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SOME NECESSARY AND SUFFICIENT CONDITIONS FOR THE OUTPUT CONTROLLABILITY OF TEMPORAL BOOLEAN CONTROL NETWORKS

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Abstract. This paper investigates the output controllability problem of temporal Boolean networks with inputs (control nodes) and outputs (controlled nodes). A temporal Boolean network is a logical dynamic system describing cellular networks with time delays. Using semi-tensor product of matrices, the temporal Boolean networks can be converted into discrete time linear dynamic systems. Some necessary and sufficient conditions on the output controllability *via* two kinds of inputs are obtained by providing corresponding reachable sets. Two examples are given to illustrate the obtained results.

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1. Introduction

The Boolean network (BN) was firstly proposed by Kauffman for modeling complex and nonlinear biological systems, see [21–23]. Since then, it has been a powerful tool in describing, analyzing, and simulating the cell networks. In this model, gene state is quantized to only two levels: true and false. Then the state of each gene is determined by the states of its neighboring genes *via* logical rules.

The control of BN is a challenging problem. So far, there are only few corresponding results because there is a shortage of systematic tool to deal with logical dynamic systems, see [6, 19]. Recently, a new matrix product, called semi-tensor product (STP) of matrices, has been proposed and a logical function can be expressed as an algebraic function, for example, see [7–11, 32]. Using STP, a logical equation can be expressed as an algebraic equation and the dynamics of a Boolean (control) network (BCN) can be converted into a linear (bilinear) discrete-time (control) system. Based on this method, the structure of attractors of a BN is investigated, and so called "rolling gears" structure is proposed in [7], which gives an explanation why tiny attractors can decide the vast order; formulas for calculating fixed points and cycles are obtained in [9]; coordinate transformation of BNs and the realization of BCNs are presented in [10]; A Mayer-type optimal control problem for BCNs with multi-input and single-input has been studied in [26, 28].

Keywords and phrases. Temporal Boolean (control) network, semi-tensor product, output controllability, time delay.

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Controllability is one of the fundamental concepts in control theory, see [17,24,39,45–47]. The state controllability problem of BCNs has been discussed by the expression of reachable set in [1,8]. [27,34,44] have presented some simple criteria to judge the controllability with respect to input-state incidence matrices of BCNs. The aim is to determine whether expressions of some selected genes (controlled nodes) can be inhibited or activated by expressions of the other gene (control nodes) in a gene regulatory network. In addition, Akutsu *et al.* proved that this problem is NP-hard in a general setting in [1].

Output controllability is a related notion for the output of the system. The output controllability describes the ability of an external input to move the output from any initial condition to any final condition in a finite time interval. A controllable system is not necessarily output controllable, and an output controllable system is not necessarily controllable. Hence, it is also an important structure property in modern control theory which reflects the dominant ability of the control inputs over outputs [12]. For switched linear systems, [41] gave a necessary and sufficient geometric type criterion for output controllability. In [25], a sufficient condition for a BCN with inputs and outputs to be output-controllable is derived by exploiting an adjacency matrix of its network topology.

It is well known that time delay phenomenon is very common in real world, and is very important in analysis and control of dynamic systems [3,18,36]. Besides, time delay happens frequently in biological and physiological systems [16,29,37,40,43]. In [4], a model for genetic regulatory networks (GRNs) with time delays was proposed and nonlinear properties of the model in terms of local stability and bifurcation was analyzed. In [2], sufficient conditions have been derived to ensure the global exponential stability of the discrete-time GRNs with delays. Many results have been obtained on the state controllability of delayed systems [13,35,42]. For BCNs with time delays, the controllability and observability are respectively investigated in [30,33].

One kind of BNs, called temporal Boolean networks (TBN) were developed to model regulatory delays, which may be caused by missing intermediary genes and spatial or biochemical delays between transcription and regulation, see [5,15,30,38]. References [5,30] investigated the controllability and the global controllability issues of μth order Boolean control networks respectively, which could be regarded as one kind of TBCNs considered in our work. Firstly, the μth order Boolean control networks were converted into new BCNs with much higher dimensions state expressed by z in both [5,30]. Then, with similar analysis as [8], authors got the controllability of new BCNs *via* two types of controls in [30]. At last, with model reconstruction, some necessary and sufficient conditions were obtained for the controllability of the original μth order Boolean control networks. The same method was also used in [5], which investigated the global controllability of the new BCNs with state z *via* Perron-Frobenius Theory as presented in [27]. Then the global controllability of the μth order Boolean control networks was deduced with the same method of model reconstruction as [30]. In [14], the problem of inferring genetic networks under the TBN model was considered. One can see that the BCN with time delays in states considered in [30,33] is a special case of temporal Boolean control network (TBCN) according to definitions. Hence, the analysis on TBCN may be much more complex and challenging. It should be noticed that TBCN is similar with higher-order Boolean control network according to Chapter 5 of [5,11,31] in which the higher-order Boolean control network can be rewritten by a BCN by using the first algebraic form of the network. Hence, the controllability analysis for higher-order Boolean control networks can be obtained from the analysis of BCNs. However, if the first algebraic form is used, the dimension of network transition matrix depending on the number of logical variables will be much larger which would make computation cost much higher. For details, please refer to Remark 3.2. Motivated by above analysis, in this paper, we will consider the output controllability of the TBCN without changing it into BCN. The main idea comes from [8,27,44].

The rest of the paper is organized as follows. Section 2 provides a brief review for the STP of matrices and the matrix expression of logical function in [6]. In Section 3, we convert the TBCNs with inputs and outputs into discrete time delay systems. Then some necessary and sufficient conditions on the output controllability *via* two kinds of inputs are present in Section 4. Examples are given to illustrate the obtained results as well. Finally, Section 5 gives a brief conclusion.

2. Preliminaries

Definition 2.1 [9]**.**

(1) Let X be a row vector of dimension np , and Y be a column vector of dimension p. Then we split X into equal-size blocks as $X^1, \dots, X^p$, which are $1 \times p$ rows. Define the STP, denoted by $\ltimes$, as

$$\begin{cases} X \ltimes Y = \sum_{i=1}^{p} X^i y_i \in R^n \\ Y^T \ltimes X^T = \sum_{i=1}^{p} y_i (X^i)^T \in R^n \end{cases}$$

(2) Let $A \in M_{m\times n}$ and $B \in M_{p\times q}$. If either n is a factor of p, say $nt = p$ and denote it as $A \prec_t B$, or p is a factor of n, say $n = pt$ and denote it as $A \succ_t B$, then we define the STP of A and B, denoted by $C = A \ltimes B$, as the follows: C consists of $m \times q$ blocks as $C = C^{ij}$ and each block is

$$C^{ij} = A^i \ltimes B_j, i = 1, \dots, m, j = 1, \dots, q$$

where A^i is the ith row of A and B_j is the jth column of B.

It is obvious that when $n = p$, $A \ltimes B = AB$. So it is a generalization of the conventional matrix product, and all the fundamental properties of conventional matrix product can be applied to the STP of matrices, *e.g.*, distributive rule, associative rule and so on. Because of this, we can omit $\ltimes$ in this paper, see [11].

