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OPTIMALITY, DUALITY AND GAP FUNCTION FOR QUASI VARIATIONAL INEQUALITY PROBLEMS

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Abstract. This paper deals with the Quasi Variational Inequality (QVI) problem on Banach spaces. Necessary and sufficient conditions for the solutions of QVI are given, using the subdifferential of distance function and the normal cone. A dual problem corresponding to QVI is constructed and strong duality is established. The solutions of dual problem are characterized according to the saddle points of the Lagrangian map. A gap function for dual of QVI is presented and its properties are established. Moreover, some applied examples are addressed.

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1. Introduction and terminology

Throughout this paper, E is a real Banach space with topological dual E^*, and $\langle .,. \rangle$ denotes the duality pairing between E and E^*, *i.e.* $\langle x^*, x\rangle = x^*(x)$ for $x^* \in E^*$ and $x \in E$. The norm of the members of E^* is defined as

$$\|x^*\| = \sup \left\{ \langle x^*, x \rangle \ : \ x \in E, \|x\| \leq 1 \right\}.$$

Let $\bar{R} = R \bigcup \{-\infty, +\infty\}$. The function $g : E \to \bar{R}$ is called proper if $g(x) > -\infty$ for all $x \in E$ and $\mathrm{dom}(g) \neq \emptyset$. For proper function g and $x \in \mathrm{dom}(g)$, the (classic) subdifferential of g at x, in the sense of convex analysis, is the convex set defined by

$$\partial g(x) := \left\{ x^* \in E^* : g(x) + \langle x^*, y - x \rangle \leq g(y) \text{ for all } y \in E \right\}.$$

The Fenchel–Moreau conjugate of g (not necessarily convex) is the function $g^* : E^* \to \bar{R}$ defined by

$$g^*(x^*) := \sup_{x \in E} \{ \langle x^*, x \rangle - g(x) \}.$$

Suppose that $C \subset E$ is nonempty and convex. The support function of C is a map from E^* into $\bar{R}$ defined by

$$\psi_C(x^*) := \sup_{c \in C} \langle x^*, c \rangle.$$

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We also adopt the following conventions

$$(+\infty)-(+\infty)=(-\infty)-(-\infty)=(+\infty)+(-\infty)=(-\infty)+(+\infty)=+\infty,$$
$$(\infty)+y=y+(\infty)=\infty,\ \text{for each } y\in R.$$

The distance function corresponding to $U\subseteq E$, denoted by $d_U:E\longrightarrow \bar{R}$, is defined by

$$d_U(x)=\inf_{u\in U}\|x-u\|, \qquad x\in E.$$

Let X be a nonempty, closed, and convex subset of E, and let $K:X\rightrightarrows X$ be a set-valued map such that for every $v\in X$, the set $K(v)$ is a nonempty and closed subset of X. Furthermore, let $A:E\rightrightarrows E^*$ be a given set-valued map. In this paper, we consider the following Quasi Variational Inequality (QVI) problem: finding $x\in K(x)$ and $t^*\in A(x)$ such that

$$\langle t^*, v-x\rangle\geq 0, \quad \text{for all } v\in K(x). \qquad (QVI). \tag{1.1}$$

The point x is said to be a solution to (QVI) and we say that the pair (x,t^*) solves (QVI).

QVIs were introduced and investigated at first by Bensoussan and Lions [4,5]. They introduced these problems in connection with impulse optimal control problems. After introducing QVIs by Bensoussan and Lions, many scholars studied these problems from different standpoints, see, *e.g.*, [3,10,12,14,22,25] and the references therein. One of the most important aspects of these problems is their connections with and applications in various well-known problems in different fields of science, engineering, and economics, including complementarity problems [14], filtration in continuum mechanics [3], contact problems with compliant obstacles [26], contact problems with Coulomb friction [6,19], game theory [15,18,24], oligopolistic markets [24,26], traffic [12], computational biology [17], *etc.* QVIs are generalized forms of classic equilibrium problems which have been frequently studied in recent years. Blum and Oettli [8] showed that equilibrium problems include optimization problems, Nash equilibria, complementarity problems, fixed point problems and variational inequalities as particular cases. Also, Iusem and Sosa [20] investigated that multiobjective optimization problems can be obtained by equilibrium problems. The results of the above-mentioned publications show that the QVIs are useful models of many practical problems.

In this paper, some necessary and sufficient conditions are given to characterize the solutions of QVIs. It is done using the subdifferential of distance function and also the normal cone. Also, a dual problem corresponding to QVI is constructed and strong duality is established. A characterization for the solutions of dual problem is proved using the saddle point notion. Furthermore, the gap function for dual of QVI is dealt with and its properties are established.

The rest of the paper unfolds as follows: Sections 2 and 3 contain characterizations of the solutions of QVI and its dual. The gap function and its properties are presented in Section 4. In Section 5, we present some applied examples.

2. Necessary and sufficient conditions

In this section, we present some necessary and sufficient conditions for the solutions of QVI. To this end, we use a penalty mechanism and some results from variational analysis and nonsmooth analysis.

The following proposition helps us in the sequel.

Proposition 2.1 ([11])**.** *Let $g:E\to R$ be Lipschitz around $\bar{x}$ with Lipschitz rank K. Assume that $\bar{x}$ is a local minimizer of g on the closed set $U\subseteq E$. Then $\bar{x}$ is a local minimizer of $g+Kd_U$ on E (unconstrained).*

The following proposition provides a necessary and sufficient condition for the solutions of QVI using the distance function.

