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OPTIMAL ∞-QUASICONFORMAL IMMERSIONS

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Abstract. For a Hamiltonian $K \in C^2(\mathbb{R}^{N\times n})$ and a map $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$, we consider the supremal functional

$$E_\infty(u,\Omega) \;:=\; \big\|K(Du)\big\|_{L^\infty(\Omega)}. \tag{1}$$

The "Euler−Lagrange" PDE associated to (1) is the quasilinear system

$$A_\infty u \;:=\; \Big(K_P \otimes K_P + K[K_P]^\perp K_{PP}\Big)(Du) : D^2u \;=\; 0. \tag{2}$$

Here K_P is the derivative and $[K_P]^\perp$ is the projection on its nullspace. (1) and (2) are the fundamental objects of vector-valued Calculus of Variations in L^∞ and first arose in recent work of the author [N. Katzourakis, *J. Differ. Eqs.* **253** (2012) 2123–2139; *Commun. Partial Differ. Eqs.* **39** (2014) 2091–2124]. Herein we apply our results to Geometric Analysis by choosing as K the dilation function

$$K(P) = |P|^2 \mathrm{det}(P^\top P)^{-1/n}$$

which measures the deviation of u from being conformal. Our main result is that appropriately defined minimisers of (1) solve (2). Hence, PDE methods can be used to study optimised quasiconformal maps. Nonconvexity of K and appearance of interfaces where $[K_P]^\perp$ is discontinuous cause extra difficulties. When $n = N$, this approach has previously been followed by Capogna−Raich [8] and relates to Teichmüller's theory. In particular, we disprove a conjecture appearing therein.

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1. Introduction

Let $\mathcal{M}_0$ be a topological submanifold of $\mathbb{R}^N$ with boundary. In this paper we are interested in the problem of finding a Riemannian manifold $(\mathcal{M}, g)$ which has *minimal dilation* and satisfies $\partial\mathcal{M} = \partial\mathcal{M}_0$. In this setting, dilation is a functional on $L^\infty(\mathcal{M}, \otimes^{(2)}T^*\mathcal{M})$, defined as the L^∞ norm of the trace of the Distortion Tensor

$$\mathbf{G} \;:=\; \frac{g}{\det(g)^{1/n}}. \tag{1.1}$$

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This problem is an extension of the classical *Teichmüller Problem* (see [5, 6, 20]). The scaling in (1.1) is such that $\mathbf{G}$ is invariant under conformal tranformations and, as we explain later, the geometric meaning of $\mathrm{tr}(\mathbf{G})$ being "minimal" is that "geometry is distorted as less as possible". As a first step, we consider a simplified problem for the case of immersions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ with prescribed boundary values on $\partial\Omega$. Then, the dilation functional for $\mathcal{M} = u(\Omega)$ becomes

$$K_\infty(u, \Omega) \ := \ \big\|K(Du)\big\|_{L^\infty(\Omega)}, \tag{1.2}$$

where K will be called the *dilation function* and is given by

$$K(P) \ := \ \begin{cases} \dfrac{|P|^2}{\det(P^\top P)^{1/n}} \ , & \text{on } S^+, \\ +\infty \ , & \text{on } \mathbb{R}^{N\times n} \setminus S^+. \end{cases} \tag{1.3}$$

In (1.3), $|P| = \mathrm{tr}(P^\top P)^{1/2}$ is the Euclidean norm on $\mathbb{R}^{N\times n}$ and

$$S^+ \ := \ \Big\{ P \in \mathbb{R}^{N\times n} \ : \ \det\big(P^\top P\big) > 0 \Big\}. \tag{1.4}$$

Important objects of Geometric Topology related to (1.2) arise for $n = N$. Homeomorphisms $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in $W^{1,n}_{loc}(\Omega)^N$ which satisfy $K_\infty(u, \Omega) < \infty$ are called *Quasiconformal Maps* and constitute a class of maps well studied in the literature; see for example [2, 7, 10, 19, 21]. L^p averages of Quasiconformal maps, that is weakly differentable homeomorphisms for which $\|K(Du)\|_{L^p(\Omega)} < \infty$ have also been systematically considered. *Conformal maps*, namely those homeomorphisms $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ in $C^1(\Omega)^N$ which satisfy $Du^\top Du = \frac{1}{n}|Du|^2 I$ form a special important class of Quasiconformal maps since for those $K(Du)$ is constant and equals n. Conformal maps preserve *angles*, but not necessarily *lengths* and hence distort the geometry of Ω in a controlled fashion. However, by Liouville's rigidity theorem, when $n \geq 3$ the only conformal maps that exist are compositions of rotations, dilations, and the inversion $x \mapsto x/|x|^2$. Hence, quasiconformal maps for which $K(Du)$ is merely bounded relax conformality but still deform Ω to $u(\Omega)$ in a fairly controlled fashion.

The problem with Quasiconformal maps is that too little information on their structure is provided by a mere norm bound, and the same holds for the *finite distortion problem* when one restricts attention to minimisers of the dilation functional. The subtle point is that (1.2) is *nonlocal*, in the sense that with respect to the Ω argument (1.2) is not a measure. Simple examples certify that minimisers over a domain with fixed boundary values are not local minimisers over subdomains and the direct method of Calculus of Variations when applied to (1.2) generally does not produce PDE solutions.

In the very recent work, Capogna and Raich [8], remedied this problem by "optimising" Quasiconformal maps. The idea is to consider an appropriate nonstandard L^∞ variational problem for (1.2) and derive a PDE governing Optimal Quasiconformal Maps that can be used as platform for their qualitative study. Motivated by the classical results of Aronsson [3, 4] on *Calculus of Variations in* L^∞, they developed an L^∞ variational approach for extremal (as they are called therein) quasiconformal maps. The essence of this approach is the following: let $Q_p u = 0$ be the Euler−Lagrange system associated to the functional $\|K(Du)\|_{L^p(\Omega)}$. Then, at least formally Q_p tends to a certain operator Q_∞ and $\|K(Du)\|_{L^p(\Omega)}$ tends to $\|K(Du)\|_{L^\infty(\Omega)}$, both as $p \to \infty$. The operator Q_∞ defines a quasilinear 2nd order system in non-divergence form. However, it is not *a priori* clear that the following rectagle "commutes"

$$\begin{array}{ccc} \|K(Du)\|_{L^p(\Omega)} & \longrightarrow & Q_p u = 0 \\ \downarrow \ p \to \infty & & \downarrow \ p \to \infty \\ \|K(Du)\|_{L^\infty(\Omega)} & \dashrightarrow & Q_\infty u = 0 \end{array} \tag{1.5}$$

so that Q_∞ has a variational structure with respect to K_∞, in the sense that appropriately defined minimisers of K_∞ $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ solve $Q_\infty u = 0$. In such an event, $Q_\infty u = 0$ will play the role of "Euler−Lagrange PDE" for the dilation functional. This turns out to be the case, though. Among other far-reaching contributions which include a deep study of dilations of extensions up to the boundary and quasiconformal gradient flows, Capogna and Raich introduced in [8] a localized minimality notion for (1.2) and proved that those local minimisers among "competitors" indeed solve the formally derived PDE.

Simultaneously and independently, the author, also inspired by Aronsson's work and the successful modern evolution of the field of Calculus of Variations in L^∞ (see for example [9]), initiated the development of vector-valued Calculus of Variations in L^∞ for general supremal functionals in [12,17] with particular emphasis to the model functional $\|Du\|_{L^\infty(\Omega)} = \text{ess sup}_\Omega |Du|$. For a Hamiltonian $H \in C^2(\mathbb{R}^{N\times n})$ and the respective supremal functional

$$E_\infty(u,\Omega) \;:=\; \|H(Du)\|_{L^\infty(\Omega)}, \tag{1.6}$$

the PDE system which plays the role of "Euler-Langrange PDE" for (1.6) is

$$A_\infty u \;:=\; \Big(H_P \otimes H_P + H[H_P]^\perp H_{PP}\Big)(Du) : D^2u \;=\; 0. \tag{1.7}$$

Here $[H_P(Du(x))]^\perp$ is the projection on the nullspace of $H_P(Du(x))^\top : \mathbb{R}^N \longrightarrow \mathbb{R}^n$, and H_P, H_{PP} denotes derivatives (for details see Preliminaries 2). The special case of $H(P) = |P|^2$ leads to the important ∞-*Laplacian*

$$\Delta_\infty u \;:=\; \Big(Du \otimes Du + |Du|^2[Du]^\perp \otimes I\Big) : D^2u \;=\; 0. \tag{1.8}$$

System (1.7) is a quasilinear 2nd order system in non-divergence form which arises in the limit of the Euler−Lagrange system of the L^p functional $\|H(Du)\|_{L^p(\Omega)}$ as $p \to \infty$. In the scalar case of $n = 1$ the normal coefficient of (1.8) $|Du|^2[Du]^\perp$ vanishes, and the same holds for submersions in general. The scalar ∞-Laplacian then becomes $Du \otimes Du : D^2u = 0$.

Unlike the scalar case of $n = 1$, in the full vector case of (1.7) intriguing phenomena appear. Except for the emergence of "singular solutions" to (1.7), a further difficulty not present in the scalar case is that (1.7) *has discontinuous coefficients* even for C^∞ solutions. There exist C^∞ solutions whose rank of $H_P(Du)$ is not constant: such an example on $\mathbb{R}^2$ for (1.8) is given by $u(x, y) = e^{ix} - e^{iy}$ which is ∞-Harmonic near the origin and has $\text{rk}(Du) = 1$ on the diagonal, but it has $\text{rk}(Du) = 2$ otherwise and hence the projection $[Du]^\perp$ is discontinuous [12]. More sophisticated examples with interfaces which have junction and corners appear in [15]. In general, ∞-*Harmonic maps present a phase separation* and on each phase the dimension of the tangent space is constant and these phases are separated by *interfaces* whereon the rank of Du "jumps" and $[Du]^\perp$ is discontinuous [12,17]. Extensions of the results of [12,13] to the subelliptic setting appear in [14]. Moreover, it has very recently been established that the celebrated scalar L^∞ uniqueness theory has no counterpart when $N \geq 2$ [16].

