HAMILTON–JACOBI EQUATIONS FOR OPTIMAL CONTROL ON JUNCTIONS AND NETWORKS *,**

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Abstract. We consider continuous-state and continuous-time control problems where the admissible trajectories of the system are constrained to remain on a network. A notion of viscosity solution of Hamilton–Jacobi equations on the network has been proposed in earlier articles. Here, we propose a simple proof of a comparison principle based on arguments from the theory of optimal control. We also discuss stability of viscosity solutions.

1. Introduction

A network (or a graph) is a set of items, referred to as vertices (or nodes/crosspoints), with connections between them referred to as edges. In the recent years there has been an increasing interest in the investigation of dynamical system and differential equation on networks, in particular in connection with problem of data transmission and traffic management (see for example Garavello–Piccoli [12], Engel *et al.* [9]). While control problems with state constrained in closures of open sets are well studied [8, 16, 22, 23] there is to our knowledge much fewer literature on problems on networks. The results of Frankowska and Plaskacz [10, 11] do apply to some closed sets with empty interior, but not to networks with crosspoints (except in very particular cases).

The literature on continuous-state and continuous-time control on networks is recent: the first two articles were published in 2012: control problems whose dynamics is constrained to a network and related Hamilton–Jacobi equations were studied in [1]: a Hamilton–Jacobi equation on the network was proposed, with a definition of viscosity solution, which reduces to the usual one if the network is a straight line (*i.e.* is composed of two parallel edges sharing an endpoint) and if the dynamics and cost are continuous; while in the interior of an edge, one can test the equation with a smooth test-function, the main difficulties arise at the vertices where the network does not have a regular differential structure. At a vertex, a notion of derivative similar to that of

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Dini's derivative (see for example [2]) was proposed: admissible test-functions are continuous functions whose restriction to each edge is $\mathcal{C}^1$. With this definition, the intrinsic geodesic distance, fixed one argument, is an admissible test-function with respect to the other argument. The Hamiltonian at a vertex depends on all directional derivatives in the directions of the edges containing the vertex (see Sect. 3.5) below. Independently, Imbert, Monneau and Zidani [14] proposed an equivalent notion of viscosity solution for studying a Hamilton–Jacobi approach to junction problems and traffic flows. There is also the work by Schieborn and Camilli [21], in which the authors focus on eikonal equations on networks and on a less general notion of viscosity solution.

Both [1,14] contain the first comparison and uniqueness results: in [1], suitably modified geodesic distances are used in the doubling variables method for proving comparison theorems under rather strong continuity assumptions. In [14], Imbert, Monneau and Zidani used a completely different argument based on the explicit solution of a related optimal control problem, which could be obtained because it was assumed that the Hamiltonians associated with each edge did not depend on the state variable.

A general comparison result has finally been obtained in the quite recent paper by Imbert–Monneau [13]. In the latter article, the Hamiltonians in the edges are completely independent from each other; the main assumption is that the Hamiltonian in each edge, say $H_i(x,p)$ for the edge indexed i, is bimonotone, *i.e.* non increasing (resp. non decreasing) for p smaller (resp. larger) than a given threshold $p_i^0(x)$. Of course, convex Hamiltonian coming from optimal control theory are bimonotone. Moreover, [13] handles more general transmission conditions than the previous articles, with an additional running cost at the junctions. In [13], the proof of the comparison result is rather involved and only uses arguments from the theory of partial differential equations: in the most simple case where all the Hamiltonians related to the edges are strictly convex and reach their minima at $p = 0$, the idea consists of doubling the variables and using a suitable test-function; then, in the general case, perturbation arguments are used for applying the results proved in the former case.

In coincidence with these research efforts about networks, Barles, Briani and Chasseigne, see [3,4], have recently studied control problems with discontinuous dynamics and costs, obtaining comparison results for some Bellman equations arising in this context, with original and elegant arguments. Related problems were also recently addressed by Rao *et al.* [19,20].

The aim of the present paper is to focus on optimal control problems with independent dynamics and running costs in the edges, and to show that the arguments in [3] can be adapted to yield a simple proof of a comparison result.

Sections 2 to 5 are devoted to the case of a junction, *i.e.* a network with one vertex only. Section 2 contains a description of the geometry and of the optimal control problem. In Section 3, a Hamilton–Jacobi equation is proposed for the value function, together with a notion of viscosity solution. It is proved that the value function is a viscosity solution of the Hamilton–Jacobi equation. Also in Section 3, Lemma 3.6 on the structure of the Hamiltonian at the vertex will be important for obtaining the comparison principle. Some important properties of viscosity sub and supersolutions are given in Section 4, and the comparison principle is proved in Section 5. In Section 6 we discuss the stability of the viscosity sub and super solution under perturbations of the Hamiltonians. In Section 7 we show that all the results can be easily extended to the case when there is an additional cost at the junction. Finally, in Section 8, the results obtained for the junction are generalized for networks with more than one vertices.

2. The junction

2.1. The geometry

Let us focus on the model case of a junction in $\mathbb{R}^d$ with N semi-infinite straight edges, $N > 1$. The edges are denoted by $(J_i)_{i=1,\ldots,N}$. The edge J_i is the closed half-line $\mathbb{R}^+ e_i$. The vectors e_i are two by two distinct unit vectors in $\mathbb{R}^d$. The half-lines J_i are glued at the origin O to form the junction $\mathcal{G}$:

$$\mathcal{G} = \bigcup_{i=1}^N J_i.$$

The geodetic distance $\mathrm{d}(x, y)$ between two points x, y of $\mathcal{G}$ is

$$\mathrm{d}(x, y) = \begin{cases} |x - y| & \text{if } x, y \text{ belong to the same edge } J_i \\ |x| + |y| & \text{if } x, y \text{ belong to different branches } J_i \text{ and } J_j. \end{cases}$$

2.2. The optimal control problem

We consider infinite horizon optimal control problems which have different dynamics and running costs in the edges. We are going to describe the assumptions on the dynamics and costs in each edge J_i. The sets of controls are denoted by A_i and the system is driven by a dynamics f_i and the running cost is given by ℓ_i. Our main assumptions are as follows:

[H0] A is a metric space (one can take $A = \mathbb{R}^m$). For $i = 1, \ldots, N$, A_i is a non empty compact subset of A and $f_i : J_i \times A_i \to \mathbb{R}$ is a continuous bounded function. The sets A_i are disjoint. Moreover, there exists $L > 0$ such that for any i, x, y in J_i and $a \in A_i$,

$$|f_i(x, a) - f_i(y, a)| \leq L|x - y|.$$

We will use the notation $F_i(x)$ for the set $\{f_i(x, a)e_i, a \in A_i\}$.

[H1] For $i = 1, \ldots, N$, the function $\ell_i : J_i \times A_i \to \mathbb{R}$ is a continuous and bounded function. There is a modulus of continuity ω_i such that for all x, y in J_i and for all $a \in A_i$, $|\ell_i(x, a) - \ell_i(y, a)| \leq \omega_i(|x - y|)$.

[H2] For $i = 1, \ldots, N$, $x \in J_i$, the non empty and closed set

$$\mathrm{FL}_i(x) \equiv \{(f_i(x, a)e_i, \ell_i(x, a)), a \in A_i\}$$

is convex.

[H3] There is a real number $\delta > 0$ such that for any $i = 1, \ldots, N$,

$$[-\delta e_i, \delta e_i] \subset F_i(O).$$

Remark 2.1. In [H0] the assumption that the sets A_i are disjoint is not restrictive: it is made only for simplifying the proof of Theorem 2.3 below. Indeed, if A_i are not disjoint, then we define $\tilde{A}_i = \{i\} \times A_i$ and $\tilde{f}_i(x, \tilde{a}) = f_i(x, a)$, $\tilde{\ell}_i(x, \tilde{a}) = \ell_i(x, a)$ if $x \in J_i$ and $\tilde{a} = (i, a)$ with $a \in A_i$.
The sets $\tilde{A}_i$ are disjoint compact subsets of $\tilde{A} = \cup_{i=1,\ldots,n} \tilde{A}_i$ which is a (compact) metric space for the distance $\tilde{d}((i, a), (j, b)) = |i - j| + d_A(a, b)$, and the functions $\tilde{f}_i$, $\tilde{\ell}_i$ inherit the properties of f_i and ℓ_i.

The assumption [H2] is not essential: it is made in order to avoid the use of relaxed controls.
Assumption [H3] is a *strong* controllability condition at the vertex (which implies the coercivity of the Hamiltonian). It has already been widely used in the framework of networks (for instance, the same assumption is made in [1,3], and the coercivity of the Hamiltonian is assumed [13]). We will see that [H3] yields the continuity of the value function. Without any controllability condition, the value function may not be continuous and the definition of the viscosity solutions should differ from the one proposed below. There are of course milder controlabilty conditions, but with them, our techniques do not seem to apply in a straightforward manner.
Here is a general version of Filippov implicit function lemma, see [17], which will be useful to prove Theorem 2.3 below.

Theorem 2.2. *Let I be an interval of $\mathbb{R}$ and $\gamma : I \to \mathbb{R}^d \times \mathbb{R}^d$ be a measurable function. Let K be a closed subset of $\mathbb{R}^d \times A$ and $\Psi : K \to \mathbb{R}^d \times \mathbb{R}^d$ be continuous. Assume that $\gamma(I) \subset \Psi(K)$, then there is a measurable function $\Phi : I \to K$ with*

$$\Psi \circ \Phi(t) = \gamma(t) \quad \text{for a.a. } t \in I.$$

Proof. See [17]. □

Let us denote by M the set:

$$M = \left\{(x,a);\ x \in \mathcal{G}, \quad a \in A_i \text{ if } x \in J_i\backslash\{O\}, \text{ and } a \in \cup_{i=1}^N A_i \text{ if } x = O\right\}. \tag{2.1}$$

The set M is closed. We also define the function f on M by

$$\forall (x,a) \in M, \qquad f(x,a) = \begin{cases} f_i(x,a)e_i & \text{if } x \in J_i\backslash\{O\}, \\ f_i(O,a)e_i & \text{if } x = O \text{ and } a \in A_i. \end{cases}$$

The function f is continuous on M because the sets A_i are disjoint. Let $\tilde{F}(x)$ be defined by

$$\tilde{F}(x) = \begin{cases} F_i(x) & \text{if } x \text{ belongs to the edge } J_i\backslash\{O\} \\ \cup_{i=1}^N F_i(O) & \text{if } x = O. \end{cases}$$

For $x \in \mathcal{G}$, the set of admissible trajectories starting from x is

$$Y_x = \left\{y_x \in Lip(\mathbb{R}^+;\mathcal{G})\ :\ \begin{matrix}\dot{y}_x(t) \in \tilde{F}(y_x(t)), & \text{for a.a. } t > 0, \\ y_x(0) = x, \end{matrix}\right\}. \tag{2.2}$$

Theorem 2.3. *Assume* [H0], [H1], [H2] *and* [H3]. *Then*

1. *For any $x \in \mathcal{G}$, Y_x is non empty.*
2. *For any $x \in \mathcal{G}$, for each trajectory y_x in Y_x, there exists a measurable function $\Phi : [0,+\infty) \to M$, $\Phi(t) = (\varphi_1(t), \varphi_2(t))$ with*
$$(y_x(t), \dot{y}_x(t)) = (\varphi_1(t), f(\varphi_1(t), \varphi_2(t))), \quad \text{for a.e. } t,$$
   which means in particular that y_x is a continuous representation of φ_1
3. *Almost everywhere in $[0,+\infty)$,*

$$\dot{y}_x(t) = \sum_{i=1}^N 1_{\{y_x(t)\in J_i\backslash\{O\}\}} f_i(y_x(t), \varphi_2(t))e_i.$$

