphi}}_i(t) \cos \omega_i t \right) = 0,$$

from which it follows that $\breve{\tilde{\theta}}_i(t) \equiv 0$ and $\breve{\tilde{\varphi}}_i(t) \equiv 0$. Thus, we have

$$S = \{(0,0,0,0,0)\}.$$

By LaSalle's invariance principle, the following result can be directly obtained:

$$\lim_{t\to\infty} \|T(t)Z_0(x)\|^2_{\mathbb{F}_2} = 0. \tag{4.10}$$

This directly yields

$$\lim_{t\to\infty} \left(\int_0^L \left(w_t^2(x,t) + a(x) w_x^2(x,t) \right) \mathrm{d}x + X_1^2(t) + \tilde{X}_2^2(t) + \sum_{i=1}^{n} \left(\tilde{\theta}_i^2(t) + \tilde{\varphi}_i^2(t) \right) \right) = 0,$$

which implies that

$$\lim_{t\to\infty} \left(\int_0^L \left(w_t^2(x,t) + a(x) w_x^2(x,t) \right) \mathrm{d}x + X_1^2(t) \right) = 0,$$

$$\lim_{t\to\infty} \tilde{X}_2^2(t) = 0,$$

and

$$\begin{cases} \lim_{t\to\infty} \hat{\theta}_i(t) &= \theta_i, \quad i = 1,\ldots,n, \\ \lim_{t\to\infty} \hat{\varphi}_i(t) &= \varphi_i, \quad i = 1,\ldots,n. \end{cases}$$

We next turn to prove that (4.4), (4.5) and (4.6) hold for any initial value $Z_0(x) \in \mathbb{F}_2$. From $\overline{\mathbb{D}}_2 = \mathbb{F}_2$, there exists a sequence $\{ Z_{j^*}(x,t) \mid Z_{j^*}(x,t) \in \mathbb{D}_2,\ j^* \in \mathbb{N}\}$ such that

$$\lim_{j^*\to\infty} Z_{j^*}(x,t) = Z_0(x),$$

by which and noting that $T(t)$ is a contraction semigroup defined on $\mathbb{F}_2$, we arrive at

$$\begin{aligned} \|T(t)Z_0(x)\|^2_{\mathbb{F}_2} &\leq \|T(t)(Z_0(x) - Z_{j^*}(x,t))\|^2_{\mathbb{F}_2} + \|T(t)Z_{j^*}(x,t)\|^2_{\mathbb{F}_2} \\ &\leq \|Z_0(x) - Z_{j^*}(x,t)\|^2_{\mathbb{F}_2} + \|T(t)Z_{j^*}(x,t)\|^2_{\mathbb{F}_2}. \end{aligned}$$

Since $Z_{j^*}(x,t) \in \mathbb{D}_2$ and (4.10) holds, we obtain

$$\lim_{t\to\infty} \|T(t)Z_0(x)\|^2_{\mathbb{F}_2} = 0,$$

and hence for any initial value $Z_0(x) \in \mathbb{F}_2$, (4.4), (4.5) and (4.6) hold. This completes the proof. □

Remark 4.3. It is worth mentioning that, if $d(t)$ in (1.1) is replaced by $d(t)+\varepsilon(t)$ where $\varepsilon(t) \in \mathbb{L}^2[0, +\infty)$ is the perturbation for $d(t)$, then for any initial value $Z_0(x) \in \mathbb{F}_2$, only the boundedness of the states $w(x,t)$, $X_1(t)$, $\tilde{X}_2(t)$ and $\Theta(t)$ of the resulting closed-loop system can be obtained under the adaptive controller consisting of (2.12) and (2.13). This is because when $d(t)$ in (1.1) is replaced by $d(t)+\varepsilon(t)$, by choosing $\alpha > \frac{1}{2K}$, function (2.8) satisfies

