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STABILIZATION AND DESTABILIZATION *VIA* TIME-VARYING NOISE FOR UNCERTAIN NONLINEAR SYSTEMS *

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Abstract. This paper considers the stochastic stabilization and destabilization for uncertain nonlinear systems. Remarkably, the systems in question allow serious parameter unknowns (which don't belong to any known constant set) and serious time-variations, and possess more general growth conditions than those in the related existing literature. The former feature makes the time-invariant scheme inapplicable, and a time-varying one is proposed, mainly to compensate the serious parameter unknowns, as well as serious time-variations. First, a time-varying stochastic noise is successfully constructed to super-exponentially stabilize the special but representative case without adverse serious time-variations. Then, for the general case and general decay rate, it suffices to find a fast enough time-varying gain for the stochastic noise. Moreover, by a time-varying method, the stochastic destabilization with general growth rate is also achieved for uncertain nonlinear systems.

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1. Introduction

Usually, stochastic noise has a negative influence on systems, and hence some proper means should be adopted to counteract the influence. But it is not always the case. Sometimes, stochastic noise can improve the performance of systems. For instance, in [8], stochastic noise was employed to stabilize a two-dimensional linear system. Recognizing this interesting fact, much attention has been paid to the stabilization/destabilization *via* stochastic noise (see *e.g.*, [2,4–6,13,18–21] and references therein). However, all the related existing results fail to be applied to the systems with serious parameter unknowns, since no compensation mechanism was taken for the serious parameter unknowns.

This paper is devoted to the stabilization and destabilization *via* stochastic noise for uncertain nonlinear systems. Remarkably, the systems in question allow serious parameter unknowns and serious time-variations, and possess more general growth conditions than those in [2,4,13,20,21]. Motivated by [10–12,16,17], time-varying technique is adopted to compensate the serious parameter unknowns. Based on this, a time-varying stochastic noise is first constructed to super-exponentially stabilize the special but representative case without

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adverse serious time-variations. Then, for the general case and general decay rate, it suffices to find a fast enough time-varying gain for the stochastic noise, which can overtake the time-variations and the desired decay rate. It is worth pointing out that, although some works, such as [6, 20], introduced time-varying stochastic noises to stabilize nonlinear systems, the time-varying noises involved can only deal with the time-variations in the systems and guarantee the desired decay rate for the perturbed stochastic systems, but cannot compensate any serious parameter unknowns. Moreover, we apply a time-varying method to the stochastic destabilization, and achieve the destabilization with general growth rate for uncertain nonlinear systems by time-varying stochastic noise.

The remainder of this paper is organized as follows. Section 2 gives some notations and preliminary knowledge. Section 3 presents the stochastic stabilization for uncertain nonlinear systems. Section 4 addresses the stochastic destabilization for uncertain nonlinear systems. Section 5 collects the proofs of two claims. Section 6 provides two simulation examples. Section 7 gives some concluding remarks.

2. Notation and preliminary knowledge

Throughout this paper, the following notation is adopted. We use $\mathbb{Z}_+$ to denote the set of all positive integers, $\mathbb{R}_+$ to denote the set of all nonnegative real numbers, $\mathbb{R}_{\geq t_0}$ to denote the set of all real numbers not less than t_0, $\mathbb{R}^n$ to denote the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ to denote the space of real $n \times m$-matrices. For a given vector or matrix X, we use X^{T} to denote its transpose. We use $\|X\|$ to denote the Euclidean norm of the vector X. For a given matrix, we use $\mathrm{tr}\,\{X\}$ to denote its trace when X is a square matrix, and $\|X\|_F$ to denote the Frobenius norm of X, that is, $\|X\|_F = \sqrt{\mathrm{tr}\,\{X^{\mathrm{T}}X\}}$. For real numbers a and b, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. For any given sets $\mathbb{U}$ and $\mathbb{V}$, we use $\mathbb{C}(\mathbb{U}, \mathbb{V})$ to denote the set of all continuous functions mapping from $\mathbb{U}$ to $\mathbb{V}$, and $\mathbb{C}^\infty(\mathbb{U}, \mathbb{V})$ to denote the set of all infinitely differentiable functions mapping from $\mathbb{U}$ to $\mathbb{V}$. We denote $\log 0 = -\infty$.

Consider the stochastic differential system

$$\mathrm{d}x \;=\; f(t,x)\,\mathrm{d}t + g(t,x)\,\mathrm{d}B(t) \tag{2.1}$$

where $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ with the initial value $x(t_0) = x_0$; $B(t)$ is an m-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with Ω being a sample space, $\mathcal{F}$ being a filtration, and $\mathbf{P}$ being a probability measure; $f : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions with $f(t, 0) \equiv 0$ and $g(t, 0) \equiv 0$. Clearly, system (2.1) admits a trivial solution $x(t) \equiv 0$.

To guarantee the existence and uniqueness of solutions for system (2.1), the following assumption is imposed on system (2.1):

Assumption 2.1. For any $T > t_0$ and any $k \in \mathbb{Z}_+$, there is a constant $C > 0$ (depending on T and k) such that

$$\|f(t,x) - f(t,y)\| \;\vee\; \|g(t,x) - g(t,y)\|_F \;\leq\; C\|x - y\|$$

for all $t \in [t_0, T]$ and $x, y \in \mathbb{R}^n$ with $\|x\| \vee \|y\| \leq k$.

Assumption 2.1 shows that the local Lipschitz condition holds for the drift and diffusion terms of system (2.1) on every finite time subinterval. By (Rem. 6.3.4 in p. 113 of [3] or Thm. 3.19 in p. 95 of [15]), we can directly obtain the following lemma:

Lemma 2.2. *Under Assumption 2.1, for any initial value $x_0 \in \mathbb{R}^n$, system (2.1) has a unique strong solution $x(t)$ on $[t_0, \tau_e)$, where τ_e is the explosion time, that is, $\tau_e = \lim_{\epsilon\to+\infty} \inf\left\{t \geq t_0 \middle| \|x(t)\| \geq \epsilon\right\}$.*

To guarantee the global existence of solutions of system (2.1), the drift and diffusion terms of system (2.1) are usually required to satisfy the linear growth condition, the monotone condition [14] or the Khas'minskii condition described by Lyapunov-like functions (see *e.g.*, Thm. 4.1 in p. 84 of [8] or Thm. 3.19 in p. 95 of [15]). It is worth pointing out that the third condition actually covers the former two conditions as special cases.

The following lemma shows that, for system (2.1) under Assumption 2.1, any solution starting from a non-zero point will never reach the origin. This lemma can be directly derived from (Lem. 5.1 in p. 164 of [15]) and hence its proof is omitted here.

Lemma 2.3. *Under Assumption 2.1, for any initial value $x_0 \neq 0$, the solution $x(t)$ of system (2.1) satisfies*

$$\mathbf{P}\big\{x(t) \neq 0 \text{ for all } t \in [t_0, \tau_e)\big\} \;=\; 1.$$

We end this section with the definitions of "*serious parameter unknowns*" and "*serious time-variations*".

Definition 2.4. System $\mathrm{d}x = f(t,x)\,\mathrm{d}t$ is said to allow *serious parameter unknowns*, if $f : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\|f(t,x)\| \;\leq\; \theta \bar{f}(t,x), \tag{2.2}$$

where θ is an unknown positive constant (which doesn't belong to any known bounded interval) and $\bar{f}(t,x)$ is a known nonnegative function. The system is said to allow *serious time-variations*, if there holds

$$\|f(t,x)\| \;\leq\; h(t)\hat{f}(x), \tag{2.3}$$

where $h(t)$ is a nonnegative unbounded function of time and $\hat{f}(x)$ is a nonnegative function.

