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MULTIPLICITY AND CONCENTRATION BEHAVIOR OF POSITIVE SOLUTIONS FOR A SCHRÖDINGER–KIRCHHOFF TYPE PROBLEM *VIA* PENALIZATION METHOD *, **

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Abstract. In this paper we are concerned with questions of multiplicity and concentration behavior of positive solutions of the elliptic problem

$$(P_\varepsilon) \qquad \begin{cases} \mathcal{L}_\varepsilon u = f(u) \ \text{ in } \ \mathbb{R}^3, \\ u > 0 \ \text{ in } \ \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where ε is a small positive parameter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathcal{L}_\varepsilon$ is a nonlocal operator defined by

$$\mathcal{L}_\varepsilon u = M\left(\frac{1}{\varepsilon}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{\varepsilon^3}\int_{\mathbb{R}^3} V(x)u^2\right)\left[-\varepsilon^2 \Delta u + V(x)u\right],$$

$M : \mathbb{R}_+ \to \mathbb{R}_+$ and $V : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions which verify some hypotheses.

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1. Introduction

In this paper we shall focus our attention on questions of multiplicity, concentration behavior and positivity of solutions for the following problem

$$(P_\varepsilon) \qquad \begin{cases} \mathcal{L}_\varepsilon u = f(u) \ \text{ in } \ \mathbb{R}^3, \\ u > 0 \ \text{ in } \ \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

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where ε is a small positive parameter, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mathcal{L}_\varepsilon$ is a nonlocal operator defined by

$$\mathcal{L}_\varepsilon u = M\left(\frac{1}{\varepsilon}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{\varepsilon^3}\int_{\mathbb{R}^3} V(x)u^2\right)\left[-\varepsilon^2 \Delta u + V(x)u\right],$$

$M : \mathbb{R}_+ \to \mathbb{R}_+$ and $V : \mathbb{R}^3 \to \mathbb{R}$ are continuous functions that satisfy some conditions which will be stated later on.

Problem (P_ε) is a natural extension of two classes of very important problems in applications, namely, Kirchhoff problems and Schrödinger problems.

a) When $\varepsilon = 1$ and $V = 0$ we are dealing with problem

$$\begin{cases} -M\left(\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(u) \ \text{ in } \ \mathbb{R}^3, \\ \qquad\qquad u > 0 \ \text{ in } \ \mathbb{R}^3, u \in H^1(\mathbb{R}^3), \end{cases}$$

which represents the stationary case of Kirchhoff model [17] for small transverse vibrations of an elastic string by considering the effect of the changes in the length during the vibrations.

In fact, since the length of string is variable during the vibrations, the tension in the string changes with time and depends of the L^2 norm of the gradient of the displacement u. More precisely, we have

$$M(t) = \frac{P_0}{h} + \frac{E}{2L}t, \quad t > 0,$$

where L is the length of the string, h is the area of cross-section, E is the Young modulus of the material and P_0 is the initial tension.

Moreover, problem (P_ε) is catch nonlocal because of the presence of the term $M\left(\int_{\mathbb{R}^3} |\nabla u|^2\right)$ which implies that the equation in (P_ε) is no longer a pointwise identity. This phenomenon causes some mathematical difficulties which makes the study of such a class of problem particularly interesting.

The version of problem (P_ε) in bounded domain began to call attention of several researchers especially after the work of Lions [20], where a functional analysis approach was proposed to attack it.

We have to point out that nonlocal problems also appear in other fields as, for example, biological systems where u describes a process which depends on the average of itself (for example, population density). See, for example, [3] and its references.

The reader may consult [1–3, 9, 10, 14, 21] and the references therein, for more informations on nonlocal problems.

b) On the other hand, when $M = 1$ we have the problem

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = f(u) \ \text{ in } \ \mathbb{R}^3, \\ \qquad u > 0 \ \text{ in } \ \mathbb{R}^3, u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.1}$$

which appear in different models, for example, it is related to the existence of standing waves of the nonlinear Schrodinger equation

$$i\varepsilon\frac{\partial \Psi}{\partial t} = -\varepsilon \Delta \Psi + (V(x) + E)\Psi - f(\Psi), \ \forall x \in \mathbb{R}^N, \tag{1.2}$$

where $f(t) = |t|^{p-2}u$ and $2 < p < 2^* = \dfrac{2N}{N-2}$. A standing wave of (1.2) is a solution of the form $\Psi(x,t) = \exp(-iEt/\varepsilon)u(x)$. In this case, u is a solution of (1.1). Existence and concentration of positive solutions for the problem (1.1) have been extensively studied in recent years, see for example the papers [7, 8, 11, 12, 15, 24] and their references.

A considerable effort has been devoted during the last years in studying problems of the type (P_ε), as can be seen in [4, 16, 18, 23, 27, 29] and references therein. This is due to their significance in applications as well as to their mathematical relevance.

Before stating our main result, we need the following hypotheses on the function M:

(M_1) There is $m_0 > 0$ such that $M(t) \geq m_0,\ \forall t \geq 0$.
(M_2) The function $t \mapsto M(t)$ is increasing.
(M_3) For each $t_1 \geq t_2 > 0$,

$$\frac{M(t_1)}{t_1} - \frac{M(t_2)}{t_2} \leq m_0 \left(\frac{1}{t_1} - \frac{1}{t_2}\right),$$

where m_0 is given in (M_1).

The potential V is a continuous function satisfying:

(V_1) There is $V_0 > 0$ such that $V_0 = \inf_{x \in \mathbb{R}^3} V(x)$.
(V_2) For each $\delta > 0$ there is a bounded and Lipschitz domain $\Omega \subset \mathbb{R}^3$ such that

$$V_0 < \min_{\partial\Omega} V, \quad \Pi = \{x \in \Omega : V(x) = V_0\} \neq \emptyset$$

and

$$\Pi_\delta = \{x \in \mathbb{R}^3 : \mathrm{dist}(x, \Pi) \leq \delta\} \subset \Omega.$$

Moreover, we assume that the continuous function f vanishes in $(-\infty, 0)$ and verifies

(f_1)

$$\lim_{t \to 0^+} \frac{f(t)}{t^3} = 0\cdot$$

(f_2) There is $q \in (4, 6)$ such that

$$\lim_{t \to \infty} \frac{f(t)}{t^{q-1}} = 0\cdot$$

(f_3) There is $\theta \in (4, 6)$ such that

$$0 < \theta F(t) \leq f(t)t,\ \forall t > 0.$$

(f_4) The application

$$t \mapsto \frac{f(t)}{t^3}$$

is non-decreasing in $(0, \infty)$.

The main result of this paper is:

Theorem 1.1. *Suppose that the function M satisfies (M_1)–(M_3), the potential V satisfies (V_1)–(V_2) and the function f satisfies (f_1)–(f_4). Then, given $\delta > 0$ there is $\overline{\varepsilon} = \overline{\varepsilon}(\delta) > 0$ such that the problem (P_ε) has at least $\mathrm{Cat}_{\Pi_\delta}(\Pi)$ positive solutions, for all $\varepsilon \in (0, \overline{\varepsilon})$. Moreover, if u_ε denotes one of these positive solutions and $\eta_\varepsilon \in R^3$ its global maximum, then*

$$\lim_{\varepsilon \to 0} V(\eta_\varepsilon) = V_0.$$

A typical example of function verifying the assumptions (M_1)–(M_3) is given by $M(t) = m_0 + bt$, where $m_0 > 0$ and $b > 0$. More generally, any function of the form $M(t) = m_0 + bt + \sum_{i=1}^{k} b_i t^{\gamma_i}$ with $b_i \geq 0$ and $\gamma_i \in (0, 1)$ for all $i \in \{1, 2, \dots, k\}$ verifies the hypotheses (M_1)–(M_3).

A typical example of function verifying the assumptions (f_1)–(f_4) is given by $f(t) = \sum_{i=1}^{k} c_i (t^+)^{q_i - 1}$ with $c_i \geq 0$ not all null and $q_i \in [\theta, 6)$ for all $i \in \{1, 2, \ldots, k\}$.

Recently some authors have considered problems of the type (P_ε). For example, He and Zou [16], by using Lusternik–Schnirelmann theory and minimax methods, proved a result of multiplicity and concentration behavior for the following equation

$$\begin{cases} -(\varepsilon^2 a + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(x)u = f(u) \ \text{ in } \ \mathbb{R}^3 \\ u > 0 \ \text{ in } \ \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \tag{1.3}$$

assuming, between others hypotheses, that $f \in C^1(\mathbb{R})$ has a subcritical growth 3-superlinear and the potential V verifies a assumption introduced by Rabinowitz [24], namely,

$$(R) \qquad V_\infty = \liminf_{|x| \to \infty} V(x) > V_0 = \inf_{\mathbb{R}^3} V(x) > 0.$$

In [27], Wang, Tian, Xu and Zhang have considered the problem

$$\begin{cases} -(\varepsilon^2 a + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2)\Delta u + V(x)u = \lambda f(u) + |u|^4 u \ \text{ in } \ \mathbb{R}^3 \\ u > 0 \ \text{ in } \ \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.4}$$

Assuming that f is only continuous, has subcritical growth 3-superlinear and the potential verifies (R), the authors showed that (1.6) has multiple positive solutions when λ is large enough, by using Lusternik–Schnirelmann theory, minimax methods and a approach as in [26] (see also [25]).

Other results for the problem Schödinger–Kirchhoff type can be seen in [4, 18, 23, 29] and references therein.

Motivated by results found in [4, 12, 16, 27], we study multiplicity via Lusternik–Schnirelmann theory and concentration behavior of solutions for the problem (P_ε). Here we use the hypotheses (V_1)–(V_2) that were first introduced by Del Pino and Felmer [12] for laplacian case. For p-laplacian case, see [5].

We emphasize that, at least in our knowledge, does not exist in the literature actually available results involving problems Schrödinger–Kirchhoff type, where the potential is like that introduced by Del Pino and Felmer [12]. This is a difficulty that occurs, possibly by competition between the growth of nonlocal term and the growth of nonlinearity.

Here, we use the same type of truncation explored in [12], however, we make a new approach and some estimates are totally different, for example, we show that solution of truncated problem is solution of the original problem with distinct arguments.

Moreover, we completed the results found in [4, 16, 27] in the following sense:

1 - Since M is a function more general than those in [16] and [27], we have a additional difficulty. In general, the weak limit of the Palais–Smale sequences is not weak solution of the autonomous problem. We overcome this difficulty with assumptions different from those found in [4].
2 - Since the function f is only continuous, we cannot use standard arguments on the Nehari manifold. Hence, our result is similar then those found in [27]. However, since the hypotheses on function V are different, our arguments are completely different. Moreover, our result is for all positive lambda.

The paper is organized as follows. In the Section 2 we show that the auxiliary problem has a positive solution and we introduce some tools needed for the multiplicity result, namely, Lemma 2.3 and Proposition 2.4. In the Section 3 we study the autonomous problem associated. This study allows us to show that the auxiliary problem has multiple solutions. In the Section 4 we prove the main result using Moser iteration method [22].

2. The auxiliary problem

Considering the change of variable $x = \varepsilon z$ in (P_ε) we obtain the modified problem

$$(\widetilde{P}_\varepsilon) \qquad \begin{cases} \widetilde{\mathcal{L}}_\varepsilon u = f(u) & \text{in } \mathbb{R}^3, \\ u > 0 & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases}$$

where

$$\widetilde{\mathcal{L}}_\varepsilon u = M\left(\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2\right)[-\Delta u + V(\varepsilon x)u],$$

which is clearly equivalent to (P_ε).

Since (f_1) and (f_4) imply that

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 0$$

and

$$t \mapsto \frac{f(t)}{t}$$

is a application increasing in $(0,\infty)$ and unbounded, we can adapt to our case the penalization method introduced by Del Pino and Felmer [12].

