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DIFFERENTIAL GAMES OF PARTIAL INFORMATION FORWARD-BACKWARD DOUBLY SDE AND APPLICATIONS*

Eddie C.M. Hui[1] and Hua Xiao[2]

Abstract. This paper addresses a new differential game problem with forward-backward doubly stochastic differential equations. There are two distinguishing features. One is that our game systems are initial coupled, rather than terminal coupled. The other is that the admissible control is required to be adapted to a subset of the information generated by the underlying Brownian motions. We establish a necessary condition and a sufficient condition for an equilibrium point of nonzero-sum games and a saddle point of zero-sum games. To illustrate some possible applications, an example of linear-quadratic nonzero-sum differential games is worked out. Applying stochastic filtering techniques, we obtain an explicit expression of the equilibrium point.

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1. Introduction

Game theory is a useful tool which helps us understand economic, social, political, and biological phenomena. Stochastic differential game problems have increasingly attracted more research attentions, and the related games approached solutions are widely used in social and behavioral sciences. Herein, we are primarily interested in stochastic differential games of forward-backward doubly stochastic differential equations (FBDSDEs, for short). This research is inspired by finding an equilibrium point of a linear-quadratic (LQ, for short) nonzero-sum differential game of backward doubly stochastic differential equations (BDSDEs, for short). Now we explain this in more detail.

Let T be a constant and $\left(\Omega, \mathscr{F}, P\right)$ be a complete probability space, on which two mutually independent standard Brownian motions $B(\cdot) \in \mathrm{R}^l$ and $W(\cdot) \in \mathrm{R}^d$ are defined. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B, \tag{1.1}$$

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[1] Department of Building and Real Estate, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, P.R. China. bscmhui@polyu.edu.hk

[2] School of Mathematics and Statistics, Shandong University, Weihai, Weihai 264209, P.R. China. xiao_hua@sdu.edu.cn

where $\mathcal{F}_t^W = \mathcal{N} \vee \sigma\{W(r) - W(0) : 0 \le r \le t\}$ and $\mathcal{F}_{t,T}^B = \mathcal{N} \vee \sigma\{B(T) - B(r) : t \le r \le T\}$. Note that the set $\mathcal{F}_t, t \in [0,T]$ is neither increasing nor decreasing, so it does not constitute a filtration. Let $\mathcal{L}_T^p(\Omega; \mathcal{S})$ denote all classes of $\mathcal{F}_T$-measurable random variables $\{\xi : \Omega \to \mathcal{S}\}$ satisfying $E|\xi|^p < \infty$, and $\mathcal{L}_{\mathcal{F}_t}^p(0,T;\mathcal{S})$ denote all classes of $\mathcal{F}_t$-adapted stochastic processes $\{x(t) : [0,T] \times \Omega \to \mathcal{S}\}$ satisfying $\mathbb{E}\big[\int_0^T |x(t)|^p \mathrm{d}t\big] < +\infty$.

There exists an interesting financial phenomenon in the market that two players work together to achieve a given goal at certain future time. Inspired by this financial phenomenon, Wang and Yu [15] studied a kind of nonzero-sum differential game in which the game system is governed by

$$\begin{cases} -\mathrm{d}Y^{v_1,v_2}(t) = g\big(t, Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t)\big)\mathrm{d}t - Z^{v_1,v_2}(t)\mathrm{d}W(t), \\ \quad Y^{v_1,v_2}(T) = \xi. \end{cases} \tag{1.2}$$

Here ξ is a given random variable denoting the future goal at the terminal time T, and $v_1(\cdot)$ and $v_2(\cdot)$ are $\mathcal{F}_t^W$-adapted control processes of Player 1 and Player 2, respectively. Note that (1.2) is a nonlinear backward stochastic differential equation (BSDE, for short) which was originally introduced by Pardoux and Peng [11].

It is well known that there may exist so called informal trading such as "*insider trading*" in the market (for more argumentations about this, see *e.g.* [1,3] and references therein). That is, the players at the current time t possess extra information of the future developing of the market from t to T that is represented by $\mathcal{F}_{t,T}^B$, as well as the accumulated information $\mathcal{F}_t^W$ from 0 to t. Clearly, Wang and Yu's model cannot capture this case. In order to make up the above-mentioned limitation of system (1.2), we introduce the following BDSDE originally discussed by Pardoux and Peng [12], whose dynamics are described by

$$\begin{cases} -\mathrm{d}Y^{v_1,v_2}(t) = g\big(t, Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t)\big)\mathrm{d}t \\ \qquad\qquad + \bar{g}\big(t, Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t)\big)\hat{\mathrm{d}}B(t) - Z^{v_1,v_2}(t)\mathrm{d}W(t), \\ \quad Y^{v_1,v_2}(T) = \xi. \end{cases}$$

Here the integral with respect to $\hat{\mathrm{d}}B(t)$ is a "backward Itô integral" and the integral with respect to $\mathrm{d}W(t)$ is a standard forward Itô integral, which are two types of particular cases of the Itô–Skorohod integral. The extra noise $B(\cdot)$ generates $\mathcal{F}_{t,T}^B$ which represents the information concerning the future market development. It is very natural that we require the control processes $v_1(\cdot)$ and $v_2(\cdot)$ to be $\mathcal{F}_t$-adapted, rather than only $\mathcal{F}_t^W$-adapted or $\mathcal{F}_{t,T}^B$-adapted.

We introduce an LQ nonzero-sum differential game of BDSDEs, which inspires us to study the differential game theory of FBDSDEs. In detail, we consider the following 1-dimensional linear BDSDE:

$$\begin{cases} -\mathrm{d}Y(t) = [A_1Y(t) + B_1Z(t) + C_1v_1(t) + D_1v_2(t)]\mathrm{d}t \\ \qquad\qquad + [A_2Y(t) + B_2Z(t) + C_2v_1(t) + D_2v_2(t)]\hat{\mathrm{d}}B(t) - Z(t)\mathrm{d}W(t), \\ \quad Y(T) = \xi, \end{cases} \tag{1.3}$$

and the performance criterion, for $i = 1, 2$,

$$\begin{aligned} J_i(v_1(\cdot), v_2(\cdot)) = -\frac{1}{2}\mathbb{E}\bigg\{ &\langle F_{i1}Y(0), Y(0)\rangle + \int_0^T \Big[\langle F_{i2}Y(t), Y(t)\rangle \\ &+ \langle F_{i3}Z(t), Z(t)\rangle + \langle F_{i4}v_1(t), v_1(t)\rangle + \langle F_{i5}v_2(t), v_2(t)\rangle\Big]\mathrm{d}t\bigg\}. \end{aligned} \tag{1.4}$$

For simplicity, we assume that ξ is a real-valued random variable, all the coefficients in (1.3) and (1.4) are 1-dimensional, $l = d = 1$, $F_{i1}, F_{i2}, F_{i3} \ge 0$, $F_{i4}, F_{i5} > 0$.

We are to seek a $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, such that

$$\begin{cases} J(u_1(\cdot), u_2(\cdot)) \ge J(v_1(\cdot), u_2(\cdot)), \\ J(u_1(\cdot), u_2(\cdot)) \ge J(u_1(\cdot), v_2(\cdot)). \end{cases}$$

Here $\mathcal{U}_1 \times \mathcal{U}_2$ is a certain admissible control set for Player 1 and Player 2. If such a $(u_1(\cdot), u_2(\cdot))$ exists, we call it an equilibrium point. For simplicity, we denote this problem by *Problem (LQNZB)*.