4. *Almost everywhere on $\{t : y_x(t) = O\}$, $f(O, \varphi_2(t)) = 0$.*

Proof. The proof of point 1 is easy, because $0 \in \tilde{F}(O)$.
The proof of point 2 is a consequence of Theorem 2.2, with $K = M$, $I = [0, +\infty)$, $\gamma(t) = (y_x(t), \dot{y}_x(t))$ and $\Psi(x,a) = (x, f(x,a))$.
From point 2, we deduce that

$$\dot{y}_x(t) = \sum_{i=1}^N 1_{\{y_x(t)\in J_i\backslash\{O\}\}} f_i(y_x(t), \varphi_2(t))e_i + 1_{\{y_x(t)=O\}} f(O, \varphi_2(t)),$$

and from Stampacchia's theorem, $f(O, \varphi_2(t)) = 0$ almost everywhere in $\{t : y_x(t) = O\}$. This yields points 3 and 4. $\qquad\square$

It is worth noticing that in Theorem 2.3, a solution y_x can be associated with several control laws $\varphi_2(\cdot)$. We introduce the set of admissible controlled trajectories starting from the initial datum x:

$$\mathcal{T}_x = \left\{\begin{matrix}(y_x, \alpha) \in L^\infty_{\text{Loc}}(\mathbb{R}^+; M) : & y_x \in Lip(\mathbb{R}^+;\mathcal{G}), \\ & y_x(t) = x + \int_0^t f(y_x(s), \alpha(s))ds & \text{in } \mathbb{R}^+ \end{matrix}\right\}. \tag{2.3}$$

Remark 2.4. If two different edges are aligned with each other, say the edges J_1 and J_2, many other assumptions can be made on the dynamics and costs:

- a trivial case in which the assumptions [H1]-[H3] are satisfied is when the dynamics and costs are continuous at the origin, *i.e.* $A_1 = A_2$; f_1 and f_2 are respectively the restrictions to $J_1 \times A_1$ and $J_2 \times A_2$ of a continuous and bounded function $f_{1,2}$ defined in $\mathbb{R}e_1 \times A_1$, which is Lipschitz continuous with respect to the first variable; ℓ_1 and ℓ_2 are respectively the restrictions to $J_1 \times A_1$ and $J_2 \times A_2$ of a continuous and bounded function $\ell_{1,2}$ defined in $\mathbb{R}e_1 \times A_1$.
- In this particular geometrical setting, one can allow some mixing (relaxation) at the vertex with several possible rules: More precisely, in [3, 4], Barles *et al.* introduce several kinds of trajectories which stay at the junction: the *regular* trajectories are obtained by mixing outgoing dynamics from J_1 and J_2, whereas *singular* trajectories are obtained by mixing strictly ingoing dynamics from J_1 and J_2. Two different value functions are obtained whether singular mixing is permitted or not.

The cost functional. The cost associated to the trajectory $(y_x, \alpha) \in \mathcal{T}_x$ is

$$J(x; (y_x, \alpha)) = \int_0^\infty \ell(y_x(t), \alpha(t)) \mathrm{e}^{-\lambda t} \mathrm{d}t, \tag{2.4}$$

where $\lambda > 0$ is a real number and the Lagrangian ℓ is defined on M by

$$\forall (x, a) \in M, \qquad \ell(x, a) = \begin{cases} \ell_i(x, a) & \text{if } x \in J_i \backslash \{O\}, \\ \ell_i(O, a) & \text{if } x = O \text{ and } a \in A_i. \end{cases}$$

The value function. The value function of the infinite horizon optimal control problem is

$$u(x) = \inf_{(y_x, \alpha) \in \mathcal{T}_x} J(x; (y_x, \alpha)). \tag{2.5}$$

Proposition 2.5. *Assume* [H0],[H1], [H2] *and* [H3]. *Then the value function v is bounded and continuous on $\mathcal{G}$.*

Proof. The proof essentially uses Assumption [H3]. Since it is classical, we skip it. □

3. The Hamilton–Jacobi equation

3.1. Test-functions

For the definition of viscosity solutions on the irregular set $\mathcal{G}$, it is necessary to first define a class of the admissible test-functions

Definition 3.1. A function $\varphi : \mathcal{G} \to \mathbb{R}$ is an admissible test-function if

- φ is continuous in $\mathcal{G}$
- for any j, $j = 1, \dots, N$, $\varphi|_{J_j} \in \mathcal{C}^1(J_j)$.

The set of admissible test-functions is noted $\mathcal{R}(\mathcal{G})$. If $\varphi \in \mathcal{R}(\mathcal{G})$ and $\zeta \in \mathbb{R}$, let $D\varphi(x, \zeta e_i)$ be defined by $D\varphi(x, \zeta e_i) = \zeta \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(x)$ if $x \in J_i \backslash \{O\}$ and $D\varphi(O, \zeta e_i) = \zeta \lim_{h \to 0+} \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(he_i)$.

Property 3.2. *If $\varphi = g \circ \psi$ with $g \in \mathcal{C}^1$ and $\psi \in \mathcal{R}(\mathcal{G})$, then $\varphi \in \mathcal{R}(\mathcal{G})$ and*

$$D\varphi(O, \zeta) = g'(\psi(O))D\psi(O, \zeta).$$

3.2. Vector fields

For $i = 1, \ldots, N$, we denote by $F_i^+(O)$ and $\mathrm{FL}_i^+(O)$ the sets

$$F_i^+(O) = F_i(O) \cap \mathbb{R}^+ e_i, \qquad \mathrm{FL}_i^+(O) = \mathrm{FL}_i(O) \cap (\mathbb{R}^+ e_i \times \mathbb{R}),$$

which are non empty thanks to assumption [H3]. Note that $0 \in \cap_{i=1}^N F_i(O)$. From assumption [H2], these sets are compact and convex. For $x \in \mathcal{G}$, the sets $F(x)$ and $\mathrm{FL}(x)$ are defined by

$$F(x) = \begin{cases} F_i(x) & \text{if } x \text{ belongs to the edge } J_i \backslash \{O\} \\ \cup_{i=1,\ldots,N} F_i^+(O) & \text{if } x = O, \end{cases}$$

and

$$\mathrm{FL}(x) = \begin{cases} \mathrm{FL}_i(x) & \text{if } x \text{ belongs to the edge } J_i \backslash \{O\} \\ \cup_{i=1,\ldots,N} \mathrm{FL}_i^+(O) & \text{if } x = O. \end{cases}$$

3.3. Definition of viscosity solutions

We now introduce the definition of a viscosity solution of

$$\lambda u(x) + \sup_{(\zeta,\xi)\in \mathrm{FL}(x)} \{-Du(x,\zeta) - \xi\} = 0 \quad \text{in } \mathcal{G}. \tag{3.1}$$

Definition 3.3. • An upper semi-continuous function $u : \mathcal{G} \to \mathbb{R}$ is a subsolution of (3.1) in $\mathcal{G}$ if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at x, then

$$\lambda u(x) + \sup_{(\zeta,\xi)\in \mathrm{FL}(x)} \{-D\varphi(x,\zeta) - \xi\} \leq 0; \tag{3.2}$$

• A lower semi-continuous function $u : \mathcal{G} \to \mathbb{R}$ is a supersolution of (3.1) if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local minimum point at x, then

$$\lambda u(x) + \sup_{(\zeta,\xi)\in \mathrm{FL}(x)} \{-D\varphi(x,\zeta) - \xi\} \geq 0; \tag{3.3}$$

• A continuous function $u : \mathcal{G} \to \mathbb{R}$ is a viscosity solution of (3.1) in $\mathcal{G}$ if it is both a viscosity subsolution and a viscosity supersolution of (3.1) in $\mathcal{G}$.

Remark 3.4. At $x \in J_i \backslash \{O\}$, the notion of sub, respectively super-solution in Definition 3.3 is equivalent to the standard definition of viscosity sub, respectively super-solution of

$$\lambda u(x) + \sup_{a \in A_i} \{-f_i(x,a) \cdot Du(x) - \ell_i(x,a)\} = 0.$$

3.4. Hamiltonians

We define the Hamiltonians $H_i : J_i \times \mathbb{R} \to \mathbb{R}$ by

$$H_i(x,p) = \max_{a \in A_i}(-pf_i(x,a) - \ell_i(x,a)) \tag{3.4}$$

and the Hamiltonian $H_O : \mathbb{R}^N \to \mathbb{R}$ by

$$H_O(p_1, \ldots, p_N) = \max_{i=1,\ldots,N} \max_{a \in A_i \text{ s.t. } f_i(O,a)\geq 0}(-p_i f_i(O,a) - \ell_i(O,a)). \tag{3.5}$$

We also define what may be called the tangential Hamiltonian at O by

$$H_O^T = -\min_{i=1,\ldots,N} \min_{a \in A_i \text{ s.t. } f_i(O,a)=0} \ell_i(O,a). \tag{3.6}$$

Thanks to the definitions of $\mathrm{FL}_i(x)$ (in particular of $\mathrm{FL}_i^+(O)$), the continuity properties of the data and the compactness of A_i, one easily notes that the following definition is equivalent to Definition 3.3.

Definition 3.5. • An upper semi-continuous function $u : \mathcal{G} \to \mathbb{R}$ is a subsolution of (3.1) in $\mathcal{G}$ if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at x, then

$$
\begin{aligned}
&\lambda u(x) + H_i\left(x, \tfrac{\mathrm{d}\varphi}{\mathrm{d}x_i}(x)\right) \leq 0 && \text{if } x \in J_i \backslash \{O\}, \\
&\lambda u(O) + H_O\left(\tfrac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \dots, \tfrac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O)\right) \leq 0.
\end{aligned}
\tag{3.7}
$$

• A lower semi-continuous function $u : \mathcal{G} \to \mathbb{R}$ is a supersolution of (3.1) if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local minimum point at x, then

$$
\begin{aligned}
&\lambda u(x) + H_i\left(x, \tfrac{\mathrm{d}\varphi}{\mathrm{d}x_i}(x)\right) \geq 0 && \text{if } x \in J_i \backslash \{O\}, \\
&\lambda u(O) + H_O\left(\tfrac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \dots, \tfrac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O)\right) \geq 0.
\end{aligned}
\tag{3.8}
$$

The Hamiltonian H_i are continuous with respect to $x \in J_i$, convex with respect to p. Moreover $p \mapsto H_i(O, p)$ is coercive, *i.e.* $\lim_{|p| \to +\infty} H_i(O, p) = +\infty$ from the controlability assumption [H3]. Following Imbert–Monneau [13], we introduce the nonempty compact interval $\mathcal{P}_0^i$

$$
\mathcal{P}_0^i = \{p_0^i \in \mathbb{R} \text{ s.t. } H_i(O, p_0^i) = \min_{p \in \mathbb{R}} H_i(O, p)\}.
\tag{3.9}
$$

Lemma 3.6. *Assume* [H0], [H1], [H2] *and* [H3], *then*

1. $p_0^i \in \mathcal{P}_0^i$ *if and only if there exists* $a^* \in A_i$ *such that* $f_i(O, a^*) = 0$ *and* $H_i(O, p_0^i) = -p_0^i f_i(O, a^*) - \ell_i(O, a^*) = -\ell_i(O, a^*)$

2.

$$
\min_{p \in \mathbb{R}} H_i(O, p) = - \min_{a \in A_i \ s.t. \ f_i(O,a)=0} \ell_i(O, a)
\tag{3.10}
$$

3. *For all* $p \in \mathbb{R}$, *if* $p \geq p_0^i$ *for some* $p_0^i \in \mathcal{P}_0^i$ *then*

$$
\max_{a \in A_i \ s.t. \ f_i(O,a) \geq 0} (-p f_i(O, a) - \ell_i(O, a)) = \min_{q \in \mathbb{R}} H_i(O, q) = - \min_{a \in A_i \ s.t. \ f_i(O,a)=0} \ell_i(O, a).
$$

Proof. The Hamiltonian H_i reaches its minimum at p_0^i if and only if $0 \in \partial H_i(O, p_0^i)$. The subdifferential of $H_i(O, \cdot)$ at p_0^i is characterized by

$$
\partial H_i(O, p_0^i) = \overline{\mathrm{co}}\{-f_i(O, a); a \in A_i \text{ s.t. } H_i(O, p_0^i) = -p_0^i f_i(O, a) - \ell_i(O, a)\},
$$

see [24]. But from [H2],

$$
\{(f_i(O, a), \ell_i(O, a)); a \in A_i \text{ s.t. } H_i(O, p_0^i) = -p_0^i f_i(O, a) - \ell_i(O, a)\}
$$

is compact and convex. Hence,

$$
\partial H_i(O, p_0^i) = \{-f_i(O, a); a \in A_i \text{ s.t. } H_i(O, p_0^i) = -p_0^i f_i(O, a) - \ell_i(O, a)\}.
$$

Therefore, $0 \in \partial H_i(O, p_0^i)$ if and only if there exists $a^* \in A_i$ such that $f_i(O, a^*) = 0$ and $H_i(O, p_0^i) = -\ell_i(O, a^*)$. We have proved point 1.

