ESAIM: COCV 23 (2017) 455–473
DOI: 10.1051/cocv/2015056

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

CONTROL PROBLEMS GOVERNED BY TIME-DEPENDENT MAXIMAL MONOTONE OPERATORS

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Abstract. The paper concerns on an infinite dimensional Hilbert space, the existence and uniqueness of absolutely continuous solutions, for Lipschitz single-valued perturbations of evolution problems involving maximal-monotone operators. This result allows us to extend to optimal control problems associated with such equations, the relaxation theorems with Young measures proved recently in [S. Saïdi, L. Thibault and M.F. Yarou, *Numer. Funct. Anal. Optim.* **34** (2013) 1156–1186].

Mathematics Subject Classification. 34A60, 49A52.

Received June 5, 2015. Revised November 27, 2015. Accepted December 22, 2015.

1. Introduction

The study of inclusions involving maximal-monotone operators constitutes an important class in set-valued analysis. These operators enjoy some properties that guarantees existence and uniqueness of solutions related to such problems. The authors in [9], were interested on an interval $I := [0, T]$, in the evolution inclusion

$$\begin{cases} -\dot{x}(t) \in A(t)x(t) & \text{a.e. } t \in I \\ x(0) = x_0 \in \mathrm{D}\,(A(0)), \end{cases}$$

where $A(t) : \mathrm{D}\,(A(t)) \subset H \rightrightarrows H$ is a maximal monotone operator in a Hilbert space H for every $t \in I$, and the dependence $t \mapsto A(t)$ is –in some sense– absolutely continuous on I. They proved the existence and uniqueness of an absolutely continuous (AC) solution to the problem with the following assumptions:

(C_1) There exists an $a \in W^{1,1}(I)$ such that

$$\mathrm{dis}\,(A(t), A(s)) \leq \|a(t) - a(s)\| \text{ for } t, s \in I, \tag{1.1}$$

where $\mathrm{dis}\,(\cdot,\cdot)$ is the pseudo-distance between maximal monotone operators introduced in [15] (see the definition below).

Keywords and phrases. Maximal monotone operators, optimal control, pseudo-distance, Lipschitz perturbation, absolutely continuous, Young measures.

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(C_2) There exists a positive constant c such that

$$\|A^{\circ}(t)x\| \leq c(1+\|x\|) \text{ for } t \in I,\ x \in \mathrm{D}\,(A(t)), \tag{1.2}$$

where $A^{\circ}(t)x$ denotes the element of minimal norm of $A(t)x$.

In this work, we deal first with the problem of the form

$$\begin{cases} -\dot{x}(t) \in A(t)x(t) + f(t,x(t)) & \text{a.e. } t \in I \\ x(0) = x_0 \in \mathrm{D}\,(A(0)), \end{cases}$$

where $f : I \times H \longrightarrow H$ is a single-valued map which is Lipschitz continuous with respect to the second variable on any bounded subset of H and which satisfies the natural growth condition

$$\|f(t,x)\| \leq \beta(t)(1+\|x\|),\ \forall (t,x) \in I \times H;$$

with $\beta(\cdot)$ a non negative function in $L^2_{\mathbb{R}}(I)$. Then, in the second part of the paper, we use this result to study the relaxation of a Bolza type problem associated with such evolution inclusion.

The main existence theorem we establish here, is inspired by the work developed in [14] dealing with the particular case of the subdifferential operator, that is,

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + f(t,x(t)) & \text{a.e. } t \in [T_0,T] \\ x(T_0) = x_0 \in \operatorname{dom}\varphi(T_0,\cdot), \end{cases}$$

where for each $t \in [T_0,T]$, $\varphi(t,\cdot)$ is a time-dependent proper lower semicontinuous (lsc) convex function of a Hilbert space H into $\mathbb{R} \cup \{+\infty\}$; $\operatorname{dom}\varphi(t,\cdot)$ denotes the effective domain of the function $\varphi(t,\cdot)$ and the perturbation f satisfies the conditions above. For $\varphi(t,\cdot)$ taken as the indicator function of a closed convex or r-prox-regular moving set $C(t)$ (the sweeping process), many results were obtained in the finite dimensional setting (see *e.g.* [3]) and in the infinite dimensional setting (see *e.g.* [6]).

The result obtained is used to study a Bolza type control problem. Let U be a compact metric space, $\Gamma : I \rightrightarrows U$ a nonempty compact-valued measurable map, and define the Lebesgue-measurable set-valued map Σ from I into $\mathcal{M}^1_+(U)$ (the set of probability measures on U) by

$$\Sigma(t) = \{P \in \mathcal{M}^1_+(U) : P(\Gamma(t)) = 1\}.$$

Denote by S_Γ (resp. S_Σ) the set of all Lebesgue measurable selections of Γ (resp. Σ). Let $g : I \times H \times U \to H$ be a map satisfying some appropriate conditions, in particular some Lipschitz property with respect to the second variable as above. For $x_0 \in D(A(0))$, $\zeta \in S_\Gamma$, and $\mu \in S_\Sigma$, let us consider the two following problems

$$(\mathcal{P}_\zeta) \begin{cases} -\dot{x}(t) \in A(t)x(t) + g(t,x(t),\zeta(t)) & \text{a.e. } t \in I \\ x(0) = x_0 \end{cases}$$

and

$$(\mathcal{P}_\mu) \begin{cases} -\dot{x}(t) \in A(t)x(t) + \int_{\Gamma(t)} g(t,x(t),u)\mu_t(\mathrm{d}u) & \text{a.e. } t \in I \\ x(0) = x_0. \end{cases}$$

We will see that both of these problems has one and only one solution. These solutions will be denoted respectively by $x_\zeta(\cdot)$ and $x_\mu(\cdot)$.

The aim of the second part of the paper is to present a Bolza-type problem for $(\mathcal{P}_\zeta)$ and $(\mathcal{P}_\mu)$. Let $J : I \times H \times U \to \mathbb{R}$ be an integrand. Then, consider the following control problems:

$$\inf_{\zeta(\cdot) \in S_\Gamma} \int_I J(t,x_\zeta(t),\zeta(t))\,\mathrm{d}t \tag{P.O}$$

and

$$\inf_{\mu \in S_\Sigma} \int_I \int_U J(t, x_\mu(t), u)\mu_t(\mathrm{d}u)\,\mathrm{d}t. \tag{P.R}$$

The latter is called the relaxed problem, we will prove that inf (P.O) = min (P.R). Several relaxation results dealing with optimal control problems have been concerned. They involved m-accretive operators $A(t),\ t \in [0,1]$ (see [4]), or the sweeping process (see [6]). More recently, the paper [14] discussed the case of time-dependant subdifferential operator in Hilbert spaces. For more details and related results, see [5,7,8,16].

The result of the second part can be used to study in the finite dimensional setting, the existence of viscosity sub-solutions of Hamilton–Jacobi–Bellman PDEs related to control problems subject to evolution inclusions involving time dependent maximal monotone operators and Young measures. This will be the object of a forthcoming paper [13].

The paper is organized as follows. In Section 2, we give the necessary background material. In Section 3, we state some results concerning the non-autonomous case of an evolution equation governed by time-dependent maximal monotone operator $A(t)x(t)$, proved in [9], along with the case $A(t)x(t) + h(t)$ with an $L^2_H(I)$ map $h(\cdot)$. In Section 4, we prove the existence and uniqueness of the absolutely continuous solution for the single valued perturbation depending on both time variable and state variable of the problem under consideration. The last Section 5 is devoted to an application of the main result of Section 4 to the relaxation problem of optimal control with Young measures.

2. Notation and preliminaries

In all the paper $I := [0,T]$ is an interval of $\mathbb{R}$ and H is a real Hilbert space whose inner product is denoted by $\langle\cdot,\cdot\rangle$ and the associated norm by $\|\cdot\|$. We denote by $B_H[x,r]$ the closed ball of center x and radius r on H, by $\mathbb{B}$ the closed unit ball, and by $\mathbf{1}_A$ the characteristic function of a set A, that is, $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise. By λ, we denote the Lebesgue measure and by $\mathcal{C}_H(I)$ the space of continuous maps $x : I \to H$ with the norm of the uniform convergence on $\mathcal{C}_H(I)$ $\|x\|_\infty = \sup_{t\in I}\|x(t)\|$. We denote by $L^p_H(I)$ the the space of dt-measurable maps $x : I \longrightarrow H$ such that $\int_I \|x(t)\|^p\,\mathrm{d}t < +\infty$ with the norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p\,\mathrm{d}t)^{\frac{1}{p}}, 1 \le p < +\infty$.

