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UNIFORM LOCAL NULL CONTROL OF THE LERAY-α MODEL *, **

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Abstract. This paper deals with the distributed and boundary controllability of the so called Leray-α model. This is a regularized variant of the Navier−Stokes system (α is a small positive parameter) that can also be viewed as a model for turbulent flows. We prove that the Leray-α equations are locally null controllable, with controls bounded independently of α. We also prove that, if the initial data are sufficiently small, the controls converge as $\alpha \to 0^+$ to a null control of the Navier−Stokes equations. We also discuss some other related questions, such as global null controllability, local and global exact controllability to the trajectories, *etc*.

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1. Introduction. The main results

Let $\Omega \subset \mathbb{R}^N (N = 2, 3)$ be a bounded connected open set whose boundary Γ is of class C^2. Let $\omega \subset \Omega$ be a (small) nonempty open set, let $\gamma \subset \Gamma$ be a (small) nonempty open subset of Γ and assume that $T > 0$. We will use the notation $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$ and we will denote by $\mathbf{n} = \mathbf{n}(\mathbf{x})$ the outward unit normal to Ω at the points $\mathbf{x} \in \Gamma$; spaces of $\mathbb{R}^N$-valued functions, as well as their elements, are represented by boldface letters.

The Navier−Stokes system for a homogeneous viscous incompressible fluid (with unit density and unit kinematic viscosity) subject to homogeneous Dirichlet boundary conditions is given by

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{y} \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in } Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\mathbf{y}$ (the velocity field) and p (the pressure) are the unknowns, $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ is a forcing term and $\mathbf{y}_0 = \mathbf{y}_0(\mathbf{x})$ is a prescribed initial velocity field.

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In order to prove the existence of a solution to the Navier−Stokes system, Leray in [26] had the idea of creating a turbulence *closure* model without enhancing viscous dissipation. Thus, he introduced a "regularized" variant of (1.1) by modifying the nonlinear term as follows:

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in } Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q, \end{cases}$$

where $\mathbf{y}$ and $\mathbf{z}$ are related by

$$\mathbf{z} = \phi_\alpha * \mathbf{y} \tag{1.2}$$

and ϕ_α is a smoothing kernel. At least formally, the Navier−Stokes equations are recovered in the limit as $\alpha \to 0^+$, so that $\mathbf{z} \to \mathbf{y}$.

In this paper, we will consider a special smoothing kernel, associated to the Stokes-like operator $\mathbf{Id} + \alpha^2\mathbf{A}$, where $\mathbf{A}$ is the Stokes operator (see Sect. 2). This leads to the following modification of the Navier−Stokes equations, called the Leray-α system (see [4]):

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in } \ Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in } \ Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{in } \ Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on } \ \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \ \Omega. \end{cases} \tag{1.3}$$

In almost all previous works found in the literature, Ω is either the N-dimensional torus and the PDE's in (1.3) are completed with periodic boundary conditions or the whole space $\mathbb{R}^N$. Then, $\mathbf{z}$ satisfies an equation of the kind

$$\mathbf{z} - \alpha^2\Delta\mathbf{z} = \mathbf{y} \tag{1.4}$$

and the model is (apparently) slightly different from (1.3). However, since $\nabla\cdot\mathbf{y} = 0$, it is easy to see that (1.4), in these cases, is equivalent to the equation satisfied by $\mathbf{z}$ and π in (1.3).

It has been shown in [4] that, at least for periodic boundary conditions, the numerical solution of the equations in (1.3) matches successfully with empirical data from turbulent channel and pipe flows for a wide range of Reynolds numbers. Accordingly, the Leray-α system has become preferable to other turbulence models, since the associated computational cost is lower and no introduction of *ad hoc* parameters is required.

In [19], the authors have compared the numerical solutions of three different α-models useful in turbulence modelling (in terms of the Reynolds number associated to a Navier−Stokes velocity field). The results improve as one passes from the Navier−Stokes equations to these models and clearly show that the Leray-α system has the best performance. Therefore, it seems quite natural to carry out a theoretical analysis of the solutions to (1.3).

We will be concerned with the following controlled systems

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in } \ Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in } \ Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{in } \ Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0} & \text{on } \ \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \ \Omega, \end{cases} \tag{1.5}$$

and

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{in } \ Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in } \ Q, \\ \nabla\cdot\mathbf{y} = 0, \ \nabla\cdot\mathbf{z} = 0 & \text{in } \ Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{h}1_\gamma & \text{on } \ \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \ \Omega, \end{cases} \tag{1.6}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ (respectively $\mathbf{h} = \mathbf{h}(\mathbf{x}, t)$) stands for the control, assumed to act only in the (small) set ω (respectively on γ) during the whole time interval $(0, T)$. The symbol 1_ω (respectively 1_γ) stands for the characteristic function of ω (respectively of γ).

In the applications, the *internal control* $\mathbf{v}1_\omega$ can be viewed as a gravitational or electromagnetic field. The *boundary control* $\mathbf{h}1_\gamma$ is the trace of the velocity field on Σ.

Remark 1.1. It is completely natural to suppose that $\mathbf{y}$ and $\mathbf{z}$ satisfy the same boundary conditions on Σ since, in the limit, we should have $\mathbf{z} = \mathbf{y}$. Consequently, we will assume that the boundary control $\mathbf{h}1_\gamma$ acts simultaneously on both variables $\mathbf{y}$ and $\mathbf{z}$.

In what follows, $(\cdot\,,\cdot)$ and $\|\cdot\|$ denote the usual L^2 scalar products and norms (in $L^2(\Omega)$, $\mathbf{L}^2(\Omega)$, $L^2(Q)$, *etc.*) and K, C, C_1, C_2, $\ldots$ denote various positive constants (usually depending on ω, Ω and T). Let us recall the definitions of some usual spaces in the context of incompressible fluids:

$$\mathbf{H} = \left\{ \mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\},$$
$$\mathbf{V} = \left\{ \mathbf{u} \in \mathbf{H}^1_0(\Omega) : \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega \right\}.$$

Note that, for every $\mathbf{y}_0 \in \mathbf{H}$ and every $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$, there exists a unique solution $(\mathbf{y}, p, \mathbf{z}, \pi)$ for (1.5) that satisfies (among other things)

$$\mathbf{y}, \mathbf{z} \in C^0([0, T]; \mathbf{H});$$

see Proposition 1.2 below. This is in contrast with the lack of uniqueness of the Navier−Stokes system when $N = 3$.

The main goals of this paper are to analyze the controllability properties of (1.5) and (1.6) and determine the way they depend on α as $\alpha \to 0^+$.

The null controllability problem for (1.5) at time $T > 0$ is the following:

For any $\mathbf{y}_0 \in \mathbf{H}$, find $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$ such that the corresponding state (the corresponding solution to (1.5)) satisfies

$$\mathbf{y}(T) = \mathbf{0} \quad \textit{in} \quad \Omega. \tag{1.7}$$

The null controllability problem for (1.6) at time $T > 0$ is the following:

For any $\mathbf{y}_0 \in \mathbf{H}$, find $\mathbf{h} \in \mathbf{L}^2(0, T; \mathbf{H}^{-1/2}(\gamma))$ with $\int_\gamma \mathbf{h} \cdot \mathbf{n} \,\mathrm{d}\Gamma = 0$ and an associated state (the corresponding solution to (1.6)) satisfying

$$\mathbf{y}, \mathbf{z} \in C^0\left([0, T]; \mathbf{L}^2(\Omega)\right)$$

and (1.7).

Recall that, in the context of the Navier−Stokes equations, Lions conjectured in [27] the global distributed and boundary approximate controllability; since then, the controllability of these equations has been intensively studied, but for the moment only partial results are known.

Thus, the global approximate controllability of the two-dimensional Navier−Stokes equations with Navier slip boundary conditions was obtained by Coron in [6]. Also, by combining results concerning global and local controllability, the global null controllability for the Navier−Stokes system on a two-dimensional manifold without boundary was established in Coron and Fursikov [7]; see also Guerrero *et al.* [24] for another global controllability result.

The local exact controllability to bounded trajectories has been obtained by Fursikov and Imanuvilov [17,18], Imanuvilov [25] and Fernández-Cara *et al.* [13] under various circumstances; see Guerrero [22] and González-Burgos *et al.* [21] for similar results related to the Boussinesq system. Let us also mention [3,8,9,14], where analogous results are obtained with a reduced number of scalar controls.

For the (simplified) one-dimensional viscous Burgers model, positive and negative results can be found in [12,20,23]; see also [11], where the authors consider the one-dimensional compressible Navier−Stokes system.

Our first main result in this paper is the following:

Theorem 1.2. *There exists* $\epsilon > 0$ (*independent of* α) *such that, for each* $\mathbf{y}_0 \in \mathbf{H}$ *with* $\|\mathbf{y}_0\| \le \epsilon$, *there exist controls* $\mathbf{v}_\alpha \in L^\infty(0,T;\mathbf{L}^2(\omega))$ *such that the associated solutions to* (1.5) *fulfill* (1.7). *Furthermore, these controls can be found satisfying the estimate*

$$\|\mathbf{v}_\alpha\|_{L^\infty(\mathbf{L}^2)} \le C, \tag{1.8}$$

where C *is also independent of* α.

Our second main result is the analog of Theorem 1.2 in the framework of boundary controllability. It is the following:

Theorem 1.3. *There exists* $\delta > 0$ (*independent of* α) *such that, for each* $\mathbf{y}_0 \in \mathbf{H}$ *with* $\|\mathbf{y}_0\| \le \delta$, *there exist controls* $\mathbf{h}_\alpha \in L^\infty(0,T;\mathbf{H}^{-1/2}(\gamma))$ *with* $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\,\mathrm{d}\Gamma = 0$ *and associated solutions to* (1.6) *that fulfill* (1.7). *Furthermore, these controls can be found satisfying the estimate*

$$\|\mathbf{h}_\alpha\|_{L^\infty(H^{-1/2})} \le C, \tag{1.9}$$

where C *is also independent of* α.

The proofs rely on suitable fixed-point arguments. The underlying idea has applied to many other nonlinear control problems. However, in the present cases, we find two specific difficulties:

- In order to find spaces and fixed-point mappings appropriate for Schauder's Theorem, the initial state $\mathbf{y}_0$ must be regular enough. Consequently, we have to establish *regularizing properties* for (1.5) and (1.6); see Lemmas 2.7 and 4.2 below.
- For the proof of the uniform estimates (1.8) and (1.9), careful estimates of the null controls and associated states of some particular linear problems are needed.

We will also prove results concerning the controllability in the limit, as $\alpha \to 0^+$. It will be shown that the null-controls for (1.5) can be chosen in such a way that they converge to null-controls for the Navier−Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in } Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \Omega. \end{cases} \tag{1.10}$$

Also, it will be seen that the null-controls for (1.6) can be chosen such that they converge to boundary null-controls for the Navier−Stokes system

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{y}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{0} & \text{in } Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{h}1_\gamma & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \Omega. \end{cases} \tag{1.11}$$

More precisely, our third and fourth main results are the following:

Theorem 1.4. *Let* $\epsilon > 0$ *be furnished by Theorem 1.2. Assume that* $\mathbf{y}_0 \in \mathbf{H}$ *and* $\|\mathbf{y}_0\| \le \epsilon$, *let* $\mathbf{v}_\alpha$ *be a null control for* (1.5) *satisfying* (1.8) *and let* $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ *be the associated state. Then, at least for a subsequence, one has*

$$\mathbf{v}_\alpha \to \mathbf{v} \text{ weakly-}* \text{ in } L^\infty(0,T;\mathbf{L}^2(\omega)),$$
$$\mathbf{z}_\alpha \to \mathbf{y} \text{ and } \mathbf{y}_\alpha \to \mathbf{y} \text{ strongly in } \mathbf{L}^2(Q),$$

as $\alpha \to 0^+$, *where* $(\mathbf{y}, \mathbf{v})$ *is, together with some* p, *a state-control pair for* (1.10) *satisfying* (1.7).