Point 2 is a direct consequence of point 1.

If p is greater than or equal to some $p_0^i \in \mathcal{P}_0^i$, then

$$
\max_{a \in A_i : f_i(O,a) \geq 0} (-p f_i(O, a) - \ell_i(O, a)) \leq \max_{a \in A_i : f_i(O,a) \geq 0} (-p_0^i f_i(O, a) - \ell_i(O, a)) = H_i(O, p_0^i)
$$

where the last identity comes from point 1.

On the other hand,

$$
\max_{a \in A_i : f_i(O,a) \geq 0} (-p f_i(O, a) - \ell_i(O, a)) \geq - \min_{a \in A_i : f_i(O,a)=0} \ell_i(O, a).
$$

Point 3 is obtained by combining the two previous observations and point 2. $\qquad \square$
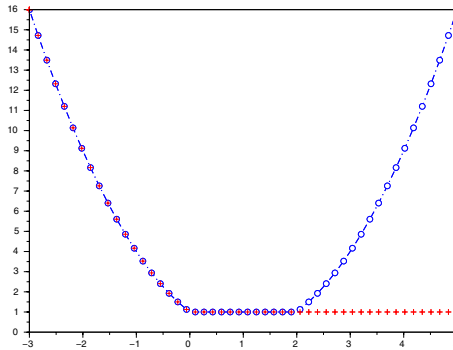
FIGURE 1. The graphs of the Hamiltonian $p \mapsto H_i(O, p)$ (with the circles) and of $p \mapsto$ $H_i^+(O, p) \equiv \max_{a \in A_i : f_i(O,a) \geq 0}(-pf_i(O,a) - \ell_i(O,a))$ (with the signs +). In the example, $\mathcal{P}_0^i = [0, 2]$.

Remark 3.7. It can also be proved that $p \leq \max(q \in \mathcal{P}_0^i)$ if and only if

$$H_i(O, p) = \max_{a \in A_i : f_i(O,a) \geq 0}(-pf_i(O,a) - \ell_i(O,a)).$$

In Figure 1, we give an example for the graphs of $p \mapsto H_i(O, p)$ and of $p \mapsto H_i^+(O, p) \equiv$ $\max_{a \in A_i : f_i(O,a) \geq 0}(-pf_i(O,a) - \ell_i(O,a))$, and the related interval $\mathcal{P}_0^i$.

3.5. Existence

Theorem 3.8. *Assume* [H0]*,* [H1]*,* [H2] *and* [H3]*. The value function v defined in* (2.5) *is a bounded viscosity solution of* (3.1) *in $\mathcal{G}$.*

The proof of Theorem 3.8 is made in several steps, namely Proposition 3.9 and Lemmas 3.10 and 3.11 below: the first step consists of proving that the value function is a viscosity solution of a Hamilton–Jacobi equation with a more general definition of the Hamiltonian: for that, we introduce larger relaxed vector fields: for $x \in \mathcal{G}$,

$$\tilde{f}(x) = \left\{ \eta \in \mathbb{R}^d : \begin{array}{l} \exists (y_{x,n}, \alpha_n)_{n \in \mathbb{N}}, \\ (y_{x,n}, \alpha_n) \in \mathcal{T}_x, \\ \exists (t_n)_{n \in \mathbb{N}} \end{array} \text{ s.t. } \begin{array}{l} t_n \to 0^+ \text{ and} \\ \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f(y_{x,n}(t), \alpha_n(t)) \mathrm{d}t = \eta \end{array} \right\}$$

and

$$\widetilde{f\ell}(x) =$$

$$\left\{ (\eta, \mu) \in \mathbb{R}^d \times \mathbb{R} : \begin{array}{l} \exists (y_{x,n}, \alpha_n)_{n \in \mathbb{N}}, \\ (y_{x,n}, \alpha_n) \in \mathcal{T}_x, \\ \exists (t_n)_{n \in \mathbb{N}} \end{array} \text{ s.t. } \left| \begin{array}{l} t_n \to 0^+, \\ \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f(y_{x,n}(t), \alpha_n(t)) \mathrm{d}t = \eta, \\ \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \ell(y_{x,n}(t), \alpha_n(t)) \mathrm{d}t = \mu \end{array} \right. \right\}.$$

Proposition 3.9. *Assume* [H0]*,* [H1]*,* [H2] *and* [H3]*. The value function v defined in* (2.5) *is a viscosity solution of*

$$\lambda u(x) + \sup_{(\zeta, \xi) \in \widetilde{f\ell}(x)} \{-Du(x, \zeta) - \xi\} = 0 \quad in \ \mathcal{G}, \tag{3.11}$$

where the definition of viscosity solution is exactly the same as Definition 3.3, replacing $\mathrm{FL}(x)$ *with* $\widetilde{f\ell}(x)$*.*

Proof. See [1]. □

For all $\varphi \in \mathcal{R}(\mathcal{G})$, it is clear that if $x \in J_i \backslash \{O\}$, then $H_i(x, D\varphi) = \sup_{(\zeta, \xi) \in \widetilde{f\ell}(x)} \{-D\varphi(x, \zeta) - \xi\}$. We are left with comparing $\sup_{(\zeta, \xi) \in \mathrm{FL}(O)} \{-D\varphi(O, \zeta) - \xi\}$ and $\sup_{(\zeta, \xi) \in \widetilde{f\ell}(O)} \{-D\varphi(O, \zeta) - \xi\}$. The two quantities are the same. This is a consequence of the following lemma

Lemma 3.10.

$$\widetilde{f\ell}(O) = \bigcup_{i=1,\ldots,N} \overline{\mathrm{co}} \left\{ \mathrm{FL}_i^+(O) \cup \bigcup_{j \neq i} \Big(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \Big) \right\}.$$

Proof. The proof being a bit long, we postpone it to the appendix. □

Lemma 3.11. *Assume* [H0], [H1], [H2] *and* [H3]. *For any function* φ *in* $\mathcal{R}(\mathcal{G})$,

$$\sup_{(\zeta, \xi) \in \widetilde{f\ell}(O)} \{-D\varphi(O, \zeta) - \xi\} = \max_{(\zeta, \xi) \in \mathrm{FL}(O)} \{-D\varphi(O, \zeta) - \xi\}. \tag{3.12}$$

Proof. It was proved in [1] that $\mathrm{FL}(O) \subset \widetilde{f\ell}(O)$. Hence

$$\max_{(\zeta, \xi) \in \mathrm{FL}(O)} \{-D\varphi(O, \zeta) - \xi\} \leq \sup_{(\zeta, \xi) \in \widetilde{f\ell}(O)} \{-D\varphi(O, \zeta) - \xi\}.$$

From the piecewise linearity of the function $(\zeta, \mu) \mapsto -D\varphi(O, \zeta) - \mu$, we infer that

$$\sup_{(\zeta, \mu) \in \overline{\mathrm{co}} \left\{ \mathrm{FL}_i^+(O) \cup \bigcup_{j \neq i} \Big(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R}) \Big) \right\}} (-D\varphi(O, \zeta) - \mu)$$

$$= \max \left(\max_{(\zeta, \mu) \in \mathrm{FL}_i^+(O)} (-D\varphi(O, \zeta) - \mu), \max_{j \neq i} \max_{(0, \mu) \in \mathrm{FL}_j(O)} -\mu \right)$$

$$\leq \max_{j=1,\ldots,N} \max_{(\zeta, \mu) \in \mathrm{FL}_j^+(O)} -D\varphi(O, \zeta) - \mu) = \max_{(\zeta, \xi) \in \mathrm{FL}(O)} \{-D\varphi(O, \zeta) - \xi\}.$$

We conclude by using Lemma 3.10. □

4. PROPERTIES OF VISCOSITY SUB AND SUPERSOLUTIONS

In this part, we study sub and supersolutions of (3.1), transposing ideas coming from Barles-Briani-Chasseigne [3, 4] to the present context.

Lemma 4.1. *Assume* [H0], [H1], [H2] *and* [H3]. *Let* R *be a positive real number such that for all* $i = 1, \ldots, N$ *and* $x \in B(O, R) \cap J_i$

$$\left[-\frac{\delta}{2} e_i, \frac{\delta}{2} e_i \right] \subset F_i(x).$$

For any bounded viscosity subsolution u *of* (3.1), *there exists a constant* $C^* > 0$ *such that* u *is a viscosity subsolution of*

$$|Du(x)| \leq C^* \quad in \ B(O, R) \cap \mathcal{G},$$

i.e. *for any* $x \in B(O, R) \cap \mathcal{G}$ *and* $\varphi \in \mathcal{R}(\mathcal{G})$ *such that* $u - \varphi$ *has a local maximum point at* x,

$$\left| \frac{d\varphi}{dx_i}(x) \right| \leq C^* \qquad if \ x \in B(O, R) \cap J_i \backslash \{O\} \tag{4.1}$$

$$\min_i \frac{d\varphi}{dx_i}(O) \geq -C^* \qquad\qquad if \ x = O. \tag{4.2}$$

Proof. Let M_u (resp M_ℓ) be an upper bound on $|u|$ (resp. ℓ_j for all $j = 1, \ldots, N$). The viscosity inequality (3.7) yields that

$$H_i \left(x, \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(x) \right) \leq \lambda M_u \quad \text{if } x \in B(O, R) \cap J_i \backslash \{O\}, \tag{4.3}$$

$$H_O \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O) \right) \leq \lambda M_u \quad \text{if } x = O. \tag{4.4}$$

From the controlability in $B(O, R) \cap J_i$, we see that H_i is coercive with respect to its second argument uniformly in $x \in B(O, R) \cap J_i$, and more precisely that $H_i(x, p) \geq \frac{\delta}{2}|p| - M_\ell$.

Thus, from (4.3), there exists a constant $C^* = 2\frac{\lambda M_u + M_\ell}{\delta}$ such that $|\frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(x)| \leq C^*$ if $x \in B(O, R) \cap J_i \backslash \{O\}$.