If $A : D(A) \rightrightarrows H$ is a maximal monotone operator, then the effective domain of A is $\mathrm{D}\,(A) = \{x \in H : Ax \neq \emptyset\}$. It is well known (see, *e.g.*, [2]) that any maximal monotone operator A satisfies the closure property, that is, if $x = \lim_{n\to\infty} x_n$ strongly in H and $y = \lim_{n\to\infty} y_n$ weakly in H, where $x_n \in \mathrm{D}\,(A)$and $y_n \in Ax_n$, then, $x \in \mathrm{D}\,(A)$ and $y \in Ax$.

The assumption (C_1) is given using the pseudo-distance between two maximal monotone operator A_1 and A_2 defined in [15], as follows

$$\mathrm{dis}\,(A_1, A_2) := \sup\{\frac{\langle y_1 - y_2, x_2 - x_1\rangle}{\|y_1\| + \|y_2\| + 1},\ x_i \in \mathrm{D}\,(A_i),\ y_i \in A_i x_i,\ i = \overline{1,2}\}$$

(the distance may be equal to $+\infty$).

The element of minimal norm of $A(t)x$, is defined by $A^\circ(t)x \in A(t)x$ such that $\|A^\circ(t)x\| = d(0, A(t)x)$. Concerning the properties of maximal monotone operators in Hilbert spaces, we refer to [1,2].

Recall now, the following consequence of Gronwall's lemma proved in [6].

Lemma 2.1. *Let $(x_n(\cdot))$ be a sequence of absolutely continuous maps from I to H. Assume that $\lim_n x_n(0) = 0$ and, for any n,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|x_n(t)\|^2) \le \beta_n(t)\|x_n(t)\|^2 + \alpha_n(t) \quad \text{a.e. } t \in I,$$

where $\alpha_n(\cdot)$ and $\beta_n(\cdot)$ are non negative functions in $L^1_{\mathbb{R}}(I)$. Assume moreover that the sequence $(\beta_n(\cdot))$ is bounded in $L^1_{\mathbb{R}}(I)$ and $\lim_n \int_0^T \alpha_n(t)\,\mathrm{d}t = 0$. Then,

$$\lim_n \|x_n(\cdot)\|_\infty = 0.$$

3. Single-valued time-dependent perturbations

We intend in this section to recall the existence result for evolution equation governed by time-dependent maximal monotone operator $A(t)x(t)$, proved in [9], and deduce some estimation of the solution for the case $A(t)x(t)+h(t)$ with an $L^2_H(I)$ map $h(\cdot)$.

A function $x(\cdot):I\to H$ is said to be a solution of the unperturbed problem, if $x(\cdot)$ is AC on I, with $x(0)=x_0\in \mathrm{D}\,(A(0))$ and $x(t)\in \mathrm{D}\,(A(t))$ for $t\in I$; and if the differential inclusion is satisfied for Lebesgue-a.e. $t\in I$.

Theorem 3.1. *Let for every $t\in I$, a maximal monotone operator $A(t):\mathrm{D}\,(A(t))\to 2^H$ be given such that (C_1) and (C_2) are satisfied. Then, the unperturbed problem*

$$\begin{cases} -\dot{x}(t)\in A(t)x(t) & \text{a.e. } t\in I\\ x(0)=x_0\in \mathrm{D}\,(A(0)), \end{cases}$$

has a unique AC solution $x(\cdot)$ on I which satisfies the following estimate

$$\int_0^T \|\dot{x}(s)\|^2\,\mathrm{d}s\le c^2\int_0^T (1+\|x(s)\|)^2\,\mathrm{d}s. \tag{3.1}$$

Proof. To prove the existence and uniqueness see ([9], Thm. 3). To get the required estimate of the derivative of the solution, we develop some arguments of Theorem 3 in [9]. □

We need the following useful application of Theorem 3.1 concerning an evolution problem with single-valued perturbation depending only on time.

Theorem 3.2. *Under the assumptions of Theorem 3.1, for any $h\in L^2_H(I)$ and $x_0\in \mathrm{D}\,(A(0))$, the differential equation*

$$\begin{cases} -\dot{x}(t)\in A(t)x(t)+h(t) & \text{a.e. in} \quad I\\ x(0)=x_0\in \mathrm{D}\,(A(0)) \end{cases}$$

admits a unique absolutely continuous solution $x(\cdot)$ satisfying

$$\|\dot{x}(\cdot)\|^2_{L^2_H(I)}\le 2\|h(\cdot)\|^2_{L^2_H(I)}+4c^2\int_0^T (1+\|x(t)\|^2)\,\mathrm{d}t, \tag{3.2}$$

with the same constant c of the previous theorem.

Proof. Set $\Phi(t)=\int_0^t h(\tau)\,\mathrm{d}\tau$ for every $t\in I$. Then, the equation

$$\begin{cases} -\dot{x}(t)\in A(t)x(t)+h(t)\\ x(0)=x_0\in \mathrm{D}\,(A(0)) \end{cases}$$

is equivalent to the following equation

$$\begin{cases} -\dot{y}(t)\in B(t)y(t)\\ y(0)=x_0\in \mathrm{D}\,(B(0)), \end{cases} \tag{3.3}$$

where

$$y(t)=x(t)+\Phi(t),\quad \forall t\in I$$
$$B(t)z=A(t)(z-\Phi(t)),\quad \forall (t,z)\in I\times H.$$

It's obvious that $B(t),\ t \in I$, is a maximal monotone operator in H. Therefore, it is sufficient to show that $B(t),\ t \in I$, satisfies the conditions of Theorem 3.1. So, it remains to examine (C_1) and (C_2).

Let us check first (C_1). Consider for any $s, t \in I,\ x_1 \in \mathrm{D}\,(B(t)),\ x_2 \in \mathrm{D}\,(B(s))$ and $y_1 \in B(t)x_1,\ y_2 \in B(s)x_2$. Then, taking the definition of $\mathrm{dis}\,(\cdot,\cdot)$, and the fact that $A(t),\ t \in I$ satisfies (C_1) into account, one has

$$\begin{aligned}\langle y_2 - y_1, x_1 - \Phi(t) - (x_2 - \Phi(s))\rangle &\leq \mathrm{dis}\,(A(t), A(s))(1 + \|y_1\| + \|y_2\|)\\ &\leq \|a(t) - a(s)\|(1 + \|y_1\| + \|y_2\|).\end{aligned}$$

Hence

$$\begin{aligned}\langle y_2 - y_1, x_1 - x_2\rangle &= \langle y_2 - y_1, x_1 - \Phi(t) - (x_2 - \Phi(s))\rangle + \langle y_2 - y_1, \Phi(t) - \Phi(s)\rangle\\ &\leq [\|a(t) - a(s)\| + \|\Phi(t) - \Phi(s)\|](1 + \|y_1\| + \|y_2\|).\end{aligned}$$

Let us put now

$$a_1(t) = \int_0^t [\|\dot{a}(\tau)\| + \|h(\tau)\|]\,\mathrm{d}\tau, \quad \forall t \in I.$$

Thus, $a_1(\cdot)$ is absolutely continuous, with

$$\langle y_2 - y_1, x_1 - x_2\rangle \leq \|a_1(t) - a_1(s)\|(1 + \|y_1\| + \|y_2\|),$$

which entails (C_1).

Now, note that for any $t \in I,\ y \in \mathrm{D}\,(B(t))$, one has

$$\begin{aligned}\|B^\circ(t)y\| &= \|A^\circ(t)(y - \Phi(t))\|\\ &\leq c(1 + \|y - \Phi(t)\|)\\ &\leq c(1 + \|y\| + \int_0^T \|h(\tau)\|\,\mathrm{d}\tau).\end{aligned}$$

Setting $c_1 = c(1 + \int_0^T \|h(\tau)\|\,\mathrm{d}\tau)$, one obtains

$$\|B^\circ(t, y)\| \leq c_1 + \frac{c_1}{1 + \int_0^T \|h(\tau)\|\,\mathrm{d}\tau}\|y\| \leq c_1(1 + \|y\|).$$

Consequently, assumptions of Theorem 3.1 are satisfied, we conclude that the perturbed problem admits a unique AC solution $x(\cdot)$.