3. Stochastic stabilization

This section is devoted to the stochastic stabilization of the uncertain nonlinear systems with serious parameter unknowns (which don't belong to any known constant set). By the time-varying technique, we first construct a time-varying stochastic noise to super-exponentially stabilize the uncertain nonlinear systems without adverse serious time-variations. Then, for the general case and general decay rate, it suffices to find a fast enough time-varying gain for the stochastic noise. Moreover, there exists a peering result for the stochastic destabilization of uncertain nonlinear systems, which will be considered in next section.

Throughout this and next sections, we focus on the following uncertain nonlinear system:

$$\mathrm{d}x \;=\; f(t,x)\,\mathrm{d}t, \tag{3.1}$$

where $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathbb{R}^n$ with the initial value $x(t_0) = x_0$; $f : \mathbb{R}_{\geq t_0} \times \mathbb{R}^n \to \mathbb{R}^n$ is an unknown continuous function which, as that in system (2.1), satisfies $f(t,0) \equiv 0$ and Assumption 2.1 with $g(t,x) \equiv 0$.

Moreover, the following additional assumption is made on system (3.1):

Assumption 3.1. There exist an unknown constant $\theta > 0$ and a known function $\gamma \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ such that, for all $t \geq t_0$ and $x \in \mathbb{R}^n$,

$$x^{\mathrm{T}} f(t,x) \;\leq\; \theta\gamma(x)\|x\|^2. \tag{3.2}$$

We next show that Assumption 3.1 not only makes system (3.1) allow serious parameter unknowns, but also possesses rather generality. In fact, (3.2) is obviously implied by (2.2) with $\bar{f}(t,x) = \gamma(x)\|x\|$. Besides (3.2), another type of growth, also often encountered in the related literature (see, *e.g.*, [9,13,17]), is the following:

$$\|f(t,x)\| \;\leq\; \theta\bar{\gamma}(x)\|x\| \tag{3.3}$$

with function $\bar{\gamma} \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$. Clearly, (3.3) implies (3.2), and hence the latter is more general than the former.

If the time-variations in $f(t,x)$ vanish or are weak, then there is no essential difference between (3.2) and (3.3), regardless of the expressions of $\gamma(x)$ and $\bar{\gamma}(x)$. However, (3.2) allows some serious time-variations in $f(t,x)$, and (3.3) does not. Consider, for example, 1-dimensional system $\mathrm{d}x = (-tx + x^3)\,\mathrm{d}t$. Obviously, term "$-tx$" makes the system have serious time-variations. Noticing $x(-tx + x^3) = -tx^2 + x^4 \leq x^4$, there holds (3.2) with

$\theta = 1$ and $\gamma(x) = x^2$ for the 1-dimensional system. But, noting $|-tx + x^3| = \sqrt{t^2x^2 - 2tx^4 + x^6}$, there doesn't hold (3.3) for the 1-dimensional system.

Although (3.3) is somewhat simple and implies (3.2) with $\gamma(x) \le \bar{\gamma}(x)$, we are still inclined to (3.2), since $\gamma(x) \le \bar{\gamma}(x)$ would lead to less conservative stochastic stabilization. For example, for 2-dimensional function $f(t, x) = [x_1 + x_1x_2^2,\, x_2 - x_1^2x_2]^{\mathrm{T}}$, it is easy to see that (3.2) holds with $\gamma(x) = 1$, and (3.3) holds with $\bar{\gamma}(x) = \sqrt{1 + x_1^2x_2^2}$. Thus, based on (3.2), one can see from (3.4) below that a more delicate gain function of stochastic noise can be found to stabilize system (3.1).

It is worth noting that Assumption 3.1 shows the presence of serious parameter unknowns, and hence system (3.1) essentially differs from those in [2, 13, 20, 21] where the growth conditions of the nonlinear systems investigated therein are precisely known. Next we adopt the time-varying technique to compensate the serious parameter unknowns, and establish the theory on the stochastic stabilization for uncertain nonlinear systems.

First, choose a function $\rho \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\lim_{\|x\| \to +\infty} \rho(x) \;=\; +\infty,$$

for example, $\rho(x) = \|x\|^\beta$ with constant $\beta > 0$. Based on this, choose a local Lipschitz function $\alpha \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\alpha(x) \ge \rho(x)\gamma^{\frac{1}{2}}(x).$$

It is necessary to point out that such function $\alpha(\cdot)$ always exists, which can be seen from (Lem. 2.1 in [9]).

Then, by introducing a scalar standard Brownian motion $B(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with Ω being a sample space, $\mathcal{F}$ being a filtration, and $\mathbf{P}$ being a probability measure, we construct the following stochastic noise for system (3.1):

$$\big(t^p + \alpha(x)\big)x\,\mathrm{d}B(t) \tag{3.4}$$

with constant $p > 0$.

Now, we have the following theorem which shows stochastic noise (3.4) can super-exponentially stabilize system (3.1) under Assumption 3.1.

Theorem 3.2. *If Assumption 3.1 holds, then the following stochastic system*

$$\mathrm{d}x \;=\; f(t,x)\,\mathrm{d}t + \big(t^p + \alpha(x)\big)x\,\mathrm{d}B(t), \tag{3.5}$$

which is system (3.1) perturbed by stochastic noise (3.4), has a unique strong solution $x(t)$ on $[t_0, +\infty)$ for any initial value $x_0 \in \mathbb{R}^n$, and is super-exponentially stable in the following sense:

$$\limsup_{t\to+\infty} \frac{1}{t^{2p+1}} \log\|x(t)\| \;\le\; -\frac{1}{4p+2} \qquad \text{a.s.} \tag{3.6}$$

Proof. By Lemma 2.2, for any initial value $x_0 \in \mathbb{R}^n$, system (3.5) has a unique strong solution $x(t)$ on $[0, \tau_e)$, where τ_e is the explosion time. Moreover, it can be verified that $\tau_e = +\infty$ a.s. (see Sect. 5 for the proof), that is, the solution $x(t)$ is defined on $[t_0, +\infty)$ a.s. Clearly, (3.6) holds for $x_0 = 0$ since $x(t) \equiv 0$. Therefore, it suffices to prove (3.6) for $x_0 \ne 0$.

By Lemma 2.3, for any initial value $x_0 \ne 0$, the solution $x(t) \ne 0$ for all $t \in [t_0, +\infty)$ a.s. Then, by Itô's formula and Assumption 3.1, and letting

$$\mathcal{M}(t) := 2\int_{t_0}^{t} \big(s^p + \alpha(x(s))\big)\,\mathrm{d}B(s),$$

we have

$$\begin{aligned}\log\|x(t)\|^2 &= \log\|x_0\|^2 + \int_{t_0}^{t} \Big(2\|x(s)\|^{-2}x^{\mathrm{T}}(s)f(s,\, x(s)) - \big(s^p + \alpha(x(s))\big)^2\Big)\,\mathrm{d}s + \mathcal{M}(t)\\ &\le \log\|x_0\|^2 + \int_{t_0}^{t} \Big(2\theta\gamma(x(s)) - \big(s^p + \alpha(x(s))\big)^2\Big)\,\mathrm{d}s + \mathcal{M}(t).\end{aligned} \tag{3.7}$$