Proposition 2.2. *Let X be a nonempty, closed, and convex subset of E. Let $x \in K(x)$ and $t^* \in A(x)$. Assume that $f : E \to R$ is a single-valued map defined by*

$$f(z) = \langle t^*, z - x \rangle, \qquad z \in E.$$

Then, the following are equivalent:

(i) *the pair (x, t^*) solves (QVI);*
(ii) *x is a minimizer of $f + \|t^*\| d_{K(x)}$.*

Proof. Assume that the pair (x, t^*) solves (QVI). Then, by definition of (QVI), the vector x is a minimizer of f on $K(x)$. It is not difficult to show that f is Lipschitz with rank $\|t^*\|$. Therefore, by Proposition 2.1, x is a minimizer of $f + \|t^*\| d_{K(x)}$.

Conversely, assume that (ii) holds. Since $x \in K(x)$, we have

$$f(x) + \|t^*\| d_{K(x)}(x) = \langle t^*, x - x \rangle + \|t^*\| d_{K(x)}(x) = 0.$$

Let $v \in K(x)$ be arbitrary. Since x is a minimizer of $f + \|t^*\| d_{K(x)}$, we get

$$\langle t, v - x \rangle = \langle t, v - x \rangle + \|t^*\| d_{K(x)}(v) \geq 0.$$

This completes the proof. □

The following theorem gives a necessary and sufficient condition for the solutions of QVI using the normal cone and distance function. For convex set $A \subseteq E$, $N_A(\bar{x})$ denotes the normal cone to A at $\bar{x}$, defined as

$$N_A(\bar{x}) = \left\{ t^* \in E^* \ : \ \langle t^*, x - \bar{x} \rangle \leq 0, \ \forall x \in A \right\}.$$

Also, $\mathbb{B}_{E^*}$ stands for the closed unit ball in E^*.

Theorem 2.3. *Let X be a nonempty, closed, and convex subset of E, and $K : X \rightrightarrows X$ be a set-valued map such that $K(v)$ is nonempty, closed and convex for each $v \in X$. Let $A : E \rightrightarrows E^*$ be a given set-valued map. Assume that $\bar{x} \in K(\bar{x})$ and $t^* \in A(\bar{x})$. Then the following are equivalent:*

(i) *the pair $(\bar{x}, t^*)$ solves (QVI).*
(ii) *$\frac{-t^*}{\|t^*\|} \in \partial d_{K(\bar{x})}(\bar{x})$.*
(iii) *$-t^* \in N_{K(\bar{x})}(\bar{x})$.*

Proof. If the pair $(\bar{x}, t^*)$ solves (QVI), then by Proposition 2.2, $\bar{x}$ is a minimizer of the function $\langle t^*, \cdot - \bar{x} \rangle + \|t^*\| d_{K(\bar{x})}(\cdot)$. Therefore, by Theorem 2.5.7 in [28],

$$0 \in \partial \left(\langle t, \cdot - \bar{x} \rangle + \|t^*\| d_{K(\bar{x})}(\cdot) \right)(\bar{x}).$$

Since the distance function is continuous and $\partial d_{K(\bar{x})}(\bar{x}) = N_{K(\bar{x})}(\bar{x}) \bigcap \mathbb{B}_{E^*}$ (see [9]), we have

$$\partial \left(\langle t^*, \cdot - \bar{x} \rangle + \|t^*\| d_{K(\bar{x})} \right)(\bar{x}) = t^* + \|t^*\| \partial d_{K(\bar{x})}(\bar{x}) = t^* + \|t^*\| \left(N_{K(\bar{x})}(\bar{x}) \bigcap \mathbb{B}_{E^*} \right). \tag{2.1}$$

Hence, $\frac{-t^*}{\|t^*\|} \in \partial d_{K(\bar{x})}(\bar{x})$.

Conversely, if $\frac{-t^*}{\|t^*\|} \in \partial d_{K(\bar{x})}(\bar{x})$, for each $y \in K(\bar{x})$,

$$0 = d_{K(\bar{x})}(y) - d_{K(\bar{x})}(\bar{x}) \geq \left\langle \frac{-t^*}{\|t^*\|}, y - \bar{x} \right\rangle.$$

Hence, we have $\langle t^*, y - \bar{x} \rangle \geq 0$ for each $y \in K(\bar{x})$, which implies that, $(\bar{x}, t^*)$ solves (QVI). Therefore, (i) and (ii) are equivalent.

The equivalence of (ii) and (iii) is clear because of (2.1) and the fact that $\frac{-t^*}{\|t^*\|} \in \mathbb{B}_{E^*}$. □

In the above theorem, we proved that, under convexity, $-t^* \in N_{K(\bar{x})}(\bar{x})$ is a necessary and sufficient condition for solvability of (QVI) by $(\bar{x}, t^*)$. The following theorem proves that it is a necessary condition even without convexity assumption if we use Clarke normal cone instead of the usual normal cone. To this end, we use the properties of Clarke generalized gradients.