To obtain the estimate (3.2), we may follow the same arguments as in Theorem 3 [9] with the necessary modifications. For this purpose, we just summarize the needed passage of the proof and omit details. We fix appropriate partitions $0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = T$ of I. Then, we construct a consequence of functions $(y_n(\cdot))_n$ satisfying

$$-\frac{\mathrm{d}^+ y_n}{\mathrm{d}t}(t) = B^\circ(\psi_n(t))y_n(t) \text{ for } t \in [t_i^n, t_{i+1}^n[,\ y_n(t) \in \mathrm{D}\,(B(\psi_n(t))),$$

and where the function $\psi_n : I \to I$ is defined by

$$\begin{cases}\psi_n(t) = t_i^n & \text{if } t \in [t_i^n, t_{i+1}^n[,\ i \in \{0, \ldots, k_n - 1\},\\ \psi_n(t_{k_n}^n) = T.\end{cases}$$

Thus, the following inequality

$$\|\dot{y}_n(t)\| \leq \|B^\circ(\psi_n(t))y_n(t)\| = \|A^\circ(\psi_n(t))(y_n(t) - \Phi_n(\psi_n(t)))\| \text{ a.e. } t \in I,$$

holds true. Notice that we choose a sequence of step mappings $h_n := \sum_{i=1}^{k_n-1} v_i \mathbf{1}_{[t_i^n, t_{i+1}^n[}$, where $\{v_i,\ i = 1,\dots,k_n-1\} \subset H$, that converges to the function h with respect to the strong topology of $L^2_H(I)$. Next, we define Φ_n by

$$\Phi_n(t) = \int_0^t h_n(\tau)\mathrm{d}\tau,\ t \in I.$$

Taking (1.2) into account, the latter inequality yields

$$\|\dot{y}_n(t)\| \le c(1 + \|y_n(t) - \Phi_n(\psi_n(t))\|).$$

Let's mention that by repeating the process used in [9] with $B(t)$ instead of $A(t)$ (to solve the problem (3.3)), the different convergence modes hold true. As a consequence, the sequence $(\dot{y}_n(\cdot))_n$ converges weakly to $\dot{y}(\cdot)_n$ in $L^2_H(I)$ and $(y_n(\cdot))$ converges strongly to $y(\cdot)$ in $L^2_H(I)$. So, passing to the limit when $n \to \infty$, entails

$$\|\dot{y}(\cdot)\|_{L^2_H(I)} \le \{c^2 \int_0^T (1 + \|y(s) - \Phi(s)\|)^2\, \mathrm{d}s\}^{\frac{1}{2}},$$

i.e.,

$$\|\dot{x}(\cdot) + h(\cdot)\|_{L^2_H(I)} \le c\{\int_0^T (1 + \|x(s)\|)^2\, \mathrm{d}s\}^{\frac{1}{2}}.$$

Hence, one has

$$\|\dot{x}(\cdot)\|^2_{L^2_H(I)} \le 2\|h(\cdot)\|^2_{L^2_H(I)} + 4c^2 \int_0^T (1 + \|x(t)\|^2)\, \mathrm{d}t,$$

which is the required inequality. □

We can now address the case of perturbation depending on both time variable and state variable. The estimate (3.2) will play an important role in the proof of the next Theorem.

4. Single-valued "measurable/Lipschitz" perturbations

This section is devoted to the study of the perturbed problem where $f(\cdot,\cdot)$ is separately measurable and Lipschitz continuous. We prove the existence and uniqueness theorem, using technics from Theorem 4.1 [14].

Theorem 4.1. *Let assumptions (C_1) and (C_2) be satisfied and let $f : I \times H \to H$ be a map such that*

(i) *f is separately measurable on I;*
(ii) *for every $\eta > 0$, there exists a non-negative function $\gamma_\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that, for all $t \in I$ and for any $x, y \in B[0,\eta]$*

$$\|f(t,x) - f(t,y)\| \le \gamma_\eta(t)\|x - y\|; \tag{4.1}$$

(iii) *there exists a non-negative function $\beta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that, for all $t \in I$ and for all $x \in H$, one has*

$$\|f(t,x)\| \le \beta(t)(1 + \|x\|). \tag{4.2}$$

Then, for any $x_0 \in \mathrm{D}\,(A(0))$, the following problem

$$\begin{cases} -\dot{x}(t) \in A(t)x(t) + f(t,x(t)) & \text{a.e. } t \in I \\ x(0) = x_0 \end{cases} \tag{4.3}$$

has one and only one absolutely continuous solution $x(\cdot)$ on I.

Proof. We use subdivisions of I and estimates depending on the initial point of each subinterval. Then, we construct a sequence of solutions $(x_n(\cdot))$ of a problem with single-valued time dependent perturbation in each subinterval by using Theorem 3.2. After that, we show the convergence of this sequence to a solution $x(\cdot)$ of (4.3).

Choose $\tau > 0$ such that

$$\begin{cases} \quad \tau < \frac{1}{2\sqrt{2}c} \\ \text{and} \\ \quad \int_0^\tau \beta^2(t)\,\mathrm{d}t < \frac{1}{8T}[1-8c^2T^2]. \end{cases} \tag{4.4}$$

First part: If $T \le \tau$

1. Construction of the sequence $(x_n(\cdot))$

Define, for every $n \in \mathbb{N}$, a partition of I with

$$t_i^n = i\frac{T}{n} \quad (0 \le i \le n).$$

Consider first the following differential inclusion on the interval $[t_0^n, t_1^n]$

$$\begin{cases} -\dot{x}(t) \in A(t)x(t) + f(t, x_0) \quad \text{a.e. } t \in [t_0^n, t_1^n] \\ \ x(t_0^n) = x_0 \in D(A(0)), \end{cases}$$

and observe that the map $f(\cdot, x_0)$ depends only on t and is in $L_H^2([t_0^n, t_1^n])$ (by assumption (iii)). By Theorem 3.2, the latter differential inclusion has one and only one absolutely continuous solution that we denote by $x_0^n(\cdot) : [t_0^n, t_1^n] \to H$. According to (3.2) this solution satisfies

$$\|\dot{x}_0^n\|^2_{L_H^2([t_0^n,t_1^n])} \le 2\|h_0^n\|^2_{L_H^2([t_0^n,t_1^n])} + 4c^2\int_{t_0^n}^{t_1^n}(1+\|x_0^n(s)\|^2)\,\mathrm{d}s;$$

where $h_0^n : t \mapsto f(t, x_0)$ for all $t \in [t_0^n, t_1^n]$.

Likewise, the differential inclusion

$$\begin{cases} -\dot{x}(t) \in A(t)x(t) + f(t, x_0^n(t_1^n)) \quad \text{a.e. } t \in [t_1^n, t_2^n] \\ \ x(t_1^n) = x_0^n(t_1^n) \in D(A(t_1^n)), \end{cases}$$

has one and only one absolutely continuous solution that we denote by $x_1^n(\cdot) : [t_1^n, t_2^n] \to H$ with $x_1^n(t_1^n) = x_0^n(t_1^n)$, and it satisfies (3.2).

And so on, for each n, there exists a finite sequence of absolutely continuous maps $x_i^n(\cdot) : [t_i^n, t_{i+1}^n] \to H$ $(0 \le i \le n-1)$ such that, for each $i \in \{0, \dots, n-1\}$,

$$\begin{cases} -\dot{x}_i^n(t) \in A(t)x_i^n(t) + f(t, x_{i-1}^n(t_i^n)) \quad \text{a.e. } t \in [t_i^n, t_{i+1}^n] \\ \ x_i^n(t_i^n) = x_{i-1}^n(t_i^n) \in D(A(t_i^n)), \end{cases}$$

with

$$\|\dot{x}_i^n\|^2_{L_H^2([t_i^n,t_{i+1}^n])} \le 2\|h_i^n\|^2_{L_H^2([t_i^n,t_{i+1}^n])} + 4c^2\int_{t_i^n}^{t_{i+1}^n}(1+\|x_i^n(s)\|^2)\,\mathrm{d}s; \tag{4.5}$$

where $x_{-1}^n(0) = x_0$, $h_i^n : t \mapsto f(t, x_i^n(t_i^n))$ for all $t \in [t_i^n, t_{i+1}^n]$.

Define $x_n : I \to H$ by

$$x_n(t) = x_i^n(t) \ \ \forall t \in [t_i^n, t_{i+1}^n]\,,\ i \in \{0, \dots, n-1\}.$$

Obviously $x_n(\cdot)$ is absolutely continuous on I, and, putting

$$\begin{cases} \theta_n(0) = 0 \\ \theta_n(t) = t_i^n \ \text{ if } t \in]t_i^n, t_{i+1}^n], \ i \in \{0, \dots, n-1\}, \end{cases}$$

one has

$$\begin{cases} -\dot{x}_n(t) \in A(t)x_n(t) + f(t, x_n(\theta_n(t))) & \text{a.e. } t \in I \\ \ \ x_n(0) = x_0. \end{cases}$$

Note that the map $f(\cdot, x_n(\theta_n(\cdot)))$ defined for $t \in I$ belongs to $L^2_H(I)$ because, by assumption (iii) for each $i \in \{0, \dots, n-1\}$, the map $f(\cdot, x_i^n(t_i^n))$ belongs to $L^2_H([t_i^n, t_{i+1}^n])$.

