ESAIM: COCV 23 (2017) 443–454
DOI: 10.1051/cocv/2015055

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

EVENTUAL DIFFERENTIABILITY OF A STRING WITH LOCAL KELVIN–VOIGT DAMPING *

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Abstract. In this paper, we study a wave equation with local Kelvin–Voigt damping, which models one-dimensional wave propagation through two segments consisting of an elastic and a viscoelastic medium. Under the assumption that the damping coefficients change smoothly near the interface, we prove that the semigroup corresponding to the system is eventually differentiable.

Mathematics Subject Classification. 35M20, 35Q72, 74D05.

Received September 16, 2014. Revised August 29, 2015. Accepted December 11, 2015.

1. Introduction

In this paper, we consider one-dimensional wave propagation through two segments consisting of an elastic and a Kelvin–Voigt medium. The latter material is a viscoelastic material having the properties both of elasticity and viscosity. The mathematical model is the following partial differential equation.

$$\begin{cases} u_{tt}(x,t) - [u'(x,t) + a(x)u_t'(x,t)]' = 0 & \text{in } (-1,1)\times\mathbb{R}^+, \\ u(t,-1) = u(t,1) = 0 & \text{in } \mathbb{R}^+, \\ u(x,t) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } [-1,1], \end{cases} \tag{1.1}$$

where prime represents partial derivative with respect to x, and the damping coefficient function $a(x)$ satisfies

$$a(x) \in C^1([-1,1]); \quad a(x) = 0 \quad \text{for } x \in [-1,0]; \qquad a(x) > 0 \quad \text{for } x \in (0,1]. \tag{A1}$$

Let $H_0^1(-1,1)$ be the space $\{u \in H^1(-1,1) \mid u(-1) = u(1) = 0\}$. We introduce a Hilbert space

$$\mathcal{H} = H_0^1(-1,1) \times L^2(-1,1),$$

Keywords and phrases. Semigroup, local Kelvin–Voigt damping, eventual differentiability of semigroup.

* *This work was supported by the National Natural Science Foundation of China (grants No. 11231007, 60974033) and Beijing Municipal Natural Science Foundation (grant No. 4132051).*

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whose inner product induced norm is given by

$$\|U\|_{\mathcal{H}} = \sqrt{\|u\|^2_{H^1_0(-1,1)} + \|v\|^2_{L^2(-1,1)}}, \qquad \forall\, U = (u,v) \in \mathcal{H}.$$

Define an unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}U = \Big(v,\ (u' + av')'\Big), \quad \forall\, U = (u,v) \in D(\mathcal{A}),$$

and

$$\mathcal{D}(\mathcal{A}) = \Big\{(u,v) \in \mathcal{H} \mid v \in H^1_0(-1,1), (u' + av')' \in L^2(-1,1)\Big\}.$$

Then system (1.1) can be written as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(u(\cdot,t), u_t(\cdot,t)\big) = \mathcal{A}\big(u(\cdot,t), u_t(\cdot,t)\big), \quad \forall\, t \ge 0, \quad \big(u(0), u_t(0)\big) = \big(u_0, u_1\big). \tag{1.2}$$

It is known [5] that $\mathcal{A}$ generates a C_0-semigroup of contractions $\exp(t\mathcal{A})$ on $\mathcal{H}$ if the coefficient function $a(\cdot) \ge 0$ is piecewisely continuous, nonnegative, and strictly positive on a subinterval of $[-1,1]$.

The Kelvin–Voigt damping is much stronger than the viscous damping (*i.e.*, the damping term is replaced by $-a(x)u_t$) in the sense that if the entire medium is of the Kelvin–Voigt type, the damping for the wave equation not only induces exponential energy decay, but also restricts the spectrum of the associated semigroup generator to a sector in the left half plane, and the associated semigroup is analytic; while if the entire medium is of the viscous type, the associated semigroup is still exponentially stable, and the spectrum of the semigroup generator has a vertical asymptote on the left half plane, hence does not have any smoothing property.

When the damping is local in an internal region of the domain, such a comparison is not valid anymore. It is well known that the local viscous damping for the one-dimensional wave equation leads to exponential energy decay even if the damping coefficient function has a jump discontinuity at the interface [4]. For the high dimensional systems, we refer to [1] for the well-known "geometric optics" condition, which guarantees the exact controllability, and consequently the exponential stability of the wave equation with local viscous damping. However, the local Kelvin–Voigt damping model has much more interesting behavior. Let us recall the following relevant results in the literature.

• In 1998, Chen, Liu and Liu [5] proved lack of the exponential stability for system (1.1) when the damping coefficient is a step function, *e.g.*, $a(x) \equiv 1$ on $(0,1]$ and satisfies condition (A1). This unexpected result reveals that the Kelvin–Voigt damping does not follow the "geometric optics" condition. It turns out that the strong damping leads to reflection of waves at the interface $x = 0$, which then fails to be effectively damped because they do not enter the region of damping. In 2005, Liu and Rao [10] proved that the solution of this model actually decays at a rate of $\frac{1}{t^2}(\ln t)^{5/2}$. The log term can be removed without any change in their proof by using the necessary and sufficient conditions for polynomial stability of a semigroup which appeared in 2010 [2].

• In 2002, it was shown in [8] that exponential energy decay still holds if the damping coefficient in system (1.1) is smooth enough, say $a(x) \in C^2[-1,1]$ and satisfies condition (A1). This indicates that the asymptotic behavior of the solution to system (1.1) depends on the regularity of the damping coefficient function, which is not the case for the viscous damping model.

• An interesting property of (1.1) was revealed by Renardy [12] in 2004. He proved that the real part of the eigenvalues of system (1.1) are not bounded below if the damping coefficient $a(x)$ satisfies condition (A1) and

$$\lim_{x\to 0^+} \frac{a'(x)}{x^\alpha} = k > 0, \qquad \text{for some } \alpha > 0, \tag{A2}$$

which implies that $a(x)$ behaves like $x^{\alpha+1}$ near the interface $x = 0$. Note that such a function is only of C^1 when $\alpha \le 1$. This property led to two conjectures on system (1.1). First, is $a(x) \in C^1[-1,1]$ good enough to ensure exponential stability? Second, does the solution of (1.1) have some kind of regularity?