Clearly, $\mathcal{M}(t)$ is a continuous local martingale with the initial value $\mathcal{M}(t_0) = 0$. Then, for any $\varepsilon \in (0,1)$ and any $k \in \mathbb{Z}_+$, using the exponential martingale inequality (see Thm. 7.4 in p. 44 of [14]) yields

$$\mathbf{P}\left\{ \sup_{t_0 \le t \le t_0+k} \left(\mathcal{M}(t) - \varepsilon \int_{t_0}^{t} \left(s^p + \alpha(x(s))\right)^2 \mathrm{d}s \right) \ > \ \frac{4}{\varepsilon} \log k \right\} \ \le \ \frac{1}{k^2}.$$

By Borel−Cantelli's lemma (see Lem. 2.4 in p. 7 of [14]), we obtain that, for almost all $\omega \in \Omega$, there is a sufficiently large integer $k_0 = k_0(\omega) > 1$ such that, for any integer $k \ge k_0$,

$$\mathcal{M}(t) \ \le \ \varepsilon \int_{t_0}^{t} \left(s^p + \alpha(x(s))\right)^2 \mathrm{d}s + \frac{4}{\varepsilon} \log k, \quad \forall t \in [t_0,\, t_0 + k]. \tag{3.8}$$

Substituting (3.8) into (3.7) and noting $\varepsilon \in (0,\,1)$ yield that, for any integer $k \ge k_0$ and all $t \in [t_0, t_0 + k]$,

$$\begin{aligned} \log \|x(t)\|^2 \le \log \|x_0\|^2 + \int_{t_0}^{t} \Big(&(\varepsilon - 1)s^{2p} + 2\theta\gamma(x(s)) \\ &+ (\varepsilon - 1)\alpha^2(x(s)) \Big) \,\mathrm{d}s + \frac{4}{\varepsilon} \log k \qquad \text{a.s.} \end{aligned} \tag{3.9}$$

By $\varepsilon \in (0,\,1)$ and $\alpha(x) \ge \rho(x)\gamma^{\frac{1}{2}}(x)$, it is deduced that, for all $x \in \mathbb{R}^n$,

$$2\theta\gamma(x) + (\varepsilon - 1)\alpha^2(x) \le (2\theta + (\varepsilon - 1)\rho^2(x))\gamma(x).$$

Moreover, by $\lim_{\|x\| \to +\infty} \rho(x) = +\infty$, there exists a constant $N > 0$ sufficiently large such that $\rho^2(x) \ge \frac{2\theta}{1-\varepsilon}$ for all $\|x\| \ge N$, which together with $\varepsilon \in (0,1)$ implies that, for all $x \in \mathbb{R}^n$,

$$2\theta\gamma(x) + (\varepsilon - 1)\alpha^2(x) \ \le \ \sup_{\|x\| \le N} \big(2\theta + (\varepsilon - 1)\rho^2(x)\big)\gamma(x) \ =: \ H.$$

Then, by (3.9), we derive that, for any integer $k \ge k_0$ and all $t \in [t_0, t_0 + k]$,

$$\log \|x(t)\|^2 \le \log \|x_0\|^2 + \frac{\varepsilon - 1}{2p + 1}\big(t^{2p+1} - t_0^{2p+1}\big) + H(t - t_0) + \frac{4}{\varepsilon} \log k \quad \text{a.s.}$$

Furthermore, for any integer $k \ge k_0$ and all $t \in [t_0 + k - 1, t_0 + k]$,

$$\frac{\log \|x(t)\|^2}{t^{2p+1}} \ \le \ \frac{\log \|x_0\|^2}{t^{2p+1}} + \frac{\varepsilon - 1}{2p + 1} \cdot \left(1 - \left(\frac{t_0}{t}\right)^{2p+1}\right) + \frac{H(t - t_0)}{t^{2p+1}} + \frac{4\log(t - t_0 + 1)}{\varepsilon t^{2p+1}} \quad \text{a.s.}$$

From this, it follows that

$$\limsup_{t \to +\infty} \frac{\log \|x(t)\|}{t^{2p+1}} \ \le \ \frac{\varepsilon - 1}{4p + 2} \qquad \text{a.s.}$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0^+$ yields that (3.6) holds.

This completes the proof. □

From the proof of Theorem 3.2, it can be seen that term $\alpha(x)x\,\mathrm{d}B(t)$ and term $t^p x\,\mathrm{d}B(t)$ play different roles in stabilizing uncertain nonlinear system (3.1). Specifically, the former is to suppress the potential explosion of system (3.1), and the later is to deal with the serious parameter unknowns of system (3.1). It is worth noting that, owing to the introduction of time-varying stochastic noise, super-exponential stability is established for the perturbed stochastic system, but this is very hard if one only concentrates on time-invariant stochastic noises.

In the stochastic noise, we have chosen the time-varying gain as t^p, partly since the unknown parameter can be overtaken by t^p as t goes to infinity. It is quite natural that when adverse serious time-variations exist in the systems as well, we would find a faster time-varying gain, which can overtake the serious time-variations as t goes to infinity, and based on which, the stochastic noise of the above structure very likely remains stabilizing the uncertain nonlinear system. Moreover, from the proof of Theorem 3.2, recognize that the the decay rate of the perturbed stochastic system is determined by the time-varying gain in the stochastic noise, and increases with accelerating the time-varying gain. With these insights, the stochastic stabilization with general decay rate will be established for system (3.1) with serious parameter unknowns and serious time-variations which is formulated by the following assumption:

Assumption 3.3. There exist an unknown constant $\theta > 0$ and known functions $h \in \mathbb{C}(\mathbb{R}_{\geq t_0}, \mathbb{R}_+)$ and $\gamma \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ such that, for all $t \geq t_0$ and $x \in \mathbb{R}^n$,

$$x^{\mathrm{T}} f(t,x) \ \leq \ \theta h(t)\gamma(x)\|x\|^2. \tag{3.10}$$

Assumption 3.3 makes system (3.1) allow serious parameter unknowns and serious time-variations, since (3.10) is obviously implied by (2.2) with $\bar{f}(t,x) = h(t)\gamma(x)\|x\|$, and also implied by (2.3) with $\hat{f}(x) = \theta\gamma(x)\|x\|$. Remark that the stochastic stabilization with general decay rate of system (3.1) has been investigated in [20] under the following one-sided polynomial growth condition:

$$x^{\mathrm{T}} f(t,x) \ \leq \ \delta(t)\big(l_0 + \sum_{i=1}^{n} l_i\|x\|^{\alpha_i}\big)\|x\|^2 \ =: \delta(t)\gamma'(x)\|x\|^2 \tag{3.11}$$

with known positive continuous function $\delta(t)$ and known nonnegative constants l_i's and α_i's. Clearly, (3.11) is a special case of Assumption 3.3, since (3.11) excludes serious parameter unknowns, and $\gamma'(x)$ cannot be non-polynomial function of $\|x\|$, which greatly limits the nonlinearities in system (3.1).

Let $\lambda(t) \in \mathbb{C}(\mathbb{R}_{\geq t_0}, \mathbb{R}_+)$ be the desired decay rate, which is strictly increasing and satisfies $\lambda(t) \to +\infty$ as $t \to +\infty$. Under Assumption 3.3, we first choose a known increasing function $L \in \mathbb{C}^{\infty}(\mathbb{R}_{\geq t_0}, \mathbb{R}_+)$ satisfying

(S1) $\lim_{t\to+\infty} \dot{L}(t)/L^2(t) = 0$;
(S2) $\lim_{t\to+\infty} \big(h(t) + \log t\big)/L(t) = 0$;
(S3) $L(t) \geq \log\lambda(t)$ for all $t \geq t_0$.

From Lemma 2.2 in [7] and its proof, we see that such $L(t)$ always exists and can be explicitly constructed. Moreover, by (S2), there clearly holds that $\lim_{t\to+\infty} L(t) = +\infty$.