Set

$$h_n(t) = f(t, x_n(\theta_n(t))), \ \forall t \in I.$$

Then, for $I_i := [t_i^n, t_{i+1}^n]$, the inequality

$$\|\dot{x}_n\|^2_{L^2_H(I_i)} \leq 2\|h_n\|^2_{L^2_H(I_i)} + 4c^2 \int_{t_i^n}^{t_{i+1}^n} (1 + \|x_n(s)\|^2) \,\mathrm{d}s$$

holds true.

Observing that

$$\|\dot{x}_n\|^2_{L^2_H(I_i)} \leq 2 \int_{t_i^n}^{t_{i+1}^n} \|f(t, x_n(\theta_n(t)))\|^2 \,\mathrm{d}t + 4c^2 \int_{t_i^n}^{t_{i+1}^n} \mathrm{d}s + 4c^2 \|x_n\|^2_{L^2_H(I_i)}, \tag{4.6}$$

and taking (iii) into account, one has

$$\begin{aligned} \|\dot{x}_n\|^2_{L^2_H(I_i)} &\leq 2(1 + \|x_n(t_i^n)\|)^2 \int_{t_i^n}^{t_{i+1}^n} \beta^2(t) \,\mathrm{d}t + 4c^2(t_{i+1}^n - t_i^n) + 4c^2 \int_{t_i^n}^{t_{i+1}^n} \|x_n(s)\|^2 \,\mathrm{d}s, \\ &\leq 4(1 + \max \|x_n(t_i^n)\|^2) \int_{t_i^n}^{t_{i+1}^n} \beta^2(t) \,\mathrm{d}t + 4c^2(t_{i+1}^n - t_i^n) + 4c^2 \int_{t_i^n}^{t_{i+1}^n} \|x_n(s)\|^2 \,\mathrm{d}s. \end{aligned} \tag{4.7}$$

This being true for any $i \in \{0, \dots, n-1\}$, one has

$$\begin{aligned} \sum_{i=0}^{n-1} \int_{t_i^n}^{t_{i+1}^n} \|\dot{x}_n(s)\|^2 \,\mathrm{d}s \leq 4(1 + \|x_n(\cdot)\|^2_\infty) \int_0^T \beta^2(t) \,\mathrm{d}t \\ + 4c^2 T + 4c^2 T \|x_n(\cdot)\|^2_\infty. \end{aligned}$$

Then,

$$\begin{aligned} \|\dot{x}_n\|^2_{L^2_H(I)} &\leq 4(1 + \|x_n(\cdot)\|^2_\infty) \int_0^T \beta^2(t) \,\mathrm{d}t + 4c^2 T + 4c^2 T \|x_n(\cdot)\|^2_\infty \\ &\leq 4[\int_0^T \beta^2(t) \,\mathrm{d}t + c^2 T] \|x_n(\cdot)\|^2_\infty + 4 \int_0^T \beta^2(t) \,\mathrm{d}t + 4c^2 T. \end{aligned}$$

Setting

$$\alpha = 4 \left[\int_0^T \beta^2(t) \,\mathrm{d}t + c^2 T \right]$$

one gets

$$\|\dot{x}_n\|^2_{L^2_H(I)} \le \alpha[\|x_n(\cdot)\|^2_\infty + 1]. \tag{4.8}$$

Using the Cauchy–Schwartz inequality and taking (4.8) into account, one has for all $s \in I$

$$\|x_n(s) - x_0\|^2 \le s\left(\int_0^s \|\dot{x}_n(t)\|^2 \mathrm{d}t\right) \le T\alpha[\|x_n(\cdot)\|^2_\infty + 1]$$

and hence

$$\|x_n(s)\|^2 \le 2\|x_0\|^2 + 2\|x_n(s) - x_0\|^2 \le 2\|x_0\|^2 + 2T\alpha[\|x_n(\cdot)\|^2_\infty + 1].$$

Consequently, for each n, we get

$$[1 - 2T\alpha]\|x_n(\cdot)\|^2_\infty \le 2[\|x_0\|^2 + T\alpha].$$

According to (4.4), that is, $2T\alpha < 1$, one has, for any t and for any n,

$$\|x_n(\cdot)\|_\infty \le M \tag{4.9}$$

and

$$\|h_n(t)\| = \|f(t, x_n(\theta_n(t)))\| \le \beta(t)(1 + M), \tag{4.10}$$

where

$$M = \left[\frac{2\|x_0\|^2 + 2T\alpha}{1 - 2T\alpha}\right]^{\frac{1}{2}}.$$

In a straight way, (4.8) and (4.9) yield

$$\sup_{n\in\mathbb{N}} \int_0^T \|\dot{x}_n(t)\|^2 \mathrm{d}t \le \alpha(1 + M^2). \tag{4.11}$$

As a result, in view of (4.10), one deduces that

$$\sup_{n\in\mathbb{N}} \|\dot{x}_n(\cdot) + f(\cdot, x_n(\theta_n(\cdot)))\|_{L^2_H(I)} < +\infty. \tag{4.12}$$

Therefore, we may suppose, without loss of generality, that $(\dot{x}_n(\cdot) + f(\cdot, x_n(\theta_n(\cdot))))$ converges weakly in $L^2_H(I)$.

It results from (4.6), for all n,

$$\|\dot{x}_n\|^2_{L^2_H(I)} \le 2\int_0^T \|f(t, x_n(\theta_n(t)))\|^2 \,\mathrm{d}t + 4c^2T + 4c^2\|x_n\|^2_{L^2_H(I)}. \tag{4.13}$$

2. Convergence of the sequence $(x_n(\cdot))$

Let p and q be arbitrary integers. We know that, for a.e. $t \in I$

$$-\dot{x}_p(t) - f(t, x_p(\theta_p(t))) \in A(t)x_p(t)$$
$$-\dot{x}_q(t) - f(t, x_q(\theta_q(t))) \in A(t)x_q(t).$$

Therefore, the monotonicity property of $A(t),\ t \in I$ ensures that

$$\langle -\dot{x}_p(t) - f(t, x_p(\theta_p(t))) + \dot{x}_q(t) + f(t, x_q(\theta_q(t))), x_p(t) - x_q(t)\rangle \ge 0$$

hence

$$\langle \dot{x}_p(t) - \dot{x}_q(t), x_p(t) - x_q(t)\rangle \le \langle -f(t, x_p(\theta_p(t))) + f(t, x_q(\theta_q(t))), x_p(t) - x_q(t)\rangle.$$

We may write

$$\langle -f(t,x_p(\theta_p(t)))+f(t,x_q(\theta_q(t))),x_p(t)-x_q(t)\rangle =\\ \langle f(t,x_q(\theta_q(t)))-f(t,x_q(t)),x_p(t)-x_q(t)\rangle+\langle f(t,x_q(t))-f(t,x_p(t)),x_p(t)\\ -x_q(t)\rangle+\langle f(t,x_p(t))-f(t,x_p(\theta_p(t))),x_p(t)-x_q(t)\rangle.$$

Thanks to assumption (ii) and (4.9), one has for the non-negative function $\gamma_M(\cdot)\in L^2_{\mathbb{R}}(I)$ and for a.e. $t\in I$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x_p(t)-x_q(t)\|^2=\langle \dot{x}_p(t)-\dot{x}_q(t),x_p(t)-x_q(t)\rangle\le\gamma_M(t)\|x_p(t)-x_q(t)\|^2\\ +\gamma_M(t)\|x_p(t)-x_q(t)\|[\|x_p(t)-x_p(\theta_p(t))\|+\|x_q(\theta_q(t))-x_q(t)\|]. \tag{4.14}$$

Next, making use of the absolute continuity of x_p and x_q, we may write

$$\|x_p(t)-x_p(\theta_p(t))\|+\|x_q(t)-x_q(\theta_q(t))\|=\left\|\int_{\theta_p(t)}^t\dot{x}_p(s)\mathrm{d}s\right\|+\left\|\int_{\theta_q(t)}^t\dot{x}_q(s)\mathrm{d}s\right\|.$$