Applying Theorem 4.1 in [5], we conclude that if the equilibrium point exists, then it is necessary to satisfy the following form:

$$\begin{cases} u_1(t) = -F_{14}^{-1}\big(C_1 y_1(t) + C_2 z_1(t)\big), \\ u_2(t) = -F_{25}^{-1}\big(D_1 y_2(t) + D_2 z_2(t)\big), \end{cases}$$

where (y_i, z_i) $(i = 1, 2)$ is the solution of the FBDSDE

$$\begin{cases} -\mathrm{d}Y(t) = \Big[A_1 Y(t) + B_1 Z(t) - C_1 F_{14}^{-1}\big(C_1 y_1(t) + C_2 z_1(t)\big) \\ \qquad\qquad - \; D_1 F_{25}^{-1}\big(D_1 y_2(t) + D_2 z_2(t)\big)\Big]\mathrm{d}t \\ \qquad\qquad + \; \Big[A_2 Y(t) + B_2 Z(t) - C_2 F_{14}^{-1}\big(C_1 y_1(t) + C_2 z_1(t)\big) \\ \qquad\qquad - \; D_2 F_{25}^{-1}\big(D_1 y_2(t) + D_2 z_2(t)\big)\hat{\mathrm{d}}B(t) - Z(t)\mathrm{d}W(t), \\ \mathrm{d}y_i(t) = \big(A_1 y_i(t) + A_2 z_i(t) + F_{i2} Y(t)\big)\mathrm{d}t + \big(B_1 y_i(t) + B_2 z_i(t) + F_{i3} Z(t)\big)\mathrm{d}W(t) \\ \qquad\qquad - \; z_i(t)\hat{\mathrm{d}}B(t), \\ Y(T) = \xi, \; y_i(0) = F_{i1} Y(0). \end{cases} \tag{1.5}$$

We note that equation (1.5) is exactly the type of time-symmetric forward-backward stochastic differential equation (FBSDE, for short) (see [13]), but with an initial coupled constraint. This kind of equations possesses fine dynamics and contains BSDEs, BDSDEs and initial coupled FBSDEs as a special case. Then, it is natural to investigate some differential game theory derived by them.

Recall that $\mathcal{F}_t$ represents the full information arising from the market, which may contain the past, present and future information. Due to this, in principle, it is not completely available to the players. However, it is possible that the players possess a subset of $\mathcal{F}_t$ which is denoted by $\mathcal{E}_t$. In order to distinguish $\mathcal{E}_t$ from $\mathcal{F}_t$, we call $\mathcal{E}_t$ a sub-information or partial information of $\mathcal{F}_t$. Note that $\mathcal{E}_t$ could be the δ-delayed information defined by $\mathcal{F}_{(t-\delta)^+}$, where δ is a given positive constant delay. Here we require the control processes $v_1(\cdot)$ and $v_2(\cdot)$ to be $\mathcal{E}_t$-adapted. This implies that the players will only depend on $\mathcal{E}_t$ to choose their control strategies. Based on the above-mentioned arguments, we are interested in initiating a study of differential games of initial coupled FBDSDEs with partial information.

Up till now, to our best knowledge, there are only two papers about optimal control of BDSDEs and initial coupled FBSDEs (see [5, 17]), and a few studies about differential games of BSDEs (see [15, 16]). For the topics about the optimal control and differential games of terminal coupled FBSDEs or FBDSDEs, refer to [2, 4, 7, 10, 14, 19, 20, 22, 23], specially the monographs [9, 21], etc. However, little or none has been done on differential games of BDSDEs, initial coupled FBSDEs and FBDSDEs, and our research can just right make up this scarcity. Also, some comparisons between our results and the existing literature are specified in conclusion section.

The rest is organized as follows. Section 2 formulates a nonzero-sum game of initial-coupled FBDSDEs with partial information. Applying classical convex variation and adjoint techniques, a maximum principle, also called a necessary condition, is established for an equilibrium point (refer to Thm. 2.1). By virtue of the concavity assumptions of certain functions, we derive a verification theorem, also called a sufficient condition, which is the main result in this paper (refer to Thm. 2.3). To illustrate the theoretical results, we work out an LQ nonzero-sum differential game. Applying Theorem 2.1, Theorem 2.3 and the stochastic filtering techniques of FBSDEs, an explicit expression of the equilibrium point is obtained. Likewise, Section 3 gives a maximum principle and a verification theorem for a saddle point of zero-sum differential games. Finally, Section 4 gives some concluding remarks.

2. Nonzero-sum differential games

2.1. Problem formulation

Let U_i be a nonempty convex subset of R^{k_i} $(i = 1, 2)$. The processes $v_1(t) = v_1(t, \omega)$ and $v_2(t) = v_2(t, \omega)$ are the control processes of Player 1 and Player 2, respectively. We always use the subscript 1 (resp. the subscript 2) to characterize the variable corresponding to Player 1 (resp. Player 2). We denote the set of all open-loop controls for Player i $(i = 1, 2)$ by

$$\mathcal{U}_i = \left\{ v_i(\cdot) : [0, T] \times \Omega \longrightarrow U_i \middle| v_i(\cdot) \text{ is } \mathcal{E}_t\text{-adapted and satisfies } \mathbb{E} \int_0^T |v_i(t)|^2 \mathrm{d}t < \infty \right\},$$

where $\mathcal{E}_t$ is a sub-information of $\mathcal{F}_t$ available to the players, *i.e.*

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \qquad \text{for all } t.$$

Each element of $\mathcal{U}_i$ is called an admissible control for Player i on [0, T] $(i = 1, 2)$. $\mathcal{U}_1 \times \mathcal{U}_2$ is called the set of open-loop admissible controls for the players.

We introduce the mappings

$$\begin{aligned}
&f : [0, T] \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2 \to \mathrm{R}^n,\\
&\bar{f} : [0, T] \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2 \to \mathrm{R}^{n\times d},\\
&g : [0, T] \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2 \to \mathrm{R}^m,\\
&\bar{g} : [0, T] \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2 \to \mathrm{R}^{m\times l},\\
&\phi : \mathrm{R}^m \to \mathrm{R}^n, \qquad \varphi, \varphi_i : \mathrm{R}^m \to \mathrm{R}^1, \qquad \gamma, \gamma_i : \mathrm{R}^n \to \mathrm{R}^1,\\
&l, l_i : [0, T] \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2 \to \mathrm{R}^1 \ (i = 1, 2).
\end{aligned}$$

Assumption (H1): For any $(y, z, Y, Z, v_1, v_2) \in \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2$, we assume that $f(\cdot, y, z, Y, Z, v_1, v_2), \bar{f}(\cdot, y, z, Y, Z, v_1, v_2), g(\cdot, Y, Z, v_1, v_2)$ and $\bar{g}(\cdot, Y, Z, v_1, v_2)$ are continuous with respect to t. Also, we assume that $f, \bar{f}, g$ and $\bar{g}$ are continuously differentiable with respect to $(y, z, Y, Z,\ v_1, v_2)$, and their derivatives with respect to (y, z, Y, Z, v_1, v_2) are uniformly bounded. $l, l_1, l_2, \varphi, \varphi_1, \varphi_2, \gamma, \gamma_1$ and γ_2 are continuously differential with respect to $(y, z, Y, Z, v_1,\ v_2)$ and their derivatives with respect to (y, z, Y, Z, v_1, v_2) are continuous and bounded by $K(1 + |y| + |z| + |Y| + |Z| + |v_1| + |v_2|)$. For any $(y_1, z_1, Y_1, Z_1, u_1, u_2)$, $(y_2, z_2, Y_2, Z_2, v_1, v_2) \in \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1 \times U_2$, there exist constants $k > 0$ and $0 < c < 1$ such that

$$\begin{aligned}
&|\bar{f}(t, y_1, z_1, Y_1, Z_1, u_1, u_2) - \bar{f}(t, y_2, z_2, Y_2, Z_2, v_1, v_2)|^2 + |\bar{g}(t, Y_1, Z_1, u_1, u_2) - \bar{g}(t, Y_2, Z_2, v_1, v_2)|^2\\
&\le k(|y_1 - y_2|^2 + |Y_1 - Y_2|^2 + |Z_1 - Z_2|^2 + |u_1 - v_1|^2 + |u_2 - v_2|^2) + c|z_1 - z_2|^2.
\end{aligned}$$