Theorem 1.5. *Let $\delta > 0$ be furnished by Theorem 1.3. Assume that $\mathbf{y}_0 \in \mathbf{H}$ and $\|\mathbf{y}_0\| \le \delta$, let $\mathbf{h}_\alpha$ be a null control for (1.6) satisfying (1.9) and let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ be the associated state. Then, at least for a subsequence, one has*

$$\mathbf{h}_\alpha \to \mathbf{h} \text{ weakly-}* \text{ in } L^\infty(0,T;H^{-1/2}(\gamma)),$$
$$\mathbf{z}_\alpha \to \mathbf{y} \text{ and } \mathbf{y}_\alpha \to \mathbf{y} \text{ strongly in } \mathbf{L}^2(Q),$$

as $\alpha \to 0^+$, where $(\mathbf{y}, \mathbf{h})$ is, together with some p, a state-control pair for (1.11) satisfying (1.7).

The rest of this paper is organized as follows. In Section 2, we will recall some properties of the Stokes operator and we will prove some results concerning the existence, uniqueness and regularity of the solution to (1.3). Section 3 deals with the proofs of Theorems 1.2 and 1.4. Section 4 deals with the proofs of Theorems 1.3 and 1.5. Finally, in Section 5, we present some additional comments and open questions.

2. Preliminaries

In this section, we will recall some properties of the Stokes operator. Then, we will prove that the Leray-α system is well-posed. Also, we will recall the Carleman inequalities and null controllability properties of the Oseen system.

2.1. The Stokes operator

Let $\mathbf{P} : \mathbf{L}^2(\Omega) \mapsto \mathbf{H}$ be the orthogonal projector, usually known as the *Leray Projector*. Recall that $\mathbf{P}$ maps $\mathbf{H}^s(\Omega)$ into $\mathbf{H}^s(\Omega) \cap \mathbf{H}$ for all $s \ge 0$.

We will denote by $\mathbf{A}$ the *Stokes operator*, *i.e.* the self-adjoint operator in $\mathbf{H}$ formally given by $\mathbf{A} = -\mathbf{P}\Delta$. For any $\mathbf{u} \in D(\mathbf{A}) := \mathbf{V} \cap \mathbf{H}^2(\Omega)$ and any $\mathbf{w} \in \mathbf{H}$, the identity $\mathbf{A}\mathbf{u} = \mathbf{w}$ holds if and only if

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{w}, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}.$$

It is well-known that $\mathbf{A} : D(\mathbf{A}) \mapsto \mathbf{H}$ can be inverted and its inverse $\mathbf{A}^{-1}$ is self-adjoint, compact and positive. Consequently, there exists a nondecreasing sequence of positive numbers λ_j and an associated orthonormal basis of $\mathbf{H}$, denoted by $(\mathbf{w}_j)_{j=1}^{\infty}$, such that

$$\mathbf{A}\mathbf{w}_j = \lambda_j \mathbf{w}_j \quad \forall j \ge 1.$$

Accordingly we can introduce the real powers of the Stokes operator. Thus, for any $r \in \mathbb{R}$, we set

$$D(\mathbf{A}^r) = \left\{ \mathbf{u} \in \mathbf{H} : \mathbf{u} = \sum_{j=1}^{\infty} u_j \mathbf{w}_j, \text{ with } \sum_{j=1}^{\infty} \lambda_j^{2r} |u_j|^2 < +\infty \right\}$$

and

$$\mathbf{A}^r \mathbf{u} = \sum_{j=1}^{\infty} \lambda_j^r u_j \mathbf{w}_j, \quad \forall\, \mathbf{u} = \sum_{j=1}^{\infty} u_j \mathbf{w}_j \in D(\mathbf{A}^r).$$

Let us present a result concerning the domains of the powers of the Stokes operator.

Theorem 2.1. *Let $r \in \mathbb{R}$ be given, with $-\frac{1}{2} < r < 1$. Then*

$$D(\mathbf{A}^{r/2}) = \mathbf{H}^r(\Omega) \cap \mathbf{H} \quad \textit{whenever} \quad -\frac{1}{2} < r < \frac{1}{2},$$
$$D(\mathbf{A}^{r/2}) = \mathbf{H}^r_0(\Omega) \cap \mathbf{H} \quad \textit{whenever} \quad \frac{1}{2} \le r \le 1.$$

Moreover, $\mathbf{u} \mapsto (\mathbf{u}, \mathbf{A}^r \mathbf{u})^{1/2}$ is a Hilbertian norm in $D(\mathbf{A}^{r/2})$, equivalent to the usual Sobolev $\mathbf{H}^r$-norm. In other words, there exist constants $c_1(r), c_2(r) > 0$ such that

$$c_1(r)\|\mathbf{u}\|_{\mathbf{H}^r} \le (\mathbf{u}, \mathbf{A}^r \mathbf{u})^{1/2} \le c_2(r)\|\mathbf{u}\|_{\mathbf{H}^r} \quad \forall \mathbf{u} \in D(\mathbf{A}^{r/2}).$$

The proof of Theorem 2.1 can be found in [16]. Notice that, in view of the interpolation K-method of Lions and Peetre, we have $D(\mathbf{A}^{r/2}) = D((-\Delta)^{r/2}) \cap \mathbf{H}$. Hence, thanks to an explicit description of $D((-\Delta)^{r/2})$, the stated result holds.

Now, we are going to recall an important property of the semigroup of contractions $\mathrm{e}^{-t\mathbf{A}}$ generated by $\mathbf{A}$, see [15]:

Theorem 2.2. *For any $r > 0$, there exists $C(r) > 0$ such that*

$$\|\mathbf{A}^r \mathrm{e}^{-t\mathbf{A}}\|_{\mathcal{L}(\mathbf{H};\mathbf{H})} \leq C(r)\, t^{-r} \quad \forall\, t > 0. \tag{2.1}$$

In order to prove (2.1), it suffices to observe that, for any $\mathbf{u} = \sum_{j=1}^{+\infty} u_j \mathbf{w}_j \in \mathbf{H}$, one has

$$\mathbf{A}^r \mathrm{e}^{-t\mathbf{A}} \mathbf{u} = \sum_{j=1}^{+\infty} \lambda_j^r \mathrm{e}^{-t\lambda_j} u_j \mathbf{w}_j.$$

Consequently,

$$\|\mathbf{A}^r \mathrm{e}^{-t\mathbf{A}} \mathbf{u}\|^2 = \sum_{j=1}^{+\infty} \left|\lambda_j^r \mathrm{e}^{-t\lambda_j} u_j\right|^2 \leq \left(\max_{\lambda\in\mathbb{R}} \lambda^r \mathrm{e}^{-t\lambda}\right)^2 \|\mathbf{u}\|^2$$

and, since $\max\limits_{\lambda\in\mathbb{R}} \lambda^r \mathrm{e}^{-t\lambda} = (r/e)^r\, t^{-r}$, we get easily (2.1).

2.2. Well-posedness for the Leray-α system

Let us see that, for any $\alpha > 0$, under some reasonable conditions on $\mathbf{f}$ and $\mathbf{y}_0$, the Leray-α system (1.3) possesses a unique global weak solution. Before this, let us introduce σ_N given by

$$\sigma_N = \begin{cases} 2 & \text{if } N = 2, \\ 4/3 & \text{if } N = 3. \end{cases}$$

Then, we have the following result:

Proposition 2.3. *Assume that $\alpha > 0$ is fixed. Then, for any $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{H}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (1.3), with*

$$\begin{gathered} \mathbf{y}_\alpha \in L^2(0,T;\mathbf{V}) \cap C^0([0,T];\mathbf{H}),\ (\mathbf{y}_\alpha)_t \in L^2(0,T;\mathbf{V}'), \\ \mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A}^{3/2})) \cap C^0([0,T];D(\mathbf{A})). \end{gathered} \tag{2.2}$$

Furthermore, the following estimates hold:

$$\begin{aligned} \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})} + \|\mathbf{y}_\alpha\|_{C^0([0,T];\mathbf{H})} &\leq CB_0(\mathbf{y}_0,\mathbf{f}), \\ \|(\mathbf{y}_\alpha)_t\|_{L^{\sigma_N}(\mathbf{V}')} &\leq CB_0(\mathbf{y}_0,\mathbf{f})(1 + B_0(\mathbf{y}_0,\mathbf{f})), \\ \|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{H})} + 2\alpha^2 \|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{V})} &\leq CB_0(\mathbf{y}_0,\mathbf{f})^2, \\ 2\alpha^2 \|\mathbf{z}_\alpha\|^2_{L^\infty(\mathbf{V})} + \alpha^4 \|\mathbf{z}_\alpha\|^2_{L^\infty(D(\mathbf{A}))} &\leq CB_0(\mathbf{y}_0,\mathbf{f})^2. \end{aligned} \tag{2.3}$$

Here, C is independent of α and we have introduced the notation

$$B_0(\mathbf{y}_0,\mathbf{f}) := \|\mathbf{y}_0\| + \|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}.$$

Proof. The proof follows classical and rather well-known arguments; see for instance [10,30]. For completeness, they will be recalled.

• EXISTENCE: We will reduce the proof to the search of a fixed point of an appropriate mapping Λ_α[3].

Thus, for each $\overline{\mathbf{y}} \in L^2(0,T;\mathbf{H})$, let $(\mathbf{z},\pi)$ be the unique solution to

$$\begin{cases} \mathbf{z} - \alpha^2 \Delta \mathbf{z} + \nabla \pi = \overline{\mathbf{y}} & \text{in } Q, \\ \nabla \cdot \mathbf{z} = 0 & \text{in } Q, \\ \mathbf{z} = \mathbf{0} & \text{on } \Sigma. \end{cases}$$

It is clear that $\mathbf{z} \in L^2(0,T;D(\mathbf{A}))$ and then, thanks to the Sobolev embedding, we have $\mathbf{z} \in L^2(0,T;\mathbf{L}^\infty(\Omega))$. Moreover, the following estimates are satisfied:

$$\|\mathbf{z}\|^2 + 2\alpha^2 \|\mathbf{z}\|^2_{L^2(\mathbf{V})} \le \|\overline{\mathbf{y}}\|^2,$$
$$2\alpha^2 \|\mathbf{z}\|^2_{L^2(\mathbf{V})} + \alpha^4 \|\mathbf{z}\|^2_{L^2(D(\mathbf{A}))} \le \|\overline{\mathbf{y}}\|^2.$$

From this $\mathbf{z}$, we can obtain the unique solution $(\mathbf{y},p)$ to the linear system of the Oseen kind

$$\begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{f} & \text{in } Q, \\ \nabla \cdot \mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \Omega. \end{cases}$$

Since $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{y}_0 \in \mathbf{H}$, it is clear that

$$\mathbf{y} \in L^2(0,T;\mathbf{V}) \cap C^0([0,T];\mathbf{H}), \quad \mathbf{y}_t \in L^2(0,T;\mathbf{V}')$$

and the following estimates hold:

$$\begin{gathered} \|\mathbf{y}\|_{C^0([0,T];\mathbf{H})} + \|\mathbf{y}\|_{L^2(\mathbf{V})} \le C_1 B_0(\mathbf{y}_0,\mathbf{f}), \\ \|\mathbf{y}_t\|_{L^2(\mathbf{V}')} \le C_2(1 + \|\mathbf{z}\|_{L^2(D(\mathbf{A}))}) B_0(\mathbf{y}_0,\mathbf{f}) \le C_2(1+\alpha^{-2}\|\overline{\mathbf{y}}\|) B_0(\mathbf{y}_0,\mathbf{f}). \end{gathered} \tag{2.4}$$

Now, we introduce the Banach space

$$\mathbf{W} = \{\mathbf{w} \in L^2(0,T;\mathbf{V}) : \mathbf{w}_t \in L^2(0,T;\mathbf{V}')\},$$

the closed ball

$$\mathbf{K} = \{\, \overline{\mathbf{y}} \in L^2(0,T;\mathbf{H}) : \|\overline{\mathbf{y}}\| \le C_1 \sqrt{T} B_0(\mathbf{y}_0,\mathbf{f}) \,\}$$

and the mapping $\tilde{\Lambda}_\alpha$, with $\tilde{\Lambda}_\alpha(\overline{\mathbf{y}}) = \mathbf{y}$, for all $\overline{\mathbf{y}} \in L^2(0,T;\mathbf{H})$. Obviously $\tilde{\Lambda}_\alpha$ is well-defined and maps continuously the whole space $L^2(0,T;\mathbf{H})$ into $\mathbf{W} \cap \mathbf{K}$.