Proposition 2.2 [9]**.**

(1) Assume $A \succ_t B$, then (where $\otimes$ is the Kronecker product, I_t is the identity matrix with dimensions t)

$$A \ltimes B = A(B \otimes I_t).$$

Assume $A \prec_t B$, then

$$A \ltimes B = (A \otimes I_t)B.$$

(2) Assume $A \in M_{m\times n}$ is given. Let $Z \in R^t$ be a row vector. Then

$$A \ltimes Z = Z \ltimes (I_t \otimes A).$$

Let $Z \in R^t$ be a column vector. Then

$$Z \ltimes A = (I_t \otimes A) \ltimes Z.$$

Furthermore, we give some notations as following:

- Define a delta set as $\Delta_k := \{\delta_k^i | i = 1, 2, \dots k\}$, where δ_k^i is the ith column of I_t.
- A matrix $A \in M_{m\times n}$ is called a logical matrix if the columns of A, denoted by $Col(A)$, satisfy $Col(A) \subset \Delta_m$.
- The set of all $m \times n$ logical matrices is denoted by $\mathcal{L}_{m\times n}$.
- If matrix $A = [\delta_m^{i_1}, \delta_m^{i_2}, \dots, \delta_m^{i_n}]$, we denote it as $A = \delta_m[i_1, i_1, \dots, i_n]$.

Definition 2.3 [9]**.** An $mn \times mn$ matrix $W_{m,n}$ is called swap matrix, if it is constructed in the following way: label its columns by $(11, 12, \dots, 1n, \dots, m1, m2, \dots, mn)$ and its rows by $(11, 21, \dots, m1,$
$\dots, 1n, 2n, \dots, mn)$. Then its element in the position $((I, J), (i, j))$ is assigned as

$$w_{(I,J),(i,j)} = \delta_{i,j}^{I,J} = \begin{cases} 1, & I = i \text{ and } J = j, \\ 0, & otherwise. \end{cases} \tag{2.1}$$

When $m = n$, we briefly denote $W_{[n]} := W_{[m,n]}$.

Proposition 2.4. *Let $X \in R^m$ and $Y \in R^n$ be two columns. Then*

$$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X, \quad W_{[n,m]} \ltimes Y \ltimes X = X \ltimes Y.$$

A logical domain, denoted by $\mathcal{D}$, is defined as $\mathcal{D} = \{T = 1, F = 0\}$. To use matrix expression we identify each element in $\mathcal{D}$ with a vector as $T \sim \delta_2^1$ and $F \sim \delta_2^2$, and denote $\varDelta := \varDelta_2 = \{\delta_2^1,\ \delta_2^2\}$. Using STP of matrices, a logical function with n arguments $f:\ \mathcal{D}^n \to \mathcal{D}$ can be expressed in its algebraic form as follows:

Lemma 2.5 [9]**.** *Any logical function $f(a_1, \dots, a_n)$ with logical arguments $a_1, \dots, a_n \in \varDelta$ can be expressed in a multi-linear form as*

$$f(a_1, \dots, a_n) = M_f a_1 a_2 \dots a_n$$

where $M_f \in \mathcal{L}_{2\times 2^n}$ is unique and called the structure matrix of f.

Given logical arguments $p, q \in \varDelta$, we have the following structure matrices for the fundamental logical functions: $\neg p = M_n p,\ p \vee q = M_d pq,\ P \wedge q = M_c pq,\ p \to q = M_i pq,\ p \leftrightarrow q = M_e pq$, where $M_n = \delta_2[2, 1],\ M_d = \delta_2[1, 1, 1, 2],\ M_c = \delta_2[1, 2, 2, 2],\ M_i = \delta_2[1, 2, 1, 1],\ M_e = \delta_2[1, 2, 2, 1]$.

Lemma 2.6 [9]**.** *Assume $p_k = a_1 a_2 \dots a_k$ with logical arguments $a_1, \dots, a_k \in \varDelta$, then*

$$p_k^2 = \varPhi_k p_k$$

where $\varPhi_k = \prod_{i=1}^k I_{2^{i-1}} \otimes [(I_2 \otimes W_{[2,2^{k-i}]})M_r],\ M_r = \delta_4[1, 4]$.

3. Algebraic form of TBCNs

A BN of a set of nodes $a_1, \dots, a_n \in \varDelta$ can be described as:

$$a_i(t+1) = f_i(a_1(t), a_2(t), \dots, a_n(t)),\ i = 1, 2, \dots, n, \tag{3.1}$$

where $f_i, i = 1, 2, \dots, n$ are logical functions, $t = 0, 1, 2, \dots$. Note that time delay phenomena are very common in nature, and it is well known that, in many cases, time delay cannot be avoided in practice. Motivated by this, we consider the TBNs [38] as follows:

$$\begin{aligned} a_i(t+1) = f_i(&a_1(t), \dots, a_n(t), a_1(t-1), \dots, a_n(t-1), \\ &\dots, a_1(t-\tau), \dots, a_n(t-\tau)),\ i = 1, 2, \dots, n, \end{aligned} \tag{3.2}$$

where τ is a positive integer delay.

Let $x(t) = \ltimes_{i=1}^n a_i(t)$ which is a bijective mapping pointed out by Cheng and Qi [9]. Using Lemma 2.5, for each logical function $f_i,\ i = 1, 2, \dots, n$, we can find its structure matrix M_i. Then system (3.2) can be converted into an algebraic form as:

$$\begin{aligned} a_i(t+1) &= M_i \ltimes_{j=1}^n a_j(t) \ltimes_{j=1}^n a_j(t-1) \dots \ltimes_{j=1}^n a_j(t-\tau) \\ &= M_i x(t) x(t-1) \dots x(t-\tau),\ i = 1, \dots, n. \end{aligned} \tag{3.3}$$

Multiplying all system in (3.3) together yields:

$$\begin{aligned} x(t+1) &= \ltimes_{i=1}^n a_i(t+1) \\ &= \ltimes_{i=1}^n [M_i x(t) x(t-1) \dots x(t-\tau)]. \end{aligned} \tag{3.4}$$

Theorem 3.1. *Equation (3.4) can be expressed as*

$$x(t+1) = L_0 x(t) x(t-1) \dots x(t-\tau), \tag{3.5}$$

where $L_0 = M_1[\ltimes_{i=2}^n I_{n(\tau+1)} \otimes M_i \varPhi_{n(\tau+1)}]$ and L_0 is called the network transition matrix of (3.2).