In the following, we specify the nonzero-sum differential game of forward-backward doubly stochastic systems. Given $\xi \in \mathcal{L}^2_T(\Omega; \mathrm{R}^m)$ and $\phi \in \mathcal{L}^2_T(\Omega; \mathrm{R}^n)$, we consider an FBDSDE

$$\begin{cases}
-\mathrm{d}Y^{v_1,v_2}(t) = g(t, Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t))\mathrm{d}t\\
\qquad\qquad + \bar{g}(t, Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t))\hat{\mathrm{d}}B(t) - Z^{v_1,v_2}(t)\mathrm{d}W(t),\\
\mathrm{d}y^{v_1,v_2}(t) = f(t, y^{v_1,v_2}(t), z^{v_1,v_2}(t), Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t))\mathrm{d}t\\
\qquad\qquad + \bar{f}(t, y^{v_1,v_2}(t), z^{v_1,v_2}(t), Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t))\mathrm{d}W(t)\\
\qquad\qquad - z^{v_1,v_2}(t)\hat{\mathrm{d}}B(t),\\
Y^{v_1,v_2}(T) = \xi, \qquad y^{v_1,v_2}(0) = \phi(Y^{v_1,v_2}(0)), \qquad 0 \le t \le T.
\end{cases} \tag{2.1}$$

Under the assumption (H1), for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, there exists a unique solution $\big(y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot), Y^{v_1,v_2}(\cdot), Z^{v_1,v_2}(\cdot)\big) \in \mathcal{L}^2_{\mathcal{F}_t}(0,T;\mathrm{R}^n) \times \mathcal{L}^2_{\mathcal{F}_t}(0,T;\mathrm{R}^{n\times l}) \times \mathcal{L}^2_{\mathcal{F}_t}(0,T;\mathrm{R}^m) \times \mathcal{L}^2_{\mathcal{F}_t}(0,T;\mathrm{R}^{m\times d})$ to equation (2.1) (see [12]).

Consider a performance criterion

$$J_i(v_1(\cdot), v_2(\cdot)) = \mathbb{E}\bigg[\int_0^T l_i\big(t, y^{v_1,v_2}(t), z^{v_1,v_2}(t), Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t)\big)\mathrm{d}t + \varphi_i(Y^{v_1,v_2}(0))\bigg] + \gamma_i(y^{v_1,v_2}(T)).$$

For any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, we assume that $l_i(\cdot, y^{v_1,v_2}(\cdot), z^{v_1,v_2}(\cdot), Y^{v_1,v_2}(\cdot), Z^{v_1,v_2}(\cdot), v_1(\cdot), v_2(\cdot)) \in \mathcal{L}^1_{\mathcal{F}_t}(0,T;\mathrm{R})$ and $\gamma_i \in \mathcal{L}^1_T(\Omega;\mathrm{R})$ $(i=1,2)$.

Problem (NZSG): Find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) \geq J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) \geq J_2(u_1(\cdot), v_2(\cdot)), \end{cases}$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$. We call $(u_1(\cdot), u_2(\cdot))$ an open-loop equilibrium point of *Problem (NZSG)* (if it does exist). It is easy to see that the existence of an open-loop equilibrium point implies that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) = \sup\limits_{v_1(\cdot)\in\mathcal{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) = \sup\limits_{v_2(\cdot)\in\mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot)). \end{cases}$$

2.2. Necessary condition

Suppose that $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of *Problem (NZSG)* with the trajectory $\Big(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot)\Big)$ of (2.1). For all $t \in [0,T]$, let $v_i(t) \in U_i$ be such that $u_i(\cdot) + v_i(\cdot) \in \mathcal{U}_i$ $(i=1,2)$. Notice that $\mathcal{U}_i$ is convex, then for $0 \leq \epsilon, \rho \leq 1, i = 1,2$,

$$u_{1\epsilon}(t) = u_1(t) + \epsilon v_1(t) \in \mathcal{U}_1,\ u_{2\rho}(t) = u_2(t) + \rho v_2(t) \in \mathcal{U}_2, \quad 0 \leq t \leq T.$$

For simplicity, we denote

$$\begin{aligned} f(t) &= f\big(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t)\big), \\ g(t) &= g\big(t, Y(t), Z(t), u_1(t), u_2(t)\big), \\ Y^{u_{1\epsilon}}(t) &= Y^{(u_1+\epsilon v_1, u_2)}(t),\ Y^{u_{2\rho}}(t) = Y^{(u_1, u_2+\rho v_2)}(t), \\ h_i(\epsilon, \rho) &= J_i(u_1 + \epsilon v_1, u_2 + \rho v_2), \end{aligned}$$

define the processes

$$\hat{Y}^1(t) = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\, Y^{u_{1\epsilon}}(t)|_{\epsilon=0},\ \ \hat{Y}^2(t) = \frac{\mathrm{d}}{\mathrm{d}\rho}\, Y^{u_{2\rho}}(t)|_{\rho=0},$$

and make the similar notations for $\bar{f}, \bar{g}, l_i, \hat{y}^i, \hat{z}^i, \hat{Z}^i, i = 1,2$. For $i = 1,2$, we have the following variational equations:

$$\begin{cases} -d\hat{Y}^i(t) = \hat{g}^i(t)\mathrm{d}t + \hat{\bar{g}}^i(t)\hat{\mathrm{d}}B(t) - \hat{Z}^i(t)\mathrm{d}W(t), \\ \quad d\hat{y}^i(t) = \hat{f}^i(t)\mathrm{d}t + \hat{\bar{f}}^i(t)\mathrm{d}W(t) - \hat{z}^i(t)\hat{\mathrm{d}}B(t), \\ \quad \hat{Y}^i(T) = 0,\ \hat{y}^i(0) = \phi_Y(Y(0))\hat{Y}^i(0) \end{cases}$$

where

$$
\begin{aligned}
\hat{g}^i(t) &= g_Y(t)\hat{Y}^i(t) + g_Z(t)\hat{Z}^i(t) + g_{v_i}(t)v_i(t),\\
\hat{\bar{g}}^i(t) &= \bar{g}_Y(t)\hat{Y}^i(t) + \bar{g}_Z(t)\hat{Z}^i(t) + \bar{g}_{v_i}(t)v_i(t),\\
\hat{f}^i(t) &= f_y(t)\hat{y}^i(t) + f_z(t)\hat{z}^i(t) + f_Y(t)\hat{Y}^i(t) + f_Z(t)\hat{Z}^i(t) + f_{v_i}(t)v_i(t),\\
\hat{\bar{f}}^i(t) &= \bar{f}_y(t)\hat{y}^i(t) + \bar{f}_z(t)\hat{z}^i(t) + \bar{f}_Y(t)\hat{Y}^i(t) + \bar{f}_Z(t)\hat{Z}^i(t) + \bar{f}_{v_i}(t)v_i(t),\\
\hat{l}^i(t) &= l_{iy}(t)\hat{y}^i(t) + l_{iz}(t)\hat{z}^i(t) + l_{iY}(t)\hat{Y}^i(t) + l_{iZ}(t)\hat{Z}^i(t) + l_{iv_i}(t)v_i(t).
\end{aligned}
$$

Next, we define the generalized *Hamiltonian function* $H_i : [0,T] \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times \mathrm{U}_1 \times \mathrm{U}_2 \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d}$ as follows:

$$
\begin{aligned}
H_i(t,y,z,Y,Z,v_1,v_2,p_i,\bar{p}_i,q_i,\bar{q}_i) \triangleq\ & \langle q_i, f(y,z,Y,Z,v_1,v_2)\rangle + \langle \bar{q}_i, \bar{f}(y,z,Y,Z,v_1,v_2)\rangle\\
& - \langle p_i, g(Y,Z,v_1,v_2)\rangle\ - \langle \bar{p}_i, \bar{g}(Y,Z,v_1,v_2)\rangle\ + l_i(y,z,Y,Z,v_1,v_2).
\end{aligned}
$$

