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ON BOUNDARY CONTROL OF A HYPERBOLIC SYSTEM WITH A VANISHING CHARACTERISTIC SPEED

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Abstract. Despite of the fact that distributed (internal) controls are usually used to obtain controllability for a hyperbolic system with vanishing characteristic speeds, this paper is, however, devoted to study the case where only boundary controls are considered. We first prove that the system is not (null) controllable in finite time. Next, we give a sufficient and necessary condition for the asymptotic stabilization of the system under a natural feedback.

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1. Introduction and main results

There are many publications concerning the exact boundary controllability for linear and nonlinear hyperbolic systems, including wave equations, Saint-Venant equations, Euler equations, *etc.* (see [5, 14, 20, 25] and the references therein).

As for the general first order hyperbolic systems, one can refer to the monograph of Li [14] for almost complete results on the exact controllability of the following 1-D quasilinear hyperbolic system (including many conservation laws and balance laws)

$$U_t + A(U)U_x = B(U) \tag{1.1}$$

in the context of classical solution. It is well known that if all the characteristic speeds of the system do not vanish, *i.e.*, all the eigenvalues of $A(U)$ are nonzero in the domain under consideration, then (1.1) is exactly controllable (at least locally in C^1 class) by boundary controls provided that the control time is sufficiently large. (see Li and Rao [15, 16]; see also Wang [26] for the nonautonomous case).

On the other hand, the controllability problems have been studied also in the context of weak entropy solutions for hyperbolic conservation laws since the early result on attainable set by Ancona and Marson [1]. In the scalar case, one can refer to Horsin [10] for boundary controllability result of Burgers equation and to Perrollaz [23] when an additional distributed control appears in the right hand side. In the case of systems, the first result on

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controllability was due to Bressan and Coclite [4]. For general strictly hyperbolic systems of conservation laws with genuinely nonlinear or linearly degenerate and non-vanishing characteristic fields, it is shown in [4] that a small BV initial state can be driven to a constant state by an open-loop control asymptotically in time (see also [2] for one-sided control). Moreover, a negative result for a class of 2×2 hyperbolic systems was also proved in [4] that an initial data with a dense distribution of shocks can not be driven to constant by using boundary control. Different from this negative result, Glass obtained the boundary controllability of 1-D isentropic Euler equation [7]. We emphasize that the above study all assume that the characteristic speeds of the system do not vanish.

A widely open question naturally arises that whether it is possible to establish exact controllability for a hyperbolic system with vanishing characteristic speeds. The situation seems different. Notice that the solution $u(t,x)$ of the degenerate case of the equation (essentially an ordinary differential equation)

$$\begin{cases} u_t = 0, \quad (t,x) \in (0,T)\times(0,2\pi), \\ u(0,x) = u_0(x), \quad x\in(0,2\pi), \end{cases} \tag{1.2}$$

obeys

$$u(t,\cdot) \equiv u_0, \quad \forall t\in[0,T]. \tag{1.3}$$

Obviously, it is impossible to change the value u by using the boundary controls acting on the end $x=0$ and/or $x=2\pi$.

There are several directions that have been attempted to answer the above open question.

One direction is to add some internal controls. For systems with identically zero characteristic speeds, a general result on exact controllability has been obtained by combining internal controls acted on the components corresponding to the zero eigenvalues and boundary controls acting on the other components [19,26]. There the internal controls depend on both variables (t,x) and are globally distributed. Later on, an exact controllability result, with some constrains concerning the compatibility between the initial and final data, was established by using only the internal controls acted on the components corresponding to nonzero eigenvalues [17]. Again the internal controls are globally distributed. It is worthy of mentioning that exact controllability for a simplified model is also realized in [17] by switching controls where the internal control is locally distributed.

Another direction is to loose the requirements of exact controllability if one considers only the boundary controls because internal controls are usually not applicable for some physical reasons. It is possible to obtain *partial* exact controllability by boundary controls, if one aims to control only the values of the components corresponding to the nonzero eigenvalues [27,28].

While the most interesting problem is whether one could establish exact controllability for a hyperbolic system with zero characteristic speeds by *only* boundary controls. In this direction, Gugat [9] proved a global exact boundary controllability result which covers the critical case in which one of the characteristic speed is zero. In spite of the degenerate model (1.2), it is still possible to prove the controllability for some nonlinear system. The return method of Coron has been applied to the many situations in [5], where the linearized system is not controllable while the nonlinearity enables the corresponding nonlinear system to be controllable. One can apply the return method to realize exact boundary controllability for quasilinear hyperbolic systems with a zero characteristic speed [6], when the possible vanishing characteristic speed can be driven nonzero after sufficiently long time. However, this approach seems to be no longer valid for the system with identically zero characteristic speeds (see [6], Rem. 1.3).

In particular, it is worthy of mentioning that Glass recently obtained the exact boundary controllability for 1-D non-isentropic Euler equation in both Eulerian and Lagrangian coordinates in the context of weak entropy solution [8]. It seems surprising because the corresponding controllability in C^1 framework is false. Actually, the physical entropy S (in Lagrangian coordinates) obeys $\partial_t S = 0$, thus S can not be changed by the boundary controls. The reason is that the systems are not equivalent in the context of weak entropy solution even under invertible transformation of the unknown functions. In the case of [8], the boundary controllability could be obtained through the interactions of the other characteristic families to act indirectly on the vanishing (vertical) characteristic family.

The purpose of this paper is to understand how the coupling in the equations affects the boundary controllability of a hyperbolic system when it possesses a vanishing characteristic speed. More precisely, we consider the following linear hyperbolic system:

$$\begin{cases} u_t = \alpha u_x + \beta v, \\ v_t = \gamma u, \end{cases} \tag{1.4}$$

where $\alpha \neq 0, \beta, \gamma \in \mathbb{R}$ are constants. Without loss of generality, we may assume that $\alpha < 0$ in the sequel. Actually, if $\alpha > 0$, by introducing new unknown variable $(\tilde{u}(t,x), \tilde{v}(t,x)) = (u(t, 2\pi - x), v(t, 2\pi - x))$, then $(\tilde{u}, \tilde{v})$ satisfies the same type of system as (1.4) with α being replaced by $-\alpha$. The results in case of $\alpha > 0$ are easy consequence of that in case of $\alpha < 0$, see Remark 1.7.

The problems that we are interested in are controllability and asymptotic stabilization problems under only boundary controls, and how the control properties are determined by the coefficients α, β, γ.