Proof. By Lemma 2.6, $[x(t)x(t-1)\dots x(t-\tau)]^2 = \Phi_{n(\tau+1)}x(t)x(t-1)\dots x(t-\tau)$. Then

$$\begin{aligned} x(t+1) &= \ltimes_{i=1}^{n}\left[M_i x(t)x(t-1)\dots x(t-\tau)\right] \\ &= M_1[(I_{2^{n(\tau+1)}}\otimes M_2)\Phi_{n(\tau+1)}]x(t)x(t-1)\dots x(t-\tau)M_3 \\ &\quad \dots M_n x(t)x(t-1)\dots x(t-\tau) \\ &= M_1[\ltimes_{i=2}^{3} I_{2^{n(\tau+1)}}\otimes M_i\Phi_{n(\tau+1)}]x(t)x(t-1)\dots x(t-\tau)M_4 \\ &\quad \dots M_n x(t)x(t-1)\dots x(t-\tau) \\ &= \dots \\ &= M_1[\ltimes_{i=2}^{n} I_{2^{n(\tau+1)}}\otimes M_i\Phi_{n(\tau+1)}]x(t)x(t-1)\dots x(t-\tau). \end{aligned} \tag{3.6}$$

□

Next, we consider TBCN with outputs as follows:

$$\begin{cases} a_i(t+1) = f_i(u_1(t),\dots u_m(t), a_1(t),\dots,a_n(t), a_1(t-1),\dots,a_n(t-1), \\ \qquad\qquad \dots, a_1(t-\tau),\dots,a_n(t-\tau)),\ i=1,\dots,n, \\ y_j(t) = h_j(a_1(t),\dots,a_n(t)),\ j=1,\dots,p, \end{cases} \tag{3.7}$$

where $u_i, i=1,2,\dots,m$ are inputs (or controls); $y_j(t),\ j=1,\dots,p$ are outputs; $f_i,\ i=1,\dots,n,\ h_j,\ j=1,\dots,p$ are logical functions. In this paper, two kinds of inputs (or controls) are considered:

(A) The controls satisfy certain logical rules, called input networks such as:

$$u_i(t+1) = g_i(u_1(t),u_2(t),\dots,u_m(t)),\ i=1,2,\dots,m, \tag{3.8}$$

where $g_i, i=1,2,\dots,m$ are logical functions, and the initial states $u_j(0), j=1,2,\dots,m$, can be arbitrarily given.

(B) The controls are free Boolean sequences (or designable).

Let $u(t)=\ltimes_{j=1}^{m}u_j(t),\ y(t)=\ltimes_{j=1}^{p}y_j(t)$. By Lemma 2.5, for every logical function $f_i,\ g_j$ and h_l, we can find its structure matrix $M_{1i},\ M_{2j}$ and $M_{3l},\ i=1,\dots,n,\ j=1,\dots,m,\ l=1,\dots,p$, respectively. Then from (3.7) and (3.8), we can obtain

$$a_i(t+1) = M_{1i}u(t)x(t)\dots x(t-\tau),\ i=1,\dots,n, \tag{3.9}$$

$$u_j(t+1) = M_{2j}u(t),\ j=1,\dots,m, \tag{3.10}$$

$$y_l(t) = M_{3l}x(t),\ l=1,\dots,p. \tag{3.11}$$

Similar with Theorem 3.1, multiplying (3.9) yields $x(t+1) = Lu(t)x(t)x(t-1)\dots x(t-\tau)$ with $L = M_{11}[\ltimes_{i=2}^{n}(I_{2^{m+n(\tau+1)}}\otimes M_{1i}\Phi_{m+n(\tau+1)})]$. Multiplying (3.10) leads to $u(t+1)=Gu(t)$ with $G = M_{21}(I_{2^m}\otimes M_{22})\Phi_m(I_{2^m}\otimes M_{23})\Phi_m\dots(I_{2^m}\otimes M_{2m})\Phi_m$. And multiplying (3.11) gives $y(t) = Hx(t)$, where $H = M_{31}[\ltimes_{l=2}^{p}(I_{2^n}\otimes M_{3l}\Phi_n)]$. Based on above analysis, a TBCN (3.7, 3.8) can be expressed in an algebraic form as follows,

$$\begin{cases} x(t+1) = Lu(t)x(t)x(t-1)\dots x(t-\tau), \\ y(t) = Hx(t), \end{cases} \tag{3.12}$$

$$\text{and } u(t+1) = Gu(t), \tag{3.13}$$

where $L,\ H$ are respectively the network transition matrices of two equations in (3.7), and G is the network transition matrix of (3.8).

Remark 3.2. It should be noticed that by using the first algebraic form of the network from Chapter 5 of [11], TBCN or μth order Boolean control network can be rewritten by a BCN with no delay. Hence, it can be a good idea to study the controllability of TBCNs by using the corresponding BCNs from the results in [8, 27], see references [5, 30]. However, if the first algebraic form is used, the dimension of network transition matrix of corresponding BCNs will be much bigger which would make computation cost much higher, see the dimension of states in system (5) of [30] and system (5) of [5]. From (3.12), it is easy to calculate that the dimension of L is $2^n \times 2^{n(\tau+1)+m}$. However, if the TBCNs are rewritten by BCNs using the first algebraic form, then the dimension of the corresponding network transition matrix of the BCNs would be $2^{n(\tau+1)} \times 2^{n(\tau+1)+m}$, which is much bigger if n or τ is a large number. Hence, though the proof of our theorems is more complex relatively, the dimension of the input-state transfer matrix is much less than the method of using the first algebraic form, and hence the cost of the computation will be less compared with [5, 30] if n and τ are large numbers. Let us take $n = 6,\ m = 1,\ \tau = 4$ for example, the dimensions of the network transition matrices are 64×2147483648 and $1073741824 \times 2147483648$, respectively. Thus, it is easy to see the computation cost would be much higher if the TBCNs are rewritten by using the above-mentioned method. Furthermore, considering the TBCNs directly, we can find the relationship between the network transition matrix (or the Boolean functions) of the TBCN and the state clearly, see (3.12). However, if the new BCN is used, the relationship of states x would not be so clear, see system (5) in [30] and system (5) in [5], where the z coming from multiplying x at different times concerning μ.

4. Output controllability of TBCNs

In this section, we consider the output controllability problem of TBCN (3.7) with inputs and outputs, equivalently (3.12), and the analysis will be given *via* two kinds of inputs. Definitions 4.1 and 4.13 are similar with the ones used in [1, 8, 33] for controllability with respect to fixed initial states. Note that the following Definitions 4.2 and 4.14 of controllability is different from them, and can be better coincide with the classical definition of controllability in linear systems theory (see, *e.g.*, [20]).

4.1. Output controllability of input Boolean networks

In this subsection, we consider case (A).

Definition 4.1. Consider the TBCN (3.12) with control (3.13). Given the finite time $s \in \mathbb{N}^+$, initial state sequence $x(-i),\ i \in \{0, 1, \ldots, \tau\}$ and the destination output $y_f \in \Delta_{2^p}$, y_f is said to be $s-$*output*$-$*controllable* (*or reachable*) from initial state sequence with fixed (designable) input structure G, if we can find control input $u(0)$ (and G), such that $y(s) = y_f$.

Definition 4.2. The TBCN (3.12) with control (3.13) is said to be $s-$*output*$-$*controllable* (*or reachable*) if for any $a_i \in \Delta_{2^n},\ i \in \{0, 1, \ldots, \tau\}$ and $b \in \Delta_{2^p}$, there exist the finite time $s \in \mathbb{N}^+$ and the control input $u(0)$ such that $x(-i) = a_i,\ i \in \{0, 1, \ldots, \tau\}$ to $y(s) = b$.

Remark 4.3. Definition 4.1 is based on fixed initial state sequence and destination output, and it describes the controllability of the destination output. Definition 4.2 is for any initial state sequence and destination output, and it represents the controllability of the system. The following controllability analysis will be given with respect to both definitions one by one.