If $x = O$, we use the fact that $H_i^+(O, p) \geq \frac{\delta}{2} \max(0, -p) - M_\ell$. The viscosity inequality (4.4) then yields that $\min_i \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O) \geq -C^*$. $\qquad \square$

Lemma 4.2. *Assume* [H0], [H1], [H2] *and* [H3]. *There exists a neighborhood of O in $\mathcal{G}$ in which any bounded viscosity subsolution u of* (3.1) *is Lipschitz continuous.*

Proof. We adapt the proof of Ishii, see [15].
Take R as in Lemma 4.1, fix $z \in B(O, R) \cap \mathcal{G}$ and set $r = (R - |z|)/4$. Fix any $y \in \mathcal{G}$ such that $\mathrm{d}(y, z) < r$. It can be checked that for any $x \in \mathcal{G}$, if $\mathrm{d}(x, y) < 3r$ then $\mathrm{d}(x, O) < R$. Choose a function $f \in \mathcal{C}^1([0, 3r))$ such that $f(t) = t$ in $[0, 2r]$ and $f'(t) \geq 1$ for all $t \in [0, 3r)$ and $\lim_{t \to 3r} f(t) = +\infty$. Fix any $\varepsilon > 0$. We are going to show that

$$u(x) \leq u(y) + (C^* + \varepsilon)f(\mathrm{d}(x, y)), \quad \forall x \in \mathcal{G} \text{ such that } \mathrm{d}(x, y) < 3r, \tag{4.5}$$

where C^* is the constant in Lemma 4.1.

Let us proceed by contradiction. Assume that (4.5) is not true. According to the properties of f, the function $x \mapsto u(x) - u(y) - (C^* + \varepsilon)f(\mathrm{d}(x, y))$ admits a maximum $\xi \in B(y, 3r) \cap \mathcal{G}$. However, from the fact (4.5) is not true, we deduce that $\xi \neq y$. Hence, it is possible to modify the function $\psi : \mathcal{G} \to \mathbb{R}$, $x \mapsto (C^* + \varepsilon)f(\mathrm{d}(x, y))$ away from a neighborhood of ξ and obtain an admissible test function that we use in the viscosity inequality satisfied by u; from (4.1) and (4.2) in Lemma 4.1 and from explicit calculatiobns concerning the derivatives of $\mathrm{d}(x, y)$ at the point ξ, we obtain that

$$(C^* + \varepsilon)f'(\mathrm{d}(\xi, y)) \leq C^*,$$

which leads to a contradiction.

If $\mathrm{d}(x, z) < r$ then $\mathrm{d}(x, y) < 2r$ and $f(\mathrm{d}(x, y)) = \mathrm{d}(x, y)$. In this case, (4.5) yields that

$$u(x) \leq u(y) + (C^* + \varepsilon)\mathrm{d}(x, y), \quad \forall x, y \in \mathcal{G} \text{ s.t } \mathrm{d}(x, z) < r, \ \mathrm{d}(y, z) < r.$$

By symmetry, we get

$$|u(x) - u(y)| \leq (C^* + \varepsilon)\mathrm{d}(x, y), \quad \forall x, y \in \mathcal{G} \text{ s.t } \mathrm{d}(x, z) < r, \ \mathrm{d}(y, z) < r,$$

and by letting ε tend to zero:

$$|u(x) - u(y)| \leq C^*\mathrm{d}(x, y), \quad \forall x, y \in \mathcal{G} \text{ s.t } \mathrm{d}(x, z) < r, \ \mathrm{d}(y, z) < r. \tag{4.6}$$

Now, for two arbitrary points x, y in $\mathcal{G} \cap B(O, R)$, we take $r = \frac{1}{4}\min(R - |x|, R - |y|)$ and choose a finite sequence $(z_j)_{j=1,\ldots,M} \in \mathcal{G}$ belonging to the segment $[x, y]$ if x and y belong to some J_i or to $[O, x] \cup [O, y]$ in the opposite case, and such that $z_1 = x$, $z_M = y$ and $\mathrm{d}(z_i, z_{i+1}) < r$ for all $i = 1, \ldots, M - 1$ and $\sum_{i=1}^{M-1} \mathrm{d}(z_i, z_{i+1}) = \mathrm{d}(x, y)$. From (4.6), we get that

$$|u(x) - u(y)| \leq C^*\mathrm{d}(x, y), \quad \forall x, y \in \mathcal{G} \cap B(O, R). \qquad \square$$

Lemma 4.3. *Assume* [H0], [H1], [H2] *and* [H3]. *Any bounded viscosity subsolution u of* (3.1) *is such that*

$$\lambda u(O) \leq -H_O^T. \tag{4.7}$$

Proof. Since, from Lemma 4.2, u is Lipschitz continuous in a neighborhood of O, we know that there exists a test-function φ in $\mathcal{R}(\mathcal{G})$ which touches u from above at O. Since u is a subsolution of (3.1), we see that $\lambda u(O) + H_O(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O)) \leq 0$, which implies that $\lambda u(O) + H_O^T \leq 0$. □

Remark 4.4. It is interesting to note that in [3] and [4], a condition similar to (4.7) is introduced to characterize a particular viscosity solution of the transmission problem studied there among all the possible solutions in the sense of Ishii, (this condition is not satisfied by all subsolutions).

In the present context, the fact that (4.7) is automatically satisfied by subsolutions seems to be linked to the richness of the space $\mathcal{R}(\mathcal{G})$: for any Lipschitz function u defined in a neighborhood of O, there exists $\varphi \in \mathcal{R}(\mathcal{G})$ such that $u - \varphi$ has a maximum at O.

The following lemma can be found in [3,4] in a different context:

Lemma 4.5. *Let $v : \mathcal{G} \to \mathbb{R}$ be a viscosity supersolution of* (3.1) *in $\mathcal{G}$ and w be a continuous viscosity subsolution of* (3.1) *in $\mathcal{G}$. Then if $x \in J_i \backslash \{0\}$, we have for all $t > 0$,*

$$v(x) \geq \inf_{\alpha_i(\cdot), \theta_i} \left(\int_0^{t \wedge \theta_i} \ell_i(y_x^i(s), \alpha_i(s)) e^{-\lambda s} \mathrm{d}s + v(y_x^i(t \wedge \theta_i)) e^{-\lambda(t \wedge \theta_i)} \right), \tag{4.8}$$

and

$$w(x) \leq \inf_{\alpha_i(\cdot), \theta_i} \left(\int_0^{t \wedge \theta_i} \ell_i(y_x^i(s), \alpha_i(s)) e^{-\lambda s} \mathrm{d}s + w(y_x^i(t \wedge \theta_i)) e^{-\lambda(t \wedge \theta_i)} \right), \tag{4.9}$$

where $\alpha_i \in L^\infty(0, \infty; A_i)$, y_x^i is the solution of $y_x^i(t) = x + \left(\int_0^t f_i(y_x^i(s), \alpha_i(s)) \mathrm{d}s \right) e_i$ and θ_i is such that $y_x^i(\theta_i) = 0$ and θ_i lies in $[\tau_i, \bar{\tau}_i]$, where τ_i is the exit time of y_x^i from $J_i \backslash \{O\}$ and $\bar{\tau}_i$ is the exit time of y_x^i from J_i.

Proof. See [3] for the detailed proof. We restrict ourselves to mentioning that the proof of (4.8) uses the results of Blanc [6,7] on the minimal supersolution of exit time control problems, whereas the proof of (4.9) uses the comparison results of Barles–Perthame [5] and the continuity of w. □

The following theorem is reminiscent of Theorem 3.3 in [3]:

Theorem 4.6. *Assume* [H0], [H1], [H2] *and* [H3]. *Let $v : \mathcal{G} \to \mathbb{R}$ be a viscosity supersolution of* (3.1), *bounded from below by $-c|x| - C$ for two positive numbers c and C. Either* [A] *or* [B] *below is true:*

[**A**] *There exists $\eta > 0$, $i \in \{1, \ldots, N\}$ and a sequence $x_k \in J_i$, $\lim_{k \to +\infty} x_k = O$ such that $\lim_{k \to +\infty} v(x_k) = v(O)$ and for each k, there exists a control law α_i^k such that the corresponding trajectory $y_{x_k}(s) \in J_i$ for all $s \in [0, \eta]$ and*

$$v(x_k) \geq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s)) e^{-\lambda s} \mathrm{d}s + v(y_{x_k}(\eta)) e^{-\lambda \eta} \tag{4.10}$$

[**B**]

$$\lambda v(O) + H_O^T \geq 0. \tag{4.11}$$

Proof. Let us assume that [B] does not hold.
For any i in $\{1, \ldots, N\}$, take for example

$$q_i = \min_{p_0^i \in \mathcal{P}_0^i} p_0^i,$$

and $q = (q_1, \ldots, q_N)$. From Lemma 3.8,

$$H_O(q) = H_O^T. \tag{4.12}$$

Consider the function

$$v(x) - q_i|x| + \frac{|x|^2}{\varepsilon^2} \quad \text{if } x \in J_i.$$

Standard arguments show that this function reaches its minimum near O and any sequence of such minimum points x_ε converges to O and that $v(x_\varepsilon)$ converges to $v(O)$.

It is not possible that x_ε be O, because since v is a viscosity supersolution of (3.1), we would have that

$$\lambda v(O) + H_O(q) \geq 0,$$

and therefore $\lambda v(O) + H_O^T \geq 0$, which is a contradiction since [B] does not hold.

Therefore, there exists $i \in \{1, \ldots, N\}$ such that, up to the extraction of a subsequence, $x_\varepsilon \in J_i \backslash \{O\}$, for all ε. We can therefore apply Lemma 4.5: for any $t > 0$,

$$v(x_\varepsilon) \geq \inf_{\alpha_i(\cdot), \theta_i} \left(\int_0^{t \wedge \theta_i} \ell_i(y_{x_\varepsilon}^i(s), \alpha_i(s)) e^{-\lambda s} ds + v(y_{x_\varepsilon}^i(t \wedge \theta_i)) e^{-\lambda(t \wedge \theta_i)} \right), \tag{4.13}$$

where y_x^i is the solution of $y_x^i(t) = x + \left(\int_0^t f_i(y_x^i(s), \alpha_i(s)) ds \right) e_i$.

Take $t = 1$ for example. From [H0] and [H2], the minimum in (4.13) is reached for some $\alpha_{i,\varepsilon}$ and $\theta_{i,\varepsilon} > 0$, see [3]:

$$v(x_\varepsilon) \geq \int_0^{1 \wedge \theta_{i,\varepsilon}} \ell_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) e^{-\lambda s} ds + v(y_{x_\varepsilon}^i(1 \wedge \theta_{i,\varepsilon})) e^{-\lambda(1 \wedge \theta_{i,\varepsilon})}. \tag{4.14}$$

Assume by contradiction that [A] does not hold, then $\lim_{\varepsilon \to 0} \theta_{i,\varepsilon} = 0$.

Since x_ε is a minimum of $v(x) - q_i|x| + \frac{|x|^2}{\varepsilon^2}$, we deduce from (4.14) that

$$0 \geq \int_0^{\theta_{i,\varepsilon}} \ell_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) e^{-\lambda s} ds + v(y_{x_\varepsilon}^i(\theta_{i,\varepsilon}))(e^{-\lambda \theta_{i,\varepsilon}} - 1) - q_i|x_\varepsilon| + \frac{|x_\varepsilon|^2}{\varepsilon^2}, \tag{4.15}$$

and therefore

$$0 \geq \int_0^{\theta_{i,\varepsilon}} \ell_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) e^{-\lambda s} ds + v(y_{x_\varepsilon}^i(\theta_{i,\varepsilon}))(e^{-\lambda \theta_{i,\varepsilon}} - 1) - q_i|x_\varepsilon|. \tag{4.16}$$

We can write (4.16) as

$$0 \leq \int_0^{\theta_{i,\varepsilon}} \left(-\ell_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) e^{-\lambda s} - q_i f_i(y_{x_\varepsilon}^i(s), \alpha_{i,\varepsilon}(s)) \right) ds - v(y_{x_\varepsilon}^i(\theta_{i,\varepsilon}))(e^{-\lambda \theta_{i,\varepsilon}} - 1). \tag{4.17}$$

Dividing by $\theta_{i,\varepsilon}$ and letting ε tend to 0, we obtain that $\lambda v(O) + H_i(O, q_i) \geq 0$. This implies that $\lambda v(O) + H_O^T \geq 0$, which is a contradiction since [B] does not hold. □

5. Comparison principle and uniqueness

Theorem 5.1. *Assume* [H0], [H1], [H2] *and* [H3]. *Let* $u : \mathcal{G} \to \mathbb{R}$ *be a bounded continuous viscosity subsolution of* (3.1), *and* $v : \mathcal{G} \to \mathbb{R}$ *be a bounded viscosity supersolution of* (3.1). *Then* $u \leq v$ *in* $\mathcal{G}$.