For this, let $K > \dfrac{2}{m_0}$, where m_0 is given in (M_1) and $a > 0$ such that $f(a) = \dfrac{V_0}{K}a$. We define

$$\widetilde{f}(t) = \begin{cases} f(t) \text{ if } t \le a, \\ \frac{V_0}{K}t \text{ if } t > a \end{cases}$$

and

$$g(x,t) = \chi_\Omega(x)f(t) + (1-\chi_\Omega(x))\widetilde{f}(t),$$

where χ is characteristic function of set Ω. From hypotheses (f_1)–(f_4) we get that g is a Carathéodory function and the following conditions are observed:

(g_1)

$$\lim_{t\to 0^+} \frac{g(x,t)}{t^3} = 0, \text{ uniformly in } x \in \mathbb{R}^3.$$

(g_2)

$$\lim_{t\to \infty} \frac{g(x,t)}{t^{q-1}} = 0, \text{ uniformly in } x \in \mathbb{R}^3,$$

(g_3) (i)

$$0 \le \theta G(x,t) < g(x,t)t, \ \forall x \in \Omega \text{ and } \forall t > 0$$

and
(ii)

$$0 \le 2G(x,t) \le g(x,t)t \le \frac{1}{K}V_0 t^2, \ \forall x \in \mathbb{R}^3\backslash\Omega \text{ and } \forall t > 0.$$

(g_4) For each $x \in \Omega$, the application $t \mapsto \frac{g(x,t)}{t^3}$ is increasing in $(0,\infty)$ and for each $x \in \mathbb{R}^3\backslash\Omega$, the application $t \mapsto \frac{g(x,t)}{t^3}$ is increasing in $(0,a)$.

Moreover, from definition of g, we have $g(x,t) \leq f(t)$, for all $t \in (0,+\infty)$ and for all $x \in \mathbb{R}^3$, $g(x,t) = 0$ for all $t \in (-\infty, 0)$ and for all $x \in \mathbb{R}^3$.

Now we study the auxiliary problem

$$(P_{\varepsilon,A}) \qquad \begin{cases} \widetilde{\mathcal{L}}_\varepsilon u = g(\varepsilon x, u), \ \mathbb{R}^3 \\ u > 0, \ \mathbb{R}^3 \\ u \in H^1(\mathbb{R}^3). \end{cases}$$

Observe that positive solutions of $(P_{\varepsilon,A})$ with $u(x) \leq a$ for each $x \in \mathbb{R}^3 \backslash \Omega$ are also positive solutions of $(\widetilde{P}_\varepsilon)$.

We obtain solutions of $(P_{\varepsilon,A})$ as critical points of the energy functional

$$J_\varepsilon(u) = \frac{1}{2}\widehat{M}\left(\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2\right) - \int_{\mathbb{R}^3} G(\varepsilon x, u),$$

where $\widehat{M}(t) = \displaystyle\int_0^t M(s)\mathrm{d}s$ and $G(x,t) = \displaystyle\int_0^t g(\varepsilon x, s)\mathrm{d}s$, which is well defined on the Hilbert space H_ε, given by

$$H_\varepsilon = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\varepsilon x)u^2 < \infty\},$$

provided of the inner product

$$(u,v)_\varepsilon = \int_{\mathbb{R}^3} \nabla u \nabla v + \int_{\mathbb{R}^3} V(\varepsilon x)uv.$$

The norm induced by inner product is denoted by

$$\|u\|_\varepsilon^2 = \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V(\varepsilon x)u^2.$$

Since M and f are continuous we have that $J_\varepsilon \in C^1(H_\varepsilon, \mathbb{R})$ and

$$J'_\varepsilon(u)v = M(\|u\|_\varepsilon^2)(u,v)_\varepsilon - \int_{\mathbb{R}^3} g(\varepsilon x, u)v, \ \forall u, v \in H_\varepsilon.$$

Now, we will fix some notations. We denote the Nehari manifold associated to J_ε by

$$\mathcal{N}_\varepsilon = \{u \in H_\varepsilon \backslash \{0\} : J'_\varepsilon(u)u = 0\}$$

and by Ω_ε the set

$$\Omega_\varepsilon = \{x \in \mathbb{R}^3 : \varepsilon x \in \Omega\},$$

by H_ε^+ the subset of H_ε given by

$$H_\varepsilon^+ = \{u \in H_\varepsilon : |\mathrm{supp}\,(u^+) \cap \Omega_\varepsilon| > 0\}$$

and by S_ε^+ the intersection $S_\varepsilon \cap H_\varepsilon^+$, where S_ε is the unit sphere of H_ε.

Lemma 2.1. *The set H_ε^+ is open in H_ε.*

Proof. Suppose by contradiction there are a sequence $\{u_n\} \subset H_\varepsilon \backslash H_\varepsilon^+$ and $u \in H_\varepsilon^+$ such that $u_n \to u$ in H_ε. Hence $|\mathrm{supp}\,(u_n^+) \cap \Omega_\varepsilon| = 0$ for all $n \in \mathbb{N}$ and $u_n^+(x) \to u^+(x)$ a.e. in $x \in \Omega_\varepsilon$. So,

$$u^+(x) = \lim_{n\to\infty} u_n^+(x) = 0, \ \text{a.e in } x \in \Omega_\varepsilon.$$

But, this contradicts the fact that $u \in H_\varepsilon^+$. Therefore H_ε^+ is open. □

From definition of S^+_ε and from Lemma 2.1 it follows that S^+_ε is a incomplete $C^{1,1}$-manifold of codimension 1, modeled on H_ε and contained in the open H^+_ε. Thus, $H_\varepsilon = T_u S^+_\varepsilon \oplus \mathbb{R}\, u$ for each $u \in S^+_\varepsilon$, where $T_u S^+_\varepsilon = \{v \in H_\varepsilon : (u,v)_\varepsilon = 0\}$.

Finally, we mean by weak solution of $(P_{\varepsilon,A})$ a function $u \in H_\varepsilon$ such that

$$M(\|u\|^2_\varepsilon)(u,v)_\varepsilon = \int_{\mathbb{R}^3} g(\varepsilon x, u)v, \ \forall v \in H_\varepsilon.$$

Therefore, critical points of J_ε are weak solutions of $(P_{\varepsilon,A})$.

Lemma 2.2. *The functional J_ε satisfies the following conditions:*

a) *There are $\alpha, \rho > 0$ such that*

$$J_\varepsilon(u) \geq \alpha, \ \textit{with } \|u\|_\varepsilon = \rho.$$

b) *There is $e \in H_\varepsilon \backslash B_\rho(0)$ with $J_\varepsilon(e) < 0$.*

Proof. The item a) follows directly from the hypotheses (M_1), (g_1) and (g_2).

On the other hand, it follows from (M_3) that there is $\gamma_1 > 0$ such that $M(t) \leq \gamma_1(1+t)$ for all $t \geq 0$. So, for each $u \in H^+_\varepsilon$ and $t > 0$ we have

$$\begin{aligned} J_\varepsilon(tu) &= \frac{1}{2}\widehat{M}(\|tu\|^2_\varepsilon) - \int_{\mathbb{R}^3} G(\varepsilon x, tu) \\ &\leq \frac{\gamma_1}{2} t^2 \|u\|^2_\varepsilon + \frac{\gamma_1}{4} t^4 \|u\|^4_\varepsilon - \int_{\Omega_\varepsilon} G(\varepsilon x, tu). \end{aligned}$$

From $(g_3)(i)$, we obtain $C_1, C_2 > 0$ such that

$$J_\varepsilon(tu) \leq \frac{\gamma_1}{2} t^2 \|u\|^2_\varepsilon + \frac{\gamma_1}{4} t^4 \|u\|^4_\varepsilon - C_1 t^\theta \int_{\Omega_\varepsilon} (u^+)^\theta + C_2 |supp(u^+) \cap \Omega_\varepsilon|.$$

Since $\theta \in (4,6)$ we conclude b). □

Once f and M are only continuous the next two results are very important, because allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ (see Lem. 2.3 (A_3) and Prop. 2.4) and the incompleteness of S^+_ε (see Lem. 2.3 (A_4)).

Lemma 2.3. *Suppose that the function M satisfies (M_1)–(M_3), the potential V satisfies (V_1)–(V_2) and the function f satisfies (f_1)–(f_4). So:*

(A_1) For each $u \in H^+_\varepsilon$, let $h : \mathbb{R}_+ \to \mathbb{R}$ be defined by $h_u(t) = J_\varepsilon(tu)$. Then, there is a unique $t_u > 0$ such that $h'_u(t) > 0$ in $(0, t_u)$ and $h'_u(t) < 0$ in (t_u, ∞).

(A_2) there is $\tau > 0$ independent on u such that $t_u \geq \tau$ for all $u \in S^+_\varepsilon$. Moreover, for each compact set $\mathcal{W} \subset S^+_\varepsilon$ there is $C_\mathcal{W} > 0$ such that $t_u \leq C_\mathcal{W}$, for all $u \in \mathcal{W}$.

(A_3) The map $\widehat{m}_\varepsilon : H^+_\varepsilon \to \mathcal{N}_\varepsilon$ given by $\widehat{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon := \widehat{m}_\varepsilon\big|_{S^+_\varepsilon}$ is a homeomorphism between S^+_ε and $\mathcal{N}_\varepsilon$. Moreover, $m^{-1}_\varepsilon(u) = \frac{u}{\|u\|_\varepsilon}$.

(A_4) If there is a sequence $(u_n) \subset S^+_\varepsilon$ such that $dist(u_n, \partial S^+_\varepsilon) \to 0$, then $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$.

Proof. To prove (A_1), it is sufficient to note that, from the Lemma 2.2, $h_u(0) = 0$, $h_u(t) > 0$ when $t > 0$ is small and $h_u(t) < 0$ when $t > 0$ is large. Since $h_u \in C^1(\mathbb{R}_+, \mathbb{R})$, there is $t_u > 0$ global maximum point of h_u and $h'_u(t_u) = 0$. Thus, $J'_\varepsilon(t_u u)(t_u u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. We see that $t_u > 0$ is the unique positive number such that $h'_u(t_u) = 0$. Indeed, suppose by contradiction that there are $t_1 > t_2 > 0$ with $h'_u(t_1) = h'_u(t_2) = 0$. Then, for $i = 1, 2$

$$t_i M(\|t_i u\|^2_\varepsilon)\|u\|^2_\varepsilon = \int_{\mathbb{R}^3} g(\varepsilon x, t_i u)u.$$

So,

$$\frac{M(\|t_i u\|_\varepsilon^2)}{\|t_i u\|_\varepsilon^2} = \frac{1}{\|u\|_\varepsilon^4}\int_{\mathbb{R}^3}\left[\frac{g(\varepsilon x, t_i u)}{(t_i u)^3}\right]u^4.$$

Therefore,

$$\frac{M(\|t_1 u\|_\varepsilon^2)}{\|t_1 u\|_\varepsilon^2} - \frac{M(\|t_2 u\|_\varepsilon^2)}{\|t_2 u\|_\varepsilon^2} = \frac{1}{\|u\|_\varepsilon^4}\int_{\mathbb{R}^3}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4.$$

It follows from (M_3) and (g_4) that

$$\begin{aligned}\frac{m_0}{\|u\|_\varepsilon^2}\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right) \geq{}& \frac{1}{\|u\|_\varepsilon^4}\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{t_2u\leq a<t_1u\}}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4\\ &+\frac{1}{\|u\|_\varepsilon^4}\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{a<t_2u\}}\left[\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^3} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^3}\right]u^4.\end{aligned}$$

By using the definition of g we obtain

$$\begin{aligned}\frac{m_0}{\|u\|_\varepsilon^2}\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right) \geq{}& \frac{1}{\|u\|_\varepsilon^4}\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{t_2u\leq a<t_1u\}}\left[\frac{V_0}{K}\frac{1}{(t_1 u)^2} - \frac{f(t_2 u)}{(t_2 u)^3}\right]u^4\\ &+\frac{1}{\|u\|_\varepsilon^4}\frac{1}{K}\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right)\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{a<t_2u\}}V_0u^2.\end{aligned}$$

Multiplying both sides by $\frac{\|u\|_\varepsilon^4}{\left(\frac{1}{t_1^2}-\frac{1}{t_2^2}\right)}$ and using the hypothesis $t_1 > t_2$, it follows that

$$\begin{aligned}m_0\|u\|_\varepsilon^2 \leq{}& \frac{t_1^2t_2^2}{t_2^2-t_1^2}\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{t_2u\leq a<t_1u\}}\left[\frac{V_0}{K}\frac{1}{(t_1 u)^2} - \frac{f(t_2 u)}{(t_2 u)^3}\right]u^4\\ &+\frac{1}{K}\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{a<t_2u\}}V_0u^2.\end{aligned}$$

Therefore,

$$\begin{aligned}m_0\|u\|_\varepsilon^2 \leq{}& -\left(\frac{t_2^2}{t_1^2-t_2^2}\right)\frac{1}{K}\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{t_2u\leq a<t_1u\}}V_0u^2\\ &+\left(\frac{t_1^2}{t_1^2-t_2^2}\right)\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{t_2u\leq a<t_1u\}}\frac{f(t_2u)}{t_2u}u^2+\frac{1}{K}\int_{(\mathbb{R}^3\setminus\Omega_\varepsilon)\cap\{a<t_2u\}}V_0u^2.\end{aligned}$$

So,

$$m_0\|u\|_\varepsilon^2 \leq \frac{1}{K}\int_{\mathbb{R}^3\setminus\Omega_\varepsilon}V_0u^2 \leq \frac{1}{K}\|u\|_\varepsilon^2.$$

Since $u \neq 0$, we have that $m_0 \leq \frac{1}{K} < m_0$, but this is a contradiction. Thus, (A_1) is proved.