In this paper we work towards the problem mentioned in the beginning by extending the theory of [8] to the case of immersions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ and in the same time we elaborate it and make it more efficient in certain respects. First of all, we allow for positive codimension $N - n$ and take into account the exterior geometry of immersions. Moreover, our maps are local diffeomorphisms onto their images, but in our analysis we do *not* impose the global topological constraint that our maps are homemorphisms onto their image and allow for self-intersections. However, *all* our results and notions are still valid and with the exact same proofs in this restricted class. For distinction, we introduce the following terminology: *an immersion* $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ *in* $C^1(\Omega)^N$ *is called p-Quasiconformal when* $\|K(Du)\|_{L^p(\Omega)} < \infty$, $1 \leq p \leq \infty$. We begin by repeating part of the program of [12,13] under the lens of [8] to the extended case. After some introductory material is Section 2, in Section 3 we calculate the PDE system which Optimal p-Quasiconformal immersions satisfy (Eqs. (3.23) and (3.24)), that is the Euler−Lagrange system of $K_p(u, \Omega) := \|K(Du)\|_{L^p(\Omega)}$. Then, in Section 4 we formally derive in the limit as $p \to \infty$ the PDE system which Optimal ∞-Quasiconformal immersions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ satisfy, that is the system associated to (1.2):

$$Q_\infty u \;:=\; \Big(K_P \otimes K_P + K[K_P]^\perp K_{PP}\Big)(Du) : D^2u \;=\; 0 \tag{1.9}$$

where the derivatives of the dilation are given by

$$K_P(Du) \;=\; 2Du\frac{g^{-1}S(g)}{\det(g)^{1/n}}, \tag{1.10}$$

$$K_{PP}(Du) \;=\; 2\left(I\otimes\frac{g^{-1}S(g)}{\det(g)^{1/n}} \;+\; Du\otimes Du : \frac{g^{-1}E}{\det(g)^{1/n}}\right) \;+\; O(Du). \tag{1.11}$$

Here $g = Du^\top Du$, S is the Ahlfors operator given by (2.7), E is a constant tensor given by (3.11) and $O(Du)$ is a tensor annihilated by $[K_P(Du)]^\perp$ and does not appear in the PDE system (1.9) (for details see Lems. 3.1, 3.2). The derivation has overlaps with the respective in [12], but is not a direct consequence since we utilise the specific structure of the Hamiltonian (1.3). By restricting ourselves to $n = N$ and employing Lemma 4.2 to relate the seemingly different system (1.9) to that of [8], we see that the derivation as $p\to\infty$ in [8] is incomplete and their PDE is only a part of (1.9). System (1.9) consists of two systems whose defining vector-valued nonlinearities are normal to each other:

$$K_P(Du)\otimes K_P(Du) : D^2u \;= 0, \tag{1.12}$$

$$[K_P(Du)]^\perp K_{PP}(Du) : D^2u \;= 0. \tag{1.13}$$

System (1.12) is the "tangential" part in (the range of the projection) $[K_P(Du)]^\top$ and system (1.13) is the "normal" part in $[K_P(Du)]^\perp$ (see Fig. 1). The reason for this terminology is that $[Du]^\top$ is (the projection on) the tangent bundle of the immersion, $[Du]^\perp$ is its normal bundle and by (1.10) we have that $[K_P(Du)]^\top \subseteq [Du]^\top$.

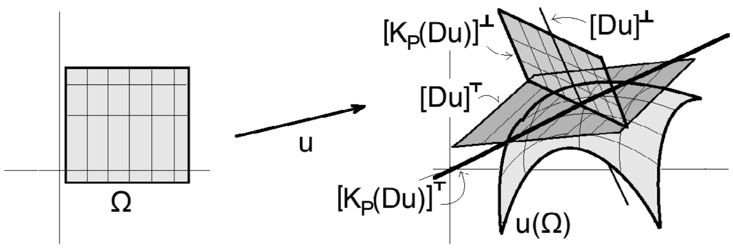

Figure 1.

The derivation in [8] has lost information along directions in $[K_P(Du)]^\perp$ and reveals only system (1.12). System (1.13) appears also in zero-codimension when $n = N$ since generally $K_P(Du)$ does not have rank equal to n, although by assumption the rank of Du equals n. More importantly, *when the rank of* $K_P(Du)$ *becomes nonconstant, the coefficients of* (1.9) *become discontinuous*. This leads to the *appearance of interfaces* whereon the projection $[K_P(Du)]^\perp$ is discontinuous. These interfaces are boundaries of the different *phases* to which Optimal ∞-Quasiconformal maps naturally separate.

In Section 5 we move to the variational structure of Optimal ∞-Quasiconformal maps. Inspired from [13], we introduce the variational notion of ∞-*Minimal Dilation*, which is Rank-One Locally Minimal Dilation with "Minimally Distorted Area" of $u(\Omega)$ (Def. 5.1). Rank-one locally minimal dilation requires that an immersion is a local minimiser for the dilation functional when the "set of competitors" is the one obtained by taking essentially scalar local variations with fixed zero boundary values (Fig. 2). Minimally distorted area means that the immersion is a local minimiser where the "set of competitors" is the one obtained by taking variations along sections of the normal vector bundle $[K_P(Du)]^\perp$ over $u(\Omega)$ with free boundary values (Fig. 3). The appearance of interfaces where the dimension of $[K_P(Du)]^\perp$ jumps causes substantial difficulties, even in the very definition of the minimality notion. Our first main result is Theorem 5.2, wherein we prove that ∞-Quasiconformal maps with ∞-Minimal Dilation are Optimal, *at least off the interfaces of discontinuities in the coefficients*. This result follows closely Theorem 2.1 in [12] and Theorem 2.2 in [13], but nonconvexity of (1.3), appearance of discontinuities in (1.9) and the necessity of restriction to specific variations create complications not present in the results just quoted. We note that the rank-one minimality notion gives rise to the tangential system and the condition on the minimality of the area gives rise to the normal system.

In Section 6 we study some geometric aspects of (1.9) and of the interfaces of its solutions. In Section 6.1 we show that system (1.9) has a "geometric" rather coordinate-free reformulation, at least off interfaces of discontinuities. More precisely, (1.12) and (1.13) are respectively equivalent to

$$S(\mathbf{G})D\big(\mathrm{tr}(\mathbf{G})\big) \;=\; 0, \tag{1.14}$$

$$\mathbb{B}^{\perp} : \big(\mathrm{tr}(\mathbf{G})\big)_P \;=\; 0, \tag{1.15}$$

where $\mathbf{G}$ is given by (1.1) for $g = Du^{\top}Du$ and $\mathbb{B}^{\perp}$ is a "generalized 2nd fundamental form" with respect to normal sections valued in $[K_P(Du)]^{\perp}$. If $K_P(Du)$ has full rank n, then $[K_P(Du)]^{\perp}$ coincides with the normal bundle $[Du]^{\perp}$ of the immersion and $\mathbb{B}^{\perp}$ reduces to the standard object. System (1.14) is quite "metrically invariant" but system (1.15) depends on the exterior geometry and measures the "shape of $u(\Omega)$". In Section 6.2, by assuming some *a priori* local C^1 regularity on the interfaces but with possible self-intersections, we prove an identity which shows that the covariant gradient of $[K_P(Du)]^{\perp}$ along the interface is differentiable when projected along $K_P(Du)$.

In Section 7 we turn our attention to the converse statement of that in Theorem 5.2, that is the sufficiency of (1.9) for the variational notion of ∞-Minimal Dilation. Nonconvexity of (1.3) and the resemblance to similar phenomena in *Minimal Surfaces* leaves little hope for system (1.13) to be sufficient for minimally distorted area. However, in Proposition 7.2 we establish that when $n = 2 \leq N$ there is a triple equivalence among solutions of (1.12), the condition the dilation (1.3) to be constant and the immersion to have rank-one locally minimal dilation. This result relates directly to the two-dimensional results in [1, 7, 11]. In particular, when $n = 2$ interfaces disappear and the coefficients of (1.9) become continuous.

Moreover, as a consequence of Example 7.5 which certifies that rank-one locally minimal dilation is *strictly weaker* than the variational notion utilized in [8] with respect to general vector-valued variations (among competitors), *we disprove the conjecture of Capogna–Raich on the sufficiency of* (1.3) *explicitely stated in p.* 855. Finally, at the end of Section 7 we loosely discuss the much more complicated case when $n \geq 3$. In this case results are less sharp. Although it is hardly conclusive, it seems that dilation may not be constant but we do believe that (1.12) is still sufficient for rank-one locally minimal dilation.

Throughout this paper, as in [8] and also in [12, 17], we restrict our analysis to the unnatural class of C^2 solutions. This is only the first step in our study and we can not go much further without an appropriate "weak" theory of nondifferentiable solutions for (1.9). The latter much deeper problem, namely defining a notion of solution for which we can also prove existence to the Dirichlet problem, will be considered in future work.

2. Preliminaries

Throughout this paper we reserve $n, N \in \mathbb{N}$ for the dimensions of Euclidean spaces and $\mathbb{S}^{N-1}$ denotes the unit sphere of $\mathbb{R}^N$. Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to N and Latin $i, j, k, \ldots$ form 1 to n. The summation convention will always be employed in repeated indices in a product. Vectors are always viewed as columns and we differentiate along rows. Hence, for $a, b \in \mathbb{R}^n$, $a^{\top}b$ is their inner product and $ab^{\top}$ equals $a \otimes b$. If $u = u_\alpha e_\alpha : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is in $C^2(\Omega)^N$, the gradient matrix Du is viewed as $D_i u_\alpha e_\alpha \otimes e_i : \Omega \longrightarrow \mathbb{R}^{N\times n}$ and the Hessian tensor D^2u as $D^2_{ij}u_\alpha e_\alpha \otimes e_i \otimes e_j : \Omega \longrightarrow \mathbb{R}^{N\times n^2}$. The Euclidean (Frobenious) norm on $\mathbb{R}^{N\times n}$ is $|P| = (P_{\alpha i}P_{\alpha i})^{1/2} = (\mathrm{tr}(P^{\top}P))^{1/2}$. We also introduce the following *contraction operation* for tensors which extends the Euclidean inner product $P : Q = \mathrm{tr}(P^{\top}Q) = P_{\alpha i}Q_{\alpha i}$ of $\mathbb{R}^{N\times n} = \mathbb{R}^N \otimes \mathbb{R}^n$. Let "$\otimes^{(r)}$" denote the r-fold tensor product. If $S \in \otimes^{(q)}\mathbb{R}^N \otimes^{(s)} \mathbb{R}^n$, $T \in \otimes^{(p)}\mathbb{R}^N \otimes^{(s)} \mathbb{R}^n$ and $q \geq p$, we define a tensor $S : T$ in $\otimes^{(q-p)}\mathbb{R}^N$ by

$$S : T \;:=\; \big(S_{\alpha_q,\ldots\alpha_p,\ldots\alpha_1\, i_s,\ldots i_1} T_{\alpha_p,\ldots\alpha_1\, i_s,\ldots i_1}\big)\, e_{\alpha_q} \otimes, \ldots \otimes e_{\alpha_{p+1}}. \tag{2.1}$$

