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ON FRACTIONAL LAPLACIANS – 3

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Abstract. We investigate the role of the noncompact group of dilations in $\mathbb{R}^n$ on the difference of the quadratic forms associated to the fractional Dirichlet and Navier Laplacians. Then we apply our results to study the Brezis–Nirenberg effect in two families of noncompact boundary value problems involving the Navier−Laplacian.

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1. Introduction

The Sobolev space $H^m(\mathbb{R}^n) = W_2^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, is the space of distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ with finite norm

$$\|u\|_m^2 = \int\limits_{\mathbb{R}^n} \left(1 + |\xi|^2\right)^m |\mathcal{F}u(\xi)|^2 \, \mathrm{d}\xi,$$

see for instance Section 2.3.3 of the monograph [13, 24]. Here $\mathcal{F}$ denotes the Fourier transform

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-i\xi\cdot x} u(x) \, \mathrm{d}x.$$

For arbitrary $m \in \mathbb{R}$ we define fractional Laplacian on $\mathbb{R}^n$ by the quadratic form

$$Q_m[u] = \langle (-\Delta)^m u, u \rangle := \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}u(\xi)|^2 \mathrm{d}\xi, \tag{1.1}$$

with domain

$$\mathrm{Dom}(Q_m) = \{u \in \mathcal{S}'(\mathbb{R}^n) \, : \, Q_m[u] < \infty\}.$$

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Let Ω be a bounded and smooth domain in $\mathbb{R}^n$. We introduce the "Dirichlet" fractional Laplacian in Ω (denoted by $(-\Delta_\Omega)^m_D$) as the restriction of $(-\Delta)^m$. More precisely, its quadratic form is given by (1.1) with domain

$$\mathrm{Dom}(Q^D_{m,\Omega}) = \{u \in \mathrm{Dom}(Q_m) \, : \, \mathrm{supp}\, u \subset \overline{\Omega}\}.$$

Also we define the "Navier" fractional Laplacian as the mth power of the conventional Dirichlet Laplacian in the sense of spectral theory. Its quadratic form reads

$$Q^N_{m,\Omega}[u] = \langle(-\Delta_\Omega)^m_N u, u\rangle := \sum\nolimits_j \lambda_j^m \cdot |\langle u, \varphi_j\rangle|^2.$$

Here, λ_j and φ_j are eigenvalues and eigenfunctions of the Dirichlet Laplacian in Ω, respectively, and $\mathrm{Dom}(Q^N_{m,\Omega})$ consists of distributions in Ω such that $Q^N_{m,\Omega}[u] < \infty$.

For $m = 1$ these operators evidently coincide: $(-\Delta_\Omega)_N = (-\Delta_\Omega)_D$. We emphasize that, in contrast to $(-\Delta_\Omega)^m_N$, the operator $(-\Delta_\Omega)^m_D$ is not the mth power of the Dirichlet Laplacian for $m \neq 1$. In the recent paper [2], the interested reader may find a thorough review of some differences between the Dirichlet and Navier Laplacians of order $m \in (0,1)$, see in particular Section 2.1 of [2] and references therein.

It is well known that for $m > 0$, the quadratic forms $Q^D_{m,\Omega}$ and $Q^N_{m,\Omega}$ generate Hilbert structures on their domains, and

$$\mathrm{Dom}(Q^D_{m,\Omega}) = \widetilde{H}^m(\Omega) \subseteq \mathrm{Dom}(Q^N_{m,\Omega}),$$

where

$$\widetilde{H}^m(\Omega) = \{u \in H^m(\mathbb{R}^n) \, : \, \mathrm{supp}\, u \subset \overline{\Omega}\}.$$

It is also easy to see that for $m \in \mathbb{N}$, $u \in \widetilde{H}^m(\Omega)$

$$Q^D_{m,\Omega}[u] = Q^N_{m,\Omega}[u].$$

In [15, 17] we compared the operators $(-\Delta_\Omega)^m_D$ and $(-\Delta_\Omega)^m_N$ for non-integer m. It turned out that the difference between their quadratic forms is positive or negative depending on the fact whether $\lfloor m \rfloor$ is odd or even. However, roughly speaking, this difference disappears as $\Omega \to \mathbb{R}^n$.

Namely, denote by $F(\Omega)$ the class of smooth and bounded domains containing Ω. For any $u \in \mathrm{Dom}(Q^D_{m,\Omega})$ the form $Q^D_{m,\Omega'}[u]$ does not depend on $\Omega' \in F(\Omega)$ while the form $Q^N_{m,\Omega'}[u]$ does depend on $\Omega' \supset \Omega$, and the following relations hold.

