ESAIM: COCV 20 (2014) 517–523
DOI: 10.1051/cocv/2013073

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

A REMARK ON THE COMPACTNESS FOR THE CAHN–HILLIARD FUNCTIONAL

Giovanni Leoni[1]

Abstract. In this note we prove compactness for the Cahn–Hilliard functional without assuming coercivity of the multi-well potential.

Mathematics Subject Classification. 49J45, 26B30.

Received July 29, 2013.
Published online March 27, 2014.

1. Introduction

The purpose of this note is to prove compactness for the Cahn–Hilliard functional (see [5, 8, 9]) without assuming coercivity of the multi-well potential W. Precisely, for $\varepsilon > 0$ consider the functional

$$F_\varepsilon : W^{1,2}\left(\Omega;\mathbb{R}^d\right) \to [0,\infty]$$

defined by

$$F_\varepsilon\left(u\right) := \int_\Omega \left(\frac{1}{\varepsilon}W(u) + \varepsilon|\nabla u|^2\right)\,\mathrm{d}x,$$

where $d \geq 1$ and the potential W satisfies the following hypotheses:

(H_1) $W : \mathbb{R}^d \to [0,\infty)$ is continuous, $W\left(z\right) = 0$ if and only if $z \in \{\alpha_1, \ldots, \alpha_\ell\}$ for some $\alpha_i \in \mathbb{R}^d$, $i = 1, \ldots, \ell$, with $\alpha_i \neq \alpha_j$ for $i \neq j$.

(H_2) There exists $L > 0$ such that

$$\inf_{|z|\geq L} W\left(z\right) > 0.$$

Then the following result holds.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be an open bounded connected set with Lipschitz boundary. Assume that the multi-well potential W satisfies conditions (H_1) and (H_2). Let $\varepsilon_n \to 0^+$ and let $\{u_n\} \subset W^{1,2}\left(\Omega;\mathbb{R}^d\right)$ be such that*

$$M := \sup_n F_{\varepsilon_n}\left(u_n\right) < \infty \tag{1.1}$$

Keywords and phrases. Singular perturbations, gamma-convergence, compactness.

[1] Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, 15213 PA, USA. giovanni@andrew.cmu.edu

Article published by EDP Sciences

and

$$\frac{1}{|\Omega|}\int_{\Omega} u_n(x)\,\mathrm{d}x = m \quad \text{for all } n\in\mathbb{N} \tag{1.2}$$

and for some $m\in\mathbb{R}^d$. Then there exist $u\in BV(\Omega;\{\alpha_1,\dots,\alpha_\ell\})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k}\to u \text{ in } L^1\left(\Omega;\mathbb{R}^d\right).$$

For a two-well potential ($\ell=2$), Theorem 1.1 has been proved in the scalar case $d=1$ by Modica [8] under the assumption

$$\frac{1}{C}|z|^p \le W(z)\le C|z|^p$$

for all $|z|$ large and for some $p>2$, and by Sternberg [9] for $p\ge 2$; while in the vectorial case $d\ge 2$, it has been proved by Fonseca and Tartar [4] under the assumption

$$\frac{1}{C}|z|\le W(z)$$

for all $|z|$ large. The case of a multi-well potential $\ell\ge 3$ has been studied by Baldo (see Props. 4.1 and 4.2 in [2]), who proved compactness of a sequence of minimizers bounded in $L^\infty(\Omega)$.

An example of a double-well potential satisfying (H_1) and (H_2) with $d=1$ but not coercive is

$$W(z)=\arctan\left[(z-\alpha)^2(z-\beta)^2\right],$$

while an example of a potential satisfying (H_1) but not (H_2) is

$$W(z)=(z-\alpha)^2(z-\beta)^2\,\mathrm{e}^{-|z|^2}.$$

In the one dimensional case $N=1$, the hypothesis (1.2) is not needed. Indeed, we have the following elementary result.