Let $(u_1,u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the solution $\big(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot)\big)$ of equation (2.1). We shall use the abbreviated notation $H_i(t)$ defined by

$$
H_i(t) \equiv H_i\big(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_i(t), \bar{p}_i(t), q_i(t), \bar{q}_i(t)\big).
$$

The adjoint equations are described by the following generalized stochastic Hamiltonian systems:

$$
\begin{cases}
dp_i(t) = -\, H_{iY}^*(t)\mathrm{d}t - H_{iZ}^*(t)\mathrm{d}W(t) - \bar{p}_i(t)\hat{\mathrm{d}}B(t),\\
-dq_i(t) = H_{iy}^*(t)\mathrm{d}t + H_{iz}^*(t)\hat{\mathrm{d}}B(t) - \bar{q}_i(t)\mathrm{d}W(t),\\
p_i(0) = -\, \varphi_{iY}^*(Y(0)) - \phi_Y^*\big(Y(0)\big)q_i(0),\\
q_i(T) = \gamma_{iy}^*\big(y(T)\big).
\end{cases}
\tag{2.2}
$$

Then we have the following maximum principle for equilibrium points of *Problem (NZSG)*.

Theorem 2.1. *Let (H1) hold and* $\Big(u_1(\cdot),\ u_2(\cdot)\Big)$ *be an equilibrium point of Problem (NZSG). Furthermore,* $\Big(y(\cdot), z(\cdot),\ Y(\cdot), Z(\cdot)\Big)$ *and* $\Big(p_i(\cdot), \bar{p}_i(\cdot), q_i(\cdot), \bar{q}_i(\cdot)\Big)$ *are the solutions of* (2.1) *and* (2.2) *corresponding to the control* $\Big(u_1(\cdot),\ u_2(\cdot)\Big)$*, respectively. Then it follows that*

$$
\Big\langle E[H_{1v_1}^*(t)|\mathcal{E}_t], v_1(t) - u_1(t)\Big\rangle \le 0 \tag{2.3}
$$

and

$$
\Big\langle E[H_{2v_2}^*(t)|\mathcal{E}_t], v_2(t) - u_2(t)\Big\rangle \le 0 \tag{2.4}
$$

are true for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$*, a.e. a.s.*

Proof: Since $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point, we have

$$
\frac{\partial h_1}{\partial \epsilon}(0,0) = \lim_{\epsilon\to 0} \frac{J_1(u_1+\epsilon v_1, u_2) - J_1(u_1,u_2)}{\epsilon} \le 0.
$$

Then

$$
\begin{aligned}
0 &\ge \frac{\partial}{\partial \epsilon} h_1(\epsilon,0)|_{\epsilon=0}\\
&= E\int_0^T \Big(l_{1y}(t)\hat{y}^1(t) + l_{1z}(t)\hat{z}^1(t) + l_{1Y}(t)\hat{Y}^1(t) + l_{1Z}(t)\hat{Z}^1(t) + l_{1v_1}(t)v_1(t)\Big)\mathrm{d}t\\
&\quad +\ E\Big(\varphi_{1Y}\big(Y(0)\big)\hat{Y}^1(0) + \gamma_{1y}\big(y(T)\big)\hat{y}^1(T)\Big).
\end{aligned}
\tag{2.5}
$$

Applying Itô's formula to $\langle p_1(t), \hat{Y}^1(t)\rangle$ and $\langle q_1(t), \hat{y}^1(t)\rangle$, and integrating from 0 to T, we have

$$
\begin{aligned}
& E\Big(\varphi_{1Y}\big(Y(0)\big)\hat{Y}^1(0)\Big) \\
=& -E\langle p_1(0)+\phi_Y^*(Y(0))q_1(0), \hat{Y}^1(0)\rangle = -\langle \phi_Y^*(Y(0))q_1(0), \hat{Y}^1(0)\rangle \\
& -E\int_0^T \Big(p_1^*(t)g_{v_1}(t)v_1(t) + q_1^*(t)f_Y(t)\hat{Y}^1(t) + \bar{q}_1^*(t)\bar{f}_Y(t)\hat{Y}^1(t) - l_{1Y}(t)\hat{Y}^1(t) \\
& + \bar{p}_1^*(t)\bar{g}_{v_1}(t)v_1(t) + q_1^*(t)f_Z(t)\hat{Z}^1(t) + \bar{q}_1^*(t)\bar{f}_Z(t)\hat{Z}^1(t) - l_{1Z}(t)\hat{Z}^1(t)\Big)\mathrm{d}t,
\end{aligned} \tag{2.6}
$$

and

$$
\begin{aligned}
& E\Big(\gamma_{1y}\big(y(T)\big)\hat{y}^1(T)\Big) = \langle \phi_Y^*(Y(0))q_1(0), \hat{Y}^1(0)\rangle \\
& + E\int_0^T \Big(q_1^*(t)f_Y(t)\hat{Y}^1(t) + q_1^*(t)f_Z(t)\hat{Z}^1(t) + q_1^*(t)f_{v_1}(t)v_1(t) - l_{1y}(t)\hat{y}^1(t) \\
& - l_{1z}(t)\hat{z}^1(t) + \bar{q}_1^*(t)\bar{f}_Y(t)\hat{Y}^1(t) + \bar{q}_1^*(t)\bar{f}_Z(t)\hat{Z}^1(t) + \bar{q}_1^*(t)\bar{f}_{v_1}(t)v_1(t)\Big)\mathrm{d}t.
\end{aligned} \tag{2.7}
$$

Substituting (2.6) and (2.7) into (2.5), for all $v_1 \in U_1$ such that $u_1(\cdot)+v_1(\cdot) \in \mathcal{U}_1$, we get

$$
\begin{aligned}
0 &\geq \frac{\partial}{\partial \epsilon} h_1(\epsilon, 0)|_{\epsilon=0} \\
&= E\int_0^T \Big(q_1^*(t)f_{v_1}(t) + \bar{q}_1^*(t)\bar{f}_{v_1}(t) + p_1^*(t)g_{v_1}(t) + \bar{p}_1^*(t)\bar{g}_{v_1}(t) + l_{1v_1}(t)\Big)v_1(t)\mathrm{d}t \\
&= E\int_0^T \Big\langle H_{1v_1}^*(t), v_1(t)\Big\rangle \mathrm{d}t = E\int_0^T E\left[\Big\langle H_{1v_1}^*(t), v_1(t)\Big\rangle \Big| \mathcal{E}_t\right]\mathrm{d}t,
\end{aligned}
$$

which implies that (2.3) is true. (2.4) can be proved by the same method as shown in proving (2.3). □

If $\big(v_1(\cdot), v_2(\cdot)\big)$ is adapted to $\mathcal{F}_t$, we have the following corollary.

Corollary 2.2. *Suppose that $\mathcal{E}_t = \mathcal{F}_t$ for all t. Let (H1) hold, and $\Big(u_1(\cdot),\ u_2(\cdot)\Big)$ be an equilibrium point of Problem (NZSG). Moreover, $\Big(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot)\Big)$ and $\Big(p_i(\cdot), \bar{p}_i(\cdot), q_i(\cdot), \bar{q}_i(\cdot)\Big)$ are the solutions of (2.1) and (2.2) corresponding to the control $\Big(u_1(\cdot),\ u_2(\cdot)\Big)$, respectively. Then it follows that*

$$
\Big\langle H_{1v_1}^*(t), v_1(t) - u_1(t)\Big\rangle \leq 0
$$

and

$$
\Big\langle H_{2v_2}^*(t), v_2(t) - u_2(t)\Big\rangle \leq 0
$$

are true for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, a.e. a.s.