• Recently, Zhang [13] made progress on the first conjecture by showing that the semigroup associated with system (1.1) is exponentially stable if the coefficient function $a(\cdot) \in C^1([-1,1])$ satisfies (A1) and the following conditions:

$$a'(0)=0,\ \int_0^x \frac{|a'(s)|^2}{a(s)}\,\mathrm{d}s \le C|a'(x)|,\ \forall x\in[0,1],\ C>0.$$

It is easy to verify that function $a(x)=x^\alpha$ ($\alpha>1$) satisfies the above assumption.

• For the corresponding system on high dimensional spacial domain, exponential stability was obtained in [9] with certain conditions.

• As for the second conjecture, it was pointed out by Renardy that the associated semigroup can never be analytic since any initial disturbance with support in the interior of the elastic part of the medium must first reach the viscoelastic part before it experiences any damping, hence there is no immediate smoothing effect. But his findings leads us to conjecture that the local Kelvin–Voigt damping satisfying conditions (A1)–(A2) may induce eventual differentiability. Numerical computation of the eigenvalues by Enbree [6] in 2008 showed that the imaginary part of the eigenvalue grows exponentially with respect to this negative real part asymptotically which is an indicator of possible eventual differentiability.

Our main result in this paper is the following.

Theorem 1.1. *Suppose function $a(\cdot)$ satisfies assumption* (A1) *and* (A2)*. Then the semigroup* $\exp(t\mathcal{A})$ *associated with system* (1.1) *is eventually differentiable,* i.e., *differentiable when $t>t_0$ for some constant $t_0>0$.*

Hence, the system has the following desired dynamical properties: (a) Vibrations modes with higher frequency decay at higher exponential rates; (b) the decay rate is determined by the spectrum of the semigroup generator.

In what follows, we denote $\rho(A)$, $\sigma(A)$ and $R(\lambda, A)$ by the resolvent set, spectrum and resolvent operator of A, respectively. Our proof is based on the following necessary and sufficient condition for eventually differentiable semigroup.

Lemma 1.2 ([11], Thm. 4.7)**.** *Let $\exp(tA)$ be a C_0 semigroup and let A be its infinitesimal generator. If $\|\exp(tA)\| \le M\exp(\omega t)$, then the following two assertions are equivalent*

1. *There exists a $t_0>0$ such that $\exp(tA)$ is differentiable for $t>t_0$.*
2. *There exist real constants κ and b, $C>0$ such that*

$$\rho(A)\supset \Theta_{\kappa,b}=\{\lambda\in\mathbb{C} : \mathrm{Re}\lambda \ge \kappa - b\ln|\mathrm{Im}\lambda|\} \tag{1.3}$$

and

$$\|R(\lambda,A)\| \le C|\mathrm{Im}\lambda|, \qquad for \quad \lambda\in\Theta_{\kappa,b}, \quad \mathrm{Re}\lambda<\omega. \tag{1.4}$$

We are going to verify (1.3) and (1.4) in the rest of the paper. The following one-dimensional Sobolev inequality will be useful.

Lemma 1.3 ([12])**.** *Let function $a(\cdot)$ satisfy* (A1) *and* (A2)*. Assume $y(\cdot)\in H^1(0,1)$ satisfies $y(1)=0$. Then there is a positive constant a_0, independent of y, such that*

$$\|a'a^{-\frac12}y\|_{L^2(-1,1)} \le a_0\|a^{\frac12}y'\|_{L^2(-1,1)}. \tag{1.5}$$

2. Spectrum property

In this section, we shall verify condition (1.3) for the operator $\mathcal{A}$.

Theorem 2.1. *Suppose that function $a(\cdot)$ satisfies* (A1) *and* (A2)*. Then, there exist $\kappa\in\mathbb{R}$ and $b>0$ such that $\Theta_{\kappa,b}\subset\rho(\mathcal{A})$.*

Proof. First, condition (1.3) is equivalent to

$$\sigma(\mathcal{A}) \subset \mathbb{C} \setminus \Theta_{\kappa,b} = \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda < \kappa - b \ln |\mathrm{Im}\lambda|\} \text{ for some } \kappa \in \mathbb{R},\ b > 0. \tag{2.1}$$

This clearly holds if $\lambda \in \sigma(\mathcal{A})$ and $|\mathrm{Im}\lambda|$ is bounded since $\mathrm{Re}\lambda < 0$ (Thm. 2.2 in [9]). Therefore, we only need to deal with the case that the imaginary part of the spectrum of $\mathcal{A}$ is unbounded. We also note the fact $\sigma(\mathcal{A}) \setminus \mathbb{R} = \sigma_p(\mathcal{A})$, where $\sigma_p(\mathcal{A})$ is the point spectrum of $\mathcal{A}$ (Lem. 4.1 in [5]). Hence, it suffices to check that for any $\lambda \in \sigma_p(\mathcal{A})$ with $|\mathrm{Im}\lambda| > 1$ (without loss of generality), there exist $\kappa \in \mathbb{R}$ and $b > 0$ such that

$$-\frac{\mathrm{Re}\lambda}{\ln |\mathrm{Im}\lambda|} > -\frac{\kappa}{\ln |\mathrm{Im}\lambda|} + b. \tag{2.2}$$

We are going to verify (2.2) by a contradiction argument. If (2.2) is false, then we have that for any $\kappa \in \mathbb{R}$ and $b > 0$, there exists an eigenvalue λ such that $-\frac{\mathrm{Re}\lambda}{\ln |\mathrm{Im}\lambda|} \le -\frac{\kappa}{\ln |\mathrm{Im}\lambda|} + b$. Especially, we take sequences of $0 < \kappa_n < M\,(M > 0)$ and $b_n \to 0$ with $b_n > 0$. As a result, there exist a sequence of eigenvalues $\lambda_n \doteq -\mu_n + \mathrm{i}\omega_n \in \mathbb{C}$ with $\mu_n > 0$ and $\omega_n \to \infty$ $(\omega_n > 1)$ such that

$$\frac{\mu_n}{\ln \omega_n} \le -\frac{\kappa_n}{\ln \omega_n} + b_n \to 0, \quad \text{as } n \to \infty. \tag{2.3}$$