More precisely, we first study the controllability problem for the system (1.4) under the boundary control $h \in L^2(0,T)$ for some $T > 0$:

$$\begin{cases} u_t = \alpha u_x + \beta v, \quad (t,x) \in (0,T) \times (0, 2\pi), \\ v_t = \gamma u, \quad (t,x) \in (0,T) \times (0, 2\pi), \\ u(t,0) - u(t, 2\pi) = h(t), \quad t \in (0,T). \end{cases} \tag{1.5}$$

Definition 1.1. Let $T > 0$, the system (1.5) is exactly controllable if: for any (u_0, v_0), $(u_1, v_1) \in (L^2(0,2\pi))^2$, there exists a function $h \in L^2(0,T)$ such that the solution of the mixed initial-boundary problem (1.5) with the following initial data

$$(u(0,x), v(0,x)) = (u_0(x), v_0(x)), \quad x \in (0, 2\pi) \tag{1.6}$$

satisfies the finial condition:

$$(u(T,x), v(T,x)) = (u_1, v_1), \quad x \in (0, 2\pi). \tag{1.7}$$

The system (1.5) is exactly null controllable if: for any $(u_0, v_0) \in (L^2(0,2\pi))^2$, there exists a function $h \in L^2(0,T)$ such that the solution of (1.5) and (1.6) satisfies

$$(u(T,x), v(T,x)) = (0,0), \quad x \in (0, 2\pi). \tag{1.8}$$

The system (1.5) is asymptotically null controllable if it is null controllable for $T = +\infty$ in the sense that

$$\lim_{t \to +\infty} (u(t,\cdot), v(t,\cdot)) = (0,0) \quad \text{in } (L^2(0,2\pi))^2. \tag{1.9}$$

Remark 1.2. The system (1.5) is exactly controllable if and only if it is exactly null controllable because the system (1.5) is linear and time revertible.

We also consider the asymptotic stabilization problem for system (1.4) under a feedback law

$$\begin{cases} u_t = \alpha u_x + \beta v, \quad (t,x) \in (0,+\infty) \times (0, 2\pi), \\ v_t = \gamma u, \quad (t,x) \in (0,+\infty) \times (0, 2\pi), \\ u(t,0) = k u(t, 2\pi), \quad t \in (0,+\infty). \end{cases} \tag{1.10}$$

Definition 1.3. The system (1.5) is asymptotically stabilizable if: there exists $k \in \mathbb{R}$ such that for any $(u_0, v_0) \in (L^2(0,2\pi))^2$, the solution of the closed-loop system (1.10) with the initial data (1.6) is asymptotically stable, *i.e.*, (1.9) holds.

The main results that we obtain in this paper are the following theorems.

Theorem 1.4. *For any given $0 < T < +\infty$, the system (1.5) is not exactly (null) controllable.*

Theorem 1.5. *The system (1.10) is asymptotically stabilizable if and only if $\beta\gamma < 0$.*

Theorem 1.6. *If $\beta\gamma < 0$, the system (1.5) is asymptotically null controllable.*

Remark 1.7. Theorems 1.4–1.6 still hold in case of $\alpha > 0$.

Remark 1.8. If $\beta\gamma < 0$, the system (1.5) is a nontrivial example which is not exactly (null) controllable in finite time but is asymptotically stabilizable (and is asymptotically null controllable).

Inspired by [21] (for BBM equation) and [24] (for wave equation), the proof of Theorem 1.4 relies on moment theory and Paley−Wiener theorem [12]. The key observation is that the spectrum of the adjoint system accumulates at a finite point in $\mathbb{C}$. To our best knowledge, it has not been noticed before for first order hyperbolic systems. The adjoint observability problem by duality is also discussed in Remark 3.1. The proof of Theorem 1.5 is based on Riesz basis method and a careful analysis of the spectrum, while the discussion about Lyapunov functionals are provided in Remark 4.3. Finally, Theorem 1.6 follows as a consequence of Theorem 1.5.

The rest of this paper is organized as follows. In section 2, we state the well-posedness of both the open-loop system (1.5) and the closed-loop system (1.10) without proof. The proofs of the Theorem 1.4, Theorems 1.5 and 1.6 are given in Sections 3, 4 and 5, respectively.

2. Well-posedness of the system (1.5) and the system (1.10)

In this section, we give the definition and well-posedness of solution to the system (1.5) and the system (1.10). The well-posedness issue is fundamental to the control problems. Here we only present the results without proof, which can be derived by classical methods, such as characteristic method [18] or by theory of semigroup of linear operators [22].

Definition 2.1. Let $0 < T \leq +\infty$, $h \in L^2(0,T)$ and $(u_0, v_0) \in (L^2(0,2\pi))^2$ be given. A solution of the initial-boundary value problem (1.5), (1.6) is a function $(u,v) \in C^0([0,T];(L^2(0,2\pi))^2)$, such that, for every $\tau \in [0,T]$ (if $T = +\infty, \tau \in [0,+\infty)$) and for every $(\varphi_1, \varphi_2) \in (C^1([0,\tau]\times[0,2\pi]))^2$ such that $\varphi_1(t,0) = \varphi_1(t,2\pi)$, $\forall t \in [0,\tau]$, one has

$$-\int_0^\tau \int_0^{2\pi} [(\varphi_{1t} - \alpha\varphi_{1x})u + \beta v\varphi_1]\,\mathrm{d}x\mathrm{d}t + \alpha\int_0^\tau h(t)\varphi_1(t,2\pi)\mathrm{d}t + \int_0^{2\pi} \varphi_1(\tau,x)u(\tau,x)\mathrm{d}x - \int_0^{2\pi} \varphi_1(0,x)u_0(x)\mathrm{d}x = 0 \tag{2.1}$$

and

$$-\int_0^\tau \int_0^{2\pi} [\varphi_{2t}v + \gamma u\varphi_2]\,\mathrm{d}x\mathrm{d}t + \int_0^{2\pi} \varphi_2(\tau,x)v(\tau,x)\mathrm{d}x - \int_0^{2\pi} \varphi_2(0,x)v_0(x)\mathrm{d}x = 0. \tag{2.2}$$

Lemma 2.2. *Let $T > 0$, $(u_0, v_0) \in (L^2(0,2\pi))^2$ and $h \in L^2(0,T)$ be given. Then the initial-boundary value problem (1.5) with (1.6) has a unique solution, which depends continuously on (u_0, v_0, h) with the following estimate*

$$\|(u(t,\cdot),v(t,\cdot))\|_{(L^2(0,2\pi))^2} \leq C(\|(u_0,v_0)\|_{(L^2(0,2\pi))^2} + \|h\|_{L^2(0,T)}), \quad \forall t \in [0,T] \tag{2.3}$$

for some constant $C > 0$.