Definition 4.4. For BCN without time delay, and controller (3.13) with fixed G, the input-state transfer matrix $\Theta_i^G,\ i \in \mathbb{N}^+$ is defined as follows: for any $u(0) \in \Delta_{2^m}$ and any $x(0) \in \Delta_{2^n}$, we have

$$x(i) = \Theta_i^G u(0)x(0),\ i \in \mathbb{N}^+. \tag{4.1}$$

From Definition 4.4, the input-state transfer matrix for BCN shows a clear relationship between the input and state. Given an initial state and an input, we can get the state at any time if the input-state transfer matrix is obtained. Similar with the statement of input-state transfer matrix for BCN, we have the corresponding input-state transfer matrix for TBCN in the following definition. For simplicity, we first denote the vector

$$X(i) = \ltimes_{j=0}^{i} x(-j) \in \Delta_{2^{n(i+1)}},\ i \in \{0, 1, \dots, \tau\}. \tag{4.2}$$

Definition 4.5. For TBCN (3.12) and controller (3.13) with fixed G, the input-state transfer matrix $L_i^G,\ i \in \mathbb{N}^+$ is defined as follows: for any $u(0) \in \Delta_{2^m}$ and any $x(-i) \in \Delta_{2^n},\ i \in \{0, 1, \dots, \tau\}$, we have

$$x(i) = L_i^G u(0) X(\tau),\ i \in \mathbb{N}^+, \tag{4.3}$$

where $X(\tau) = \ltimes_{j=0}^{\tau} x(-j) \in \Delta_{2^{n(\tau+1)}}$ from (4.2).

We start from the case of fixing s and fixing G. Theorem 4.6 and Corollary 4.8 will present the controllability for TBCNs with respect to given initial $x(-i),\ i \in \{0, 1, \dots, \tau\}$ and y_f, *i.e.*, in the sense of Definition 4.1.

Theorem 4.6. *Consider the TBCN (3.12) with control (3.13), where G is fixed. y_f is s-output-reachable from $x(-i),\ i \in \{0, 1, \dots, \tau\}$, if and only if $y_f^\top H L_s^G W_{[2^{n(\tau+1)}, 2^m]} X(\tau) \neq (\underbrace{0, \dots, 0}_{2^m})^\top$ or equivalently, there exists at least one entry of $y_f^\top H L_s^G W_{[2^{n(\tau+1)}, 2^m]} X(\tau)$ equaling 1, where*

$$L_t^G = \begin{cases} L, & t = 1, \\ LG[(I_{2^m} \otimes L_1^G)\Phi_m][I_{2^m} \otimes W_{2^{n\tau}, 2^{n(\tau+1)}} \Phi_{n(\tau)}], & t = 2, \\ LG^{s-1}[(I_{2^m} \otimes L_{s-1}^G)\Phi_m][\ltimes_{i=s-2}^{1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})] & \\ \qquad \ltimes [I_{2^m} \otimes W_{[2^{n(\tau-s+2)}, 2^{n(\tau+1)}]} \Phi_{n(\tau-t+2)}], & s = 3, \dots, \tau+1, \\ LG^{s-1}[(I_{2^m} \otimes L_{s-1}^G)\Phi_m][\ltimes_{i=s-2}^{s-\tau-1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})], & s > \tau+1. \end{cases} \tag{4.4}$$

If it is the ith entry, then $u(0) = \delta_{2^m}^i$.

Proof. A straightforward computation gives that

$$\begin{aligned} x(1) &= Lu(0)X(\tau) \triangleq L_1^G u(0) X(\tau), \\ x(2) &= Lu(1)x(1)X(\tau-1) \\ &= LGu(0)L_1^G u(0)X(\tau)X(\tau-1) \\ &= LG[(I_{2^m} \otimes L_1^G)\Phi_m]u(0)X(\tau)X(\tau-1) \\ &= LG[(I_{2^m} \otimes L_1^G)\Phi_m]u(0)W_{[2^{n\tau}, 2^{n(\tau+1)}]}\Phi_{n\tau}X(\tau) \\ &= LG[(I_{2^m} \otimes L_1^G)\Phi_m][I_{2^m} \otimes W_{[2^{n\tau}, 2^{n(\tau+1)}]}\Phi_{n\tau}]u(0)X(\tau) \\ &\triangleq L_2^G u(0)X(\tau), \\ x(3) &= Lu(2)x(2)x(1)X(\tau-2) \\ &= LG^2 u(0)L_2^G u(0)X(\tau)L_1^G u(0)X(\tau)X(\tau-2) \\ &= LG^2[(I_{2^m} \otimes L_2^G)\Phi_m]u(0)X(\tau)L_1^G u(0)X(\tau)X(\tau-2) \\ &= LG^2[(I_{2^m} \otimes L_2^G)\Phi_m][(I_{2^{m+n(\tau+1)}} \otimes L_1^G)\Phi_{m+n(\tau+1)}]u(0)X(\tau)X(\tau-2) \\ &= LG^2[(I_{2^m} \otimes L_2^G)\Phi_m][(I_{2^{m+n(\tau+1)}} \otimes L_1^G)\Phi_{m+n(\tau+1)}] \\ &\quad \ltimes [I_{2^m} \otimes W_{[2^{n(\tau-1)}, 2^{n(\tau+1)}]}\Phi_{n(\tau-1)}]u(0)X(\tau) \\ &\triangleq L_3^G u(0)X(\tau). \end{aligned}$$

For $3 \leq t \leq \tau$, we assume that

$$\begin{aligned}L_t^G &= LG^{t-1}[(I_{2^m} \otimes L_{t-1}^G)\Phi_m][\ltimes_{i=t-2}^{1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})] \\ &\quad \ltimes [I_{2^m} \otimes W_{[2^{n(\tau-t+2)},2^{n(\tau+1)}]}\Phi_{n(\tau-t+2)}].\end{aligned}$$

Then for $s = t+1$, we have that

$$\begin{aligned}x(t+1) &= Lu(t)x(t)\dots x(1)X(0) \\ &= LG^t u(0)[\ltimes_{i=t}^{1} L_i^G u(0)X(\tau)]X(0) \\ &= LG^t[(I_{2^m} \otimes L_t^G)\Phi_m][\ltimes_{i=t-1}^{1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})] \\ &\quad \ltimes [I_{2^m} \otimes W_{[2^{n(\tau-t+1)},2^{n(\tau+1)}]}\Phi_{n(\tau-t+1)}],\end{aligned}$$

and

$$\begin{aligned}L_{t+1}^G &= LG^t[(I_{2^m} \otimes L_t^G)\Phi_m][\ltimes_{i=t-1}^{1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})] \\ &\quad \ltimes [I_{2^m} \otimes W_{[2^{n(\tau-t+1)},2^{n(\tau+1)}]}\Phi_{n(\tau-t+1)}].\end{aligned}$$

By mathematical induction, one can get that

$$\begin{aligned}L_1^G &= L, \\ L_2^G &= LG[(I_{2^m} \otimes L_1^G)\Phi_m][I_{2^m} \otimes W_{2^{n\tau},2^{n(\tau+1)}}\Phi_{n(\tau)}], \\ L_t^G &= LG^{t-1}[(I_{2^m} \otimes L_{t-1}^G)\Phi_m][\ltimes_{i=t-2}^{1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})] \\ &\quad \ltimes [I_{2^m} \otimes W_{[2^{n(\tau-t+2)},2^{n(\tau+1)}]}\Phi_{n(\tau-t+2)}],\ t = 3,\dots,\tau+1.\end{aligned} \tag{4.5}$$