Taking the normalized eigenfunction $U_n = (u_n, v_n)$, we have

$$\lambda_n u_n - v_n = 0, \tag{2.4}$$

$$\lambda_n v_n - T_n' = 0, \quad T_n \doteq u_n' + a v_n'. \tag{2.5}$$

Then, by the dissipativeness of the operator $\mathcal{A}$, we conclude that

$$\frac{1}{\ln \omega_n} \mathrm{Re}\Big((\lambda_n I - \mathcal{A}) U_n, U_n \Big)_{\mathcal{H}} = -\frac{\mu_n}{\ln \omega_n} \|U_n\|^2 + \frac{1}{\ln \omega_n} \|a^{\frac{1}{2}} v_n'\|^2_{L^2(-1,1)} = 0. \tag{2.6}$$

Noting that $\|U_n\|_{\mathcal{H}} = 1$ and $\lim\limits_{n\to\infty} \frac{\mu_n}{\ln \omega_n} = 0$, we obtain from (2.6) that

$$\lim_{n\to\infty} \big(\ln \omega_n\big)^{-\frac{1}{2}} \|a^{\frac{1}{2}} v_n'\|_{L^2(-1,1)} = 0. \tag{2.7}$$

The rest of the proof is to show that $\lim\limits_{n\to\infty} \|U_n\|_{\mathcal{H}} = 0$, which contradicts to the normality of $\|U_n\|_{\mathcal{H}}$. The main idea and the majority of the work are to find a sequence $\xi_n \to 0^+$ such that $\lim\limits_{n\to\infty} \|v_n\|_{L^2(-1,\xi_n)} = 0$. Then, we can obtain the desired contradiction by showing that $\lim\limits_{n\to\infty} \|v_n\|_{L^2(\xi_n,1)} = 0$ and $\lim\limits_{n\to\infty} \|u_n'\|_{L^2(-1,1)} = 0$. For the clarity of presentation, the proof is divided into the following three steps.

Step 1. Firstly, we shall prove the following estimation based on (2.3)–(2.7).

$$\lim_{n\to\infty} \omega_n (\ln \omega_n)^{-\frac{1}{2}} \|a^{\frac{1}{2}} u_n'\|_{L^2(-1,1)} = 0, \tag{2.8}$$

$$\lim_{n\to\infty} \omega_n (\ln \omega_n)^{-\frac{1}{2}} \|a^{\frac{1}{2}} v_n\|_{L^2(-1,1)} = 0. \tag{2.9}$$

In fact, from (2.4), we have

$$\omega_n \big(a u_n', u_n'\big)_{L^2(-1,1)} - \mathrm{Im}\big(a v_n', u_n'\big)_{L^2(-1,1)} = 0. \tag{2.10}$$

It follows that

$$\omega_n\|a^{\frac{1}{2}}u_n'\|_{L^2(-1,1)} \le \|a^{\frac{1}{2}}v_n'\|_{L^2(-1,1)}. \tag{2.11}$$

Then, (2.8) is reached by plugging (2.7) into (2.11).

Moreover, we multiply (2.5) by av_n to get

$$\int_{-1}^{1}\lambda_n a|v_n|^2\mathrm{d}x + \int_{-1}^{1}T_n\overline{(av_n)'}\mathrm{d}x = 0. \tag{2.12}$$

Taking the imaginary part of (2.12) yields

$$\omega_n\int_{-1}^{1}a|v_n|^2\mathrm{d}x + \mathrm{Im}\int_{-1}^{1}(a'u_n'\overline{v_n} + aa'v_n'\overline{v_n} + au_n'\overline{v_n'})\mathrm{d}x = 0. \tag{2.13}$$

Note that we have the following estimations by using Lemma 1.3 and (2.7),

$$(\ln\omega_n)^{-\frac{1}{2}}\Big|\int_{-1}^{1}a'u_n'\overline{v_n}\mathrm{d}x\Big| \le a_0(\ln\omega_n)^{-\frac{1}{2}}\|a^{\frac{1}{2}}u_n'\|_{L^2(-1,1)}\|a^{\frac{1}{2}}v_n'\|_{L^2(-1,1)} \to 0. \tag{2.14}$$

Furthermore, from (2.4), (2.8) and (2.10),

$$\lim_{n\to\infty}(\ln\omega_n)^{-\frac{1}{2}}\Big|\mathrm{Im}\int_{-1}^{1}au_n'\overline{v_n'}\mathrm{d}x\Big| = \lim_{n\to\infty}\omega_n(\ln\omega_n)^{-\frac{1}{2}}\int_{-1}^{1}a|u_n'|^2\mathrm{d}x = 0. \tag{2.15}$$

Therefore, plugging (2.14) and (2.15) into (2.13) and using (2.7), we get

$$\omega_n(\ln\omega_n)^{-\frac{1}{2}}\int_{-1}^{1}a|v_n|^2\mathrm{d}x \le \Big[\max_{x\in[-1,1]}a'(x)\Big](\ln\omega_n)^{-\frac{1}{2}}\|a^{\frac{1}{2}}v_n'\|\|a^{\frac{1}{2}}v_n\| \to 0. \tag{2.16}$$

Then, (2.9) is proved.