(A_2) Now, let $u \in S_\varepsilon^+$. From (M_1), (g_1), (g_2) and from the Sobolev embeddings

$$m_0t_u \leq M(t_u^2)t_u = \int_{\mathbb{R}^3}g(\varepsilon x, t_uu)u \leq \frac{\xi}{4}C_1t_u^4 + \frac{C_\xi}{q}C_2t_u^q.$$

From previous inequality we obtain $\tau > 0$, independent on u, such that $t_u \geq \tau$.

Finally, if $\mathcal{W} \subset S_\varepsilon^+$ is compact, suppose by contradiction that there is $\{u_n\} \subset \mathcal{W}$ such that $t_n = t_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there is $u \in \mathcal{W}$ with $u_n \to u$ in H_ε. It follows from the arguments used in the proof of item b) of the Lemma 2.2 that

$$J_\varepsilon(t_nu_n) \to -\infty. \tag{2.1}$$

On the other hand, note that if $v \in \mathcal{N}_\varepsilon$, then by $(g_3)(i)$

$$\begin{aligned} J_\varepsilon(v) &= J_\varepsilon(v) - \frac{1}{\theta} J'_\varepsilon(v)v \\ &\geq \frac{1}{2}\widehat{M}(\|v\|_\varepsilon^2) - \frac{1}{\theta} M(\|v\|_\varepsilon^2)\|v\|_\varepsilon^2 + \frac{1}{\theta}\int_{\mathbb{R}^3\setminus\Omega_\varepsilon} [g(\varepsilon x, v)v - \theta G(\varepsilon x, v)]\,. \end{aligned}$$

From $(g_3)(ii)$ we have

$$J_\varepsilon(v) \geq \frac{1}{2}\widehat{M}(\|v\|_\varepsilon^2) - \frac{1}{\theta} M(\|v\|_\varepsilon^2)\|v\|_\varepsilon^2 - \left(\frac{\theta-2}{2\theta}\right)\frac{1}{K}\int_{\mathbb{R}^3\setminus\Omega_\varepsilon} V(\varepsilon x)v^2,$$

and so

$$J_\varepsilon(v) \geq \frac{1}{2}\widehat{M}(\|v\|_\varepsilon^2) - \frac{1}{\theta} M(\|v\|_\varepsilon^2)\|v\|_\varepsilon^2 - \left(\frac{\theta-2}{2\theta}\right)\frac{1}{K}\|v\|_\varepsilon^2.$$

By using the hypothesis (M_3), we derive $\widehat{M}(t) \geq \dfrac{[M(t)+m_0]}{2}t$, for all $t \geq 0$. Then,

$$J_\varepsilon(v) \geq \left(\frac{\theta-4}{4\theta}\right) M(\|v\|_\varepsilon^2)\|v\|_\varepsilon^2 + \frac{m_0}{4}\|v\|_\varepsilon^2 - \left(\frac{\theta-2}{2\theta}\right)\frac{1}{K}\|v\|_\varepsilon^2.$$

From (M_1), we conclude

$$J_\varepsilon(v) \geq \left(\frac{\theta-2}{2\theta}\right)\left(m_0 - \frac{1}{K}\right)\|v\|_\varepsilon^2.$$

Once $\{t_n u_n\} \subset \mathcal{N}_\varepsilon$ the previous inequality contradicts (2.1). Therefore (A_2) is true.

(A_3) First of all we observe that $\widehat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. In fact, by (A_1), for each $u \in H_\varepsilon^+$, there is a unique $m_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$ then $u \in H_\varepsilon^+$. Otherwise, we have $|supp(u^+) \cap \Omega_\varepsilon| = 0$ and by $(g_3)(ii)$

$$0 < M(\|u\|_\varepsilon^2)\|u\|_\varepsilon^2 = \int_{\mathbb{R}^3} g(\varepsilon x, u)u = \int_{\mathbb{R}^3\setminus\Omega_\varepsilon} g(\varepsilon x, u^+)u^+ \leq \frac{1}{K}\int_{\mathbb{R}^3\setminus\Omega_\varepsilon} V(\varepsilon x)u^2.$$

Hence, from (M_1)

$$0 < \left(m_0 - \frac{1}{K}\right)\|u\|_\varepsilon^2 \leq 0,$$

a contradiction. Consequently $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in S_\varepsilon^+$, m_ε^{-1} is well defined and it is a continuous function. Since,

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{t_u\|u\|_\varepsilon} = u, \ \forall\, u \in S_\varepsilon^+,$$

we conclude that m_ε is a bijection. To show that $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous, let $\{u_n\} \subset H_\varepsilon^+$ and $u \in H_\varepsilon^+$ be such that $u_n \to u$ in H_ε. Thus $u_n/\|u_n\|_\varepsilon \to u/\|u\|_\varepsilon$ in H_ε and from (A_2), there is $t_0 > 0$ such that $t_{(\frac{u_n}{\|u_n\|_\varepsilon})} \to t_0$. Since, $t_{(\frac{u_n}{\|u_n\|_\varepsilon})}(u_n/\|u_n\|_\varepsilon) \in \mathcal{N}_\varepsilon$, we obtain

$$M\left(t^2_{(\frac{u_n}{\|u_n\|_\varepsilon})}\right) t_{(\frac{u_n}{\|u_n\|_\varepsilon})} = \frac{1}{\|u_n\|_\varepsilon}\int_{\mathbb{R}^3} g\left(\varepsilon x, t_{(\frac{u_n}{\|u_n\|_\varepsilon})}\frac{u_n}{\|u_n\|_\varepsilon}\right)u_n, \ \forall\, n \in \mathbb{N}.$$

Passing to the limit $n \to \infty$, it follows that

$$M(t_0^2)t_0 = \frac{1}{\|u\|_\varepsilon}\int_{\mathbb{R}^3} g\left(\varepsilon x, t_0\frac{u}{\|u\|_\varepsilon}\right)u.$$

Hence $t_0\frac{u}{\|u\|_\varepsilon} \in \mathcal{N}_\varepsilon$ and, by (A_1), $t_{(\frac{u}{\|u\|_\varepsilon})} = t_0$, showing that $\widehat{m}_\varepsilon(u_n) = \widehat{m}_\varepsilon(\frac{u_n}{\|u_n\|_\varepsilon}) \to \widehat{m}_\varepsilon(\frac{u}{\|u\|_\varepsilon}) = \widehat{m}_\varepsilon(u)$ in H_ε. So, $\widehat{m}_\varepsilon$ and m_ε are continuous functions and (A_3) is proved.

(A_4) Finally, let $\{u_n\} \subset S_\varepsilon^+$ be a sequence such that $dist(u_n, \partial S_\varepsilon^+) \to 0$. Since, for each $v \in \partial S_\varepsilon^+$ and $n \in \mathbb{N}$, we have

$$u_n^+(x) \le |u_n(x) - v(x)| \text{ a.e in } x \in \Omega_\varepsilon,$$

it follows that

$$\int_{\Omega_\varepsilon} (u_n^+)^s \le \inf_{v \in \partial S_\varepsilon^+} \int_{\Omega_\varepsilon} |u_n - v|^s, \ \forall \ n \in \mathbb{N} \text{ and } \forall s \in [2,6]. \tag{2.2}$$

Hence, from (V_1), (V_2) and Sobolev's embedding, there is $C(s) > 0$ such that

$$\begin{aligned} \int_{\Omega_\varepsilon} (u_n^+)^s &\le C(s) \inf_{v \in \partial S_\varepsilon^+} \left\{ \int_{\Omega_\varepsilon} \left[|\nabla (u_n - v)|^2 + V(\varepsilon x)(u_n - v)^2 \right] \right\}^{s/2} \\ &\le C(s) dist(u_n, \partial S_\varepsilon^+)^s, \ \forall \ n \in \mathbb{N}. \end{aligned}$$

From (g_1), (g_2) and $(g_3)(ii)$, there are positive constants C_1 and C_2, such that, for each $t > 0$

$$\begin{aligned} \int_{\mathbb{R}^3} G(\varepsilon x, tu_n) &\le \int_{\Omega_\varepsilon} F(tu_n) + \frac{t^2}{K} \int_{\mathbb{R}^3 \setminus \Omega_\varepsilon} V(\varepsilon x) u_n^2 \\ &\le C_1 t^4 \int_{\Omega_\varepsilon} (u_n^+)^4 + C_2 t^q \int_{\Omega_\varepsilon} (u_n^+)^q + \frac{1}{K} t^2 \|u_n\|_\varepsilon^2 \\ &\le C_1 C(4) t^4 dist(u_n, \partial S_\varepsilon^+)^4 \\ &\quad + C_2 C(q) t^q dist(u_n, \partial S_\varepsilon^+)^q + \frac{1}{K} t^2. \end{aligned}$$

Therefore,

$$\limsup_{n \to \infty} \int_{\mathbb{R}^3} G(\varepsilon x, tu_n) \le \frac{1}{K} t^2, \ \forall t > 0.$$

From definition of m_ε, we have

$$\liminf_{n \to \infty} J_\varepsilon(m_\varepsilon(u_n)) \ge \liminf_{n \to \infty} J_\varepsilon(tu_n) \ge \frac{1}{2} \widehat{M}(t^2) - \frac{1}{K} t^2, \ \forall \ t > 0.$$

It follows from (M_1) and from the particular choice of K, that

$$\lim_{n \to \infty} J_\varepsilon(m_\varepsilon(u_n)) = \infty.$$

Since $\frac{1}{2}\widehat{M}(t_{u_n}^2) \ge J_\varepsilon(m_\varepsilon(u_n))$, for each $n \in \mathbb{N}$, we conclude from (M_3) that $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ as $n \to \infty$. The Lemma is proved. □

We set the applications

$$\widehat{\Psi}_\varepsilon : H_\varepsilon^+ \to \mathbb{R} \text{ and } \Psi_\varepsilon : S_\varepsilon^+ \to \mathbb{R},$$

by $\widehat{\Psi}_\varepsilon(u) = J_\varepsilon(\widehat{m}_\varepsilon(u))$ and $\Psi_\varepsilon := (\widehat{\Psi}_\varepsilon)_{|_{S_\varepsilon^+}}$.

The next proposition is a direct consequence of the Lemma 2.3. The details can be seen in the relevant material from [26]. For the convenience of the reader, here we do a sketch of the proof.