$$\dot{V}(t) \leq -\frac{K}{2}\left(a(L)w_x(L,t)+X_1(t)\right)^2 - \frac{1}{2}\left(\alpha - \frac{1}{2K}\right)\tilde{X}_2^2(t) + \frac{K}{2K\alpha-1}\varepsilon^2(t) \leq \frac{K}{2K\alpha-1}\varepsilon^2(t),$$

which can be derived similarly to the inequality in (2.14). Noting $V(t)=\|Z(x,t)\|^2_{\mathbb{F}_2}$, we have

$$V(t) \leq \|Z_0(x)\|^2_{\mathbb{F}_2} + \frac{K}{2K\alpha-1}\int_0^t \varepsilon^2(t)\,\mathrm{d}t.$$

Then, by the definition of $V(t)$ and noting $\varepsilon(t)\in\mathbb{L}^2[0,+\infty)$, it follows that states $X_1(t)$, $\tilde{X}_2(t)$ and $\Theta(t)$ of the resulting closed-loop system are bounded, and moreover, $w(x,t)$ satisfies (3.13). From (3.13) and the boundedness of $X_1(t)$, we conclude that $w(x,t)$ is bounded by using Poincaré's inequality and Agmon's inequality.

5. Simulation results

In this section, numerical results are given to illustrate the effectiveness of the theoretical results for the following system:

$$\begin{cases} w_{tt}(x,t) - (a(x)w_x(x,t))_x = 0, \\ w_x(0,t) = 0.03w_t(0,t), \\ w(L,t) = X_1(t), \\ \dot{X}_1(t) = X_2(t), \\ \dot{X}_2(t) = \lambda a(L)w_x(L,t) + \frac{1}{M}u(t) + d(t), \end{cases} \tag{5.1}$$

with scale initial conditions

$$w_0(x) = w(x,0) = 0.5(L-x)$$

and

$$w_1(x) = w_t(x,0) = \begin{cases} 0.1, & 0 \leq x \leq \frac{L}{2}, \\ 0, & \text{else}. \end{cases}$$

The values of the parameters are assumed as: $L=5\text{m}$, $M=25\text{kg}$, $m=2000\text{kg}$, $\rho=5\text{kg/m}$, $\mathrm{g}=9.8\text{m/s}^2$, $i=2$, $\omega_1=1$, $\omega_2=\frac{\pi}{4}$, $\theta_1=1.8$, $\varphi_1=0.5$, $\theta_2=1.2$ and $\varphi_2=0.3$. Then, from (1.3), we have

$$d(t) = 1.8\sin t + 0.5\cos t + 1.2\sin\frac{\pi}{4}t + 0.3\cos\frac{\pi}{4}t.$$

From [2,4], by taking $a(x)$ and λ as

$$\begin{cases} a(x) = \mathrm{g}x + \frac{\mathrm{g}m}{\rho}, \\ \lambda = \frac{(m+\rho L)\mathrm{g}}{Ma(L)}, \end{cases}$$

we obtain

$$\begin{cases} a(x) = 9.8x + 3920, \\ \lambda = 0.2. \end{cases}$$

Moreover, the initial values of the estimates of unknown parameters are chosen as: $\hat{\theta}_1(0)=1.5$, $\hat{\varphi}_1(0)=0.2$, $\hat{\theta}_2(0)=0.8$ and $\hat{\varphi}_2(0)=0.6$.

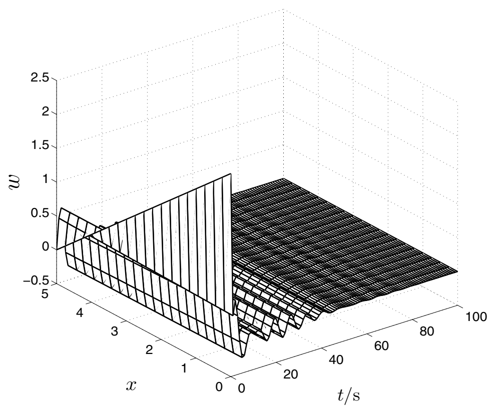

FIGURE 1. Trajectory of $w(x,t)$ when $K = 0.1$.