Furthermore,

$$\begin{aligned}x(\tau+2) &= Lu(\tau+1)x(\tau+1)\dots x(1) \\ &= LG^{\tau+1}u(0)[\ltimes_{i=\tau+1}^{1} L_i^G u(0)X(\tau)] \\ &= LG^{\tau+1}[(I_{2^m} \otimes L_{\tau+1}^G)\Phi_m][\ltimes_{i=\tau}^{1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})]u(0)X(\tau) \\ &\triangleq L_{\tau+2}^G u(0)X(\tau), \\ x(\tau+3) &= Lu(\tau+2)x(\tau+2)\dots x(2) \\ &= LG^{\tau+2}u(0)[\ltimes_{i=\tau+2}^{2} L_i^G u(0)X(\tau)] \\ &= LG^{\tau+2}[(I_{2^m} \otimes L_{\tau+2}^G)\Phi_m][\ltimes_{i=\tau+1}^{2}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})]u(0)X(\tau) \\ &\triangleq L_{\tau+3}^G u(0)X(\tau).\end{aligned}$$

For $t > \tau+1$, we assume that

$$L_t^G = LG^{t-1}[(I_{2^m} \otimes L_{t-1}^G)\Phi_m][\ltimes_{i=t-2}^{s-\tau-1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})]. \tag{4.6}$$

Then for $s = t+1$, one can obtain that

$$\begin{aligned}x(t+1) &= Lu(t)x(t)\dots x(t-\tau) \\ &= LG^t u(0)[\ltimes_{i=t}^{s-\tau} L_i^G u(0)X(\tau)] \\ &= LG^t[(I_{2^m} \otimes L_t^G)\Phi_m][\ltimes_{i=t-1}^{t-\tau}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})]u(0)X(\tau),\end{aligned}$$

and

$$L_{t+1}^G = LG^t[(I_{2^m} \otimes L_t^G)\Phi_m][\ltimes_{i=t-1}^{t-\tau}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})]u(0)X(\tau).$$

By mathematical induction, we have

$$L_s^G = LG^{s-1}[(I_{2^m} \otimes L_{s-1}^G)\Phi_m][\ltimes_{i=s-2}^{s-\tau-1}(I_{2^{m+n(\tau+1)}} \otimes L_i^G \Phi_{m+n(\tau+1)})], \ s > \tau+1.$$

Hence, we get the explicit expression of L_s^G as (4.4). Moreover,

$$x(s) = L_s^G u(0)X(\tau) = L_s^G W_{[2^{n(\tau+1)},2^m]} X(\tau)u(0), \ s \in \mathbb{N}^+. \tag{4.7}$$

It follows from (3.12) and (4.7) that

$$y(s) = HL_s^G W_{[2^{n(\tau+1)},2^m]} X(\tau)u(0), \ s \in \mathbb{N}^+.$$

It is noticed that $HL_s^G W_{[2^{n(\tau+1)},2^m]} X(\tau)$ is an $2^p \times 2^m$ matrix whose columns are elements in Δ_{2^p}. Since $y_f \in \Delta_{2^p}$, $y_f^\top HL_s^G W_{[2^{n(\tau+1)},2^m]} X(\tau) \neq [\underbrace{0,\ldots,0}_{2^m}]^\top$ means that at least one column of the matrix $HL_s^G W_{[2^{n(\tau+1)},2^m]} X(\tau)$ equals to y_f. Then we can get the conclusion by $u(0) \in \Delta_{2^m}$. □

Remark 4.7. When the time delay $\tau = 0$, then the TBCN (3.12), (3.13) become a Boolean control network. In this case, it can be induced from (4.4) that

$$L_t^G = \begin{cases} L, & t = 1, \\ LG^{t-1}[(I_{2^m} \otimes L_{t-1}^G)\Phi_m], & t > 1. \end{cases} \tag{4.8}$$

Then Theorems 9 of [8] on the controllability of the BCNs respects to Definition 4.1 can be directly deduced from Theorem 4.6.

Now, we consider the case where s is fixed and G is designable. Notice that there are $(2^m)^{2^m}$ possible distinct Gs. Each G can be expressed in the condensed form and ordered in *increasing order*, see [9]. When $m = 2$, we have $G_1 = \delta_4[1111]$, $G_2 = \delta_4[1112]$, $G_3 = \delta_4[1113], \ldots, G_{256} = \delta_4[4444]$. In general, we consider a subset $\Lambda \subset \{1, 2, \ldots, (2^m)^{2^m}\}$, and allow G to be chosen from the admissible set $\{G_\lambda | \lambda \in \Lambda\}$. The following result can be obtained immediately from Theorem 4.6.

Corollary 4.8. *Consider the TBCN (3.12) with control (3.13), where $G \in \{G_\lambda | \lambda \in \Lambda\}$. Then y_f is s-output-reachable from $x(-i)$, $i \in \{0, 1, \ldots, \tau\}$, if and only if there exists at least one entry of $\{y_f^\top HL_s^{G_\lambda} W_{[2^{n(\tau+1)},2^m]} X(\tau) | \lambda \in \Lambda\}$ equaling 1, where L_s^G is given by (4.4).*

For TBCNs without time delays, it is noticed that the matrix M in [44] which equals to Q in [27] is induced to show the controllability of BCNs with respect to any initial states and destination states. Motivated by [44] and [27], in the following, we will give necessary and sufficient conditions on the controllability of TBCNs (3.12) in terms of Definition 4.2.

Proposition 4.9. *The number of different controls $u(0)$ that steer TBCNs (3.12) with control (3.13) from $x(-i)$, $i \in \{0, 1, \ldots, \tau\}$ to $y(s) = y_f$ in s time steps is*

$$l(s; X(\tau), y_f) = y_f^\top Q_s X(\tau), \ s \in \mathbb{N}^+,$$

where $Q_s = HL_s^G 1_{2^m}$, $1_{2^m} = [\underbrace{1,\ldots,1}_{2^m}]^\top$ and L_s^G is given by (4.4).

Proof. Let $w^1(0), \ w^2(0), \ldots, w^{l(s;X(\tau),y_f)}(0)$ be the different control steer $x(-i)$, $i \in \{0, 1, \ldots, \tau\}$ to $y(s) = y_f$, *i.e.*,

$$y_f = HL_s^G w^i(0)X(\tau), \ i = 1, 2, \ldots, l(s; X(\tau), y_f), \tag{4.9}$$

from (4.7). Since each control $w^i(0) \in \Delta_{2^m}$, this implies that there exist $t(s; X(\tau), y_f) = 2^m - l(s; X(\tau), y_f)$ different control $v^j(0) \in \Delta_{2^m}$ such that

$$y_f \neq HL_s^G v^j(0)X(\tau), \; j = 1, 2, \ldots, t(s; X(\tau), y_f). \tag{4.10}$$

Multiplying (4.9) and (4.10) from the left by $y_f^\top$ yields

$$\begin{aligned} 1 &= y_f^\top HL_s^G w^i(0)X(\tau), \; i = 1, 2, \ldots, l(s; X(\tau), y_f), \\ 0 &= y_f^\top HL_s^G v^j(0)X(\tau), \; j = 1, 2, \ldots, t(s; X(\tau), y_f). \end{aligned} \tag{4.11}$$

Summing up this set of 2^m equations yields

$$l(s; X(\tau), y_f) = y_f^\top HL_s^G 1_{2^m} X(\tau) = y_f^\top Q_s X(\tau). \tag{4.12}$$

The proof is completed. □

Theorem 4.10. *The TBCN (3.12) with control (3.13) is s-output-controllable if and only if all the entries of Q_s are different from zero.*

Proof. Necessity: Suppose that entry (i, j) in Q_s is zero, then

$$\delta_{2^p}^i Q_s \delta_{2^{n(\tau+1)}}^j = 0.$$

From Theorem 4.9, there is no control $u(0)$ such that $X(\tau) = \delta_{2^{n(\tau+1)}}^j$ and $y(s) = \delta_{2^p}^i$. Hence, the TBCN is not s-output-controllable.