Step 2. By introducing variables

$$z_{n,\pm}(x) = \pm v_n(x) + \sqrt{1+\lambda_n a(x)}u_n'(x), \quad x\in[-1,1], \tag{2.17}$$

we deduce from (2.4) and (2.5) that

$$\begin{aligned} z_{n,\pm}' &== \pm\lambda_n u_n' + \frac{\lambda_n v_n}{\sqrt{1+\lambda_n a}} - \frac{\lambda_n a'}{2\sqrt{1+\lambda_n a}}u_n' \\ &= \pm\frac{\lambda_n}{\sqrt{1+\lambda_n a(x)}}z_{n,\pm}(x) - \frac{\lambda_n a'(x)}{4(1+\lambda_n a(x))}\big(z_{n,+}(x)+z_{n,-}(x)\big). \end{aligned} \tag{2.18}$$

Here complex square root $\sqrt{1+\lambda_n a(x)}$ is well defined because $1+\lambda_n a(x)$ is in the right half-plane when n is large enough. Consequently, for $x\in[-1,1]$,

$$z_{n,\pm}(x) = \exp\big[\pm\big(q_n(x)-q_n(\xi_n)\big)\big]z_{n,\pm}(\xi_n) - \int_{\xi_n}^{x}\exp\big[\pm\big(q_n(x)-q_n(s)\big)\big]\phi_n(s)\mathrm{d}s, \tag{2.19}$$

where $\xi_n\in(0,1]$ will be specified later and

$$q_n(x) \doteq \int_{-1}^{x}\frac{\lambda_n}{\sqrt{1+\lambda_n a(s)}}\mathrm{d}s, \tag{2.20}$$

$$\phi_n(x) \doteq \frac{\lambda_n a'(x)}{4(1+\lambda_n a(x))}\big[z_{n,+}(x)+z_{n,-}(x)\big]. \tag{2.21}$$

We are going to verify that $\lim_{n\to\infty} \|z_{n,\pm}\|_{L^1(-1,\xi_n)} = 0$ by choosing a proper sequence $\xi_n \to 0^+$ and estimating each terms on the right hand side of (2.19).

i) First, to deal with $z_{n,\pm}(\xi_n)$, we pick constant $\gamma > 0$ such that

$$\max\left\{\frac{1}{1+\alpha},\ \frac{2}{3+\alpha}\right\} < \gamma < \frac{2}{2+\alpha}. \tag{2.22}$$

Then,

$$1 + b_n - \gamma(1+\alpha),\ 1 - \frac{\gamma}{2}(3+\alpha),\ b_n + \frac{\gamma}{2}(2+\alpha) - 1 < 0, \quad \text{for n large enough.} \tag{2.23}$$

It follows that

$$\lim_{n\to\infty} \omega_n^{1+b_n-\gamma(1+\alpha)} = \lim_{n\to\infty} \omega_n^{1-\frac{\gamma}{2}(3+\alpha)} = 0, \tag{2.24}$$

$$\lim_{n\to\infty} \omega_n^{b_n+\frac{\gamma}{2}(2+\alpha)-1} (\ln \omega_n)^{\frac{1}{2}} = 0. \tag{2.25}$$

Then, we have the following estimation on the interval $\left[\frac{1}{2}\omega_n^{-\gamma},\ \omega_n^{-\gamma}\right]$:

$$\min_{x\in\left[\frac{1}{2}\omega_n^{-\gamma},\ \omega_n^{-\gamma}\right]} |v_n(x)| \le \sqrt{2}\omega_n^{\frac{\gamma}{2}} \left(\max_{x\in\left[\frac{1}{2}\omega_n^{-\gamma},\omega_n^{-\gamma}\right]} \left|a^{-\frac{1}{2}}(x)\right|\right) \|a^{\frac{1}{2}} v_n\|_{L^2(\frac{1}{2}\omega_n^{-\gamma},\ \omega_n^{-\gamma})}. \tag{2.26}$$

Substituting (2.9) into (2.26) and using the fact that $a(x)$ approximately equals to a constant multiple of $x^{1+\alpha}$ near $x = 0$ lead us to

$$\min_{x\in[\frac{1}{2}\omega_n^{-\gamma},\ \omega_n^{-\gamma}]} |v_n(x)| = \omega_n^{\frac{\gamma}{2}(2+\alpha)-1} (\ln \omega_n)^{\frac{1}{2}}\ o(1), \qquad \text{as}\ \ n \to \infty. \tag{2.27}$$

Noting that from (2.24),

$$\lim_{n\to\infty} |\lambda_n a(x)| \le C \lim_{n\to\infty} \omega_n (\omega_n^{-\gamma})^{1+\alpha} = 0, \tag{2.28}$$

for $x \in \left[\frac{1}{2}\omega_n^{-\gamma},\ \omega_n^{-\gamma}\right]$. Thus, by using (2.8) and a similar estimate as (2.26), we obtain

$$\min_{x\in\left[\frac{1}{2}\omega_n^{-\gamma},\ \omega_n^{-\gamma}\right]} |\sqrt{1+\lambda_n a(x)} u_n'(x)| = \omega_n^{\frac{\gamma}{2}(\alpha+2)-1} (\ln \omega_n)^{\frac{1}{2}}\ o(1). \tag{2.29}$$

Therefore, from (2.27) and (2.29), there exists $\xi_n \in [\frac{1}{2}\omega_n^{-\gamma},\ \omega_n^{-\gamma}]$ such that

$$|z_{n,\pm}(\xi_n)| = \omega_n^{\frac{\gamma}{2}(\alpha+2)-1} (\ln \omega_n)^{\frac{1}{2}}\ o(1). \tag{2.30}$$

In the rest of the proof, we shall use the sequence $\{\xi_n\}$ chosen above.

ii) We estimate the function $\exp[\pm q_n(\cdot)]$. First, it is clear that

$$\operatorname{Re} q_n(x) = -\mu_n(x+1) < 0, \quad \forall\, x \in (-1, 0]. \tag{2.31}$$