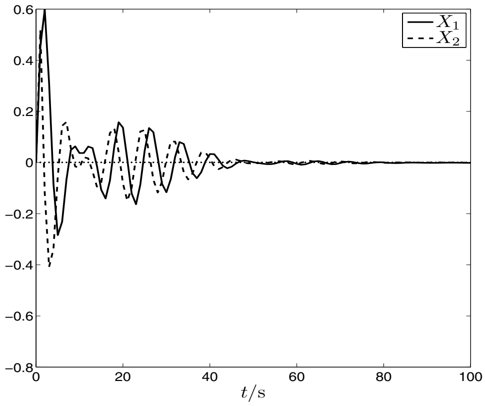

FIGURE 2. Trajectories of $X_1(t)$ and $X_2(t)$ when $K = 0.1$.

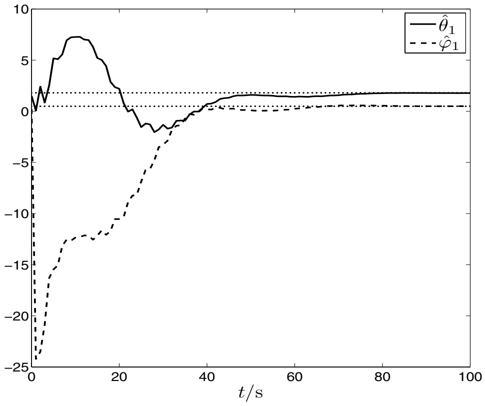

FIGURE 3. Trajectories of $\hat{\theta}_1(t)$ and $\hat{\varphi}_1(t)$ when $K = 0.1$.

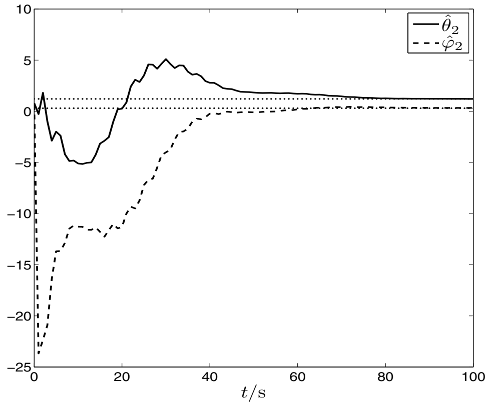

FIGURE 4. Trajectories of $\hat{\theta}_2(t)$ and $\hat{\varphi}_2(t)$ when $K = 0.1$.

To derive the explicit form of the adaptive controller for system (5.1), we find suitable design parameters $K = 0.1$ and $\alpha = 6.65$. Then, by (2.12) and (2.13), we design the following adaptive controller:

$$\begin{cases} u(t) = -25\left(396.9w_{xt}(L,t) - 0.01\left(3969w_x(L,t) + X_1(t)\right) + 6.75\tilde{X}_2(t)\right) \\ \qquad\qquad -25\left(\hat{\theta}_1(t)\sin t + \hat{\varphi}_1(t)\cos t + \hat{\theta}_2(t)\sin\frac{\pi}{4}t + \hat{\varphi}_2(t)\cos\frac{\pi}{4}t\right) - 19845w_x(L,t), \\ \dot{\hat{\theta}}_1(t) = \tilde{X}_2(t)\sin t, \\ \dot{\hat{\varphi}}_1(t) = \tilde{X}_2(t)\cos t, \\ \dot{\hat{\theta}}_2(t) = \tilde{X}_2(t)\sin\frac{\pi}{4}t, \\ \dot{\hat{\varphi}}_2(t) = \tilde{X}_2(t)\cos\frac{\pi}{4}t. \end{cases}$$