Sufficiency: From the Proof of Proposition 4.9, we can observe that if all the entries of Q_s are different from zero, then they are all positive. Hence $\delta_{2^p}^i Q_s \delta_{2^{n(\tau+1)}}^j > 0$ for any $i \in \{1, 2, \ldots, 2^p\}$ and $j \in \{1, 2, \ldots, 2^{n(\tau+1)}\}$, and further the TBCN is s-output-controllable. □

Now we give an algorithm to find a control, which drives given $x(-i), \; i \in \{0, 1, \ldots, \tau\}$ to $y(s) = y_f$ in s time steps. Since the trajectory from $X(\tau)$ to $y(s)$ is in general not unique, see Proposition 4.9, we only try to find one of them. A similar way can produce all the required trajectories. Assume $X(\tau) = \delta_{2^{n(\tau+1)}}^i$ and $y(s) = \delta_{2^p}^j$. We give the following algorithm.

Algorithm 4.11. *Assume the TBCN is given with its logical expression as (3.7) and input networks as (3.8).*

(A) Convert (3.7) and (3.8) into a linear discrete time delay system as (3.12) and (3.13) such that G, L, H can be expressed by matrices.
(B) Compute L_s^G by (4.4).
(C) Get $l(s; X(\tau), y(s)) = y(s)^\top Q_s X(\tau)$ to see the number of different controls $u(0)$ that steers the TBCN from $X(\tau)$ to $y(s)$. If $l(s; X(\tau), y(s)) = 0$, it means there is no existence of such $u(0)$, then stop.
(D) Find which entry of vector $y(s)^\top HL_s^G W_{[2^{n(\tau+1)}, 2^m]} X(\tau)$ equals 1. If it is the 1st one, then $u(0) = \delta_{2^m}^1$. Similarly, if the ith one, then $u(0) = \delta_{2^m}^i$.

Example 4.12. Consider a simple TBCN as follows,

$$\begin{cases} A(t+1) = u_1(t) \wedge A(t) \leftrightarrow B(t-1), \\ B(t+1) = u_2(t) \vee B(t-1) \rightarrow B(t-2), \\ y_1(t) = B(t), \\ y_2(t) = A(t), \end{cases} \tag{4.13}$$

with control satisfying

$$\begin{cases} u_1(t+1) = \neg u_2(t), \\ u_2(t+1) = u_1(t). \end{cases} \tag{4.14}$$

Let $s = 4,\ \tau = 2,\ x(t) = A(t)B(t)$ and $u(t) = u_1(t)u_2(t)$. Now assume $A(0) = \delta_2^1,\ A(-1) = \delta_2^2,\ A(-2) = \delta_2^1, B(0) = \delta_2^2,\ B(-1) = \delta_2^2,\ B(-2) = \delta_2^1,\ y(4) = \delta_4^1$, then $X(\tau) = \delta_{64}^{29}$.

(A) Express (4.13), (4.14) as (3.12), (3.13) respectively with $G = M_n W_{[2]} = \delta_4[3,1,4,2]$, $H = W_{[2]}$ and $L = M_e M_c(I_{2^3} \otimes M_i M_d)(I_{2^2} \otimes W_{[2]})(I_2 \otimes W_{[2]})(I_{2^2} \otimes E_d W_{[2]})(I_{2^4} \otimes M_r E_d)(I_{2^6} \otimes E_d)$. For the details of E_d, see [9].

(B) Formula (4.4) yields $L_4^G \in \mathcal{L}_{4\times 256}$ as $L_4^G = LG^3[(I_{2^2} \otimes L_3^G)\Phi_2][\ltimes_{i=2}^1 (I_{2^8} \otimes L_i^G \Phi_8)]$.

(C) $l(4; \delta_{64}^{29}, \delta_4^1) = 2 > 0$.

(D) $y(s)^\top H L_s^G W_{[2^{n(\tau+1)},2^m]} X(\tau) = [0,1,1,0]$. Hence, $u(0) = \delta_4^2$ or $u(0) = \delta_4^3$.

4.2. Control *via* free Boolean sequence

In the following, we consider the case (B) where the controls are free Boolean sequences.

Definition 4.13. Given initial state $x(-i),\ i \in \{0,1,\ldots,\tau\}$, the destination output $y_f \in \Delta_{2^p}$ and the finite time $s \in \mathbb{N}^+$, the TBCN (3.12) is said to be $s-output-controllable$ ($or\ reachable$) from initial state $x(-i), (i \in 0,1,\ldots,\tau)$ to y_f (by free Boolean sequence), if we can find the control inputs $\{u(0), u(1), \ldots, u(s-1)\}$ such that $y(s) = y_f$.

Definition 4.14. The TBCN (3.12) is said to be $s-output-controllable$ ($or\ reachable$) if for any $a_i \in \Delta_{2^n},\ i \in \{0,1,\ldots,\tau\}$ and $b \in \Delta_{2^p}$, there exist the finite time $s \in \mathbb{N}^+$ and the control input $u(t)$ steers the TBCN from $x(-i) = a_i,\ i \in \{0,1,\ldots,\tau\}$ to $y(s) = b$.

For simplicity, we denote matrix $\tilde{L} = LW_{[2^{n(\tau+1)},2^m]}$, vectors $U(i) = \ltimes_{j=0}^{i} u(j) \in \Delta_{2^{m(i+1)}},\ i \in \mathbb{N}$. Then the first equation of (3.12) can be expressed as

$$x(t+1) = \tilde{L}x(t)x(t-1)\ldots x(t-\tau)u(t). \tag{4.15}$$

Theorem 4.15. *Consider TBCN (3.12). y_f is s-output-reachable from $x(-i),\ i \in \{0,1,\ldots,\tau\}$ by controls of Boolean sequences $U(s-1)$ of length s if and only if there exists at least one entry of $y_f^\top H\tilde{L}_s X(\tau)$ equaling 1, where*

$$\tilde{L}_s = \begin{cases} \tilde{L}, & s = 1 \\ \tilde{L}\tilde{L}_1 W_{[2^{n\tau},2^{m+n(\tau+1)}]}\Phi_{n\tau}, & s = 2, \\ \tilde{L}\tilde{L}_{s-1}[\ltimes_{i=s-2}^{1}(W_{[2^n,2^{(s-1)m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})] & \\ \qquad\qquad \ltimes W_{[2^{(\tau+2-s)n},2^{(s-1)m+n(\tau+1)}]}\Phi_{(\tau+2-s)n}, & s = 3,\ldots,\tau+1. \\ \tilde{L}\tilde{L}_{s-1}[\ltimes_{i=s-2}^{s-\tau-1}(W_{[2^n,2^{(s-1)m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})], & s > \tau+1. \end{cases} \tag{4.16}$$

If it is the ith one, then $U(s-1) = \delta_{2^{sm}}^i$.