Theorem 1.2. *Assume that the multi-well potential W satisfies conditions (H_1) and (H_2). Let $\varepsilon_n\to 0^+$ and let $\{u_n\}\subset W^{1,2}\left((a,b);\mathbb{R}^d\right)$ be such that (1.1) holds. Then there exist $u\in BV((a,b);\{\alpha_1,\dots,\alpha_\ell\})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that*

$$u_{n_k}\to u \text{ in } L^1\left((a,b);\mathbb{R}^d\right).$$

On the other hand, when (1.2) holds, then condition (H_2) can be weakened to:

(H_3) $\int_0^\infty \sqrt{V(s)}\,\mathrm{d}s=\infty$, where for every $s\ge 0$,

$$V(s):=\min_{|z|=s} W(z). \tag{1.3}$$

Note that (H_2) implies that $\sqrt{V(s)}\ge \inf_{|z|\ge L}\sqrt{W(z)}>0$ for all $s\ge L$, and so (H_3) is satisfied. On the other hand, if

$$W(z)\sim\frac{c}{|z|^q}$$

as $|z|\to\infty$ for some $c>0$ and $0<q\le 2$, then (H_3) holds but not (H_1).

Theorem 1.3. *Assume that the multi-well potential W satisfies conditions (H_1) and (H_3). Let $\varepsilon_n\to 0^+$ and let $\{u_n\}\subset W^{1,2}\left((a,b);\mathbb{R}^d\right)$ be such that (1.1) and (1.2) hold. Then there exist $u\in BV((a,b);\{\alpha_1,\dots,\alpha_\ell\})$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and such that*

$$u_{n_k}\to u \text{ in } L^1\left((a,b);\mathbb{R}^d\right).$$

The next simple example shows that compactness fails without (1.2) or (H_2).

Example 1.4. If condition (H_2) does not hold, then there exists $\{z_n\} \subset \mathbb{R}^d$ such that $|z_n| \to \infty$ and

$$\lim_{n\to\infty} W(z_n) = 0.$$

Find a sequence $\varepsilon_n \to 0$ such that

$$\frac{1}{\varepsilon_n} W(z_n) \to 0,$$

(*e.g.* $\varepsilon_n := \sqrt{W(z_n)}$) and consider the sequence of functions $u_n(x) :\equiv z_n$. Then

$$F_{\varepsilon_n}(u_n) = \frac{1}{\varepsilon_n} W(z_n)(b-a) \to 0$$

but no subsequence of $\{u_n\}$ converge in $L^1((a,b))$.

Remark 1.5. I have not been able to determine if Theorems 1.2 and 1.3 hold in dimension $N \geq 2$ or if (H_3) is needed in Theorem 1.3.

2. Proof of Theorems 1.1 and 1.2

The proof of Theorem 1.1 will make use of the following auxiliary results. For a proof of the following theorem see, *e.g.*, Proposition 16.21 in [6].

Theorem 2.1. *Let $u \in W^{1,1}(\mathbb{R}^N)$, $N \geq 2$. Then*

$$\sup_{s>0} s\left[\mathcal{L}^N\left(\left\{x \in \mathbb{R}^N : |u(x)| \geq s\right\}\right)\right]^{\frac{N-1}{N}} \leq \frac{1}{\alpha_N^{1/N}} \int_{\mathbb{R}^N} |\nabla u(x)|\, \mathrm{d}x.$$

For a proof of the next theorem, see Lemma 2.6 in [1].

Theorem 2.2. *Let $A, \Omega \subset \mathbb{R}^N$ be open sets and let $1 \leq p < \infty$. Assume that A is bounded and that Ω is connected and has Lipschitz boundary at each point of $\partial\Omega \cap \overline{A}$. Then there exists a linear and continuous operator $T : W^{1,p}(\Omega) \to W^{1,p}(A)$ such that, for every $u \in W^{1,p}(\Omega)$,*