By using the implicit backward Euler method and explicit central difference method (see *e.g.*, p. 407 and 415 of [32], respectively) with the grid size is taken as $N = 20$ and time step $dt = 10^{-3}$, four figures are obtained for the closed-loop system signals. Specifically, Figures 1–2 show the trajectories of the PDE subsystem state $w(x,t)$, the ODE subsystem states $X_1(t)$ and $X_2(t)$, respectively, from which we can see that the closed-loop system (3.1) is asymptotically stable; Figure 3 shows the parameter estimates $\hat{\theta}_1(t)$ and $\hat{\varphi}_1(t)$, and Figure 4

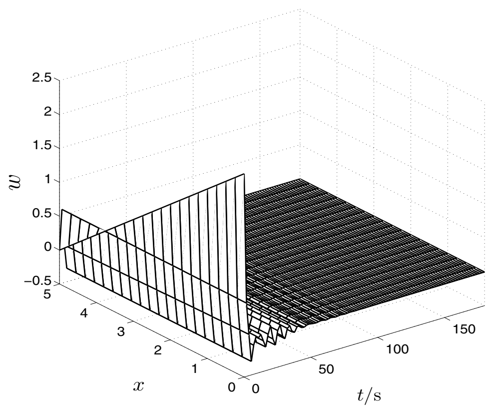

FIGURE 5. Trajectory of $w(x,t)$ when $K = 0.08$.

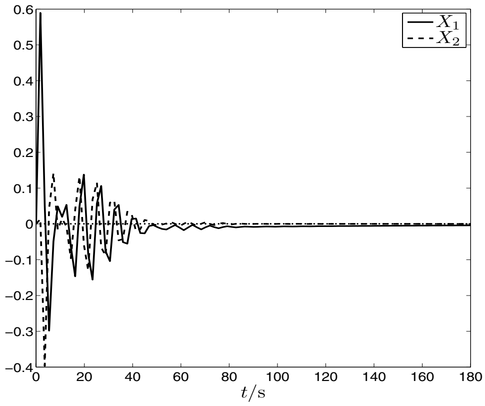

FIGURE 6. Trajectories of $X_1(t)$ and $X_2(t)$ when $K = 0.08$.

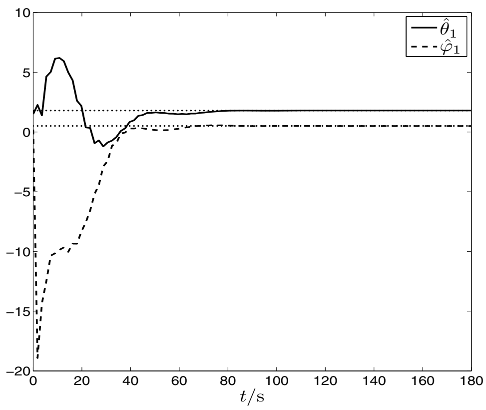

FIGURE 7. Trajectories of $\hat{\theta}_1(t)$ and $\hat{\varphi}_1(t)$ when $K = 0.08$.

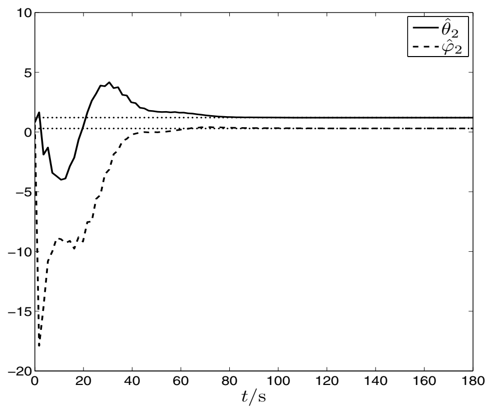

FIGURE 8. Trajectories of $\hat{\theta}_2(t)$ and $\hat{\varphi}_2(t)$ when $K = 0.08$.

shows the parameter estimates $\hat{\theta}_2(t)$ and $\hat{\varphi}_2(t)$, from the two figures we can see that $\hat{\theta}_1(t)$, $\hat{\varphi}_1(t)$, $\hat{\theta}_2(t)$ and $\hat{\varphi}_2(t)$ ultimately converge to their own real values $\theta_1 = 1.8$, $\varphi_1 = 0.5$, $\theta_2 = 1.2$ and $\varphi_2 = 0.3$ (dashed lines), respectively.