Proposition 2.4. *Suppose that the function M satisfies (M_1)–(M_3), the potential V satisfies (V_1)–(V_2) and the function f satisfies (f_1)–(f_4). Then:*

(a) $\widehat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_\varepsilon'(u)v = \frac{\|\widehat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} J_\varepsilon'(\widehat{m}_\varepsilon(u))v, \ \forall u \in H_\varepsilon^+ \text{ and } \forall v \in H_\varepsilon.$$

(b) *$\Psi_\varepsilon \in C^1(S_\varepsilon^+, \mathbb{R})$ and*

$$\Psi'_\varepsilon(u)v = \|m_\varepsilon(u)\|_\varepsilon J'_\varepsilon(m_\varepsilon(u))v, \ \forall v \in T_u S_\varepsilon^+.$$

(c) *If $\{u_n\}$ is a $(PS)_d$ sequence for Ψ_ε then $\{m_\varepsilon(u_n)\}$ is a $(PS)_d$ sequence for J_ε. If $\{u_n\} \subset \mathcal{N}_\varepsilon$ is a bounded $(PS)_d$ sequence for J_ε then $\{m_\varepsilon^{-1}(u_n)\}$ is a $(PS)_d$ sequence for Ψ_ε.*

(d) *u is a critical point of Ψ_ε if, and only if, $m_\varepsilon(u)$ is a nontrivial critical point of J_ε. Moreover, corresponding critical values coincide and*

$$\inf_{S_\varepsilon^+} \Psi_\varepsilon = \inf_{\mathcal{N}_\varepsilon} J_\varepsilon.$$

Proof. (a) Consider $u \in H_\varepsilon^+$ and $v \in H_\varepsilon$. From definition of $\widehat{\Psi}_\varepsilon$, definition of t_u and mean value Theorem,

$$\begin{aligned}
\widehat{\Psi}_\varepsilon(u+sv) - \widehat{\Psi}_\varepsilon(u) &= J_\varepsilon(t_{u+sv}(u+sv)) - J_\varepsilon(t_u u) \\
&\le J_\varepsilon(t_{u+sv}(u+sv)) - J_\varepsilon(t_{u+sv} u) \\
&= J'_\varepsilon(t_{u+sv}(u+\tau sv)) t_{u+sv} sv,
\end{aligned}$$

where $|s|$ is small sufficient and $\tau \in (0,1)$. On the other hand,

$$\widehat{\Psi}_\varepsilon(u+sv) - \widehat{\Psi}_\varepsilon(u) \ge J_\varepsilon(t_u(u+sv)) - J_\varepsilon(t_u u) = J'_\varepsilon(t_u(u+\varsigma sv)) t_u sv,$$

where $\varsigma \in (0,1)$. Since $u \mapsto t_u$ is a continuous application, it follows from previous inequalities that

$$\lim_{s\to 0} \frac{\widehat{\Psi}_\varepsilon(u+sv) - \widehat{\Psi}_\varepsilon(u)}{s} = t_u J'_\varepsilon(t_u u)v = \frac{\|\widehat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} J'_\varepsilon(\widehat{m}_\varepsilon(u))v.$$

Since $J_\varepsilon \in C^1$, it follows that the Gateaux derivative of $\widehat{\Psi}_\varepsilon$ is linear, bounded on v and it is continuous on u. From ([28], Prop. 1.3), $\widehat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+, \mathbb{R})$ and

$$\widehat{\Psi}'_\varepsilon(u)v = \frac{\|\widehat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} J'_\varepsilon(\widehat{m}_\varepsilon(u))v, \ \forall u \in H_\varepsilon^+ \text{ and } \forall v \in H_\varepsilon.$$

The item (a) is proved.

(b) The item (b) is a direct consequence of the item (a).

(c) Once $H_\varepsilon = T_u S_\varepsilon^+ \oplus \mathbb{R}\, u$ for each $u \in S_\varepsilon^+$, the linear projection $P : H_\varepsilon \to T_u S_\varepsilon^+$ defined by $P(v+tu) = v$ is continuous, namely, there is $C > 0$ such that

$$\|v\|_\varepsilon \le C\|v+tu\|_\varepsilon, \ \forall\, u \in S_\varepsilon^+, v \in T_u S_\varepsilon^+ \text{ and } t \in \mathbb{R}\,. \tag{2.3}$$

From item (a), we obtain

$$\|\Psi'_\varepsilon(u)\|_* = \sup_{\substack{v\in T_u S_\varepsilon^+ \\ \|v\|_\varepsilon = 1}} \Psi'_\varepsilon(u)v = \|w\|_\varepsilon \sup_{\substack{v\in T_u S_\varepsilon^+ \\ \|v\|_\varepsilon = 1}} J'_\varepsilon(w)v, \tag{2.4}$$

where $w = m_\varepsilon(u)$. Since $w \in \mathcal{N}_\varepsilon$, we conclude that

$$J'_\varepsilon(w)u = J'_\varepsilon(w)\frac{w}{\|w\|_\varepsilon} = 0. \tag{2.5}$$

By (2.4), we have

$$\|\Psi'_\varepsilon(u)\|_* \le \|w\|_\varepsilon \|J'_\varepsilon(w)\| = \|w\|_\varepsilon \sup_{\substack{v\in T_u S_\varepsilon^+, t\in\mathbb{R} \\ v+tu\neq 0}} \frac{J'_\varepsilon(w)(v+tu)}{\|v+tu\|_\varepsilon}.$$

Hence, from (2.3) and (2.5)

$$\|\Psi_\varepsilon'(u)\|_* \leq \|w\|_\varepsilon \|J_\varepsilon'(w)\| \leq C\|w\|_\varepsilon \sup_{v\in T_u S_\varepsilon^+\setminus\{0\}} \frac{J_\varepsilon'(w)(v)}{\|v\|_\varepsilon} = C\|\Psi_\varepsilon'(u)\|_*,$$

showing that,

$$\|\Psi_\varepsilon'(u)\|_* \leq \|w\|_\varepsilon \|J_\varepsilon'(w)\| \leq C\|\Psi_\varepsilon'(u)\|_*, \ \forall\ u \in S_\varepsilon^+. \tag{2.6}$$

Since $w \in \mathcal{N}_\varepsilon$, we have $\|w\| \geq \tau > 0$. Therefore, the inequality in (2.6) together with $J_\varepsilon(w) = \Psi_\varepsilon(u)$ imply the item (c).

(d) It follows from (2.6) that $\Psi_\varepsilon'(u) = 0$ if, and only if, $J_\varepsilon'(w) = 0$. The remainder follows from definition of Ψ_ε. □

By using (M_1)–(M_3) we have, as in [26], the following variational characterization of the infimum of J_ε over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u\in H_\varepsilon^+} \max_{t>0} J_\varepsilon(tu) = \inf_{u\in S_\varepsilon^+} \max_{t>0} J_\varepsilon(tu). \tag{2.7}$$

The main feature of the modified functional is that it satisfies the Palais–Smale condition, as we can see from the next results.

Lemma 2.5. *Let $\{u_n\}$ be a $(PS)_d$ sequence for J_ε. Then $\{u_n\}$ is bounded.*

Proof. Since $\{u_n\}$ a $(PS)_d$ sequence for J_ε, then there is $C > 0$ such that

$$C + \|u_n\|_\varepsilon \geq J_\varepsilon(u_n) - \frac{1}{\theta} J_\varepsilon'(u_n)u_n, \ \forall n \in \mathbb{N}.$$

From (M_3) and (g_3), we obtain

$$C + \|u_n\|_\varepsilon \geq \left(\frac{\theta-2}{2\theta}\right)\left(m_0 - \frac{1}{K}\right)\|u_n\|_\varepsilon^2, \ \forall n \in \mathbb{N}.$$

Therefore $\{u_n\}$ is bounded in H_ε. □

Lemma 2.6. *Let $\{u_n\}$ be a $(PS)_d$ sequence for J_ε. Then for each $\xi > 0$, there is $R = R(\xi) > 0$ such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] < \xi.$$

Proof. Let $\eta_R \in C^\infty(\mathbb{R}^3)$ such that

$$\eta_R(x) = \begin{cases} 0 & se \quad x \in B_R(0) \\ 1 & se \quad x \notin B_{2R}(0), \end{cases}$$

where $0 \leq \eta_R(x) \leq 1$, $|\nabla \eta_R| \leq \dfrac{C}{R}$ and C is a constant independent on R. Note that $\{\eta_R u_n\}$ is bounded in H_ε. From definition of J_ε

$$\int_{\mathbb{R}^3} \eta_R M(\|u_n\|_\varepsilon^2)\left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] = J_\varepsilon'(u_n)(u_n\eta_R) + \int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n\eta_R$$
$$- \int_{\mathbb{R}^3} M(\|u_n\|_\varepsilon^2)u_n \nabla u_n \nabla \eta_R.$$

Choosing $R > 0$ such that $\Omega_\varepsilon \subset B_R(0)$ and by using (M_1) and $(g_3)(ii)$, we have

$$m_0 \int_{\mathbb{R}^3} \eta_R \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] \leq J'_\varepsilon(u_n)(u_n\eta_R) + \int_{\mathbb{R}^3} \frac{1}{K} V(\varepsilon x)u_n^2\eta_R - \int_{\mathbb{R}^3} M(\|u_n\|_\varepsilon^2)u_n\nabla u_n\nabla\eta_R.$$

Therefore,

$$\left(m_0 - \frac{1}{K}\right)\int_{\mathbb{R}^3} \eta_R \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] \leq |J'_\varepsilon(u_n)(u_n\eta_R)| + \int_{\mathbb{R}^3} M(\|u_n\|_\varepsilon^2)u_n|\nabla u_n(\nabla\eta_R)|.$$

By using Cauchy–Schwarz inequality in $\mathbb{R}^3$, definition of η_R, Holder's inequality and the boundedness of $\{u_n\}$ in H_ε, we conclude that

$$\int_{\mathbb{R}^3\setminus B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] \leq C|J'_\varepsilon(u_n)(u_n\eta_R)| + \frac{C}{R}.$$

Since $\{u_n\eta_R\}$ is bounded in H_ε and $\{u_n\}$ is a $(PS)_d$ sequence for J_ε, passing to the upper limit of $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] \leq \frac{C}{R} < \xi,$$

whenever $R = R(\xi) > C/\xi$. □

The next result does not appear in [12], however, since we are working with the Kirchhoff problem type, it is required here.

Lemma 2.7. *Let $\{u_n\}$ be a $(PS)_d$ sequence for J_ε such that $u_n \rightharpoonup u$, then*

$$\lim_{n\to\infty} \int_{B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] = \int_{B_R} \left[|\nabla u|^2 + V(\varepsilon x)u^2\right],$$

for all $R > 0$.

Proof. We can assume that $\|u_n\|_\varepsilon \to t_0$, thus, we have $\|u\|_\varepsilon \leq t_0$. Let $\eta_\rho \in C^\infty(\mathbb{R}^3)$ such that

$$\eta_\rho(x) = \begin{cases} 1 & se \quad x \in B_\rho(0) \\ 0 & se \quad x \notin B_{2\rho}(0). \end{cases}$$

with $0 \leq \eta_\rho(x) \leq 1$. Let,

$$P_n(x) = M(\|u_n\|_\varepsilon^2)\left[|\nabla u_n - \nabla u|^2 + V(\varepsilon x)(u_n - u)^2\right].$$

For each $R > 0$ fixed, choosing $\rho > R$ we obtain

$$\int_{B_R} P_n = \int_{B_R} P_n\eta_\rho \leq M(\|u_n\|_\varepsilon^2)\int_{\mathbb{R}^3} \left[|\nabla u_n - \nabla u|^2 + V(\varepsilon x)(u_n - u)^2\right]\eta_\rho.$$

By expanding the inner product in $\mathbb{R}^3$,

$$\begin{aligned}\int_{B_R} P_n \leq{}& M(\|u_n\|_\varepsilon^2)\int_{\mathbb{R}^3} \left[|\nabla u_n|^2 + V(\varepsilon x)(u_n)^2\right]\eta_\rho \\ &- 2M(\|u_n\|_\varepsilon^2)\int_{\mathbb{R}^3} \left[\nabla u_n\nabla u + V(\varepsilon x)u_n u\right]\eta_\rho \\ &+ M(\|u_n\|_\varepsilon^2)\int_{\mathbb{R}^3} \left[|\nabla u|^2 + V(\varepsilon x)u^2\right]\eta_\rho.\end{aligned}$$

Setting

$$I^1_{n,\rho} = M(\|u_n\|^2_\varepsilon)\int_{\mathbb{R}^3}\left[|\nabla u_n|^2 + V(\varepsilon x)(u_n)^2\right]\eta_\rho - \int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n\eta_\rho,$$

$$I^2_{n,\rho} = M(\|u_n\|^2_\varepsilon)\int_{\mathbb{R}^3}\left[\nabla u_n\nabla u + V(\varepsilon x)u_n u\right]\eta_\rho - \int_{\mathbb{R}^3} g(\varepsilon x, u_n)u\eta_\rho,$$

$$I^3_{n,\rho} = -M(\|u_n\|^2_\varepsilon)\int_{\mathbb{R}^3}\left[\nabla u_n\nabla u + V(\varepsilon x)u_n u\right]\eta_\rho + M(\|u_n\|^2_\varepsilon)\int_{\mathbb{R}^3}\left[|\nabla u|^2 + V(\varepsilon x)u^2\right]\eta_\rho$$

and

$$I^4_{n,\rho} = \int_{\mathbb{R}^3} g(\varepsilon x, u_n)u_n\eta_\rho - \int_{\mathbb{R}^3} g(\varepsilon x, u_n)u\eta_\rho.$$