Let f be a function from E into R, locally Lipschitzian at $x \in E$. The Clarke directional derivative of f at x in direction $h \in E$, denoted by $f^\circ(x;h)$, is defined by

$$f^\circ(x;h) = \limsup_{(t,y)\to(0^+,x)} \frac{f(y+th)-f(y)}{t}\cdot$$

The Clarke subdifferential of f at x, denoted by $\partial^c f(x)$, is defined by

$$\partial^c f(x) = \{x^* \in E^* : \langle x^*, h\rangle \le f^\circ(x;h), \quad \text{for all } h \in E\}.$$

Suppose that A is a subset of E. The Clarke tangent cone to A at $x \in A$ is defined as

$$\begin{aligned} T^c(A;x) = \{h \in E : \forall\big(\{x_n\} \subseteq A,\ \{t_n\} \subseteq R\big);\ x_n \to x,\ t_n \downarrow 0, \\ \exists \{h_n\} \subseteq E; h_n \to h,\ x_n + t_n h_n \in A \ \ \forall n\}. \end{aligned}$$

The Clarke normal cone to A at $x \in A$ is defined as

$$N^c(A;x) = \{x^* \in E^* : \langle x^*, h\rangle \le 0, \quad \text{for all } h \in T^c(A;x)\}.$$

Now, we present the final result of this section. This theorem results from Proposition 2.1 and Corollary 2.4.3 in [11].

Theorem 2.4. *Let X be a nonempty, closed, and convex subset of E, and $K : X \rightrightarrows X$ be a set-valued map such that $K(v)$ is nonempty and closed for each $v \in X$. Let $A : E \rightrightarrows E^*$ be a given set-valued map. Assume that $\bar{x} \in K(\bar{x})$ and $t^* \in A(\bar{x})$. If the pair $(\bar{x}, t^*)$ solves (QVI), then $-t^* \in N^c(K(\bar{x}); \bar{x})$.*

3. Duality in reflexive Banach spaces and saddle point

In some results of this section, we assume that E is a reflexive Banach space. Moreover, in the remainder sections, assume that for every $v \in X$, the set $K(v)$ is nonempty, closed and convex.

Our aim in this section is giving a dual problem corresponding to (QVI). By the next lemma, we obtain the Fenchel–Moreau conjugate of the distance function $d_C(\cdot)$ where C is closed and convex. This result may exist in the literature, but here we present its proof from a different standpoint.

Lemma 3.1. *Let $f = d_C$, where C is closed and convex. Then for $t^* \in \mathbb{B}_{E^*}$, we have $f^*(t^*) = \sup_{c\in C}\langle t^*, c\rangle$. Also, for $t^* \notin \mathbb{B}_{E^*}$, we have $f^*(t^*) = +\infty$.*

Proof.

$$\begin{aligned} f^*(t^*) &= \sup_{x\in E}\{\langle t^*, x\rangle - \inf_{c\in C}\|x-c\|\} \\ &= \sup_{x\in E, c\in C}\{\langle t^*, x\rangle - \|x-c\|\} \\ &= \sup_{y\in E, c\in C}\{\langle t^*, y+c\rangle - \|y\|\} \\ &= \sup_{c\in C}\langle t^*, c\rangle + sup_{y\in E}\{\langle t^*, y\rangle - \|y\|\}. \end{aligned}$$

If $t^* \in \mathbb{B}_{E^*}$, then $sup_{y\in E}\{\langle t^*, y\rangle - \|y\|\} = 0$, and desired result is derived. In the case $t^* \notin \mathbb{B}_{E^*}$, there is $y \in E$ such that $\langle t^*, y\rangle > \|y\|$. Thus for each $\alpha > 0$, we have $\langle t^*, \alpha y\rangle > \|\alpha y\|$ and hence $\alpha\langle t^*, y\rangle > \alpha\|y\|$. Consequently, the result follows. $\square$

Now, we introduce the following dual problem corresponding to (QVI). This problem is denoted by (DQVI). Recall that $\psi_A(.)$ denotes the support function corresponding to the set A.

DQVI: finding $\bar{x} \in E$ and $t^* \in A(\bar{x})$ such that

$$\psi_{K(\bar{x})}\left(-\frac{t^*}{2\|t^*\|}\right) + \left\langle \frac{t^*}{2\|t^*\|}, \bar{x} \right\rangle \leq \psi_{K(\bar{x})}(h^*) - \langle h^*, \bar{x}\rangle, \quad \text{for all } h^* \in \mathbb{B}_{E^*} \qquad (DQVI). \tag{3.1}$$

In this case, we say that the pair $(\bar{x}, t^*)$ solves (DQVI).

The following lemmas will be needed in the sequel.

Lemma 3.2 ([13]). *Assume that Y is a reflexive Banach space. If $f : Y \to \bar{R}$ is a proper lower semi-continuous convex function, then $x^* \in \partial f(\bar{x})$ if and only if*

$$f(\bar{x}) + f^*(x^*) = \langle x^*, \bar{x}\rangle.$$

Lemma 3.3 ([13]). *Assume that Y is a reflexive Banach space, and $f : Y \to \bar{R}$ is a lower semi-continuous convex function. Then $x^* \in \partial f(x)$ if and only if $x \in \partial f^*(x^*)$.*

Now, we are ready to state one of the main results of this section.

Theorem 3.4. *Let X be a nonempty, closed, and convex subset of E. Assume that E is reflexive. The pair $(\bar{x}, t^*)$ solves (QVI) if and only if $(\bar{x}, t^*)$ solves $(DQVI)$.*

Proof. Let the pair $(\bar{x}, t^*)$ solve (QVI). Following a manner similar to the proofs of Proposition 2.2 and Theorem 2.3, we get $-\frac{t^*}{2\|t^*\|} \in \partial d_{K(\bar{x})}(\bar{x})$. Since the map K is closed- and convex-valued, $d_{K(\bar{x})}$ is a lower semi-continuous convex function. Thus, we may use Lemma 3.3 to conclude that $\bar{x} \in \partial d^*_{K(\bar{x})}(-\frac{t^*}{2\|t^*\|})$. This implies,

$$d^*_{K(\bar{x})}(h^*) - d^*_{K(\bar{x})}\left(-\frac{t^*}{2\|t^*\|}\right) \geq \left\langle h^* - \left(-\frac{t^*}{2\|t^*\|}\right), \bar{x}\right\rangle, \quad \text{for all } h^* \in E^*. \tag{3.2}$$