Notice that any bounded set of $\mathbf{W}$ is relatively compact in the space $L^2(0,T;\mathbf{H})$, in view of the classical results of the Aubin−Lions kind, see for instance [28].

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\tilde{\Lambda}_\alpha$. Then, thanks to (2.4), Λ_α maps $\mathbf{K}$ into itself. Moreover, it is clear that $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisfies the hypotheses of Schauder's Theorem. Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$.

This immediately achieves the proof of the existence of a solution satisfying (2.2).

The estimates $(2.3)_a$, $(2.3)_c$ and $(2.3)_d$ are obvious. On the other hand,

$$\begin{aligned} \|(\mathbf{y}_\alpha)_t\|_{L^{\sigma_N}(\mathbf{V}')} &\le C\left(\|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})} + \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})} + \|(\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha\|_{L^{\sigma_N}(\mathbf{H}^{-1})}\right) \\ &\le C\left(B_0(\mathbf{y}_0,\mathbf{f}) + \|\mathbf{z}_\alpha\|_{L^{s_N}(\mathbf{L}^4)} \|\mathbf{y}_\alpha\|_{L^{s_N}(\mathbf{L}^4)}\right) \\ &\le C\left[B_0(\mathbf{y}_0,\mathbf{f}) + \left(\|\mathbf{z}_\alpha\|_{L^\infty(\mathbf{H})} + \|\mathbf{z}_\alpha\|_{L^2(\mathbf{V})}\right)\left(\|\mathbf{y}_\alpha\|_{L^\infty(\mathbf{H})} + \|\mathbf{y}_\alpha\|_{L^2(\mathbf{V})}\right)\right] \\ &\le C B_0(\mathbf{y}_0,\mathbf{f})(1 + B_0(\mathbf{y}_0,\mathbf{f})), \end{aligned}$$

[3] Alternatively, we can prove the existence of a solution by introducing adequate Galerkin approximations and applying (classical) compactness arguments.

where $s_N = 2\,\sigma_N$. Here, the third inequality is a consequence of the continuous embedding

$$L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V}) \hookrightarrow L^{s_N}(0,T;\mathbf{L}^4(\Omega)).$$

This estimate completes the proof of (2.3).

• Uniqueness: Let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ and $(\mathbf{y}'_\alpha, p'{}_\alpha, \mathbf{z}'_\alpha, \pi'{}_\alpha)$ be two solutions to (1.3) and let us introduce $\mathbf{u} := \mathbf{y}_\alpha - \mathbf{y}'_\alpha$, $q = p_\alpha - p'_\alpha$, $\mathbf{m} := \mathbf{z}_\alpha - \mathbf{z}'_\alpha$ and $h = \pi_\alpha - \pi'_\alpha$. Then

$$\begin{cases} \mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{z}_\alpha \cdot \nabla)\mathbf{u} + \nabla q = -(\mathbf{m}\cdot\nabla)\mathbf{y}'_\alpha & \text{in } Q, \\ \mathbf{m} - \alpha^2\Delta\mathbf{m} + \nabla h = \mathbf{u} & \text{in } Q, \\ \nabla\cdot\mathbf{u} = 0,\ \nabla\cdot\mathbf{m} = 0 & \text{in } Q, \\ \mathbf{u} = \mathbf{m} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{u}(0) = \mathbf{0} & \text{in } \Omega. \end{cases}$$

Since $\mathbf{u} \in L^\infty(0,T;\mathbf{H})$, we have $\mathbf{m} \in L^\infty(0,T;D(\mathbf{A}))$ (where the estimate of this norm depends on α). Therefore, we easily deduce from the first equation of the previous system that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|^2 \le \|\mathbf{m}\|_\infty \|\nabla\mathbf{y}'_\alpha\|\|\mathbf{u}\|$$

for all t. Since $\|\mathbf{m}\|_\infty \le C\|\mathbf{m}\|_{D(\mathbf{A})} \le C\alpha^{-2}\|\mathbf{u}\|$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 + \|\nabla\mathbf{u}\|^2 \le C\alpha^{-2}\|\nabla\mathbf{y}'_\alpha\|\|\mathbf{u}\|^2.$$

Therefore, in view of Gronwall's Lemma, we see that $\mathbf{u} \equiv 0$. Accordingly, we also have $\mathbf{m} \equiv \mathbf{0}$ and uniqueness holds. □

We are now going to present some results concerning the existence and uniqueness of a strong solution. We start with a global result in the two-dimensional case.

Proposition 2.4. *Assume that $N = 2$ and $\alpha > 0$ is fixed. Then, for any $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0 \in \mathbf{V}$, there exists exactly one solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (1.3), with*

$$\begin{gathered} \mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}),\ (\mathbf{y}_\alpha)_t \in L^2(0,T;\mathbf{H}), \\ \mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A}^2)) \cap C^0([0,T];D(\mathbf{A}^{3/2})). \end{gathered} \tag{2.5}$$

Furthermore, the following estimates hold:

$$\begin{gathered} \|(\mathbf{y}_\alpha)_t\| + \|\mathbf{y}_\alpha\|_{C^0([0,T];\mathbf{V})} + \|\mathbf{y}_\alpha\|_{L^2(D(\mathbf{A}))} \le B_1(\|\mathbf{y}_0\|_\mathbf{V}, \|\mathbf{f}\|), \\ \|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{C^0([0,T];D(\mathbf{A}))} \le \|\mathbf{y}_\alpha\|^2_{C^0([0,T];\mathbf{V})}, \end{gathered} \tag{2.6}$$

where we have introduced the notation

$$B_1(r,s) := (r+s)\left[1 + (r+s)^2\right] \mathrm{e}^{C(r^2+s^2)^2}.$$

Proof. First, thanks to Proposition 2.3, we see that there exists a unique weak solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ satisfying (2.2), and (2.3). In particular, $\mathbf{z}_\alpha \in L^2(0,T;\mathbf{V})$ and we have

$$\|\mathbf{z}_\alpha(t)\| \le \|\mathbf{y}_\alpha(t)\| \quad \text{and} \quad \|\mathbf{z}_\alpha(t)\|_\mathbf{V} \le \|\mathbf{y}_\alpha(t)\|_\mathbf{V}, \quad \forall t \in [0,T].$$

As usual, we will just check that good estimates can be obtained for $\mathbf{y}_\alpha$, $(\mathbf{y}_\alpha)_t$ and $\mathbf{z}_\alpha$. Thus, we assume that it is possible to multiply by $-\Delta\mathbf{y}_\alpha$ the motion equation satisfied by $\mathbf{y}_\alpha$. Taking into account that $N=2$, we obtain:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{y}_\alpha\|^2+\|\Delta\mathbf{y}_\alpha\|^2 = &-(\mathbf{f},\Delta\mathbf{y}_\alpha)+((\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha,\Delta\mathbf{y}_\alpha)\\ &\le \|\mathbf{f}\|^2+\frac{1}{4}\|\Delta\mathbf{y}_\alpha\|^2+\|\mathbf{z}_\alpha\|^{1/2}\|\mathbf{z}_\alpha\|_{\mathbf{V}}^{1/2}\|\mathbf{y}_\alpha\|_{\mathbf{V}}^{1/2}\|\Delta\mathbf{y}_\alpha\|^{3/2}\\ &\le \|\mathbf{f}\|^2+\frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2+C\|\mathbf{z}_\alpha\|^2\|\mathbf{z}_\alpha\|_{\mathbf{V}}^2\|\mathbf{y}_\alpha\|_{\mathbf{V}}^2.\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{y}_\alpha\|^2+\|\Delta\mathbf{y}_\alpha\|^2\le C\left[\|\mathbf{f}\|^2+(\|\mathbf{y}_\alpha\|^2\|\mathbf{y}_\alpha\|_{\mathbf{V}}^2)\|\nabla\mathbf{y}_\alpha\|^2\right].$$

In view of Gronwall's Lemma and the estimates in Proposition 2.3, we easily deduce (2.5) and (2.6). □

Notice that, in this two-dimensional case, the strong estimates for $\mathbf{y}_\alpha$ in (2.6) are independent of α; obviously, we cannot expect the same when $N=3$.

In the three-dimensional case, what we obtain is the following:

Proposition 2.5. *Assume that $N=3$ and $\alpha>0$ is fixed. Then, for any $\mathbf{f}\in L^2(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0\in\mathbf{V}$, there exists exactly one solution $(\mathbf{y}_\alpha,p_\alpha,\mathbf{z}_\alpha,\pi_\alpha)$ to (1.3), with*

$$\begin{gathered}\mathbf{y}_\alpha\in L^2(0,T;D(\mathbf{A}))\cap C^0([0,T];\mathbf{V}),\ \ (\mathbf{y}_\alpha)_t\in L^2(0,T;\mathbf{H}),\\ \mathbf{z}_\alpha\in L^2(0,T;D(\mathbf{A}^2))\cap C^0([0,T];D(\mathbf{A}^{3/2})).\end{gathered}$$

Furthermore, the following estimates hold:

$$\begin{gathered}\|\mathbf{y}_\alpha\|_{L^\infty(\mathbf{V})}+\|\mathbf{y}_\alpha\|_{L^2(D(\mathbf{A}))}+\|(\mathbf{y}_\alpha)_t\|\le B_2(\|\mathbf{y}_0\|_{\mathbf{V}},\|\mathbf{f}\|,\alpha),\\ \|\mathbf{z}_\alpha\|_{L^\infty(\mathbf{V})}^2+2\alpha^2\|\mathbf{z}_\alpha\|_{L^\infty(D(\mathbf{A}))}^2\le\|\mathbf{y}_\alpha\|_{L^\infty(\mathbf{V})}^2,\end{gathered}\tag{2.7}$$

where we have introduced

$$B_2(r,s,\alpha):=C(r+s)\mathrm{e}^{C\alpha^{-4}(r+s)^2}.$$

Proof. Thanks to Proposition 2.3, there exists a unique weak solution $(\mathbf{y}_\alpha,p_\alpha,\mathbf{z}_\alpha,\pi_\alpha)$ satisfying (2.2) and (2.3).