Proposition 1.1 ([17], Thm. 2)**.** *Let $m > -1$, $m \notin \mathbb{N}_0$. If $u \in \mathrm{Dom}(Q^D_{m,\Omega})$, then*

$$Q^D_{m,\Omega}[u] = \inf_{\Omega' \in F(\Omega)} Q^N_{m,\Omega'}[u], \text{ if } 2k < m < 2k+1, \quad k \in \mathbb{N}_0; \tag{1.2}$$

$$Q^D_{m,\Omega}[u] = \sup_{\Omega' \in F(\Omega)} Q^N_{m,\Omega'}[u], \text{ if } 2k-1 < m < 2k, \quad k \in \mathbb{N}_0. \tag{1.3}$$

The main result of our paper is a quantitative version of Proposition 1.1.

Theorem 1.2. *Assume that $m > 0$, $m \notin \mathbb{N}$. Let $u \in \widetilde{H}^m(\Omega)$, and let* $\mathrm{supp}(u) \subset B_r \subset B_R \subset \Omega$*. Then*

$$Q^N_{m,\Omega}[u] \le Q^D_{m,\Omega}[u] + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}, \quad \textit{if } \lfloor m \rfloor \ \textit{ is even}; \tag{1.4}$$

$$Q^D_{m,\Omega}[u] \le Q^N_{m,\Omega}[u] + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}, \quad \textit{if } \lfloor m \rfloor \ \textit{ is odd}. \tag{1.5}$$

The Proof of Theorem 1.2 is given in Section 2. In Section 3 we apply this result for studying the equations[4]

$$(-\Delta_\Omega)^m_N u = \lambda(-\Delta_\Omega)^s_N u + |u|^{2^*_m-2}u \quad \text{in } \Omega, \tag{1.6}$$

$$(-\Delta_\Omega)^m_N u = \lambda|x|^{-2s}u + |u|^{2^*_m-2}u \quad \text{in } \Omega, \tag{1.7}$$

where $0 \le s < m < \frac{n}{2}$ and $2^*_m = \frac{2n}{n-2m}$. By solution u of (1.6) or (1.7) we mean a weak solution from $\mathrm{Dom}(Q^N_{m,\Omega})$, see Section 3 for details.

In the basic paper [3] by Brezis and Nirenberg a remarkable phenomenon was discovered for the problem

$$-\Delta u = \lambda u + |u|^{\frac{4}{n-2}}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.8}$$

which coincides with (1.6) and (1.7) with $n > 2$, $m = 1$, $s = 0$. Namely, the existence of a nontrivial solution for any small $\lambda > 0$ holds if $n \ge 4$; in contrast, for $n = 3$ non-existence phenomena for any sufficiently small $\lambda > 0$ can be observed. For this reason, the dimension $n = 3$ has been named *critical* for problem (1.8) (compare with [10, 19]).

As was pointed out in [16], the Brezis–Nirenberg effect is a nonlinear analog of the so-called zero-energy resonance for the Schrödinger operators (see, *e.g.*, [26] and ([27], pp. 287–288)).

After [3], a large number of papers have been focussed on studying the effect of lower order linear perturbations in noncompact variational problems, see for instance the list of references included in ([10], Chap. 7) about the case $m \in \mathbb{N}$, $s = 0$. The Dirichlet case with non-integer m was considered in the recent paper [16], see also [20,21] for $m \in (0,1)$ and $s = 0$. As concerns the Navier case with non-integer m, the only papers we know consider $m \in (0,1)$ and $s = 0$, see [1, 23]. We mention also the recent paper [7] and references therein for nonlinear lower-order perturbations.

We study the general case and prove the following result (see Sect. 3 for a more precise statement), that corresponds to ([16], Thm. 4.2).

Theorem 1.3. *Let $0 \le s < m < \frac{n}{2}$. If $s \ge 2m - \frac{n}{2}$ then n is not a critical dimension for (1.6) and (1.7). This means that both these equations have ground state solutions for all sufficiently small $\lambda > 0$.*

Let us recall some notation. B_R is the ball with radius R centered at the origin, $\mathbb{S}_R$ is its boundary. We denote by c with indices all explicit constants while C without indices stand for all inessential positive constants. To indicate that C depends on some parameter a, we write $C(a)$.

2. Proof of Theorem 1.2

Notice that we can assume $u \in \mathcal{C}^\infty_0(\Omega)$, the general case being covered by approximation.

Proof of (1.4). Let $m = 2k + \sigma$, $k \in \mathbb{N}_0$, $\sigma \in (0,1)$. Denote by $w^D(x,y)$, $x \in \mathbb{R}^n$, $y > 0$, the Caffarelli–Silvestre extension of $(-\Delta)^k u$ (see [5]), that is the solution of the boundary value problem

$$-\mathrm{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad w\big|_{y=0} = (-\Delta)^k u,$$

given by the generalized Poisson formula

$$w^D(x,y) = c_1(n,\sigma) \int\limits_{\mathbb{R}^n} \frac{y^{2\sigma}\,(-\Delta)^k u(\xi)}{(|x-\xi|^2 + y^2)^{\frac{n+2\sigma}{2}}}\,\mathrm{d}\xi. \tag{2.1}$$

[4] We assume that $0 \in \Omega$.