When $x \in (0, \xi_n]$,

$$\int_{-1}^{x} \frac{-\mu_n}{\sqrt{1+\lambda_n a(s)}} \mathrm{d}s = -\mu_n - \mu_n \int_0^x \frac{\exp(\mathrm{i}\varphi_n(s))}{\left[(1-\mu_n a(s))^2 + (\omega_n a(s))^2\right]^{\frac{1}{4}}} \mathrm{d}s, \tag{2.32}$$

where

$$\varphi_n(x) = -\frac{1}{2} \arg(1 + \lambda_n a(x)) - k\pi, \quad k = 0, 1.$$

By setting $n \to \infty$ and using (2.24), we have that $\omega_n a(x) \to 0$ when $x \in (0, \xi_n]$. Consequently,

$$\Big|\mathrm{Re}\int_0^x \frac{\cos\varphi_n(s)}{\big[(1-\mu_n a(s))^2 + (\omega_n a(s))^2\big]^{\frac{1}{4}}}\mathrm{d}s\Big| = \mathcal{O}\Big(\omega_n^{-\gamma}\Big), \quad \forall\, x \in (0, \xi_n].$$

Therefore, it follows from (2.32) that

$$\mathrm{Re}\int_{-1}^x \frac{-\mu_n}{\sqrt{1+\lambda_n a(s)}}\mathrm{d}s \le -\frac{\mu_n}{2}, \quad \forall\, x \in (0, \xi_n]. \tag{2.33}$$

Furthermore, it follows that for any $x \in (0, \xi_n]$,

$$\begin{aligned}\mathrm{Re}\int_{-1}^x \frac{\mathrm{i}\omega_n}{\sqrt{1+\lambda_n a(s)}}\mathrm{d}s &= \mathrm{Re}\int_0^x \frac{\mathrm{i}\omega_n}{\sqrt{1+\lambda_n a(s)}}\mathrm{d}s \\ &= -\int_0^x \frac{\omega_n \sin\varphi_n(s)}{\big[(1-\mu_n a(s))^2 + (\omega_n a(s))^2\big]^{\frac{1}{4}}}\mathrm{d}s.\end{aligned} \tag{2.34}$$

Note that

$$|\sin\varphi_n(x)| = \Big|\sin\Big[\frac{1}{2}\arg(1+\lambda_n a(x))\Big]\Big| = \sqrt{\frac{1}{2} - \frac{1-\mu_n a(x)}{2\big[(1-\mu_n a(x))^2 + (\omega_n a(x))^2\big]^{\frac{1}{2}}}}. \tag{2.35}$$

Thus, for any $x \in (0, \xi_n]$, we let $n \to \infty$ and use (2.24) to obtain

$$|\sin\varphi_n(x)| = \Big(\frac{\mu_n}{2}\Big)^{\frac{1}{2}}\mathcal{O}\Big(\omega_n^{-\frac{\gamma(\alpha+1)}{2}}\Big). \tag{2.36}$$

By substituting (2.36) into (2.34) and noting $\xi_n \in [\frac{1}{2}\omega_n^{-\gamma},\ \omega_n^{-\gamma}]$, we have that

$$\Big|\mathrm{Re}\int_{-1}^x \frac{\mathrm{i}\omega_n}{\sqrt{1+\lambda_n a(s)}}\mathrm{d}s\Big| = \mathcal{O}\Big(\omega_n^{1-\gamma-\frac{\gamma(\alpha+1)}{2}}\Big), \quad \forall\, x \in (0, \xi_n]. \tag{2.37}$$

Combining (2.24) and (2.37) yields

$$\lim_{n\to\infty}\Big|\mathrm{Re}\int_{-1}^x \frac{\mathrm{i}\omega_n}{\sqrt{1+\lambda_n a(s)}}\mathrm{d}s\Big| = 0, \quad \forall\, x \in (0, \xi_n]. \tag{2.38}$$

Finally, from (2.31), (2.33) and (2.38), we conclude that

$$\mathrm{Re}\, q_n(x) < 0, \quad \forall\, x \in (-1, \xi_n]. \tag{2.39}$$

Therefore, by (2.39) and the fact that $q_n(-1) = 0$, we get that

$$\overline{\lim_{n\to\infty}}\,|\exp[q_n(x)]| \le 1, \quad \forall\, x \in [-1, \xi_n]. \tag{2.40}$$

On the other hand, by the similar argument, we have that there exists a positive constant c such that

$$-\mathrm{Re}\, q_n(x) \le \mu_n + c, \quad \forall\, x \in (-1, \xi_n]. \tag{2.41}$$

Consequently, it follows from (2.3) and (2.41) that

$$|\exp[-q_n(x)]| \le C'|\exp\mu_n| < C\omega_n^{b_n}, \quad \forall\, x \in [-1, \xi_n]. \tag{2.42}$$

iii) Now, by substituting (2.30), (2.40) and (2.42) into (2.19), we conclude that for any n large enough,

$$|z_{n,\pm}(x)| \le \omega_n^{b_n+\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac{1}{2}}\, o(1) + C\omega_n^{b_n} m_n \int_{-1}^{\xi_n} \Big|\frac{\lambda_n a'(s)}{4(1+\lambda_n a(s))}\Big| \mathrm{d}s, \tag{2.43}$$

where $x \in [-1, \xi_n]$ and

$$m_n = \max_{x\in[-1,\xi_n]} \Big(|z_{n,+}(x)| + |z_{n,-}(x)|\Big).$$

From (A1) and (A2), we deduce that $|\lambda_n a(x)| = \mathcal{O}\Big(\omega_n^{1-\gamma(\alpha+1)}\Big)$ and $|\lambda_n a'(x)| = \mathcal{O}\Big(\omega_n^{1-\gamma\alpha}\Big)$ for $n \to \infty$ and $x \in (0, \xi_n]$. Consequently, by (2.24),

$$\begin{aligned}\lim_{n\to\infty} \omega_n^{b_n} \int_{-1}^{\xi_n} \Big|\frac{\lambda_n a'(s)}{4(1+\lambda_n a(s))}\Big| \mathrm{d}s &= \lim_{n\to\infty} \omega_n^{b_n} \int_{0}^{\xi_n} \Big|\frac{\lambda_n a'(s)}{4(1+\lambda_n a(s))}\Big| \mathrm{d}s \\ &\le \frac{1}{2} \lim_{n\to\infty} \omega_n^{b_n-\gamma-\gamma\alpha+1}.\end{aligned} \tag{2.44}$$

Substituting (2.24) into (2.44) yields

$$\lim_{n\to\infty} \omega_n^{b_n} \int_{-1}^{\xi_n} \Big|\frac{\lambda_n a'(s)}{4(1+\lambda_n a(s))}\Big| \mathrm{d}s = 0, \tag{2.45}$$

It then follows from (2.43) and (2.45) that

$$m_n = \omega_n^{b_n+\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac{1}{2}}\, o(1), \tag{2.46}$$

which further leads to

$$\lim_{n\to\infty} m_n = 0, \tag{2.47}$$

due to (2.25).