In particular, we obtain that $\mathbf{z}_\alpha\in\mathbf{L}^\infty(Q)$, with

$$\|\mathbf{z}_\alpha\|_\infty\le\frac{C}{\alpha^2}\left(\|\mathbf{y}_0\|_{\mathbf{H}}+\|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}\right).$$

On the other hand, $\mathbf{y}_0\in\mathbf{V}$. Hence, from the usual (parabolic) regularity results for Oseen systems, the solution to (1.3) is more regular, *i.e.* $\mathbf{y}_\alpha\in L^2(0,T;D(\mathbf{A}))\cap C^0([0,T];\mathbf{V})$ and $(\mathbf{y}_\alpha)_t\in L^2(0,T;\mathbf{H})$. Moreover, $\mathbf{y}_\alpha$ verifies the first estimate in (2.7). This achieves the proof. □

Let us now provide a result concerning three-dimensional strong solutions corresponding to small data, with estimates independent of α:

Proposition 2.6. *Assume that $N=3$. There exists $C_0>0$ such that, for any $\alpha>0$, any $\mathbf{f}\in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and any $\mathbf{y}_0\in\mathbf{V}$ with*

$$M:=\max\left\{\|\nabla\mathbf{y}_0\|^2,\ \|\mathbf{f}\|_{L^\infty(\mathbf{L}^2)}^{2/3}\right\}<\frac{1}{\sqrt{2(1+C_0)T}},\tag{2.8}$$

the Leray-α system (1.3) *possesses a unique solution* $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ *satisfying*

$$\mathbf{y}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}),\ (\mathbf{y}_\alpha)_t \in L^2(0,T;\mathbf{H}),$$
$$\mathbf{z}_\alpha \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}).$$

Furthermore, in that case, the following estimates hold:

$$\begin{aligned} \|\mathbf{y}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + \|\mathbf{y}_\alpha\|^2_{L^2(D(\mathbf{A}))} &\le B_3(M,T),\\ \|\mathbf{z}_\alpha\|^2_{C^0([0,T];\mathbf{V})} + 2\alpha^2\|\mathbf{z}_\alpha\|^2_{L^2(D(\mathbf{A}))} &\le \|\mathbf{y}_\alpha\|^2_{L^\infty(\mathbf{V})}, \end{aligned} \tag{2.9}$$

where we have introduced

$$B_3(M,T) := 2\left[M^3 + M + C_0T\left(\frac{M}{\sqrt{1-2(1+C_0)M^2T}}\right)^3\right].$$

Proof. The proof is very similar to the proof of the existence of a local in time strong solution to the Navier−Stokes system; see for instance [5,30].

As before, there exists a unique weak solution $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ and this solution satisfies (2.2) and (2.3).

By multiplying by $\Delta\mathbf{y}_\alpha$ the motion equation satisfied by $\mathbf{y}_\alpha$, we see that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{y}_\alpha\|^2 + \|\Delta\mathbf{y}_\alpha\|^2 &= (\mathbf{f}, \Delta\mathbf{y}_\alpha) - ((\mathbf{z}_\alpha\cdot\nabla)\mathbf{y}_\alpha, \Delta\mathbf{y}_\alpha)\\ &\le \frac{1}{2}\|\mathbf{f}\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 + \|\mathbf{z}_\alpha\|_{\mathbf{L}^6}\|\nabla\mathbf{y}_\alpha\|_{\mathbf{L}^3}\|\Delta\mathbf{y}_\alpha\|\\ &\le \frac{1}{2}\|\mathbf{f}\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 + C\|\mathbf{z}_\alpha\|_{\mathbf{V}}\|\mathbf{y}_\alpha\|^{1/2}_{\mathbf{V}}\|\Delta\mathbf{y}_\alpha\|^{3/2}. \end{aligned}$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\mathbf{y}_\alpha\|^2 + \frac{1}{2}\|\Delta\mathbf{y}_\alpha\|^2 \le \|\mathbf{f}\|^2 + C_0\|\nabla\mathbf{y}_\alpha\|^6, \tag{2.10}$$

for some $C_0 > 0$.

Let us see that, under the assumption (2.8), we have

$$\|\nabla\mathbf{y}_\alpha\|^2 \le \frac{M}{\sqrt{1-2(1+C_0)M^2T}}, \quad \forall t\in[0,T]. \tag{2.11}$$

Indeed, let us introduce the real-valued function ψ given by

$$\psi(t) = \max\left\{M, \|\nabla\mathbf{y}_\alpha(t)\|^2\right\}, \quad \forall t\in[0,T].$$

Then, ψ is almost everywhere differentiable and, in view of (2.8) and (2.10), one has

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} \le (1+C_0)\psi^3, \quad \psi(0) = M.$$

Therefore,

$$\psi(t) \le \frac{M}{\sqrt{1-2(1+C_0)M^2t}} \le \frac{M}{\sqrt{1-2(1+C_0)M^2T}}$$

and, since $\|\nabla\mathbf{y}_\alpha\|^2 \le \psi$, (2.11) holds. From this estimate, it is very easy to deduce (2.9). □

The following lemma is inspired by a result by Constantin and Foias for the Navier−Stokes equations, see [5]:

Lemma 2.7. *There exists a continuous function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$, with $\phi(s) \to 0$ as $s \to 0^+$, satisfying the following properties:*

a) *For $\mathbf{f} = \mathbf{0}$, any $\mathbf{y}_0 \in \mathbf{H}$ and any $\alpha > 0$, there exist arbitrarily small times $t^* \in (0, T/2)$ such that the corresponding solution to (1.3) satisfies $\|\mathbf{y}_\alpha(t^*)\|^2_{D(\mathbf{A})} \leq \phi(\|\mathbf{y}_0\|)$.*
b) *The set of these t^* has positive measure.*

Proof. We are only going to consider the three-dimensional case; the proof in the two-dimensional case is very similar and even easier.

The proof consists of several steps:

- Let us first see that, for any $k > 3/2$ and any $\tau \in (0, T/2]$, the set

$$R_\alpha(k,\tau) := \left\{ t \in [0,\tau] : \|\nabla \mathbf{y}_\alpha(t)\|^2 \leq \frac{k}{\tau}\, \|\mathbf{y}_0\|^2 \right\}$$

is non-empty and its measure $|R_\alpha(k,\tau)|$ satisfies $|R_\alpha(k,\tau)| \geq \tau/k$.
Obviously, we can assume that $\mathbf{y}_0 \neq \mathbf{0}$. Now, if we suppose that $|R_\alpha(k,\tau)| < \tau/k$, we have:

$$\begin{aligned}\int_0^\tau \|\nabla \mathbf{y}_\alpha(t)\|^2 \,\mathrm{d}t \geq \int_{R_\alpha(k,\tau)^c} \|\nabla \mathbf{y}_\alpha(t)\|^2 \,\mathrm{d}t &\geq \left(\tau - \frac{\tau}{k}\right) \frac{k}{\tau} \|\mathbf{y}_0\|^2 \\ &= (k-1)\|\mathbf{y}_0\|^2 > \frac{1}{2}\|\mathbf{y}_0\|^2.\end{aligned}$$

But, since $\mathbf{f} = 0$ in (1.3), we also have the following estimate:

$$\int_0^\tau \|\nabla \mathbf{y}_\alpha(t)\|^2 \,\mathrm{d}t \leq \frac{1}{2}\|\mathbf{y}_\alpha(\tau)\|^2 + \int_0^\tau \|\nabla \mathbf{y}_\alpha(t)\|^2 \,\mathrm{d}t = \frac{1}{2}\|\mathbf{y}_0\|^2.$$

So, we get a contradiction and, necessarily, $|R_\alpha(k,\tau)| \geq \tau/k$.
- Let us choose $\tau \in (0, T/2]$, $k > 3/2$, $t_{0,\alpha} \in R_\alpha(k,\tau)$ and

$$\overline{T}_\alpha \in \left[t_{0,\alpha} + \frac{\tau^2}{4((1+C_0)k^2\|\mathbf{y}_0\|^4}, t_{0,\alpha} + \frac{3\tau^2}{8((1+C_0)k^2\|\mathbf{y}_0\|^4} \right],$$

where C_0 is the constant furnished by Proposition 2.6. Since $\|\nabla \mathbf{y}_\alpha(t_{0,\alpha})\|^2 \leq \frac{k}{\tau}\, \|\mathbf{y}_0\|^2$, there exists exactly one strong solution to (1.3) in $[t_{0,\alpha}, \overline{T}_\alpha]$ starting from $\mathbf{y}_\alpha(t_{0,\alpha})$ at time $t_{0,\alpha}$ and satisfying

$$\|\nabla \mathbf{y}_\alpha(t)\|^2 \leq \frac{2k}{\tau}\|\mathbf{y}_0\|^2, \quad \forall t \in [t_{0,\alpha}, \overline{T}_\alpha].$$

Obviously, it can be assumed that $\overline{T}_\alpha < T$.
Let us introduce the set

$$G_\alpha(t_{0,\alpha}, k, \tau) := \left\{ t \in [t_{0,\alpha}, \overline{T}_\alpha] : \quad \|\varDelta \mathbf{y}_\alpha(t)\|^2 \leq 65(1+C_0) \left(\frac{k}{\tau}\right)^3 \|\mathbf{y}_0\|^6 \right\}.$$

Then, again $G_\alpha(t_{0,\alpha}, k, \tau)$ is non-empty and possesses positive measure. More precisely, one has

$$|G_\alpha(t_{0,\alpha}, k, \tau)| \geq \frac{\tau^2}{8(1+C_0)k^2\|\mathbf{y}_0\|^4}. \tag{2.12}$$

Indeed, otherwise we would get

$$\begin{aligned}\frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha}\|\Delta\mathbf{y}_\alpha(t)\|^2\,\mathrm{d}t &\geq \frac{1}{2}\int_{G_\alpha(t_{0,\alpha},k,\tau)^c}\|\Delta\mathbf{y}_\alpha(t)\|^2\,\mathrm{d}t\\ &\geq 65\left(\overline{T}_\alpha - t_{0,\alpha} - \frac{\tau^2}{8(1+C_0)k^2\|\mathbf{y}_0\|^4}\right)(1+C_0)\left(\frac{k}{\tau}\right)^3\|\mathbf{y}_0\|^6\\ &\geq \frac{65k}{16\tau}\|\mathbf{y}_0\|^2 > 4\frac{k}{\tau}\|\mathbf{y}_0\|^2.\end{aligned}$$

However, arguing as in the proof of Proposition 2.6, we also have

$$\begin{aligned}\frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha}\|\Delta\mathbf{y}_\alpha(t)\|^2\,\mathrm{d}t &\leq \|\nabla\mathbf{y}_\alpha(\overline{T}_\alpha)\|^2 + \frac{1}{2}\int_{t_{0,\alpha}}^{\overline{T}_\alpha}\|\Delta\mathbf{y}_\alpha(t)\|^2\,\mathrm{d}t\\ &\leq \|\nabla\mathbf{y}_\alpha(t_{0,\alpha})\|^2 + C_0\int_{t_{0,\alpha}}^{\overline{T}_\alpha}\|\nabla\mathbf{y}_\alpha(t)\|^6\,\mathrm{d}t\\ &\leq \frac{k}{\tau}\|\mathbf{y}_0\|^2 + 8\left(\frac{k}{\tau}\|\mathbf{y}_0\|^2\right)^3(\overline{T}_\alpha - t_{0,\alpha}) \leq 4\frac{k}{\tau}\|\mathbf{y}_0\|^2.\end{aligned}$$

Consequently, we arrive again to a contradiction and this proves (2.12).

- Let us fix $\tau \in (0, T/2]$ and $k > 3/2$. We can now define $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as follows:

$$\phi(s) := 65(1 + C_0)\frac{k}{\tau}^3 s^6.$$

Then, as a consequence of the previous steps, the set

$$\{\, t^* \in [0, T/2] : \|\mathbf{A}\mathbf{y}_\alpha(t^*)\|^2 \leq \phi(\|\mathbf{y}_0\|)\,\}$$

is non-empty and it measure is bounded from below by a positive quantity independent of α. This ends the proof. □

We will end this section with some estimates:

Lemma 2.8. *Let $s \in [1, 2]$ be given, and let us assume that $\mathbf{f} \in \mathbf{H}^s(\Omega)$. Then there exist unique functions $\mathbf{u} \in D(\mathbf{A}^{s/2})$ and $\pi \in H^{s-1}$ (π is unique up to a constant) such that*

$$\begin{cases}\mathbf{u} - \alpha^2\Delta\mathbf{u} + \nabla\pi = \alpha^2\Delta\mathbf{f} & \text{in } \Omega,\\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega,\\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma\end{cases} \tag{2.13}$$

and there exists a constant $C = C(s, \Omega)$ independent of α such that

$$\|\mathbf{u}\|_{D(\mathbf{A}^{s/2})} \leq C\|\mathbf{f}\|_{\mathbf{H}^s(\Omega)}. \tag{2.14}$$

Moreover, by interpolation arguments, $\mathbf{f} \in \mathbf{H}^s(\Omega)$, $s \in (m, m+1)$ then there exist unique functions $\mathbf{u} \in D(\mathbf{A}^{s/2})$ and $\pi \in H^{s-1}(\Omega)$ (π is unique up to a constant) which are solution of the problem above and there exists a constant $C = C(m, \Omega)$ such that

$$\|\mathbf{u}\|_{D(\mathbf{A}^{s/2})} \leq C\|\mathbf{f}\|_{\mathbf{H}^s(\Omega)}. \tag{2.15}$$

When s is an integer ($s = 1$ or $s = 2$), the proof can be obtained by adapting the proof of Proposition 2.3 in [30]. For other values of s, it suffices to use a classical interpolation argument (see [29]).