We have that,

$$0 \le \int_{B_R} P_n \le |I^1_{n,\rho}| + |I^2_{n,\rho}| + |I^3_{n,\rho}| + |I^4_{n,\rho}|. \tag{2.8}$$

Observe that

$$I^1_{n,\rho} = J'_\varepsilon(u_n)(u_n\eta_\rho) - M(\|u_n\|^2_\varepsilon)\int_{\mathbb{R}^3} u_n\nabla u_n\nabla\eta_\rho.$$

Since $\{u_n\eta_\rho\}$ is bounded in H_ε, we have $J'_\varepsilon(u_n)(u_n\eta_\rho) = o_n(1)$. Moreover, from a straightforward computation

$$\lim_{\rho\to\infty}\left[\limsup_{n\to\infty}\left|M(\|u_n\|^2_\varepsilon)\int_{\mathbb{R}^3} u_n\nabla u_n\nabla\eta_\rho\right|\right] = 0.$$

Then,

$$\lim_{\rho\to\infty}\left[\limsup_{n\to\infty}|I^1_{n,\rho}|\right] = 0. \tag{2.9}$$

We see also that

$$I^2_{n,\rho} = J'_\varepsilon(u_n)(u\eta_\rho) - M(\|u_n\|^2_\varepsilon)\int_{\mathbb{R}^3} u\nabla u_n\nabla\eta_\rho.$$

By arguing in the same way as in the previous case,

$$J'_\varepsilon(u_n)(u\eta_\rho) = o_n(1)$$

and

$$\lim_{\rho\to\infty}\left[\limsup_{n\to\infty}\left|M(\|u_n\|^2_\varepsilon)\int_{\mathbb{R}^3} u\nabla u_n\nabla\eta_\rho\right|\right] = 0.$$

Therefore,

$$\lim_{\rho\to\infty}\left[\limsup_{n\to\infty}|I^2_{n,\rho}|\right] = 0. \tag{2.10}$$

On the other hand, from the weak convergence

$$\lim_{n\to\infty}|I^3_{n,\rho}| = 0,\ \forall\ \rho > R. \tag{2.11}$$

Finally, from

$$u_n \to u,\ \text{in}\ L^s_{loc}(\mathbb{R}^3), 1 \le s < 6,$$

we conclude that

$$\lim_{n\to\infty}|I^4_{n,\rho}| = 0,\ \forall\ \rho > R. \tag{2.12}$$

From (2.8), (2.9), (2.10), (2.11) and (2.12), we obtain

$$0 \leq \limsup_{n\to\infty} \int_{B_R} P_n \leq 0.$$

Hence, $\lim_{n\to\infty} \int_{B_R} P_n = 0$ and consequently

$$\lim_{n\to\infty} \int_{B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] = \int_{B_R} \left[|\nabla u|^2 + V(\varepsilon x)u^2\right]. \qquad \square$$

Proposition 2.8. *The functional J_ε verifies the $(PS)_d$ condition in H_ε.*

Proof. Let $\{u_n\}$ be a $(PS)_d$ sequence for J_ε. From Lemma 2.5 we know that $\{u_n\}$ is bounded in H_ε. Passing to a subsequence, we obtain

$$u_n \rightharpoonup u, \text{ in } H_\varepsilon.$$

From Lemma 2.6, it follows that for each $\xi > 0$ given there is $R = R(\xi) > C/\xi$ with $C > 0$ independent on ξ such that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] < \xi.$$

Therefore, from Lemma 2.7,

$$\begin{aligned}
\|u\|_\varepsilon^2 &\leq \liminf_{n\to\infty} \|u_n\|_\varepsilon^2 \leq \limsup_{n\to\infty} \|u_n\|_\varepsilon^2 \\
&= \limsup_{n\to\infty} \left\{ \int_{B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] + \int_{\mathbb{R}^3\setminus B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] \right\} \\
&= \int_{B_R} \left[|\nabla u|^2 + V(\varepsilon x)u^2\right] + \limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus B_R} \left[|\nabla u_n|^2 + V(\varepsilon x)u_n^2\right] \\
&< \int_{B_R} \left[|\nabla u|^2 + V(\varepsilon x)u^2\right] + \xi,
\end{aligned}$$

where $R = R(\xi) > C/\xi$. Passing to the limit of $\xi \to 0$ we have $R \to \infty$, which implies

$$\|u\|_\varepsilon^2 \leq \liminf_{n\to\infty} \|u_n\|_\varepsilon^2 \leq \limsup_{n\to\infty} \|u_n\|_\varepsilon^2 \leq \|u\|_\varepsilon^2,$$

and so $\|u_n\|_\varepsilon \to \|u\|_\varepsilon$ and consequently $u_n \to u$ in H_ε. $\square$

Since f is only continuous and V has geometry of the Del Pino and Felmer type [12], in the next result (which is required for the multiplicity result) we use arguments that don't appear in [12] and [27].

Corollary 2.9. *The functional Ψ_ε verifies the $(PS)_d$ condition on S_ε^+.*

Proof. Let $\{u_n\} \subset S_\varepsilon^+$ be a $(PS)_d$ sequence for Ψ_ε. Thus,

$$\Psi_\varepsilon(u_n) \to d$$

and

$$\|\Psi_\varepsilon'(u_n)\|_* \to 0,$$

where $\|.\|_*$ is the norm in the dual space $(T_{u_n}S_\varepsilon^+)'$. It follows from Proposition 2.4(c) that $\{m_\varepsilon(u_n)\}$ is a $(PS)_d$ sequence for J_ε in H_ε. From Proposition 2.8 we conclude there is $u \in S_\varepsilon^+$ such that, passing to a subsequence,

$$m_\varepsilon(u_n) \to m_\varepsilon(u) \text{ in } H_\varepsilon.$$

From Lemma 2.3 (A_3), it follows that

$$u_n \to u \text{ in } S_\varepsilon^+.$$

□

Theorem 2.10. *Suppose that the function M satisfies (M_1)–(M_3), the potential V satisfies (V_1)–(V_2) and the function f satisfies (f_1)–(f_4). Then, the auxiliary problem $(P_{\varepsilon,A})$ has a positive ground-state solution for all $\varepsilon > 0$.*

Proof. This result follows from Lemma 2.2, Proposition 2.8 and maximum principle. □

3. Multiplicity of solutions of auxiliary problem

3.1. The autonomous problem

Since we are interested in giving a multiplicity result for the auxiliary problem, we start by considering the limit problem associated to $(\widetilde{P}_\varepsilon)$, namely, the problem

$$(P_0) \qquad \begin{cases} \mathfrak{L}_0 u = f(u), & \mathbb{R}^3 \\ u > 0, & \mathbb{R}^3 \\ u \in H^1(\mathbb{R}^3) \end{cases}$$

where

$$\mathfrak{L}_0 u = M\left(\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V_0 u^2\right)[-\Delta u + V_0 u],$$

which has the following associated functional

$$I_0(u) = \frac{1}{2}\widehat{M}\left(\int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V_0 u^2\right) - \int_{\mathbb{R}^3} F(u).$$

This functional is well defined on the Hilbert space $H_0 = H^1(\mathbb{R}^3)$ with the inner product

$$(u,v)_0 = \int_{\mathbb{R}^3} \nabla u \nabla v + \int_{\mathbb{R}^3} V_0 uv$$

and norm

$$\|u\|_0^2 = \int_{\mathbb{R}^3} |\nabla u|^2 + \int_{\mathbb{R}^3} V_0 u^2$$

fixed. We denote the Nehari manifold associated to I_0 by

$$\mathcal{N}_0 = \{u \in H_0 \backslash \{0\} : I_0'(u)u = 0\}.$$

We denote by H_0^+ the open subset of H_0 given by

$$H_0^+ = \{u \in H_0 : |\text{supp}\,(u^+)| > 0\},$$

and $S_0^+ = S_0 \cap H_0^+$, where S_0 is the unit sphere of H_0.

As in the section 2, S_0^+ is a incomplete $C^{1,1}$-manifold of codimension 1, modeled on H_0 and contained in the open H_0^+. Thus, $H_0 = T_u S_0^+ \oplus \mathbb{R}\, u$ for each $u \in S_0^+$, where $T_u S_0^+ = \{v \in H_0 : (u,v)_0 = 0\}$.

Next we enunciate without proof one Lemma and one Proposition, which allow us to prove the Lemma 3.7. The proofs follow from a similar argument to that used in the proofs of Lemma 2.3 and Proposition 2.4.

Lemma 3.1. *Suppose that the function M satisfies (M_1)–(M_3) and the function f satisfies (f_1)–(f_4). So:*

(A_1) For each $u \in H_0^+$, let $h : \mathbb{R}_+ \to \mathbb{R}$ be defined by $h_u(t) = I_0(tu)$. Then, there is a unique $t_u > 0$ such that $h_u'(t) > 0$ in $(0, t_u)$ and $h_u'(t) < 0$ in (t_u, ∞).

(A_2) there is $\tau > 0$ independent on u such that $t_u \geq \tau$ for all $u \in S_0^+$. Moreover, for each compact set $\mathcal{W} \subset S_0^+$ there is $C_{\mathcal{W}} > 0$ such that $t_u \leq C_{\mathcal{W}}$, for all $u \in \mathcal{W}$.

(A_3) The map $\widehat{m} : H_0^+ \to \mathcal{N}_0$ given by $\widehat{m}(u) = t_u u$ is continuous and $m := \widehat{m}_{|_{S_0^+}}$ is a homeomorphism between S_0^+ and $\mathcal{N}_0$. Moreover, $m^{-1}(u) = \frac{u}{\|u\|_0}$.

(A_4) If there is a sequence $(u_n) \subset S_0^+$ such that $dist(u_n, \partial S_0^+) \to 0$, then $\|m(u_n)\|_0 \to \infty$ and $I_0(m(u_n)) \to \infty$.

We set the applications

$$\widehat{\Psi}_0 : H_0^+ \to \mathbb{R} \text{ and } \Psi_0 : S_0^+ \to \mathbb{R},$$

by $\widehat{\Psi}_0(u) = I_0(\widehat{m}(u))$ and $\Psi_0 := (\widehat{\Psi}_0)_{|_{S_0^+}}$.

Proposition 3.2. *Suppose that the function M satisfies (M_1)–(M_3) and the function f satisfies (f_1)–(f_4). So:*

(a) $\widehat{\Psi}_0 \in C^1(H_0^+, \mathbb{R})$ *and*

$$\widehat{\Psi}_0'(u)v = \frac{\|\widehat{m}(u)\|_0}{\|u\|_0} I_0'(\widehat{m}(u))v, \ \forall u \in H_0^+ \text{ and } \forall v \in H_0.$$

(b) $\Psi_0 \in C^1(S_0^+, \mathbb{R})$ *and*

$$\Psi_0'(u)v = \|m(u)\|_0 I_0'(m(u))v, \ \forall v \in T_u S_0^+.$$

(c) *If $\{u_n\}$ is a $(PS)_d$ sequence for Ψ_0 then $\{m(u_n)\}$ is a $(PS)_d$ sequence for I_0. If $\{u_n\} \subset \mathcal{N}_0$ is a bounded $(PS)_d$ sequence for I_0 then $\{m^{-1}(u_n)\}$ is a $(PS)_d$ sequence for Ψ_0.*

(d) *u is a critical point of Ψ_0 if, and only if, $m(u)$ is a nontrivial critical point of I_0. Moreover, corresponding critical values coincide and*

$$\inf_{S_0^+} \Psi_0 = \inf_{\mathcal{N}_0} I_0.$$

Remark 3.3. As in the section 2, there holds

$$c_0 = \inf_{u \in \mathcal{N}_0} I_0(u) = \inf_{u \in H_0^+} \max_{t > 0} I_0(tu) = \inf_{u \in S_0^+} \max_{t > 0} I_0(tu). \tag{3.1}$$

The next Lemma allows us to assume that the weak limit of a $(PS)_d$ sequence is non-trivial.