In [5] it is also proved that

$$Q^D_{m,\Omega}[u] = Q^D_{\sigma,\Omega}[(-\Delta)^k u] = c_2(n,\sigma)\int\limits_0^\infty\int\limits_{\mathbb{R}^n} y^{1-2\sigma}|\nabla w^D|^2\,\mathrm{d}x\mathrm{d}y. \tag{2.2}$$

Integrating by parts (2.1), we arrive at following estimates for $|x|>r$:

$$|w^D(x,y)| \le \frac{C(n,m)\,y^{2\sigma}\,\|u\|_{L_1(\Omega)}}{((|x|-r)^2+y^2)^{\frac{n+m+\sigma}{2}}};\qquad |\nabla w^D(x,y)| \le \frac{C(n,m)\,y^{2\sigma-1}\,\|u\|_{L_1(\Omega)}}{((|x|-r)^2+y^2)^{\frac{n+m+\sigma}{2}}}. \tag{2.3}$$

Following ([15], Thm. 3), we define, for $x\in\overline{B}_R$ and $y\ge 0$, the function

$$\widetilde{w}(x,y) = w^D(x,y) - \widetilde{\phi}(x,y),$$

where $\widetilde{\phi}(\cdot,y)$ is the harmonic extension of $w^D(\cdot,y)$ on B_R, that is,

$$-\Delta_x\widetilde{\phi}(\cdot,y) = 0 \quad \text{in } B_R; \qquad \widetilde{\phi}(\cdot,y) = w^D(\cdot,y) \quad \text{on } \mathbb{S}_R.$$

Clearly, $\widetilde{w}\big|_{y=0} = (-\Delta)^k u$ and $\widetilde{w}\big|_{x\in\mathbb{S}_R} = 0$. Further, we have

$$\begin{aligned}\int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla\widetilde{w}|^2\,\mathrm{d}x\mathrm{d}y &= \int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}(|\nabla w^D|^2 - 2\nabla w^D\cdot\nabla\widetilde{\phi} + |\nabla\widetilde{\phi}|^2)\,\mathrm{d}x\mathrm{d}y\\ &= \int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla w^D|^2\,\mathrm{d}x\mathrm{d}y - 2\int\limits_0^\infty\int\limits_{\mathbb{S}_R} y^{1-2\sigma}(\nabla w^D\cdot\mathbf{n})\,\widetilde{\phi}\,d\mathbb{S}_R(x)\mathrm{d}y\\ &\quad + \int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla\widetilde{\phi}(x,y)|^2\,\mathrm{d}x\mathrm{d}y.\end{aligned} \tag{2.4}$$

Since $\widetilde{\phi}(\cdot,y) = w^D(\cdot,y)$ on $\mathbb{S}_R$, we can use (2.3) to get

$$\left|\int\limits_0^\infty\int\limits_{\mathbb{S}_R} y^{1-2\sigma}(\nabla w^D\cdot\mathbf{n})\,\widetilde{\phi}\,d\mathbb{S}_R(x)\mathrm{d}y\right| \le \frac{C(n,m)\,R^{n-1}}{(R-r)^{2n+2m-1}}\cdot\|u\|^2_{L_1(\Omega)}.$$

Now we estimate the last integral in (2.4). It is easy to see that $|\nabla\widetilde{\phi}(\cdot,y)|^2$ is subharmonic in B_R and thus the function

$$\rho\mapsto\frac{1}{\rho^{n-1}}\int\limits_{\mathbb{S}_\rho}|\nabla\widetilde{\phi}(x,y)|^2\,d\mathbb{S}_\rho(x)$$

is nondecreasing for $\rho\in(0,R)$. This implies

$$\begin{aligned}\int\limits_{B_R}|\nabla\widetilde{\phi}(x,y)|^2\,\mathrm{d}x &= \int\limits_0^R\int\limits_{\mathbb{S}_\rho}|\nabla\widetilde{\phi}(x,y)|^2\,d\mathbb{S}_\rho(x)d\rho\\ &\le \frac{R}{n}\int\limits_{\mathbb{S}_R}(|\nabla_x\widetilde{\phi}(x,y)|^2 + |\partial_y\widetilde{\phi}(x,y)|^2)\,d\mathbb{S}_R(x).\end{aligned}$$

Using the fact that $\partial_y\widetilde{\phi}(x,y) = \partial_y w^D(x,y)$ for $x \in \mathbb{S}_R$, and the well known estimate

$$\int\limits_{\mathbb{S}_R} |\nabla_x\widetilde{\phi}(x,y)|^2\, d\mathbb{S}_R(x) \le C(n) \int\limits_{\mathbb{S}_R} |\nabla_x w^D(x,y)|^2\, d\mathbb{S}_R(x),$$

we can apply (2.3) to arrive at

$$\int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla\widetilde{\phi}(x,y)|^2\, \mathrm{d}x\mathrm{d}y \le \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}.$$