$$T(u)(x) = u(x) \quad \textit{for } \mathcal{L}^N \textit{ a.e. } x \in \Omega \cap A,$$
$$\int_A |T(u)(x)|^p\, \mathrm{d}x \leq C \int_\Omega |u(x)|^p\, \mathrm{d}x,$$
$$\int_A |\nabla T(u)(x)|^p\, \mathrm{d}x \leq C \int_\Omega |\nabla u(x)|^p\, \mathrm{d}x,$$

where $C > 0$ depends only on N, p, A, and Ω.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. In view of (1.1) and (H_2) for every $n \in \mathbb{N}$, we have

$$M \geq \frac{1}{2}\int_\Omega \sqrt{W(u_n(x))}|\nabla u_n(x)|\, \mathrm{d}x \tag{2.1}$$
$$\geq c\int_{\{|u_n|\geq L\}} |\nabla u_n(x)|\, \mathrm{d}x,$$

where $c := \frac{1}{2}\inf_{|z|\geq L}\sqrt{W(z)} > 0$. Construct a C^1 function $f : \mathbb{R}^d \to \mathbb{R}^d$ such that $f(z) = z$ if $|z| \geq 2L$ and $f(z) = 0$ if $|z| < L$. By the chain rule, for every $n \in \mathbb{N}$ the function $v_n := f \circ u_n$ belongs to $W^{1,2}(\Omega;\mathbb{R}^d)$ and for all $i = 1, \dots, N$ and for $\mathcal{L}^N$-a.e. $x \in \Omega$,

$$\frac{\partial v_n}{\partial x_i}(x) = \sum_{j=1}^{d} \frac{\partial f}{\partial z^{(j)}}(u_n(x)) \frac{\partial (u_n)^{(j)}}{\partial x_i}(x),$$

where we write $z = \left(z^{(1)}, \dots, z^{(d)}\right)$. Since $\frac{\partial f}{\partial z^{(j)}}(z) = 0$ if $|z| < L$, it follows that

$$\begin{aligned}\int_\Omega |\nabla v_n(x)|\,\mathrm{d}x &= \int_{\{|u_n|\geq L\}} |\nabla v_n(x)|\,\mathrm{d}x \\ &\leq \operatorname{Lip} f \int_{\{|u_n|\geq L\}} |\nabla u_n(x)|\,\mathrm{d}x \leq c^{-1} M \operatorname{Lip} f.\end{aligned} \tag{2.2}$$

Let $r > 0$ be so large that $\overline{\Omega} \subset B(0,r)$ and set $A := B(0,2r)$. By Theorem 2.2 we may extend each function v_n to a function in $W^{1,1}\left(A;\mathbb{R}^d\right)$, still denoted v_n, in such a way that

$$\int_A |v_n(x)|\,\mathrm{d}x \leq C \int_\Omega |v_n(x)|\,\mathrm{d}x, \tag{2.3}$$

$$\int_A |\nabla v_n(x)|\,\mathrm{d}x \leq C \int_\Omega |\nabla v_n(x)|\,\mathrm{d}x \leq C c^{-1} M \operatorname{Lip} f, \tag{2.4}$$

where C depends only on r, N, and Ω. By the Poincaré inequality,

$$\int_A |v_n(x) - c_n|\,\mathrm{d}x \leq C \int_A |\nabla v_n(x)|\,\mathrm{d}x, \tag{2.5}$$

where $c_n := \frac{1}{|\Omega|}\int_\Omega v_n(x)\,\mathrm{d}x$ and again C depends only on r, N, and Ω. Note that, since $f(z) = z$ if $|z| \geq 2L$,

$$\begin{aligned}|c_n| &= \frac{1}{|\Omega|}\left|\int_\Omega f \circ u_n\,\mathrm{d}x\right| = \frac{1}{|\Omega|}\left|\int_{\{|u_n|>2L\}} u_n\,\mathrm{d}x + \int_{\{|u_n|\leq 2L\}} f \circ u_n\,\mathrm{d}x\right| \\ &= \left| m + \frac{1}{|\Omega|}\int_{\{|u_n|\leq 2L\}} (f \circ u_n - u_n)\,\mathrm{d}x\right| \leq |m| + 4L.\end{aligned}$$