For example, for $s = q = 2$ and $p = 1$, the tensor $S : T$ of (2.1) is a vector with components $S_{\alpha\beta ij}T_{\beta ij}$ with free index α and the indices β, i, j are contracted. In particular, in view of (2.1), the 2nd order linear system

$$A_{\alpha i\beta j}D^2_{ij}u_\beta \;+\; B_{\alpha\gamma k}D_k u_\gamma + C_{\alpha\delta}u_\delta \;=\; f_\alpha, \tag{2.2}$$

can be compactly written as $A{:}D^2u + B{:}Du + Cu = f$, where the meaning of ":" in the respective dimensions is made clear by the context. Let now $P : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be linear map. The identity map of $\mathbb{R}^N$ splits as $I = [P]^\top \oplus [P]^\perp$, where $[P]^\top$ and $[P]^\perp$ denote orthogonal projection on range $R(P)$ and nullspace $N(P^\top)$ respectively. Moreover, for the dilation function (1.3), we have $K(P) \geq n$ and $K(P) = n$ if and only if $P^\top P = \lambda I$ with $\lambda = \frac{1}{n}|P|^2$. This property of K is a simple consequence of the inequality of arithmetic-geometric mean applied to the n eigenvalues of $P^\top P$ by utilising the Spectral Theorem. Let us now recall some elementary properties of determinants. If $A = A_{ij}e_i \otimes e_j \in \mathbb{R}^n \otimes \mathbb{R}^n$, we have

$$\mathrm{cof}(A)_{ij} \;:= (-1)^{i+j}\det\Big(\sum_{k\neq i, l\neq j} A_{kl}e_k \otimes e_l\Big), \tag{2.3}$$

$$\mathrm{cof}(A) \;:= \mathrm{cof}(A)_{ij}e_i \otimes e_j, \tag{2.4}$$

$$A\,\mathrm{cof}(A)^\top \;= \mathrm{cof}(A)^\top A \;=\; \det(A)I, \tag{2.5}$$

$$D_{A_{ij}}\big(\det(A)\big) \;\equiv\; \big(\det(A)\big)_{A_{ij}} \;=\; \mathrm{cof}(A)_{ij}. \tag{2.6}$$

Obviously, subscript denotes partial derivative. The *Ahlfors operator* is defined by

$$S(A) \;:=\; \frac{1}{2}\big(A + A^\top\big) \;-\; \frac{1}{n}\mathrm{tr}(A)I \tag{2.7}$$

and has the property that for any A, $S(A)$ is symmetric and traceless, that is $\mathrm{tr}(S(A)) = 0$. Let now $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^1(\Omega)^N$. Then, the rank of Du satisfies $\mathrm{rk}(Du) = n \leq N$. u is *Conformal* when there is $f \in C^0(\Omega)$ such that $Du^\top Du = f^2 I$ on Ω, that is $D_i u_\alpha D_j u_\alpha = f^2\delta_{ij}$. For immersions, the Riemannian metric on $u(\Omega)$ induced from $\mathbb{R}^N$ is $g := Du^\top Du$ and g^{-1} denotes the pointwise inverse of the positive symmetric tensor g. Since $S(g) = g - \frac{1}{n}\mathrm{tr}(g)I$, we have the commutativity relation

$$g^{-1}S(g) \;=\; S(g)g^{-1} \;=\; I \;-\; \frac{\mathrm{tr}(g)}{n}g^{-1} \tag{2.8}$$

which will be tacitly used in the sequel. In view of these conventions, the PDE system describing Optimal Quasiconformal immersions in index form reads

$$\Big(K_{P_{\alpha i}}K_{P_{\beta j}} \;+\; K[K_P]^\perp_{\alpha\gamma}K_{P_{\gamma i}P_{\beta j}}\Big)(Du)\,D^2_{ij}u_\beta \;=\; 0. \tag{2.9}$$

The derivatives K_P, K_{PP} of K appearing here and in (1.10), (1.11) are given in index form by (3.2), (3.10). Finally, we will use the notation "Γ" for sections of vector bundles. We note that our terminology of "p-Quasiconformal" slightly deviates from the usage of this term in the literature, but its purpose is to avoid the less elegant term "L^p-Quasiconformal". Since we are only interested in the extreme case of $p = \infty$, there will be no confusion. We conclude by observing thatwhen $\Omega \Subset \mathbb{R}^n$, all immersions $u : \overline{\Omega} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ in $C^1(\overline{\Omega})^N$ are p-Quasiconformal for all $p \in [1, \infty]$.

3. Derivation of the Euler–Lagrange PDE system governing Optimal p-Quasiconformal immersions

In this section we calculate the specific form of the Euler–Lagrange system associated to the functional $\|K(Du)\|^p_{L^p(\Omega)}$ which Optimal p-Quasiconformal immersions satisfy. We begin by calculating first and second derivatives of (1.3).

Lemma 3.1. *Let K be given by (1.3). Then, $K \in C^1(S^+)$ and its derivative is given by*

$$K_P(P) \;=\; 2P\frac{\big(P^\top P\big)^{-1}S\big(P^\top P\big)}{\det\big(P^\top P\big)^{1/n}}\cdot \tag{3.1}$$

In index form (3.1) can be written as

$$K_{P_{\alpha i}}(P) \;=\; 2P_{\alpha m}\left(\frac{\delta_{mi}-\frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{mi}}{\det\big(P^\top P\big)^{1/n}}\right)\cdot \tag{3.2}$$

Proof of Lemma 3.1. We begin by observing the triviality that for $P \in S^+$, the matrix $P^\top P$ is positive symmetric on $\mathbb{R}^n$ and also

$$\big(P^\top P\big)^{-1,\top} \;=\; \big(P^\top P\big)^{\top,-1} \;=\; \big(P^\top P\big)^{-1}. \tag{3.3}$$

By differentiation of (1.3), we have

$$\begin{aligned} K_{P_{\alpha i}}(P) \;&=\; \frac{2P_{\alpha i}\det\big(P^\top P\big)^{\frac{1}{n}} \;-\; \dfrac{|P|^2}{n}\det\big(P^\top P\big)^{\frac{1}{n}-1}\mathrm{cof}\big(P^\top P\big)_{kl}(P_{\beta k}P_{\beta l})_{P_{\alpha i}}}{\det\big(P^\top P\big)^{2/n}} \\ &=\; \frac{2P_{\alpha i} \;-\; \dfrac{|P|^2}{n\det\big(P^\top P\big)}\mathrm{cof}\big(P^\top P\big)_{kl}\big(\delta_{\alpha\beta}\delta_{ik}P_{\beta l} \;+\; \delta_{\alpha\beta}\delta_{il}P_{\beta k}\big)}{\det\big(P^\top P\big)^{1/n}}\cdot \end{aligned} \tag{3.4}$$

Thus,

$$\begin{aligned} K_{P_{\alpha i}}(P) \;&=\; \frac{2P_{\alpha i} \;-\; \dfrac{|P|^2}{n\det\big(P^\top P\big)}\Big(\mathrm{cof}\big(P^\top P\big)_{il}P_{\alpha l} + \mathrm{cof}\big(P^\top P\big)_{ki}P_{\alpha k}\Big)}{\det\big(P^\top P\big)^{1/n}} \\ &= 2P_{\alpha m}\,\frac{\delta_{mi} \;-\; \dfrac{|P|^2}{n\det\big(P^\top P\big)}\dfrac{1}{2}\Big(\mathrm{cof}\big(P^\top P\big)_{im} + \mathrm{cof}\big(P^\top P\big)_{mi}\Big)}{\det\big(P^\top P\big)^{1/n}}\cdot \end{aligned} \tag{3.5}$$

Hence, (3.5) gives

$$K_P(P) \;=\; \frac{2P}{\det\big(P^\top P\big)^{1/n}}\left(I \;-\; \frac{|P|^2}{n}\left(\frac{\mathrm{cof}\big(P^\top P\big)^\top + \mathrm{cof}\big(P^\top P\big)}{2\det\big(P^\top P\big)}\right)\right) \tag{3.6}$$

and by using that

$$\mathrm{cof}\big(P^\top P\big)^\top \;=\; \mathrm{cof}\big(P^\top P\big) \;=\; \big(P^\top P\big)^{-1}\det\big(P^\top P\big), \tag{3.7}$$

equation (3.6) gives

$$\begin{aligned} K_P(P) \;&=\; \frac{2P}{\det\big(P^\top P\big)^{1/n}}\left(I \;-\; \frac{|P|^2}{n}\big(P^\top P\big)^{-1}\right) \\ &= 2P\frac{\big(P^\top P\big)^{-1}}{\det\big(P^\top P\big)^{1/n}}\left(P^\top P \;-\; \frac{|P|^2}{n}I\right)\cdot \end{aligned} \tag{3.8}$$

In view of (3.8), formula (3.1) has been established. □

Lemma 3.2. *Let K be given by (1.3). Then, $K \in C^2(S^+)$ and its 2nd derivative is given by*

$$K_{PP}(P) \;=\; 2I \otimes \frac{\big(P^\top P\big)^{-1} S\big(P^\top P\big)}{\det\big(P^\top P\big)^{1/n}} \;+\; 2P \otimes P : \frac{\big(P^\top P\big)^{-1} E}{\det\big(P^\top P\big)^{1/n}} \;+\; O(P) \tag{3.9}$$

which in index form can be written as

$$K_{P_{\alpha i}P_{\beta j}}(P) \;=\; 2\delta_{\alpha\beta}\left(\frac{\big(P^\top P\big)^{-1}_{ik}\big(P_{\gamma k}P_{\gamma j} - \frac{1}{n}|P|^2\delta_{kj}\big)}{\det\big(P^\top P\big)^{1/n}}\right) +\; 2P_{\alpha m}P_{\beta l}\left(\frac{\big(P^\top P\big)^{-1}_{ik}E_{kjlm}}{\det\big(P^\top P\big)^{1/n}}\right) \;+\; O_{\alpha i\beta j}(P). \tag{3.10}$$

Here $O_{\alpha i\beta j}(P)$ is a tensor of the form $K_{P_{\alpha m}}(P)A_{m\beta ij}(P)$ and is annihilated by $[K_P(P)]^\perp_{\gamma\alpha}$, that is $[K_P(P)]^\perp O(P) = 0$. E is a constant 4th order tensor whose components E_{kjlm} are given by

$$E_{kjlm} \;:=\; \delta_{ml}\delta_{jk} + \delta_{mj}\delta_{kl} - \frac{2}{n}\delta_{mk}\delta_{jl}. \tag{3.11}$$

The explicit form of the tensor $O_{\alpha i\beta j}(P)$ is a complicated formula which follows by the Proof of Lemma 3.2, but we do not need this formula because is "killed" by $[K_P(P)]^\perp$ and doe not appear in (1.9).