Then, choose a local Lipschitz function $\alpha \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\alpha(x) \geq \rho(x)\gamma^{\frac{1}{2}}(x)$$

with function $\rho \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ satisfying $\lim_{\|x\|\to+\infty} \rho(x) = +\infty$. Furthermore, by introducing a scalar standard Brownian motion $B(t)$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with Ω being a sample space, $\mathcal{F}$ being a filtration, and $\mathbf{P}$ being a probability measure, we construct the following stochastic noise:

$$\big(L(t) + h^{\frac{1}{2}}(t)\alpha(x)\big)x\,\mathrm{d}B(t). \tag{3.12}$$

Now, we have the following theorem which shows system (3.1) can be stabilized with decay rate $\lambda(t)$ by stochastic noise (3.12) under Assumption 3.3.

Theorem 3.4. *If Assumption 3.3 holds, then the following stochastic system*

$$\mathrm{d}x \ = \ f(t,x)\,\mathrm{d}t + \big(L(t) + h^{\frac{1}{2}}(t)\alpha(x)\big)x\,\mathrm{d}B(t), \tag{3.13}$$

which is system (3.1) perturbed by stochastic noise (3.12), has a unique strong solution $x(t)$ on $[t_0, +\infty)$ for any initial value $x_0 \in \mathbb{R}^n$, and is stable with decay rate $\lambda(t)$ in the following sense:

$$\limsup_{t\to+\infty} \frac{\log\|x(t)\|}{\log\lambda(t)} \ < \ 0 \qquad \text{a.s.} \tag{3.14}$$

Proof. By Lemma 2.2, for any initial value $x_0 \in \mathbb{R}^n$, system (3.13) has a unique strong solution $x(t)$ on $[0, \tau_e)$. Furthermore, we can show that $\tau_e = +\infty$ a.s. (see Sect. 5 for the proof). Clearly, (3.14) holds for $x_0 = 0$ since $x(t) \equiv 0$. Therefore, it suffices to prove (3.14) for $x_0 \neq 0$.

By Lemma 2.3, for any initial value $x_0 \neq 0$, the solution $x(t) \neq 0$ for all $t \in [t_0, +\infty)$ a.s. Then, from Itô's formula and Assumption 3.3, it follows that

$$\begin{aligned}\log \|x(t)\|^2 &= \log \|x_0\|^2 + \int_{t_0}^{t} \Big(2\|x(s)\|^{-2} x^{\mathrm{T}}(s) f(s, x(s)) - \big(L(s) + h^{\frac{1}{2}}(s)\alpha(x(s))\big)^2\Big)\, \mathrm{d}s + \mathcal{M}(t)\\ &\leq \log \|x_0\|^2 + \int_{t_0}^{t} \Big(2\theta h(s)\gamma(x(s)) - \big(L(s) + h^{\frac{1}{2}}(s)\alpha(x(s))\big)^2\Big)\, \mathrm{d}s + \mathcal{M}(t)\end{aligned}$$

with $\mathcal{M}(t) = 2\int_{t_0}^{t} \big(L(t) + h^{\frac{1}{2}}(t)\alpha(x)\big)\, \mathrm{d}B(s)$.

Similar to the proof of Theorem 3.2, we can derive that, for any $\varepsilon \in (0, 1)$ and almost all $\omega \in \Omega$, there is a sufficiently large integer $k_0 = k_0(\omega) > 1$ such that, for any integer $k \geq k_0$ and all $t \in [t_0, t_0 + k]$,

$$\begin{aligned}\log \|x(t)\|^2 \leq \log \|x_0\|^2 + \int_{t_0}^{t} \Big(&2\theta h(s)\gamma(x(s)) + (\varepsilon - 1)h(s)\alpha^2(x(s))\\ &+(\varepsilon - 1)L^2(s)\Big)\, \mathrm{d}s + \frac{4}{\varepsilon}\log k \qquad \text{a.s.}\end{aligned} \tag{3.15}$$

In terms of the proof of Theorem 3.2, there exists $H > 0$ such that, for all $x \in \mathbb{R}^n$,

$$2\theta\gamma(x) + (\varepsilon - 1)\alpha^2(x) \;\leq\; H,$$

which together with (3.15), implies that, for any integer $k \geq k_0$ and all $t \in [t_0, t_0 + k]$,

$$\log \|x(t)\|^2 \leq \log \|x_0\|^2 + \int_{t_0}^{t} \Big((\varepsilon - 1)L^2(s) + Hh(s)\Big)\, \mathrm{d}s + \frac{4}{\varepsilon}\log k \quad \text{a.s.} \tag{3.16}$$

By properties (S1) and (S2) of $L(t)$, there exists $T > t_0$ sufficiently large such that $L(t) > 0$ and

$$(\varepsilon - 1)L^2(t) + Hh(t) \;\leq\; -\mu_1 L^2(t) \;\leq\; -\mu_2 \dot{L}(t), \qquad \forall t \geq T,$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are some constants. From this and (3.16), it follows that, for any integer $k \geq \max\{k_0, T - t_0\}$ and all $t \in [T, t_0 + k]$,

$$\begin{aligned}\log \|x(t)\|^2 &\leq \log \|x(t_0)\|^2 + \int_{t_0}^{T} \Big((\varepsilon - 1)L^2(s) + Hh(s)\Big)\, \mathrm{d}s - \mu_2 \int_{T}^{t} \dot{L}(s)\, \mathrm{d}s + \frac{4}{\varepsilon}\log k\\ &\leq \Delta(t_0, T) - \mu_2 L(t) + \frac{4}{\varepsilon}\log k \qquad \text{a.s.,}\end{aligned}$$

where $\Delta(t_0, T) = \mu_2 L(T) + \log \|x(t_0)\|^2 + \int_{t_0}^{T} \big((\varepsilon - 1)L^2(s) + Hh(s)\big)\, \mathrm{d}s$. Hence, for any integer $k \geq \max\{k_0, T - t_0\}$ and all $t \in [t_0 + k - 1, t_0 + k]$,

$$\frac{\log \|x(t)\|^2}{L(t)} \;\leq\; \frac{\Delta(t_0, T)}{L(t)} - \mu_2 + \frac{4\log(t - t_0 + 1)}{\varepsilon L(t)} \qquad \text{a.s.,}$$

which together with property (S2) of $L(t)$ and $\lim_{t\to+\infty} L(t) = +\infty$, implies

$$\limsup_{t\to+\infty} \frac{\log \|x(t)\|^2}{L(t)} \;\leq\; -\mu_2 \qquad \text{a.s.}$$

Then, applying property (S3) of $L(t)$ again yields

$$\limsup_{t\to+\infty} \frac{\log \|x(t)\|}{\log \lambda(t)} \;<\; 0 \qquad \text{a.s.}$$

This completes the proof. $\square$

4. Stochastic destabilization

This section turns to exploiting the opposite effect of stochastic noise, *i.e.*, destabilizing uncertain nonlinear systems. We shall propose the time-varying scheme for stochastic destabilization, and achieve the destabilization with general growth rate for system (3.1) with serious parameter unknowns and serious time-variations.

We concentrate on the multidimensional case of system (3.1). This is because scalar counterexamples exist which cannot be destabilized by any equilibrium-preserving stochastic noise of type $g(t,x)\,\mathrm{d}B(t)$. For example, consider the following scalar system:

$$\dot{x} \;=\; -x - x^3,$$

whose zero solution is clearly exponentially stable. However, for any perturbed stochastic system

$$\mathrm{d}x \;=\; -(x + x^3)\,\mathrm{d}t + g(t,x)\,\mathrm{d}B(t)$$

with Assumption 2.1 and $g(t,0) \equiv 0$, similar to the proof of Theorem 3.2, it can be proved that all the solutions of the perturbed stochastic system converge to zero exponentially.