Consider a cut-off function $\varphi \in C_c^\infty(A;[0,1])$ such that $\varphi = 1$ in $B(0,r)$ and define

$$w_n := \varphi(v_n - c_n).$$

Then $w_n \in W^{1,1}\left(\mathbb{R}^N\right)$ and by (2.5),

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla w_n(x)|\,\mathrm{d}x &\leq \operatorname{Lip}\varphi \int_A |v_n - c_n|\,\mathrm{d}x + \int_A |\nabla v_n(x)|\,\mathrm{d}x \\ &\leq (C \operatorname{Lip}\varphi + 1)\int_A |\nabla v_n(x)|\,\mathrm{d}x.\end{aligned} \tag{2.6}$$

Applying Theorem 2.1 to $|w_n|$, we obtain

$$\begin{aligned}\sup_{s>0} s\left[\mathcal{L}^N\left(\left\{x \in \mathbb{R}^N : |w_n|(x) \geq s\right\}\right)\right]^{\frac{N-1}{N}} &\leq \frac{1}{\alpha_N^{1/N}} \int_{\mathbb{R}^N} |\nabla |w_n|(x)|\,\mathrm{d}x \\ &\leq C_1 \int_{\{|u_n|\geq L\}} |\nabla u_n(x)|\,\mathrm{d}x \leq C_2,\end{aligned}$$

where we have used (2.2), (2.4), and (2.6).

Fix $s_1 > 2(|m| + 4L) + 1$. Using the facts that $\varphi = 1$ in $B(0,r)$, that $f(z) = z$ if $|z| \geq 2L$, and that $|c_n| \leq |m| + 4L$, for $s \geq s_1$ we have

$$\{x \in \Omega : |u_n(x)| \geq s\} = \{x \in \Omega : |v_n(x)| \geq s\} \subset \left\{x \in \Omega : |v_n(x) - c_n| \geq \frac{s}{2}\right\}$$
$$\subset \left\{x \in \mathbb{R}^N : |w_n(x)| \geq s\right\},$$

and so

$$\mathcal{L}^N(\{x \in \Omega : |u_n(x)| \geq s\}) \leq \frac{C}{s^{\frac{N}{N-1}}}$$

for all $s \geq s_1$. Hence,

$$\int_{\{|u_n|>s_1\}} |u_n(x)| \, \mathrm{d}x = \int_{s_1}^{\infty} \mathcal{L}^N \{x \in \Omega : |u_n(x)| \geq s\} \, \mathrm{d}s$$
$$\leq C \int_{s_1}^{\infty} \frac{1}{s^{\frac{N}{N-1}}} \, \mathrm{d}s = \frac{N-1}{s_{(1)}^{\frac{1}{N-1}}},$$

which shows that $\{u_n\}$ is bounded in $L^1(\Omega; \mathbb{R}^d)$ and equi-integrable.

In view of Vitali's convergence theorem, it remains to show that a subsequence converges in measure to some function $u \in BV(\Omega; \{\alpha_1, \ldots, \alpha_\ell\})$. This is classical (see *e.g.* [2] or [4]). □

Remark 2.3. Theorem 1.1 continues to hold if in place of (1.2) we assume that

$$u_n = g \quad \text{on } \partial\Omega \tag{2.7}$$

for all $n \in \mathbb{N}$ and for some function $g \in L^1(\partial\Omega; \{\alpha_1, \ldots, \alpha_\ell\})$. In this case, by Gagliardo's trace theorem (see, *e.g.* Thm. 15.10 in [6]) there exists a function $w \in W^{1,1}(\mathbb{R}^N \setminus \Omega; \mathbb{R}^d)$ such that $w = g$ on $\partial\Omega$. Extend each u_n to be w outside Ω. We can now apply Theorem 2.1 directly to $f \circ u_n \in W^{1,1}(\mathbb{R}^N; \mathbb{R}^d)$ without introducing the constants c_n, the function φ, and without using Theorem 2.2.

We now turn to the Proof of Theorem 1.2. The following argument is likely well-known. We present it here for the convenience of the reader.