Hence,

$$d^*_{K(\bar{x})}(h^*) - d^*_{K(\bar{x})}\left(-\frac{t^*}{2\|t^*\|}\right) \geq \left\langle h^* - \left(-\frac{t^*}{2\|t^*\|}\right), \bar{x}\right\rangle, \quad \text{for all } h^* \in \mathbb{B}_{E^*}. \tag{3.3}$$

Now, using Lemma 3.1, we have

$$\psi_{K(\bar{x})}(h^*) - \psi_{K(\bar{x})}\left(-\frac{t^*}{2\|t^*\|}\right) \geq \left\langle h^* - \left(-\frac{t^*}{2\|t^*\|}\right), \bar{x}\right\rangle, \quad \text{for all } h^* \in \mathbb{B}_{E^*}. \tag{3.4}$$

Thus $(\bar{x}, t^*)$ solves (DQVI).

Conversely, if $(\bar{x}, t^*)$ solves (DQVI), then inequality (3.4) holds, and hence by Lemma 3.1, inequality (3.2) holds. Hence, by Lemma 3.3, $-\frac{t^*}{2\|t^*\|} \in \partial d_{K(\bar{x})}(\bar{x})$. Since $\| - \frac{t^*}{2\|t^*\|}\| < 1$, by Theorem 1 in [9], we have $\bar{x} \in K(\bar{x})$ and the proof is completed. □

In the following propositions, $K(\bar{x})^+$ denotes the nonnegative dual cone corresponding to $K(\bar{x})$, defined by

$$K(\bar{x})^+ = \{x^* \in E^* \ : \ \langle x^*, v\rangle \geq 0, \quad \text{for each } v \in K(\bar{x})\}.$$

Proposition 3.5. *Assume that X is a nonempty, closed, and convex subset of E, and the values of the set-valued mapping K are cones. Also, let $\bar{x} \in K(\bar{x})$ and $t^* \in A(\bar{x})$. The pair $(\bar{x}, t^*)$ solves (QVI) if and only if $t^* \in K(\bar{x})^+$ and $t^*(\bar{x}) = 0$.*

Proof. Let the pair $(\bar{x}, t^*)$ solve (QVI). Then

$$\langle t^*, v\rangle \geq \langle t^*, \bar{x}\rangle, \quad \text{for all } v \in K(\bar{x}).$$

Since $K(\bar{x})$ is a cone, this inequality implies $\langle t^*, v\rangle \geq 0 \geq \langle t^*, \bar{x}\rangle$, for all $v \in K(\bar{x})$. Hence, $t^* \in K(\bar{x})^+$. Furthermore, $\bar{x} \in K(\bar{x})$ implies $t^*(\bar{x}) = 0$.

Conversely, assume $t^* \in K(\bar{x})^+$ and $t^*(\bar{x}) = 0$. Hence

$$\langle t^*, v - \bar{x}\rangle = \langle t^*, v\rangle \geq 0 \quad \text{for all } v \in K(\bar{x}).$$

Also, $\bar{x} \in K(\bar{x})$ by assumption. Hence, $(\bar{x}, t^*)$ solves (QVI). □

Our analysis in the rest of this section is inspired by the approach used in [7]. Let us consider the optimization problem

$$\inf\{\langle t^*, z - \bar{x}\rangle \ : \ z \in K(\bar{x})\}, \tag{3.5}$$

where $t^* \in A(\bar{x})$, $\bar{x} \in K(\bar{x})$ and the maps A, K are as considered before. The map K is closed- and convex-valued.

Definition 3.6 is close to Definition 2.1 in [7]. A function $f : E \longrightarrow R$ is called closed if *epi* $f = \{(x, \alpha) \in E \times R \ : \ f(x) \leq \alpha\}$ is a closed set in $E \times R$.

Definition 3.6. Assume that Ω is a real Banach space. The function $L : E \times \Omega \to [-\infty, +\infty]$ is said to be a Lagrangian representation of (3.5) if for each $z \in E$ the function $-L(z, \cdot)$ is closed, the function $L(z, \cdot)$ is concave, and

$$\sup\{L(z, \lambda); \quad \lambda \in \Omega\} = \langle t^*, z - \bar{x}\rangle + \|t^*\| d_{K(\bar{x})}(z). \tag{3.6}$$

If (3.6) holds, then we have the equivalence between Problem (3.5) and the following problem

$$\inf_{z \in E} \sup_{\lambda \in \Omega} L(z, \lambda).$$

The dual problem of (3.5) is defined as

$$\sup_{\lambda \in \Omega} \inf_{z \in E} L(z, \lambda). \tag{3.7}$$

A pair $(\bar{x}, \bar{\theta})$ is called a saddle point of a Lagrangian map L if

$$L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\theta}) \leq L(z, \bar{\theta}) \quad \text{for all } z \in E, \lambda \in \Omega.$$

Theorem 3.7 helps us in sequel.