Moreover, the following additional assumption is imposed on system (3.1):

Assumption 4.1. There exist an unknown constant $\theta > 0$ and functions $h \in \mathbb{C}(\mathbb{R}_{\geq t_0}, \mathbb{R}_+)$ and $\gamma \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ such that, for all $t \geq t_0$ and $x \in \mathbb{R}^n$,

$$x^{\mathrm{T}} f(t,x) \;\geq\; -\theta h(t)\gamma(x)\|x\|^2. \tag{4.1}$$

Assumption 4.1 makes system (3.1) allow serious parameter unknowns and serious time-variations, since (4.1) is obviously implied by (2.2) with $\bar{f}(t,x) = h(t)\gamma(x)\|x\|$, and also implied by (2.3) with $\hat{f}(x) = \theta\gamma(x)\|x\|$.

Let $\lambda(t) \in \mathbb{C}\big(\mathbb{R}_{\geq t_0}, \mathbb{R}_+\big)$ be the desired growth rate, which is strictly increasing and satisfies $\lambda(t) \to +\infty$ as $t \to +\infty$. Under Assumption 4.1, by Lemma 2.2 in [7] and its proof, a known increasing function $L \in \mathbb{C}^{\infty}(\mathbb{R}_{\geq t_0}, \mathbb{R}_+)$ can be explicitly constructed such that

(D1) $\lim_{t\to+\infty} \dot{L}(t)/L^2(t) = 0$;
(D2) $\lim_{t\to+\infty} h(t)/L(t) = 0$;
(D3) $L(t) \geq \log\lambda(t)$ for all $t \geq t_0$.

From (D3), we see that $\lim_{t\to+\infty} L(t) = +\infty$. Such $L(t)$ will be adopted as the time-varying gain of stochastic noise. Then, choose a local Lipschitz function $\alpha \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\alpha(x) \geq \rho(x)\gamma^{\frac{1}{2}}(x)$$

with function $\rho \in \mathbb{C}(\mathbb{R}^n, \mathbb{R}_+)$ satisfying $\lim_{\|x\|\to+\infty} \rho(x) = +\infty$. Furthermore, motivated by [2], we introduce an n-dimensional standard Brownian motion $B(t)$ defined on a probability space $(\mathit{\Omega}, \mathcal{F}, \mathbf{P})$ with $\mathit{\Omega}$ being a sample space, $\mathcal{F}$ being a filtration, and $\mathbf{P}$ being a probability measure, and construct the following stochastic noise:

$$\big(L(t) + h^{\frac{1}{2}}(t)\alpha(x)\big)\,\mathbf{diag}(x_2,\, x_3, \ldots, x_n,\, x_1)\,\mathrm{d}B(t). \tag{4.2}$$

Now, we have the following theorem which shows system (3.1) can be destabilized with growth rate $\lambda(t)$ by stochastic noise (4.2) under Assumption 4.1.

Theorem 4.2. *If Assumption 4.1 holds, then the following stochastic system*

$$\mathrm{d}x \;=\; f(t,x)\,\mathrm{d}t + \big(L(t) + h^{\frac{1}{2}}(t)\alpha(x)\big)\,\mathbf{diag}(x_2, x_3, \ldots, x_n, x_1)\,\mathrm{d}B(t), \tag{4.3}$$

which is system (3.1) perturbed by stochastic noise (4.2), has a unique strong solution $x(t)$ on $[t_0, \tau_e)$ for any initial value $x_0 \in \mathbb{R}^n$, and is unstable with growth rate $\lambda(t)$ in the following sense:

$$\liminf_{t\to\tau_e} \frac{\log\|x(t)\|}{\log\lambda(t)} \;>\; 0 \qquad \text{a.s.} \tag{4.4}$$

for $x_0 \neq 0$, where τ_e is the explosion time of the solution $x(t)$.

Proof. By Lemmas 2.2 and 2.3, for any initial value $x_0 \neq 0$, system (4.3) has a unique solution $x(t) \neq 0$ on $[t_0, \tau_e)$. Define $\Omega_1 = \{\omega \in \Omega \big| \tau_e(\omega) = +\infty\}$. Clearly, (4.4) holds on $\Omega \setminus \Omega_1$. Hence, it suffices to prove (4.4) on Ω_1.

By Itô's formula and letting

$$\mathcal{M}(t) \ := \ \sum_{i=1}^{n} \int_{t_0}^{t} 2\|x(s)\|^{-2}\big(L(s) + h^{\frac{1}{2}}(s)\alpha(x(s))\big)x_i(s)x_{i+1}(s)\,\mathrm{d}B_i(s)$$

with $x_{n+1} = x_1$, we obtain that

$$\begin{aligned}\log\|x(t)\|^2 = \log\|x_0\|^2 + \int_{t_0}^{t} \Big(&2\|x(s)\|^{-2}x^{\mathrm{T}}(s)f(s,x(s)) \\ &+ \Big(L(s) + h^{\frac{1}{2}}(s)\alpha(x(s))\Big)^2 \left(1 - 2\|x(s)\|^{-4}\sum_{i=1}^{n} x_i^2(s)x_{i+1}^2(s)\right)\Big)\,\mathrm{d}s + \mathcal{M}(t).\end{aligned}$$

Then, by Assumption 4.1 and noting

$$3\sum_{i=1}^{n} x_i^2 x_{i+1}^2 \ \le \ 2\sum_{i=1}^{n} x_i^2 x_{i+1}^2 + \sum_{i=1}^{n} x_i^4 \ \le \ \|x\|^4, \tag{4.5}$$

it is deduced that

$$\log\|x(t)\|^2 \ge \log\|x_0\|^2 + \int_{t_0}^{t}\left(-2\theta h(s)\gamma(x(s)) + \frac{1}{3}\left(L(s) + h^{\frac{1}{2}}(s)\alpha(x(s))\right)^2\right)\mathrm{d}s + \mathcal{M}(t),$$

which together with $\alpha(x) \ge \rho(x)\gamma^{\frac{1}{2}}(x)$, implies

$$\begin{aligned}\log\|x(t)\|^2 \ge \log\|x_0\|^2 &+ \frac{1}{6}\int_{t_0}^{t}\big(L^2(s) + h(s)\gamma(x(s))\big(\rho^2(x(s)) - 12\theta\big)\big)\,\mathrm{d}s \\ &+ \frac{1}{6}\int_{t_0}^{t}\big(L(s) + h^{\frac{1}{2}}(s)\alpha(x(s))\big)^2\,\mathrm{d}s + \mathcal{M}(t).\end{aligned} \tag{4.6}$$

By $\lim_{\|x\|\to+\infty}\rho(x) = +\infty$, there exists a constant $N > 0$ sufficiently large such that $\rho^2(x) \ge 12\theta$ for all $\|x\| \ge N$, which implies that, for all $x \in \mathbb{R}^n$,

$$\gamma(x)\big(\rho^2(x) - 12\theta\big) \ \ge \ \inf_{\|x\|\le N}\gamma(x)\big(\rho^2(x) - 12\theta\big) \ := \ \delta.$$