Definition 2.3. Let $k \in \mathbb{R}$ and $(u_0, v_0) \in (L^2(0,2\pi))^2$ be given. A solution of the initial-boundary value problem (1.10) and (1.6) is a function $(u,v) \in C^0([0,+\infty);(L^2(0,2\pi))^2)$, such that, for every $\tau \in [0,\infty)$ and for every $(\varphi_1,\varphi_2) \in (C^1([0,\tau]\times[0,2\pi]))^2$ such that $k\varphi_1(t,0) = \varphi_1(t,2\pi)$, $\forall t \in [0,\tau]$, one has

$$-\int_0^\tau \int_0^{2\pi} [(\varphi_{1t} - \alpha\varphi_{1x})u + \beta v\varphi_1]\,\mathrm{d}x\mathrm{d}t + \int_0^{2\pi} \varphi_1(\tau,x)u(\tau,x)\mathrm{d}x - \int_0^{2\pi} \varphi_1(0,x)u_0(x)\mathrm{d}x = 0,$$

and

$$-\int_0^\tau\int_0^{2\pi}[\varphi_{2t}v+\gamma u\varphi_2]\,\mathrm{d}x\mathrm{d}t+\int_0^{2\pi}\varphi_2(\tau,x)v(\tau,x)\mathrm{d}x-\int_0^{2\pi}\varphi_2(0,x)v_0(x)\mathrm{d}x=0.$$

Lemma 2.4. *Let $k\in\mathbb{R}$ and $(u_0,v_0)\in(L^2(0,2\pi))^2$ be given. Then the initial-boundary value problem (1.10) with (1.6) has a unique solution, which depends continuously on (u_0,v_0) with the following estimate*

$$\|(u(t,\cdot),v(t,\cdot))\|_{(L^2(0,2\pi))^2}\le C\mathrm{e}^{\omega t}\|(u_0,v_0)\|_{(L^2(0,2\pi))^2},\quad \forall t\in[0,+\infty) \tag{2.4}$$

for some constants $C>0$ and $\omega\in\mathbb{R}$.

3. Proof of Theorem 1.4

We divide the proof of Theorem 1.4 into two cases: $\beta\gamma=0$ and $\beta\gamma\neq0$.

Case 1: $\boldsymbol{\beta\gamma=0}$. In Case 1, the result is trivial in the sense that (at least) one of the two equations in (1.5) is decoupled from the other one.

Case 1.1: $\boldsymbol{\gamma=0}$. Since $\gamma=0$, the second equation in (1.5) becomes

$$v_t=0,\quad (t,x)\in(0,T)\times(0,2\pi), \tag{3.1}$$

which implies that $v(t,\cdot)=v_0(\cdot)\in L^2(0,2\pi)$ for all $t\in[0,T]$. Therefore, it is impossible to drive v from $v_0\neq0\in L^2(0,2\pi)$ to zero at time T no matter what the control h is.

Case 1.2: $\boldsymbol{\beta=0}$ and $\boldsymbol{\gamma\neq0}$. Since $\beta=0$, the first equation in (1.5) becomes

$$u_t=\alpha u_x, \tag{3.2}$$

which does not depend on v. Suppose that the system (1.5) is (null) controllable. Let

$$(u_0(x),v_0(x))\equiv(1,0),\quad x\in(0,2\pi), \tag{3.3}$$

then there exists $h\in L^2(0,2\pi)$ driving (u,v) from (u_0,v_0) to $(0,0)$ at time T. Multiplying (3.2) by $\mathrm{e}^{\mathrm{i}nx}$ $(n\in\mathbb{Z})$ and integrating on $(0,T)\times(0,2\pi)$, we get from $\gamma\neq0$ and (1.5), (1.8) and (3.3) that

$$\begin{aligned}0&=\int_0^T\int_0^{2\pi}(u_t-\alpha u_x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x\mathrm{d}t\\&=\int_0^{2\pi}\Big[u\mathrm{e}^{\mathrm{i}nx}\Big]_{t=0}^{t=T}\mathrm{d}x-\alpha\int_0^T\Big[u\mathrm{e}^{\mathrm{i}nx}\Big]_{x=0}^{x=2\pi}\mathrm{d}t+\mathrm{i}n\alpha\int_0^T\int_0^{2\pi}u\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x\mathrm{d}t\\&=-\int_0^{2\pi}u_0(x)\mathrm{e}^{\mathrm{i}nx}\mathrm{d}x+\alpha\int_0^T h(t)\,\mathrm{d}t+\frac{\mathrm{i}n\alpha}{\gamma}\int_0^{2\pi}\Big[v\mathrm{e}^{\mathrm{i}nx}\Big]_{t=0}^{t=T}\mathrm{d}x\mathrm{d}t\\&=-\int_0^{2\pi}\mathrm{e}^{\mathrm{i}nx}\mathrm{d}x+\alpha\int_0^T h(t)\,\mathrm{d}t,\quad\forall n\in\mathbb{Z}.\end{aligned} \tag{3.4}$$

Therefore $\int_0^T h(t)\mathrm{d}t=0$ as $n\neq0$, while $\int_0^T h(t)\mathrm{d}t=\frac{2\pi}{\alpha}\neq0$ as $n=0$. This contradiction concludes the proof of Theorem 1.4 for the Case 1.2. $\beta\gamma=0$.

Case 2. $\boldsymbol{\beta\gamma\neq0}$. In this case, we will prove Theorem 1.4 by moment theory and contradiction arguments. The proof is inspired by [21,24].

Let us assume that there exists $h\in L^2(0,2\pi)$ such that the solution (u,v) of (1.5) with (1.6) satisfies (1.8). We introduce the adjoint system of the system (1.5):

$$\begin{cases}p_t=\alpha p_x-\gamma q,\\ q_t=-\beta p,\\ p(t,0)=p(t,2\pi).\end{cases} \tag{3.5}$$

Then for the above solution (u, v) of the system (1.5) satisfying (1.8), it holds, by duality, that for every solution (p, q) of the adjoint system (3.5),

$$
\begin{aligned}
\int_0^{2\pi} u_0(x)p(0,x) + v_0(x)q(0,x)\mathrm{d}x &= -\left[\left\langle \binom{u}{v}, \binom{p}{q} \right\rangle_{L^2(0,2\pi)}\right]_{t=0}^{t=T} \\
&= -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \binom{u}{v}, \binom{p}{q} \right\rangle_{L^2(0,2\pi)} \mathrm{d}t \\
&= -\int_0^T \left\langle \binom{u_t}{v_t}, \binom{p}{q} \right\rangle_{L^2(0,2\pi)} + \left\langle \binom{u}{v}, \binom{p_t}{q_t} \right\rangle_{L^2(0,2\pi)} \mathrm{d}t \\
&= -\int_0^T \Big[\alpha u(t,x)p(t,x)\Big]_{x=0}^{x=2\pi} \mathrm{d}t \\
&= \alpha \int_0^T h(t)p(t,0)\mathrm{d}t.
\end{aligned} \tag{3.6}
$$

Next we look for the solutions of the adjoint system (3.5). Clearly, the corresponding eigenvalue λ and the eigenvector (ξ, η) obey

$$
\begin{cases}
\lambda\xi = \alpha\xi' - \gamma\eta \\
\lambda\eta = -\beta\xi \\
\xi(0) = \xi(2\pi).
\end{cases} \tag{3.7}
$$

By canceling η, (3.7) is reduced to

$$
\alpha\lambda\xi'(x) = (\lambda^2 - \beta\gamma)\xi(x) \quad \text{with} \;\; \xi(0) = \xi(2\pi). \tag{3.8}
$$

It is easy to see that $\lambda = 0$ happens only when $\xi(x) = \eta(x) \equiv 0$ since $\beta\gamma \neq 0$. Hence (3.7) possesses only nonzero eigenvalues.