Proof of Theorem 1.2. Without loss of generality, we can assume that each function u_n is absolutely continuous.

Since the set $A_n := \{x \in (a,b) : |u_n(x)| > L\}$ is open, we may write it as

$$A_n = \bigcup_k (a_{k,n}, b_{k,n}).$$

Moreover, by (1.1) and (H_2), for every $n \in \mathbb{N}$, we have

$$M\varepsilon_n \geq \int_a^b W(u_n(x)) \, \mathrm{d}x \geq |A_n| \inf_{|z| \geq L} W(z),$$

and so its complement $(a,b) \setminus A_n$ is nonempty for all n sufficiently large. Fix any such n. If A_n is empty, then $|u_n(x)| \leq L$ for all $x \in (a,b)$. Otherwise, let $x \in (a_{k,n}, b_{k,n})$. Then at least one of the endpoints, say $a_{k,n}$, is not an endpoint of (a,b) and so $|u_n(a_{k,n})| = L$. By the fundamental theorem of calculus,

$$u_n(x) = u_n(a_{k,n}) + \int_{a_{k,n}}^{x} u'(t) \, \mathrm{d}t.$$

Hence,

$$\sup_{x \in (a_{k,n}, b_{k,n})} |u_n(x)| \leq L + \int_{\{|u_n| \geq L\}} |u_n'(t)| \, \mathrm{d}t \leq L + c^{-1}M,$$

where we have used (2.1). This shows that $\{u_n\}$ is bounded in $L^\infty((a,b); \mathbb{R}^d)$. We can now continue as in Lemma 6.2 in [3]. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. Without loss of generality, we can assume that each function u_n is absolutely continuous. In view of (1.1) and (1.3), for every $n \in \mathbb{N}$ we have

$$M \geq \frac{1}{2}\int_a^b \sqrt{W\left(u_n\left(x\right)\right)}|u_n'\left(x\right)|\,\mathrm{d}x \geq \frac{1}{2}\int_a^b \sqrt{V\left(|u_n|\left(x\right)\right)|}\,|u_n|'\left(x\right)|\,\mathrm{d}x.$$

Using the area formula for absolutely continuous functions (see, *e.g.*, Thm. 3.65 in [6]), we obtain

$$\begin{aligned} M &\geq \frac{1}{2}\int_a^b \sqrt{V\left(|u_n|\left(x\right)\right)|}\,|u_n|'\left(x\right)|\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}} \sqrt{V\left(s\right)}\,\mathrm{card}\,|u_n|^{-1}\left(\{s\}\right)\,\mathrm{d}s \\ &\geq \frac{1}{2}\int_{\min|u_n|}^{\max|u_n|} \sqrt{V\left(s\right)}\,\mathrm{d}s, \end{aligned}$$

where card is the cardinality and $|u_n|^{-1}\left(\{s\}\right) = \{x \in (a, b : |u_n\left(x\right)| = s)\}$. By (1.2) and the intermediate value theorem, there exists $x_n \in (a, b)$ such that

$$u_n\left(x_n\right) = \frac{1}{b-a}\int_a^b u_n\left(x\right)\,\mathrm{d}x = \frac{m}{b-a}\cdot$$

Hence, $|u_n\left(x_n\right)| = \frac{|m|}{b-a}$, which implies that

$$M \geq \frac{1}{2}\int_{\frac{|m|}{b-a}}^{\max|u_n|} \sqrt{V\left(s\right)}\,\mathrm{d}s.$$

By (H_3) there exists $R > 0$ such that $\int_{\frac{|m|}{b-a}}^{R} \sqrt{V\left(s\right)}\,\mathrm{d}s > 2M$. In turn, $|u_n\left(x\right)| < R$ for all $x \in (a, b)$ and all $n \in \mathbb{N}$. This shows that $\{u_n\}$ is bounded in $L^\infty\left((a, b)\,; \mathbb{R}^d\right)$. □

Remark 2.4. Observe that in Theorems 1.2 and 1.3 we can replace (H_1) with the weaker hypothesis

(H_4) $W : \mathbb{R}^d \to [0, \infty)$ is continuous and for every $r > 0$ the set

$$\{z \in B\left(0, r\right) :\, W\left(z\right) = 0\}$$

has finitely many elements.