2.3. Carleman inequalities and null controllability

In this subsection, we will recall some Carleman inequalities and a null controllability result for the Oseen system

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{h}\cdot\nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in } Q, \\ \nabla\cdot\mathbf{y} = 0 & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \Omega, \end{cases} \tag{2.16}$$

where $\mathbf{h} = \mathbf{h}(\mathbf{x}, t)$ is given. The null controllability problem for (2.16) at time $T > 0$ is the following:

For any $\mathbf{y}_0 \in \mathbf{H}$, find $\mathbf{v} \in \mathbf{L}^2(\omega \times (0, T))$ such that the associated solution to (2.16) satisfies (1.7).

We have the following result from [13] (see also [25]):

Theorem 2.9. *Assume that $\mathbf{h} \in \mathbf{L}^\infty(Q)$ and $\nabla\cdot\mathbf{h} = 0$. Then, the linear system (2.16) is null-controllable at any time $T > 0$. More precisely, for each $\mathbf{y}_0 \in \mathbf{H}$ there exists $\mathbf{v} \in L^\infty(0, T; \mathbf{L}^2(\omega))$ such that the corresponding solution to (2.16) satisfies (1.7). Furthermore, the control $\mathbf{v}$ can be chosen satisfying the estimate*

$$\|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))} \le \mathrm{e}^{K(1+\|\mathbf{h}\|_\infty^2)}\|\mathbf{y}_0\|, \tag{2.17}$$

where K only depends on Ω, ω and T.

The proof is a consequence of an appropriate Carleman inequality for the adjoint system of (2.16).
More precisely, let us consider the backwards in time system

$$\begin{cases} -\boldsymbol{\varphi}_t - \Delta\boldsymbol{\varphi} - (\mathbf{h}\cdot\nabla)\boldsymbol{\varphi} + \nabla q = \mathbf{G} & \text{in } Q, \\ \nabla\cdot\boldsymbol{\varphi} = 0 & \text{in } Q, \\ \boldsymbol{\varphi} = \mathbf{0} & \text{on } \Sigma, \\ \boldsymbol{\varphi}(T) = \boldsymbol{\varphi}_0, & \text{in } \Omega. \end{cases} \tag{2.18}$$

The following result is established in [13]:

Proposition 2.10. *Assume that $\mathbf{h} \in \mathbf{L}^\infty(Q)$ and $\nabla\cdot\mathbf{h} = 0$. There exist positive continuous functions α, α^*, $\hat\alpha$, ξ, ξ^* and $\hat\xi$ and positive constants $\hat s$, $\hat\lambda$ and $\widehat{C}$, only depending on Ω and ω, such that, for any $\boldsymbol{\varphi}_0 \in \mathbf{H}$ and any $\mathbf{G} \in \mathbf{L}^2(Q)$, the solution to the adjoint system (2.18) satisfies:*

$$\begin{aligned} &\iint_Q \mathrm{e}^{-2s\alpha}\left[s^{-1}\xi^{-1}(|\boldsymbol{\varphi}_t|^2 + |\Delta\boldsymbol{\varphi}|^2) + s\xi\lambda^2|\nabla\boldsymbol{\varphi}|^2 + s^3\xi^3\lambda^4|\boldsymbol{\varphi}|^2\right]\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ &\quad\le \widehat{C}(1+T^2)\left(s^{15/2}\lambda^{20}\iint_Q \mathrm{e}^{-4s\hat\alpha+2s\alpha^*}\xi^{*\,15/2}|\mathbf{G}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right. \\ &\qquad\left.+ s^{16}\lambda^{40}\iint_{\omega\times(0,T)} \mathrm{e}^{-8s\hat\alpha+6s\alpha^*}\xi^{*\,16}|\boldsymbol{\varphi}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right), \end{aligned} \tag{2.19}$$

for all $s \ge \hat s(T^4 + T^8)$ and for all $\lambda \ge \hat\lambda\left(1 + \|\mathbf{h}\|_\infty + \mathrm{e}^{\hat\lambda T\|\mathbf{h}\|_\infty^2}\right)$.

Now, we are going to construct the a null-control for (2.16) like in [13]. First, let us introduce the auxiliary extremal problem

$$\begin{cases} \text{Minimize } \dfrac{1}{2}\left\{\displaystyle\iint_Q \hat\rho^2|\mathbf{y}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \iint_{\omega\times(0,T)} \hat\rho_0^2|\mathbf{v}|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right\} \\ \text{Subject to } (\mathbf{y}, \mathbf{v}) \in \mathcal{M}(\mathbf{y}_0, T), \end{cases} \tag{2.20}$$

where the linear manifold $\mathcal{M}(\mathbf{y}_0, T)$ is given by

$$\mathcal{M}(\mathbf{y}_0, T) = \{\, (\mathbf{y}, \mathbf{v}) : \mathbf{v} \in \mathbf{L}^2(\omega \times (0,T)),\ (\mathbf{y}, p) \text{ solves } (2.16) \,\}$$

and $\hat{\rho}$, $\hat{\rho}_0$ are respectively given by

$$\hat{\rho} = s^{-15/4}\lambda^{-10}\mathrm{e}^{2s\hat{\alpha}-s\alpha^*}\xi^{*-15/4}, \quad \hat{\rho}_0 = s^{-8}\lambda^{-20}\mathrm{e}^{4s\hat{\alpha}-3s\alpha^*}\xi^{*-8}.$$

It can be proved that (2.20) possesses exactly one solution $(\mathbf{y}, \mathbf{v})$ satisfying

$$\|\mathbf{v}\|_{L^2(\mathbf{L}^2(\omega))} \le \mathrm{e}^{K(1+\|\mathbf{h}\|_\infty^2)}\|\mathbf{y}_0\|,$$

where K only depends on Ω, ω and T.

Moreover, thanks to the Euler−Lagrange characterization, the solution to the extremal problem (2.20) is given by

$$\mathbf{y} = \hat{\rho}^{-2}(-\boldsymbol{\varphi}_t - \Delta\boldsymbol{\varphi} - (\mathbf{h}\cdot\nabla)\boldsymbol{\varphi} + \nabla q) \ \text{ and } \ \mathbf{v} = -\hat{\rho}_0^{-2}\boldsymbol{\varphi}1_\omega.$$

From the Carleman inequality (2.19), we can conclude that $\rho_2^{-1}\boldsymbol{\varphi} \in L^\infty(0,T;\mathbf{L}^2(\Omega))$ and

$$\|\rho_2^{-1}\boldsymbol{\varphi}\|_{L^\infty(\mathbf{L}^2)} \le C\|\hat{\rho}_0^{-1}\boldsymbol{\varphi}\|_{L^2(\mathbf{L}^2(\omega))},$$

where $\rho_2 = s^{1/2}\xi^{1/2}\mathrm{e}^{s\alpha}$.

Hence,

$$\mathbf{v} = -(\hat{\rho}_0)^{-2}\boldsymbol{\varphi}1_\omega = -(\hat{\rho}_0^{-2}\rho_2)(\rho_2^{-1}\boldsymbol{\varphi}1_\omega) \in L^\infty(0,T;\mathbf{L}^2(\Omega))$$

and, therefore,

$$\|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))} \le C\|\mathbf{v}\|_{L^2(\mathbf{L}^2(\omega))} \le \mathrm{e}^{K(1+\|\mathbf{h}\|_\infty^2)}\|\mathbf{y}_0\|.$$

3. The distributed case: Theorems 1.2 and 1.4

This section is devoted to prove the local null controllability of (1.5) and the uniform controllability property in Theorem 1.4.

Proof of Theorem 1.2. We will use a fixed point argument. Contrarily to the case of the Navier−Stokes equations, it is not sufficient to work here with controls in $\mathbf{L}^2(\omega \times (0,T))$. Indeed, we need a space $\mathbf{Y}$ for $\mathbf{y}$ that ensures $\mathbf{z}$ in $\mathbf{L}^\infty(Q)$ and a space $\mathbf{X}$ for $\mathbf{v}$ guaranteeing that the solution to (2.16) with $\mathbf{h} = \mathbf{z}$ belongs to a compact set of $\mathbf{Y}$. Furthermore, we want estimates in $\mathbf{Y}$ and $\mathbf{X}$ independent of α.

In view of Lemma 2.7, in order to prove Theorem 1.2, we just need to consider the case in which the initial state $\mathbf{y}_0$ belongs to $D(\mathbf{A})$ and possesses a sufficiently small norm in $D(\mathbf{A})$.

Let us fix σ with $N/4 < \sigma < 1$. Then, for each $\tilde{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$, let $(\mathbf{z}, \pi)$ be the unique solution to

$$\begin{cases} \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \tilde{\mathbf{y}} & \text{in } Q, \\ \nabla\cdot\mathbf{z} = 0 & \text{in } Q, \\ \mathbf{z} = \mathbf{0} & \text{on } \Sigma. \end{cases}$$

Since $\tilde{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$, it is clear that $\mathbf{z} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$. Then, thanks to Theorem 2.1, we have $\mathbf{z} \in \mathbf{L}^\infty(Q)$ and the following is satisfied:

$$\begin{aligned} \|\mathbf{z}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))} + 2\alpha^2\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1/2+\sigma}))} &\le \|\tilde{\mathbf{y}}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))}, \\ 2\alpha^2\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1/2+\sigma}))} + \alpha^4\|\mathbf{z}\|^2_{L^\infty(D(\mathbf{A}^{1+\sigma}))} &\le \|\tilde{\mathbf{y}}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))}. \end{aligned} \tag{3.1}$$

In particular, we have:

$$\|\mathbf{z}\|_{L^\infty(0,T;D(\mathbf{A}^\sigma))} \le \|\tilde{\mathbf{y}}\|_{L^\infty(0,T;D(\mathbf{A}^\sigma))}.$$

Let us consider the system (2.16) with $\mathbf{h}$ replaced by $\mathbf{z}$. In view of Theorem 2.9, we can associate to $\mathbf{z}$ the null control $\mathbf{v}$ of minimal norm in $L^\infty(0,T;\mathbf{L}^2(\omega))$ and the corresponding solution $(\mathbf{y},p)$ to (2.16).