Theorem 3.7 ([7,27])**.** *A pair $(\bar{x}, \bar{\theta}) \in E \times \Omega$ is a saddle point of L if and only if $\bar{x}$ solves* (3.5)*, $\bar{\theta}$ solves* (3.7) *and the two problems have the same optimal value.*

Remark 3.8 ([7])**.** Consider the following Optimization problem:

$$\inf\{g_1(x) + g_2(x) : x \in E\}, \tag{3.8}$$

where g_1 and g_2 are two given proper functions. The Fenchel dual is defined by

$$\sup\{-g_1^*(x^*) - g_2^*(-x^*) : x^* \in E^*\}.$$

Due to (3.7), this dual problem may be obtained invoking the Lagrangian representation [7]:

$$L(x, x^*) = g_1(x) - \langle x^*, x\rangle - g_2^*(-x^*). \tag{3.9}$$

Definition 3.9. Considering $t^* \in A(\bar{x})$ and $\bar{x} \in K(\bar{x})$, the pair $(\bar{x}, t^*)$ satisfies condition (Δ) if $-t^* \in \|t^*\|\partial d_{K(\bar{x})}(\bar{x})$.

Assume that $\bar{x} \in K(\bar{x})$. Because of Proposition 2.2, the pair $(\bar{x}, t^*)$ solves the (QVI) if and only if $\bar{x}$ solves the following optimization problem:

$$\inf \left\{ \langle t^*, z - \bar{x} \rangle + \|t^*\| d_{K(\bar{x})}(z); \quad z \in E \right\}.$$

Hence, $(\bar{x}, t^*)$ solves the (QVI) if and only if $\bar{x}$ solves

$$\inf \left\{ \left\langle \frac{t^*}{\|t^*\|}, z - \bar{x} \right\rangle + d_{K(\bar{x})}(z); \quad z \in E \right\}. \tag{3.10}$$

From Remark 3.8, one can easily see that the following map gives a Lagrangian representation for the optimization Problem (3.10) where $\bar{x} \in K(\bar{x})$ and $t^* \in A(\bar{x})$:

$$L_q(x, x^*) = \left\langle \frac{t^*}{\|t^*\|}, x - \bar{x} \right\rangle - \langle x^*, x \rangle - d^*_{K(\bar{x})}(-x^*). \tag{3.11}$$

By the next theorem, we show that condition (Δ) has a connection with the saddle points of the Lagrangian function (3.11).

Theorem 3.10. *Assume that E is a reflexive real Banach space, $\bar{x} \in K(\bar{x})$ and $t^* \in A(\bar{x})$. Then the following statements are equivalent:*

(i) *the pair $(\bar{x}, t^*)$ satisfies condition (Δ);*
(ii) *$(\bar{x}, \frac{t^*}{\|t^*\|})$ is a saddle point of L_q.*

Proof. Assume that (i) holds. Then $-t^* \in \|t^*\|\partial d_{K(\bar{x})}(\bar{x})$. By Lemma 3.3, we get

$$\bar{x} \in \partial d^*_{K(\bar{x})} \left(-\frac{t^*}{\|t^*\|} \right).$$

This is equivalent to

$$d^*_{K(\bar{x})}(-y^*) - d^*_{K(\bar{x})} \left(-\frac{t^*}{\|t^*\|} \right) \geq \left\langle \frac{t^*}{\|t^*\|} - y^*, \bar{x} \right\rangle, \qquad \text{for all } y^* \in E^*. \tag{3.12}$$

On the other hand,

$$L_q(\bar{x}, y^*) = -\langle y^*, \bar{x} \rangle - d^*_{K(\bar{x})}(-y^*),$$
$$L_q \left(\bar{x}, \frac{t^*}{\|t^*\|} \right) = -\left\langle \frac{t^*}{\|t^*\|}, \bar{x} \right\rangle - d^*_{K(\bar{x})} \left(-\frac{t^*}{\|t^*\|} \right).$$

Therefore, by (3.12), we have $L_q(\bar{x}, y^*) \leq L_q(\bar{x}, \frac{t^*}{\|t^*\|})$ for each $y^* \in E^*$.

Moreover,

$$L_q \left(x, \frac{t^*}{\|t^*\|} \right) = -\left\langle \frac{t^*}{\|t^*\|}, \bar{x} \right\rangle - d^*_{K(\bar{x})} \left(-\frac{t^*}{\|t^*\|} \right), \quad \text{for all } x \in E.$$

In other words, $L_q(., \frac{t^*}{\|t^*\|})$ is a constant function, and so $L_q(x, \frac{t^*}{\|t^*\|}) = L_q(\bar{x}, \frac{t^*}{\|t^*\|})$. Therefore,

$$L_q(\bar{x}, y^*) \leq L_q \left(\bar{x}, \frac{t^*}{\|t^*\|} \right) \leq L_q \left(x, \frac{t^*}{\|t^*\|} \right) \qquad \text{for all } x \in E, \ y^* \in E^*.$$

Thus $(\bar{x}, \frac{t^*}{\|t^*\|})$ is a saddle point of L_q.

Conversely, assume that $(\bar{x}, \frac{t^*}{\|t^*\|})$ is a saddle point of L_q. Hence, inequality (3.12) holds and this implies $\bar{x} \in \partial d^*_{K(\bar{x})}(-\frac{t^*}{\|t^*\|})$. Applying Lemma 3.3, we get $-t^* \in \|t^*\|\partial d_{K(\bar{x})}(\bar{x})$ and the proof is completed. □

If one considers

$$\inf\left\{\langle t^*, z-x\rangle + 2\|t^*\| d_{K(\bar{x})}(z);\ z\in E\right\}$$

and

$$L_q^{'}(x,x^*) = \left\langle \frac{t^*}{2\|t^*\|}, x-\bar{x}\right\rangle - \langle x^*, x\rangle - d^*_{K(\bar{x})}(-x^*),$$

instead of (3.10) and (3.11), respectively, then the following result (Thm. 3.11) can be proved similar to Theorem 3.10.