In conclusion, from (2.4) we infer

$$\int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla\widetilde{w}|^2\, \mathrm{d}x\mathrm{d}y \le \int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla w^D|^2\, \mathrm{d}x\mathrm{d}y + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}. \tag{2.5}$$

Now we use the Stinga–Torrea characterization of $Q^N_{\sigma,\Omega}$. Their general result stated in Theorem 1.1 of [22] (see also the last example in Sect. 2 therein) and integration by parts imply that

$$Q^N_{m,\Omega}[u] = Q^N_{\sigma,\Omega}[(-\Delta)^k u] = c_2(n,\sigma) \inf_{\substack{w|_{x\in\partial\Omega}=0 \\ w|_{y=0}=(-\Delta)^k u}} \int\limits_0^\infty \int\limits_{\Omega} y^{1-2\sigma} |\nabla w|^2\, \mathrm{d}x\mathrm{d}y. \tag{2.6}$$

Relations (2.6), (2.5) and (2.2) give us

$$\begin{aligned} Q^N_{m,\Omega}[u] \le Q^N_{m,B_R}[u] &\le c_2(n,\sigma) \int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla\widetilde{w}|^2\, \mathrm{d}x\mathrm{d}y \\ &\le c_2(n,\sigma) \int\limits_0^\infty \int\limits_{B_R} y^{1-2\sigma} |\nabla w^D|^2\, \mathrm{d}x\mathrm{d}y + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)} \\ &\le Q^D_{m,\Omega}[u] + \frac{C(n,m)\, R^n}{(R-r)^{2n+2m}} \cdot \|u\|^2_{L_1(\Omega)}, \end{aligned}$$

and (1.4) follows. □

Proof of (1.5). Let $m = 2k - \sigma$, $k \in \mathbb{N}$, $\sigma \in (0,1)$. Denote by $w^{-D}(x,y)$, $x \in \mathbb{R}^n$, $y > 0$, the "dual" Caffarelli–Silvestre extension of $(-\Delta)^k u$ (see [4,17]), that is the solution of the boundary value problem

$$-\mathrm{div}(y^{1-2\sigma}\nabla w) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}_+; \qquad y^{1-2\sigma}\partial_y w\big|_{y=0} = -(-\Delta)^k u,$$

given by the formula

$$w^{-D}(x,y) = c_3(n,\sigma) \int\limits_{\mathbb{R}^n} \frac{(-\Delta)^k u(\xi)}{(|x-\xi|^2 + y^2)^{\frac{n-2\sigma}{2}}}\, \mathrm{d}\xi. \tag{2.7}$$

Note that the representation (2.7) is true also for $n = 1 < 2\sigma$ while for $n = 1$, $\sigma = 1/2$ it should be rewritten as follows:

$$w^{-D}(x,y) = c_3(1,1/2) \int\limits_{\mathbb{R}^n} (-\Delta)^k u(\xi) \ln(|x-\xi|^2 + y^2)\, \mathrm{d}\xi.$$

It is also shown in [17] that

$$Q^D_{m,\Omega}[u] = Q^D_{-\sigma,\Omega}[(-\Delta)^k u] = \frac{1}{c_2(n,\sigma)}\left(2\int\limits_{\mathbb{R}^n}(-\Delta)^k u(x) w^{-D}(x,0)\,\mathrm{d}x - \int\limits_0^\infty\int\limits_{\mathbb{R}^n} y^{1-2\sigma}|\nabla w^{-D}|^2\,\mathrm{d}x\mathrm{d}y\right). \tag{2.8}$$

Integrating by parts (2.7), we arrive at following estimates for $|x| > r$:

$$|w^{-D}(x,y)| \le \frac{C(n,m)\,\|u\|_{L_1(\Omega)}}{((|x|-r)^2+y^2)^{\frac{n+m-\sigma}{2}}}; \qquad |\nabla w^{-D}(x,y)| \le \frac{C(n,m)\,\|u\|_{L_1(\Omega)}}{((|x|-r)^2+y^2)^{\frac{n+m+1-\sigma}{2}}}. \tag{2.9}$$

Now we define, as in ([17], Thm. 2),

$$\widehat{w}(x,y) = w^{-D}(x,y) - \widehat{\phi}(x,y), \qquad x \in \overline{B}_R,\ y \ge 0,$$

where

$$-\Delta_x\widehat{\phi}(\cdot,y) = 0 \quad \text{in } B_R; \qquad \widehat{\phi}(\cdot,y) = w^{-D}(\cdot,y) \quad \text{on } \mathbb{S}_R.$$