Proof. A straightforward computation along (4.15) gives the following:

$$\begin{aligned}
x(1) &= \tilde{L}X(\tau)u(0) = \tilde{L}_1 X(\tau)U(0),\\
x(2) &= \tilde{L}x(1)X(\tau-1)u(1)\\
&= \tilde{L}\tilde{L}_1 X(\tau)U(0)X(\tau-1)u(1)\\
&= \tilde{L}\tilde{L}_1 W_{[2^{n\tau},2^{m+n(\tau+1)}]}\Phi_{n\tau}X(\tau)U(1)\\
&= \tilde{L}_2 X(\tau)U(1),\\
x(3) &= \tilde{L}x(2)x(1)X(\tau-2)u(2)\\
&= \tilde{L}\tilde{L}_2 X(\tau)U(1)\tilde{L}_1 X(\tau)U(0)X(\tau-2)u(2)\\
&= \tilde{L}\tilde{L}_2 W_{[2^n,2^{2m+n(\tau+1)}]}\tilde{L}_1\Phi_{m+n(\tau+1)}X(\tau)U(1)X(\tau-2)u(2)\\
&= \tilde{L}\tilde{L}_2 W_{[2^n,2^{2m+n(\tau+1)}]}\tilde{L}_1\Phi_{m+n(\tau+1)}W_{[2^{n(\tau-1)},2^{2m+n(\tau+1)}]}\Phi_{n(\tau-1)}X(\tau)U(2)\\
&= \tilde{L}_3 X(\tau)U(2).
\end{aligned}$$

For $3 \leq t \leq \tau$, we assume that

$$\begin{aligned}
\tilde{L}_t = &\tilde{L}\tilde{L}_{t-1}[\ltimes_{i=t-2}^{1}(W_{[2^n,2^{(t-1)m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})]\\
&\ltimes W_{[2^{(\tau+2-t)n},2^{(t-1)m+n(\tau+1)}]}\Phi_{(\tau+2-t)n}.
\end{aligned}\tag{4.17}$$

Then we have for $s = t+1$ that

$$\begin{aligned}
x(t+1) &= \tilde{L}x(t)\ldots x(1)X(0)u(\tau)\\
&= \tilde{L}[\ltimes_{i=t}^{1}\tilde{L}_i X(\tau)U(i-1)]X(0)u(\tau)\\
&= \tilde{L}\tilde{L}_t[\ltimes_{i=t-1}^{1}(W_{[2^n,2^{\tau m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})]W_{[2^{(\tau+1-t)n},2^{tm+n(\tau+1)}]}\Phi_{(\tau+1-t)n}X(\tau)U(\tau),
\end{aligned}$$

and

$$\tilde{L}_{t+1} = \tilde{L}\tilde{L}_t[\ltimes_{i=t-1}^{1}(W_{[2^n,2^{\tau m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})]W_{[2^{(\tau+1-t)n},2^{tm+n(\tau+1)}]}\Phi_{(\tau+1-t)n}.$$

By mathematical induction, one can get that

$$\begin{aligned}
\tilde{L}_1 &= \tilde{L},\\
\tilde{L}_2 &= \tilde{L}\tilde{L}_1 W_{[2^{n\tau},2^{m+n(\tau+1)}]}\Phi_{n\tau},\\
\tilde{L}_s &= \tilde{L}\tilde{L}_{s-1}[\ltimes_{i=s-2}^{1}(W_{[2^n,2^{(s-1)m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})]\\
&\quad\ltimes W_{[2^{(\tau+2-s)n},2^{(s-1)m+n(\tau+1)}]}\Phi_{(\tau+2-s)n},\ \ s = 3,\ldots,\tau+1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
x(\tau+2) &= \tilde{L}x(\tau+1)\ldots x(1)u(\tau+1)\\
&= \tilde{L}[\ltimes_{i=\tau+1}^{1}\tilde{L}_i X(\tau)U(i-1)]u(\tau+1)\\
&= \tilde{L}\tilde{L}_{\tau+1}[\ltimes_{i=\tau}^{1}(W_{[2^n,2^{(\tau+1)m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})]X(\tau)U(\tau+1)\\
&= \tilde{L}_{\tau+2}X(\tau)U(\tau+1),\\
x(\tau+3) &= \tilde{L}x(\tau+2)\ldots x(2)u(\tau+2)\\
&= \tilde{L}[\ltimes_{i=\tau+2}^{2}\tilde{L}_i X(\tau)U(i-1)]u(\tau+2)\\
&= \tilde{L}\tilde{L}_{\tau+2}[\ltimes_{i=\tau+1}^{2}(W_{[2^n,2^{(\tau+2)m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})]X(\tau)U(\tau+2)\\
&= \tilde{L}_{\tau+3}X(\tau)U(\tau+2).
\end{aligned}\tag{4.18}$$

For $t > \tau + 1$, we assume that

$$\tilde{L}_t = \tilde{L}\tilde{L}_{t-1}[\ltimes_{i=t-2}^{t-\tau-1}(W_{[2^n,2^{(t-1)m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})].$$

Then we have for $s = t + 1$ that

$$\begin{aligned} x(t+1) &= \tilde{L}x(t)\dots x(t-\tau)u(t) \\ &= \tilde{L}[\ltimes_{i=t}^{t-\tau}\tilde{L}_iX(\tau)U(i-1)]u(t) \\ &= \tilde{L}\tilde{L}_t[\ltimes_{i=t-1}^{t-\tau}(W_{[2^n,2^{tm+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})]X(\tau)U(t), \end{aligned}$$

and

$$\tilde{L}_{t+1} = \tilde{L}\tilde{L}_t[\ltimes_{i=t-1}^{t-\tau}(W_{[2^n,2^{tm+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})].$$

By mathematical induction, one can get that

$$\tilde{L}_s = \tilde{L}\tilde{L}_{s-1}[\ltimes_{i=s-2}^{s-\tau-1}(W_{[2^n,2^{(s-1)m+n(\tau+1)}]}\tilde{L}_i\Phi_{im+n(\tau+1)})],\ s > \tau + 1.$$

It follows from (3.12) that

$$y(s) = H\tilde{L}_sX(\tau)U(s-1),\ s \in \mathbb{N}^+, \tag{4.19}$$

where $\tilde{L}_s$ is given by (4.16). From the form of $H\tilde{L}_sX(\tau)$ and $U(s-1) \in \Delta_{2^{sm}}$, where $H\tilde{L}_sX(\tau)$ is a $2^p \times 2^{sm}$ matrix whose columns are elements in Δ_{2^p}, we can drive the conclusion by using the similar analysis as Theorem 4.6. □

Remark 4.16. Especially, from Theorems 4.6 and 4.15, when $\tau = 1$, the third explicit expressions of L_s^G in (4.4) and $\tilde{L}_s$ in (4.16) for $s = 3, \dots, \tau + 1$ should be omitted.

By (4.19), one can rewrite $y(s)$ as follow:

$$y(s) = H\tilde{L}_sW_{[2^{sm},2^{n(\tau+1)}]}U(s-1)X(\tau),\ s \in \mathbb{N}^+. \tag{4.20}$$

Then similar with Proposition 4.9 and Theorem 4.10, we will give the necessary and sufficient conditions on the controllability of TBCNs (3.12) in terms of Definition 4.2.