Theorem 3.11. *Let X be a nonempty, closed, and convex subset of E. Assume that E is reflexive, and $t^* \in A(\bar{x})$. Then the following statements are equivalent:*

(i) *the pair $(\bar{x}, t^*)$ solves (DQVI);*
(ii) *$(\bar{x}, \frac{t^*}{2\|t^*\|})$ is a saddle point of $L_q^{'}$.*

4. Gap function for dual quasi variational inequality

A set-valued map $A : E \rightrightarrows E^*$ is said injective if for any $r, s \in E$ with $r \neq s$, $A(r) \bigcap A(s) = \emptyset$. In this situation, the inverse map of A, *i.e.* A^{-1} : Range$A \to E$, is such that $x \in A^{-1}(u)$ if and only if $u \in A(x)$. It is easy to see that if a set-valued map A is injective, then A^{-1} is single-valued. The adjoint map $A' : -$Range$A \to E$ of A is defined as $A'(u) = -A^{-1}(-u)$, for $u \in -$Range$A = $ domA'.

In this section, we are going to introduce a gap function for $(DQVI)$, and then getting necessary and sufficient conditions for the existence of solutions to (QVI) and its dual.

The following set, which is defined corresponding to the set-valued map K, is used is sequel. It is assumed that this set is nonempty:

$$X_1 = \{x \in X \mid x \in K(x)\}.$$

The definition below is well-known in reference [2].

Definition 4.1. A set-valued mapping $\eta : X_1 \to R$ is said to be a gap function for (QVI) if it satisfies the following properties:

(i) $\eta(x) \geq 0$, for each $x \in X_1$,
(ii) $\eta(\hat{x}) = 0$ if and only if $\hat{x}$ solves (QVI).

In the sequel, we assume that the set-valued map A is injective and compact-valued. The Auslender gap function for (QVI) [2] is

$$g(x) = \inf_{t^*\in A(x)} \sup_{x'\in K(x)} \langle t^*, x-x'\rangle, \qquad \text{for all } x \in X_1.$$

Let A' be the adjoint map of A defined as above.

Definition 4.2. A set-valued map $\xi : E^* \to R$ is said to be a gap function for $(DQVI)$ if it satisfies the following properties:

(i) $\xi(s^*) \geq 0$, for each $s^* \in E^*$;
(ii) $\xi(t^*) = 0$ if and only if $(A'(t^*), t^*)$ solves $(DQVI)$. Notice that when $t^* = 0$, we replace $(A'(t^*), t^*)$ with $(A'(0), 0)$ in the definition.

Along the lines of [23], we define a gap function $G : E^* \to R$ for the dual quasi variational inequality problem (DQVI) as follows: for $t^* = 0$,

$$G(0) = \sup_{h^*\in\mathbb{B}_{E^*}} \left(d^*_{K(A'(0))}(0) - d^*_{K(A'(0))}(h^*) + \langle h^*, A'(0)\rangle \right)$$

and for $t^* \neq 0$,

$$G(t^*) := \sup_{h^* \in \mathbb{B}_{E^*}} \left(d^*_{K(A'(t^*))} \left(-\frac{t^*}{2\|t^*\|} \right) + \left\langle \frac{t^*}{2\|t^*\|}, A'(t^*) \right\rangle - d^*_{K(A'(t^*))}(h^*) + \langle h^*, A'(t^*) \rangle \right). \tag{4.1}$$

Theorem 4.3. *Let X be a nonempty, closed, and convex subset of E. The mapping G is a gap function for the problem $(DQVI)$.*

Proof. Assume that t^* is an arbitrary element of E^*. We assume that $t^* \neq 0$ (in the case $t^* = 0$ we apply a similar analysis). Then we have

$$\sup_{h^* \in \mathbb{B}_{E^*}} \left(d^*_{K(A'(t^*))} \left(-\frac{t^*}{2\|t^*\|} \right) + \left\langle \frac{t^*}{2\|t^*\|}, A'(t^*) \right\rangle - \psi_{K(A'(t^*))}(h^*) + \langle h^*, A'(t^*) \rangle \right)$$
$$\geq d^*_{K(A'(t^*))} \left(-\frac{t^*}{2\|t^*\|} \right) + \left\langle \frac{t^*}{2\|t^*\|}, A'(t^*) \right\rangle - \psi_{K(A'(t^*))} \left(-\frac{t^*}{2\|t^*\|} \right) + \left\langle -\frac{t^*}{2\|t^*\|}, A'(t^*) \right\rangle = 0.$$

Hence, we have $G(t^*) \geq 0$ for each $t^* \in E^*$.