Step 3. By (2.47), we conclude

$$\lim_{n\to\infty} \|v_n\|_{L^2(-1,\xi_n)} = \frac{1}{2} \lim_{n\to\infty} \|z_{n,+} - z_{n,-}\|_{L^2(-1,\xi_n)} \le C \lim_{n\to\infty} m_n = 0. \tag{2.48}$$

From (2.9) and (2.25), it is easy to get that

$$\begin{aligned}\|v_n(x)\|_{L^2(\xi_n,1)} &\le \|a^{\frac{1}{2}}(x) v_n(x)\|_{L^2(\xi_n,1)} \max_{x\in(\xi_n,1)} \Big[a^{-\frac{1}{2}}(x)\Big] \\ &= \omega_n^{-1+\frac{\gamma}{2}(\alpha+1)}\big(\ln\omega_n\big)^{\frac{1}{2}} o(1) = o(1).\end{aligned} \tag{2.49}$$

Combining (2.48) and (2.49) yields

$$\lim_{n\to\infty} \|v_n\|_{L^2(-1,1)} = 0. \tag{2.50}$$

On the other hand, it is easy to prove that

$$\lim_{n\to\infty} \|u_n'\|_{L^2(-1,1)} = \lim_{n\to\infty} \|v_n\|_{L^2(-1,1)}. \tag{2.51}$$

In fact, we multiply (2.5) by v_n to get

$$\lambda_n \|v_n\|^2_{L^2(-1,1)} + \big(u_n' + a v_n', v_n'\big)_{L^2(-1,1)} = 0. \tag{2.52}$$

Similarly, multiplying (2.4) by u_n' yields

$$\overline{\lambda}_n\|u_n'\|^2_{L^2(-1,1)} - \big(u_n', v_n'\big)_{L^2(-1,1)} = 0. \tag{2.53}$$

Therefore, we obtain (2.51) by adding (2.52) to (2.53) and taking the imaginary part of the result. Finally, it follows from (2.50) and (2.51) that $\lim_{n\to\infty}\|U_n\|_{\mathcal{H}} = 0$, which contradicts to the assumption $\|U_n\|_{\mathcal{H}} = 1$. The proof of Theorem 2.1 is completed. □

3. Estimate for the resolvent operator

In the last section, we prove that $\Theta_{\kappa,b} \subset \rho(\mathcal{A})$ for some $\kappa \in \mathbb{R}$ and $b > 0$. It is clear that $\Theta_{\kappa,b'} \subset \rho(\mathcal{A})$ for any $0 < b' < b$. In this section, we shall prove condition (1.3), *i.e.*, there exists a positive constant C such that

$$\|(\lambda I - A)^{-1}\| \le C\big|\mathrm{Im}\lambda\big|, \tag{3.1}$$

for any $\lambda \in \Theta_{\kappa,b'}$ with $\mathrm{Re}\lambda \le \omega$, where ω is the growth type of $\exp(t\mathcal{A})$. We will specify the choice of b' later.

If condition (3.1) is false, then there exist a sequence of $\lambda_n \doteq -\mu_n + \mathrm{i}\omega_n \in \rho(\mathcal{A})$ with $\mu_n > 0$, $\omega_n \to \infty$ (assume $\omega_n > 1$) and a sequence of vector $\{U_n\}_{n=1}^\infty = \{(u_n, v_n)\}_{n=1}^\infty \subset D(\mathcal{A})$ with $\|U_n\|_{\mathcal{H}} = 1$ such that

$$\lim_{n\to\infty} \omega_n\|(\lambda_n I - \mathcal{A})U_n\|_{\mathcal{H}} = 0, \tag{3.2}$$

i.e., as $n \to \infty$,

$$f_n \doteq \omega_n\big(\lambda_n u_n - v_n\big) \to 0, \qquad \text{in } H_0^1(-1,1), \tag{3.3}$$

$$g_n \doteq \omega_n\big(\lambda_n v_n - T_n'\big) \to 0, \qquad \text{in } L^2(-1,1), \tag{3.4}$$

where T_n is defined by (2.5). Note that for convenience, here we use the same notation $\{U_n\}_{n=1}^\infty = \{(u_n, v_n)\}_{n=1}^\infty$ as in Section 2, but it isn't the sequence of eigenvectors. From $\omega_n\mathrm{Re}\big((\lambda_n I - \mathcal{A})U_n, U_n\big) \to 0$, we conclude that as $n \to \infty$,

$$-\omega_n\mu_n\|U_n\|^2_{\mathcal{H}} + \omega_n\|a^{\frac{1}{2}}v_n'\|^2_{L^2(-1,1)} \to 0.$$

Consequently,

$$\lim_{n\to\infty}\frac{1}{\ln\omega_n}\|a^{\frac{1}{2}}v_n'\|^2_{L^2(-1,1)} = \lim_{n\to\infty}\frac{\mu_n}{\ln\omega_n}\cdot \tag{3.5}$$

By using (1.3), we substitute $\mu_n \le -\kappa + b'\ln\omega_n$ into (3.5) to get

$$\lim_{n\to\infty}(\ln\omega_n)^{-\frac{1}{2}}\|a^{\frac{1}{2}}v_n'\|_{L^2(-1,1)} \le \sqrt{b'}. \tag{3.6}$$

Similar to the Step 1 in the proof of Theorem 2.1, we have the following estimates under assumptions (A1), (A2) and (3.6).

$$\lim_{n\to\infty}\omega_n(\ln\omega_n)^{-\frac{1}{2}}\|a^{\frac{1}{2}}u_n'\|_{L^2(-1,1)} < C, \tag{3.7}$$

$$\lim_{n\to\infty}\omega_n(\ln\omega_n)^{-\frac{1}{2}}\|a^{\frac{1}{2}}v_n\|_{L^2(-1,1)} < C, \tag{3.8}$$

where C is a positive constant depending on $a(\cdot)$ and b.