Proof of Lemma 3.2. We begin by calculating the derivative $\big(\big(P^\top P\big)^{-1}_{mi}\big)_{P_{\beta j}}$. We have

$$\big(P^\top P\big)^{-1}_{mi}\big(P^\top P\big)_{ik} \;=\; \delta_{mk} \tag{3.12}$$

which gives

$$\begin{aligned} \big(\big(P^\top P\big)^{-1}_{mi}\big)_{P_{\beta j}}\big(P^\top P\big)_{ik} \;&=\; -\big(P^\top P\big)^{-1}_{mi}(P_{\gamma i}P_{\gamma k})_{P_{\beta j}} \\ &=\; -\big(P^\top P\big)^{-1}_{mi}[\delta_{\beta\gamma}\delta_{ij}P_{\gamma k} + P_{\gamma i}\delta_{\beta\gamma}\delta_{kj}] \\ &=\; -\big(P^\top P\big)^{-1}_{ml}[P_{\beta k}\delta_{lj} + P_{\beta l}\delta_{kj}]. \end{aligned} \tag{3.13}$$

Hence, we have

$$\big(\big(P^\top P\big)^{-1}_{mi}\big)_{P_{\beta j}} \;=\; -\big(P^\top P\big)^{-1}_{ml}[P_{\beta k}\delta_{lj} + P_{\beta l}\delta_{kj}]\big(P^\top P\big)^{-1}_{ki}. \tag{3.14}$$

Now we differentiate (3.2):

$$\begin{aligned} K_{P_{\alpha i}P_{\beta j}}(P) \;=\; 2\delta_{\alpha\beta}\delta_{mj}&\left(\frac{\delta_{mi} - \frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{mi}}{\det\big(P^\top P\big)^{1/n}}\right) \;-\; 2P_{\alpha m}\left(\frac{\big(|P|^2\big(P^\top P\big)^{-1}_{mi}\big)_{P_{\beta j}}}{n\det\big(P^\top P\big)^{1/n}}\right) \\ &-\;\left[2P_{\alpha m}\left(\frac{\delta_{mi} - \frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{mi}}{\det\big(P^\top P\big)^{1/n}}\right)\right]\frac{\big(\det\big(P^\top P\big)^{1/n}\big)_{P_{\beta j}}}{\det\big(P^\top P\big)^{1/n}}. \end{aligned} \tag{3.15}$$

In view of (3.2), the last summand in (3.15) is annihilated by the projection $[K_P(P)]^\perp_{\gamma\alpha}$. We rewrite (3.15) as

$$K_{P_{\alpha i}P_{\beta j}}(P) \;=2\delta_{\alpha\beta}\left(\frac{\delta_{ij} - \frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{ij}}{\det\big(P^\top P\big)^{1/n}}\right) -\; 2P_{\alpha m}\left(\frac{\big(|P|^2\big(P^\top P\big)^{-1}_{mi}\big)_{P_{\beta j}}}{n\det\big(P^\top P\big)^{1/n}}\right) \;+\; O_{\alpha i\beta j}(P). \tag{3.16}$$

By using (3.14) in (3.16), we have

$$K_{P_{\alpha i}P_{\beta j}}(P) \;=\; 2\delta_{\alpha\beta}\left(\frac{\delta_{ij} - \frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{ij}}{\det\big(P^\top P\big)^{1/n}}\right) \;+S_{\alpha i\beta j}(P) \;+\; O_{\alpha i\beta j}(P), \tag{3.17}$$

where we have set

$$S_{\alpha i\beta j}(P) \;:=\; \frac{2}{n}P_{\alpha m}\frac{2P_{\beta j}\big(P^\top P\big)^{-1}_{mi} - |P|^2\big(P^\top P\big)^{-1}_{ml}[P_{\beta k}\delta_{lj} + P_{\beta l}\delta_{kj}]\big(P^\top P\big)^{-1}_{ki}}{\det\big(P^\top P\big)^{1/n}}. \tag{3.18}$$

Equation (3.18) gives

$$\begin{aligned} S_{\alpha i\beta j}(P) \;=\; &-\; \frac{4}{n}P_{\alpha m}P_{\beta j}\frac{\big(P^\top P\big)^{-1}_{mi}}{\det\big(P^\top P\big)^{1/n}} \\ &+\; 2P_{\alpha m}\left(\frac{\frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{mj}}{\det\big(P^\top P\big)^{1/n}}\right)\big(P^\top P\big)^{-1}_{ki}P_{\beta k} \\ &+\; 2P_{\alpha m}\left(\frac{\frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{mk}}{\det\big(P^\top P\big)^{1/n}}\right)\big(P^\top P\big)^{-1}_{ij}P_{\beta k}. \end{aligned} \tag{3.19}$$

We rewrite (3.19) as

$$\begin{aligned} S_{\alpha i\beta j}(P) \;=\; &-\; \frac{4}{n}P_{\alpha m}P_{\beta j}\frac{\big(P^\top P\big)^{-1}_{mi}}{\det\big(P^\top P\big)^{1/n}} \\ &+\; 2P_{\alpha m}\left(\frac{-\delta_{mj} + \frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{mj}}{\det\big(P^\top P\big)^{1/n}} \;+\; \frac{\delta_{mj}}{\det\big(P^\top P\big)^{1/n}}\right)\big(P^\top P\big)^{-1}_{ki}P_{\beta k} \\ &+\; 2P_{\alpha m}\left(\frac{-\delta_{mk} + \frac{1}{n}|P|^2\big(P^\top P\big)^{-1}_{mk}}{\det\big(P^\top P\big)^{1/n}} \;+\; \frac{\delta_{mk}}{\det\big(P^\top P\big)^{1/n}}\right)\big(P^\top P\big)^{-1}_{ij}P_{\beta k} \end{aligned} \tag{3.20}$$

and observe that in view of (3.2), $[K_P(Du)]^\perp_{\gamma\alpha}$ annihilates the first summands in the brackets of (3.20) and $S_{\alpha i\beta j}(P)$ simplifies to

$$\begin{aligned} S_{\alpha i\beta j}(P) \;=\; &2\frac{P_{\alpha k}P_{\beta k}\big(P^\top P\big)^{-1}_{ij} + P_{\alpha j}P_{\beta k}\big(P^\top P\big)^{-1}_{ki} - \frac{2}{n}P_{\alpha m}P_{\beta j}\big(P^\top P\big)^{-1}_{mi}}{\det\big(P^\top P\big)^{1/n}} \\ &+\; O_{\alpha i\beta j}(P), \end{aligned} \tag{3.21}$$

for some tensor $O_{\alpha i\beta j}(P)$ annihilated by $[K_P(Du)]^\perp_{\gamma\alpha}$. We rewrite (3.21) as

$$S_{\alpha i\beta j}(P) \;=\; 2P_{\alpha m}P_{\beta l}\big(P^\top P\big)^{-1}_{ki}\left(\frac{\delta_{ml}\delta_{jk} + \delta_{mj}\delta_{kl} - \frac{2}{n}\delta_{mk}\delta_{jl}}{\det\big(P^\top P\big)^{1/n}}\right) + O_{\alpha i\beta j}(P). \tag{3.22}$$

In view of (3.22), (3.18), (3.17) and (3.11), equation (3.10) follows. □

In view of Lemma 3.1, the Euler−Lagrange system describing Optimal p-Quasiconformal immersions $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is

$$Q_p u \;:=\; \mathrm{Div}\Big(K(Du)^{p-1}K_P(Du)\Big) \;=\; 0. \tag{3.23}$$

In view of (3.1), (3.23) can be written in index form as

$$D_i\left(\left(\frac{\mathrm{tr}(g)}{\det(g)^{1/n}}\right)^{p-1} D_k u_\alpha\,\frac{g^{-1}_{km}S(g)_{mi}}{\det(g)^{1/n}}\right) \;=\; 0, \tag{3.24}$$

where $g = Du^\top Du$ is the Riemannian metric and S is the Ahlfors operator of (2.7).

4. Derivation of the PDE system governing Optimal ∞-Quasiconformal immersions

The derivation we perform is this section can be deduced by a reworking of our results in [12, 13] and application of Lemmas 3.1 and 3.2 proved previously, but for the reader's convenience it is best to argue at the outset. Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$. By distributing derivatives in (3.23), we have

$$(p-1)K^{p-2}K_{P_{\alpha i}}(Du)K_{P_{\beta j}}(Du)D^2_{ij}u_\beta \ + \ K^{p-1}K_{P_{\alpha i}P_{\beta j}}(Du)D^2_{ij}u_\beta \ = \ 0. \tag{4.1}$$

For each $x \in \Omega$, $K_P\big((Du)(x)\big) : \mathbb{R}^n \longrightarrow \mathbb{R}^N$ is a linear map. We define the orthogonal projections

$$[K_P(Du)]^\perp \ := \ \mathrm{Proj}_{N((K_P(Du))^\top)}, \tag{4.2}$$

$$[K_P(Du)]^\top \ := \ \mathrm{Proj}_{R(K_P(Du))}, \tag{4.3}$$

which are the projections on nullspace of $(K_P(Du))^\top$ and range of $K_P(Du)$ respectively. We rewrite (4.1) by applying the expansion $I = [K_P(Du)]^\perp + [K_P(Du)]^\top$ of the identity of $\mathbb{R}^N$ and contract the derivative in the left hand side to obtain

$$\begin{aligned} K_P(Du)D\big(K(Du)\big) \ &+ \frac{K}{p-1}[K_P(Du)]^\top K_{PP}(Du) : D^2u \\ &= \ -\frac{K}{p-1}[K_P(Du)]^\perp K_{PP}(Du) : D^2u. \end{aligned} \tag{4.4}$$

The left hand side is a vector valued in $[K_P(Du)]^\top$ and the right hand side is a vector valued in $[K_P(Du)]^\perp$. By orthogonality, left and right hand side vanish and actually (4.4) decouples to two systems. We rescale the right hand side of (4.4) by multiplying by $p-1$ and rearrange to obtain

$$\begin{aligned} K_P(Du)\otimes K_P(Du) : D^2u \ &+ \ K[K_P(Du)]^\perp K_{PP}(Du) : D^2u \\ &= \ -\frac{K(Du)}{p-1}[K_P(Du)]^\top K_{PP}(Du) : D^2u. \end{aligned} \tag{4.5}$$

We rewrite as

$$\Big(K_P\otimes K_P + K[K_P]^\perp K_{PP}\Big)(Du) : D^2u \ = \ -\frac{K[K_P]^\top K_{PP}}{p-1}(Du) : D^2u. \tag{4.6}$$

As $p\to\infty$, (4.6) leads to (1.9).

Remark 4.1. We note that we can also remove the dilation function K from the normal coefficient $[K_P]^\perp K_{PP}$ with the renormalisation because it is strictly positive: $K(Du) \geq n > 0$. We do not have this option in the case of the general system (1.7), because $|H(Du)|$ may vanish. However, when $n = 2 \leq N$ and $H(P) = |P|^2$, in [17] we show that non-constant ∞-Harmonic maps have no interior gradient zeros: either $|Du| > 0$ or $|Du| \equiv 0$.

The next differential identity relates our system (1.9) with the seemingly different Aronsson PDE system of Capogna$-$Raich in [8]. In particular, it follows that even when $n = N$ the PDE system derived in [8] is only a projection of (1.9) along $[K_P(Du)]^\top$. Hence, the PDE system in [8] seems to fail to encapsulate all the information of optimised quasiconformal maps.

Lemma 4.2. *Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a local diffeomorphism in $C^1(\Omega)^n$. Then, we have the identity*

$$K_P(Du) \ = \ -\frac{2K(Du)}{n}\left((Du)^{-1,\top} - n\frac{Du}{|Du|^2}\right) \tag{4.7}$$

where K and K_P are given by (1.3) and (3.1).