Lemma 3.4. *Let $\{u_n\} \subset H_0$ be a $(PS)_d$ sequence for I_0 with $u_n \rightharpoonup 0$. Then, only one of the alternatives below holds:*

a) *$u_n \to 0$ in H_0*

b) *there is a sequence $(y_n) \subset \mathbb{R}^3$ and constants $R, \beta > 0$ such that*

$$\liminf_{n \to \infty} \int_{B_R(y_n)} u_n^2 \geq \beta > 0.$$

Proof. Suppose that b) doesn't hold. It follows that for all $R > 0$ we have

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} u_n^2 = 0.$$

Since $\{u_n\}$ is bounded in H_0, we conclude from ([28], Lem. 1.21) that

$$u_n \to 0 \text{ in } L^s(\mathbb{R}^3), 2 < s < 6.$$

From (M_1), (f_1) and (f_2),

$$0 \leq m_0 \|u_n\|_0 \leq \int_{\mathbb{R}^3} f(u_n)u_n + o_n(1) = o_n(1).$$

Therefore the item $a)$ is true. □

Remark 3.5. As it has been mentioned, if u is the weak limit of a $(PS)_{c_0}$ sequence $\{u_n\}$ for the functional I_0, then we can assume $u \neq 0$, otherwise we would have $u_n \rightharpoonup 0$ and, once it doesn't occur $u_n \to 0$, we conclude from the Lemma 3.4 that there are $\{y_n\} \subset \mathbb{R}^3$ and $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} u_n^2 \geq \beta > 0.$$

Set $v_n(x) = u_n(x + y_n)$, making a change of variable, we can prove that $\{v_n\}$ is a $(PS)_{c_0}$ sequence for the functional I_0, it is bounded in H_0 and there is $v \in H_0$ with $v_n \rightharpoonup v$ in H_0 with $v \neq 0$.

In the next Proposition we obtain a positive ground-state solution for the autonomous problem (P_0).

Theorem 3.6. *Let $\{u_n\} \subset H_0$ be a $(PS)_{c_0}$ sequence for I_0. Then there is $u \in H_0\backslash\{0\}$ with $u \geq 0$ such that, passing a subsequence, we have $u_n \to u$ in H_0. Moreover, u is a positive ground-state solution for the problem (P_0).*

Proof. Arguing as Lemma 2.5, we have that $\{u_n\}$ is bounded in H_0. Thus, passing a subsequence if necessary, we obtain

$$u_n \rightharpoonup u \text{ em } H_0, \tag{3.2}$$

$$u_n \to u \text{ em } L^s_{loc}(\mathbb{R}^3), 1 \leq s < 6 \tag{3.3}$$

and

$$\|u_n\|_0 \to t_0. \tag{3.4}$$

So, from (3.2) we conclude that

$$(u_n, v)_0 \to (u, v)_0, \ \forall v \in H_0. \tag{3.5}$$

On the other hand, due to density of $C_0^\infty(\mathbb{R}^3)$ in H_0 and from convergence in (3.3), it results that

$$\int_{\mathbb{R}^3} f(u_n)v \to \int_{\mathbb{R}^3} f(u)v, \forall v \in H_0. \tag{3.6}$$

Now, from convergence in (3.2) and (3.4), occurs

$$\|u\|_0^2 \leq \liminf_{n\to\infty} \|u_n\|_0^2 = t_0^2,$$

and from (M_2) it follows that $M(\|u\|_0^2) \leq M(t_0^2)$.

Since (M_3) implies that the function $t \mapsto \frac{1}{2}\widehat{M}(t) - \frac{1}{4}M(t)t$ is non-decreasing, we can argue as in [4] and to prove that $M(t_0^2) = M(\|u\|_0^2)$ and the theorem now follows from fact that functional I_0 has the mountain pass geometry and from ([28], Thm. 1.15). □

The next lemma is a compactness result on the autonomous problem which we will use later.

Lemma 3.7. *Let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^3)$ such that $I_0(u_n) \to c_0$ and $\{u_n\} \subset \mathcal{N}_0$. Then, $\{u_n\}$ has a convergent subsequence in $H^1(\mathbb{R}^3)$.*

Proof. Since $\{u_n\} \subset \mathcal{N}_0$, it follows from Lemma 3.1 (A_3), Proposition 3.2(d) and Remark 3.1 that

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in S_0^+, \ \forall n \in \mathbb{N} \tag{3.7}$$

and

$$\Psi_0(v_n) = I_0(u_n) \to c_0 = \inf_{S_0^+} \Psi_0.$$

Although S_0^+ is incomplete, due to Lemma 3.1 (A_4), we can still apply the Ekeland's variational principle ([13], Thm. 1.1) to the functional $\xi_0 : V \to \mathbb{R} \cup \{\infty\}$ defined by $\xi_0(u) = \Psi_0(u)$ if $u \in S_0^+$ and $\xi_0(u) = \infty$ if $u \in \partial S_0^+$, where $V = \overline{S_0^+}$ is a complete metric space equipped with the metric $d(u,v) = \|u - v\|_0$. In fact, from Lemma 3.1 (A_4), $\xi_0 \in C(V, \mathbb{R} \cup \{\infty\})$ and, from Proposition 3.2(d), ξ_0 is bounded from below. Thus, we can conclude there is a sequence $\{\widehat{v}_n\} \subset S_0^+$ such that $\{\widehat{v}_n\}$ is a $(PS)_{c_0}$ sequence for Ψ_0 on S_0^+ and

$$\|\widehat{v}_n - v_n\|_0 = o_n(1). \tag{3.8}$$

The remainder of the proof follows by using Proposition 3.2, Theorem 3.6 and arguing as in the proof of Corollary 2.9. □

In the next subsection we will relate the number of positive solutions of $(P_{\varepsilon,A})$ to topology of Π, for this we need some preliminary results.

3.2. Technical results

Let $\delta > 0$ fixed and $\Pi_\delta \subset \Omega$. Let $\eta \in C_0^\infty([0,\infty))$ be such that $0 \le \eta(t) \le 1$, $\eta(t) = 1$ if $0 \le t \le \delta/2$ and $\eta(t) = 0$ if $t \ge \delta$. We denote by w a positive ground-state solution of the problem (P_0) (see Thm. 3.6).

For each $y \in \Pi = \{x \in \Omega : V(x) = V_0\}$, we define the function

$$\widetilde{\Upsilon}_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) w\left(\frac{\varepsilon x - y}{\varepsilon}\right).$$

Let $t_\varepsilon > 0$ be the unique positive number such that

$$\max_{t \ge 0} J_\varepsilon(t\widetilde{\Upsilon}_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon \widetilde{\Upsilon}_{\varepsilon,y}).$$

By noticing that $t_\varepsilon \widetilde{\Upsilon}_{\varepsilon,y} \in \mathcal{N}_\varepsilon$, we can now define the continuous function

$$\begin{aligned} \Upsilon_\varepsilon : \Pi &\longrightarrow \mathcal{N}_\varepsilon \\ y &\longmapsto \Upsilon_\varepsilon(y) = t_\varepsilon \widetilde{\Upsilon}_{\varepsilon,y}. \end{aligned}$$

Lemma 3.8. *Let* $\Pi \subset \Omega$. *Then,*

$$\lim_{\varepsilon \to 0} J_\varepsilon(\Upsilon_\varepsilon(y)) = c_0 \ \textit{uniformly in} \ y \in \Pi.$$

Proof. Arguing by contradiction, we suppose that there exist $\delta_0 > 0$ and a sequence $\{y_n\} \subset \Pi$ verifying

$$| J_{\varepsilon_n}(\Upsilon_{\varepsilon_n}(y_n)) - c_0 | \ge \delta_0 \text{ where } \varepsilon_n \to 0 \text{ when } \ n \to \infty. \tag{3.9}$$

From definition of $\Upsilon_{\varepsilon_n}(y_n)$, we have

$$J_{\varepsilon_n}(\Upsilon_{\varepsilon_n}(y_n)) = \frac{1}{2}\widehat{M}\left(t_{\varepsilon_n}^2 \|\widetilde{\Upsilon}_{\varepsilon_n,y_n}\|_{\varepsilon_n}^2\right) - \int_{\mathbb{R}^3} G\left(\varepsilon_n x, t_{\varepsilon_n} \widetilde{\Upsilon}_{\varepsilon_n,y_n}\right) \tag{3.10}$$

and

$$J'_{\varepsilon_n}(\Upsilon_{\varepsilon_n}(y_n))\Upsilon_{\varepsilon_n}(y_n) = 0. \tag{3.11}$$

Using definition of $\Upsilon_{\varepsilon_n}(y_n)$ again and making the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, we have

$$J_{\varepsilon_n}(\Upsilon_{\varepsilon_n}(y_n)) = \frac{1}{2}\widehat{M}\left(t_{\varepsilon_n}^2\left(\int_{\mathbb{R}^3}|\nabla(\eta(|\varepsilon_n z|)w(z))|^2 + \int_{\mathbb{R}^3}V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^2\right)\right)$$
$$-\int_{\mathbb{R}^3}G(\varepsilon_n z + y_n, t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)).$$

Moreover, putting

$$\Lambda_n^2 = \int_{\mathbb{R}^3}|\nabla(\eta(|\varepsilon_n z|)w(z))|^2 + \int_{\mathbb{R}^3}V(\varepsilon_n z + y_n)(\eta(|\varepsilon_n z|)w(z))^2,$$

the equality in (3.11) yields

$$\frac{M(t_{\varepsilon_n}^2\Lambda_n^2)}{t_{\varepsilon_n}^2\Lambda_n^2} = \frac{1}{\Lambda_n^4}\int_{\mathbb{R}^3}\left[\frac{g(\varepsilon_n z + y_n, t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))}{(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))^3}\right](\eta(|\varepsilon_n z|)w(z))^4.$$

For each $n \in \mathbb{N}$ and for all $z \in B_{\frac{\delta}{\varepsilon_n}}(0)$, we have $\varepsilon_n z \in B_\delta(0)$. So,

$$\varepsilon_n z + y_n \in B_\delta(y_n) \subset \Pi_\delta \subset \Omega.$$

Since $G = F$ in Ω, it follows from (3.10) that

$$J_{\varepsilon_n}(\Upsilon_{\varepsilon_n}(y_n)) = \frac{1}{2}\widehat{M}(t_{\varepsilon_n}^2\Lambda_n^2) - \int_{\mathbb{R}^3}F(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)) \tag{3.12}$$

and

$$\frac{M(t_{\varepsilon_n}^2\Lambda_n^2)}{t_{\varepsilon_n}^2\Lambda_n^2} = \frac{1}{\Lambda_n^4}\int_{\mathbb{R}^3}\left[\frac{f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))}{(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))^3}\right](\eta(|\varepsilon_n z|)w(z))^4. \tag{3.13}$$

From the Lebesgue's theorem, when $n \to \infty$

$$\|\widetilde{\Upsilon}_{\varepsilon_n, y_n}\|_{\varepsilon_n}^2 = \Lambda_n^2 \to \|w\|_0^2, \tag{3.14}$$

$$\int_{\mathbb{R}^3}f(\eta(|\varepsilon_n z|)w(z))\eta(|\varepsilon_n z|)w(z) \to \int_{\mathbb{R}^3}f(w)w$$

and

$$\int_{\mathbb{R}^3}F(\eta(|\varepsilon_n z|)w(z)) \to \int_{\mathbb{R}^3}F(w). \tag{3.15}$$

We see that there is a subsequence, still denoted by $\{t_{\varepsilon_n}\}$, with $t_{\varepsilon_n} \to 1$. In fact, since $\eta = 1$ in $B_{\frac{\delta}{2}}(0)$ and $B_{\frac{\delta}{2}}(0) \subset B_{\frac{\delta}{2\varepsilon_n}}(0)$ for n large enough, it follows from (3.13) that

$$\frac{M(t_{\varepsilon_n}^2\Lambda_n^2)}{t_{\varepsilon_n}^2\Lambda_n^2} \geq \frac{1}{\Lambda_n^4}\int_{B_{\frac{\delta}{2}}(0)}\left[\frac{f(t_{\varepsilon_n}w(z))}{(t_{\varepsilon_n}w(z))^3}\right]w(z)^4.$$

From continuity of w (which follows from standard regularity theory), there is $\widehat{z} \in \mathbb{R}^3$ such that $w(\widehat{z}) = \min_{\overline{B}_{\frac{\delta}{2}}(0)} w(z)$. So, from (f_4)

$$\frac{1}{\Lambda_n^4}\frac{f(t_{\varepsilon_n}w(\widehat{z}))}{(t_{\varepsilon_n}w(\widehat{z}))^3}\int_{B_{\frac{\delta}{2}}(0)}w(z)^4 \leq \frac{M(t_{\varepsilon_n}^2\Lambda_n^2)}{t_{\varepsilon_n}^2\Lambda_n^2}. \tag{3.16}$$

Suppose by contradiction that there is a subsequence $\{t_{\varepsilon_n}\}$ with $t_{\varepsilon_n} \to \infty$. Thus, passing to the limit as $n \to \infty$ in (3.16), we conclude, from (M_3) and (f_3), that the left side converges to infinity and the right side is bounded, which is a contradiction. Therefore, $\{t_{\varepsilon_n}\}$ is bounded and passing to a subsequence we have $t_{\varepsilon_n} \to t_0$ with $t_0 \geq 0$.