2.3. Sufficient condition

In what follows, we aim to establish a verification theorem, also called a sufficient condition, for an equilibrium point. For this, we introduce an additional condition as follows.

Assumption (H2): $\phi(Y) = MY$, where M is a non-zero constant matrix with order $n \times m$.

Note that this is a standard assumption in the optimal control theory of forward-backward stochastic systems (see [10], etc.).

Theorem 2.3. *Let (H1) and (H2) hold. Let (y, z, Y, Z) and $(p_i, \bar{p}_i, q_i, \bar{q}_i)$ be the solutions of equations (2.1) and (2.2) corresponding to the admissible control $\big(u_1(\cdot), u_2(\cdot)\big)$, respectively. Suppose that φ_i and γ_i are concave in Y and y $(i = 1, 2)$ respectively, and that for all $(t, y, z, Y, Z,) \in [0, T] \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d}$,*

$$(y, z, Y, Z, v_1) \longrightarrow H_1\big(t, y, z, Y, Z, v_1, u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)\big), \tag{2.8}$$

$$(y, z, Y, Z, v_2) \longrightarrow H_2\big(t, y, z, Y, Z, u_1(t), v_2, p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)\big) \tag{2.9}$$

are concave. Moreover,

$$\begin{aligned}
&\mathbb{E}\Big[H_1\big(t,\ y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)\big)\big|\mathcal{E}_t\Big] \\
&= \sup_{v_1\in\ U_1} \mathbb{E}\Big[H_1\big(t, y(t), z(t), Y(t), Z(t), v_1, u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)\big)\big|\mathcal{E}_t\Big], \qquad (2.10)\\
&\mathbb{E}\Big[H_2\big(t,\ y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)\big)\big|\mathcal{E}_t\Big] \\
&= \sup_{v_2\in\ U_2} \mathbb{E}\Big[H_2\big(t, y(t), z(t), Y(t), Z(t), u_1(t), v_2, p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)\big)\big|\mathcal{E}_t\Big]. \qquad (2.11)
\end{aligned}$$

Then $\big(u_1(\cdot), u_2(\cdot)\big)$ is an equilibrium point of Problem (NZSG).

Proof : Let $(v_1(\cdot), u_2(\cdot))$ and $(u_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the corresponding solutions $(y^{v_1}, z^{v_1}, Y^{v_1}, Z^{v_1})$ and $(y^{v_2}, z^{v_2}, Y^{v_2}, Z^{v_2})$ to equation (2.1). We define the following terms

$$\begin{aligned}
H_1(t) &= H_1(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)),\\
H_1^{v_1}(t) &= H_1(t, y^{v_1}(t), z^{v_1}(t), Y^{v_1}(t), Z^{v_1}(t), v_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)),\\
H_1^{v_2}(t) &= H_1(t, y^{v_2}(t), z^{v_2}(t), Y^{v_2}(t), Z^{v_2}(t), u_1(t), v_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)),\\
f^{v_1}(t) &= f(t, y^{v_1}(t), z^{v_1}(t), Y^{v_1}(t), Z^{v_1}(t), v_1(t), u_2(t)),\\
f^{v_2}(t) &= f(t, y^{v_2}, z^{v_2}, Y^{v_2}, Z^{v_2}, u_1(t), v_2(t)),
\end{aligned}$$

and similar notations are made for $\bar{f}^{v_1}, \bar{f}^{v_2}, \ldots$

By virtue of the concavity property of φ_1 and γ_1, we have for $\forall\ v_1(\cdot) \in \mathcal{U}_1$

$$J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \leq I_1 + I_2 + I_3 \tag{2.12}$$

with

$$\begin{aligned}
I_1 &= \mathbb{E}\left[\gamma_{1y}(y(T))(y^{v_1}(T) - y(T))\right],\\
I_2 &= \mathbb{E}\left[\varphi_{1Y}(Y(0))(Y^{v_1}(0) - Y(0))\right],\\
I_3 &= \mathbb{E}\int_0^T \Big(l_1^{v_1}(t) - l_1(t)\Big)\mathrm{d}t.
\end{aligned}$$

Applying Itô's formula to $\langle q_1(t), y^{v_1}(t) - y(t)\rangle$ and $\langle p_1(t), Y^{v_1}(t) - Y(t)\rangle$,

$$\begin{aligned} I_1 = &\ \mathbb{E}[\langle q_1(0), M(Y^{v_1}(0) - Y(0))\rangle] \\ &+ \mathbb{E}\int_0^T \Big(\langle q_1(t), f^{v_1}(t) - f(t)\rangle - \langle H_{1y}^*(t), y^{v_1}(t) - y(t)\rangle \\ &+ \langle \bar{q}_1(t), \bar{f}^{v_1}(t) - \bar{f}(t)\rangle - \langle H_{1z}^*(t), z^{v_1}(t) - z(t)\rangle\Big)\mathrm{d}t, \end{aligned} \tag{2.13}$$

$$\begin{aligned} I_2 = &- \mathbb{E}[\langle q_1(0), M(Y^{v_1}(0) - Y(0))\rangle] \\ &- \mathbb{E}\int_0^T \Big(\langle p_1(t), g^{v_1}(t) - g(t)\rangle + \langle H_{1Y}^*(t), Y^{v_1}(t) - Y(t)\rangle \\ &+ \langle \bar{p}_1(t), \bar{g}^{v_1}(t) - \bar{g}(t)\rangle + \langle H_{1Z}^*(t), Z^{v_1}(t) - Z(t)\rangle\Big)\mathrm{d}t, \end{aligned} \tag{2.14}$$

$$\begin{aligned} I_3 = &\ \mathbb{E}\int_0^T \Bigg(H_1^{v_1}(t) - H_1(t) - \langle q_1(t), f^{v_1}(t) - f(t)\rangle - \langle \bar{q}_1(t), \bar{f}^{v_1}(t) - \bar{f}(t)\rangle \\ &+ \langle \bar{p}_1(t), \bar{g}^{v_1}(t) - \bar{g}(t)\rangle + \langle p_1(t), g^{v_1}(t) - g(t)\rangle\Bigg)\mathrm{d}t. \end{aligned} \tag{2.15}$$

Substituting (2.13)–(2.15) into (2.12), it follows immediately that

$$\begin{aligned} &J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \\ \le &\ \mathbb{E}\int_0^T \Bigg(H_1^{v_1}(t) - H_1(t) - \langle H_{1Y}^*(t), Y^{v_1}(t) - Y(t)\rangle - \langle H_{1Z}^*(t), Z^{v_1}(t) - Z(t)\rangle \\ &- \langle H_{1y}^*(t), y^{v_1}(t) - y(t)\rangle - \langle H_{1z}^*(t), z^{v_1}(t) - z(t)\rangle\Bigg)\mathrm{d}t. \end{aligned} \tag{2.16}$$

Since $v_1 \to \mathbb{E}\Big[H_1\big(t, y(t), z(t), Y(t), Z(t), v_1, u_2(t), p_1(t), q_1(t), \bar{q}_1(t)\big)\big|\mathcal{E}_t\Big]$ is maximum for $v_1 = u_1$ and since $v_1(t)$ and $u_1(t)$ are $\mathcal{E}_t$-measurable, we get

$$\begin{aligned} &\mathbb{E}\left[\frac{\partial}{\partial v_1}H_1\big(t,\ y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)\big)\big(v_1(t) - u_1(t)\big)\big|\mathcal{E}_t\right] \\ =&\mathbb{E}\left[\frac{\partial}{\partial v_1}H_1\big(t,\ y(t), z(t), Y(t), Z(t), v_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)\big)\big|\mathcal{E}_t\right]_{v_1 = u_1}\big(v_1(t) - u_1(t)\big) \\ \le& 0. \end{aligned} \tag{2.17}$$