To show the effects of design parameters K and α on the theoretical results, we take $K = 0.08$ while α remains the same, then we obtain Figures 5–8. These figures together with Figures 1–4 (especially Figs. 2 and 6) show that the convergence time when $K = 0.1$ ($t < 80$ s in Fig. 2) is shorter than that when $K = 0.08$ ($t > 100$ s in Fig. 6).

6. Concluding remarks

In this paper, the asymptotic stabilization has been investigated for a class of PDE-ODE cascade systems with uncertain harmonic disturbances. A non-adaptive controller $X_2^*(t)$ is first designed for a PDE subsystem of (1.1). Then, based on the obtained controller and the adaptive technique, an adaptive feedback controller is successfully constructed for the entire system which guarantees that not only the error $X_2(t) - X_2^*(t)$ converges to zero, but also the closed-loop system is asymptotically stable and the parameter estimates converge to their own real values. Moreover, the well-posedness of the closed-loop system is established through the semigroup approach. Our further research is to the output feedback boundary control design for the case of more general unknown harmonic disturbances, such as those with not only unknown amplitudes but also unknown frequencies.

Appendix A.

A.1. Proof of Proposition 3.2

Define the following functional $J(\Gamma)$ on $\mathbb{H}^1(0,L)$:

$$J(\Gamma)=\frac{1}{2}\int_0^L\left(a\Gamma_x^2+\Gamma^2\right)\mathrm{d}x-\int_0^L(p^*+\eta^*)\Gamma\,\mathrm{d}x+\frac{qa(0)}{2}\left(\Gamma(0)-p^*(0)\right)^2$$
$$+\frac{K}{2(K+1)}\left(\left(1+\frac{1}{K}\right)\Gamma(L)-\frac{1}{K}\left(r^*+\frac{v^*+\Psi^{*\mathrm{T}}\Omega}{1+\alpha+\Omega^{\mathrm{T}}\Omega}\right)\right)^2, \tag{A.1}$$

which together with Young's Inequality yields

$$J(\Gamma)\geq\frac{1}{2}\int_0^L a\Gamma_x^2\,\mathrm{d}x+\frac{1}{2}\int_0^L\Gamma^2\,\mathrm{d}x-\frac{1}{2}\int_0^L\left((p^*+\eta^*)^2+\Gamma^2\right)\mathrm{d}x+\frac{qa(0)}{2}\left(\Gamma(0)-p^*(0)\right)^2$$
$$+\frac{K}{2(K+1)}\left(\left(1+\frac{1}{K}\right)\Gamma(L)-\frac{1}{K}\left(r^*+\frac{v^*+\Psi^{*\mathrm{T}}\Omega}{1+\alpha+\Omega^{\mathrm{T}}\Omega}\right)\right)^2$$
$$\geq\frac{1}{2}\int_0^L a\Gamma_x^2\,\mathrm{d}x-\frac{1}{2}\int_0^L(p^*+\eta^*)^2\,\mathrm{d}x+\frac{qa(0)}{2}\left(\Gamma(0)-p^*(0)\right)^2$$
$$+\frac{K}{2(K+1)}\left(\left(1+\frac{1}{K}\right)\Gamma(L)-\frac{1}{K}\left(r^*+\frac{v^*+\Psi^{*\mathrm{T}}\Omega}{1+\alpha+\Omega^{\mathrm{T}}\Omega}\right)\right)^2. \tag{A.2}$$