Since $\alpha < 0$, the solution of (3.8) is given by

$$
\xi(x) = \mathrm{e}^{\frac{\lambda^2 - \beta\gamma}{\lambda\alpha}x}\xi(0), \tag{3.9}
$$

where the eigenvalue λ is determined by

$$
\mathrm{e}^{\frac{\lambda^2 - \beta\gamma}{\lambda\alpha}2\pi} = 1, \tag{3.10}
$$

i.e.

$$
\frac{\lambda^2 - \beta\gamma}{\lambda\alpha} = \mathrm{i}n, \quad n \in \mathbb{Z}. \tag{3.11}
$$

Therefore, we obtain the eigenvalues of the equations (3.7):

$$
\lambda_n^{\pm} = \begin{cases}
\dfrac{\mathrm{i}n\alpha \pm \sqrt{4\beta\gamma - n^2\alpha^2}}{2}, & \text{if } n^2 < \dfrac{4\beta\gamma}{\alpha^2}, n \in \mathbb{Z} \\
\dfrac{\mathrm{i}n\alpha \pm \mathrm{i}\sqrt{n^2\alpha^2 - 4\beta\gamma}}{2}, & \text{if } n^2 \geq \dfrac{4\beta\gamma}{\alpha^2}, n \in \mathbb{Z}
\end{cases} \tag{3.12}
$$

and the corresponding eigenvectors:

$$
(\xi_n^{\pm}(x), \eta_n^{\pm}(x)) = \mathrm{e}^{\mathrm{i}nx}\left(1, \frac{-\beta}{\lambda_n^{\pm}}\right), \quad n \in \mathbb{Z}. \tag{3.13}
$$

Obviously for all $n\in\mathbb{Z}$, $(p_n^{\pm}(t,x),q_n^{\pm}(t,x)) = \mathrm{e}^{\lambda_n^{\pm}t}(\xi_n^{\pm}(x),\eta_n^{\pm}(x)) = \mathrm{e}^{\lambda_n^{\pm}t+\mathrm{i}nx}(1,\frac{-\beta}{\lambda_n^{\pm}})$ satisfies (3.5). Substituting $(p_n^{\pm}(t,x),q_n^{\pm}(t,x))$ into (3.6) yields that the control $h\in L^2(0,T)$ driving (u_0,v_0) to $(0,0)$ satisfies that

$$\int_0^T h(t)\mathrm{e}^{\lambda_n^{\pm}t}\mathrm{d}t = \frac{1}{\alpha\lambda_n^{\pm}}\int_0^{2\pi}(\lambda_n^{\pm}u_0(x)-\beta v_0(x))\mathrm{e}^{\mathrm{i}nx}\mathrm{d}x,\quad \forall n\in\mathbb{Z}. \tag{3.14}$$

Let

$$(u_0(x),v_0(x))\equiv(c_1,c_2),\quad x\in(0,2\pi), \tag{3.15}$$

where $c_1,c_2\in\mathbb{R}$ are two constants such that

$$\lambda_0^{\pm}c_1-\beta c_2\neq 0. \tag{3.16}$$

Thus

$$\int_0^T h(t)\mathrm{e}^{\lambda_n^{\pm}t}\mathrm{d}t = \frac{\lambda_n^{\pm}c_1-\beta c_2}{\alpha\lambda_n^{\pm}}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}nx}\mathrm{d}x = 0,\quad \forall n\in\mathbb{Z}\backslash\{0\}, \tag{3.17}$$

$$\int_0^T h(t)\mathrm{e}^{\lambda_0^{\pm}t}\mathrm{d}t = \frac{2\pi(\lambda_0^{\pm}c_1-\beta c_2)}{\alpha\lambda_0^{\pm}}\neq 0. \tag{3.18}$$

Let

$$F(z)=\int_0^T h(t)\mathrm{e}^{-\mathrm{i}zt}\mathrm{d}t,\quad z\in\mathbb{C}. \tag{3.19}$$

Then it follows that $F(\mathrm{i}\lambda_0^{\pm})\neq 0$ and $F(\mathrm{i}\lambda_n^{\pm})=0$ for all $n\in\mathbb{Z}\backslash\{0\}$. On the other hand, F is an entire function on $\mathbb{C}$ due to Paley$-$Wiener theorem [12]. It is easy to get from $\alpha<0$ and (3.12) that

$$\lim_{n\to+\infty}\lambda_n^{+}=0. \tag{3.20}$$

Therefore F is an entire function vanishing on the set $\{z=\mathrm{i}\lambda_n^{\pm}|n\in\mathbb{Z}\backslash\{0\}\}$ which has an accumulation point $z=0\in\mathbb{C}$. This simply implies $F\equiv 0$, however, this contradicts with $F(\mathrm{i}\lambda_0^{+})\neq 0$. Consequently, there exists no function $h\in L^2(0,T)$ such that (3.14) holds for the initial data given by (3.15). This concludes the proof of Theorem 1.4 for the Case 2. $\beta\gamma\neq 0$. □

Remark 3.1. The key observation is that the spectrum of the adjoint system (3.5) accumulates at a finite point in $\mathbb{C}$. Thanks to the fact of (3.20), it is easy to find a sequence of nontrivial solution $\{(p_n,q_n)\}_{n=1}^{+\infty}$ to (3.5) which satisfies that $\|(p_n(0,\cdot),q_n(0,\cdot))\|_{(L^2(0,2\pi))^2}\geq c_0$ for some constant $c_0>0$ and $\|p_n(\cdot,0)\|_{L^2(0,T)}\to 0$ as $n\to+\infty$. For instance, one can take

$$(p_n(t,x),q_n(t,x)) = \mathrm{e}^{\lambda_n^{+}t+\mathrm{i}nx}(\lambda_n^{+},-\beta)-\mathrm{e}^{\lambda_{n+1}^{+}t+\mathrm{i}(n+1)x}(\lambda_{n+1}^{+},-\beta), \tag{3.21}$$

which gives $\|(p_n(0,\cdot),q_n(0,\cdot))\|^2_{(L^2(0,2\pi))^2}\geq 4\pi\beta^2>0$ and $\|p_n(\cdot,0)\|^2_{L^2(0,T)}=\int_0^T|\mathrm{e}^{\lambda_n^{+}t}\lambda_n^{+}-\mathrm{e}^{\lambda_{n+1}^{+}t}\lambda_{n+1}^{+}|^2\mathrm{d}t\to 0$. Then by duality argument, it follows immediately that the system (1.5) is not exactly (null) controllable, as the conclusion of Theorem 1.4.