For each p and any t, one has

$$\|x_p(t)-x_p(\theta_p(t))\|\le\int_{\theta_p(t)}^t\|\dot{x}_p(s)\|\mathrm{d}s.$$

As a result, using (4.9) and (4.14), for almost every $t\in I$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x_p(t)-x_q(t)\|^2\le\gamma_M(t)\|x_p(t)-x_q(t)\|^2\\ +2M\gamma_M(t)\left(\int_{\theta_p(t)}^t\|\dot{x}_p(s)\|\mathrm{d}s+\int_{\theta_q(t)}^t\|\dot{x}_q(s)\|\mathrm{d}s\right).$$

Recall that by construction, $0\le t-\theta_p(t)\le T/p$ for any $t\in I$ and any $p\in\mathbb{N}$. Consequently, for any $0\le t\le T$, one has

$$\int_{\theta_p(t)}^t\|\dot{x}_p(s)\|\mathrm{d}s\le(t-\theta_p(t))^{\frac{1}{2}}\left(\int_0^T\|\dot{x}_p(s)\|^2\mathrm{d}s\right)^{\frac{1}{2}}\\ \le\left(\frac{T}{p}\right)^{\frac{1}{2}}\left(\int_0^T\|\dot{x}_p(s)\|^2\mathrm{d}s\right)^{\frac{1}{2}}.$$

Since by (4.11)

$$\mathcal{S}=\sup_{n\in\mathbb{N}}\|\dot{x}_n\|_{L^2_H(I)}<+\infty,$$

we obtain that, for a.e. $t\in I$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x_p(t)-x_q(t)\|^2\le\gamma_M(t)\|x_p(t)-x_q(t)\|^2\\ +2M\mathcal{S}\gamma_M(t)\left[\left(\frac{T}{p}\right)^{\frac{1}{2}}+\left(\frac{T}{q}\right)^{\frac{1}{2}}\right].$$

Let us put

$$G_{p,q}(t)=2M\mathcal{S}\gamma_M(t)\left[\left(\frac{T}{p}\right)^{\frac{1}{2}}+\left(\frac{T}{q}\right)^{\frac{1}{2}}\right].$$

Because $\gamma_M(\cdot) \in L^1_{\mathbb{R}}(I)$ and $\frac{T}{p}, \frac{T}{q} \to 0$ when $p, q \to +\infty$, one has

$$\lim_{p,q\to\infty} G_{p,q}(t) = 0 \text{ a.e. } t \in I.$$

Once more, since $\gamma_M(\cdot) \in L^1_{\mathbb{R}}(I)$, it follows from the definition of $G_{p,q}$ that

$$\lim_{p,q\to\infty} \int_0^T G_{p,q}(s)\mathrm{d}s = 0. \tag{4.15}$$

This, along with the fact that $\|x_p(0) - x_q(0)\| = 0$, entails, *via* Lemma 2.1,

$$\lim_{p,q\to\infty} \|x_p(\cdot) - x_q(\cdot)\|_\infty = 0.$$

Then, the uniform Cauchy's criterion guarantees that the sequence $(x_n(\cdot))$ converges uniformly on I to some map $x(\cdot) \in \mathcal{C}_H(I)$.

In addition, for $0 \le s \le t \le T$,

$$\|x_n(t) - x_n(s)\| = \| \int_s^t \dot{x}_n(\tau)\mathrm{d}\tau\| \le (t-s)^{\frac{1}{2}} \left(\int_0^T \|\dot{x}_n(\tau)\|^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \le (t-s)^{\frac{1}{2}} \mathcal{S}.$$

Further, observing that

$$\begin{aligned}\|x_n(\theta_n(t)) - x(t)\| &\le \|x_n(\theta_n(t)) - x_n(t)\| + \|x_n(t) - x(t)\| \\ &\le (t - \theta_n(t))^{\frac{1}{2}} \mathcal{S} + \|x_n(t) - x(t)\|\end{aligned}$$

and then

$$\|x_n(\theta_n(t)) - x(t)\| \le \left(\frac{T}{n}\right)^{\frac{1}{2}} \mathcal{S} + \|x_n(t) - x(t)\|,$$

we conclude that $\|x_n(\theta_n(t)) - x(t)\| \longrightarrow 0$, when $n \to \infty$, for any $t \in I$. Thus, the Lipschitz behavior of $f(t, \cdot)$ for each fixed t in I leads to

$$\lim_{n\to\infty} \|f(t, x_n(\theta_n(t))) - f(t, x(t))\| = 0,$$

along with

$$\lim_{n\to\infty} \int_0^T \|f(t, x_n(\theta_n(t))) - f(t, x(t))\|^2 \mathrm{d}t = 0, \tag{4.16}$$

by Lebesgue's convergence theorem because of (4.2) and (4.9) and hence

$$\lim_{n\to\infty} \int_0^T \|f(t, x_n(\theta_n(t)))\|^2 \mathrm{d}t = \int_0^T \|f(t, x(t))\|^2 \mathrm{d}t. \tag{4.17}$$

Further, letting n tends to infinity in (4.9) and (4.10), one also has

$$\|x(\cdot)\|_\infty \le M \tag{4.18}$$

and

$$\|f(t, x(t))\| \le \beta(t)(1 + M). \tag{4.19}$$

Furthermore, in view of (4.11), one knows that the sequence of velocities is bounded in $L^2_H(I)$ so that, up to a subsequence that we do not relabel, we may suppose that $(\dot{x}_n)_n$ converges weakly in $L^2_H(I)$ to some element $z(\cdot)$.

For any integer n and any $y \in H$ and for $0 \le s \le t \le T$, relying on the absolute continuity of $(x_n(\cdot))_n$, we can write

$$\int_0^T \langle y\mathbf{1}_{[s,t]}(\tau), \dot{x}_n(\tau)\rangle \mathrm{d}\tau = \langle y, x_n(t) - x_n(s)\rangle.$$

Next, passing to the limit in the equality yields

$$\langle y, \int_s^t z(\tau)\mathrm{d}\tau\rangle = \langle y, x(t) - x(s)\rangle.$$

Therefore, given any $s, t \in I$ with $s \le t$, we get $\int_s^t z(\tau)\mathrm{d}\tau = x(t) - x(s)$, and hence $x(\cdot)$ is absolutely continuous and $z(\cdot)$ coincides almost everywhere in I with $\dot{x}(\cdot)$. Thus, $\dot{x} \in L^2_H(I)$ and

$$\dot{x}_n \to \dot{x} \quad \text{weakly in } L^2_H(I). \tag{4.20}$$

Then, taking the superior limit on n in (4.13) and using the uniform convergence of $x_n(\cdot)$ to $x(\cdot)$, (4.17) and (4.20) yield

$$\|\dot{x}\|^2_{L^2_H(I)} \le 2\int_0^T \|f(t, x(t))\|^2 \,\mathrm{d}t + 4c^2T + 4c^2\|x\|^2_{L^2_H(I)}. \tag{4.21}$$

3. The map $x(\cdot)$ is a solution of (4.3).
Recall that, for each $n \in \mathbb{N}$

$$\begin{cases} -\dot{x}_n(t) \in A(t)x_n(t) + f(t, x_n(\theta_n(t))) \ \text{ a.e. } t \in I. \\ \ x_n(0) = x_0. \end{cases}$$

Then using the closure property of maximal monotone operator entails

$$-\dot{x}(t) \in A(t)x(t) + f(t, x(t)) \quad \text{a.e. } t \in I.$$

Second part: If $T > \tau$

We choose a partition of I such that for each subinterval J of I with length$(J) \le \tau$ one has $\int_J \beta^2(s)\,\mathrm{d}s \le \frac{1}{8T}[1 - 8c^2T^2]$ and we also fix some integer N such that $T/N \le \tau$. Put $T_i := \frac{i}{N}T$ for $i = 0, \ldots, N$ and observe that for each $i = 1, \ldots, N$ we have

$$\int_{T_{i-1}}^{T_i} \beta^2(s)\,\mathrm{d}s < \frac{1}{8T}[1 - 8c^2T^2] < \frac{1}{8(T_i - T_{i-1})}[1 - 8c^2(T_i - T_{i-1})^2]$$

and hence (4.4) relative to the interval $[T_{i-1}, T_i]$ is fulfilled. Consequently me may apply what precedes to the intervals $[0, T_1]$, $[T_1, T_2]$, $\ldots$, and $[T_{N-1}, T]$, and we obtain absolutely continuous solutions $y_1(\cdot)$ on $[0, T_1]$ with $y_1(0) = x_0$, $y_2(\cdot)$ on $[T_1, T_2]$ with $y_2(T_1) = y_1(T_1)$, $\ldots$, $y_N(\cdot)$ on $[T_{N-1}, T]$ with $y_N(T_{N-1}) = y_{N-1}(T_{N-1})$. So the mapping $x(\cdot)$ from $I = I$ into H defined by $x(t) = y_i(t)$ for all $t \in [T_{i-1}, T_i]$, $i = 1, 2, \ldots, N$ is obviously an absolutely continuous solution on I of (4.3).