Now, suppose that $G(\hat{t}^*) = 0$. We assume that $\hat{t}^* \neq 0$ (in the case $\hat{t}^* = 0$ one can apply the same manner). We have

$$\psi_{K\left(A'(\hat{t}^*)\right)} \left(-\frac{\hat{t}^*}{2\|\hat{t}^*\|} \right) + \left\langle \frac{\hat{t}^*}{2\|\hat{t}^*\|}, A'(\hat{t}^*) \right\rangle \leq \psi_{K\left(A'(\hat{t}^*)\right)}(h^*) - \langle h^*, A'\left(\hat{t}^*\right) \rangle, \qquad \text{for all } h^* \in \mathbb{B}_{E^*}.$$

So, the pair $(A'(\hat{t}^*), \hat{t}^*)$ solves $(DQVI)$. Also, one can easily check that if the pair $(A'(\hat{t}^*), \hat{t}^*)$ solves $(DQVI)$, then $G(\hat{t}^*) = 0$. □

From Theorems 3.4 and 4.3, we may obtain the following result:

Theorem 4.4. *Let X be a nonempty, closed, and convex subset of E. Assume that E is reflexive and A is injective. Let g and G be gap functions for (QVI) and $(DQVI)$, as in the above. Suppose that $t^* \in E^*$. Then the following assertions are equivalent:*

(i) $G(t^*) = 0$;
(ii) $A'(t^*) \in X_1,\ g(A'(t^*)) = 0$;
(iii) $d_{K(A'(t^*))}(A'(t^*)) + d^*_{K(A'(t^*))}(\frac{t^*}{2\|t^*\|}) = \langle \frac{t^*}{2\|t^*\|}, A'(t^*) \rangle$.

Proof. Since g is a gap function for (QVI), the results follow from Theorem 4.3 and Lemma 3.2. □

5. Examples

In this section, we provide some practical examples to clarify the main results of the paper.

Example 1. We consider a variational inequality with obstacles [16, 21, 25]. Let Ω be a bounded domain in $R^n,\ n \geq 2$, with Lipschitz continuous boundary.

We use the standard notations:

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \ \frac{\partial u}{\partial x_i} \in L^2(\Omega), \quad i = 1, \dots, n \right\}$$

in which $\frac{\partial u}{\partial x_i}$ (for $1 \leq i \leq n$) is the first generalized derivative of the map u. Here, $H^1(\Omega)$ is equipped with the norm:

$$\| u \|^2_{H^1(\Omega)} = \| u \|^2_{0,2} + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{0,2}$$

where $\|\cdot\|_{0,2}$ is the usual norm on $L^2(\Omega)$. Note that $H^1(\Omega)$ is a separable and reflexive Banach space (see [1] for more information). Also,

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : \quad u|_{\partial\Omega} \equiv 0\}$$

is a reflexive Banach space ($\partial\Omega$ stands for the boundary of Ω).

Let ϕ and ψ be two elements of $H_0^1(\Omega)$. The inequality $\phi \le \psi$ means $\phi(x) \le \psi(x)$ for almost every x in Ω.

Let M be a mapping from $H_0^1(\Omega)$ into itself, and let $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ be the so-called Laplacian operator, where $H^{-1}(\Omega)$ is the topological dual space of the Sobolev space $H_0^1(\Omega)$. Consider the following quasi-variational inequality problem: find $u \in H_0^1(\Omega)$ such that $u \le M(u)$ in $H_0^1(\Omega)$ and

$$\langle -\Delta u, v - u\rangle \ge 0, \qquad \forall v \in H_0^1(\Omega), \quad v \le M(u). \tag{5.1}$$

If u solves (5.1), then

$$\langle \triangle u, M(u) - u\rangle = 0.$$

We construct the corresponding dual problem and we prove the above equality by duality results given in the present paper.

Setting $A(f) := -\Delta(f)$ and $K(f) := \{g \in H_0^1(\Omega) : \ g \le M(f)\}$ for each $f \in H_0^1(\Omega)$, we may construct the following dual problem and apply our main results.

Find $g \in H_0^1(\Omega)$ such that

$$\psi_{K(g)}\left(\frac{\Delta(g)}{2\|\Delta(g)\|}\right) - \left\langle \frac{\Delta(g)}{2\|\Delta(g)\|}, g\right\rangle \le \psi_{K(g)}(h^*) - \langle h^*, g\rangle, \quad \text{for all } h^* \in \mathbb{B}_{H^{-1}(\Omega)}. \qquad (DQVI) \tag{5.2}$$

Let $\Theta := \{\varphi \in H_0^1(\Omega) : \varphi \ge 0, \ a.e. \text{ on } \Omega\}$. It is clear that Θ is a cone in $H_0^1(\Omega)$ with vertex at the origin. We denote the polar cone of Θ by Θ^+ which is defined as follows

$$\Theta^+ := \{t \in H^{-1}(\Omega) : \quad \langle t, \varphi\rangle \ge 0, \quad \forall \varphi \in \Theta\}.$$

We are going to describe the associated dual problem. First, note that if $v \in K(g)$, then there is an element $\tau \in \Theta$ such that $v = M(g) - \tau$. Now, let $h^* \in H^{-1}(\Omega)$. Then we have

$$\psi_{K(g)}(h^*) = \sup_{v \in K(g)} \langle v, h^*\rangle = \langle M(g), h^*\rangle + \sup_{\tau \in \Theta}\langle -\tau, h^*\rangle.$$

Note that $\sup_{\tau\in\Theta}\langle -\tau, h^*\rangle = 0$ for $h^* \in \Theta^+$. Hence,

$$\psi_{K(g)}(h^*) = \langle M(g), h^*\rangle + I_{\Theta^+}(h^*),$$

where

$$I_{\Theta^+}(h^*) = \begin{cases} 0, & h^* \in \Theta^+ \\ +\infty, & h^* \notin \Theta^+ \end{cases}$$

Hence, by Theorem 3.4, we deduce that the quasi variational problem (5.1) has a solution if and only if the following variational problem has a solution

$$\begin{cases} \text{find } \ t^* \in H^{-1}(\Omega), \quad g \in A^{-1}(t^*) \quad s.t. \quad -t^* \in \Theta^+, \\ \qquad \left\langle -\frac{t^*}{2\|t^*\|}, M(g)\right\rangle + \left\langle \frac{t^*}{2\|t^*\|}, g\right\rangle \le \langle s^*, M(g)\rangle - \langle s^*, g\rangle, \quad \forall s^* \in \Theta^+ \cap \mathbb{B}_{H^{-1}(\Omega)}. \end{cases}$$

Now, assume that the pair (g, t^*) solves the last variational problem. Setting $s^* = 0$, we get

$$\left\langle -\frac{t^*}{2\|t^*\|}, M(g)\right\rangle + \left\langle \frac{t^*}{2\|t^*\|}, g\right\rangle \le 0.$$

On the other hand, $-t^* \in \Theta^+$. Hence, we conclude that $\langle t^*, M(g) - g\rangle = 0$.