Combining (2.8), (2.16) with (2.17), we conclude that

$$J_1(v_1(\cdot), u_2(\cdot)) - J_1(u_1(\cdot), u_2(\cdot)) \le 0, \tag{2.18}$$

for all $v_1(\cdot) \in \mathcal{U}_1$. Repeating the similar proceeding as shown in deriving (2.18), we can prove that

$$J_2(u_1(\cdot), v_2(\cdot)) - J_2(u_1(\cdot), u_2(\cdot)) \le 0.$$

Based on the arguments above, $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of *Problem (NZSG)*. □

Corollary 2.4. *Suppose that $\mathcal{E}_t = \mathcal{F}_t$ for all t and that (H1), (H2), (2.8) and (2.9) hold. Suppose that φ_i and γ_i are concave in Y and y $(i = 1, 2)$, respectively. Let (y, z, Y, Z) and $(p_i, \bar{p}_i, q_i, \bar{q}_i)$ be the solutions of equations (2.1)* *and* (2.2) *corresponding to the admissible control* $\big(u_1(\cdot), u_2(\cdot)\big)$*, respectively. Moreover,*

$$\begin{aligned}
&H_1\big(t,\ y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)\big)\\
&= \sup_{v_1 \in\ U_1} H_1\big(t, y(t), z(t), Y(t), Z(t), v_1, u_2(t), p_1(t), \bar{p}_1(t), q_1(t), \bar{q}_1(t)\big)\\
&H_2\big(t,\ y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)\big)\\
&= \sup_{v_2 \in\ U_2} H_2\big(t, y(t), z(t), Y(t), Z(t), u_1(t), v_2, p_2(t), \bar{p}_2(t), q_2(t), \bar{q}_2(t)\big).
\end{aligned}$$

Then $(u_1(\cdot), u_2(\cdot))$ *is an equilibrium point of nonzero-sum differential games.*

2.4. An example

In this section, we first work out an LQ nonzero-sum differential game, and then specify how to apply the foregoing theoretical results to find an explicit expression of the equilibrium point.

Consider the linear FBDSDE

$$\begin{cases}
-\mathrm{d}Y^{v_1,v_2}(t) = \big[a_0(t) + a_1(t)Y^{v_1,v_2}(t) + a_2(t)Z^{v_1,v_2}(t) + a_3(t)v_1(t) + a_4(t)v_2(t)\big]\mathrm{d}t\\
\qquad\qquad + b_0(t)\hat{\mathrm{d}}B(t) - Z^{v_1,v_2}(t)\mathrm{d}W(t),\\
\mathrm{d}y^{v_1,v_2}(t) = \Big[c_0(t) + c_1(t)y^{v_1,v_2}(t) + c_2(t)Y^{v_1,v_2}(t) + c_3(t)Z^{v_1,v_2}(t)\Big]\mathrm{d}t\\
\qquad\qquad + d_0(t)\mathrm{d}W(t) - z^{v_1,v_2}(t)\hat{\mathrm{d}}B(t),\\
Y^{v_1,v_2}(T) = \xi, \quad y^{v_1,v_2}(0) = \ MY^{v_1,v_2}(0),
\end{cases} \tag{2.19}$$

and the performance criterion, for $i = 1, 2$,

$$\begin{aligned}
J_i\big(v_1(\cdot), v_2(\cdot)\big) = -\frac{1}{2}\mathbb{E}\Bigg[\int_0^T \Big(&\langle e_{i1}(t)y^{v_1,v_2}(t), y^{v_1,v_2}(t)\rangle + \langle e_{i2}(t)z^{v_1,v_2}(t), z^{v_1,v_2}(t)\rangle\\
&+ \langle e_{i3}(t)Y^{v_1,v_2}(t), Y^{v_1,v_2}(t)\rangle + \langle e_{i4}(t)Z^{v_1,v_2}(t), Z^{v_1,v_2}(t)\rangle + \langle e_{i7}(t)v_i(t), v_i(t)\rangle\Big)\mathrm{d}t\\
&+ \langle e_{i5}(T)y^{v_1,v_2}(T), y^{v_1,v_2}(T)\rangle + \langle e_{i6}(0)Y^{v_1,v_2}(0), Y^{v_1,v_2}(0)\rangle\Bigg].
\end{aligned} \tag{2.20}$$

Here, we assume that all the coefficients in (2.19) and (2.20) are bounded and deterministic functions of t; $e_{i1}, \ldots, e_{i6}$ are symmetric nonnegative definite; and e_{i7} is symmetric uniformly positive definite. The set of admissible controls is defined by

$$\begin{aligned}
\mathcal{U}_i = \{v_i(\cdot) \mid v_i(\cdot) \text{ is an } \mathrm{R}^{k_i}\text{-valued } \mathcal{E}_t\text{-adapted process}\\
\text{and satisfies } \mathbb{E}\int_0^T v_i^2(t)\mathrm{d}t < \infty\}, i = 1, 2.
\end{aligned}$$

Here

$$\mathcal{E}_t = \mathcal{N} \vee \sigma\big\{W(r) : 0 \le r \le t\big\}.$$

For simplicity, we only deal with the case of 1-dimensional coefficients. Our task is to find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, such that

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) = \sup\limits_{v_1(\cdot)\in\mathcal{U}_1} J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) = \sup\limits_{v_2(\cdot)\in\mathcal{U}_2} J_2(u_1(\cdot), v_2(\cdot)). \end{cases}$$

Solving: we find the equilibrium point by three steps.

(i) Seek candidate equilibrium points.

Let $\tilde{q}_i(t)$ denote the filtering of $q_i(\cdot)$ with respect to $\mathcal{E}_t$, *i.e.* $\tilde{q}_i(t) = \mathbb{E}\big(q_i(t)\big|\mathcal{E}_t\big)$. The similar notations are made for $\tilde{\bar{q}}_i(t), \tilde{p}_i(t), \tilde{\bar{p}}_i(t), \ldots, i = 1, 2$. We write down the concrete *Hamiltonian* function:

$$\begin{aligned} H_i(t, y, z, Y, Z, v_1, v_2, p_i, \bar{p}_i, q_i, \bar{q}_i) \triangleq\ & \langle q_i, c_0(t) + c_1 y + c_2(t)Y + c_3(t)Z\rangle + \langle \bar{q}_i, d_0(t)\rangle \\ & - \langle p_i, a_0(t) + a_1(t)Y + a_2(t)Z + a_3(t)v_1 + a_4(t)v_2\rangle \ - \langle \bar{p}_i, b_0(t)\rangle \\ & - \frac{1}{2}\Big(\langle e_{i1}(t)y, y\rangle + \langle e_{i2}(t)z, z\rangle + \langle e_{i3}(t)Y, Y\rangle + \langle e_{i4}(t)Z, Z\rangle + \langle e_{i7}(t)v_i, v_i\rangle\Big). \end{aligned} \tag{2.21}$$