From (3.13), (3.14), (M_1) and (f_4) we have that $t_0 > 0$. Thus, passing to the limit as $n \to \infty$ in (3.13), we have

$$M(t_0^2\|w\|_0^2)\|w\|_0^2 t_0 = \int_{\mathbb{R}^3} f(t_0 w)w. \tag{3.17}$$

Since $w \in \mathcal{N}_0$, we obtain $t_0 = 1$. So, passing to the limit of $n \to \infty$ in (3.12) and using (3.14) and (3.15) we obtain

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Upsilon_{\varepsilon_n}(y_n)) = I_0(w) = c_0,$$

which is a contradiction with (3.9). □

Let's consider the specific subset of the Nehari manifold

$$\widetilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leq c_0 + h_1(\varepsilon)\},$$

where $h_1 : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\Upsilon_\varepsilon(\Pi) \subset \widetilde{\mathcal{N}}_\varepsilon$ and $\lim_{\varepsilon\to 0} h_1(\varepsilon) = 0$. Observe that h_1 exists due to the Lemma 3.8. In particular, $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for all small $\varepsilon > 0$.

Now we consider $\rho > 0$ such that $\Pi_\delta \subset B_\rho(0)$ and $\chi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$\chi(x) = \begin{cases} x & se \quad |x| \leq \rho \\ \dfrac{\rho x}{|x|} & se \quad |x| \geq \rho. \end{cases}$$

We also consider the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \longrightarrow \mathbb{R}^3$ given by

$$\beta_\varepsilon(u) = \frac{\displaystyle\int_{\mathbb{R}^3} \chi(\varepsilon x)u(x)^2}{\displaystyle\int_{\mathbb{R}^3} u(x)^2}.$$

Since $\Pi \subset B_\rho(0)$, the definition of χ and Lebesgue's theorem imply that

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Upsilon_\varepsilon(y)) = y \text{ uniformly in } y \in \Pi. \tag{3.18}$$

The next result is fundamental to show that the solutions of the auxiliary problem are solutions of the original problem. Moreover, it allows us to show the behavior of such solutions in the norm $|.|_{L^\infty(\mathbb{R}^3\setminus\Omega_\varepsilon)}$.

Proposition 3.9. *Let $\{u_n\}$ be a sequence in $H^1(\mathbb{R}^3)$ such that*

$$J_{\varepsilon_n}(u_n) \to c_0 \Omega$$

and

$$J'_{\varepsilon_n}(u_n)(u_n) = 0, \ \forall n \in \mathbb{N}$$

with $\varepsilon_n \to 0$ when $n \to \infty$. Then, there is a subsequence $\{\widetilde{y}_n\} \subset \mathbb{R}^3$ such that the sequence $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in $H^1(\mathbb{R}^3)$. Moreover, passing to a subsequence,

$$y_n \to \widetilde{y} \text{ with } y \in \Pi,$$

where $y_n = \varepsilon_n \widetilde{y}_n$.

Proof. We can always consider $u_n \geq 0$ and $u_n \neq 0$. As in Lemma 2.5 and arguing as Remark 3.5 we have that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and there are $(\widetilde{y}_n) \subset \mathbb{R}^3$ and positive constants R and α such that

$$\liminf_{n\to\infty} \int_{B_R(\widetilde{y}_n)} u_n^2 \geq \alpha > 0. \tag{3.19}$$

Considering $v_n(x) = u_n(x + \widetilde{y}_n)$ we conclude that $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and therefore, passing to a subsequence, we get

$$v_n \rightharpoonup v, \text{ in } H^1(\mathbb{R}^3)$$

with $v \neq 0$. For each $n \in \mathbb{N}$, let $t_n > 0$ such that $\widetilde{v}_n = t_n v_n \in \mathcal{N}_0$ (see Lem. 3.1 (A_1)). We have that

$$\begin{aligned} c_0 \leq I_0(\widetilde{v}_n) &= \frac{1}{2}\widehat{M}(t_n^2\|u_n\|_0^2) - \int_{\mathbb{R}^3} F(t_n u_n) \\ &\leq \frac{1}{2}\widehat{M}(t_n^2\|u_n\|_{\varepsilon_n}^2) - \int_{\mathbb{R}^3} G(\varepsilon_n x, t_n u_n). \end{aligned}$$

Hence,

$$c_0 \leq I_0(\widetilde{v}_n) \leq J_\varepsilon(t_n u_n) \leq J_\varepsilon(u_n) = c_0 + o_n(1), \tag{3.20}$$

which implies,

$$I_0(\widetilde{v}_n) \to c_0 \text{ and } \{\widetilde{v}_n\} \subset \mathcal{N}_0. \tag{3.21}$$

Thus, $\{\widetilde{v}_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $\widetilde{v}_n \rightharpoonup \widetilde{v}$. From well-known arguments we can assume that $t_n \to t_0$ with $t_0 > 0$. So, from uniqueness of the weak limit we have $\widetilde{v} = t_0 v$, $v \neq 0$. From Lemma 3.7 we obtain,

$$\widetilde{v}_n \to \widetilde{v} \text{ in } H^1(\mathbb{R}^3). \tag{3.22}$$

This convergence implies

$$v_n \to \frac{\widetilde{v}}{t_0} = v \text{ in } H^1(\mathbb{R}^3)$$

and

$$I_0(\widetilde{v}) = c_0 \text{ and } I_0'(\widetilde{v})\widetilde{v} = 0. \tag{3.23}$$

Now, we will show that $\{y_n\}$ is bounded, where $y_n = \varepsilon_n \widetilde{y}_n$. In fact, otherwise, there exists a subsequence $\{y_n\}$ with $|y_n| \to \infty$. Observe that

$$m_0\|v_n\|_0^2 \leq \int_{\mathbb{R}^3} g(\varepsilon_n z + y_n, v_n)v_n.$$

Let $R > 0$ such that $\Omega \subset B_R(0)$. Since we may suppose that $|y_n| \geq 2R$, for each $z \in B_{\frac{R}{\varepsilon_n}}(0)$ we have

$$|\varepsilon_n z + y_n| \geq |y_n| - |\varepsilon_n z| \geq 2R - R = R.$$

Thus,

$$m_0\|v_n\|_0^2 \leq \int_{B_{\frac{R}{\varepsilon_n}}(0)} \widetilde{f}(v_n)v_n + \int_{\mathbb{R}^3\setminus B_{\frac{R}{\varepsilon_n}}(0)} f(v_n)v_n.$$

Since $v_n \to v$ in $H^1(\mathbb{R}^3)$, it follows from Lebesgue's theorem that

$$\int_{\mathbb{R}^3\setminus B_{\frac{R}{\varepsilon_n}}(0)} f(v_n)v_n = o_n(1).$$

On the other hand, since $\widetilde{f}(v_n) \leq \frac{V_0}{K} v_n$, we obtain

$$m_0 \|v_n\|_0^2 \leq \frac{1}{K} \int_{B_{\frac{R}{\varepsilon_n}}(0)} V_0 v_n^2 + o_n(1),$$

and therefore,

$$\left(m_0 - \frac{1}{K}\right) \|v_n\|_0 \leq o_n(1),$$

which is a contradiction. Hence, $\{y_n\}$ is bounded and we can assume $y_n \to \overline{y}$ in $\mathbb{R}^3$. We see that $\overline{y} \in \overline{\Omega}$ because if $\overline{y} \notin \overline{\Omega}$, we can proceed as above and conclude that $\|v_n\|_0 \leq o_n(1)$.

In order to prove that $V(\overline{y}) = V_0$, we suppose by contradiction that $V_0 < V(\overline{y})$. Consequently, from (3.22), Fatou's Lemma and the invariance of $\mathbb{R}^3$ by translations, we obtain

$$\begin{aligned} c_0 &< \liminf_{n\to\infty} \left[\frac{1}{2}\widehat{M}\left(\int_{\mathbb{R}^3} |\nabla \widetilde{v}_n|^2 + \int_{\mathbb{R}^3} V(\varepsilon_n z + y_n)\widetilde{v}_n^2\right) - \int_{\mathbb{R}^3} F(\widetilde{v}_n)\right] \\ &\leq \liminf_{n\to\infty} J_{\varepsilon_n}(t_n u_n) \\ &\leq \liminf_{n\to\infty} J_{\varepsilon_n}(u_n) = c_0, \end{aligned}$$

which is a contradiction and the proof is finished. □

Corollary 3.10. *Assume the same hypotheses of Proposition 3.9. Then, for any given $\gamma_2 > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\int_{B_R(\widetilde{y}_n)^c} \left(|\nabla u_n|^2 + |u_n|^2\right) < \gamma_2, \quad \textit{for all } n \geq n_0.$$

Proof. By using the same notation of the proof of Proposition 3.9, we have for any $R > 0$

$$\int_{B_R(\widetilde{y}_n)^c} \left(|\nabla u_n|^2 + |u_n|^2\right) = \int_{B_R(0)^c} \left(|\nabla v_n|^2 + |v_n|^2\right).$$

Since (v_n) strongly converges in $H^1(\mathbb{R}^N)$ the result follows. □

Lemma 3.11. *Let $\delta > 0$ and $\Pi_\delta = \{x \in \mathbb{R}^3 : dist(x, M) \leq \delta\}$. Then,*

$$\lim_{\varepsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} \inf_{y\in\Pi_\delta} |\beta_\varepsilon(u) - y| = \lim_{\varepsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u), \Pi_\delta) = 0.$$

Proof. The proof of this Lemma follows from well-known arguments and can be found in [5], Lemma 3.7. □

3.3. Multiplicity of solutions for $(\mathsf{P}_{\varepsilon,\mathsf{A}})$

Next we prove our multiplicity result for the problem $(P_{\varepsilon,A})$, by using arguments slightly different to those in [27], in fact, since S_ε^+ is a incomplete metric space, we cannot use (directly) an abstract result as in ([11], Thm. 2.1), instead, we invoke the category abstract result in [26].

Theorem 3.12. *Suppose that the function M satisfies (M_1)–(M_3), the potential V satisfies (V_1)–(V_2) and the function f satisfies (f_1)–(f_4). Then, given $\delta > 0$ there is $\overline{\varepsilon} = \overline{\varepsilon}(\delta) > 0$ such that the auxiliary problem $(P_{\varepsilon,A})$ has at least $\mathrm{Cat}_{\Pi_\delta}(\Pi)$ positive solutions, for all $\varepsilon \in (0, \overline{\varepsilon})$.*

Proof. For each $\varepsilon > 0$, we define the function $\zeta_\varepsilon : \Pi \to S_\varepsilon^+$ by

$$\zeta_\varepsilon(y) = m_\varepsilon^{-1}(\Upsilon_\varepsilon(y)),\ \forall y \in \Pi.$$

From the Lemma 3.8, we have

$$\lim_{\varepsilon\to 0} \Psi_\varepsilon(\zeta_\varepsilon(y)) = \lim_{\varepsilon\to 0} J_\varepsilon(\Upsilon_\varepsilon(y)) = c_0,\ \text{uniformly in } y \in \Pi.$$

Thus, the set

$$\widetilde{S}_\varepsilon^+ = \{u \in S_\varepsilon^+ : \Psi_\varepsilon(u) \leq c_0 + h_1(\varepsilon)\},$$

is nonempty, for all $\varepsilon \in (0, \overline{\varepsilon})$, because $\zeta_\varepsilon(\Pi) \subset \widetilde{S}_\varepsilon^+$, where the function h_1 was already introduced in the definition of the set $\widetilde{\mathcal{N}}_\varepsilon$.