For $\Gamma\in\mathbb{H}^1(0,L)$, define the norm $\|\Gamma\|_{\mathbb{H}^1(0,L)}=\sqrt{\int_0^L\Gamma^2(x)\,\mathrm{d}x}+\sqrt{\int_0^L\Gamma_x^2(x)\,\mathrm{d}x}$. Then, by Poincaré's inequality, we have $\Gamma(0)\to\infty$ or $\int_0^L\Gamma_x^2(x)\,\mathrm{d}x\to\infty$ as $\|\Gamma\|_{\mathbb{H}^1(0,L)}\to\infty$. Hence, when $\|\Gamma\|_{\mathbb{H}^1(0,L)}\to\infty$, from (A.2), there holds $J(\Gamma)\to\infty$, which implies that $J(\Gamma)$ is coercive.

Moreover, from (A.1), it is easy to verify that $J(\Gamma)$ is convex and continuous on $\mathbb{H}^1(0,L)$. Noting that $\mathbb{H}^1(0,L)$ is a closed, convex and reflexive space, by Proposition 38.15(a) on page 155 of [33] and choosing $\mathbb{M}=\mathbb{H}^1(0,L)$, there exists a function $p\in\mathbb{H}^1(0,L)$ such that

$$J(p)=\inf_{\Gamma\in\mathbb{H}^1(0,L)}J(\Gamma),$$

which implies that the function $\Upsilon:\mu\to\Upsilon(\mu)=J(p+\mu\Gamma)$ attained a minimum at the point $\mu=0$. Thus, for any $\Gamma\in\mathbb{H}^1(0,L)$, there holds

$$\left.\frac{\mathrm{d}\left(J(p+\mu\Gamma)\right)}{\mathrm{d}\mu}\right|_{\mu=0}=0. \tag{A.3}$$

From (A.1) and (A.3), it follows that

$$\begin{aligned}\frac{\mathrm{d}\left(J(p+\mu\Gamma)\right)}{\mathrm{d}\mu} &= \frac{\mathrm{d}}{\mathrm{d}\mu}\Bigg(\frac{1}{2}\int_0^L \left(a\left(p_x+\mu\Gamma_x\right)^2+\left(p+\mu\Gamma\right)^2\right)\mathrm{d}x-\int_0^L\left(p^*+\eta^*\right)\left(p+\mu\Gamma\right)\mathrm{d}x\\&\quad+\frac{qa(0)}{2}\left(p(0)+\mu\Gamma(0)-p^*(0)\right)^2+\frac{K}{2(K+1)}\Bigg(\left(1+\frac{1}{K}\right)\left(p(L)+\mu\Gamma(L)\right)\\&\quad-\frac{1}{K}\left(r^*+\frac{v^*+\Psi^{*\mathrm{T}}\Omega}{1+\alpha+\Omega^{\mathrm{T}}\Omega}\right)\Bigg)^2\Bigg)\\&=\int_0^L\left(a\left(p_x+\mu\Gamma_x\right)\Gamma_x+\left(p+\mu\Gamma\right)\Gamma\right)\mathrm{d}x-\int_0^L\left(p^*+\eta^*\right)\Gamma\,\mathrm{d}x\\&\quad+qa(0)\left(p(0)+\mu\Gamma(0)-p^*(0)\right)\Gamma(0)+\Bigg(\left(1+\frac{1}{K}\right)\left(p(L)+\mu\Gamma(L)\right)\\&\quad-\frac{1}{K}\left(r^*+\frac{v^*+\Psi^{*\mathrm{T}}\Omega}{1+\alpha+\Omega^{\mathrm{T}}\Omega}\right)\Bigg)\Gamma(L),\end{aligned}$$

and hence

$$\begin{aligned}\left.\frac{\mathrm{d}\left(J(p+\mu\Gamma)\right)}{\mathrm{d}\mu}\right|_{\mu=0}&=\int_0^L\left(ap_x\Gamma_x+p\Gamma\right)\mathrm{d}x-\int_0^L\left(p^*+\eta^*\right)\Gamma\,\mathrm{d}x+qa(0)\left(p(0)-p^*(0)\right)\Gamma(0)\\&\quad+\left(\left(1+\frac{1}{K}\right)p(L)-\frac{1}{K}\left(r^*+\frac{v^*+\Psi^{*\mathrm{T}}\Omega}{1+\alpha+\Omega^{\mathrm{T}}\Omega}\right)\right)\Gamma(L).\end{aligned}\tag{A.4}$$