According to Theorem 2.1, we confirm that the candidate equilibrium points must satisfy the following form:

$$\begin{cases} u_1(t) = -e_{17}^{-1}(t)a_3(t)\tilde{p}_1(t), \\ u_2(t) = -e_{27}^{-1}(t)a_4(t)\tilde{p}_2(t), \end{cases} \tag{2.22}$$

where $\big(p_i(\cdot), \bar{p}_i(\cdot), q_i(\cdot), \bar{q}_i(\cdot)\big)$, for $i = 1, 2$, is the solution of the following adjoint equation:

$$\begin{cases} dp_i(t) = \Big(e_{i3}(t)Y(t) + a_1 p_i(t) - c_2(t)q_i(t)\Big)\mathrm{d}t \\ \qquad\qquad + \Big(e_{i4}(t)Z(t) + a_2(t)p_i(t) - c_3(t)q_i(t)\Big)\mathrm{d}W(t) - \bar{p}_i(t)\hat{\mathrm{d}}B(t), \\ -dq_i(t) = \Big(-e_{i1}(t)y(t) + c_1(t)q_i(t)\Big)\mathrm{d}t - \Big(e_{i2}(t)z(t)\Big)\hat{\mathrm{d}}B(t) - \bar{q}_i(t)\mathrm{d}W(t), \\ p_i(0) = e_{i6}Y(0) - Mq_i(0), \quad q_i(T) = -e_{i5}(T)y(T), \end{cases} \tag{2.23}$$

and $\big(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot)\big)$ is the solution of the following state equation:

$$\begin{cases} -\mathrm{d}Y(t) = \big[a_0(t) + a_1(t)Y(t) + a_2(t)Z(t) - a_3(t)^2 e_{17}^{-1}(t)\tilde{p}_1(t) \\ \qquad\qquad - a_4(t)^2 e_{27}^{-1}(t)\tilde{p}_2(t)\big]\mathrm{d}t + b_0(t)\hat{\mathrm{d}}B(t) - Z(t)\mathrm{d}W(t), \\ \mathrm{d}y(t) = \Big[c_0(t) + c_1(t)y(t) + c_2(t)Y(t) + c_3(t)Z(t)\Big]\mathrm{d}t + d_0(t)\mathrm{d}W(t) - z(t)\hat{\mathrm{d}}B(t), \\ Y(T) = \xi, \quad y(o) = \ MY(0). \end{cases} \tag{2.24}$$

(ii) Optimal filtering with $\mathcal{E}_t = \mathcal{N} \vee \sigma\big\{W(r); 0 \le r \le t\big\}$.

Equation (2.23) together with (2.24) constitutes a triple dimensional FBDSDE. In order to find the explicit expression of the candidate equilibrium point, we need to compute the optimal filters $\tilde{p}_1(\cdot)$ and $\tilde{p}_2(\cdot)$ of $p_1(\cdot)$

and $p_2(\cdot)$, respectively. Applying Lemma 5.4 in [18] to (2.23) and (2.24), we conclude that $\tilde{p}_1(\cdot)$ and $\tilde{p}_2(\cdot)$ satisfy the following triple dimensional FBSDE:

$$\left\{\begin{aligned} d\begin{pmatrix}\tilde{y}(t)\\ \tilde{p}_1(t)\\ \tilde{p}_2(t)\end{pmatrix} =& \left\{\begin{pmatrix} c_2(t) & 0 & 0\\ e_{13}(t) & -c_2(t) & 0\\ e_{23}(t) & 0 & -c_2(t)\end{pmatrix}\begin{pmatrix}\tilde{Y}(t)\\ \tilde{q}_1(t)\\ \tilde{q}_2(t)\end{pmatrix} + \begin{pmatrix} c_1(t) & 0 & 0\\ 0 & a_1(t) & 0\\ 0 & 0 & a_1(t)\end{pmatrix}\begin{pmatrix}\tilde{y}(t)\\ \tilde{p}_1(t)\\ \tilde{p}_2(t)\end{pmatrix}\right.\\ &\left. + \begin{pmatrix} c_3(t) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{Z}(t)\\ \tilde{\tilde{q}}_1(t)\\ \tilde{\tilde{q}}_2(t)\end{pmatrix} + \begin{pmatrix} c_0(t)\\ 0\\ 0\end{pmatrix}\right\}\mathrm{d}t\\ & + \left\{\begin{pmatrix} 0 & 0 & 0\\ 0 & -c_3(t) & 0\\ 0 & 0 & -c_3(t)\end{pmatrix}\begin{pmatrix}\tilde{Y}(t)\\ \tilde{q}_1(t)\\ \tilde{q}_2(t)\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0\\ e_{14}(t) & 0 & 0\\ e_{24}(t) & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{Z}(t)\\ \tilde{\tilde{q}}_1(t)\\ \tilde{\tilde{q}}_2(t)\end{pmatrix}\right.\\ &\left. + \begin{pmatrix} 0 & 0 & 0\\ 0 & a_2(t) & 0\\ 0 & 0 & a_2(t)\end{pmatrix}\begin{pmatrix}\tilde{y}(t)\\ \tilde{p}_1(t)\\ \tilde{p}_2(t)\end{pmatrix} + \begin{pmatrix} d_0(t)\\ 0\\ 0\end{pmatrix}\right\}\mathrm{d}W(t),\\ \begin{pmatrix}\tilde{y}(0)\\ \tilde{p}_1(0)\\ \tilde{p}_2(0)\end{pmatrix} =& \begin{pmatrix} M & 0 & 0\\ e_{16}(t) & -M & 0\\ e_{26}(t) & 0 & -M\end{pmatrix}\begin{pmatrix}\tilde{Y}(0)\\ \tilde{q}_1(0)\\ \tilde{q}_2(0)\end{pmatrix}, \end{aligned}\right. \tag{2.25a}$$

and

$$\left\{\begin{aligned} -d\begin{pmatrix}\tilde{Y}(t)\\ \tilde{q}_1(t)\\ \tilde{q}_2(t)\end{pmatrix} =& \left\{\begin{pmatrix} a_1(t) & 0 & 0\\ 0 & c_1(t) & 0\\ 0 & 0 & c_1(t)\end{pmatrix}\begin{pmatrix}\tilde{Y}(t)\\ \tilde{q}_1(t)\\ \tilde{q}_2(t)\end{pmatrix}\right.\\ & + \begin{pmatrix} 0 & -a_3(t)^2 e_{17}^{-1}(t) & -a_4(t)^2 e_{28}^{-1}(t)\\ -e_{11}(t) & 0 & 0\\ -e_{21}(t) & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{y}(t)\\ \tilde{p}_1(t)\\ \tilde{p}_2(t)\end{pmatrix}\\ &\left. + \begin{pmatrix} a_2(t) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\tilde{Z}(t)\\ \tilde{\tilde{q}}_1(t)\\ \tilde{\tilde{q}}_2(t)\end{pmatrix} + \begin{pmatrix} a_0(t)\\ 0\\ 0\end{pmatrix}\right\}\mathrm{d}t - \begin{pmatrix}\tilde{Z}(t)\\ \tilde{\tilde{q}}_1(t)\\ \tilde{\tilde{q}}_2(t)\end{pmatrix}\mathrm{d}W(t),\\ \begin{pmatrix}\tilde{Y}(T)\\ \tilde{q}_1(T)\\ \tilde{q}_2(T)\end{pmatrix} =& \begin{pmatrix}\mathbb{E}[\xi|\mathcal{E}_T]\\ -e_{15}(T)\tilde{y}(T)\\ -e_{25}(T)\tilde{y}(T)\end{pmatrix}. \end{aligned}\right. \tag{2.25b}$$

Note that (2.25a) is a *forward* stochastic differential filtering equation, while (2.25b) is a *backward* stochastic differential filtering equation (2.25a) together with (2.25b) constitutes a coupled forward-backward stochastic differential filtering equation denoted by (2.25), which is distinguished from the classical filtering literature (see *e.g.* [8]). Now, we obtain an explicit candidate equilibrium point for the foregoing LQ nonzero-sum differential game.

(iii) Verify that $\big(u_1(\cdot), u_2(\cdot)\big)$ denoted by (2.22) is indeed an equilibrium point.

We can check that the system (2.19) and the performance criterion (2.20) satisfy the assumptions (H1) and (H2), that $\gamma_i(y) = -e_{i5}y^2$ and $\varphi_i(Y) = -e_{i6}Y^2$ are concave with respect to y and Y respectively, and that the Hamiltonian H_i $(i = 1, 2)$ denoted by (2.21) satisfies the conditions (2.8)–(2.11). Then, from Theorem 2.3, we conclude that $\big(u_1(\cdot), u_2(\cdot)\big)$ denoted by (2.22) is indeed an equilibrium point.

3. Zero-sum differential games

In this section, we study a zero-sum version of *Problem (NZSG)*.