Proof. It is a simple matter to check that there exists a positive real number M such that the function $\psi(x) = -|x|^2 - M$ is a viscosity subsolution of (3.1). For $0 < \mu < 1$, μ close to 1, the function $u_\mu = \mu u + (1 - \mu)\psi$ is a viscosity subsolution of (3.1), which tends to $-\infty$ as $|x|$ tends to $+\infty$. Let M_μ be the maximal value of $u_\mu - v$ which is reached at some point $\bar{x}_\mu$.

We want to prove that $M_\mu \leq 0$.

1. If $\bar{x}_\mu \neq O$, then we introduce the function $u_\mu(x) - v(x) - d^2(x, \bar{x}_\mu)$, which has a strict maximum at $\bar{x}_\mu$, and we double the variables, *i.e.* for $0 < \varepsilon \ll 1$, we consider

$$u_\mu(x) - v(y) - \mathrm{d}^2(x, \bar{x}_\mu) - \frac{\mathrm{d}^2(x, y)}{\varepsilon^2}.$$

Classical arguments then lead to the conclusion that $u_\mu(\bar{x}_\mu) - v(\bar{x}_\mu) \leq 0$, thus $M_\mu \leq 0$.

2. If $\bar{x}_\mu = O$. We use Theorem 4.6; we have two possible cases:

   [B] $\lambda v(O) \geq -H_O^T$.
   From Lemma 4.3, $\lambda u(O) + H_O^T \leq 0$. Therefore, we obtain that $u_\mu(O) \leq v(O)$, thus $M_\mu \leq 0$.

   [A] With the notations of Theorem 4.6, we have that

   $$v(x_k) \geq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s))\mathrm{e}^{-\lambda s}\mathrm{d}s + v(y_{x_k}(\eta))\mathrm{e}^{-\lambda \eta}.$$

   Moreover, from Lemma 4.5,

   $$u_\mu(x_k) \leq \int_0^\eta \ell_i(y_{x_k}(s), \alpha_i^k(s))\mathrm{e}^{-\lambda s}\mathrm{d}s + u_\mu(y_{x_k}(\eta))\mathrm{e}^{-\lambda \eta}.$$

   Therefore

   $$u_\mu(x_k) - v(x_k) \leq (u_\mu(y_{x_k}(\eta)) - v(y_{x_k}(\eta)))\mathrm{e}^{-\lambda \eta}.$$

   Letting k tend to $+\infty$, we find that $M_\mu \leq M_\mu \mathrm{e}^{-\lambda \eta}$, which implies that $M_\mu \leq 0$

We conclude by letting μ tend to 1. $\qquad\square$

Remark 5.2. Under the assumptions [H0], [H1], [H2] and [H3], it is possible to obtain the following more general comparison principle, see [18]:
Let $u : \mathcal{G} \to \mathbb{R}$ be a bounded viscosity subsolution of (3.1), and $v : \mathcal{G} \to \mathbb{R}$ be a bounded viscosity supersolution of (3.1). Then $u \leq v$ in $\mathcal{G}$.
The proof, given in [18] is more technical and is done two steps:

1. using Lemma 4.2, *i.e.* the Lipschitz continuity of any subsolution in a neighborhood of O, prove a local comparison principle
2. obtain the global comparison result by a localization argument.

Corollary 5.3. *Assume* [H0], [H1], [H2] *and* [H3]. *The value function u of the optimal control problem* (2.5) *is the unique bounded viscosity solution of* (3.1).

6. STABILITY

We now study the stability of sub and super solutions with respect to the uniform convergence of the costs and dynamics.

6.1. Assumptions

We consider a family (indexed by $\varepsilon \in [0, 1]$) of optimal control problems on the network whose dynamics and costs are denoted $(f_i^\varepsilon, \ell_i^\varepsilon)$ for $i = 1, \ldots, N$. As above, A is a metric space (one can take $A = \mathbb{R}^m$) and

for $i = 1, \ldots, N$, A_i are nonempty disjoint compact subsets of A. Hereafter, we suppose that the following properties hold uniformly with respect to ε:

[H0$^\varepsilon$] The functions $f_i^\varepsilon : J_i \times A_i \to \mathbb{R}$ are continuous and bounded uniformly w.r.t. $\varepsilon \in [0, 1]$; in particular, there exists $M > 0$ such that $|f_i^\varepsilon(x, a)| \leq M$ for any $\varepsilon \in [0, 1]$, $i = 1, \ldots, N$, $x \in J_i$, $a \in A_i$. Moreover, there exists $L > 0$ such that for any ε, i, x, y in J_i and $a \in A_i$,

$$|f_i^\varepsilon(x, a) - f_i^\varepsilon(y, a)| \leq L|x - y|.$$

We will use the notation $F_i^\varepsilon(x)$ for the set $\{f_i^\varepsilon(x, a)e_i, a \in A_i\}$.

[H1$^\varepsilon$] For $i = 1, \ldots, N$, the functions $\ell_i^\varepsilon : J_i \times A_i \to \mathbb{R}$ are continuous and bounded uniformly w.r.t. $\varepsilon \in [0, 1]$; we may assume that $|\ell_i^\varepsilon(x, a)| \leq M$ for any $\varepsilon \in [0, 1]$, $i = 1, \ldots, N$, $x \in J_i$, $a \in A_i$ with the same constant M as above. There is a modulus of continuity ω_i such that for all $\varepsilon \in [0, 1]$, x, y in J_i and $a \in A_i$, $|\ell_i^\varepsilon(x, a) - \ell_i^\varepsilon(y, a)| \leq \omega_i(|x - y|)$.

[H2$^\varepsilon$] For $i = 1, \ldots, N$, $x \in J_i$, the non empty and closed set

$$\mathrm{FL}_i^\varepsilon(x) \equiv \{(f_i^\varepsilon(x, a)e_i, \ell_i^\varepsilon(x, a)), a \in A_i\}$$

is convex.

[H3$^\varepsilon$] There is a real number $\delta > 0$ such that for any $\varepsilon \in [0, 1]$, $i = 1, \ldots, N$,

$$[-\delta e_i, \delta e_i] \subset F_i^\varepsilon(O).$$

We also assume the local uniform convergence of f_i^ε to f_i^0 and ℓ_i^ε to ℓ_i^0 as $\varepsilon \to 0$: for all $i = 1, \ldots, N$ and $R > 0$,

[H4$^\varepsilon$]

$$\lim_{\varepsilon \to 0} \max_{x \in B(O,R), a \in A_i} |f_i^\varepsilon(x, a) - f_i^0(x, a)| = 0.$$

[H5$^\varepsilon$]

$$\lim_{\varepsilon \to 0} \max_{x \in B(O,R), a \in A_i} |\ell_i^\varepsilon(x, a) - \ell_i^0(x, a)| = 0.$$

6.2. Convergence of the Hamiltonian at the vertex as $\varepsilon \to 0$

Lemma 6.1. *For ε fixed in $[0, 1]$ and $i \in \{1, \ldots, N\}$, let $a^* \in A_i$ be such that $f_i^\varepsilon(O, a^*) \geq 0$. There exists a sequence $a_n^* \in A_i$ such that*

$$f_i^\varepsilon(O, a_n^*) \geq \frac{\delta}{n} > 0, \tag{6.1}$$

$$|f_i^\varepsilon(O, a_n^*) - f_i^\varepsilon(O, a^*)| \leq \frac{2M}{n}, \tag{6.2}$$

$$|\ell_i^\varepsilon(O, a_n^*) - \ell_i^\varepsilon(O, a^*)| \leq \frac{2M}{n}. \tag{6.3}$$

Proof. From [H3$^\varepsilon$] there exists $a_\delta \in A_i$ such that $f_i^\varepsilon(O, a_\delta) = \delta$. From [H2$^\varepsilon$],

$$\lambda(f_i^\varepsilon(O, a_\delta), \ell_i^\varepsilon(O, a_\delta)) + (1 - \lambda)(f_i^\varepsilon(O, a^*), \ell_i^\varepsilon(O, a^*)) \in \mathrm{FL}_i^\varepsilon(O)$$

for any $\lambda \in [0, 1]$. In particular, for $\lambda = \frac{1}{n}$, there exists $a_n^* \in A_i$ such that

$$\frac{1}{n}(f_i^\varepsilon(O, a_\delta), \ell_i^\varepsilon(O, a_\delta)) + (1 - \frac{1}{n})(f_i^\varepsilon(O, a^*), \ell_i^\varepsilon(O, a^*)) = (f_i^\varepsilon(O, a_n^*), \ell_i^\varepsilon(O, a_n^*))$$

which yields (6.1). The statements (6.2) (6.3) follow from [H0$^\varepsilon$] and [H1$^\varepsilon$]. □

Corollary 6.2. *For any $\varepsilon \in [0,1]$, $i \in \{1, \ldots, N\}$ and $p_i \in \mathbb{R}$,*

$$\max_{a \in A_i \ s.t. \ f_i^\varepsilon(O,a) \geq 0} (-p_i f_i^\varepsilon(O,a) - \ell_i^\varepsilon(O,a)) = \sup_{a \in A_i \ s.t. \ f_i^\varepsilon(O,a) > 0} (-p_i f_i^\varepsilon(O,a) - \ell_i^\varepsilon(O,a)). \tag{6.4}$$

As in the previous sections, we define the Hamiltonians

$$H_i^\varepsilon(x,p) = \max_{a \in A_i} (-p f_i^\varepsilon(x,a) - \ell_i^\varepsilon(x,a)), \tag{6.5}$$

$$H_O^\varepsilon(p_1, \ldots, p_N) = \max_{i=1,\ldots,N} \max_{a \in A_i \ s.t. \ f_i^\varepsilon(O,a) \geq 0} (-p_i f_i^\varepsilon(O,a) - \ell_i^\varepsilon(O,a)). \tag{6.6}$$

With

$$H_{O,i}^\varepsilon(p_i) = \max_{a \in A_i \ s.t. \ f_i^\varepsilon(O,a) \geq 0} (-p_i f_i^\varepsilon(O,a) - \ell_i^\varepsilon(O,a)), \tag{6.7}$$

we can write $H_O^\varepsilon(p_1, \ldots, p_N) = \max_{i=1,\ldots,N} H_{O,i}^\varepsilon(p_i)$. Finally, we define

$$H_O^{T,\varepsilon} = - \min_{i=1,\ldots,N} \min_{a \in A_i \ s.t. \ f_i^\varepsilon(x,a)=0} \ell_i^\varepsilon(O,a). \tag{6.8}$$

Proposition 6.3. *For any $p \in \mathbb{R}^N$,*

$$\lim_{\varepsilon \to 0^+} H_O^\varepsilon(p) = H_O^0(p). \tag{6.9}$$

Proof. Let us first prove that

$$\limsup_{\varepsilon \to 0^+} H_O^\varepsilon(p) \leq H_O^0(p). \tag{6.10}$$

For any $i \in \{1, \ldots, N\}$, let $(a^\varepsilon)_\varepsilon$ be a family of points in A_i such that $f_i^\varepsilon(O, a_\varepsilon) \geq 0$. Up to the extraction of subsequence, we can assume that there exists $a^0 \in A_i$ such that $\lim_{\varepsilon \to 0^+} a^\varepsilon = a^0$. Then $f_i^0(O, a^0) \geq 0$ and

$$(-p_i f_i^\varepsilon(O, a^\varepsilon) - \ell_i^\varepsilon(O, a^\varepsilon)) = (-p_i f_i^0(O, a^0) - \ell_i^0(O, a^0)) + o(1).$$