Proof of Lemma 4.2. By observing that for any invertible $A \in \mathbb{R}^n \otimes \mathbb{R}^n$ there holds $A^{-1,\top} = A^{\top,-1}$, we have

$$\big(Du^\top Du\big)^{-1} \;=\; (Du)^{-1}(Du)^{\top,-1} \;=\; (Du)^{-1}(Du)^{-1,\top}. \tag{4.8}$$

Thus, we obtain

$$\begin{aligned}(Du)^{-1,\top} - n\frac{Du}{|Du|^2} \;&=\; -\frac{n}{|Du|^2}\left(Du - \frac{|Du|^2}{n}(Du)^{-1,\top}\right)\\ &=\; -\frac{n}{|Du|^2}\left(Du - \frac{|Du|^2}{n}Du(Du)^{-1}(Du)^{-1,\top}\right)\\ &=\; -\frac{n}{|Du|^2}Du\left(I - \frac{|Du|^2}{n}(Du)^{-1}(Du)^{-1,\top}\right).\end{aligned} \tag{4.9}$$

Consequently, by (4.8) and (4.9), we obtain

$$\begin{aligned}-\frac{|Du|^2}{n}\left((Du)^{-1,\top} - n\frac{Du}{|Du|^2}\right) \;&= Du\left(I - \frac{|Du|^2}{n}\big(Du^\top Du\big)^{-1}\right)\\ &= Du\left(Du^\top Du\right)^{-1}\left(Du^\top Du - \frac{|Du|^2}{n}I\right).\end{aligned} \tag{4.10}$$

Hence, by (3.1) and (1.3) we have

$$\begin{aligned}-\frac{2K(Du)}{n}\left((Du)^{-1,\top} - n\frac{Du}{|Du|^2}\right) \;&= 2Du\left(Du^\top Du\right)^{-1}\left(\frac{Du^\top Du - \frac{|Du|^2}{n}I}{\det\left(Du^\top Du\right)^{1/n}}\right)\\ &= K_P(Du).\end{aligned} \tag{4.11}$$

The desired identity follows. □

5. Variational structure of Optimal ∞-Quasiconformal immersions

We begin by introducing a minimality notion of vector-valued Calculus of Variations in L^∞ for the supremal dilation functional (1.2). Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^1(\Omega)^N$. In view of (3.1), we have the identity

$$K_P(Du) \;=\; \left(2\frac{Du\big(Du^\top Du\big)^{-1}}{\det\left(Du^\top Du\right)^{1/n}}\right)S\big(Du^\top Du\big). \tag{5.1}$$

Generally, the rank of $K_P(Du)$ may not be constant throughout Ω, although by assumption $\mathrm{rk}(Du) = \mathrm{rk}(Du^\top Du) \equiv n$, because possibly $\mathrm{rk}(S(Du^\top Du)) < n$ on certain regions of Ω. We set

$$\Omega_k \;:=\; \mathrm{int}\Big\{\mathrm{rk}\big(S(Du^\top Du)\big) \;=\; k\Big\}\;, \quad k \;= 0, 1, \ldots, n, \tag{5.2}$$

where "int" denotes topological interior. The $n+1$ open sets Ω_k are the *"phases"* of the immersion u. Their complement in Ω

$$\mathcal{S} \;:=\; \Omega \setminus (\cup_0^n \Omega_k) \tag{5.3}$$

is the set of *"interfaces"* and is closed in Ω with empty interior. We will also need the *"augmented phases"*

$$\Omega_k^* \;:=\; \Big\{\mathrm{rk}\big(S(Du^\top Du)\big) \;=\; k\Big\}\;, \quad k \;=\; 0,\, 1\,, \ldots,\, n. \tag{5.4}$$

Obviously, $\{\Omega_0^*, \dots, \Omega_n^*\}$ is a partition of Ω to disjoint phases and $\mathcal{S}$ can be written as $\mathcal{S} = \cup_0^n(\Omega_k^* \setminus \Omega_k)$. The extreme cases of Ω_0^* and Ω_n^* are particularly important. Ω_0^* is the *conformality set* of the immersion and is closed in Ω. Hence,

$$\Omega_0^* \;=\; \left\{ Du^\top Du = \frac{|Du|^2}{n} I \right\}. \tag{5.5}$$

Similarly, by Corollary 6.3 that follows, if u solves $K_P(Du) \otimes K_P(Du) : D^2u = 0$, then Ω_n^* is the *constant dilation set* of the immersion and coincides with Ω_n:

$$\Omega_n^* \;=\; \left\{ \frac{|Du|^2}{\det(Du^\top Du)^{1/n}} = \text{const.} \right\}. \tag{5.6}$$

If Ω_n is not connected, then the constants may differ in connected cmponents.

Definition 5.1. Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^1(\Omega)^N$.

(i) We say that u has *Rank-One Locally Minimal Dilation* when for all compactly contained subdomains D of Ω, all functions g over D vanishing on ∂D and all directions ξ, u is a minimiser on D with respect to essentially scalar variations $u + f\xi$:

$$\left.\begin{array}{l} D \subset\subset \Omega, \\ f \in C_0^1(D), \\ \xi \in \mathbb{S}^{N-1} \end{array}\right\} \implies K_\infty(u, \Omega) \;\le\; K_\infty(u + f\xi, \Omega). \tag{5.7}$$

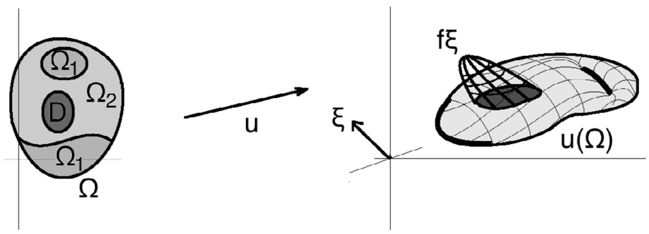

FIGURE 2.

(ii) We say that $u(\Omega)$ *has Minimally Distorted Area* when for all compactly contained subdomains D *off the interfaces*, all functions h on $\bar{D}$ (not only vanishing on ∂D) and all vector fields ν along u normal to $K_P(Du)$, u is a minimiser on D with respect to normal free variations $u + h\nu$:

$$\left.\begin{array}{l} D \subset\subset \Omega \setminus \mathcal{S}, \\ h \in C^1(\bar{D}), \\ \nu \in \Gamma([K_P(Du)]^\perp) \end{array}\right\} \implies K_\infty(u, \Omega) \;\le\; K_\infty(u + h\nu, \Omega). \tag{5.8}$$

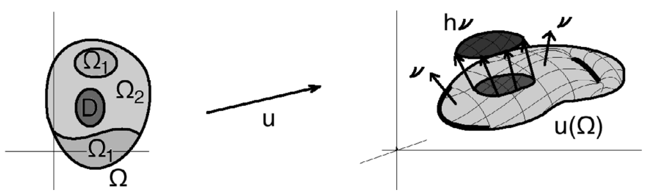

FIGURE 3.

(iii) We call u *Minimal* ∞*-Quasiconformal Immersion* when u is has Rank-One Locally Minimal Dilation with Minimally Distorted Area of $u(\Omega) \subseteq \mathbb{R}^N$.

By employing the previous minimality notion, we have the next

Theorem 5.2 (Variational structure of Optimal ∞-Quasiconformal immersions)**.** *Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$. Then, if u is Minimal ∞-Quasiconformal, it follows that u solves*

$$K_P(Du) \otimes K_P(Du) : D^2u \;=\; 0, \quad \text{on } \Omega, \tag{5.9}$$

$$[K_P(Du)]^{\perp} K_{PP}(Du) : D^2u \;=\; 0, \quad \text{on } \Omega \setminus \mathcal{S}, \tag{5.10}$$

where $\mathcal{S}$ is the set of interfaces of rank discontinuities of $S(Du^\top Du)$.

We note that by the results of Section 6 that follows, in the case $n = 2 \leq N$ Theorem 5.2 can be strengthend to the following

Corollary 5.3 (2-Dimensional Optimal ∞-Quasiconformal immersions)**.** *Let $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$. If u is Minimal ∞-Quasiconformal, it follows that u is Optimal ∞-Quasiconformal.*

The point in Corollary 5.3 is that (5.10) is satisfied on Ω and not only on $\Omega \setminus \mathcal{S}$. Actually, *when $n = 2$ then the set of interfaces is empty: $\mathcal{S} = \emptyset$.*

The proof of Theorem 5.2 is split in two lemmas.

Lemma 5.4. *Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$. If u has Rank-One Locally Minimal Dilation, then u solves $K_P(Du) \otimes K_P(Du) : D^2u = 0$ on Ω.*

The proof of Lemma 5.4 follows by Theorem 2.1 in [12] and relates to Lemma 2.3 in [13], but we present a simplified more direct proof for the reader's convenience.

Proof of Lemma 5.4. Fix $x \in \Omega$, $0 < \varepsilon < \operatorname{dist}(x, \partial\Omega)$, $\delta > 0$ and $\xi \in \mathbb{S}^{N-1}$. Choose $D := \mathbb{B}_\varepsilon(x)$ and $f \in C^1_0(D)$ given by

$$f(z) \;:=\; \frac{1}{2}\big(\varepsilon^2 - |z-x|^2\big). \tag{5.11}$$

Since $\mathrm{rk}(Du) = n$ on Ω and $Df(z) = -(z-x)$, by restricting δ sufficiently we obtain that $\mathrm{rk}(Du + \delta\xi \otimes Df) = n$ on $\mathbb{B}_\varepsilon(x)$. By Taylor expansions of $K(Du)$ and $K(Du + \delta\xi \otimes Df)$ at x we have

$$K(Du(z)) \;=\; K(Du(x)) \;+\; D\big(K(Du)\big)(x)^\top (z-x) \;+\; o(|z-x|), \tag{5.12}$$

as $z \to x$, and also by using that $D^2 f = -I$ and $Df(x) = 0$ we have

$$\begin{aligned} K\big((Du + \delta\xi \otimes Df)(z)\big) \;&=\; K\big((Du + \delta\xi \otimes Df)(x)\big) \\ &\quad\; + \; D\big(K(Du + \delta\xi \otimes Df)\big)(x)^\top (z-x) \;+\; o(|z-x|) \\ &=\; K(Du(x)) \;+\; K_P(Du(x))^\top \big(D^2u(x) - \delta\xi \otimes I\big)(z-x) \\ &\quad\; + \; o(|z-x|) \\ &=\; K(Du(x)) + \Big(D\big(K(Du)\big)^\top - \delta\xi^\top K_P(Du)\Big)(x)(z-x) \\ &\quad\; + \; o(|z-x|), \end{aligned} \tag{5.13}$$

as $z \to x$. By (5.12) we have the estimate

$$\begin{aligned} K_\infty\big(u, \mathbb{B}_\varepsilon(x)\big) \;&\geq\; K(Du(x)) \;+\; \max_{\{|z-x| \leq \varepsilon\}} \Big\{D\big(K(Du)\big)(x)^\top (z-x)\Big\} \;+\; o(\varepsilon) \\ &=\; K(Du(x)) \;+\; \varepsilon\big|D\big(K(Du)\big)(x)\big| \;+\; o(\varepsilon), \end{aligned} \tag{5.14}$$