Moreover, by properties (D1) and (D2) of $L(t)$, there exists $T > t_0$ sufficiently large such that, for all $t \ge T$ and all $x \in \mathbb{R}^n$,

$$L^2(t) + h(t)\gamma(x)\big(\rho^2(x) - 12\theta\big) \ \ge \ L^2(t) + \delta h(t) \ \ge \ \mu_1 L^2(t) \ \ge \ \mu_2\dot{L}(t),$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are some constants. Substituting this into (4.6) yields that, for all $t \ge T$,

$$\begin{aligned}\log\|x(t)\|^2 &\ge \log\|x_0\|^2 + \frac{1}{6}\int_{t_0}^{T}\Big(L^2(s) + h(s)\gamma(x(s))\big(\rho^2(x(s)) - 12\theta\big)\Big)\,\mathrm{d}s \\ &\quad + \frac{\mu_2}{6}\int_{T}^{t}\dot{L}(s)\,\mathrm{d}s + \frac{1}{6}\int_{t_0}^{t}\big(L(s) + h^{\frac{1}{2}}(s)\alpha(x(s))\big)^2\,\mathrm{d}s + \mathcal{M}(t) \\ &= \Delta(t_0, T) + \frac{\mu_2}{6}L(t) + \frac{1}{6}\int_{t_0}^{t}\big(L(s) + h^{\frac{1}{2}}(s)\alpha(x(s))\big)^2\,\mathrm{d}s + \mathcal{M}(t) \quad \text{a.s. on } \Omega_1,\end{aligned} \tag{4.7}$$

where $\Delta(t_0, T) = \log\|x_0\|^2 + \frac{1}{6}\int_{t_0}^{T}\big(L^2(s) + h(s)\gamma(x(s))(\rho^2(x(s)) - 12\theta)\big)\,\mathrm{d}s - \frac{\mu_2}{6}L(T)$.

Note that the quadratic variation of $\mathcal{M}(t)$

$$\langle \mathcal{M}, \mathcal{M}\rangle_t \ = \ \sum_{i=1}^{n}\int_{t_0}^{t} 4\|x(s)\|^{-4}\big(L(s)+h^{\frac{1}{2}}(s)\alpha(x(s))\big)^2 x_i^2(s)x_{i+1}^2(s)\,\mathrm{d}s \tag{4.8}$$

is continuous, increasing and nonnegative a.s. on Ω_1. Then we consider the following two mutually exclusive cases:

Case 1. On $\Omega_2 = \big\{\omega \in \Omega_1 \,\big|\, \lim_{t\to+\infty}\langle \mathcal{M}, \mathcal{M}\rangle_t < +\infty\big\}$.

By the definition of the quadratic variation of local martingale (see p. 12 of [14]), we have that, for almost all $\omega \in \Omega_2$,

$$\mathcal{M}^2(t) \ = \ \langle \mathcal{M}, \mathcal{M}\rangle_t + \overline{\mathcal{M}}(t)$$

with some continuous local martingale $\overline{\mathcal{M}}(t)$. Then, by (Thm. 3.9 in p. 14 of [14]), we see that $\lim_{t\to+\infty}\mathcal{M}^2(t)$ exists and is finite a.s. on Ω_2, which together with the continuity of $\mathcal{M}(t)$, implies that $\mathcal{M}(t)$ is bounded a.s. on Ω_2.

Then, by (4.7), $\lim_{t\to+\infty} L(t) = +\infty$ and the boundedness of $\mathcal{M}(t)$ on Ω_2, we obtain that

$$\liminf_{t\to+\infty}\frac{\log\|x(t)\|^2}{L(t)} \ \geq \ \liminf_{t\to+\infty}\left(\frac{\mu_2}{6}+\frac{\mathcal{M}(t)}{L(t)}+\frac{\Delta(t_0,T)}{L(t)}\right) \ > \ 0 \quad \text{a.s. on } \Omega_2,$$

which together with property (D3) of $L(t)$, implies that (4.4) holds with $\tau_e = +\infty$ a.s. on Ω_2.

Case 2. On $\Omega_1 \setminus \Omega_2 = \big\{\omega \in \Omega_1 \,\big|\, \lim_{t\to+\infty}\langle \mathcal{M}, \mathcal{M}\rangle_t = +\infty\big\}$.

By (4.5) and (4.8), we obtain

$$\langle \mathcal{M}, \mathcal{M}\rangle_t \ \leq \ \frac{4}{3}\int_{t_0}^{t}\big(L(s)+h^{\frac{1}{2}}(s)\alpha(x(s))\big)^2\,\mathrm{d}s \ =: \ \mathcal{A}(t). \tag{4.9}$$

Clearly, $\lim_{t\to+\infty}\mathcal{A}(t) = +\infty$ a.s. on $\Omega_1\setminus\Omega_2$. Moreover, by the strong law of large numbers (see Thm. 3.4 in p. 12 of [14]), we derive

$$\lim_{t\to+\infty}\frac{\mathcal{M}(t)}{\langle \mathcal{M}, \mathcal{M}\rangle_t} \ = \ 0 \qquad \text{a.s. on } \Omega_1\setminus\Omega_2,$$

which together with (4.9), implies that

$$\lim_{t\to+\infty}\frac{\mathcal{M}(t)}{\mathcal{A}(t)} \ = \ 0 \qquad \text{a.s. on } \Omega_1\setminus\Omega_2.$$

Then, by (4.7) and (4.9), we derive

$$\liminf_{t\to+\infty}\frac{\log\|x(t)\|^2}{\mathcal{A}(t)} \ \geq \ \liminf_{t\to+\infty}\left(\frac{1}{8}+\frac{\mathcal{M}(t)}{\mathcal{A}(t)}+\frac{\Delta(t_0,T)}{\mathcal{A}(t)}\right) \ > \ 0 \quad \text{a.s. on } \Omega_1\setminus\Omega_2. \tag{4.10}$$

By properties (D1) and (D3) of $L(t)$, there exists $T' \geq t_0$ sufficiently large such that, for all $t \geq T'$,

$$\begin{aligned}\mathcal{A}(t) \ &\geq \ \frac{4}{3}\int_{t_0}^{t} L^2(s)\,\mathrm{d}s \geq \mu_3\int_{T'}^{t}\dot{L}(s)\,\mathrm{d}s\,\mathrm{d}s \\ &= \mu_3 L(t) - \mu_3 L(T') \ \geq \ \mu_4\log\lambda(t) \quad \text{a.s. on } \Omega_1\setminus\Omega_2,\end{aligned}$$

where $\mu_3 > 0$ and $\mu_4 > 0$ are some constants. From this and (4.10), it follows that (4.4) holds with $\tau_e = +\infty$ a.s. on $\Omega_1\setminus\Omega_2$.

This completes the proof. □

5. Detailed proofs of two claims

Proof of "$\tau_e = +\infty$ a.s." in Theorem 3.2. Let $V(x) = \|x\|^\nu$ with $\nu \in (0,1)$. Then, by (3.5) and Itô's formula, we derive

$$\mathcal{L}V(x) \ = \ \frac{\nu}{2}\Big(2\|x\|^{\nu-2}x^{\mathrm{T}}f(t,x) + (\nu-1)\|x\|^{\nu}\big(t^p+\alpha(x)\big)^2\Big),$$

which together with Assumption 3.1, $\nu \in (0,1)$ and $\alpha(x) \geq \rho(x)\gamma^{\frac{1}{2}}(x)$, implies that, for all $t \geq t_0$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}\mathcal{L}V(x) &\leq \frac{\nu}{2}\Big(2\theta\|x\|^{\nu}\gamma(x) + (\nu-1)\|x\|^{\nu}\alpha^2(x)\Big)\\ &\leq \frac{\nu}{2}\big(2\theta + (\nu-1)\rho^2(x)\big)\gamma(x)V(x).\end{aligned}$$