Example 2 (Nash equilibria under constraint)**.** Let $C = [0, +\infty) \subset R$. Let $\rho_i : R^n \to R,\ \ i = 1, \ldots, n$ be some given maps. Let $f_1, f_2, \ldots, f_n$ be real-valued functions on R^n in which each f_i is convex in the ith variable.

Furthermore, we assume that for $i = 1, \ldots, n$ the map f_i is Gateaux differentiable. The Gateaux derivative of f_i at x is denoted by $d_G f_i(x)$. A Nash equilibrium is any vector $u = (u_1, \ldots, u_n) \in R^n$ which satisfies the following conditions

$$\begin{cases} u_i \le \rho_i(u), \\ f_i(u_1, \ldots, u_i, \ldots, u_n) \le f_i(u_1, \ldots, y_i, \ldots, u_n), \qquad \forall y_i \le \rho_i(u), \end{cases}$$

for each $i = 1, \ldots, n$. We set $Q_i(u) := \{w \in R :\ w \le \rho_i(u)\}$ for $i = 1, \ldots, n$. Then, the above problem takes the following form: find $u = (u_1, \ldots, u_n) \in R^n$ with the following properties:

$$\begin{cases} u_i \in Q_i(u), \\ f_i(u_1, \ldots, u_i, \ldots, u_n) \le f_i(u_1, \ldots, y_i, \ldots, u_n); \quad \forall y_i \in Q_i(u), \end{cases}$$

for each $i = 1, \ldots, n$.

Since each function f_i is convex in the ith variable, it is easy to see that for each $i \in \{1, 2, \ldots, n\}$ and $u \in Q_i(u)$, the above inequality holds true if and only if

$$\langle d_G h_i(u_i), y_i - u_i\rangle \ge 0, \qquad \forall y_i \in Q_i(u), \tag{5.3}$$

in which $h_i : R \to R$ is defined by $h_i(x) = f_i(u_1, \ldots, u_{i-1}, x, u_{i+1}, \ldots, u_n)$. Now, setting $\Psi := d_G(h_1) \times d_G(h_2) \times \ldots \times d_G(h_n)$ and $Q(u) = Q_1(u) \times Q_2(u) \times \ldots \times Q_n(u)$, the above problem reduces to the following QVI :

$$\begin{cases} find\ u \in Q(u) \qquad s.t. \\ \qquad\quad \langle \Psi(u), y - u\rangle \ge 0, \quad \forall y \in Q(u). \end{cases}$$

Hence, by an argument similar to Example 1, it can be seen that the corresponding $(DQVI)$ takes the following form:

$$\begin{cases} find\ t^* \in R^n, \quad r_i \in (d_G(h_i))^{-1}(t_i^*), \quad i = 1, 2, \ldots, n \ \ s.t.\ t^* \in -C^n, \\ \displaystyle\sum_{i=1}^{n} \left\langle -\frac{t_i^*}{2\|t_i^*\|}, \rho_i(r) \right\rangle + \sum_{i=1}^{n} \left\langle \frac{t_i^*}{2\|t_i^*\|}, r_i \right\rangle \le \sum_{i=1}^{n} \langle s_i^*, \rho_i(r)\rangle - \sum_{i=1}^{n} \langle s_i^*, r_i\rangle, \\ \forall s^* \in C^n \cap \mathbb{B}_{R^n}. \end{cases}$$

If the pair (t^*, r) is a solution of this variational inequality, then setting $s^* = 0$ we get

$$\langle t_i^*, r_i - \rho_i(r)\rangle = 0,$$

for $i = 1, \ldots, n$.

We close the paper by a numerical example.

Example 3. Let $K : [1, -1] \rightrightarrows R$ and $A : [1, -1] \to R$ be two maps defined by $K(x) := [-x^2, x^2]$ and $A(x) := -x$ for each $x \in [-1, 1]$. Consider the following quasi variational inequality: find $x \in [-1, 1]$ such that $x \in K(x)$ and

$$\langle -x, v - x\rangle \ge 0, \qquad \text{for all } v \in K(x). \tag{5.4}$$

Let x be an arbitrary element of the interval $[-1, 1]$. It is obvious that $-1 \in K(-1)$; and $K(-1)$ is closed and convex; and that $-1 \in N_{K(-1)}(-1) = N_{[-1,1]}(-1)$. Therefore we obtain that $-1 \in \partial d_{K(-1)}(-1)$. Hence, Theorem 2.3 implies that the pair $(-1, 1)$ solves the above QVI problem. Also, by Theorem 3.10 the pair $(-1, 1)$ is a saddle point of the Lagrangian map

$$\begin{aligned} L_q(x, x^*) &= \langle 1, x + 1\rangle - \langle x^*, x\rangle - d^*_{[-1,1]}(-x^*) \\ &= (x + 1) - x^* x - |x^*|. \end{aligned}$$

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