Clearly, $\widehat{w}\big|_{x\in\mathbb{S}_R} = 0$. Arguing as for (1.4) and using (2.9) instead of (2.3), we obtain

$$\int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla\widehat{w}|^2\,\mathrm{d}x\mathrm{d}y \le \int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla w^{-D}|^2\,\mathrm{d}x\mathrm{d}y + \frac{C(n,m)\,R^n}{(R-r)^{2n+2m}}\cdot\|u\|^2_{L_1(\Omega)}. \tag{2.10}$$

We can use the "dual" Stinga–Torrea characterization of $Q^N_{-\sigma,\Omega}$. It was proved in [17] that

$$\begin{aligned} Q^N_{m,\Omega}[u] &= Q^N_{-\sigma,\Omega}[(-\Delta)^k u] \\ &= \frac{1}{c_2(n,\sigma)}\sup_{w|_{x\in\partial\Omega}=0}\left(\int\limits_\Omega(-\Delta)^k u(x)w(x,0)\,\mathrm{d}x - \int\limits_0^\infty\int\limits_\Omega y^{1-2\sigma}|\nabla w|^2\,\mathrm{d}x\mathrm{d}y\right). \end{aligned} \tag{2.11}$$

Relations (2.11), (2.10), (2.8) and the evident equality

$$\int\limits_{B_R}(-\Delta)^k u(x)\widehat{\phi}(x,0)\,\mathrm{d}x = 0\,,$$

give us

$$\begin{aligned} Q^N_{m,\Omega}[u] \ge Q^N_{m,B_R}[u] &\ge \frac{1}{c_2(n,\sigma)}\left(2\int\limits_{B_R}(-\Delta)^k u(x)\widehat{w}(x,0)\,\mathrm{d}x - \int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla\widehat{w}|^2\,\mathrm{d}x\mathrm{d}y\right) \\ &\ge \frac{1}{c_2(n,\sigma)}\left(2\int\limits_{B_R}(-\Delta)^k u(x)w^{-D}(x,0)\,\mathrm{d}x - \int\limits_0^\infty\int\limits_{B_R} y^{1-2\sigma}|\nabla w^{-D}|^2\,\mathrm{d}x\mathrm{d}y\right) \\ &\quad - \frac{C(n,m)\,R^n}{(R-r)^{2n+2m}}\cdot\|u\|^2_{L_1(\Omega)} = Q^D_{m,\Omega}[u] - \frac{C(n,m)\,R^n}{(R-r)^{2n+2m}}\cdot\|u\|^2_{L_1(\Omega)}, \end{aligned}$$

and (1.5) follows. The proof is complete. □

Remark 2.1. It can be seen from the proof that the estimates (1.3) and (1.4) are sharp in order of decay as $R\to\infty$.

3. The Brezis–Nirenberg effect for Navier fractional Laplacians

We recall the Sobolev and Hardy inequalities

$$Q_m[u] \geq \mathcal{S}_m \left(\int\limits_{\mathbb{R}^n} |u|^{2^*_m} \, \mathrm{d}x \right)^{2/2^*_m} \tag{3.1}$$

$$Q_m[u] \geq \mathcal{H}_m \int\limits_{\mathbb{R}^n} |x|^{-2m} |u|^2 \, \mathrm{d}x \,, \tag{3.2}$$

that hold for any $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $0 < m < \frac{n}{2}$. The *best Sobolev constant* $\mathcal{S}_m$ and the *best Hardy constant* $\mathcal{H}_m$ were explicitly computed in [8] (see also [6]), and in [12], respectively.

It is well known that $\mathcal{H}_m$ is not attained, that is, there are no functions with finite left and right-hand sides of (3.2) providing equality in (3.2). In contrast, it has been proved in [8] that $\mathcal{S}_m$ is attained by a unique family of functions, all of them being obtained from

$$\phi(x) = (1 + |x|^2)^{\frac{2m-n}{2}} \tag{3.3}$$

by translations, dilations in $\mathbb{R}^n$ and multiplication by constants.

A standard dilation argument implies that

$$\inf_{\substack{u \in \mathrm{Dom}(Q^D_{m,\Omega}) \\ u \neq 0}} \frac{Q^D_{m,\Omega}[u]}{\left(\int\limits_{\Omega} |u|^{2^*_m} \, \mathrm{d}x \right)^{2/2^*_m}} = \mathcal{S}_m .$$

The key fact used in further considerations is the equality

$$\inf_{\substack{u \in \mathrm{Dom}(Q^N_{m,\Omega}) \\ u \neq 0}} \frac{Q^N_{m,\Omega}[u]}{\left(\int\limits_{\Omega} |u|^{2^*_m} \, \mathrm{d}x \right)^{2/2^*_m}} = \mathcal{S}_m , \tag{3.4}$$

that has been established in [18] (see also earlier results [11,25] for $m = 2$, [10] for $m \in \mathbb{N}$ and [15] for $0 < m < 1$). Clearly, the Sobolev constant $\mathcal{S}_m$ is never achieved on $\mathrm{Dom}(Q^N_{m,\Omega})$.