For any $\zeta_n \in [-1,1]$, we integrate (3.4) on (x,ζ_n) and combine the result with (3.3) to get

$$\left(\frac{\lambda_n}{\sqrt{1+\lambda_n a(x)}}\right)^2\int_x^{\zeta_n} v_n(\tau)\mathrm{d}\tau + v_n'(x) - \frac{\lambda_n}{1+\lambda_n a}T_n(\zeta_n) = F_n(x,\zeta_n), \tag{3.9}$$

where

$$F_n(x,\zeta_n) \doteq -\frac{f_n'(x)}{\omega_n(1+\lambda_n a(x))} + \frac{\lambda_n}{\omega_n(1+\lambda_n a(x))}\int_x^{\zeta_n} g_n(\tau)\mathrm{d}\tau.$$

Let

$$w_{n,\pm}(x) \doteq \frac{\lambda_n}{\sqrt{1+\lambda_n a(x)}}\int_x^{\zeta_n} v_n(\tau)\mathrm{d}\tau \pm v_n(x).$$

Then,

$$w_{n,\pm}'(x) = \frac{\mp\lambda_n w_{n,\pm}(x)}{\sqrt{1+\lambda_n a(x)}} - \frac{\lambda_n a'(x)}{4(1+\lambda_n a(x))}\big(w_{n,+}(x)+w_{n,-}(x)\big) \pm \frac{\lambda_n T_n(\zeta_n)}{1+\lambda_n a(x)} \pm F_n(x,\zeta_n). \tag{3.10}$$

Consequently,

$$\begin{aligned} w_{n,\pm}(x) = \exp\big[\mp\big(q_n(x)-q_n(\zeta_n)\big)\big]w_{n,\pm}(\zeta_n) - \int_{\zeta_n}^x \exp\big[\mp\big(q_n(x)-q_n(s)\big)\big]\tilde{\phi}_n(s)\mathrm{d}s \\ \pm\int_{\zeta_n}^x \exp\big[\mp\big(q_n(x)-q_n(s)\big)\big]\Big[\frac{\lambda_n T_n(\zeta_n)}{1+\lambda_n a(s)} + F_n(s,\zeta_n)\Big]\mathrm{d}s, \end{aligned} \tag{3.11}$$

where q_n is the same variable defined in (2.20) and $\tilde{\phi}_n$ is defined by:

$$\tilde{\phi}_n(x) \doteq \frac{\lambda_n a'(x)}{4(1+\lambda_n a(x))}\big[w_{n,+}(x)+w_{n,-}(x)\big].$$

In what follows, we shall estimate each terms in the right hand side of (3.11).

First, Let $\zeta_n \in \big[\frac{1}{2}\omega_n^{-\gamma}, \omega_n^{-\gamma}\big]$, where γ is a positive constant satisfying (2.22). We choose b' small enough. Then, by the same idea as (2.23) and (2.25), we conclude that

$$\lim_{n\to\infty}\omega_n^{1+b'-\gamma(1+\alpha)} = \lim_{n\to\infty}\omega_n^{1-\frac{\gamma}{2}(3+\alpha)} = 0, \tag{3.12}$$

$$\lim_{n\to\infty}\omega_n^{b'+\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac{1}{2}} = 0. \tag{3.13}$$

Consequently, by the same argument as Step 2(ii) in the proof of Theorem 2.1, we obtain that when $x\in[-1,\zeta_n]$,

$$\begin{aligned} \overline{\lim_{n\to\infty}}\big|\exp[q_n(x)]\big| \le 1, \\ \big|\exp[-q_n(x)]\big| \le C\omega_n^{b'}, \quad C>0. \end{aligned} \tag{3.14}$$

Furthermore, it is easy to obtain

$$\begin{aligned} \min_{x\in\left[\frac{1}{2}\omega_n^{-\gamma},\,\omega_n^{-\gamma}\right]} \Big(|v_n(x)| + |u_n'(x)| + |a(x)v_n'(x)|\Big) \\ \le\ C\omega_n^{\frac{\gamma}{2}(\alpha+2)}\Big(\|a^{\frac{1}{2}}v_n\|_{L^2(-1,1)} + \|a^{\frac{1}{2}}u_n'\|_{L^2(-1,1)}\Big) + C\omega_n^{-\frac{\gamma\alpha}{2}}\|a^{\frac{1}{2}}v_n'\|_{L^2(-1,1)}. \end{aligned} \tag{3.15}$$

Replacing (3.6), (3.7) and (3.8) into (3.15) and noting that $\frac{\gamma}{2}(\alpha+2)-1 > -\frac{\gamma\alpha}{2}$ from (3.12), we have that there exists a positive constant C depend on κ, b and $a(\cdot)$ such that for n large enough

$$\min_{x\in\left[\frac{1}{2}\omega_n^{-\gamma},\,\omega_n^{-\gamma}\right]} \Big(|v_n(x)| + |u_n'(x)| + |a(x)v_n'(x)|\Big) \le C\omega_n^{\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac{1}{2}}. \tag{3.16}$$

Therefore, there exists $\zeta_n \in \big[\frac{1}{2}\omega_n^{-\gamma},\, \omega_n^{-\gamma}\big]$ such that

$$|w_{n,\pm}(\zeta_n)| + |T_n(\zeta_n)| \le C\omega_n^{\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac{1}{2}}. \tag{3.17}$$