as $\varepsilon \to 0$, and also by (5.13) we have

$$
\begin{aligned}
K_\infty\big(u+\delta f\xi,\mathbb{B}_\varepsilon(x)\big) \ &\leq\ K(Du(x)) \ +\ \max_{\{|z-x|\leq\varepsilon\}}\Big\{D\big(K(Du)\big)^\top \\
&\qquad\qquad - \delta\xi^\top K_P(Du)\big)(x)(z-x)\Big\} \ +\ o(\varepsilon) \\
&=\ K(Du(x)) \ +\ \varepsilon\big|D\big(K(Du)\big) - \delta\xi^\top K_P(Du)\big|(x) \ +\ o(\varepsilon),
\end{aligned}
\tag{5.15}
$$

as $\varepsilon \to 0$. Then, since u has Rank-One Locally Minimal Dilation, by (5.14) and (5.15) we have

$$
\begin{aligned}
0 \ &\leq\ K_\infty\big(u+\delta f\xi,\mathbb{B}_\varepsilon(x)\big) \ -\ K_\infty\big(u,\mathbb{B}_\varepsilon(x)\big) \\
&\leq\ \varepsilon\Big(\big|D\big(K(Du)\big) - \delta\xi^\top K_P(Du)\big| \ -\ \big|D\big(K(Du)\big)\big|\Big)(x) \ +\ o(\varepsilon),
\end{aligned}
\tag{5.16}
$$

as $\varepsilon \to 0$. Suppose first $D\big(K(Du)\big)(x) = 0$. Since

$$
K_P(Du)\otimes K_P(Du) : D^2u \ =\ K_P(Du)D\big(K(Du)\big)
\tag{5.17}
$$

we obtain that $\big(K_P(Du)\otimes K_P(Du) : D^2u\big)(x) = 0$ as desired. If $D\big(K(Du)\big)(x) \neq 0$, then Taylor expansion of the function

$$
p \ \mapsto\ \big|D\big(K(Du)\big)(x) + p\big| - \big|D\big(K(Du)\big)(x)\big|
\tag{5.18}
$$

at $p_0 = 0$ and evaluated at $p = -\,\delta\xi^\top K_P(Du(x))$, (5.16) implies after letting $\varepsilon \to 0$ that

$$
0 \ \leq\ -\delta\,\xi^\top K_P(Du(x))\left(\frac{D\big(K(Du)\big)}{\big|D\big(K(Du)\big)\big|}\right)(x) \ +\ o(\delta).
\tag{5.19}
$$

By letting $\delta \to 0$ in (5.19) we obtain $\xi^\top\big(K_P(Du)\otimes K_P(Du) : D^2u\big)(x) \geq 0$ for any direction ξ. Since ξ and x are arbitrary we get $K_P(Du)\otimes K_P(Du) : D^2u = 0$ on Ω. The lemma follows. $\square$

Lemma 5.5. *Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$ with Minimally Distorted Area of $u(\Omega)$. Then, u solves $[K_P(Du)]^\perp K_{PP}(Du) : D^2u = 0$ on $\Omega\setminus\mathcal{S}$.*

Proof of Lemma 5.5. Fix $x \in \Omega\setminus\mathcal{S}$. Then, x belongs to some phase Ω_k of constant rank and $\mathrm{rk}\big(S(Du^\top Du)\big) \equiv k$ thereon. We choose $0 < \varepsilon < \frac{1}{2}\mathrm{dist}(x,\partial\Omega_k)$ and $0 < \delta < 1$. By the Rank Theorem (see *e.g.* [18]) and application of the Gram−Schmidt procedure to a local frame field adapted to the immersion near $u(x)$, we can construct a local frame of sections $\{\nu^1,\ldots,\nu^{N-k}\}$ spanning $\Gamma\big(\underline{[K_P(Du)]^\perp},\mathbb{B}_{2\varepsilon}(x)\big)$ for ε small enough. Let ν be a linear combination of these sections and choose an $h \in C^1\big(\overline{\mathbb{B}_\varepsilon(x)}\big)$. Since $\mathrm{rk}(Du) = n$ on Ω, by restricting δ sufficiently we obtain $\mathrm{rk}\big(D(u+\delta h\nu)\big) = n$ on $\mathbb{B}_\varepsilon(x)$. By differentiating $\nu^\top K_P(Du) = 0$ we obtain

$$
D_k\nu_\alpha K_{P_{\alpha i}}(Du) \ =\ -\nu_\alpha K_{P_{\alpha i}P_{\beta j}}(Du)D^2_{kj}u_\beta
\tag{5.20}
$$

and by putting $i = k$ and summing, we get

$$
D_i\nu_\alpha K_{P_{\alpha i}}(Du) \ =\ -\nu_\alpha K_{P_{\alpha i}P_{\beta j}}(Du)D^2_{ij}u_\beta
\tag{5.21}
$$

that is

$$
D\nu : K_P(Du) \ =\ -\nu^\top K_{PP}(Du) : D^2u.
\tag{5.22}
$$

By Taylor expansion of the dilation and usage of $\nu^\top K_P(Du) = 0$, we obtain

$$
\begin{aligned}
K\big(D(u+\delta h\nu)\big) \ &=\ K(Du) \ +\ K_P\big(Du\big) : D(\delta h\nu) \ +\ o(\delta|h\nu|) \\
&=\ K(Du) \ +\ \delta K_P\big(Du\big) : \big(hD\nu \ +\ \nu\otimes Dh\big) \ +\ o(\delta) \\
&=\ K(Du) \ +\ \delta\Big(hD\nu : K_P\big(Du\big) \ +\ \nu^\top K_P\big(Du\big)Dh\Big) \ +\ o(\delta) \\
&=\ K(Du) \ +\ \delta hD\nu : K_P\big(Du\big) \ +\ o(\delta)
\end{aligned}
\tag{5.23}
$$

as $\delta \to 0$. By (5.23) and (5.22) we have

$$K\big(D(u+\delta h\nu)\big) \;=\; K(Du) \;-\; 2\delta h\big(\nu^\top K_{PP}(Du) : D^2u\big) \;+\; o(\delta), \tag{5.24}$$

as $\delta \to 0$. Hence, since $u(\Omega)$ has minimally distorted area, by (5.24) we have

$$\begin{aligned} K_\infty\big(u,\mathbb{B}_\varepsilon(x)\big) \;&\leq\; K_\infty\big(u+\delta h\nu,\mathbb{B}_\varepsilon(x)\big) \\ &=\; \sup_{\mathbb{B}_\varepsilon(x)} \Big\{K(Du) \;-\; 2\delta h\big(\nu^\top K_{PP}(Du) : D^2u\big) \;+\; o(\delta)\Big\} \end{aligned} \tag{5.25}$$

as $\delta \to 0$, which gives

$$\begin{aligned} K_\infty\big(u,\mathbb{B}_\varepsilon(x)\big) \;&\leq \sup_{\mathbb{B}_\varepsilon(x)} K(Du) \;-\; 2\delta \min_{\mathbb{B}_\varepsilon(x)} \Big\{h\big(\nu^\top K_{PP}(Du) : D^2u\big)\Big\} \;+\; o(\delta) \\ &= K_\infty\big(u,\mathbb{B}_\varepsilon(x)\big) \;-\; 2\delta \min_{\mathbb{B}_\varepsilon(x)} \Big\{h\big(\nu^\top K_{PP}(Du) : D^2u\big)\Big\} \;+\; o(\delta). \end{aligned} \tag{5.26}$$

Hence, by passing to the limit as $\delta \to 0$, (5.26) gives

$$\min_{\mathbb{B}_\varepsilon(x)} \Big\{h\big(\nu^\top K_{PP}(Du) : D^2u\big)\Big\} \;\leq\; 0. \tag{5.27}$$

We now choose as h the constant function

$$h \;:=\; \mathrm{sgn}\,\big(\nu^\top K_{PP}(Du) : D^2u\big)\,(x) \tag{5.28}$$

and by (5.27) as $\varepsilon \to 0$ we get $\big|\nu^\top K_{PP}(Du) : D^2u\big|(x) = 0$. Since ν is an arbitrary normal section and x is an arbitrary point on $\Omega \setminus \mathcal{S}$, we get $([K_P]^\perp K_{PP})(Du) : D^2u = 0$ on $\Omega \setminus \mathcal{S}$ and the lemma follows. □

6. Geometric properties of Optimal ∞-Quasiconformal immersions

6.1. Geometric form of the PDE system

In this subsection we show that system (1.1) decouples to two system one normal to to other which can be written in geometric rather coordinate-free fashion, at least within the phases of solutions whereon the coefficients of the system are continuous.

Proposition 6.1. *Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$. If K is the dilation (1.3) and its derivatives are given by (3.1) and (3.9), then the Aronsson system*

$$Q_\infty u \;=\; \Big(K_P \otimes K_P + [K_P]^\perp K_{PP}\Big)(Du) : D^2u \;=\; 0 \tag{6.1}$$

is equivalent on each phase $\Omega_k = int\{rk(S(Du^\top Du)) = k\}$ to the pair of systems

$$S(\mathbf{G})D\big(\mathrm{tr}(\mathbf{G})\big) \;=\; 0, \tag{6.2}$$

$$\mathbb{B}^\perp : \big(\mathrm{tr}(\mathbf{G})\big)_P \;=\; 0, \tag{6.3}$$

where $\mathbf{G}$ is given by (1.1), $g = Du^\top Du$ is the Riemannian metric on $u(\Omega)$, S is the Ahlfors operator and $\mathbb{B}^\perp$ is the "generalized 2nd fundamental form", defined for every local normal section $\nu \in \Gamma([K_P(Du)]^\perp, D)$ over $D \subseteq \Omega \setminus \mathcal{S}$ as $(\mathbb{B}^\perp)_\nu := D\nu$. Moreover, (6.2) is valid on all of Ω.

We observe that system (6.2) can also be written as

$$S(g)D\left(\frac{\mathrm{tr}(g)}{\det(g)^{1/n}}\right) \;=\; 0 \tag{6.4}$$

and hence depends only on the metric structure of the immersion. System (6.2) is the "tangential system". On the other hand, (6.3) can be written also as

$$\mathbb{B}^{\perp} : \left(\frac{\mathrm{tr}(g)}{\det(g)^{1/n}}\right)_P \;=\; 0 \tag{6.5}$$

and depends on the exterior geometry as well, the "shape" of $u(\Omega)$. System (6.3) is the "normal system".