In addition to Theorem 1.4, we have the following proposition which might be important for the extension to nonlinear systems (see [3]).

Proposition 3.2. *The reachable set from $(u_0,v_0)=(0,0)\in\mathcal{H}$ with control $h\in L^2(0,T)$ is $\mathcal{R}=\mathcal{F}(L^2(0,T))$ where $\mathcal{F}:L^2(0,T)\mapsto\mathcal{H}$ is defined by $\mathcal{F}(h)=(u(T,\cdot),v(T,\cdot))$. Moreover, if $\beta\gamma\neq 0$, for every $(u_1,v_1)\in\mathcal{R}$, there exists a unique control $h\in L^2(0,T)$ that steers the system (1.5) from $(0,0)$ to (u_1,v_1).*

Proof. The first statement is obvious. Concerning the second statement, it is sufficient to prove that if $\beta\gamma \neq 0$ and $\mathcal{F}(h) = (0,0) \in \mathcal{H}$, then $h = 0$. Following the proof of Theorem 1.4, we have $\int_0^T h(t)p(t,0)\mathrm{d}t = 0$ for every solution (p,q) of the adjoint system (3.5). Then the function $F(z)$ defined by (3.19) is an entire function on $\mathbb{C}$ and it vanishes on the set $\{\mathrm{i}\lambda_n^\pm\}_{n\in\mathbb{Z}}$ which has an accumulation point $0 \in \mathbb{C}$. This implies that $F \equiv 0$ on $\mathbb{C}$ and thus $h \equiv 0$. □

Remark 3.3. The controllable set of the initial data which can be driven to $(0,0) \in \mathcal{H}$ with control $h \in L^2(0,T)$ can be described in a similar way as the reachable set $\mathcal{R}$ since the system (1.5) is time reversible.

4. Proof of Theorem 1.5

In this section, we will prove that $k \in \mathbb{R}$ does exist such that the closed-loop system (1.10) is asymptotically stable if and only if $\beta\gamma < 0$. The proof relies on the Riesz basis method and a careful analysis of the spectrum.

Proposition 4.1. *All the the eigenvalues of the closed loop system* (1.10) *have a strictly negative real part if and only if* $\beta\gamma < 0$ *and* $0 < |k| < 1$.

Proof of Proposition 4.1. Clearly, the corresponding eigenvalue λ and the eigenvector (ϕ,ψ) satisfy

$$\begin{cases} \lambda(\phi,\psi)^{tr} = \mathcal{A}(\phi,\psi)^{tr} \\ \phi(0) = k\phi(2\pi), \end{cases} \tag{4.1}$$

where

$$\mathcal{A} = \begin{pmatrix} \alpha\partial_x & \beta \\ \gamma & 0 \end{pmatrix} \tag{4.2}$$

is the infinitesimal generator of the semigroup corresponding to the system (1.10). By canceling ψ, (4.1) is reduced to

$$\alpha\lambda\phi'(x) = (\lambda^2 - \beta\gamma)\phi(x) \quad \text{with} \quad \phi(0) = k\phi(2\pi). \tag{4.3}$$

It is easy to see that $\lambda = 0$ happens only when $\phi(x) = \psi(x) \equiv 0$ since $\beta\gamma \neq 0$. Hence (4.1) has no zero eigenvalues.

Since $\alpha < 0$, the solution of (4.3) is given by

$$\phi(x) = \mathrm{e}^{\frac{\lambda^2-\beta\gamma}{\lambda\alpha}x}\phi(0), \tag{4.4}$$

where the eigenvalue λ is determined by

$$\mathrm{e}^{\frac{\lambda^2-\beta\gamma}{\lambda\alpha}2\pi} = \frac{1}{k}, \quad k \neq 0. \tag{4.5}$$

For every $n \in \mathbb{Z}$, let

$$\tilde{n} = \begin{cases} n, & \text{if } k > 0, \\ n + \frac{1}{2}, & \text{if } k < 0, \end{cases} \tag{4.6}$$

then

$$\frac{\lambda^2 - \beta\gamma}{\lambda\alpha}2\pi = \mathrm{i}2\tilde{n}\pi - \ln|k|, \tag{4.7}$$

or equivalently,

$$\lambda^2 - \frac{\alpha}{2\pi}(\mathrm{i}2\tilde{n}\pi - \ln|k|)\lambda - \beta\gamma = 0. \tag{4.8}$$

Therefore, for any $n \in \mathbb{Z}$ and $k \neq 0$, (4.8) has two roots $\lambda^{\pm}_{n,k} \in \mathbb{C}$ such that

$$\left[\lambda^{\pm}_{n,k} - \frac{\alpha}{4\pi}(\mathrm{i}2\tilde{n}\pi - \ln|k|)\right]^2 = \beta\gamma + \frac{\alpha^2}{16\pi^2}(\mathrm{i}2\tilde{n}\pi - \ln|k|)^2. \tag{4.9}$$

Direct computations give us the real and imaginary part of $\lambda^{\pm}_{n,k}$:

$$\Re(\lambda^{\pm}_{n,k}) = \frac{-\alpha\ln|k|}{4\pi} \pm \sqrt{\frac{c_{n,k} + \sqrt{c^2_{n,k} + 4d^2_{n,k}}}{2}}, \tag{4.10}$$

$$\Im(\lambda^{\pm}_{n,k}) = \frac{\alpha\tilde{n}}{2} \mp \operatorname{sgn}(\tilde{n}\ln|k|)\sqrt{\frac{-c_{n,k} + \sqrt{c^2_{n,k} + 4d^2_{n,k}}}{2}}, \tag{4.11}$$

where

$$c_{n,k} = \beta\gamma + \frac{\alpha^2}{16\pi^2}\left(\ln^2|k| - 4\tilde{n}^2\pi^2\right), \tag{4.12}$$

$$d_{n,k} = \frac{\alpha^2\ln|k|\,\tilde{n}}{8\pi}\cdot \tag{4.13}$$

Since $\alpha < 0$, it is obvious to see that if $|k| \geq 1$, $\Re(\lambda^{+}_{n,k}) \geq 0$. On the other hand, if $0 < |k| < 1$, we get easily from (4.10) and (4.12)–(4.13) that

$$\Re(\lambda^{\pm}_{n,k}) < 0, \quad \forall n \in \mathbb{Z} \tag{4.14}$$

is equivalent to

$$\frac{\alpha^2\ln^2|k|}{16\pi^2} > \frac{c_{n,k} + \sqrt{c^2_{n,k} + 4d^2_{n,k}}}{2}, \quad \forall n \in \mathbb{Z} \tag{4.15}$$

or further to,

$$-\beta\gamma + \frac{\alpha^2(\ln^2|k| + 4\tilde{n}^2\pi^2)}{16\pi^2} > \sqrt{c^2_{n,k} + 4d^2_{n,k}}, \quad \forall n \in \mathbb{Z} \tag{4.16}$$

then to

$$-\frac{\beta\gamma\alpha^2\ln^2|k|}{4\pi^2} > 0, \quad \forall n \in \mathbb{Z} \tag{4.17}$$

and finally to $\beta\gamma < 0$. This ends the proof of Proposition 4.1. □

In addition to Proposition 4.1, we can obtain, after careful computations to (4.10)–(4.11), the asymptotic behavior of the eigenvalues $\lambda^{\pm}_{n,k}$ as $n \to \infty$:

$$\lim_{n\to\infty} \lambda^{+}_{n,k} = 0 \quad \text{and} \quad \lim_{n\to\infty} |\lambda^{-}_{n,k}| = +\infty. \tag{4.18}$$