This implies that

$$\limsup_{\varepsilon \to 0} \max_{a \in A_i \ s.t \ f_i^\varepsilon(O,a) \geq 0} (-p_i f_i^\varepsilon(O, a^\varepsilon) - \ell_i^\varepsilon(O, a^\varepsilon)) \leq \max_{a \in A_i \ s.t \ f_i^0(O,a) \geq 0} (-p_i f_i^0(O, a^0) - \ell_i^0(O, a^0))$$

i.e. (6.10).
We are left with proving that

$$\liminf_{\varepsilon \to 0^+} H_O^\varepsilon(p) \geq H_O^0(p). \tag{6.11}$$

For a positive integer n, call $A_{i,n,\delta}^\varepsilon$ the set

$$A_{i,n,\delta}^\varepsilon = \{a \in A_i \ s.t. \ f_i^\varepsilon(O,a) \geq \frac{\delta}{n}\}.$$

The set $A_{i,n,\delta}^0$ is compact and from [H4$^\varepsilon$], there exists $\bar{\varepsilon}_n$ such that for any $\varepsilon \leq \bar{\varepsilon}_n$,

$$A_{i,n,\delta}^0 \subset A_{i,2n,\delta}^\varepsilon \subset \{a \in A_i \ s.t. \ f_i^\varepsilon(O,a) \geq 0\}.$$

This implies that

$$\max_{a \in A_{i,n,\delta}^0} (-p_i f_i^0(O,a) - \ell_i^0(O,a)) \leq \max_{a \in A_i \ s.t. \ f_i^\varepsilon(O,a) \geq 0} (-p_i f_i^\varepsilon(O,a) - \ell_i^\varepsilon(O,a)) + o(1)$$

and letting $\varepsilon \to 0$

$$\max_{i=1,\ldots,N} \max_{a \in A^0_{i,n,\delta}} (-p_i f^0_i(O,a) - \ell^0_i(O,a))$$

$$\leq \liminf_{\varepsilon \to 0^+} \max_{i=1,\ldots,N} \max_{a \in A_i \text{ s.t. } f^\varepsilon_i(O,a) \geq 0} (-p_i f^\varepsilon_i(O,a) - \ell^\varepsilon_i(O,a)).$$

Therefore, for any positive integer n,

$$\max_{i=1,\ldots,N} \max_{a \in A^0_{i,n,\delta}} (-p_i f^0_i(O,a) - \ell^0_i(O,a)) \leq \liminf_{\varepsilon \to 0^+} H^\varepsilon_O(p) \tag{6.12}$$

Consider now $a^0 \in A_i$ such that

$$-p_i f^0_i(O,a^0) - \ell^0_i(O,a^0) = H^0_O(p)$$

$$= \max_{j=1,\ldots,N} \max_{a \in A_j \text{ s.t. } f^0_i(O,a) \geq 0} (-p_j f^0_j(O,a) - \ell^0_j(O,a)).$$

From Lemma 6.1, there exists a sequence $(a^0_n)_{n>0}$ such that $a^0_n \in A_{i,n,\delta}$ and

$$\lim_{n \to \infty} (-p_i f^0_i(O,a^0_n) - \ell^0_i(O,a^0_n)) = (-p_i f^0_i(O,a^0) - \ell^0_i(O,a^0)) = H^0_O(p).$$

From (6.12),

$$(-p_i f^0_i(O,a^0_n) - \ell^0_i(O,a^0_n)) \leq \liminf_{\varepsilon \to 0^+} H^\varepsilon_O(p) \tag{6.13}$$

which yields (6.11) by letting $n \to \infty$. $\qquad\square$

Remark 6.4. Note that for proving Proposition 6.3, only [H2^0], [H3^0] are needed, (in addition to [H0$^\varepsilon$], [H1$^\varepsilon$], [H4$^\varepsilon$] and [H5$^\varepsilon$]).

Remark 6.5. It is possible to prove under the hypotheses of the Proposition 6.3 that for any $p_i \in \mathbb{R}$,

$$\lim_{\varepsilon \to 0} H^\varepsilon_{O,i}(p_i) = H^0_{O,i}(p_i). \tag{6.14}$$

The proof is very much like that of Proposition 6.3.

6.3. Convergence of the sub or super solutions as $\varepsilon \to 0$

We consider the family of Hamilton–Jacobi equations depending on the parameter ε:

$$\lambda u(x) + \sup_{(\zeta,\xi) \in \mathrm{FL}^\varepsilon(x)} \{-Du(x,\zeta) - \xi\} = 0 \quad \text{in } \mathcal{G}, \tag{6.15}$$

$$\lambda u(x) + \sup_{(\zeta,\xi) \in \mathrm{FL}^0(x)} \{-Du(x,\zeta) - \xi\} = 0 \quad \text{in } \mathcal{G}. \tag{6.16}$$

Theorem 6.6. *Let u^ε be a sequence of uniformly Lipschitz subsolutions of (6.15) converging to u^0 as $\varepsilon \to 0$ locally uniformly on $\mathcal{G}$. Then u^0 is a subsolution of (6.16).*

Proof. Consider $x_0 \in \mathcal{G}$ and $\varphi \in \mathcal{R}(\mathcal{G})$ such that x_0 is a strict local maximum point of $u^0 - \varphi$; we wish to prove that

$$\lambda u^0(x_0) + H^0_i\left(x_0, \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(x_0)\right) \leq 0 \quad \text{if } x_0 \in J_i \backslash \{O\},$$

$$\lambda u^0(O) + H^0_O\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O)\right) \leq 0 \quad \text{if } x_0 = O.$$

The proof is standard if $x_0 \neq O$. Let us assume that $x_0 = O$. We have to prove that

$$\lambda u^0(O) + \max_{i=1,\dots,N} \max_{a \in A_i \text{ s.t. } f_i^0(O,a) \geq 0} \left(-\frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O) f_i^0(O,a) - \ell_i^0(O,a) \right) \leq 0. \tag{6.17}$$

Having fixed $i \in \{1 \dots N\}$, define

$$d_i(y) = \begin{cases} 0 & \text{if } y \in J_i, \\ |y| & \text{otherwise.} \end{cases}$$

Let $\bar{L}$ be an uniform bound of the Lipschitz constant of $u^\varepsilon - \varphi$. Take $C = \bar{L} + 1$.

The function $y \mapsto u^0(y) - \varphi(y) - Cd_i(y)$ reaches a strict local maximum point at O, say in $B(O, R)$. Thanks to the local uniform convergence of u^ε, there exists a sequence of local maximum points y^ε in $B(O, R)$ of $y \mapsto u^\varepsilon(y) - \varphi(y) - Cd_i(y)$ which converges to O as $\varepsilon \to 0$.

Moreover $y^\varepsilon \in J_i$, because if it was not the case, then

$$u^\varepsilon(y^\varepsilon) - \varphi(y^\varepsilon) - u^\varepsilon(O) - \varphi(O) \leq \bar{L}|y^\varepsilon| = \bar{L}d_i(y^\varepsilon),$$

would imply

$$u^\varepsilon(y^\varepsilon) - \varphi(y^\varepsilon) - Cd_i(y^\varepsilon) \leq u^\varepsilon(O) - \varphi(O) - d_i(y^\varepsilon) < u^\varepsilon(O) - \varphi(O),$$

which would contradict the definition of y^ε.

Then, take $y \mapsto \varphi(y) + Cd_i(y)$ as a test function in the viscosity inequality satisfied by u^ε. We make out two cases:

Case 1: $y^\varepsilon \in J_i \backslash \{O\}$. We obtain

$$\lambda u^\varepsilon(y^\varepsilon) + H_i^\varepsilon \left(y^\varepsilon, \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(y^\varepsilon) \right) \leq 0,$$

and letting $\varepsilon \to 0$

$$\lambda u^0(O) + H_i^0 \left(O, \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O) \right) \leq 0. \tag{6.18}$$

Case 2: $y^\varepsilon = O$.

$$\lambda u^\varepsilon(O) + \max_{j=1,\dots,N} \max_{a \in A_j \text{ s.t. } f_j^\varepsilon(O,a) \geq 0} (-p_j f_j^\varepsilon(O,a) - \ell_j^\varepsilon(O,a)) \leq 0,$$

where $p_j = \frac{\mathrm{d}\varphi}{\mathrm{d}x_j}(O) + C$ if $j \neq i$ and $p_i = \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O)$. Hence,

$$\lambda u^\varepsilon(O) + \max_{a \in A_i \text{ s.t. } f_i^\varepsilon(O,a) \geq 0} \left(-\frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O) f_i^\varepsilon(O,a) - \ell_i^\varepsilon(O,a) \right) = \lambda u^\varepsilon(O) + H_{O,i}^\varepsilon \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O) \right) \leq 0.$$

From (6.14), we deduce that

$$\lambda u^0(O) + H_{O,i}^0 \left(\frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O) \right) \leq 0. \tag{6.19}$$

Summarizing, we have (6.19) in all cases, because (6.18) implies (6.19). We have proved (6.17).   □

Theorem 6.7. *Let* $(u^\varepsilon)_\varepsilon$ *be a sequence of supersolutions of* (6.15) *such that*

- *there exist a real number* $C > 0$ *s.t. for all* ε *and* $x \in \mathcal{G}$, $|u^\varepsilon(x)| \leq C(1 + |x|)$
- *the sequence* u^ε *converges to* u^0 *locally uniformly on* $\mathcal{G}$ *as* $\varepsilon \to 0$.

Then u^0 *is a supersolution of* (6.16).

Proof. Consider $x_0 \in \mathcal{G}$ and $\varphi \in \mathcal{R}(\mathcal{G})$ such that x_0 is a strict local minimum point of $u^0 - \varphi$; if $x_0 \neq O$, the proof that $\lambda u^0(x_0) + H_i^0(x_0, \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(x_0)) \geq 0$ is standard. We therefore focus on the case when $x_0 = O$.

We consider two cases:

First case: for any $i = 1, \ldots, N$, $\frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O) \leq \max(q : q \in \mathcal{P}_0^i)$ and $H_i^0(O, \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O)) = H_O^0(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O))$. In this case, we can use the standard stability argument: there exists a sequence (x^ε) such that x^ε is a local minimum point of $u^\varepsilon - \varphi$ and such that x^ε converges to O and $u^\varepsilon(x^\varepsilon)$ converges to $u^0(O)$. If for a subsequence ε_n, $x^{\varepsilon_n} = O$, then the viscosity inequality is

$$\lambda u^{\varepsilon_n}(O) + H_O^{\varepsilon_n}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O)\right) \geq 0$$

and by passing to the limit as $n \to \infty$ thanks to Proposition 6.3,

$$\lambda u^0(O) + H_O^0\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O)\right) \geq 0, \tag{6.20}$$

which is the desired viscosity inequality for u^0. If there does not exists such a subsequence, we can assume that for a subsequence ε_n, $x^{\varepsilon_n} \in J_i \backslash \{O\}$. The viscosity inequality is

$$\lambda u^{\varepsilon_n}(x^{\varepsilon_n}) + H_i^{\varepsilon_n}(x^{\varepsilon_n}, \frac{\mathrm{d}\varphi}{dx_i}(x^{\varepsilon_n}) \geq 0,$$

and by passing to the limit as $n \to \infty$,

$$\lambda u^0(O) + H_i^0\left(O, \frac{\mathrm{d}\varphi}{dx_i}(O)\right) \geq 0.$$

Then (6.20) is obtained since $H_i^0(O, \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O)) = H_O^0(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O))$.