By $\lim_{\|x\|\to+\infty}\rho(x) = +\infty$, there exists $r > 0$ sufficiently large such that $\rho^2(x) \geq \frac{2\theta}{1-\nu}$ for all $\|x\| \geq r$, which together with $\nu \in (0,1)$ implies that, for all $x \in \mathbb{R}^n$,

$$\frac{\nu}{2}\big(2\theta + (\nu-1)\rho^2(x)\big)\gamma(x) \ \leq \ \sup_{\|x\|\leq r}\frac{\nu}{2}\left(2\theta + (\nu-1)\rho^2(x)\right)\gamma(x) \ =: \ c.$$

From this, it follows that

$$\mathcal{L}V(x) \ \leq \ cV(x),$$

which together with Theorem 4.1 in page 84 of [8], implies $\tau_e = +\infty$ a.s. for any initial value. $\square$

Proof of "$\tau_e = +\infty$ a.s." in Theorem 3.4. For contradiction, suppose that for a solution $x(t)$, there holds $\mathbf{P}\big\{\tau_e < +\infty\big\} > 0$. Then, there exist $\varepsilon \in (0,1)$ and $T > t_0$ such that

$$\mathbf{P}\big\{\tau_k \leq T\big\} \geq \varepsilon, \qquad \forall k \geq N \tag{5.1}$$

for $N > 0$ sufficiently large, where $\tau_k = \inf\big\{t \geq t_0 \big| \|x(t)\| \geq k\big\}$, $k \in \mathbb{Z}_+$.

Let $\nu \in (0,1)$. By Itô's formula, we obtain that, for any $k \geq N$,

$$\mathbf{E}\big(\|x(T\wedge\tau_k)\|^{\nu}\big) \ = \ \|x_0\|^{\nu} + \mathbf{E}\left(\int_{t_0}^{T\wedge\tau_k}\frac{\nu}{2}q\left(s,x(s)\right)\,\mathrm{d}s\right),$$

where $q(t,x) = 2\|x\|^{\nu-2}x^{\mathrm{T}}f(t,x) + (\nu-1)\big(L(t) + h^{\frac{1}{2}}(t)\alpha(x)\big)^2$. By Assumption 3.3 and $\alpha(x) \geq \rho(x)\gamma^{\frac{1}{2}}(x)$, it is deduced that, for all $t \geq t_0$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}q(t,\,x) &\leq 2\theta h(t)\gamma(x)\|x\|^{\nu} + (\nu-1)h(t)\|x\|^{\nu}\alpha^2(x)\\ &\leq h(t)\big(2\theta + (\nu-1)\rho^2(x)\big)\|x\|^{\nu}\gamma(x).\end{aligned}$$

Moreover, by $\lim_{\|x\|\to+\infty}\rho(x) = +\infty$, there exists a constant $r > 0$ sufficiently large such that $\rho^2(x) \geq \frac{2\theta}{1-\nu}$ for all $\|x\| \geq r$, which together with $\nu \in (0,\,1)$ implies that, for $x \in \mathbb{R}^n$,

$$\big(2\theta + (\nu-1)\rho^2(x)\big)\|x\|^{\nu}\gamma(x) \ \leq \ \sup_{\|x\|\leq r}\big(2\theta + (\nu-1)\rho^2(x)\big)\|x\|^{\nu}\gamma(x) \ =: \ \delta.$$

Then, we derive that

$$q(t,x) \ \leq \ \delta h(t), \qquad \forall t \geq t_0,\ \forall x \in \mathbb{R}^n.$$

From this, it follows that, for any $k \geq N$,

$$\mathbf{E}\big(\|x(T\wedge\tau_k)\|^{\nu}\big) \ \leq \ \|x_0\|^{\nu} + \frac{\delta\nu}{2}\int_{t_0}^{T}h(s)\,\mathrm{d}s \ < \ +\infty. \tag{5.2}$$

However, from (5.1), it follows that, for any $k \geq N$,

$$\mathbf{E}\big(\|x(T \wedge \tau_k)\|^\nu\big) \ \geq\ k^\nu \mathbf{P}\{\tau_k \leq T\} \ \geq\ k^\nu \varepsilon,$$

which implies $\lim_{k\to+\infty} \mathbf{E}\big(\|x(T \wedge \tau_k)\|^\nu\big) = +\infty$. Clearly, this contradicts (5.2). Hence, for any initial value, there holds $\mathbf{P}\{\tau_e < +\infty\} = 0$. □

6. Simulation results

In this section, two examples are given to further show the effectiveness and correctness of our results.

Example 6.1. Consider the stochastic stabilization for the following 2-dimensional nonlinear system:

$$\begin{cases} \mathrm{d}x_1 &=& (\theta_1 x_1 + x_1 x_2^2)\,\mathrm{d}t, \\ \mathrm{d}x_2 &=& (\theta_2 x_2 - x_1^2 x_2)\,\mathrm{d}t, \end{cases} \tag{6.1}$$

where θ_1 and θ_2 are unknown constants. In what follows, let $t_0 = 0$.

It is easy to verify that system (6.1) satisfies Assumption 3.1 with $\gamma(x) \equiv 1$ and $\theta = \max\{|\theta_1|, |\theta_2|\}$. According to the construction procedure of the stochastic noise in Theorem 3.2, we introduce the following stochastic noise:

$$\big(t + (1 + \|x\|^2)^{\frac{1}{16}}\big) x \,\mathrm{d}B(t),$$

which perturbs system (6.1) into the following stochastic system:

$$\begin{cases} \mathrm{d}x_1 &=& (\theta_1 x_1 + x_1 x_2^2)\,\mathrm{d}t + \big(t + (1 + \|x\|^2)^{\frac{1}{16}}\big) x_1 \,\mathrm{d}B(t), \\ \mathrm{d}x_2 &=& (\theta_2 x_2 - x_1^2 x_2)\,\mathrm{d}t + \big(t + (1 + \|x\|^2)^{\frac{1}{16}}\big) x_2 \,\mathrm{d}B(t), \end{cases} \tag{6.2}$$

where $B(t)$ is a scalar standard Brownian motion.

Let $\theta_1 = 1$, $\theta_2 = 5$. Using MATLAB, Figures 1–4 are obtained to exhibit the trajectories of the states of system (6.2) with $x(0) = [5, -2]^{\mathrm{T}}$, $x(0) = [3, 4]^{\mathrm{T}}$, $x(0) = [-7, -3]^{\mathrm{T}}$ and $x(0) = [-4, 6]^{\mathrm{T}}$, respectively.

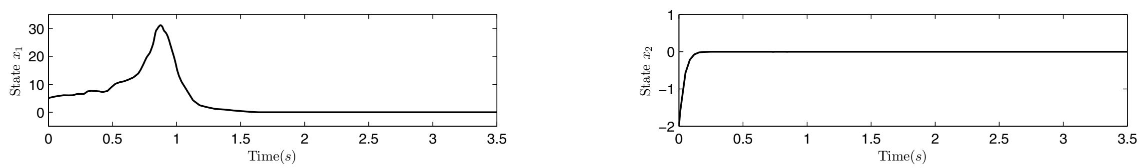

FIGURE 1. Trajectories with $x(0) = [5, -2]^{\mathrm{T}}$.

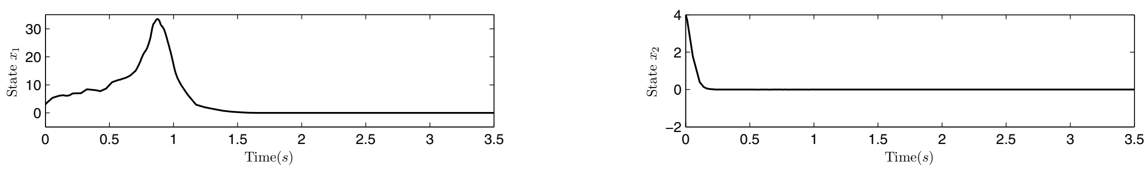

FIGURE 2. Trajectories with $x(0) = [3, 4]^{\mathrm{T}}$.