Proposition 4.17. *The number of different controls $u(t)$ that steer TBCN (3.12) from $x(-i),\ i \in \{0, 1, \dots, \tau\}$ to $y(s) = y_f$ in s time steps is*

$$l'(s; X(\tau), y_f) = y_f^\top P_sX(\tau),\ s \in \mathbb{N}^+,$$

where $P_s = H\tilde{L}_sW_{[2^{sm},2^{n(\tau+1)}]}1_{2^{sm}}$ and $\tilde{L}_s$ is given by (4.16).

Theorem 4.18. *The TBCN (3.12) is $s-output-controllable$ (or reachable) if and only if all the entries of P_s are different from zero.*

Remark 4.19. As a special case, when $\tau = 0$, then from the Proof of Theorem 4.15, we have $\tilde{L}_s = \tilde{L}^s$ for $s > 0$. In this case, Theorem 4.15 deduces Theorem 18 in [8], and

$$P_s = H\tilde{L}^sW_{[2^{sm},2^n]}1_{2^{sm}} = H(L1_{2^m})^s.$$

In fact, the last equality can be proved as follows. For any $a \in \Delta_{2^n}$,

$$\begin{aligned}
\tilde{L}^s W_{[2^{sm},2^n]} 1_{2^{sm}} a &= \tilde{L}^s a 1_{2^{sm}} \\
&= (LW_{[2^n,2^m]})^s a \ltimes_{i=1}^{s} 1_{2^m} \\
&= (LW_{[2^n,2^m]})^{s-1} LW_{[2^n,2^m]} a 1_{2^m} \ltimes_{i=1}^{s-1} 1_{2^m} \\
&= (LW_{[2^n,2^m]})^{s-1} (L1_{2^m} a) \ltimes_{i=1}^{s-1} 1_{2^m} \\
&= (LW_{[2^n,2^m]})^{s-2} LW_{[2^n,2^m]} (L1_{2^m} a) 1_{2^m} \ltimes_{i=1}^{s-2} 1_{2^m} \\
&= (LW_{[2^n,2^m]})^{s-2} (L1_{2^m} (L1_{2^m} a)) \ltimes_{i=1}^{s-2} 1_{2^m} \\
&\quad \vdots \\
&= (L1_{2^m})^s a,
\end{aligned} \tag{4.21}$$

which means that any corresponding columns of $\tilde{L}^s W_{[2^{sm},2^n]} 1_{2^{sm}}$ and $(L1_{2^m})^s$ are equal. Hence, $H\tilde{L}^s W_{[2^{sm},2^n]} 1_{2^{sm}} = H(L1_{2^m})^s$. It is noticed that both of the matrix M given by (14) in [44], and Q defined by Theorem 3 of [27] equal to $L1_{2^m}$. Then it indicates that Theorem 4.18 in this paper generalizes the controllability criteria of Corollary 3.6 in [44], and Corollary 2 in [27] to the case of TBCNs.

The following algorithm is to find a control driving given $x(-i),\ i \in \{0, 1, \dots, \tau\}$ to $y(s) = y_f$ in s time steps with free controls. Similar with Algorithm 4.11, we assume $X(\tau) = \delta^i_{2^{n(\tau+1)}}$ and $y(s) = \delta^j_{2^p}$.

Algorithm 4.20. *If the TBCN is given with its logical expression as (3.7).*

(A) *Convert (3.7) into a linear discrete time delay system as (3.12) such that $L,\ H$ can be expressed by matrices.*
(B) *Compute $\tilde{L}_s$ by (4.16).*
(C) *Get $l'(s; X(\tau), y(s)) = y_f^\top P_s X(\tau)$ to see the number of different controls $\{u(0), u(1), \dots, u(s-1)\}$ that steer the TBCN from $X(\tau)$ to $y(s)$. If $l(s; X(\tau), y(s)) = 0$, it means there is no such control, then stop.*
(D) *Find which entry of vector $y(s)^\top H\tilde{L}_s X(\tau)$ equals 1. If it is the ith one, then $U(s-1) = \delta^i_{2^{sm}}$.*

Example 4.21. Consider the following TBCN

$$\begin{cases}
A(t+1) = C(t) \wedge B(t-1), \\
B(t+1) = A(t) \vee C(t-1), \\
C(t+1) = \neg u(t) \to A(t-1), \\
y_1(t) = A(t) \wedge C(t), \\
y_2(t) = B(t).
\end{cases} \tag{4.22}$$

Let $s = 3,\ \tau = 1,\ x(t) = A(t)B(t)C(t)$. Assume $A(0) = \delta^1_2,\ A(-1) = \delta^2_2, B(0) = \delta^1_2,\ B(-1) = \delta^2_2, C(0) = \delta^2_2,\ C(-1) = \delta^1_2,\ y(3) = \delta^4_4$, then $X(\tau) = A(0)B(0)C(0)A(-1)B(-1)$
$C(-1) = \delta^{15}_{64}$.

(A) We can express (4.22) with $H = M_c(I_2 \otimes W_{[2]})$ and $\tilde{L} = M_c(I_{2^2} \otimes M_d)(I_{2^4} \otimes M_i M_n)W_{[2,2^5]}$
$W_{[2]}E_d(I_{2^2} \otimes W_{[2,2^2]})W_{[2,2^2]}$ as (4.15).
(B) From Remark 4.16, formula (4.16) yields $\tilde{L}_3 \in \mathcal{L}_{8\times 512}$ as $\tilde{L}_3 = \tilde{L}\tilde{L}_2[W_{[2^3,2^8]}\tilde{L}_1\Phi_7]$.
(C) $l'(3; \delta^{15}_{64}, \delta^4_4) = 2 > 0$.
(D) $y(s)^\top H\tilde{L}_s X(\tau) = [0,0,0,0,0,1,0,1]$. Hence, $U(2) = \delta^6_8$ or δ^8_8. We choose for example, $u(0)u(1)u(2) = \delta^8_8$, which means that the corresponding controls are

$$u(0) = u(1) = u(2) = \delta^2_2.$$

One can see that the BCN with time delays in states considered in [30,33] is a special case of TBCN. Hence, results in [8,27,30,44] are not applicable to judge the controllability of both examples. In fact, we can also use Algorithm 4.20 to consider the controllability for more complex TBCNs, *e.g.*, time-delayed regulation networks for human HeLa cell cycling [29]. The detail is omitted here.

Remark 4.22. The obtained results in this paper are theoretical, and the model as well as examples we have considered here are idealized. There are many constraints on the genetic network in the real world, which may be difficult to be expressed by simple BCNs or TBCNs. Based on STP method, it is found that many existing basic results on BCNs can hardly be directly used in real world, see [7–11, 26, 28, 32]. However, the proposed theoretical research on BCNs or TBCNs would contribute to analyzing the related actual biological systems. We will consider more practical factors of biological systems in our future research, and attempt to verify the effectiveness of the obtained results with the data of real biological systems.

5. Conclusion

In this paper, some necessary and sufficient conditions for a TBCN model to be output-controllable have been derived. Based on semi-tensor product of matrices and the matrix expression of logic, we have converted the TBCNs into discrete time systems with time delays. Two kinds of definitions of output controllability have been proposed and further investigated *via* different controls. The obtained results generalize some existing results in [8,27,44]. Finally, two examples are given to illustrate the efficiency of the proposed results.

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