Indeed, if $\{u_n\} \subset W^{1,2}\left((a, b)\,; \mathbb{R}^d\right)$ is such that (1.1) holds, then by Theorem 1.2 or 1.3, there exists $R > 0$ such that $|u_n\left(x\right)| < R$ for all $x \in (a, b)$ and all $n \in \mathbb{N}$. Find $S \in (R, 2R)$ such that $V\left(S\right) > 0$. Note that such S exists, since otherwise we would have $V\left(s\right) = 0$ for all $s \in (R, 2R)$, which would imply that $\{z \in B\left(0, 2R\right) :\, W\left(z\right) = 0\}$ has infinitely many elements and would contradict (H_4). Define

$$W_1\left(z\right) := \begin{cases} W\left(z\right) & \text{if } |z| < S, \\ W\left(\frac{z}{|z|}S\right) & \text{if } |z| \geq S. \end{cases}$$

Since $|u_n\left(x\right)| < R < S$ for all $x \in (a, b)$ and all $n \in \mathbb{N}$, we have that

$$M \geq F_{\varepsilon_n}\left(u_n\right) = \int_a^b \left(\frac{1}{\varepsilon_n}W_1(u_n) + \varepsilon_n|u_n'|^2\right)\,\mathrm{d}x.$$

The function W_1 satisfies hypotheses (H_1) and (H_2). Hence, we may now apply Theorem 1.2 to find $u \in BV\left((a, b)\,; \{\alpha_1, \ldots, \alpha_\ell\}\right)$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \to u \text{ in } L^1\left((a, b)\,; \mathbb{R}^d\right).$$

Here $\{\alpha_1, \ldots, \alpha_\ell\}$ are the zeros of W in $B\left(0, s\right)$.

In view of the previous remark, we can prove a compactness result for $N = 1$ and bounded domains for the functional studied in the classical paper of Modica and Mortola [7].

Corollary 2.5. *Let* $\varepsilon_n \to 0^+$ *and let* $\{u_n\} \subset W^{1,2}\left((a,b);\mathbb{R}^d\right)$ *be such that*

$$\int_a^b \left(\frac{1}{\varepsilon_n}\sin^2\left(\pi u_n\right) + \varepsilon_n |u_n'\left(x\right)|^2\right) \mathrm{d}x \leq M$$

and (1.2) *hold. Then there exist* $u \in BV\left((a,b);\{\alpha_1,\ldots,\alpha_\ell\}\right)$ *and a subsequence* $\{u_{n_k}\}$ *of* $\{u_n\}$ *such that*

$$u_{n_k} \to u \text{ in } L^1\left(a,b\right).$$

Here, $\{\alpha_1,\ldots,\alpha_\ell\} \subset \mathbb{Z}$.

Proof. It is enough to observe that the function $W\left(z\right) = \sin^2\left(\pi z\right)$ satisfies (H_3) and (H_4). □

Remark 2.6. I am not aware of any compactness result for $N \geq 2$ for the functional

$$\int_\Omega \left(\frac{1}{\varepsilon}\sin^2\left(\pi u\right) + \varepsilon|\nabla u|^2\right) \mathrm{d}x,$$

when (1.2) holds. Note that $W\left(z\right) = \sin^2\left(\pi z\right)$ satisfies (H_3) and (H_4) but not (H_1) and (H_2).

Acknowledgements. The research of G. Leoni was partially funded by the National Science Foundation under Grant No. DMS-1007989. G. Leoni also acknowledges the Center for Nonlinear Analysis (NSF Grant No. DMS-0635983, PIRE Grant No. OISE-0967140), where part of this research was carried out. The author wishes to thank Massimiliano Morini for useful conversations on the subject of this paper.

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