Since $\mathbf{y}_0 \in D(\mathbf{A})$, $\mathbf{z} \in \mathbf{L}^\infty(Q)$ and $\mathbf{v} \in L^\infty(0,T;\mathbf{L}^2(\omega))$, we have

$$\mathbf{y} \in L^2(0,T;D(\mathbf{A})) \cap C^0([0,T];\mathbf{V}),\ \mathbf{y}_t \in L^2(0,T;\mathbf{H})$$

and the following estimate holds:

$$\|\mathbf{y}_t\|^2_{L^2(\mathbf{H})} + \|\mathbf{y}\|^2_{L^2(D(\mathbf{A}))} + \|\mathbf{y}\|^2_{L^\infty(\mathbf{V})} \le C\left(\|\mathbf{y}_0\|^2_{\mathbf{V}} + \|\mathbf{v}\|^2_{L^\infty(\mathbf{L}^2(\omega))}\right)\mathrm{e}^{C\|\mathbf{z}\|^2_\infty}. \tag{3.2}$$

We will use the following result:

Lemma 3.1. *One has $\mathbf{y} \in L^\infty(0,T;D(\mathbf{A}^{\sigma'}))$, for all $\sigma' \in (\sigma,1)$, with*

$$\|\mathbf{y}\|_{L^\infty(D(\mathbf{A}^{\sigma'}))} \le C(\|\mathbf{y}_0\|_{D(\mathbf{A})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})\mathrm{e}^{C\|\tilde{\mathbf{y}}\|^2_{L^\infty(D(\mathbf{A}^\sigma))}}.$$

Proof. In view of (2.16), $\mathbf{y}$ solves the following abstract initial value problem:

$$\begin{cases} \mathbf{y}_t = -\mathbf{A}\mathbf{y} - \mathbf{P}((\mathbf{z}\cdot\nabla)\mathbf{y}) + \mathbf{P}(\mathbf{v}1_\omega) \text{ in } [0,T], \\ \mathbf{y}(0) = \mathbf{y}_0. \end{cases}$$

This system can be rewritten as the nonlinear integral equation

$$\mathbf{y}(t) = \mathrm{e}^{-t\mathbf{A}}\mathbf{y}_0 - \int_0^t \mathrm{e}^{-(t-s)\mathbf{A}}\mathbf{P}((\mathbf{z}\cdot\nabla)\mathbf{y})(s)\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)\mathbf{A}}\mathbf{P}(\mathbf{v}1_\omega)(s)\,\mathrm{d}s.$$

Consequently, applying the operator $\mathbf{A}^{\sigma'}$ to both sides, we have

$$\mathbf{A}^{\sigma'}\mathbf{y}(t) = \mathbf{A}^{\sigma'}\mathrm{e}^{-t\mathbf{A}}\mathbf{y}_0 + \int_0^t \mathbf{A}^{\sigma'}\mathrm{e}^{-(t-s)\mathbf{A}}\left[-P((\mathbf{z}\cdot\nabla)\mathbf{y})(s) + P(\mathbf{v}1_\omega)(s)\right]\mathrm{d}s.$$

Taking norms in both sides and using Theorem 2.2, we see that

$$\begin{aligned} \|\mathbf{A}^{\sigma'}\mathbf{y}\|(t) &\le \|\mathbf{y}_0\|_{D(A^{\sigma'})} + \int_0^t (t-s)^{-\sigma'}\left[\|\mathbf{z}(s)\|_\infty\|\nabla\mathbf{y}(s)\| + \|\mathbf{v}(s)1_\omega\|\right]\mathrm{d}s \\ &\le C\|\mathbf{y}_0\|_{D(\mathbf{A})} + (\|\mathbf{z}\|_\infty\|\mathbf{y}\|_{L^\infty(\mathbf{V})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})\int_0^t (t-s)^{-\sigma'}\,\mathrm{d}s. \end{aligned}$$

Now, using (3.1) and (3.2) and taking into account that $\sigma' < 1$, we easily obtain that

$$\|\mathbf{A}^{\sigma'}\mathbf{y}\|(t) \le C(\|\mathbf{y}_0\|_{D(\mathbf{A})} + \|\mathbf{v}\|_{L^\infty(\mathbf{L}^2(\omega))})\left[1 + \|\tilde{\mathbf{y}}\|_{L^\infty(D(\mathbf{A}^\sigma))}\mathrm{e}^{C\|\tilde{\mathbf{y}}\|^2_{L^\infty(D(\mathbf{A}^\sigma))}}\right].$$

This ends the proof. ∎

Now, let us set

$$\mathbf{W} = \{\,\mathbf{w} \in L^\infty(0,T;D(\mathbf{A}^{\sigma'})) : \mathbf{w}_t \in L^2(0,T;\mathbf{H})\,\}$$

and let us consider the closed ball

$$\mathbf{K} = \{\,\tilde{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma)) : \|\tilde{\mathbf{y}}\|_{L^\infty(D(\mathbf{A}^\sigma))} \le 1\,\}$$

and the mapping $\tilde{\Lambda}_\alpha$, with $\tilde{\Lambda}_\alpha(\tilde{\mathbf{y}}) = \mathbf{y}$ for all $\tilde{\mathbf{y}} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$. Obviously, $\tilde{\Lambda}_\alpha$ is well-defined; furthermore, in view of Lemma 3.1 and equation (3.2), it maps the whole space $L^\infty(0,T;D(\mathbf{A}^\sigma))$ into $\mathbf{W}$.

Notice that, if $\mathbf{U}$ is bounded set of $\mathbf{W}$ then it is relatively compact in the space $L^\infty(0,T;D(\mathbf{A}^\sigma))$, in view of the classical results of the Aubin−Lions kind, see for instance [28].

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\tilde{\Lambda}_\alpha$. Then, thanks to Lemma 3.1 and (2.17), if $\|\mathbf{y}_0\|_{D(\mathbf{A})} \le \varepsilon$ (independent of α!) Λ_α maps $\mathbf{K}$ into itself. Moreover, it is clear that $\Lambda_\alpha : \mathbf{K} \mapsto \mathbf{K}$ satisfies the hypotheses of Schauder's Theorem. Indeed, this nonlinear mapping is continuous and compact (the latter is a consequence of the fact that, if $\mathbf{B}$ is bounded in $L^\infty(0,T;D(\mathbf{A}^\sigma))$, then $\tilde{\Lambda}_\alpha(\mathbf{B})$ is bounded in $\mathbf{W}$). Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$, and this ends the proof of Theorem 1.2. $\square$

Proof of Theorem 1.4. Let $\mathbf{v}_\alpha$ be a null control for (1.5) satisfying (1.8) and let $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ be the state associated to $\mathbf{v}_\alpha$. From (1.8) and the estimates (2.3) for the solutions $\mathbf{y}_\alpha$, there exist $\mathbf{v} \in L^\infty\left(0,T;\mathbf{L}^2(\omega)\right)$ and $\mathbf{y} \in L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ with $\mathbf{y}_t \in L^{\sigma_N}(0,T;\mathbf{V}')$ such that, at least for a subsequence

$$
\begin{aligned}
\mathbf{v}_\alpha &\to \mathbf{v} \text{ weakly-}\star \text{ in } L^\infty\left(0,T;\mathbf{L}^2(\omega)\right),\\
\mathbf{y}_\alpha &\to \mathbf{y} \text{ weakly-}\star \text{ in } L^\infty\left(0,T;\mathbf{H}\right) \text{ and weakly in } L^2\left(0,T;\mathbf{V}\right),\\
(\mathbf{y}_\alpha)_t &\to \mathbf{y}_t \text{ weakly in } L^{\sigma_N}(0,T;\mathbf{V}').
\end{aligned}
$$

Since $\mathbf{W} := \{\mathbf{m} \in L^2\left(0,T;\mathbf{V}\right) :\ \mathbf{m}_t \in L^{\sigma_N}(0,T;\mathbf{V}')\}$ is continuously and compactly embedded in $\mathbf{L}^2(Q)$, we have that

$$
\mathbf{y}_\alpha \to \mathbf{y} \text{ in } \mathbf{L}^2(Q) \text{ and } a.e.
$$

This is sufficient to pass to the limit in the equations satisfied by $\mathbf{y}_\alpha$, $\mathbf{v}_\alpha$ and $\mathbf{z}_\alpha$. We conclude that $\mathbf{y}$ is, together with some pressure p, a solution to the Navier−Stokes equations associated to a control $\mathbf{v}$ and satisfies (1.7). $\square$

4. The boundary case: Theorems 1.3 and 1.5

This section is devoted to prove the local boundary null controllability of (1.6) and the uniform controllability property in Theorem 1.5.

Proof of Theorem 1.3. Again, we will use a fixed point argument. Contrarily to the case of distributed controllability, we will have to work in a space $\tilde{\mathbf{Y}}$ of functions defined in an extended domain.

Let $\widetilde{\Omega}$ be given, with $\Omega \subset \widetilde{\Omega}$ and $\partial\widetilde{\Omega} \cap \Gamma = \Gamma \setminus \gamma$ such that $\partial\widetilde{\Omega}$ is of class C^2 (see Fig. 1). Let $\omega \subset \widetilde{\Omega} \setminus \overline{\Omega}$ be a non-empty open subset and let us introduce $\widetilde{Q} := \widetilde{\Omega} \times (0,T)$ and $\widetilde{\Sigma} := \partial\widetilde{\Omega} \times (0,T)$. The spaces and operators associate to the domain $\widetilde{\Omega}$ will be denoted by $\widetilde{\mathbf{H}}$, $\widetilde{\mathbf{V}}$, $\widetilde{\mathbf{A}}$, *etc.*

Remark 4.1. In view of Lemma 2.7, for the Proof of Theorem 1.3 we just need to consider the case in which the initial state $\mathbf{y}_0$ belongs to $\mathbf{V}$ and possesses a sufficiently small norm in $\mathbf{V}$. Indeed, we only have to take initially $\mathbf{h}_\alpha \equiv \mathbf{0}$ and apply Lemma 2.7 to the solution to (1.6). $\square$

Let $\mathbf{y}_0 \in \mathbf{V}$ be given and let us introduce the extension by zero $\tilde{\mathbf{y}}_0$ of $\mathbf{y}_0$. Then $\tilde{\mathbf{y}}_0 \in \widetilde{\mathbf{V}}$.

We will use the following result, similar to Lemma 2.7, whose proof is postponed to the end of the section:

Lemma 4.2. *There exists a continuous function $\phi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ satisfying $\phi(s) \to 0$ as $s \to 0^+$ with the following property:*

a) *For any $\mathbf{y}_0 \in \mathbf{V}$ and any $\alpha > 0$, there exist times $T_0 \in (0,T)$, controls $\mathbf{h}_\alpha \in L^2(0,T_0;\mathbf{H}^{1/2}(\Gamma))$ with $\int_\gamma \mathbf{h}_\alpha \cdot \mathbf{n}\, \mathrm{d}\Gamma \equiv 0$, associated solutions $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha)$ to (1.6) in $\Omega \times (0,T_0)$ and arbitrarily small times $t^* \in (0,T/2)$ such that the $\mathbf{y}_\alpha$ can be extended to $\widetilde{\Omega} \times (0,T_0)$ and the extensions satisfy $\|\tilde{\mathbf{y}}_\alpha(t^*)\|^2_{D(\widetilde{\mathbf{A}})} \le \phi(\|\mathbf{y}_0\|_{\mathbf{V}})$.*
b) *The set of these t^* has positive measure.*
c) *The controls $\mathbf{h}_\alpha$ are uniformly bounded,* i.e.

$$
\|\mathbf{h}_\alpha\|_{L^\infty(0,T_0;\mathbf{H}^{1/2}(\gamma))} \le C.
$$

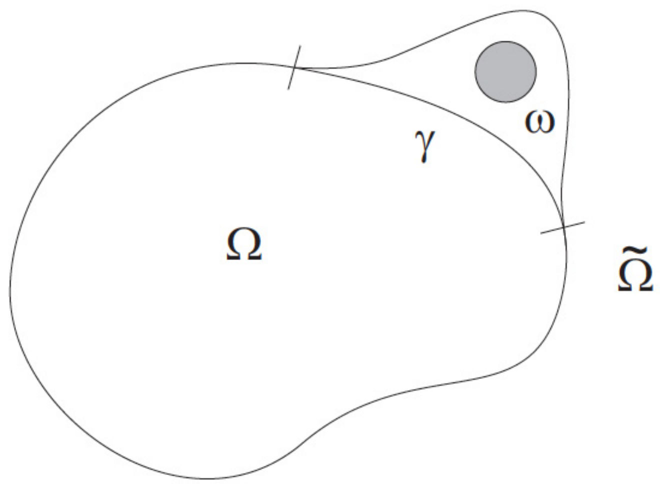

FIGURE 1. The domain $\widetilde{\Omega}$.