The uniqueness follows from the monotonicity property of the maximal monotone operator $A(t)$, $t \in I$, and the Lipschitz condition on f. Then, the proof is complete. □

As a consequence, we have the following properties

Proposition 4.2. *The absolutely continuous solution $x(\cdot)$ of (4.3) satisfies*

$$\int_0^T \|\dot{x}(t)\|^2 \,\mathrm{d}t \le 2\int_0^T \|f(t, x(t))\|^2 \,\mathrm{d}t + 4c^2T + 4c^2\int_0^T \|x(t)\|^2 \,\mathrm{d}t. \tag{4.22}$$

Moreover, one has $\|x(\cdot)\|_\infty \leq K$. *Then*

$$\|f(t, x(t))\| \leq \beta(t)(1+K) \quad \text{a.e. } t \in I \tag{4.23}$$

and

$$\int_0^T \|\dot{x}(t)\|^2 \, \mathrm{d}t \leq 2(1+K)^2 \int_0^T \beta^2(t) \, \mathrm{d}t + 4c^2 T(1+K^2), \tag{4.24}$$

with

$$K = \|x_0\| + [\xi(T)]^{\frac{1}{2}}$$

and where $\xi(\cdot)$ *is the increasing, continuous, and non-negative function defined on* I *by*

$$\xi(s) = b(s) + 4s \int_0^s b(\tau)[\beta^2(\tau) + 2c^2] \exp\left(\int_\tau^s [\beta^2(\theta) + 2c^2]c(\theta)d\theta\right) \mathrm{d}\tau$$

and for each $t \in I$

$$b(t) = t[4 \int_0^t \beta^2(\tau)(1 + \|x_0\|)^2 \, \mathrm{d}\tau + 4c^2 T(2\|x_0\|^2 + 1)].$$

Proof. Let $x(\cdot)$ be the unique solution of (4.3), it's easy to see that (4.21) holds true in the whole interval I.

Owing to (4.21) and assumption (iii) and making use of the absolute continuity of $x(\cdot)$ on I, we may write for $0 \leq s < T$

$$\begin{aligned}\|x(s) - x_0\|^2 &\leq s \int_0^s \|\dot{x}(\tau)\|^2 \mathrm{d}\tau \\ &\leq s\left[2\int_0^s \beta^2(\tau)(1 + \|x(\tau)\|)^2 \, \mathrm{d}\tau + 4c^2 T + 4c^2 \int_0^s \|x(\tau)\|^2 \, \mathrm{d}\tau\right].\end{aligned}$$

Hence, for any $s \in I$

$$\begin{aligned}\|x(s) - x_0\|^2 &\leq s\left[4\int_0^s \beta^2(\tau)(1 + \|x_0\|)^2 \, \mathrm{d}\tau + 8c^2 \int_0^s \|x_0\|^2 \, \mathrm{d}\tau 4c^2 T + 4\int_0^s (\beta^2(\tau) + 2c^2)\|x(\tau) - x_0\|^2 \, \mathrm{d}\tau\right] \\ &\leq s\left[4\int_0^s \beta^2(\tau)(1 + \|x_0\|)^2 \, \mathrm{d}\tau + 8c^2 s\|x_0\|^2 + 4c^2 T + 4\int_0^s (\beta^2(\tau) + 2c^2)\|x(\tau) - x_0\|^2 \, \mathrm{d}\tau\right] \\ &\leq s\left[4\int_0^s \beta^2(\tau)(1 + \|x_0\|)^2 \, \mathrm{d}\tau + 4c^2 T(2\|x_0\|^2 + 1) + 4\int_0^s (\beta^2(\tau) + 2c^2)\|x(\tau) - x_0\|^2 \, \mathrm{d}\tau\right].\end{aligned}$$

Applying Gronwall's inequality entails that given $s \in I$, one has

$$\|x(s) - x_0\|^2 \leq \xi(s) \tag{4.25}$$

where

$$\xi(s) = b(s) + c(s) \int_0^s b(\tau)[\beta^2(\tau) + 2c^2] \exp\left(\int_\tau^s [\beta^2(\theta) + 2c^2]c(\theta)d\theta\right) \mathrm{d}\tau$$

with

$$\begin{aligned}b(t) &= t\left[4\int_0^t \beta^2(\tau)(1 + \|x_0\|)^2 \, \mathrm{d}\tau + 4c^2 T(2\|x_0\|^2 + 1)\right], \\ c(t) &= 4t.\end{aligned}$$

Clearly such functions $b(\cdot)$, $c(\cdot)$ and $\xi(\cdot)$ are increasing and continuous on I.

Finally, as a straight consequence of (4.25) and the finiteness of T, one has

$$\|x(\cdot)\|_\infty \leq K,$$

where $K = \|x_0\| + [\xi(T)]^{\frac{1}{2}}$. Consequently, (4.23) and (4.24) hold true. □

The following proposition gives a topological result concerning the map $a \mapsto x_a(\cdot)$ which associates with each $a \in D(A(0))$ the unique solution of the foregoing problem with the initial condition a.

Proposition 4.3. *Under the assumptions of Theorem 4.1, for each $a \in D(A(0))$, let $x_a(\cdot)$ be the unique solution of the problem*

$$\begin{cases} -\dot{x}(t) \in A(t)x(t) + f(t, x(t)) & \text{a.e. } t \in I \\ x(0) = a. \end{cases}$$

Then, the map $\psi : a \mapsto x_a(\cdot)$ from $D(A(0))$ to the space $\mathcal{C}_H(I)$ endowed with the uniform convergence norm is Lipschitz continuous on any bounded subset of $D(A(0))$.

Proof. Let M be any fixed positive real number. We are going to prove that the map ψ is Lipschitz continuous on $D(A(0)) \cap M\mathbb{B}$.

According to Theorem 4.1 and Proposition 4.2, there exists a real number M_1 depending only on M such that, for all $y \in D(A(0)) \cap M\mathbb{B}$ and, for all $t \in I$,

$$\|f(t, x_y(t))\| \leq (1 + M_1)\beta(t) \tag{4.26}$$

and

$$\int_0^T \|\dot{x}_y(t)\|^2 \, \mathrm{d}t \leq 2(1 + M_1)^2 \int_0^T \beta^2(t) \, \mathrm{d}t + 4c^2 T(1 + M_1^2).$$

Thanks to this last inequality, for some $\eta_1 > 0$ depending only on M, for all $y \in D(A(0)) \cap M\mathbb{B}$ and, for all $t \in I$

$$x_y(t) \in B[0, \eta_1]. \tag{4.27}$$

Fix any $a, b \in D(A(0)) \cap M\mathbb{B}$. By the monotonicity property of the operator $A(t)$, we have, for almost all $t \in I$,

$$\langle -\dot{x}_a(t) - f(t, x_a(t)) + \dot{x}_b(t) + f(t, x_b(t)), x_a(t) - x_b(t)\rangle \geq 0$$

and then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|x_a(t) - x_b(t)\|^2) \leq \|f(t, x_a(t)) - f(t, x_b(t))\| \|x_a(t) - x_b(t)\|. \tag{4.28}$$

By virtue of assumption (ii) of Theorem 4.1, there is a non-negative function $\gamma_{\eta_1}(\cdot) \in L^2_{\mathbb{R}}(I)$ such that $f(t, \cdot)$ is $\gamma_{\eta_1}(t)$-Lipschitz on $B[0, \eta_1]$ (this function depends only on M), the above inequality, along with (4.27), entails that for almost all $t \in I$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|x_a(t) - x_b(t)\|^2) \leq 2\gamma_{\eta_1}(t)\|x_a(t) - x_b(t)\|^2.$$

Applying Gronwall's lemma, we get, for any $t \in I$

$$\sup_{t \in I} \|x_a(t) - x_b(t)\|^2 \leq \|a - b\|^2 \exp\left\{2\int_0^T \gamma_{\eta_1}(s)\mathrm{d}s\right\}.$$

Therefore,

$$\|x_a(\cdot) - x_b(\cdot)\|_\infty \leq B\|a - b\|$$

where

$$B := \exp\left\{\int_0^T \gamma_{\eta_1}(s)\mathrm{d}s\right\},$$

the proof is then complete. □

Now, we use the main existence result of this section to the study of a Bolza control problem (see also [6,14] for further results).