Proof of Proposition 6.1. By applying the orthogonal projections (4.2) and (4.3) to (6.1), we decouple it to

$$K_P(Du)\otimes K_P(Du) : D^2u \;=\; 0, \tag{6.6}$$

$$[K_P(Du)]^{\perp}K_{PP}(Du) : D^2u \;=\; 0. \tag{6.7}$$

In view of (3.1), we rewrite (6.6) as

$$Dug^{-1}S(g)D\big(K(Du)\big) \;=\; 0. \tag{6.8}$$

By using that $K(Du) = \mathrm{tr}(\mathbf{G})$ and that Dug^{-1} has constant rank equal to n and hence is left invertible, we obtain

$$\big(Dug^{-1}\big)^{-1}Dug^{-1}S(g)D\big(\mathrm{tr}(\mathbf{G})\big) \;=\; S(g)D\big(\mathrm{tr}(\mathbf{G})\big) \;=\; 0. \tag{6.9}$$

Since $g = \det(g)^{1/n}\mathbf{G}$, system (6.9) leads to (6.2). To obtain (6.3), we observe that (6.7) is equivalent to

$$\nu^{\top}K_{PP}(Du) : D^2u \;=\; 0, \tag{6.10}$$

for all local normal sections $\nu \in \Gamma([K_P(Du)]^{\perp}, D)$, $D \subseteq \Omega \setminus \mathcal{S}$. By (5.22), equation (6.10) is equivalent to $-D\nu : K_P(Du) = 0$. Hence, we rewrite it as

$$-D\nu : \big(\mathrm{tr}(\mathbf{G})\big)_P \;=\; 0. \tag{6.11}$$

By definition of $\mathbb{B}^{\perp}$, system (6.11) leads to (6.3) and the proposition follows. $\square$

Remark 6.2. We will later show that the 2-dimensional case $n = 2 \leq N$ is prominent. In this case, interfaces of discontinuities of the coefficients disappear and $\mathbb{B}^{\perp}$ conicides with the standard 2nd fundamental form.

Corollary 6.3 (Constant dilation on Ω_n)**.** *Let $u : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$ solving $K_P(Du) \otimes K_P(Du) : D^2u = 0$. Then, on the n-phase Ω_n given by (5.2), u has constant dilation on each connected component of Ω_n.*

Proof of Corollary 6.3. By (5.2) and (6.9), we have that $S(g)$ is invertible on Ω_n and consequently we get $D\big(K(Du)\big) = 0$ on Ω_n. $\square$

6.2. A Geometric property of interfaces of solutions

We begin with a differential identity valid *on the interfaces* of discontinuity, under a local regularity assumption on the interface. We assume only C^1 regularity, but we allow for possibly complicated topology and self-intersections.

Proposition 6.4 (Covariant derivatives on interfaces)**.** *Let* $u\colon \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ *be an immersion in* $C^2(\Omega)^N$. *Suppose the set of interfaces* $\mathcal{S}$ *inside* Ω *given by* (5.3) *contains a* C^1 *immersed submanifold* M *and let* ∇^M *be its Riemannian gradient. Then, we have the identity*

$$\begin{aligned}\nabla^M\big([K_P(Du)]^\perp\big) : K_P(Du) \ &= \ -\big([K_P]^\perp K_{PP}\big)(Du) : D^2 u \\ &\quad + \ \big([K_P]^\perp K_{PP}\big)(Du) : \nabla^{M^\perp} Du,\end{aligned} \tag{6.12}$$

valid on $M \subseteq \mathcal{S}$, *where* $\nabla^{M^\perp}$ *is the orthogonal complement of* ∇^M *in* $\mathbb{R}^n$.

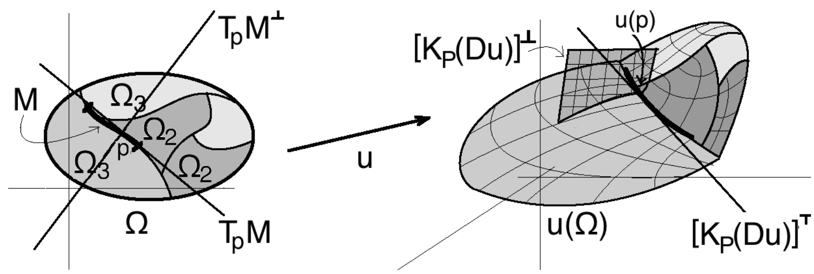

FIGURE 4.

Remark 6.5. The point in (6.12) is that $[K_P(Du)]^\perp$ has covariantly differentiable contraction with $K_P(Du)$ along (part of the interface) M, *without having assumed that* $S(Du^\top Du)$ *has constant rank on* M and hence without having assumed that $[K_P(Du)]^\perp$ is differentiable on $M \subseteq \Omega$.

Proof of Proposition 6.4. By assuming as we can that M is immersed by the inclusion into Ω, we fix a point $p \in M \subseteq \Omega$ and consider coordinates near p adapted to the immersion. Let $\{\nabla^M_1, \dots, \nabla^M_n\}$ denote the n components of ∇^M with respect to the standard coordinates of $\mathbb{R}^n$. By differentiating covariantly near p the identity

$$[K_P(Du)]^\perp_{\alpha\beta} K_{P_{\beta j}}(Du) \ = \ 0 \tag{6.13}$$

we obtain

$$\begin{aligned}\nabla^M_i\big([K_P(Du)]^\perp_{\alpha\beta}\big) K_{P_{\beta j}}(Du) \ &= - \ [K_P(Du)]^\perp_{\alpha\beta} \nabla^M_i\big(K_{P_{\beta j}}(Du)\big) \\ &= - \ [K_P(Du)]^\perp_{\alpha\beta} K_{P_{\beta j} P_{\gamma k}}(Du) \nabla^M_i D_k u_\gamma.\end{aligned} \tag{6.14}$$

By applying the expansion $\nabla^M = D - \nabla^{M^\perp}$, putting $i = j$ and summing, (6.14) implies (6.12) and the proposition follows. □

The previous identity readily implies the next

Corollary 6.6. *In the setting of Proposition 6.4 above, if* u *solves the system* $([K_P]^\perp K_{PP})(Du) : D^2 u = 0$, *then we have*

$$\nabla^M\big([K_P(Du)]^\perp\big) : K_P(Du) \ = \ \big([K_P]^\perp K_{PP}\big)(Du) : \nabla^{M^\perp} Du. \tag{6.15}$$

In particular, the vector field

$$\nabla^M\big([K_P(Du)]^\perp\big) : K_P(Du) \ : \ M \longrightarrow \mathbb{R}^N \tag{6.16}$$

is "normal"to $u(M)$, *namely, it is valued in* $[K_P(Du)]^\perp$:

$$[K_P(Du)]^\top \Big(\nabla^M\big([K_P(Du)]^\perp\big) : K_P(Du)\Big) \ = \ 0. \tag{6.17}$$

Proof of Corollary 6.6. Since the immersion u solves $[K_P(Du)]^{\perp}K_{PP}(Du) : D^2u = 0$, (6.12) gives (6.15). By applying the projection $[K_P(Du)]^{\top}$ to the latter, (6.17) follows. Hence, the vector field $\nabla^M\big([K_P(Du)]^{\perp}\big) : K_P(Du)$ equals its projection on $[K_P(Du)]^{\perp}$ and the corollary follows. $\square$

7. Sufficiency of $K_P(Du) \otimes K_P(Du) : D^2u = 0$ for Rank-One Locally Minimal Dilation when $n = 2 \leq N$

In this section we show that in the case of 2-dimensional immersions when $n = 2 \leq N$, the tangential system $K_P(Du) \otimes K_P(Du) : D^2u = 0$ is sufficient for the minimality notion of Rank-One Locally Minimal Dilation. This follows as a corollary of the fact that when $n = 2$, solutions of this system necessarily have constant dilation. In particular, the rank of $S(Du^{\top}Du)$ is constant throughout the domain and interfaces of discontinuity on the coefficents of the normal system $([K_P]^{\perp}K_{PP})(Du) : D^2u = 0$ disappear.

As a corollary, we show that when $n = N = 2$, the conjecture of Capogna$-$Raich in [8] on the sufficiency of system $(K_P \otimes K_P)(Du) : D^2u = 0$ for their stronger local minimality notion is false. This follows by Example 7.5 below in which we construct a diffeomorphism with constant dilation on a domain of the plane which has the same boundary values with the identity.

Lemma 7.1 (Constant dilation)**.** *Let u: $\Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$ which solves $K_P(Du) \otimes K_P(Du) : D^2u = 0$ on Ω. Suppose Ω is connected and let $\Omega^*_0, \dots, \Omega^*_n$ be the augmented $n+1$ phases of the immersion given by* (5.4)*. Then:*

(i) *$S(Du^{\top}Du)$ has nowhere rank equal to one:*

$$\Omega^*_1 \;=\; \emptyset. \tag{7.1}$$

(ii) *If moreover $n = 2$, then $\Omega^*_0 \in \{\emptyset, \Omega\}$. That is, Ω^*_0 is either empty or equals the whole Ω. Hence, u has constant dilation everywhere on Ω:*

$$K(Du) \;\equiv\; k \;\geq\; 2. \tag{7.2}$$

If it happens that $\Omega^*_0 \neq \emptyset$, then $k = 2$ and in this case u is conformal on Ω.

Proof of Lemma 7.1.

(i) On Ω^*_1 we have $\mathrm{rk}(S(Du^{\top}Du)) = 1$ and also $S(Du^{\top}Du) = S(Du^{\top}Du)^{\top}$. Since $S(Du^{\top}Du)$ is a rank-one symmetric matrix, there exist $\lambda : \Omega^*_1 \longrightarrow \mathbb{R}$ and $a : \Omega^*_1 \longrightarrow \mathbb{R}^n$ such that $\lambda > 0$, $|a| = 1$ and $S(Du^{\top}Du) = \lambda\, a \otimes a$. Hence, we obtain

$$\lambda \;=\; \lambda\,|a|^2 \;=\; \mathrm{tr}(\lambda\, a \otimes a) \;=\; \mathrm{tr}\big(S(Du^{\top}Du)\big) \;=\; 0. \tag{7.3}$$

Consequently, $\Omega^*_1 = \emptyset$.