From up above considerations, together with Lemma 3.8, Lemma 2.3 (A_3), equality (3.18) and Lemma 3.11, there is $\overline{\varepsilon} = \overline{\varepsilon}(\delta) > 0$, such that the diagram of continuous applications bellow is well defined for $\varepsilon \in (0, \overline{\varepsilon})$

$$\Pi \xrightarrow{\Upsilon_\varepsilon} \Upsilon_\varepsilon(\Pi) \xrightarrow{m_\varepsilon^{-1}} \zeta_\varepsilon(\Pi) \xrightarrow{m_\varepsilon} \Upsilon_\varepsilon(\Pi) \xrightarrow{\beta_\varepsilon} \Pi_\delta.$$

We conclude from (3.18) that there is a function $\lambda(\varepsilon, y)$ with $|\lambda(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in \Pi$, for all $\varepsilon \in (0, \overline{\varepsilon})$, such that $\beta_\varepsilon(\Upsilon_\varepsilon(y)) = y + \lambda(\varepsilon, y)$ for all $y \in \Pi$. Hence, the application $H : [0,1] \times \Pi \to \Pi_\delta$ defined by $H(t, y) = y + (1-t)\lambda(\varepsilon, y)$ is a homotopy between $\alpha_\varepsilon \circ \zeta_\varepsilon = \beta_\varepsilon \circ \Upsilon_\varepsilon$ and the inclusion $i : \Pi \to \Pi_\delta$, where $\alpha_\varepsilon = \beta_\varepsilon \circ m_\varepsilon$. Therefore,

$$\mathrm{cat}_{\zeta_\varepsilon(\Pi)} \zeta_\varepsilon(\Pi) \geq \mathrm{cat}_{\Pi_\delta}(\Pi). \tag{3.24}$$

It follows from Corollary 2.9 and from category abstract theorem in [26], with $c = c_\varepsilon \leq c_0 + h_1(\varepsilon) = d$ and $K = \zeta_\varepsilon(\Pi)$, that Ψ_ε has at least $\mathrm{cat}_{\zeta_\varepsilon(\Pi)}\zeta_\varepsilon(\Pi)$ critical points on $\widetilde{S}_\varepsilon^+$. So, from item (d) of the Proposition 2.4 and from (3.24), we conclude that J_ε has at least $\mathrm{cat}_{\Pi_\delta}(\Pi)$ critical points in $\widetilde{\mathcal{N}}_\varepsilon$. □

4. Proof of Theorem 1.1

In this section we prove our main theorem. The idea is to show that the solutions obtained in Theorem 3.12 verify the following estimate $u_\varepsilon(x) \leq a\ \forall x \in \Omega_\varepsilon^c$ for ε small enough. This fact implies that these solutions are in fact solutions of the original problem $(\widetilde{P}_\varepsilon)$. The key ingredient is the following result, whose proof uses an adaptation of the arguments found in [19], which are related to the Moser iteration method [22].

Lemma 4.1. *Let $\varepsilon_n \to 0^+$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of $(P_{\varepsilon_n, A})$. Then $J_{\varepsilon_n}(u_n) \to c_0$ and $u_n \in L^\infty(\mathbb{R}^3)$. Moreover, for any given $\gamma > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$|u_n|_{L^\infty(B_R(\tilde{y}_n)^c)} < \gamma, \qquad \textit{for all } n \geq n_0, \tag{4.1}$$

where $\tilde{y}_n$ is given by Proposition 3.9.

Proof. Since $J_{\varepsilon_n}(u_n) \leq c_0 + h_1(\varepsilon_n)$ with $\lim\limits_{n\to\infty} h_1(\varepsilon_n) = 0$, we can argue as in the proof of the inequality (3.20) to conclude that $J_{\varepsilon_n}(u_n) \to c_0$. Thus, we may invoke Proposition 3.9 to obtain a sequence $(\widetilde{y}_n) \subset \mathbb{R}^3$ satisfying the conclusions of that proposition.

Fix $R > 1$ and consider $\eta_R \in C^\infty(\mathbb{R}^3)$ such that $0 \leq \eta_R \leq 1$, $\eta_R \equiv 0$ in $B_{R/2}(0)$, $\eta_R \equiv 1$ in $B_R(0)^c$ and $|\nabla \eta_R| \leq C/R$. For each $n \in \mathbb{N}$ and $L > 0$, we define $\eta_n(x) := \eta_R(x - \tilde{y}_n)$, $u_{L,n} \in H^1(\mathbb{R}^3)$ and $z_{L,n} \in H_\varepsilon$ by

$$u_{L,n}(x) := \min\{u_n(x), L\}, \qquad z_{L,n} := \eta_n^2 u_{L,n}^{2(\beta-1)} u_n,$$

with $\beta > 1$ to be determined later.

From definition of $z_{L,n}$ and $J'_{\varepsilon_n}(u_n)z_{L,n}=0$, we have

$$m_0\left[\int_{\mathbb{R}^3}\eta_n^2u_{L,n}^{2(\beta-1)}|\nabla u_n|^2+2\int_{\mathbb{R}^3}\eta_nu_nu_{L,n}^{2(\beta-1)}\nabla\eta_n\cdot\nabla u_n\right] \leq\int_{\mathbb{R}^3}\left(g(\varepsilon_nx,u_n)-m_0V(\varepsilon_nx)u_n\right)\eta_n^2u_nu_{L,n}^{2(\beta-1)}.$$

Now, the result follows by arguing as in [6], Lemma 4.1. □

We are now ready to prove the main result of the paper.

4.1. Proof of Theorem 1.1

Suppose that $\delta>0$ is such that $\Pi_\delta\subset\Omega$. We first claim that there exists $\widetilde{\varepsilon}_\delta>0$ such that, for any $0<\varepsilon<\widetilde{\varepsilon}_\delta$ and any solution $u\in\widetilde{\mathcal{N}}_\varepsilon$ of the problem $(P_{\varepsilon,A})$, there holds

$$|u|_{L^\infty(\mathbb{R}^3\setminus\Omega_\varepsilon)}<a. \tag{4.2}$$

In order to prove the claim we argue by contradiction. So, suppose that for some sequence $\varepsilon_n\to0^+$ we can obtain $u_n\in\widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $J'_{\varepsilon_n}(u_n)=0$ and

$$|u_n|_{L^\infty(\mathbb{R}^3\setminus\Omega_{\varepsilon_n})}\geq a. \tag{4.3}$$

As in Lemma 4.1, we have that $J_{\varepsilon_n}(u_n)\to c_0$ and therefore we can use Proposition 3.9 to obtain a sequence $(\widetilde{y}_n)\subset\mathbb{R}^3$ such that $\varepsilon_n\widetilde{y}_n\to y_0\in\Pi$.

If we take $r>0$ such that $B_r(y_0)\subset B_{2r}(y_0)\subset\Omega$ we have that

$$B_{r/\varepsilon_n}(y_0/\varepsilon_n)=(1/\varepsilon_n)B_r(y_0)\subset\Omega_{\varepsilon_n}.$$

Moreover, for any $z\in B_{r/\varepsilon_n}(\widetilde{y}_n)$, there holds

$$\left|z-\frac{y_0}{\varepsilon_n}\right|\leq|z-\widetilde{y}_n|+\left|\widetilde{y}_n-\frac{y_0}{\varepsilon_n}\right|<\frac{1}{\varepsilon_n}(r+o_n(1))<\frac{2r}{\varepsilon_n},$$

for n large. For these values of n we have that $B_{r/\varepsilon_n}(\widetilde{y}_n)\subset\Omega_{\varepsilon_n}$ or, equivalently, $\mathbb{R}^3\setminus\Omega_{\varepsilon_n}\subset\mathbb{R}^3\setminus B_{r/\varepsilon_n}(\widetilde{y}_n)$. On the other hand, it follows from Lemma 4.1 with $\gamma=a$ that, for any $n\geq n_0$ such that $r/\varepsilon_n>R$, there holds

$$|u_n|_{L^\infty(\mathbb{R}^3\setminus\Omega_{\varepsilon_n})}\leq|u_n|_{L^\infty(\mathbb{R}^3\setminus B_{r/\varepsilon_n}(\widetilde{y}_n))}\leq|u_n|_{L^\infty(\mathbb{R}^3\setminus B_R(\widetilde{y}_n))}<a,$$

which contradicts (4.3) and proves the claim.

Let $\widehat{\varepsilon}_\delta>0$ given by Theorem 3.12 and set $\varepsilon_\delta:=\min\{\widehat{\varepsilon}_\delta,\widetilde{\varepsilon}_\delta\}$. We shall prove the theorem for this choice of ε_δ. Let $0<\varepsilon<\varepsilon_\delta$ be fixed. By applying Theorem 3.12 we obtain $\operatorname{cat}_{\Pi_\delta}(\Pi)$ nontrivial solutions of the problem $(P_{\varepsilon,A})$. If $u\in H_\varepsilon$ is one of these solutions we have that $u\in\widetilde{\mathcal{N}}_\varepsilon$, and therefore we can use (4.2) and the definition of g to conclude that $g_\varepsilon(\cdot,u)\equiv f(u)$. Hence, u is also a solution of the problem $(\widetilde{P}_\varepsilon)$. An easy calculation shows that $\widehat{u}(x):=u(x/\varepsilon)$ is a solution of the original problem (P_ε). Then, (P_ε) has at least $\operatorname{cat}_{\Pi_\delta}(\Pi)$ nontrivial solutions.

We now consider $\varepsilon_n\to0^+$ and take a sequence $u_n\in H_{\varepsilon_n}$ of solutions of the problem $(\widetilde{P}_{\varepsilon_n})$ as above. In order to study the behavior of the maximum points of u_n, we first notice that, by (g_1), there exists $\gamma>0$ such that

$$g(\varepsilon x,s)s\leq\frac{V_0}{K}s^2,\quad\text{for all } x\in\mathbb{R}^3,\ s\leq\gamma. \tag{4.4}$$

By applying Lemma 4.1 we obtain $R>0$ and $(\widetilde{y}_n)\subset\mathbb{R}^3$ such that

$$|u_n|_{L^\infty(B_R(\widetilde{y}_n))^c}<\gamma, \tag{4.5}$$

Up to a subsequence, we may also assume that

$$|u_n|_{L^\infty(B_R(\widetilde{y}_n))} \geq \gamma. \tag{4.6}$$

Indeed, if this is not the case, we have $|u_n|_{L^\infty(\mathbb{R}^3)} < \gamma$, and therefore it follows from $J'_{\varepsilon_n}(u_n) = 0$ and (4.4) that

$$m_0\|u_n\|^2_{\varepsilon_n} \leq \int_{\mathbb{R}^3} g(\varepsilon_n x, u_n)u_n \leq \frac{V_0}{K}\int_{\mathbb{R}^3} u_n^2.$$

The above expression implies that $\|u_n\|_{\varepsilon_n} = 0$, which does not make sense. Thus, (4.6) holds.

By using (4.5) and (4.6) we conclude that the maximum point $p_n \in \mathbb{R}^3$ of u_n belongs to $B_R(\widetilde{y}_n)$. Hence $p_n = \widetilde{y}_n + q_n$, for some $q_n \in B_R(0)$. Recalling that the associated solution of (P_{ε_n}) is of the form $\widehat{u}_n(x) = u_n(x/\varepsilon_n)$, we conclude that the maximum point η_n of $\widehat{u}_n$ is $\eta_n := \varepsilon_n\widetilde{y}_n + \varepsilon_n q_n$. Since $(q_n) \subset B_R(0)$ is bounded and $\varepsilon_n\widetilde{y}_n \to y_0 \in \Pi$ (according to Proposition 3.9), we obtain

$$\lim_{n\to\infty} V(\eta_{\varepsilon_n}) = V(y_0) = V_0,$$

which concludes the proof of the theorem.

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