5. Application to a Bolza-type problem

From now on, the Hilbert space H is assumed to be *separable.* We will apply the theorem of existence and uniqueness (Thm. 4.1) to an optimal control problem for differential inclusion governed by time-dependent-maximal monotone operator *via* Young measures. Let's recall first, some preliminary results about Young measures. For more details, we refer the reader to [5, 6, 14].

5.1. Young measures

Let $(S, \mathcal{S}, \sigma)$ be a complete measure space with a non-negative finite measure σ and let U be a complete separable metric space. One calls Young measure on $S \times U$, any non-negative finite measure ν on $(S \times U, \mathcal{S} \otimes \mathcal{B}(U))$ that satisfies

$$\forall A \in \mathcal{S},\ \nu(A \times U) = \sigma(A).$$

In other words, a Young measure on $\mathcal{S} \otimes \mathcal{B}(U)$ is a non-negative measure whose projection on S (that is, its image by the map $(s, u) \mapsto s$) is equal to σ. The set of Young measures will be denoted by $\mathcal{Y}(S, \sigma, U)$.

In other respects, let $\mathcal{M}^1_+(U)$ be the set of all probability measures on $(U, \mathcal{B}(U))$. Following [5], we denote by $\mathcal{Y}_{\mathrm{dis}}(S, \sigma, U)$ the set of all maps $\mu : S \to \mathcal{M}^1_+(U)$ (up to σ-almost everywhere equality) which are λ-measurable in the sense that, for any $B \in \mathcal{B}(U)$, the function $s \mapsto \mu_s(B)$ is $\mathcal{S}$-measurable.

Remark 5.1. If $\mu \in \mathcal{Y}_{\mathrm{dis}}(S, \sigma, U),\ A \in \mathcal{S} \otimes \mathcal{B}(U)$ and if $\mathbf{1}_A$ is the characteristic function of A (that is, $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise), then the function $s \mapsto \int_U \mathbf{1}_A(s, u)\mu_s(\mathrm{d}u)$ is $\mathcal{S}$-measurable on S and the set function ν defined by

$$\nu(A) = \int_S \int_U \mathbf{1}_A(s, u)\, \mu_s(\mathrm{d}u)\sigma(\mathrm{d}s) \tag{5.1}$$

for all $A \in \mathcal{S} \otimes \mathcal{B}(U)$ is a Young measure on $S \times U$. Accordingly, any member of $\mathcal{Y}_{\mathrm{dis}}(S, \sigma, U)$ is called a disintegrable Young measure.

Conversely, under the above assumptions on S and U, any Young measure on $S \times U$ is associated with some $\mu \in \mathcal{Y}_{\mathrm{dis}}(S, \sigma, U)$ in the way above.

Remark 5.2.

(1) If ν is the Young measure corresponding to the member $\mu \in \mathcal{Y}_{\mathrm{dis}}(S, \sigma, U)$, *i.e.*, the Young measure defined by (5.1), then, for any function $\psi : S \times U \to \mathbb{R} \cup \{-\infty, +\infty\}$ which is $\mathcal{S} \otimes \mathcal{B}(U)$-measurable and non-negative (resp. ν-integrable), the function $s \mapsto \int_U \psi(s, u)\mu_s(\mathrm{d}u)$ is σ-measurable (resp. is σ-integrable) and one has

$$\int_{S \times U} \psi\, \mathrm{d}\nu = \int_S \int_U \psi(s, u)\, \mu_s(\mathrm{d}u)\sigma(\mathrm{d}s).$$

(2) If ν is a Young measure associated with some $\mu \in \mathcal{Y}_{\mathrm{dis}}(S, \sigma, U)$ we will make no distinction between ν and μ, that is, for all $s \in S$, we will write ν_s instead of μ_s.

(3) Any $\mathcal{S}$-measurable map $u(\cdot) : S \to U$ defines a Young measure on $S \times U$ called the Young measure associated with $u(\cdot)$. This is the Young measure corresponding to the member $\mu \in \mathcal{Y}_{\mathrm{dis}}(S, \sigma, U)$ defined by $\mu_s := \delta_{u(s)}$, where $\delta_{u(s)}$ is the Dirac mass at the point $u(s)$, *i.e.*, for any $B \in \mathcal{B}(U)$, $\delta_{u(s)}(B) = 1$ if $u(s) \in B$ and 0 otherwise.

5.2. Carathéodory integrands and narrow convergence

One calls integrand any function $\psi : S \times U \to \mathbb{R} \cup \{-\infty, +\infty\}$ that is $\mathcal{S} \otimes \mathcal{B}(U)$-measurable. An integrand is said to be of Carathéodory type if, for any $s \in S$, the partial function $\psi(s, \cdot)$ is continuous and takes finite values on U. An integrand ψ is said to be L^1-bounded if there exists some non-negative function $\gamma \in L^1_{\mathbb{R}}(S, \sigma)$ such that $|\psi(s, u)| \le \gamma(s)$ for all $(s, u) \in S \times U$.

The set $\mathcal{Y}(S,\sigma,U)$ will be endowed with the narrow topology. Recall that a sequence (ν^n) of $\mathcal{Y}(S,\sigma,U)$ converges to ν in $\mathcal{Y}(S,\sigma,U)$ if, for any L^1-bounded Carathéodory integrand ψ,

$$\lim_n \int_{S\times U} \psi \, \mathrm{d}\nu^n = \int_{S\times U} \psi \, \mathrm{d}\nu. \tag{5.2}$$

In the same vein, one says that a sequence (μ^n) of $\mathcal{Y}_{\mathrm{dis}}(S,\sigma,U)$ converges in $\mathcal{Y}_{\mathrm{dis}}(S,\sigma,U)$ to μ if the sequence of the corresponding Young measures converges in $\mathcal{Y}(S,\sigma,U)$. It amounts to saying that, for any L^1-bounded Carathéodory integrand ψ,

$$\lim_n \int_S \int_U \psi(s,u)\mu_s^n(\mathrm{d}u)\sigma(\mathrm{d}s) = \int_S \int_U \psi(s,u)\mu_s(\mathrm{d}u)\sigma(\mathrm{d}s). \tag{5.3}$$

We recall also some important results proved in [5]

Proposition 5.3. *Let $h_n(\cdot),\ h(\cdot) \in \mathcal{C}_H(I)$ $(n \geq 1)$ and $\mu^n,\ \mu \in \mathcal{Y}_{\mathrm{dis}}(I,\lambda,U)$. Assume that $(h_n(\cdot))$ converges uniformly to $h(\cdot)$ and (μ^n) converges to μ in $\mathcal{Y}_{\mathrm{dis}}(I,\lambda,U)$. Let $\theta^n \in \mathcal{Y}(I,\lambda,H\times U)$ be defined by $\theta^n_t := \delta_{h_n(t)}\otimes\mu^n_t$. Then, θ^n converges in $\mathcal{Y}(I,\lambda,H\times U)$ to the Young measure $\theta \in \mathcal{Y}(I,\lambda,H\times U)$ defined by $\theta_t := \delta_{h(t)} \otimes \mu_t$.*

Recall that a sequence of functions $(f_n(\cdot))$ is said to be uniformly integrable in $L^1_{\mathbb{R}}(I)$, if it is bounded in $L^1_{\mathbb{R}}(I)$, and

$$\lim_{\lambda(A)\to 0} \sup_n \int_A |f_n(t)| \, \mathrm{d}t = 0.$$

Proposition 5.4. *Let $u_n(\cdot) : I \longrightarrow U$ $(n \geq 1)$ be measurable maps. Assume that the sequence of the associated Young measures (ν^n) (that is, $\nu^n_t = \delta_{u_n(t)}$) converges to ν in $\mathcal{Y}(I,\lambda,U)$. Let $\psi : I\times U \to \mathbb{R}$ be a Carathéodory integrand. Assume that the sequence $(\psi(\cdot,u_n(\cdot)))_n$ is uniformly integrable in $L^1_{\mathbb{R}}(I)$. Then, ψ is ν-integrable and*

$$\int_{I\times U} \psi \, \mathrm{d}\nu = \lim_n \int_I \psi(t,u_n(t)) \, \mathrm{d}t.$$

The space $\mathcal{Y}_{\mathrm{dis}}(I,\lambda,U)$ has the following compactness property (see [5,8]).