Now, substituting (3.14) and (3.17) into (3.11) yields

$$\begin{aligned}|w_{n,\pm}(x)| \le{}& C\omega_n^{b'}\left[\omega_n^{\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac{1}{2}} + \tilde m_n \int_{\zeta_n}^x \left|\frac{\lambda_n a'(s)}{4(1+\lambda_n a(s))}\right| \mathrm{d}s\right] \\ &+ \left|\int_{\zeta_n}^x \big[I_1(x,s,\zeta_n) + I_2(x,s) + I_3(x,s,\zeta_n)\big]\mathrm{d}s\right|, \quad x\in[-1,\zeta_n],\end{aligned} \tag{3.18}$$

where

$$\tilde m_n \doteq \max_{x\in[-1,\zeta_n]}\Big(|w_{n,+}(x)| + |w_{n,-}(x)|\Big),$$

$$I_1(x,s,\zeta_n) \doteq \exp\big[\mp\big(q_n(x)-q_n(s)\big)\big]\frac{\lambda_n T_n(\zeta_n)}{1+\lambda_n a(s)},$$

$$I_2(x,s) \doteq -\exp\big[\mp\big(q_n(x)-q_n(s)\big)\big]\frac{f_n'(s)}{\omega_n(1+\lambda_n a(s))},$$

$$I_3(x,s,\zeta_n) \doteq \exp\big[\mp\big(q_n(x)-q_n(s)\big)\big]\frac{\lambda_n}{\omega_n(1+\lambda_n a(s))}\int_s^{\zeta_n} g_n(\tau)\mathrm{d}\tau.$$

When $x\in(-1,0)$, it is easy to obtain

$$\int_0^x I_1\mathrm{d}s = \int_0^x \exp\big[\mp\lambda_n(x-s)\big]\lambda_n T_n(\zeta_n)\mathrm{d}s = \mp\Big[\exp\big(\mp\lambda_n x\big)-1\Big]T_n(\zeta_n), \tag{3.19}$$

and

$$\lim_{n\to\infty}\int_{\zeta_n}^0 I_1\mathrm{d}s = \mp\lim_{n\to\infty}\Big[1-\exp\big(\mp\lambda_n\zeta_n\big)\Big]T_n(\zeta_n). \tag{3.20}$$

Noting that $\mu_n \le -\kappa + b'\ln\omega_n$. Therefore, plugging (3.17) into (3.19) and (3.20) yields

$$\lim_{n\to\infty}\left|\int_{\zeta_n}^x I_1\mathrm{d}s\right| \le C\lim_{n\to\infty}\omega_n^{b'+\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac{1}{2}}. \tag{3.21}$$

Similarly, by (3.14),

$$\left|\int_{\zeta_n}^x I_2\mathrm{d}s\right| \le \int_{\zeta_n}^x \exp\big[\mp\big(\mu_n(x-s)\big)\big]\left|\frac{f_n'(s)}{\omega_n}\right|\mathrm{d}s = \omega_n^{b'-1}o(1). \tag{3.22}$$

Finally, for n large enough,

$$\begin{aligned}\left|\int_{\zeta_n}^x I_3\mathrm{d}s\right| &= \left|\frac{\lambda_n}{\omega_n}\int_{\zeta_n}^x\int_x^\tau \exp\big[\mp\lambda_n(x-s)\big]g_n(\tau)\mathrm{d}s\mathrm{d}\tau\right| \\ &= \frac{1}{\omega_n}\left|\int_{\zeta_n}^x\Big[\exp\big(\mp\lambda_n(x-\tau)\big)-1\Big]g_n(\tau)\mathrm{d}\tau\right| \\ &= \omega_n^{b'-1}o(1).\end{aligned} \tag{3.23}$$

For the case $x=-1$ and $x\in[0,\zeta_n]$, we deduce the same estimation for the last integral in (3.18) by similar argument. Thus, plugging (3.21) and (3.23) into (3.18), we obtain that for any $x\in[-1,\zeta_n]$,

$$|w_{n,\pm}(x)| \le C\omega_n^{b'+\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac12} + C\omega_n^{b'}\tilde{m}_n\int_{\zeta_n}^{x}\left|\frac{\lambda_n a'(s)}{4(1+\lambda_n a(s))}\right|\mathrm{d}s + \omega_n^{b'-1}o(1). \tag{3.24}$$

By the same argument as (2.45) in (3.24),

$$\tilde{m}_n \le C\lim_{n\to\infty}\omega_n^{b'+\frac{\gamma}{2}(\alpha+2)-1}(\ln\omega_n)^{\frac12} + \omega_n^{b'-1}o(1). \tag{3.25}$$

Hence, we conclude that

$$\|v_n\|_{L^2(-1,\zeta_n)} \to 0 \tag{3.26}$$

by choosing b' and γ satisfies (3.12) and (3.13). Similar to (2.51), we can prove that

$$\lim_{n\to\infty}\|u_n'\|_{L^2(-1,1)} = \lim_{n\to\infty}\|v_n\|_{L^2(-1,1)}, \tag{3.27}$$

Combining (3.8), (3.26) and (3.27) yields $\|U_n\|_{\mathcal{H}}\to 0$, which contradicts to the assumption. Thus, we have finished the proof for Theorem 1.1.

Remark 3.1. It follows from Theorem 1.1 that semigroup $\exp(t\mathcal{A})$ is differentiable for $t>t_0$. Therefore, for $t>nt_0$ $(n=1,2,\dots)$, $\exp(t\mathcal{A})$: $\mathcal{H}\to D(\mathcal{A}^n)$ and $[\exp(t\mathcal{A})]^{(n)}=\mathcal{A}^n\exp(t\mathcal{A})$ is a bounded linear operator. This implies that the solution u of system (1.1) satisfies $(u(\cdot,t),\ u_t(\cdot,t))\in D(\mathcal{A}^n)$ for any $(u_0,u_1)\in\mathcal{H}$ and $t>nt_0$, $n=1,2,\dots$

Acknowledgements. We would like to thank the referees for their thoughtful suggestions which have greatly increased the readability of this paper.

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