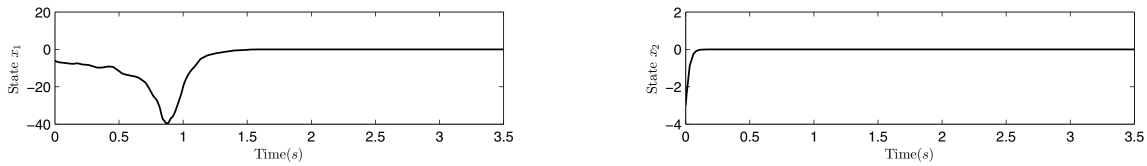

FIGURE 3. Trajectories with $x(0) = [-7, -3]^{\mathrm{T}}$.

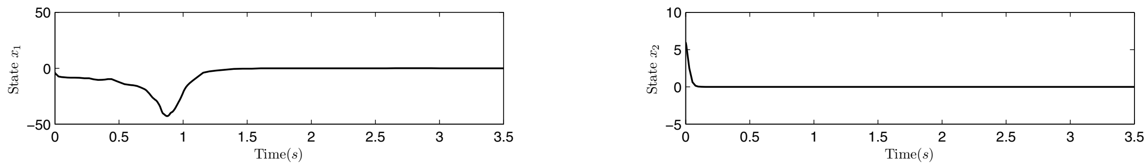

FIGURE 4. Trajectories with $x(0) = [-4, 6]^{\mathrm{T}}$.

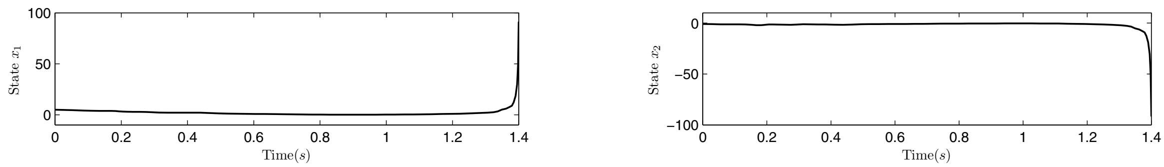

FIGURE 5. Trajectories with $x(0) = [5, -1]^{\mathrm{T}}$.

Example 6.2. Consider the stochastic stabilization for the following 2-dimensional system:

$$\begin{cases} \mathrm{d}x_1 &= (a_{11}x_1 + a_{12}x_2)\,\mathrm{d}t, \\ \mathrm{d}x_2 &= (a_{21}x_1 + a_{22}x_2)\,\mathrm{d}t, \end{cases} \tag{6.3}$$

where a_{ij}'s are unknown constants. In what follows, let $t_0 = 0$.

It is easy to verify that system (6.3) satisfies Assumption 4.1 with $\gamma(x) \equiv 1$, $h(t) \equiv 1$, and $\theta = \max\{|a_{11}| + (|a_{12}| + |a_{21}|)/2,\ |a_{22}| + (|a_{12}| + |a_{21}|)/2\}$.

Let $\lambda(t) = t$. Then, according to the construction procedure of the stochastic noise in Theorem 4.2, we introduce the following stochastic noise:

$$\big((t + 0.1)^{\frac{1}{2}} + \|x\|\big)\mathbf{diag}\{x_2,\, x_1\}\,\mathrm{d}B(t),$$

which perturbs system (6.3) into the following stochastic system:

$$\begin{cases} \mathrm{d}x_1 &= (a_{11}x_1 + a_{12}x_2)\,\mathrm{d}t + \big((t + 0.1)^{\frac{1}{2}} + \|x\|\big)x_2\,\mathrm{d}B_1(t), \\ \mathrm{d}x_2 &= (a_{21}x_1 + a_{22}x_2)\,\mathrm{d}t + \big((t + 0.1)^{\frac{1}{2}} + \|x\|\big)x_1\,\mathrm{d}B_2(t), \end{cases} \tag{6.4}$$

where $B(t) = [B_1(t),\, B_2(t)]^{\mathrm{T}}$ is a 2-dimensional standard Brownian motion.

Let $a_{11} = -2$, $a_{12} = 1$, $a_{21} = -1$, $a_{22} = -2$. Using MATLAB, Figures 5–8 are obtained to exhibit the trajectories of the states of system (6.4) with $x(0) = [5, -1]^{\mathrm{T}}$, $x(0) = [4, 2]^{\mathrm{T}}$, $x(0) = [-3, -6]^{\mathrm{T}}$ and $x(0) = [-5, 10]^{\mathrm{T}}$, respectively.

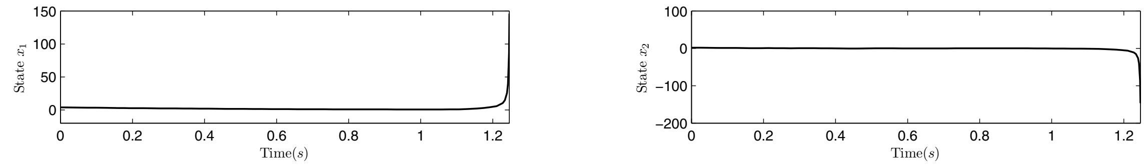

FIGURE 6. Trajectories with $x(0) = [4, 2]^{\mathrm{T}}$.

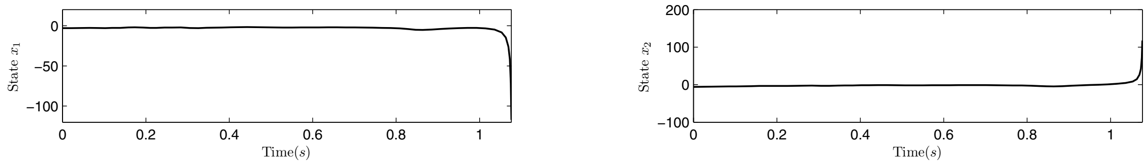

FIGURE 7. Trajectories with $x(0) = [-3, -6]^{\mathrm{T}}$.

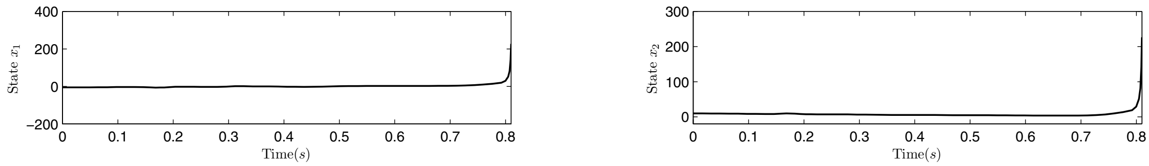

FIGURE 8. Trajectories with $x(0) = [-5, 10]^{\mathrm{T}}$.

7. Concluding remarks

In this paper, we have considered the stochastic stabilization and destabilization of the uncertain nonlinear systems with serious parameter unknowns. The time-varying technique is adopted to effectively compensate the serious unknowns, and based on this, a time-varying stochastic noise is introduced to establish the stabilization/destabilization with general decay/growth rate of uncertain nonlinear systems. To our knowledge, there have been some works on stochastic stabilization/destabilization of switching systems and functional differential systems (see *e.g.*, [1,22] and references therein), but the systems therein don't allow serious parameter unknowns. Therefore, a further research is to extend the results of this paper to the switching systems and functional differential systems with serious parameter unknowns, and to accomplish the stochastic stabilization/destabilization with general decay/growth rate.

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