For any $\Gamma\in\mathbb{C}_0^\infty(0,L)$ which implies that $\Gamma(0)=0$ and $\Gamma(L)=0$, by (A.3) and (A.4), we have

$$\int_0^L ap_x\Gamma_x\,\mathrm{d}x+\int_0^L p\Gamma\,\mathrm{d}x-\int_0^L\left(p^*+\eta^*\right)\Gamma\,\mathrm{d}x=0,$$

which implies that

$$(ap_x)_x-p=-p^*-\eta^*,\ \ p\in\mathbb{H}^2(0,L).\tag{A.5}$$

Moreover, using integration by parts for (A.4), we obtain

$$\begin{aligned}\left.\frac{\mathrm{d}\left(J(p+\mu\Gamma)\right)}{\mathrm{d}\mu}\right|_{\mu=0}&=-\int_0^L(ap_x)_x\Gamma\,\mathrm{d}x+\int_0^L p\Gamma\,\mathrm{d}x-\int_0^L\left(p^*+\eta^*\right)\Gamma\,\mathrm{d}x\\&\quad-a(0)\left(p_x(0)-q\left(p(0)-p^*(0)\right)\right)\Gamma(0)+\Bigg(a(L)p_x(L)\\&\quad+\left(1+\frac{1}{K}\right)p(L)-\frac{1}{K}\left(r^*+\frac{v^*+\Psi^{*\mathrm{T}}\Omega}{1+\alpha+\Omega^{\mathrm{T}}\Omega}\right)\Bigg)\Gamma(L),\end{aligned}$$

by which and (A.5) as well as the fact that Γ is arbitrary, there holds

$$\begin{cases}p_x(0)\;=\;q(p(0)-p^*(0)),\\ a(L)p_x(L)\;=\;-\left(1+\frac{1}{K}\right)p(L)+\frac{1}{K}\left(r^*+\frac{v^*+\Psi^{*\mathrm{T}}\Omega}{1+\alpha+\Omega^{\mathrm{T}}\Omega}\right).\end{cases}$$

Therefore, the existence of solutions of system (3.6) is proved. □

A.2. Useful inequalities

The following two lemmas provide several useful inequalities. The proofs of these two lemmas are similar to those of Lemmas B.1 and B.2 in [30] and hence are omitted here.

Lemma A.1 (Poincaré's inequality)**.** *For any $w \in \mathbb{H}^1(0, L)$, the following inequalities hold:*

$$\begin{cases} \int_0^L w^2(x)\mathrm{d}x \le 2Lw^2(0) + 4L^2 \int_0^L w_x^2(x)\mathrm{d}x, \\ \int_0^L w^2(x)\mathrm{d}x \le 2Lw^2(L) + 4L^2 \int_0^L w_x^2(x)\mathrm{d}x. \end{cases}$$

Lemma A.2 (Agmon's inequality)**.** *For any $w \in \mathbb{H}^1(0, L)$, the following inequalities hold:*

$$\begin{cases} \max_{x\in[0,L]} w^2(x) \le w^2(0) + 2\sqrt{\int_0^L w^2(x)\mathrm{d}x \int_0^L w_x^2(x)\mathrm{d}x}, \\ \max_{x\in[0,L]} w^2(x) \le w^2(L) + 2\sqrt{\int_0^L w^2(x)\mathrm{d}x \int_0^L w_x^2(x)\mathrm{d}x}. \end{cases}$$

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