(ii) When $n = 2$, by (i) we have that $\Omega = \Omega^*_0 \cup \Omega^*_2$. On Ω^*_0 the immersion u is conformal. By Corollary 6.3, on Ω^*_2 u has constant dilation. Hence, u has constant dilation on each connected component of $\Omega^*_0 \cup \Omega^*_2 = \Omega$. This means that $K(Du)$ is piecewise constant on Ω. By assumption, Ω is connected and also $K(Du) \in C^0(\Omega)$. As a result, necessarily either $\Omega^*_0 = \emptyset$ or $\Omega^*_0 = \Omega$. If $\Omega^*_0 \neq \emptyset$, then u is conformal on Ω. The lemma follows. $\square$

Proposition 7.2 (Equivalences in the 2-Dimensional case)**.** *Let $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$. Then, the following are equivalent:*

(i) *u has Rank-One Locally Minimal Dilation on Ω.*

(ii) *u solves $K_P(Du) \otimes K_P(Du) : D^2u = 0$ on Ω.*

(iii) *u has constant dilation on connected components of Ω.*

Proof of Proposition 7.2. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) have already been estabished, so it suffices to prove (iii) $\Rightarrow$ (i). For, suppose u has constant dilation on connected components of Ω. Fix $D \subset\subset \Omega$, $f \in C^1_0(D)$ and $\xi \in \mathbb{S}^{N-1}$. We may assume D is connected and that $\mathrm{rk}(Du + \xi \otimes Df) = n$ on D. Then, since $f|_{\partial D} \equiv 0$, there exists an interior critical point $\bar{x} \in D$ of f. By using that $Df(\bar{x}) = 0$, we estimate

$$
\begin{aligned}
K_\infty(u + f\xi, D) &= \sup_D K\big(Du + \xi \otimes Df\big) \\
&\geq K\big(Du(\bar{x}) + \xi \otimes Df(\bar{x})\big) \\
&= K(Du(\bar{x})) \\
&= \sup_D K(Du) \\
&= K_\infty(u, D).
\end{aligned}
\tag{7.4}
$$

Hence, u has rank-one locally minimal dilation and the proposition follows. $\square$

Directly from Proposition 7.2 we obtain the following

Corollary 7.3 (Absence of interfaces in the 2-Dimensional case)**.** *Let $u : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$ which solves $Q_\infty u = 0$ on the connected set Ω. Then the rank of $S(Du^\top Du)$ is constant on Ω, and equals either 0 or 2. If $\mathrm{rk}\big(S(Du^\top Du)\big) = 0$ then u satisfies*

$$K(Du) \equiv 2, \tag{7.5}$$

$$K_{PP}(Du) : D^2 u = 0. \tag{7.6}$$

The condition $K(Du) \equiv 2$ is equivalent to Conformality: $Du^\top Du = \frac{1}{n}|Du|^2 I$. If $\mathrm{rk}\big(S(Du^\top Du)\big) = 2$, then u satisfies

$$K(Du) \equiv \text{const.} > 2, \tag{7.7}$$

$$[Du]^\perp K_{PP}(Du) : D^2 u = 0. \tag{7.8}$$

Remark 7.4. Since the dilation (1.3) fails to be convex, it seems that sufficiency of the normal system $[K_P(Du)]^\perp K_{PP}(Du) : D^2 u = 0$ for minimally distorted area does not hold. In particular, the respective convexity arguments used in the case of the ∞-Laplacian in [13] fail.

The following example certifies that the variational notion of rank-one locally minimal dilation is genuinely weaker than the respective notion of "locally minimal dilation" used in [8], where general vector-valued variations with the same boundary values are considered.

Example 7.5 (Rank-One locally minimal dilation is strictly weaker notion)**.** (cf. [8], Cor. 1.6(2)) Let $\Omega := \mathbb{D}^2 \setminus \{0\} \subseteq \mathbb{R}^2$ be the punctured unit disc on the plane. Fix $\gamma > -1$ and consider the maps $u, u^\gamma : \Omega \longrightarrow \Omega$ where $u(x) := x$ and $u^\gamma(x) := |x|^\gamma x$. Then, $u = u^\gamma$ on $\partial\Omega = \mathbb{S}^1 \cup \{0\}$ and u is conformal on Ω while u^γ is quasiconformal but has constant strictly greater dilation:

$$K(Du) \equiv 2 < 2 + \frac{\gamma^2}{\gamma + 1} \equiv K(Du^\gamma). \tag{7.9}$$

For completeness, we provide some details of our calculations. We readily have

$$Du^\gamma(x) = |x|^\gamma \left(I + \gamma \frac{x}{|x|} \otimes \frac{x}{|x|} \right) \tag{7.10}$$

and by setting $\frac{x}{|x|} = (a,b)^\top$ we obtain

$$Du^\gamma(x) \;=\; |x|^\gamma \begin{bmatrix} 1+\gamma a^2 & \gamma ab \\ \gamma ba & 1+\gamma b^2 \end{bmatrix}. \tag{7.11}$$

By using that $a^2+b^2=1$, we have

$$\begin{aligned} K(Du^\gamma) \;&=\; \frac{|Du^\gamma|^2}{(\det(Du^\gamma)\det(Du^\gamma))^{1/2}} \\ &=\; \frac{|x|^{2\gamma}\big[(1\,+\,\gamma a^2)^2\,+\,(1+\gamma b^2)^2\,+\,2(\gamma ab)^2\big]}{|x|^{2\gamma}\big[(1\,+\,\gamma a^2)(1+\gamma b^2)-(\gamma ab)^2\big]} \\ &=\; 2\,+\,\frac{\gamma^2}{\gamma+1}. \end{aligned} \tag{7.12}$$

As a conclusion, in view of Corollary 7.2, u^γ has rank-one minimal dilation over Ω, but does not have minimal dilation over Ω since it has the same boundary values on $\partial\Omega$ with a conformal map. If moreover $\gamma > 0$, then both u, u^γ are in $C^1(\overline{\Omega})^2$.

7.1. On the sufficiency of $K_P(Du)\otimes K_P(Du) : D^2u = 0$ for Rank-One Locally Minimal Dilation in the case of dimensions $3 \leq n \leq N$

In this subsection we loosely discuss the much more complicated case of dimensions $n \geq 3$. In this case results are less sharp since Lemma 7.1 generally fails when $n > 2$.

To begin with, let $u : \Omega \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^N$ be an immersion in $C^2(\Omega)^N$. Obviously, we have $\mathrm{rk}(Du) = 3 \leq N$. By Lemma 3.1 and Proposition 6.1, we may rewrite system $K_P(Du)\otimes K_P(Du) : D^2u = 0$ as

$$g^{-1}S(g)D\big(K(Du)\big) \;=\; 0, \tag{7.13}$$

where $g = Du^\top Du$. We recall that in the case of $n=2$, Lemma 7.1 asserts that $S(g)$ either has two nonzero opposite eigenvalues (and hence has a saddle structure), or it vanishes. In the two-dimensional case this covers all possible values of rank and it follows that the dilation is constant throughout connected domains.

When $n=3$, Lemma 7.1 still works with the same proof, but now asserts only that

(i) there is no one-dimensional phase Ω_1^*, and

(ii) $\Omega = \Omega_0^* \cup \Omega_2^* \cup \Omega_3^*$ with $K(Du)$ constant on connected components of the set $\Omega_0^* \cup \Omega_3^*$.

When $n = 3$ no information is provided for the two-dimensional phase Ω_2^*. Let us analyse more closely what happens in this case when $\Omega_2^* \neq \emptyset$ and nontrivial interfaces of discontinuities may appear, where $\Omega_2^* = \{\mathrm{rk}(S(g)) = 2\}$. Let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3$ be the eigenvalue functions on Ω of the Riemannian metric g. Then, the spectrum of $S(g)$ is

$$\begin{aligned} \sigma\big(S(g)\big) \;&=\; \sigma(g) \;-\; \frac{\mathrm{tr}(g)}{3} \\ &=\; \left\{\lambda_1 - \frac{\lambda_1+\lambda_2+\lambda_3}{3},\, \lambda_2 - \frac{\lambda_1+\lambda_2+\lambda_3}{3},\, \lambda_3 - \frac{\lambda_1+\lambda_2+\lambda_3}{3}\right\} \\ &=\; \left\{\frac{2\lambda_1-\lambda_2-\lambda_3}{3},\, \frac{2\lambda_2-\lambda_3-\lambda_1}{3},\, \frac{2\lambda_3-\lambda_2-\lambda_1}{3}\right\}. \end{aligned} \tag{7.14}$$

We distinguish the following cases:

(a) $0 < \lambda_1 = \lambda_2 = \lambda_3 =: \lambda$. Then, by (7.14) we have that $S(g) = 0$.

(b) $0<\lambda_1=\lambda_2=:\lambda<\lambda_3$. Then, by (7.14) we have that

$$\sigma\big(S(g)\big) \;=\; \{-\mu,-\mu,2\mu\} \tag{7.15}$$

where $\mu:=\frac{\lambda_3-\lambda}{3}>0$. By the Spectral Theorem, there is an orthonormal frame $\{a_1,a_2,a_3\}$ of $\mathbb{R}^3$ such that

$$S(g) \;=\; -\mu\big(a_1\otimes a_1 \;+\; a_2\otimes a_2\big) \;+\; 2\mu\, a_3\otimes a_3 \tag{7.16}$$

and $S(g)$ has rank three.

(c) $0<\lambda_1<\lambda_2=\lambda_3$. Again as before $S(g)$ has rank three.

(d) $0<\lambda_1<\lambda_2<\lambda_3$. This is the only case where rank equal to two may appear. Since $\lambda_2+\lambda_3>2\lambda_1$ and $\lambda_1+\lambda_2<2\lambda_3$, we get

$$\mu_1 \;:=\; \frac{2\lambda_1-\lambda_2-\lambda_3}{3} \;<\; 0\;, \quad \mu_3 \;:=\; \frac{2\lambda_3-\lambda_2-\lambda_1}{3} \;>\; 0 \tag{7.17}$$

but it may happen that

$$\mu_2 \;:=\; \frac{2\lambda_2-\lambda_3-\lambda_1}{3} \tag{7.18}$$

vanishes, like for example in the extremal quasiconformal map $u:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ given by $u(x,y,z):=(e^x,\sqrt{2}ye^x,\sqrt{3}ze^x)^\top$. We have

$$Du^\top Du\,(x,y,z) \;=\; e^{2x}\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix} \tag{7.19}$$

and hence we get $(\lambda_1,\lambda_2,\lambda_3)=(e^{2x},2e^{2x},3e^{2x})$, which implies $\mu_2=0$. Generally, the set of interfaces of a three-dimensional optimal quasiconformal map is given by

$$\mathcal{S} \;=\; \partial\{\mu_2\;=\;0\} \tag{7.20}$$

and the two-dimensional phase of u is given by

$$\varOmega_2 \;=\; \mathrm{int}\{\mu_2=0\}. \tag{7.21}$$

Since $S(g)$ is traceless, the condition $\mathrm{tr}(S(g))=0$ implies $-\mu_1=\mu_3=:\mu>0$ and hence $\sigma\big(S(g)\big)=\{-\mu,0,\mu\}$. By the Spectral Theorem, there exists an orthonormal frame $\{a,b,c\}$ of $\mathbb{R}^3$ such that

$$S(g) \;=\; -\mu\left(a\otimes a \;-\; c\otimes c\right). \tag{7.22}$$

By (7.13), we have that $D\big(K(Du)\big)$ is perpendicular to $\{a,c\}$ and hence

$$D\big(K(Du)\big) \;=\; b\otimes b\, D\big(K(Du)\big) \tag{7.23}$$

which implies that the dilation of u varies only in the direction of b. Consequently, $K(Du)$ depends only on b through a certain function k:

$$K\big(Du(x)\big) \;=\; k\big(b(x)\big). \tag{7.24}$$

Unlike the case $n=2$, when $n=3$ we do *not* obtain that the dilation of three-dimensional optimal quasiconformal immersions is constant, at least not by the previous reasoning.

However, by Theorem 5.2 in all dimensions $2\leq n\leq N$ Rank-One Locally Minimal Dilation implies solvability of $K_P(Du)\otimes K_P(Du):D^2u=0$ and by the higher-dimensional extension of Example 7.5, rank-one locally minimal dilation is genuinely weaker than locally minimal dilation. Although it seems reasonable that $K_P(Du)\otimes K_P(Du):D^2u=0$ is sufficient for rank-one locally minimal dilation, we can not definitely conclude for the validity of the conjecture of Capogna$-$Raich in [8] for $n\geq 3$.

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