The corresponding equality for the Hardy constant, that is,

$$\inf_{\substack{u \in \mathrm{Dom}(Q^N_{m,\Omega}) \\ u \neq 0}} \frac{Q^N_{m,\Omega}[u]}{\int\limits_{\Omega} |x|^{-2m} |u|^2 \, \mathrm{d}x} = \mathcal{H}_m \,, \tag{3.5}$$

was proved in [18] as well (see also [9,14] for $m \in \mathbb{N}$).

We point out that the infima

$$\varLambda_1(m,s) := \inf_{\substack{u \in \mathrm{Dom}(Q^N_{m,\Omega}) \\ u \neq 0}} \frac{Q^N_{m,\Omega}[u]}{Q^N_{s,\Omega}[u]}, \qquad \widetilde{\varLambda}_1(m,s) := \inf_{\substack{u \in \mathrm{Dom}(Q^N_{m,\Omega}[u]) \\ u \neq 0}} \frac{Q^N_{m,\Omega}[u]}{\int\limits_{\Omega} |x|^{-2s} |u|^2 \, \mathrm{d}x} \tag{3.6}$$

are positive and achieved. Since $\mathrm{Dom}(Q^N_{m,\Omega})$ is compactly embedded into $\mathrm{Dom}(Q^N_{s,\Omega})$, this fact is well known for $\varLambda_1(m,s)$ and follows from (3.5) for $\widetilde{\varLambda}_1(m,s)$.

Weak solutions to (1.6), (1.7) can be obtained as suitably normalized critical points of the functionals

$$\mathcal{R}^{\Omega}_{\lambda,m,s}[u] = \frac{Q^N_{m,\Omega}[u] - \lambda Q^N_{s,\Omega}[u]}{\left(\int\limits_{\Omega} |u|^{2^*_m}\,\mathrm{d}x\right)^{2/2^*_m}}, \tag{3.7}$$

$$\widetilde{\mathcal{R}}^{\Omega}_{\lambda,m,s}[u] = \frac{Q^N_{m,\Omega}[u] - \lambda \int\limits_{\Omega} |x|^{-2s}|u|^2\,\mathrm{d}x}{\left(\int\limits_{\Omega} |u|^{2^*_m}\,\mathrm{d}x\right)^{2/2^*_m}}, \tag{3.8}$$

respectively. It is easy to see that both functionals are well defined on $\mathrm{Dom}(Q^N_{m,\Omega}) \setminus \{0\}$.

In fact, we prove the existence of ground states for functionals (3.7) and (3.8). We introduce the quantities

$$\mathcal{S}^{\Omega}_{\lambda}(m,s) = \inf_{\substack{u\in \mathrm{Dom}(Q^N_{m,\Omega})\\ u\neq 0}} \mathcal{R}^{\Omega}_{\lambda,m,s}[u]; \qquad \widetilde{\mathcal{S}}^{\Omega}_{\lambda}(m,s) = \inf_{\substack{u\in \mathrm{Dom}(Q^N_{m,\Omega})\\ u\neq 0}} \widetilde{\mathcal{R}}^{\Omega}_{\lambda,m,s}[u].$$

By standard arguments we have $\mathcal{S}^{\Omega}_{\lambda}(m,s) \le \mathcal{S}_m$, argue for instance as in ([16], Lem. 4.1). In addition, if $\lambda \le 0$ then $\mathcal{S}^{\Omega}_{\lambda}(m,s) = \mathcal{S}_m$ and it is not achieved. Similar statements hold for $\widetilde{\mathcal{S}}^{\Omega}_{\lambda}(m,s)$.

We are in position to prove our existence result that includes Theorem 1.3 in the introduction.

Theorem 3.1. *Assume $s \ge 2m - \frac{n}{2}$.*

i) *For any $0 < \lambda < \Lambda_1(m,s)$ the infimum $\mathcal{S}^{\Omega}_{\lambda}(m,s)$ is achieved and (1.6) has a nontrivial solution in $\mathrm{Dom}(Q^N_{m,\Omega})$.*
ii) *For any $0 < \lambda < \widetilde{\Lambda}_1(m,s)$ the infimum $\widetilde{\mathcal{S}}^{\Omega}_{\lambda}(m,s)$ is achieved and (1.7) has a nontrivial solution in $\mathrm{Dom}(Q^N_{m,\Omega})$.*

Proof. We prove i), the proof of the second statement is similar. Using the relation (3.4) and arguing for instance as in ([16], Lem. 4.1) one has that if $0 < \mathcal{S}^{\Omega}_{\lambda}(m,s) < \mathcal{S}_m$, then $\mathcal{S}^{\Omega}_{\lambda}(m,s)$ is achieved.