In view of (4.4) and(4.7), the corresponding eigenvectors can be chosen as

$$E^{+}_{n,k} := (\phi^{+}_{n,k}(x), \psi^{+}_{n,k}(x))^{tr} = \mathrm{e}^{(\frac{-\ln|k|}{2\pi} + \mathrm{i}\tilde{n})x}(\lambda^{+}_{n,k}, \gamma)^{tr}, \quad n \in \mathbb{Z}, \tag{4.19}$$

$$E^{-}_{n,k} := (\phi^{-}_{n,k}(x), \psi^{-}_{n,k}(x))^{tr} = \mathrm{e}^{(\frac{-\ln|k|}{2\pi} + \mathrm{i}\tilde{n})x}\left(1, \frac{\gamma}{\lambda^{-}_{n,k}}\right)^{tr}, \quad n \in \mathbb{Z}. \tag{4.20}$$

In order to prove Theorem 1.5, we first prove that if $\beta\gamma \geq 0$ and $k \in \mathbb{R}$, the system (1.10) is not asymptotically stable. According to Proposition 4.1, if $\beta\gamma \geq 0$, then for any $k \in \mathbb{R}$, there exists $n \in \mathbb{Z}$, such that $\Re(\lambda^{+}_{n,k}) \geq 0$. Let the initial data be the corresponding eigenvector: $(u_0, v_0)^{tr} = E^{+}_{n,k} \in (L^2(0, 2\pi))^2$, then the corresponding solution of the system (1.10) is given by $(u(t,\cdot), v(t,\cdot)) = \mathrm{e}^{\lambda^{+}_{n,k}t}(u_0, v_0)$, which is obviously not stable.

Now it remains to prove that if $\beta\gamma < 0$, there exists suitable $k \in \mathbb{R}$ such that the system (1.10) is asymptotically stable. More precisely, we apply the Riesz basis approach to prove that the solution of the system (1.10) satisfies (1.9) if $\beta\gamma < 0$ and $0 < |k| < 1$.

Let us emphasize that it is possible that some eigenvalues may coincide. Actually, the occurrence of an eigenvalue with multiplicity greater than one happens if and only if

$$\beta\gamma = -\frac{\alpha^2 \ln^2 k}{16\pi^2} < 0 \quad \text{and} \quad 0 < k < 1. \tag{4.21}$$

Consequently, the only multiple eigenvalue is given by

$$\lambda^+_{0,k} = \lambda^-_{0,k} = \lambda_0 := -\frac{\alpha \ln k}{4\pi} < 0. \tag{4.22}$$

In this case, the dimension of the eigenspace of λ_0 is one and the corresponding eigenvector can be chosen as

$$E^+_{0,k} := \mathrm{e}^{\frac{-\ln k}{2\pi}x}(\lambda_0, \gamma)^{tr}. \tag{4.23}$$

Let $E^-_{0,k}$ be the root vector corresponding to λ_0:

$$\mathcal{A}E^-_{0,k} = \lambda_0 E^-_{0,k} + E^+_{0,k} \tag{4.24}$$

where the operator $\mathcal{A}$ is given by (4.2). Then by (4.21), (4.22), it is easy to find a typical root vector

$$E^-_{0,k} := \mathrm{e}^{\frac{-\ln k}{2\pi}x}(\lambda_0 + 1, \gamma)^{tr} \tag{4.25}$$

which satisfies (4.24) and is linearly independent of $E^+_{0,k}$.

Now we claim the following proposition:

Proposition 4.2. *For any fixed $\beta\gamma < 0$ and $0 < |k| < 1$, $\{(E^+_{n,k}, E^-_{n,k}),\ n \in \mathbb{Z}\}$ forms a Riesz basis of the complex Hilbert space $(L^2(0, 2\pi))^2$.*

Proof of Proposition 4.2. We introduce the classical orthogonal basis $\{(e^+_{n,k}, e^-_{n,k}),\ n \in \mathbb{Z}\}$ of $(L^2(0, 2\pi))^2$, where

$$e^+_{n,k} = \mathrm{e}^{\mathrm{i}\tilde{n}x}(1, 0)^{tr}, \quad e^-_{n,k} = \mathrm{e}^{\mathrm{i}\tilde{n}x}(0, 1)^{tr}, \quad n \in \mathbb{Z}. \tag{4.26}$$

Clearly, we have

$$\langle e^+_{n,k}, e^-_{m,k}\rangle_{(L^2(0,\pi))^2} = 0, \quad \forall n, m \in \mathbb{Z}, \tag{4.27}$$

$$\langle e^+_{n,k}, e^+_{m,k}\rangle_{(L^2(0,\pi))^2} = \langle e^-_{n,k}, e^-_{m,k}\rangle_{(L^2(0,\pi))^2} = 2\pi\delta_{nm}, \quad \forall n, m \in \mathbb{Z}, \tag{4.28}$$

where δ_{nm} stands for the Kronecker's delta. The proof will be divided into two cases due to the possibility of the occurrence of multiple eigenvalues.

Case 1: (4.21) is not true. In this case, all the eigenvalues are distinct. Since $\lambda^+_{n,k} \neq \lambda^-_{n,k}$ for all $n \in \mathbb{Z}$, the matrix $\begin{pmatrix} \lambda^+_{n,k} & 1 \\ \gamma & \frac{\gamma}{\lambda^-_{n,k}} \end{pmatrix}$ is invertible. Then it is easy to get from

$$(E^+_{n,k}, E^-_{n,k}) = \mathrm{e}^{\frac{-\ln|k|}{2\pi}x}(e^+_{n,k}, e^-_{n,k}) \begin{pmatrix} \lambda^+_{n,k} & 1 \\ \gamma & \frac{\gamma}{\lambda^-_{n,k}} \end{pmatrix}, \tag{4.29}$$

that $\{(E^+_{n,k}, E^-_{n,k}),\ n \in \mathbb{Z}\}$ forms a basis of the complex Hilbert space $(L^2(0, 2\pi))^2$.