Proposition 5.5. *If U is a compact metric space, then any sequence in $\mathcal{Y}_{\mathrm{dis}}(I,\lambda,U)$ has a subsequence which converges in $\mathcal{Y}_{\mathrm{dis}}(I,\lambda,U)$.*

In order to establish the existence result for our Bolza problem (P.O), we consider an another optimal control problem called the relaxed problem. We will prove that the latter has an optimal solution and that its optimal value equals the infimum in the problem (P.O).

Let U be a compact metric space and $\varGamma : I \rightrightarrows U$ be a Lebesgue-measurable set-valued map with nonempty compact values. Let us consider the set-valued map $\varSigma(\cdot)$ defined on I by

$$\varSigma(t) = \{P \in \mathcal{M}^1_+(U) : P(\varGamma(t)) = 1\}.$$

Denote by $S_\varGamma$ (resp. $S_\varSigma$) the set of all Lebesgue-measurable selections (up to almost everywhere equality) of $\varGamma$ (resp. $\varSigma$).

The following Proposition holds true (see, *e.g.*, [6]).

Proposition 5.6. *Let $\varGamma : I \rightrightarrows U$ be a Lebesgue-measurable set-valued map with nonempty compact values. The set-valued map $\varSigma(\cdot)$ defined on I by*

$$\varSigma(t) = \{P \in \mathcal{M}^1_+(U) : P(\varGamma(t)) = 1\}$$

is Lebesgue-measurable with nonempty compact convex values and the set $S_\varSigma$ is nonempty and sequentially closed in $\mathcal{Y}_{\mathrm{dis}}(I,\lambda,U)$.

The members of S_Γ are called original controls and those of S_Σ relaxed controls. Clearly, $S_\Gamma \subset S_\Sigma$ in the sense that, for any $\zeta \in S_\Gamma$, the Young measure μ with $(\mu_t := \delta_{\zeta(t)})_{t\in I}$ satisfies $\mu \in S_\Sigma$.

Let $g : I \times H \times U \longrightarrow H$ be a map satisfying:

(i) for any $t \in I$, $g(t, \cdot, \cdot)$ is continuous on $H \times U$;
(ii) for each $(x, u) \in H \times U$, $g(\cdot, x, u)$ is λ-measurable on I;
(iii) for every $\eta > 0$, there exists a non-negative function $\gamma_\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that, for all $(t, u) \in I \times U$ and for all $x, y \in B[0, \eta]$
$$\|g(t, x, u) - g(t, y, u)\| \le \gamma_\eta(t)\|x - y\|;$$
(iv) there exists a non-negative function $\beta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that, for all $(t, x, u) \in I \times H \times U$, one has
$$\|g(t, x, u)\| \le \beta(t)(1 + \|x\|).$$

Now, assume that (C_1) and (C_2) are satisfied. Given $x_0 \in D(A(0))$, $\zeta \in S_\Gamma$, $\mu \in S_\Sigma$, consider the two following perturbed problems:
$$(\mathcal{P}_\zeta)\begin{cases} -\dot{x}(t) \in A(t)x(t) + g(t, x(t), \zeta(t)) & \text{a.e. } t \in I \\ x(0) = x_0 \end{cases}$$
and
$$(\mathcal{P}_\mu)\begin{cases} -\dot{x}(t) \in A(t)x(t) + \int_{\Gamma(t)} g(t, x(t), u)\mu_t(\mathrm{d}u) & \text{a.e. } t \in I \\ x(0) = x_0. \end{cases}$$
The map
$$h_\mu(t, x) = \int_{\Gamma(t)} g(t, x, u)\mu_t(\mathrm{d}u) = \int_U g(t, x, u)\mu_t(\mathrm{d}u),$$
is separately Lebesgue-measurable on I. Moreover, thanks to the assumptions on g and the fact that $\mu_t(\Gamma(t)) = \mu_t(U) = 1$, we have

– for every $\eta > 0$, for all $t \in I$ and for all $x, y \in B[0, \eta]$,
$$\|h_\mu(t, x) - h_\mu(t, y)\| \le \gamma_\eta(t)\|x - y\|;$$
– for all $(t, x) \in I \times H$, one has
$$\|h_\mu(t, x)\| \le \beta(t)(1 + \|x\|). \tag{5.4}$$

Consequently, by Theorem 4.1, for any $\zeta \in S_\Gamma$ and for any $\mu \in S_\Sigma$, each of the perturbed problem $(\mathcal{P}_\zeta)$ and $(\mathcal{P}_\mu)$ has one and only one solution, which will be denoted by $x_\zeta(\cdot)$ and $x_\mu(\cdot)$ respectively.

5.3. The relaxation result

The Bolza-type optimal control problem to be relaxed in the present work is the following:
$$\inf_{\zeta(\cdot)\in S_\Gamma} \int_0^T J(t, x_\zeta(t), \zeta(t))\,\mathrm{d}t \tag{P.O}$$
where $x_\zeta(\cdot)$ is the unique absolutely continuous solution of $(\mathcal{P}_\zeta)$ and the cost functional $J : I \times H \times U \longrightarrow \mathbb{R}$ is an integrand such that for each $t \in I$, $J(t, \cdot, \cdot)$ is continuous on $H \times U$, J is also bounded from below.

The relaxed control problem is
$$\inf_{\mu\in S_\Sigma} \int_0^T \int_U J(t, x_\mu(t), u)\,\mu_t(\mathrm{d}u)\mathrm{d}t \tag{P.R}$$
with $x_\mu(\cdot)$ being the absolutely continuous solution of $(\mathcal{P}_\mu)$.

The question is finding situations when (P.R) has an optimal solution and when the equality $\inf\,$(P.O) $=$ $\min\,$(P.R) holds true?

Now, we give the following results. The proofs are similar to those in [6] (see also [14]).

Remark 5.7.

(1) The cost functional J in (P.O) and (P.R) takes values in $[m_J T, +\infty]$, where $m_J := \inf J$.
(2) Because of the boundedness assumption on J and the inclusion

$$\emptyset \neq \{x_\zeta(\cdot) : \ \zeta \in S_\Gamma\} \subset \{x_\mu(\cdot) : \ \mu \in S_\Sigma\},$$

it is straightforward that

$$-\infty < \inf(\text{P.R}) \leq \inf(\text{P.O}) \leq +\infty. \tag{5.5}$$

Proposition 5.8. *Under the assumptions above, suppose that, (C_3) for any bounded sequence $(x_n(\cdot))$ in $(\mathcal{C}_H(I), \|\cdot\|_\infty)$ and for any sequence $(\zeta_n(\cdot))$ in S_Γ, the sequence $(J(\cdot, x_n(\cdot), \zeta_n(\cdot)))$ is uniformly integrable in $L^1_{\mathbb{R}}(I)$.*

Let $\mu \in S_\Sigma$. Then, the function $t \mapsto \int_U J(t, x_\mu(t), u)\mu_t(\mathrm{d}u)$ belongs to $L^1_{\mathbb{R}}(I)$. Moreover, for any $(\zeta_n(\cdot))$ in S_Γ such that the sequence of the associated Young measures converges in $\mathcal{Y}(I, \lambda, U)$ to μ, the sequence $(x_{\zeta_n}(\cdot))$ converges uniformly in $\mathcal{C}_H(I)$ to $x_\mu(\cdot)$ and

$$\int_0^T \int_U J(t, x_\mu(t), u)\mu_t(\mathrm{d}u)\mathrm{d}t = \lim_n \int_0^T J(t, x_{\zeta_n}(t), \zeta_n(t))\mathrm{d}t.$$

Remark 5.9.

(1) Both of inf (P.O) and inf (P.R) are real numbers, under assumption (C_3).
(2) The cost functional in (P.R) is finite under assumption (C_3).

Consider the two following control problems

$$\inf_{\zeta(\cdot)\in S_\Gamma} \int_0^T J(t, x_\zeta(t), \zeta(t))\,\mathrm{d}t \tag{P.O}$$

and

$$\inf_{\mu\in S_\Sigma} \int_0^T \int_U J(t, x_\mu(t), u)\,\mu_t(\mathrm{d}u)\mathrm{d}t. \tag{P.R}$$

Theorem 5.10. *Under the assumptions above, the control problem (P.R) has an optimal solution. Furthermore, one has*

$$\min(\text{P.R}) = \inf(\text{P.O}).$$

Acknowledgements. The authors would like to thank Professor Lionel Thibault for his valuable comments and suggestions.

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