In view of Lemma 4.2, for the proof of Theorem 1.3, we just need to consider the case in which the initial state $\mathbf{y}_0$ is such that its extension $\tilde{\mathbf{y}}_0$ to $\widetilde{\Omega}$ belongs to $D(\tilde{\mathbf{A}})$ and possesses a sufficiently small norm in $D(\tilde{\mathbf{A}})$.

We will prove that there exists $(\tilde{\mathbf{y}}_\alpha, \tilde{p}_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \tilde{\mathbf{v}})$, with $\tilde{\mathbf{v}} \in L^\infty(0,T;\mathbf{L}^2(\omega))$, satisfying

$$\begin{cases} \tilde{\mathbf{y}}_t - \Delta\tilde{\mathbf{y}} + (\tilde{\mathbf{z}}\cdot\nabla)\tilde{\mathbf{y}} + \nabla\tilde{p} = \tilde{\mathbf{v}}1_\omega & \text{in } \widetilde{Q}, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \tilde{\mathbf{y}} & \text{in } Q, \\ \nabla\cdot\tilde{\mathbf{y}} = 0 & \text{in } \widetilde{Q}, \\ \nabla\cdot\mathbf{z} = 0 & \text{in } Q, \\ \tilde{\mathbf{y}} = \mathbf{0} & \text{on } \widetilde{\Sigma}, \\ \mathbf{z} = \tilde{\mathbf{y}} & \text{on } \Sigma, \\ \tilde{\mathbf{y}}(0) = \tilde{\mathbf{y}}_0 & \text{in } \widetilde{\Omega} \end{cases} \tag{4.1}$$

and $\tilde{\mathbf{y}}(T) = \mathbf{0}$ in $\widetilde{\Omega}$, where $\tilde{\mathbf{z}}$ is the extension by zero of $\mathbf{z}$. Obviously, if this were the case, the restriction $(\mathbf{y}, p)$ of $(\tilde{\mathbf{y}}, \tilde{p})$ to Q, the couple $(\mathbf{z}, \pi)$ and the lateral trace $\mathbf{h} := \tilde{\mathbf{y}}|_{\gamma\times(0,T)}$ would satisfy (1.6) and (1.7).

Let us fix σ with $N/4 < \sigma < \beta < 1$. Then, for each $\overline{\mathbf{y}} \in L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))$, let $\mathbf{w} = \mathbf{w}(\mathbf{x},t)$ and $\pi = \pi(\mathbf{x},t)$ be the unique solution to

$$\begin{cases} \mathbf{w} - \alpha^2\Delta\mathbf{w} + \nabla\pi = \alpha^2\Delta\overline{\mathbf{y}} & \text{in } Q, \\ \nabla\cdot\mathbf{w} = 0 & \text{in } Q, \\ \mathbf{w} = \mathbf{0} & \text{on } \Sigma. \end{cases}$$

Since $\overline{\mathbf{y}} \in L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))$, its restriction to Q belongs to $L^\infty(0,T;\mathbf{H}^{2\sigma}(\Omega))$. Then, Lemma 2.8 implies $\mathbf{w} \in L^\infty(0,T;D(\mathbf{A}^\sigma))$ and, thanks to Theorem 2.1, we also have $\mathbf{w} \in \mathbf{L}^\infty(Q)$ and

$$\|\mathbf{w}\|^2_{L^\infty(0,T;D(\mathbf{A}^\sigma))} \le C\|\overline{\mathbf{y}}\|^2_{L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))},$$

where C is independent of α.

Let $\tilde{\mathbf{w}}$ be the extension by zero of $\mathbf{w}$ and let us set $\tilde{\mathbf{z}} := \overline{\mathbf{y}} + \tilde{\mathbf{w}}$. Let us consider the system (2.16) with $\mathbf{h}$ replaced by $\tilde{\mathbf{z}}$ and Ω replaced by $\widetilde{\Omega}$. In view of Theorem 2.9, we can associate to $\tilde{\mathbf{z}}$ the null control $\tilde{\mathbf{v}}$ of minimal norm in $L^\infty(0,T;\mathbf{L}^2(\tilde\omega))$ and the corresponding solution $(\tilde{\mathbf{y}},\tilde p)$ to (2.16). Since $\tilde{\mathbf{y}}_0\in D(\tilde{\mathbf{A}})$, $\tilde{\mathbf{z}}\in\mathbf{L}^\infty(\widetilde Q)$ and $\tilde{\mathbf{v}}\in L^\infty(0,T;\mathbf{L}^2(\tilde\omega))$, we have

$$\tilde{\mathbf{y}}\in L^2(0,T;D(\tilde{\mathbf{A}}))\cap C^0([0,T];\tilde{\mathbf{V}}),\ \tilde{\mathbf{y}}_t\in L^2(0,T;\tilde{\mathbf{H}})$$

and the following estimate holds:

$$\|\tilde{\mathbf{y}}_t\|^2_{L^2(\tilde{\mathbf{H}})}+\|\tilde{\mathbf{y}}\|^2_{L^2(D(\tilde{\mathbf{A}}))}+\|\tilde{\mathbf{y}}\|^2_{L^\infty(\tilde{\mathbf{V}})}\le C\left(\|\tilde{\mathbf{y}}_0\|^2_{\tilde{\mathbf{V}}}+\|\tilde{\mathbf{v}}\|^2_{L^\infty(\mathbf{L}^2(\tilde\omega))}\right)\mathrm{e}^{C\|\tilde{\mathbf{z}}\|^2_\infty}. \tag{4.2}$$

Also, in account of Lemma 3.1, one has $\tilde{\mathbf{y}}\in L^\infty(0,T;D(\tilde{\mathbf{A}}^\beta))$ and

$$\|\tilde{\mathbf{y}}\|_{L^\infty(D(\tilde{\mathbf{A}}^\beta))}\le C(\|\tilde{\mathbf{y}}_0\|_{D(\tilde{\mathbf{A}})}+\|\tilde{\mathbf{v}}\|_{L^\infty(\mathbf{L}^2(\tilde\omega))})\mathrm{e}^{C\|\overline{\mathbf{y}}\|_{L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))}}).$$

Now, let us set

$$\mathbf{W}=\{\,\mathbf{m}\in L^\infty(0,T;D(\tilde{\mathbf{A}}^\beta)):\mathbf{m}_t\in L^2(0,T;\tilde{\mathbf{H}})\,\},$$

and let us consider the closed ball

$$\mathbf{K}=\{\,\overline{\mathbf{y}}\in L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma)):\|\overline{\mathbf{y}}\|_{L^\infty(D(\tilde{\mathbf{A}}^\sigma))}\le 1\,\}$$

and the mapping $\tilde\Lambda_\alpha$, with $\tilde\Lambda_\alpha(\overline{\mathbf{y}})=\tilde{\mathbf{y}}$ for all $\overline{\mathbf{y}}\in L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))$. Obviously, $\tilde\Lambda_\alpha$ is well-defined and maps the whole space $L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))$ into $\mathbf{W}$. Furthermore, any bounded set $\mathbf{U}\subset\mathbf{W}$ then it is relatively compact in $L^\infty(0,T;D(\tilde{\mathbf{A}}^\sigma))$.

Let us denote by Λ_α the restriction to $\mathbf{K}$ of $\tilde\Lambda_\alpha$. Thanks to Lemma 3.1 and (2.17), there exists $\varepsilon>0$ (independent of α) such that if $\|\tilde{\mathbf{y}}_0\|_{D(\tilde{\mathbf{A}})}\le\varepsilon$, Λ_α maps $\mathbf{K}$ into itself and it is clear that $\Lambda_\alpha:\mathbf{K}\mapsto\mathbf{K}$ satisfies the hypotheses of Schauder's Theorem. Consequently, Λ_α possesses at least one fixed point in $\mathbf{K}$ and (4.1) possesses a solution.This ends the Proof of Theorem 1.3.

Proof of Theorem 1.5. The proof is easy, in view of the previous uniform estimates. It suffices to adapt the argument in the Proof of Theorem 1.4 and deduce the existence of subsequences that converge (in an appropriate sense) to a solution to (1.11) satisfying (1.7). For brevity, we omit the details. □

Proof of Lemma 4.2. For instance, let us only consider the case $N=3$. We will reduce the proof to the search of a fixed point of another mapping Φ_α.

For any $\mathbf{y}_0\in\mathbf{V}$, any $T_0\in(0,T)$ and any $\overline{\mathbf{y}}\in L^4(0,T_0;\widetilde{\mathbf{V}}))$, let $(\mathbf{w},\pi)$ be the unique solution to

$$\begin{cases}\mathbf{w}-\alpha^2\Delta\mathbf{w}+\nabla\pi=\alpha^2\Delta\overline{\mathbf{y}} & \text{in } \ \Omega\times(0,T_0),\\ \nabla\cdot\mathbf{w}=0 & \text{in } \ \Omega\times(0,T_0),\\ \mathbf{w}=\mathbf{0} & \text{on } \Gamma\times(0,T_0),\end{cases}$$

let $\tilde{\mathbf{w}}$ be the extension by zero of $\mathbf{w}$, let us set $\tilde{\mathbf{z}}:=\overline{\mathbf{y}}+\tilde{\mathbf{w}}$ and let us introduce the Oseen system

$$\begin{cases}\tilde{\mathbf{y}}_t-\Delta\tilde{\mathbf{y}}+(\tilde{\mathbf{z}}\cdot\nabla)\tilde{\mathbf{y}}+\nabla\tilde p=\mathbf{0} & \text{in } \ \widetilde\Omega\times(0,T_0),\\ \nabla\cdot\tilde{\mathbf{y}}=0 & \text{in } \ \widetilde\Omega\times(0,T_0),\\ \tilde{\mathbf{y}}=\mathbf{0} & \text{on } \partial\widetilde\Omega\times(0,T_0),\\ \tilde{\mathbf{y}}(0)=\tilde{\mathbf{y}}_0 & \text{in } \ \widetilde\Omega.\end{cases}$$

It is clear that the restriction of $\overline{\mathbf{y}}$ to $\Omega\times(0,T_0)$ belongs to $L^4(0,T_0;\mathbf{H}^1(\Omega))$, whence we have from Lemma 2.8 that $\mathbf{w}\in L^4(0,T_0;\mathbf{V})$ and

$$\|\mathbf{w}\|_{L^4(0,T_0;\mathbf{V})}\le C\|\overline{\mathbf{y}}\|_{L^4(0,T_0;\widetilde{\mathbf{V}})}.$$

It is also clear that we can get estimates like those in the proof of Proposition 2.6 for $\widetilde{\mathbf{y}}$. In other words, for any $\mathbf{y}_0 \in \mathbf{V}$, we can find a sufficiently small $T_0 > 0$ such that

$$\widetilde{\mathbf{y}} \in L^2(0, T_0; D(\widetilde{\mathbf{A}})) \cap C^0([0, T_0]; \widetilde{\mathbf{V}}), \quad \widetilde{\mathbf{y}}_t \in L^2(0, T_0; \widetilde{\mathbf{H}})$$

and

$$\|\widetilde{\mathbf{y}}\|_{L^2(0,T_0;D(\widetilde{\mathbf{A}}))} + \|\widetilde{\mathbf{y}}\|_{C^0([0,T_0];\widetilde{\mathbf{V}})} + \|\widetilde{\mathbf{y}}_t\|_{L^2(0,T_0;\widetilde{\mathbf{H}})} \le C\left(T_0, \|\mathbf{y}_0\|_{\mathbf{V}}, \|\overline{\mathbf{y}}\|_{L^4(0,T_0;\widetilde{\mathbf{V}})}\right),$$

where C is nondecreasing with respect to all arguments and goes to zero as $\|\mathbf{y}_0\|_{\mathbf{V}} \to 0$.