Second case: $\mathcal{I} \neq \{1, \ldots, N\}$, where $\mathcal{I}$ is the (possibly empty) set of indices i such that $\frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O) \leq \max(q : q \in \mathcal{P}_0^i)$ and $H_i^0(O, \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O)) = H_O^0(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O))$. It is always possible to find a function $\psi \in \mathcal{R}(\mathcal{G})$ such that

1. $\psi(O) = \varphi(O)$
2. $H_O^0(\frac{d\psi}{\mathrm{d}x_1}(O), \ldots, \frac{d\psi}{\mathrm{d}x_N}(O)) = H_O^0(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O))$
3. if $i \in \mathcal{I}$, then $\psi|_{J_i}$ coincides with $\varphi|_{J_i}$
4. if $i \notin \mathcal{I}$, then $\frac{d\psi}{\mathrm{d}x_i}(O) < \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}(O)$ is such that $\frac{d\psi}{\mathrm{d}x_i}(O) \leq \max(q : q \in \mathcal{P}_0^i)$ and $H_i^0(O, \frac{d\psi}{\mathrm{d}x_i}(O)) = H_O^0(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O))$.

Then, since ψ touches φ at O from below, O is still a strict minimum point of $u^0 - \psi$, and for all i, $\frac{d\psi}{\mathrm{d}x_i}(O) \leq \max(q : q \in \mathcal{P}_0^i)$ and

$$H_i^0\left(O, \frac{d\psi}{\mathrm{d}x_i}(O)\right) = H_O^0\left(\frac{d\psi}{\mathrm{d}x_1}(O), \ldots, \frac{d\psi}{\mathrm{d}x_N}(O)\right) = H_O^0\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x_1}(O), \ldots, \frac{\mathrm{d}\varphi}{\mathrm{d}x_N}(O)\right). \tag{6.21}$$

We can apply the result proved in the first case to the function ψ, $i.e.$

$$\lambda u^0(O) + H_O^0\left(\frac{d\psi}{\mathrm{d}x_1}(O), \ldots, \frac{d\psi}{\mathrm{d}x_N}(O)\right) \geq 0,$$

and we get (6.20) from (6.21).

$\square$

7. EXTENSION TO A MORE GENERAL FRAMEWORK WITH AN ADDITIONAL COST AT THE JUNCTION

It is possible to extend all the results presented above to the case when there is an additional cost at the junction. Such problems are also studied in [13]. We keep the setting used above except that we take into account an additional subset A_0 of A (it is enough to suppose that A_0 is a singleton and that it is disjoint from the other sets A_i), on which the running cost is the constant ℓ_0. We define

$$M = \left\{(x,a);\ x \in \mathcal{G}, \quad a \in A_i \text{ if } x \in J_i\backslash\{O\}, \text{ and } a \in \cup_{i=0}^N A_i \text{ if } x = O\right\},$$

the dynamics

$$\forall (x,a) \in M, \qquad f(x,a) = \begin{cases} f_i(x,a)e_i & \text{if } x \in J_i\backslash\{O\}, \\ f_i(O,a)e_i & \text{if } x = O \text{ and } a \in A_i, i > 0, \\ 0 & \text{if } x = O \text{ and } a \in A_0, \end{cases}$$

and the running cost

$$\forall (x,a) \in M, \qquad \ell(x,a) = \begin{cases} \ell_i(x,a) & \text{if } x \in J_i\backslash\{O\}, \\ \ell_i(O,a) & \text{if } x = O \text{ and } a \in A_i, i > 0, \\ \ell_0 & \text{if } x = O \text{ and } a \in A_0. \end{cases}$$

The infinite horizon optimal control problem is then given by (2.5) and (2.4). We obtain that the value function v is continuous in the same manner as above and that v is a viscosity solution of (3.1) with the new definition of FL(x):

$$\text{FL}(x) = \begin{cases} \text{FL}_i(x) & \text{if } x \text{ belongs to the edge } J_i\backslash\{O\} \\ \{0, -\ell_0\} \cup \bigcup_{i=1,\ldots,N} \text{FL}_i^+(O) & \text{if } x = O. \end{cases}$$

The viscosity sub and supersolutions can be also defined as in (3.7) and (3.8) with the new definition of $H_O : \mathbb{R}^N \to \mathbb{R}$:

$$H_O(p_1,\ldots,p_N) = \max\left(-\ell_0, \max_{i=1,\ldots,N} \max_{a \in A_i \text{ s.t. } f_i(O,a) \geq 0}(-p_i f_i(O,a) - \ell_i(O,a))\right),$$

and the definition of the constant H_O^T is modified accordingly:

$$H_O^T = -\min\left(\ell_0, \min_{i=1,\ldots,N} \min_{a \in A_i \text{ s.t. } f_i(O,a)=0} \ell_i(O,a)\right).$$

With these new definitions, all the results proved in Section 4, 5 and 6 hold with obvious modifications of the proofs. In particular,

- a subsolution of the presently defined problem is also a subsolution of the former problem (without the additional cost) so it is Lipschitz continuous in a neighborhood of O, and Lemmas 4.2 and 4.3 hold.
- The proofs of Lemma 4.5 and Theorem 4.6 are unchanged. In particular, with the choice of $q = (q_i)_{i=1,\ldots,N}$ made in the proof of Theorem 4.6, we still have the identity $H_O(q) = H_O^T$.
- The proof of the comparison principle is unchanged.

8. THE CASE OF A NETWORK

8.1. The geometrical setting and the optimal control problem

We consider a network in $\mathbb{R}^d$ with a finite number of edges and vertices. A network in $\mathbb{R}^d$ is a pair $(\mathcal{V}, \mathcal{E})$ where

(i) $\mathcal{V}$ is a finite subset of $\mathbb{R}^d$ whose elements are said vertices

(ii) $\mathcal{E}$ is a finite set of edges, which are either closed straight line segments between two vertices, or a closed straight half-lines whose endpoint is a vertex. The intersection of two edges is either empty or a vertex of the network. The union of the edges in $\mathcal{E}$ is a connected subset of $\mathbb{R}^d$. For a given edge $e \in \mathcal{E}$, the notation ∂e is used for the set of endpoints of e, and $e^* = e \backslash \partial e$ stands for the interior of e. Let also u_e be a unit vector aligned with e. There are two possible such vectors: if the boundary of e is made of one vertex x only, then u_e will be oriented from x to the interior of e; if the boundary of e is made of two vertices, then the choice of the orientation is arbitrary.

We say that two vertices are adjacent if they are connected by an edge. For a given vertex x, we denote by $\mathcal{E}_x$ the set of the edges for which x is an endpoint, and N_x the cardinality of $\mathcal{E}_x$. We denote by $\mathcal{G}$ the union of all the edges in $\mathcal{E}$.

We consider infinite horizon optimal control problems which have different dynamics and running cost in the edges. We are going to describe the assumptions on the dynamics and costs in each edge e. The sets of controls are denoted by A_e and the system is driven by a dynamics f_e and the running cost is given by ℓ_e. Our main assumptions are as follows

[**H0$_n$**] A is a metric space (one can take $A = \mathbb{R}^m$). For $e \in \mathcal{E}$, A_e is a non empty compact subset of A and $f_e : e \times A_e \to \mathbb{R}$ is a continuous bounded function. The sets A_e are disjoint. Moreover, there exists $L > 0$ such that for any $e \in \mathcal{E}$, x, y in e and $a \in A_e$,

$$|f_e(x, a) - f_e(y, a)| \le L|x - y|.$$

We will use the notation $F_e(x)$ for the set $\{f_e(x, a)u_e, a \in A_e\}$.

[**H1$_n$**] For $e \in \mathcal{E}$, the function $\ell_e : e \times A_e \to \mathbb{R}$ is a continuous and bounded function. There is a modulus of continuity ω_e such that for all x, y in e and for all $a \in A_e$, $|\ell_e(x, a) - \ell_e(y, a)| \le \omega_e(|x - y|)$.

[**H2$_n$**] For $e \in \mathcal{E}$, $x \in e$, the non empty and closed set $\text{FL}_e(x) \equiv \{(f_e(x, a)u_e, \ell_e(x, a)), a \in A_e\}$ is convex.

[**H3$_n$**] There is a real number $\delta > 0$ such that for any $e \in \mathcal{E}$, for all endpoints x of e,

$$[-\delta u_e, \delta u_e] \subset F_e(x).$$

Let us denote by M the set:

$$M = \{(x, a); \; x \in \mathcal{G}, \quad a \in A_e \text{ if } x \in e^*, \text{ and } a \in \cup_{e \in \mathcal{E}_x} A_e \text{ if } x \in \mathcal{V}\}. \tag{8.1}$$

The set M is closed. We also define the function f on M by

$$\forall (x, a) \in M, \qquad f(x, a) = \begin{cases} f_e(x, a)u_e & \text{if } x \in e^*, \\ f_e(x, a)u_e & \text{if } x \in \mathcal{V} \text{ and } a \in A_e \text{ for } e \in \mathcal{E}_x. \end{cases}$$

The set of admissible controlled trajectories starting from the initial datum $x \in \mathcal{G}$ can be defined by

$$\mathcal{T}_x = \left\{ \begin{array}{l} (y_x, \alpha) \in L^\infty_{\text{loc}}(\mathbb{R}^+; M) : \quad y_x \in Lip(\mathbb{R}^+; \mathcal{G}), \\ \qquad\qquad y_x(t) = x + \int_0^t f(y_x(s), \alpha(s))\mathrm{d}s \quad \text{in } \mathbb{R}^+ \end{array} \right\}, \tag{8.2}$$

exactly as in Section 2.2.

The cost associated to the trajectory $(y_x, \alpha) \in \mathcal{T}_x$ is

$$J(x; (y_x, \alpha)) = \int_0^\infty \ell(y_x(t), \alpha(t))\mathrm{e}^{-\lambda t}\mathrm{d}t,$$

where $\lambda > 0$ is a real number and the Lagrangian ℓ is defined on M by

$$\forall (x, a) \in M, \qquad \ell(x, a) = \begin{cases} \ell_e(x, a) & \text{if } x \in e^*, \\ \ell_e(x, a) & \text{if } x \in \mathcal{V} \text{ and } a \in A_e \text{ for } e \in \mathcal{E}_x. \end{cases}$$

The value function of the infinite horizon optimal control problem is

$$v(x) = \inf_{(y_x, \alpha) \in \mathcal{T}_x} J(x; (y_x, \alpha)). \tag{8.3}$$

8.2. The Hamilton–Jacobi equation

For each edge e, $x \in e^*$, let x_e be the coordinate of x in the system (O_e, u_e) where O_e is an arbitrary origin on e.

For the definition of viscosity solutions on the irregular set $\mathcal{G}$, it is necessary to first define a class of the admissible test-functions

Definition 8.1. A function $\varphi : \mathcal{G} \to \mathbb{R}$ is an admissible test-function if

- φ is continuous in $\mathcal{G}$
- for any e, $\varphi|_e \in \mathcal{C}^1(e)$.

The set of admissible test-function is noted $\mathcal{R}(\mathcal{G})$. If $\varphi \in \mathcal{R}(\mathcal{G})$ and $\zeta \in \mathbb{R}$, let $D\varphi(x, \zeta u_e)$ be defined by $D\varphi(x, \zeta u_e) = \zeta \frac{d\varphi}{dx_e}(x)$ if $x \in e^*$, and $D\varphi(x, \zeta u_e) = \zeta \lim_{y \to x, y \in e^*} \frac{d\varphi}{dx_e}(y)$, if x is an endpoint of e.