For any $(f,g) \in (L^2(0,2\pi))^2$, there is a series of complex pairs $\{(\alpha_n^+, \alpha_n^-) \in \mathbb{C}^2\}_{n\in\mathbb{Z}}$ such that

$$(f,g)^{tr} = \sum_{n\in\mathbb{Z}} (\alpha_n^+ E_{n,k}^+ + \alpha_n^- E_{n,k}^-) = \sum_{n\in\mathbb{Z}} (E_{n,k}^+, E_{n,k}^-)(\alpha_n^+, \alpha_n^-)^{tr}. \tag{4.30}$$

By (4.29), it follows that

$$(f,g)^{tr} = \sum_{n\in\mathbb{Z}} (\beta_n^+ e_{n,k}^+ + \beta_n^- e_{n,k}^-) \mathrm{e}^{-\frac{\ln|k|}{2\pi}x}, \tag{4.31}$$

where

$$\begin{pmatrix} \beta_n^+ \\ \beta_n^- \end{pmatrix} = \begin{pmatrix} \lambda_{n,k}^+ & 1 \\ \gamma & \frac{\gamma}{\lambda_{n,k}^-} \end{pmatrix} \begin{pmatrix} \alpha_n^+ \\ \alpha_n^- \end{pmatrix}. \tag{4.32}$$

Noting that $0 < |k| < 1$ and $\mathrm{e}^{\ln|k|} \le \mathrm{e}^{\frac{\ln|k|}{2\pi}x} \le 1, \ \forall x \in [0,2\pi]$. Then we get, from (4.26) and (4.31) that

$$2\pi \sum_{n\in\mathbb{Z}} (|\beta_n^+|^2 + |\beta_n^-|^2) \le \|(f,g)\|^2_{(L^2(0,2\pi))^2} \le \mathrm{e}^{-2\ln|k|} 2\pi \sum_{n\in\mathbb{Z}} (|\beta_n^+|^2 + |\beta_n^-|^2). \tag{4.33}$$

Moreover, it is easy to see from (4.32) and (4.18) that there exists a constant $C_1 > 0$ such that

$$|\beta_n^+|^2 + |\beta_n^-|^2 \le C_1(|\alpha_n^+|^2 + |\alpha_n^-|^2), \quad \forall n \in \mathbb{Z}. \tag{4.34}$$

On the other hand, we can also derive from (4.32) that

$$\begin{aligned} \begin{pmatrix} \alpha_n^+ \\ \alpha_n^- \end{pmatrix} &= \begin{pmatrix} \lambda_{n,k}^+ & 1 \\ \gamma & \frac{\gamma}{\lambda_{n,k}^-} \end{pmatrix}^{-1} \begin{pmatrix} \beta_n^+ \\ \beta_n^- \end{pmatrix} \\ &= \begin{pmatrix} a_{n,k}^+ & b_{n,k}^+ \\ a_{n,k}^- & b_{n,k}^- \end{pmatrix} \begin{pmatrix} \beta_n^+ \\ \beta_n^- \end{pmatrix} \end{aligned} \tag{4.35}$$

where

$$a_{n,k}^+ = \frac{1}{\lambda_{n,k}^+ - \lambda_{n,k}^-}, \quad a_{n,k}^- = \frac{-\lambda_{n,k}^-}{\lambda_{n,k}^+ - \lambda_{n,k}^-}, \tag{4.36}$$

$$b_{n,k}^+ = \frac{-\lambda_{n,k}^-}{\gamma(\lambda_{n,k}^+ - \lambda_{n,k}^-)}, \quad b_{n,k}^- = \frac{\lambda_{n,k}^+ \lambda_{n,k}^-}{\gamma(\lambda_{n,k}^+ - \lambda_{n,k}^-)}. \tag{4.37}$$

Thanks to (4.18), we get additionally that

$$\lim_{n\to\infty} a_{n,k}^+ = 0, \quad \lim_{n\to\infty} a_{n,k}^- = 1, \quad \lim_{n\to\infty} b_{n,k}^+ = \frac{1}{\gamma}, \quad \lim_{n\to\infty} b_{n,k}^- = 0. \tag{4.38}$$

Therefore we can see that there exists a constant $C_2 > 0$ such that

$$|\alpha_n^+|^2 + |\alpha_n^-|^2 \le C_2(|\beta_n^+|^2 + |\beta_n^-|^2), \quad \forall n \in \mathbb{Z}. \tag{4.39}$$

Obviously, (4.32)−(4.33) and (4.39) show that $\{(E_{n,k}^+, E_{n,k}^-), \ n \in \mathbb{Z}\}$ forms a Riesz basis of $(L^2(0,2\pi))^2$ in Case 1.

Case 2: (4.21) is true. In this case, we still have $\lambda_{n,k}^+ \ne \lambda_{n,k}^-$ and (4.29) for all $n \in \mathbb{Z}\backslash\{0\}$. While for $n = 0$, we get from (4.23), (4.25) and (4.26) that

$$(E_{0,k}^+, E_{0,k}^-) = \mathrm{e}^{\frac{-\ln k}{2\pi}x} (e_{0,k}^+, e_{0,k}^-) \begin{pmatrix} \lambda_0 & \lambda_0 + 1 \\ \gamma & \gamma \end{pmatrix}. \tag{4.40}$$

Obviously, the matrix $\begin{pmatrix} \lambda_0 & \lambda_0 + 1 \\ \gamma & \gamma \end{pmatrix}$ is invertible since $\gamma \ne 0$. Then it is similar to prove that $\{(E_{n,k}^+, E_{n,k}^-), \ n \in \mathbb{Z}\}$ still forms a Riesz basis of $(L^2(0,2\pi))^2$ as in Case 1. This ends the proof of Proposition 4.2. □

Thanks to Proposition 4.2, the solution (u, v) of the initial-boundary problem (1.10) can be decomposed with respect to the Riesz's basis $\{(E^+_{n,k}, E^-_{n,k}),\ n \in \mathbb{Z}\}$. Actually, for any given initial data $(u_0, v_0)^{tr} \in (L^2(0, 2\pi))^2$, there exists a series $\{(c^+_{n,k}, c^-_{n,k}) \in \mathbb{C}^2\}_{n\in\mathbb{Z}}$ such that

$$(u_0, v_0)^{tr} = \sum_{n\in\mathbb{Z}}(c^+_{n,k}E^+_{n,k} + c^-_{n,k}E^-_{n,k}). \tag{4.41}$$

In order to decompose the solution in terms of the Riesz's basis $\{(E^+_{n,k}, E^-_{n,k}),\ n \in \mathbb{Z}\}$, we have to discuss the various cases whether the eigenvalues are distinct or not.