Since $0 < \lambda < \Lambda_1(m,s)$, then $\mathcal{S}^{\Omega}_{\lambda}(m,s) > 0$ by (3.6).

To obtain the strict inequality $\mathcal{S}^{\Omega}_{\lambda}(m,s) < \mathcal{S}_m$ we follow [3], and we take advantage of the computations in [16].

Let ϕ be the extremal of the Sobolev inequality (3.1) given by (3.3). In particular,

$$M := Q_m[\phi] = \mathcal{S}_m \left(\int\limits_{\mathbb{R}^n} |\phi|^{2^*_m}\,\mathrm{d}x\right)^{2/2^*_m}. \tag{3.9}$$

Fix a cutoff function $\varphi \in \mathcal{C}^\infty_0(\Omega)$, such that $\varphi \equiv 1$ on the ball $\{|x| < \delta\}$ and $\varphi \equiv 0$ outside the ball $\{|x| < 2\delta\}$.

If $\varepsilon > 0$ is small enough, the function

$$u_\varepsilon(x) := \varepsilon^{2m-n}\varphi(x)\phi\left(\frac{x}{\varepsilon}\right) = \varphi(x)\left(\varepsilon^2 + |x|^2\right)^{\frac{2m-n}{2}}$$

has compact support in Ω.

From ([16], Lem. 3.1) we conclude

$$\begin{aligned}
\mathfrak{A}_m^\varepsilon &:= Q^D_{m,\Omega}[u_\varepsilon] && \le \varepsilon^{2m-n}\left(M + C(\delta)\,\varepsilon^{n-2m}\right)\\
\mathcal{A}_s^\varepsilon &:= \int\limits_\Omega |x|^{-2s}|u_\varepsilon|^2\,\mathrm{d}x && \ge \begin{cases} C(\delta)\,\varepsilon^{4m-n-2s} & \text{if} \quad s > 2m-\frac{n}{2} \\ C(\delta)\,|\log\varepsilon| & \text{if} \quad s = 2m-\frac{n}{2}\end{cases}\\
\widetilde{\mathfrak{A}}_s^\varepsilon &:= Q^N_{s,\Omega}[u_\varepsilon] && \ge \mathcal{H}_s\,\mathcal{A}_s^\varepsilon \qquad [\text{ see (3.5) }]\\
\mathcal{B}^\varepsilon &:= \int\limits_\Omega |u_\varepsilon|^{2^*_m}\,\mathrm{d}x && \ge \varepsilon^{-n}\left((M\mathcal{S}_m^{-1})^{2^*_m/2} - C(\delta)\,\varepsilon^n\right).
\end{aligned}$$

If m is an integer or if $\lfloor m\rfloor$ is odd, then by (1.3)

$$\widetilde{\mathfrak{A}}_m^\varepsilon := Q^N_{m,\Omega}[u_\varepsilon] \le \mathfrak{A}_m^\varepsilon,$$

and we obtain

$$\mathcal{R}^\Omega_{\lambda,m,s}[u_\varepsilon] \le \mathcal{S}_m\,\frac{1 + C(\delta)\,\varepsilon^{n-2m} - \lambda C(\delta)\,\varepsilon^{2m-2s}}{1 - C(\delta)\,\varepsilon^n}, \qquad \text{if} \quad s > 2m - \frac{n}{2} \tag{3.10}$$

$$\mathcal{R}^\Omega_{\lambda,m,s}[u_\varepsilon] \le \mathcal{S}_m\,\frac{1 + C(\delta)\,\varepsilon^{n-2m} - \lambda C(\delta)\,\varepsilon^{n-2m}|\log\varepsilon|}{1 - C(\delta)\,\varepsilon^n}, \quad \text{if} \;\; s = 2m - \frac{n}{2}\cdot \tag{3.11}$$

Thus, for any sufficiently small $\varepsilon > 0$ we have that $\mathcal{R}^\Omega_{\lambda,m,s}[u_\varepsilon] < \mathcal{S}_m$, and the statement follows.

It remains to consider the case when $\lfloor m\rfloor$ is even. Since $\|u_\varepsilon\|_{L_1(\Omega)} \le C(\delta)$, the estimate (1.4) implies

$$\widetilde{\mathfrak{A}}_m^\varepsilon \le \mathfrak{A}_m^\varepsilon + C(\delta) = \varepsilon^{2m-n}\left(M + C(\delta)\,\varepsilon^{n-2m}\right),$$

and we again arrive at (3.10), (3.11). □

For the case $s < 2m - \frac{n}{2}$ we limit ourselves to point out the next simple existence result. Its standard proof can be obtained as for Theorem 4.3 in [16]. We omit details.