We define the Hamiltonians $H_e : e \times \mathbb{R} \to \mathbb{R}$ by

$$H_e(x, p) = \max_{a \in A_e} (-p f_e(x, a) - \ell_e(x, a)). \tag{8.4}$$

For a vertex $x \in \mathcal{V}$, for a given indexing of $\mathcal{E}_x$: $\mathcal{E}_x = \{e_1, \ldots, e_{N_x}\}$, we use the notation $A_i = A_{e_i}$, $f_i = f_{e_i}$, $\ell_i = \ell_{e_i}$ for simplicity. Let also σ_i be 1 if u_{e_i} is oriented from x to the interior of e_i and -1 in the opposite case. The Hamiltonian $H_x : \mathbb{R}^{N_x} \to \mathbb{R}$ is defined by

$$H_x(p_1, \ldots, p_{N_x}) = \max_{i=1,\ldots,N_x} \max_{a \in A_i \text{ s.t. } \sigma_i f_i(x,a) \geq 0} (-p_i f_i(x, a) - \ell_i(x, a)). \tag{8.5}$$

We wish to define viscosity solutions of the following equations

$$\lambda v(x) + H_e(x, Dv(x)) = 0 \qquad \text{if } x \in e^*, \tag{8.6}$$
$$\lambda v(x) + H_x(Dv(x)) = 0 \qquad \text{if } x \in \mathcal{V}. \tag{8.7}$$

Definition 8.2. • An upper semi-continuous function $w : \mathcal{G} \to \mathbb{R}$ is a subsolution of (8.6)–(8.7) in $\mathcal{G}$ if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $w - \varphi$ has a local maximum point at x, then

$$\lambda w(x) + H_e\left(x, \frac{d\varphi}{dx_e}(x)\right) \leq 0 \qquad \text{if } x \in e^*,$$
$$\lambda w(x) + H_x\left(\frac{d\varphi}{dx_1}(x), \ldots, \frac{d\varphi}{dx_{N_x}}(x)\right) \leq 0 \qquad \text{if } x \in \mathcal{V}, \tag{8.8}$$

where in the last case, $\frac{d\varphi}{dx_i}(x) = D\varphi(x, u_{e_i}(x))$, for $i = 1, \ldots, N_x$.
- A lower semi-continuous function $w : \mathcal{G} \to \mathbb{R}$ is a supersolution of (8.6)–(8.7) if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $w - \varphi$ has a local minimum point at x, then

$$\lambda w(x) + H_e(x, \frac{d\varphi}{dx_e}(x)) \geq 0 \qquad \text{if } x \in e^*,$$
$$\lambda w(x) + H_x\left(\frac{d\varphi}{dx_1}(x), \ldots, \frac{d\varphi}{dx_{N_x}}(x)\right) \geq 0 \qquad \text{if } x \in \mathcal{V}. \tag{8.9}$$

8.3. Comparison principle

Since all the arguments used in the junction case are local, we can replicate them in the case of a network and obtain:

Theorem 8.3. *Assume* [H0$_n$], [H1$_n$], [H2$_n$] *and* [H3$_n$]. *Let* $v : \mathcal{G} \to \mathbb{R}$ *be a bounded continuous viscosity subsolution of* (8.6)–(8.7), *and* $w : \mathcal{G} \to \mathbb{R}$ *be a bounded viscosity supersolution of* (8.6)–(8.7). *Then* $v \leq w$ *in* $\mathcal{G}$.

8.4. Existence and uniqueness

By the same arguments as in the junction case, we can prove that v is a bounded viscosity solution of (8.6)–(8.7). From the Theorem 8.3, it is the unique bounded viscosity solution.

Proposition 8.4. *Assume* [H0$_n$], [H1$_n$], [H2$_n$] *and* [H3$_n$]. *The value function v of the optimal control problem* (8.3) *is the unique bounded viscosity solution of* (8.6)–(8.7).

Remark 8.5. The stability results of Section 6 for junctions can be easily generalized to networks.

APPENDIX A. PROOF OF LEMMA 3.10

For any $i \in \{1, \ldots, N\}$, the inclusion $\overline{\mathrm{co}}\left\{\mathrm{FL}_i^+(O) \cup \bigcup_{j \neq i}\left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)\right\} \subset \widetilde{f\ell}(O)$ is proved by explicitly constructing trajectories, see [1]. We skip this part. This leads to

$$\bigcup_{i=1,\ldots,N} \overline{\mathrm{co}}\left\{\mathrm{FL}_i^+(O) \cup \bigcup_{j \neq i}\left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)\right\} \subset \widetilde{f\ell}(O).$$

We now prove the other inclusion. For any $(\zeta, \mu) \in \widetilde{f\ell}(O)$, there exists a sequence of admissible trajectories $(y_n, \alpha_n) \in \mathcal{T}_O$ and a sequence of times $t_n \to 0+$ such that

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f(y_n(t), \alpha_n(t))\mathrm{d}t = \zeta, \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \ell(y_n(t), \alpha_n(t))\mathrm{d}t = \mu.$$

- If $\zeta \neq 0$, then there must exist an index i in $\{1, \ldots, N\}$ such that $\zeta = |\zeta|e_i$: in this case, $y_n(t_n) \in J_i \backslash \{O\}$. Hence,

$$y_n(t_n) = \int_0^{t_n} f(y_n(t), \alpha_n(t))\mathrm{d}t = \sum_{j=1}^N e_j \int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_j \backslash \{O\}} \mathrm{d}t \quad (A.1)$$

with

$$\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_j \backslash \{O\}} \mathrm{d}t = 0 \qquad \text{if } j \neq i,$$
$$\int_0^{t_n} f_i(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_i \backslash \{O\}} \mathrm{d}t = |y_n(t_n)|.$$

These identities are a consequence of Stampacchia's theorem: consider for example $j \neq i$ and the function $\kappa_j : y \mapsto |y| 1_{y \in J_j}$. It is easy to check that $t \mapsto \kappa_j(y_n(t))$ belongs to $W_0^{1,\infty}(0, t_n)$ and that its weak derivative coincides almost everywhere with $t \mapsto f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_j \backslash \{O\}}$. Hence,

$$\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_j \backslash \{O\}} \mathrm{d}t = 0.$$

For $j = 1, \ldots, N$, let $T_{j,n}$ be defined by

$$T_{j,n} = \left|\left\{t \in [0, t_n] : y_n(t) \in J_j \backslash \{O\}\right\}\right|.$$

If $j \neq i$ and $T_{j,n} > 0$ then

$$\frac{1}{T_{j,n}}\left(\int_0^{t_n} f_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_j \backslash \{O\}} \mathrm{d}t, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t)) 1_{y_n(t) \in J_j \backslash \{O\}} \mathrm{d}t\right)$$
$$= \frac{1}{T_{j,n}}\left(\int_0^{t_n} f_j(O, \alpha_n(t)) 1_{y_n(t) \in J_j \backslash \{O\}} \mathrm{d}t, \int_0^{t_n} \ell_j(O, \alpha_n(t)) 1_{y_n(t) \in J_j \backslash \{O\}} \mathrm{d}t\right) + o(1)$$

where $o(1)$ is a vector tending to 0 as $n \to \infty$. Therefore, the distance of

$$\frac{1}{T_{j,n}}\left(e_j \int_0^{t_n} f_j(y_n(t), \alpha_n(t))1_{y_n(t)\in J_j\setminus\{O\}}\mathrm{d}t, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t))1_{y_n(t)\in J_j\setminus\{O\}}\mathrm{d}t\right)$$

to the set $\mathrm{FL}_j(O)$ tends to 0. Moreover, $\int_0^{t_n} f_j(y_n(t), \alpha_n(t))1_{y_n(t)\in J_j\setminus\{O\}}\mathrm{d}t = 0$. Hence, the distance of

$$\frac{1}{T_{j,n}}\left(e_j \int_0^{t_n} f_j(y_n(t), \alpha_n(t))1_{y_n(t)\in J_j\setminus\{O\}}\mathrm{d}t, \int_0^{t_n} \ell_j(y_n(t), \alpha_n(t))1_{y_n(t)\in J_j\setminus\{O\}}\mathrm{d}t\right)$$

to the set $\left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)$ tends to zero as n tends to ∞.

If the set $\{t : y_n(t) = O\}$ has a nonzero measure, then

$$\left(0, \frac{1}{|\{t : y_n(t) = O\}|}\int_0^{t_n} \ell(O, \alpha_n(t))1_{\{t:y_n(t)=O\}}\mathrm{d}t\right) \in \overline{\mathrm{co}}\left\{\bigcup_{j=1}^N \left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)\right\}.$$

Finally, we know that $T_{i,n} > 0$.

$$\frac{1}{T_{i,n}}\left(\int_0^{t_n} f_i(y_n(t), \alpha_n(t))1_{y_n(t)\in J_i\setminus\{O\}}\mathrm{d}t, \int_0^{t_n} \ell_i(y_n(t), \alpha_n(t))1_{y_n(t)\in J_i\setminus\{O\}}\mathrm{d}t\right)$$
$$= \frac{1}{T_{i,n}}\left(\int_0^{t_n} f_i(O, \alpha_n(t))1_{y_n(t)\in J_i\setminus\{O\}}\mathrm{d}t, \int_0^{t_n} \ell_i(O, \alpha_n(t))1_{y_n(t)\in J_i\setminus\{O\}}\mathrm{d}t\right) + o(1)$$

so the distance of $\frac{1}{T_{i,n}}\left(e_i \int_0^{t_n} f_i(y_n(t), \alpha_n(t))1_{y_n(t)\in J_i\setminus\{O\}}\mathrm{d}t, \int_0^{t_n} \ell_i(y_n(t), \alpha_n(t))1_{y_n(t)\in J_i\setminus\{O\}}\mathrm{d}t\right)$ to the set $\mathrm{FL}_i^+(O)$ tends to zero as n tends to ∞.

Combining all the observations above, we see that the distance of

$$\left(\frac{1}{t_n}\int_0^{t_n} f(y_n(t), \alpha_n(t))\mathrm{d}t, \frac{1}{t_n}\int_0^{t_n} \ell(y_n(t), \alpha_n(t))\mathrm{d}t\right)$$

to $\overline{\mathrm{co}}\left\{\mathrm{FL}_i^+(O) \cup \bigcup_{j\neq i}\left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)\right\}$ tends to 0 as $n \to \infty$.

Therefore $(\zeta, \mu) \in \overline{\mathrm{co}}\left\{\mathrm{FL}_i^+(O) \cup \bigcup_{j\neq i}\left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)\right\}$.

- If $\zeta = 0$, either there exists i such that $y_n(t_n) \in J_i\setminus\{O\}$ or $y_n(t_n) = O$:
  - If $y_n(t_n) \in J_i\setminus\{O\}$, then we can make exactly the same argument as above and conclude that $(\zeta, \mu) \in \overline{\mathrm{co}}\left\{\mathrm{FL}_i^+(O) \cup \bigcup_{j\neq i}\left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)\right\}$. Since $\zeta = 0$, we have in fact that $(\zeta, \mu) \in \overline{\mathrm{co}}\bigcup_{j=1}^N\left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)$.
  - If $y_n(t_n) = O$, we have that $\int_0^{t_n} f_j(y_n(t), \alpha_n(t))1_{y_n(t)\in J_j\setminus\{O\}}\mathrm{d}t = 0$ for all $j = 1, \ldots, N$. We can repeat the argument above, and obtain that $(\zeta, \mu) \in \overline{\mathrm{co}}\left\{\bigcup_{j=1}^N\left(\mathrm{FL}_j(O) \cap (\{0\} \times \mathbb{R})\right)\right\}$.

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