Case 1: (4.21) is not true. In this case, all the eigenvalues are distinct. Since $E^\pm_{n,k}$ is the eigenvector corresponding to the eigenvalue $\lambda^\pm_{n,k}$, it follows from (4.41) that

$$(u(t, \cdot), v(t, \cdot))^{tr} = \sum_{n\in\mathbb{Z}} \left(c^+_{n,k}\mathrm{e}^{\lambda^+_{n,k}t}E^+_{n,k} + c^-_{n,k}\mathrm{e}^{\lambda^-_{n,k}t}E^-_{n,k}\right). \tag{4.42}$$

Then by Proposition 4.1, we get that in the case of $\beta\gamma < 0$,

$$|\mathrm{e}^{\lambda^\pm_{n,k}t}| < 1, \quad \forall n \in \mathbb{Z} \tag{4.43}$$

and

$$\lim_{t\to+\infty} \left|\mathrm{e}^{\lambda^\pm_{n,k}t}\right| \to 0, \quad \forall n \in \mathbb{Z}. \tag{4.44}$$

Therefore, there exists $C_3 > 0$ independent of (u_0, v_0) such that

$$\begin{aligned}\|(u(t, \cdot), v(t, \cdot))\|^2_{(L^2(0,2\pi))^2} \leq& \sum_{n\in\mathbb{Z}} \left\|c^+_{n,k}E^+_{n,k} + c^-_{n,k}E^-_{n,k}\right\|^2_{(L^2(0,2\pi))^2}\\ \leq& C_3\|(u_0, v_0)\|^2_{(L^2(0,2\pi))^2},\end{aligned} \tag{4.45}$$

which implies that the series on the right hand side of (4.42) converges uniformly and strongly in $(L^2(0, 2\pi))^2$. Then taking the $(L^2(0, 2\pi))^2$ norm of (4.42) and letting $t \to +\infty$, we easily conclude by (4.44) that

$$\lim_{t\to+\infty}(u(t, \cdot), v(t, \cdot))^{tr} = \sum_{n\in\mathbb{Z}} \lim_{t\to+\infty} \left(c^+_{n,k}\mathrm{e}^{\lambda^+_{n,k}t}E^+_{n,k} + c^-_{n,k}\mathrm{e}^{\lambda^-_{n,k}t}E^-_{n,k}\right) = (0, 0). \tag{4.46}$$

This ends the proof that if $\beta\gamma < 0$, then the system (1.10) can be stabilized by taking the tuning parameter k such that $0 < |k| < 1$.

Case 2: (4.21) is true. In this case, the only eigenvalue, whose multiplicity is more than one, is λ_0 given by (4.22). We get from (4.23), (4.25), (4.26) and (4.41) that

$$\begin{aligned}(u(t, \cdot), v(t, \cdot))^{tr} =& \mathrm{e}^{\lambda_0 t}[(c^-_{0,k}t + c^+_{0,k})E^+_{0,k} + c^-_{0,k}E^-_{0,k}]\\ &+ \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(c^+_{n,k}\mathrm{e}^{\lambda^+_{n,k}t}E^+_{n,k} + c^-_{n,k}\mathrm{e}^{\lambda^-_{n,k}t}E^-_{n,k}\right).\end{aligned} \tag{4.47}$$

Again by Proposition 4.1, we have (4.44) and

$$\lim_{t\to+\infty} \left|\mathrm{e}^{\lambda_0 t}t\right| \to 0. \tag{4.48}$$

Then we easily conclude the asymptotic stability of the closed-loop system (1.10), similarly as in Case 1. This completes the proof of Theorem 1.5. $\square$

Remark 4.3. In case of $\beta\gamma < 0$ and $|k| < 1$, one can adopt the Lyapunov's approach to prove that 0 is asymptotically stable for the closed-loop system (1.10). Actually, let

$$L(t) = \int_0^{2\pi} \mathrm{e}^{-\eta x}(u^2(t,x) + \xi v^2(t,x))\mathrm{d}x \tag{4.49}$$

where $\eta > 0, \xi > 0$ are chosen such that $\mathrm{e}^{-2\pi\eta} > k^2$ and $\beta + \xi\gamma = 0$. Since $\alpha < 0$, then

$$\dot{L}(t) = \alpha(\mathrm{e}^{-2\pi\eta} - k^2)u^2(t,2\pi) + \alpha\eta \int_0^{2\pi} \mathrm{e}^{-\eta x}u^2(t,x)\mathrm{d}x \leq 0 \tag{4.50}$$

Clearly, $\dot{L}(t) \equiv 0$ implies $u \equiv 0$ and next $v \equiv 0$ thanks to the first equation of (1.10). In order to apply LaSalle Invariance Principle [13], it remains to prove that all the trajectories $\{(u(t,\cdot), v(t,\cdot))\}_{t\geq 0}$ of (1.10) are precompact in $\mathcal{H}$. Using the regularity result of semigroup of linear operator [22] and the estimate (4.50) for $\mathcal{A}(u,v)^{tr}$, it is easy to get that for every $(u_0, v_0) \in \mathcal{D}(\mathcal{A}) := \{(f,g) \in H^1(0,2\pi) \times L^2(0,2\pi) \,|\, f(0) = kf(2\pi)\}$, the set $\{(u(t,\cdot), v(t,\cdot))\}_{t\geq 0}$ is bounded in $\mathcal{D}(\mathcal{A})$, therefore precompact in $\mathcal{H}$ since $\mathcal{D}(\mathcal{A})$ is compactly embedded in $\mathcal{H}$. Then we follow the proof of ([11], p. 114, Thm. 8.13) to conclude by LaSalle Invariance Principle that $(u(t,\cdot), v(t,\cdot)) \to (0,0) \in \mathcal{H}$ as $t \to +\infty$ for every $(u_0, v_0) \in \mathcal{D}(\mathcal{A})$, and finally for every $(u_0, v_0) \in \mathcal{H}$ by the continuity of the solution with respect to the initial data.

Remark 4.4. In case of $\beta\gamma < 0$ and $0 < |k| < 1$, the closed-loop system (1.10) is not exponentially stable (although asymptotically stable) since there exists a sequence of eigenvalues converges to zero, see (4.18).

Proof of Theorem 1.6. Thanks to Theorem 1.5, one has that if $\beta\gamma < 0$, there exists $k \in \mathbb{R}$ such that the system (1.10) is asymptotically stable, *i.e.*, (1.9) is true for all initial data $(u_0, v_0) \in (L^2(0,2\pi))^2$. Let $(u(t,x), v(t,x))$ be the solution of the closed-loop system (1.10) with (1.6). Then according to the uniqueness of solution, it is clear that the system (1.5) is asymptotically (null) controllable by taking the control as:

$$h(t) = u(t,0) - u(t,2\pi).$$

This concludes the Proof of Theorem 1.6. □

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