Theorem 3.2. *Assume $s < 2m - \frac{n}{2}$.*

(i) *There exists $\lambda^* \in [0, \Lambda_1(m,s))$ such that for any $\lambda \in (\lambda^*, \Lambda_1(m,s))$ the infimum $\mathcal{S}^\Omega_\lambda(m,s)$ is attained, and hence (1.6) has a nontrivial solution.*

(ii) *There exists $\widetilde{\lambda}^* \in [0, \widetilde{\Lambda}_1(m,s))$ such that for any $\lambda \in (\widetilde{\lambda}^*, \widetilde{\Lambda}_1(m,s))$ the infimum $\widetilde{\mathcal{S}}^\Omega_\lambda(m,s)$ is attained, and hence (1.7) has a nontrivial solution.*

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References

[1] B. Barrios, E. Colorado, A. de Pablo and U. Sánchez, On some critical problems for the fractional Laplacian operator. *J. Differ. Equ.* **252** (2012) 6133–6162.

[2] M. Bonforte, Y. Sire and J.L. Vazquez, Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains. Preprint arXiv:1404.6195 (2014).

[3] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Commun. Pure Appl. Math.* **36** (1983) 437–477.

[4] X. Cabré and Y. Sire, Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **31** (2014) 23–53.

[5] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian. *Commun. Part. Differ. Equ.* **32** (2007) 1245–1260.

[6] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation. *Commun. Pure Appl. Math.* **59** (2006) 330–343.

[7] E. Colorado, A. de Pablo and U. Sánchez, Perturbations of a critical fractional equation. *Pacific J. Math.* **271** (2014) 65–84.

[8] A. Cotsiolis and N.K. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives. *J. Math. Anal. Appl.* **295** (2004) 225–236.

[9] F. Gazzola, H.-C. Grunau and E. Mitidieri, Hardy inequalities with optimal constants and remainder terms. *Trans. Amer. Math. Soc.* **356** (2004) 2149–2168.

[10] F. Gazzola, H.-C. Grunau and G. Sweers, Polyharmonic Boundary Value Problems. Vol. 1991 of *Lect. Notes Math.* Springer, Berlin (2010).

[11] Y. Ge, Sharp Sobolev inequalities in critical dimensions. *Michigan Math. J.* **51** (2003) 27–45.

[12] I.W. Herbst, Spectral theory of the operator $(p^2+m^2)^{1/2} - Ze^2/r$. *Commun. Math. Phys.* **53** (1977) 285–294.

[13] J.-L. Lions and E. Magenes, Non-homogeneous Boundary Value Problems and Applications. Vol. I, translated from the French by P. Kenneth. Springer, New York (1972).

[14] E. Mitidieri, A simple approach to Hardy inequalities. *Mat. Zametki* **67** (2000) 563–572 (in Russian). *English transl.: Math. Notes* **67** (2000) 479–486.

[15] R. Musina and A.I. Nazarov, On fractional Laplacians. *Commun. Partial Differ. Equ.* **39** (2014) 1780–1790.

[16] R. Musina and A.I. Nazarov, Non-critical dimensions for critical problems involving fractional Laplacians. *Rev. Mat. Iberoamer.* **32** (2016) 257–266.

[17] R. Musina and A.I. Nazarov, On fractional Laplacians − 2. Preprint arXiv:1408.3568 (2014).

[18] R. Musina and A.I. Nazarov, On the Sobolev and Hardy constants for the fractional Navier Laplacian. *Nonlin. Anal.* **121** (2015) 123–129.

[19] P. Pucci and J. Serrin, Critical exponents and critical dimensions for polyharmonic operators. *J. Math. Pures Appl.* (9) **69** (1990) 55–83.

[20] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian. *Trans. Amer. Math. Soc.* **367** (2015) 67–102.

[21] R. Servadei and E. Valdinoci, A Brezis-Nirenberg result for non-local critical equations in low dimension. *Commun. Pure Appl. Anal.* **12** (2013) 2445–2464.

[22] P.R. Stinga and J.L. Torrea, Extension problem and Harnack's inequality for some fractional operators. *Commun. Partial Differ. Equ.* **35** (2010) 2092–2122.

[23] J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian. *Calc. Var. Partial Differ. Equ.* **42** (2011) 21–41.

[24] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators. Deutscher Verlag Wissensch. Berlin (1978).

[25] R.C.A.M. Van der Vorst, Best constant for the embedding of the space $H^2 \cap H_0^1(\Omega)$ into $L^{2N/(N-4)}(\Omega)$. *Differ. Integral Equ.* **6** (1993) 259–276.

[26] D.R. Yafaev, On the theory of the discrete spectrum of the three-particle Schrödinger operator. *Mat. Sbornik* **94(136)** (1974) 567–593 (Russian); English transl.: *Math. USSR Sbornik* **23** (1974) 535–559.

[27] D.R. Yafaev, Mathematical Scattering Theory: Analytic Theory, Vol. 158 of *Math. Surv. Monogr.* AMS (2010).