Now, let us introduce the mapping $\Phi_\alpha : L^4(0, T_0; \widetilde{\mathbf{V}}) \mapsto L^4(0, T_0; \widetilde{\mathbf{V}})$, with $\Phi_\alpha(\overline{\mathbf{y}}) = \widetilde{\mathbf{y}}$ for all $\overline{\mathbf{y}} \in L^4(0, T; \widetilde{\mathbf{V}})$. This is a continuous and compact mapping. Indeed, from well-known interpolation results, we have that the embedding

$$L^2(0, T_0; D(\widetilde{\mathbf{A}})) \cap L^\infty(0, T_0; \widetilde{\mathbf{V}}) \hookrightarrow L^4(0, T_0; D(\widetilde{\mathbf{A}}^{3/4}))$$

is continuous and this shows that, if $\widetilde{\mathbf{y}}$ is bounded in $L^2(0, T_0; D(\widetilde{\mathbf{A}})) \cap C^0([0, T_0]; \widetilde{\mathbf{V}})$ and $\widetilde{\mathbf{y}}_t$ is bounded in $L^2(0, T_0; \widetilde{\mathbf{H}})$, then $\widetilde{\mathbf{y}}$ belongs to a compact set of $L^4(0, T_0; \widetilde{\mathbf{V}})$.

Then, as in the Proofs of Theorems 1.2 and 1.3, we immediately deduce that, whenever $\|\mathbf{y}_0\|_{\mathbf{V}} \le \delta$ (for some δ independent of α), Φ_α possesses at least one fixed point. This shows that the nonlinear system (4.1) is solvable for $\tilde{\mathbf{v}} \equiv 0$ and $\|\mathbf{y}_0\|_{\mathbf{V}} \le \delta$.

Now, the argument in the proof of Lemma 2.7 can be applied in this framework and, as a consequence, we easily deduce Lemma 4.2. $\square$

5. Additional comments and questions

5.1. Controllability problems for semi-Galerkin approximations

Let $\{\mathbf{w}^1, \mathbf{w}^2, \dots\}$ be a basis of the Hilbert space $\mathbf{V}$. For instance, we can consider the orthogonal base formed by the eigenvectors of the Stokes operator $\mathbf{A}$. Together with (1.5), we can consider the following semi-Galerkin approximated problems:

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}^m \cdot \nabla)\mathbf{y} + \nabla p = \mathbf{v}1_\omega & \text{in } Q, \\ (\mathbf{z}^m(t), \mathbf{w}) + \alpha^2(\nabla\mathbf{z}^m(t), \mathbf{w}) = (\mathbf{y}(t), \mathbf{w}) \ \forall \mathbf{w} \in \mathbf{V}_m, \ \mathbf{z}^m(t) \in \mathbf{V}_m, & t \in (0, T), \\ \nabla \cdot \mathbf{y} = 0, & \text{in } Q, \\ \mathbf{y} = \mathbf{0} & \text{on } \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

where $\mathbf{V}_m$ denotes the space spanned by $\mathbf{w}^1, \dots, \mathbf{w}^m$.

Arguing as in the proof of Theorem 1.2, it is possible to prove a local null controllability result for (5.1). More precisely, for each $m \ge 1$, there exists $\varepsilon_m > 0$ such that, if $\|\mathbf{y}_0\| \le \varepsilon_m$, we can find controls $\mathbf{v}^m$ and associated states $(\mathbf{y}^m, p^m, \mathbf{z}^m)$ satisfying (1.7). Notice that, in view of the equivalence of norms in $\mathbf{V}_m$, the fixed point argument can be applied in this case without any extra regularity assumption on $\mathbf{y}_0$; in other words, Lemma 2.7 is not needed here.

On the other hand, it can also be checked that the maximal ε_m are bounded from below by some positive quantity independent of m and α and the controls $\mathbf{v}^m$ can be found uniformly bounded in $L^\infty(0, T; \mathbf{L}^2(\omega))$. As a consequence, at least for a subsequence, the controls converge weakly-$*$ in that space to a null control for (1.5).

However, it is unknown whether the problems (5.1) are *globally* null-controllable; see below for other considerations concerning global controllability.

5.2. Another strategy: Applying an inverse function theorem

There is another way to prove the local null controllability of (1.5) that relies on *Liusternik's Inverse Function Theorem,* see for instance [1]. This strategy has been introduced in [18] and has been applied successfully to the controllability of many semilinear and nonlinear PDE's. In the framework of (1.5), the argument is as follows:

1. Introduce an appropriate Hilbert space $\mathbf{Y}$ of *state-control pairs* $(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha)$ satisfying (1.7).
2. Introduce a second Hilbert space $\mathbf{Z}$ of right hand sides and initial data and a well-defined mapping $\mathbf{F} : \mathbf{Y} \mapsto \mathbf{Z}$ such that the null controllability of (1.5) with state-controls in $\mathbf{Y}$ is equivalent to the solution of the nonlinear equation
$$\mathbf{F}(\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha) = (\mathbf{0}, \mathbf{y}_0), \quad (\mathbf{y}_\alpha, p_\alpha, \mathbf{z}_\alpha, \pi_\alpha, \mathbf{v}_\alpha) \in \mathbf{Y}. \tag{5.2}$$
3. Prove that $\mathbf{F}$ is C^1 in a neighborhood of $(\mathbf{0}, 0, \mathbf{0}, 0, \mathbf{0})$ and $\mathbf{F}'(\mathbf{0}, 0, \mathbf{0}, 0, \mathbf{0})$ is onto.

Arguing as in [13], all this can be accomplished satisfactorily. As a result, (5.2) can be solved for small initial data $\mathbf{y}_0$ and the local null controllability of (1.5) holds.

5.3. On global controllability properties

It is unknown whether a general global null controllability result holds for (1.5). This is not surprising, since the same question is also open for the Navier−Stokes system.

What can be proved (as well as for the Navier−Stokes system) is the null controllability for large time: for any given $\mathbf{y}_0 \in \mathbf{H}$, there exists $T_* = T_*(\|\mathbf{y}_0\|)$ such that (1.5) can be driven exactly to zero with controls $\mathbf{v}_\alpha$ uniformly bounded in $L^\infty(0, T_*; \mathbf{L}^2(\omega))$.

Indeed, let ε be the constant furnished by Theorem 1.2 corresponding to the time $T = 1$ (for instance). Let us first take $\mathbf{v}_\alpha \equiv \mathbf{0}$. Then, since the solution to (1.3) with $\mathbf{f} = \mathbf{0}$ satisfies $\|\mathbf{y}_\alpha(t)\| \searrow 0$, there exists T_0 (depending on $\|\mathbf{y}_0\|$ but not on α) such that $\|\mathbf{y}_\alpha(T_0)\| \le \varepsilon$. Therefore, there exist controls $\mathbf{v}'_\alpha \in L^\infty(T_0, T_0 + 1; \mathbf{L}^2(\omega))$ such that the solution to (1.5) that starts from $\mathbf{y}_\alpha(T_0)$ at time T_0 satisfies $\mathbf{y}_\alpha(T_0 + 1) = \mathbf{0}$. Hence, the assertion is fulfilled with $T_* = T_0 + 1$ and
$$\mathbf{v}_\alpha = \begin{cases} \mathbf{0} & \text{for} \quad 0 \le t < T_0, \\ \mathbf{v}'_\alpha & \text{for} \quad T_0 \le t \le T_*. \end{cases}$$

A similar argument leads to the null controllability of (1.5) for large α. In other words, it is also true that, for any given $\mathbf{y}_0 \in \mathbf{H}$ and $T > 0$, there exists $\alpha_* = \alpha_*(\|\mathbf{y}_0\|, T)$ such that, if $\alpha \ge \alpha_*$, then (1.5) can be driven exactly to zero at time T.

5.4. The Burgers-α system

There exist similar results for a regularized version of the Burgers equation, more precisely the Burgers-α system
$$\begin{cases} y_t - y_{xx} + z y_x = v 1_{(a,b)} & \text{in} \quad (0, L) \times (0, T), \\ z - \alpha^2 z_{xx} = y & \text{in} \quad (0, L) \times (0, T), \\ y(0, t) = y(L, t) = z(0, t) = z(L, t) = 0 & \text{on} \quad (0, T), \\ y(x, 0) = y_0(x) & \text{in} \quad (0, L). \end{cases} \tag{5.3}$$

These have been proved in [2].

This system can be viewed as a toy or preliminary model of (1.5). There are, however, several important differences between (1.5) and (5.3):

- The solution to (5.3) satisfies a maximum principle that provides a useful L^∞-estimate.
- There is no apparent energy decay for the uncontrolled solutions. As a consequence, the large time null controllability of (5.3) is unknown.

- It is known that, in the limit $\alpha = 0$, *i.e.* for the Burgers equation, global null controllability does not hold; consequently, in general, the null controllability of (5.3) with controls bounded independently of α is impossible.

We refer to [2] for further details.

5.5. Local exact controllability to the trajectories

It makes sense to consider not only null controllability but also *exact to the trajectories* controllability problems for (1.5). More precisely, let $\hat{\mathbf{y}}_0 \in \mathbf{H}$ be given and let $(\hat{\mathbf{y}}, \hat{p}, \hat{\mathbf{z}}, \hat{\pi})$ a sufficiently regular solution to (1.3) for $\mathbf{f} \equiv \mathbf{0}$ and $\mathbf{y}_0 = \hat{\mathbf{y}}_0$. Then the question is whether, for any given $\mathbf{y}_0 \in \mathbf{H}$, there exist controls $\mathbf{v}$ such that the associated states, *i.e.* the associated solutions to (1.5), satisfy

$$\mathbf{y}(T) = \hat{\mathbf{y}}(T) \text{ in } \Omega.$$

The change of variables

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{u},\ \mathbf{z} = \hat{\mathbf{z}} + \mathbf{w},$$

allows to rewrite this problem as the null controllability of a system similar, but not identical, to (1.5). It is thus reasonable to expect that a local result holds.

5.6. Controlling with few scalar controls

The local null controllability with $N-1$ or even less scalar controls is also an interesting question.

In view of the achievements in [3] and [9] for the Navier−Stokes equations, it is reasonable to expect that results similar to Theorems 1.2 and 1.4 hold with controls $\mathbf{v}$ such that $v_i \equiv 0$ for some i; under some geometrical restrictions, it is also expectable that local exact controllability to the trajectories holds with controls of the same kind, see [14].

5.7. Other related controllability problems

There are many other interesting questions concerning the controllability of (1.5) and related systems.

For instance, we can consider questions like those above for the Leray-α equations completed with other boundary conditions: Navier, Fourier or periodic conditions for $\mathbf{y}$ and $\mathbf{z}$, conditions of different kinds on different parts of the boundary, *etc.* We can also consider Boussinesq-α systems, *i.e.* systems of the form

$$\begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + (\mathbf{z}\cdot\nabla)\mathbf{y} + \nabla p = \theta\mathbf{k} + \mathbf{v}1_\omega & \text{in} \quad Q, \\ \theta_t - \Delta\theta + \mathbf{z}\cdot\nabla\theta = \mathbf{w}1_\omega & \text{in} \quad Q, \\ \mathbf{z} - \alpha^2\Delta\mathbf{z} + \nabla\pi = \mathbf{y} & \text{in} \quad Q, \\ \nabla\cdot\mathbf{y} = 0,\ \nabla\cdot\mathbf{z} = 0 & \text{in} \quad Q, \\ \mathbf{y} = \mathbf{z} = \mathbf{0},\ \theta = 0 & \text{on} \quad \Sigma, \\ \mathbf{y}(0) = \mathbf{y}_0,\ \theta(0) = \theta_0 & \text{in} \quad \Omega. \end{cases}$$

Some of these results will be analyzed in a forthcoming paper.

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