We introduce the performance criterion

$$J(v_1(\cdot), v_2(\cdot)) = \mathbb{E}\bigg[\int_0^T l\big(t, y^{v_1,v_2}(t), z^{v_1,v_2}(t), Y^{v_1,v_2}(t), Z^{v_1,v_2}(t), v_1(t), v_2(t)\big)\mathrm{d}t + \varphi(y^{v_1,v_2}(T))\bigg] + \gamma(Y^{v_1,v_2}(0)).$$

For any $\big(v_1(\cdot),\, v_2(\cdot)\big) \in \mathcal{U}_1 \times \mathcal{U}_2$, we assume that $l(\cdot, y^{v_1,v_2}(\cdot), Y^{v_1,v_2}(\cdot), Z^{v_1,v_2}(\cdot), v_1(\cdot), v_2(\cdot)) \in \mathcal{L}^1_{\mathcal{F}_t}(0, T; \mathrm{R})$ and $\gamma \in \mathcal{L}^1_T\ (\Omega; \mathrm{R})$.

Problem (ZSG): Find $(u_1(\cdot), u_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$ such that

$$J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)) \leq J(v_1(\cdot), u_2(\cdot)),$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$. We call $(u_1(\cdot), u_2(\cdot))$ an open-loop saddle point of *Problem (ZSG)* (if it exists).

It is well known that the zero-sum game can be regarded as a special case of the foregoing nonzero-sum game. In fact, we let

$$-J_1 = J_2 = J.$$

If $(u_1(\cdot), u_2(\cdot))$ is an equilibrium point of *Problem (NZSG)*, we have

$$\begin{cases} J_1(u_1(\cdot), u_2(\cdot)) \geq J_1(v_1(\cdot), u_2(\cdot)), \\ J_2(u_1(\cdot), u_2(\cdot)) \geq J_2(u_1(\cdot), v_2(\cdot)), \end{cases}$$

which implies that

$$J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)) \leq J(v_1(\cdot), u_2(\cdot)).$$

We define a new *Hamiltonian function* $H : [0, T] \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times \mathrm{U}_1 \times \mathrm{U}_2 \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d}$ as follows:

$$\begin{aligned} H(t, y, z, Y, Z, v_1, v_2, p, \bar{p}, q, \bar{q}) \triangleq\ & \langle q, f(y, z, Y, Z, v_1, v_2)\rangle + \langle \bar{q}, \bar{f}(y, z, Y, Z, v_1, v_2)\rangle \\ & - \langle p, g(Y, Z, v_1, v_2)\rangle\ - \langle \bar{p}, \bar{g}(Y, Z, v_1, v_2)\rangle\ + l(y, z, Y, Z, v_1, v_2). \end{aligned}$$

Let $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ with the solution $\big(y(\cdot), z(\cdot), Y(\cdot), Z(\cdot)\big)$ of equation (2.1). We shall use the abbreviated notation $H(t)$ defined by

$$H(t) \equiv H\big(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)\big).$$

The adjoint equations are described by the following generalized stochastic Hamiltonian systems:

$$\begin{cases} \mathrm{d}p(t) = -\, H^*_Y(t)\mathrm{d}t - H^*_Z(t)\mathrm{d}W(t) - \bar{p}(t)\hat{\mathrm{d}}B(t), \\ -\mathrm{d}q(t) = H^*_y(t)\mathrm{d}t + H^*_z(t)\hat{\mathrm{d}}B(t) - \bar{q}(t)\mathrm{d}W(t), \\ p(0) = -\, \varphi^*_Y(Y(0)) - \phi^*_Y\big(Y(0)\big)q(0), \\ q(T) = \gamma^*_y\big(y(T)\big). \end{cases} \tag{3.1}$$

Based on the above arguments, we can derive the following maximum principle of *Problem (ZSG)*. Since some mathematical deductions are parallel to those of Section 2, then we will omit the detailed proof.

Theorem 3.1. *Let (H1) hold and $(u_1(\cdot), u_2(\cdot))$ be a saddle point of Problem (ZSG). Let $\Big(y(\cdot), z(\cdot),\, Y(\cdot), Z(\cdot)\Big)$ and $\Big(p(\cdot), \bar{p}(\cdot), q(\cdot),\, \bar{q}(\cdot)\Big)$ be the solutions of (2.1) and (3.1) corresponding to the control $(u_1(\cdot), u_2(\cdot))$, respectively. Then it follows that*

$$\Big\langle E[H^*_{v_1}(t)|\mathcal{E}_t], v_1(t) - u_1(t)\Big\rangle \geq 0$$

and

$$\Big\langle E[H^*_{v_2}(t)|\mathcal{E}_t], v_2(t)-u_2(t)\Big\rangle \leq 0$$

are true for any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, *a.e. a.s.*

Remark 3.2. If $\big(u_1(\cdot), u_2(\cdot)\big)$ is an equilibrium point (resp. a saddle point) of nonzero-sum (resp. zero-sum) differential games and $\big(u_1(t), u_2(t)\big)$ is an interior point of $U_1 \times U_2$ a.s. for all $t \in [0,T]$, then the inequalities in Theorem 2.2 (resp. Thm. 3.1) are equivalent to the following equations

$$E[H^*_{iv_i}(t)|\mathcal{E}_t] = 0, i=1,2 \ \big(resp.\ E[H^*_{v_j}(t)|\mathcal{E}_t]=0, j=1,2\big).$$

In the sequel, we give a verification theorem for a saddle point of zero-sum games.

Theorem 3.3. *Let* (H_1) *hold and* $\phi(Y) = MY$, *where* M *is a non-zero constant matrix with order* $n \times m$. *Let* (y, z, Y, Z) *and* $(p, \bar{p}, q, \bar{q})$ *be the solutions of equations* (2.1) *and* (3.1) *corresponding to the admissible control* $\big(u_1(\cdot), u_2(\cdot)\big)$, *respectively. Suppose that the Hamiltonian function* H *satisfies the following conditional mini-maximum principle:*

$$\begin{aligned}
&E\Big[H\big(t, y(t), z(t), Y(t), Z(t), u_1(t), u_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)\big)\big|\mathcal{E}_t\Big]\\
&= \inf_{v_1(\cdot)\in\mathcal{U}_1} E\Big[H(t, y(t), z(t), Y(t), Z(t), v_1(t), u_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)\big)\big|\mathcal{E}_t\Big]\\
&= \sup_{v_2(\cdot)\in\mathcal{U}_2} E\Big[H(t, y(t), z(t), Y(t), Z(t), u_1(t), v_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)\big)\big|\mathcal{E}_t\Big].
\end{aligned}$$

(a) Suppose that both φ *and* γ *are concave, and*

$$(t, y, z, Y, Z, v_2) \longrightarrow H\big(t, y, z, Y, Z, u_1(t), v_2, p(t), \bar{p}(t), q(t), \bar{q}(t)\big)$$

is concave, for all $(t, y, z, Y, Z, v_2) \in [0,T] \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_2$. *Then we have*

$$J(u_1(\cdot), v_2(\cdot)) \leq J(u_1(\cdot), u_2(\cdot)), \qquad \textit{for all } v_2(\cdot) \in \mathcal{U}_2,$$

and

$$J(u_1(\cdot), u_2(\cdot)) = \sup_{v_2(\cdot)\in\mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)).$$

(b) Suppose that both φ *and* γ *are convex, and*

$$(t, y, z, Y, Z, v_1) \longrightarrow H\big(t, y, z, Y, Z, v_1, u_2(t), p(t), \bar{p}(t), q(t), \bar{q}(t)\big)$$

is convex, for all $(t, y, z, Y, Z, v_1) \in [0,T] \times \mathrm{R}^n \times \mathrm{R}^{n\times l} \times \mathrm{R}^m \times \mathrm{R}^{m\times d} \times U_1$. *Then we have*

$$J(u_1(\cdot), u_2(\cdot)) \leq J(v_1(\cdot), u_2(\cdot)), \qquad \textit{for all } v_1(\cdot) \in \mathcal{U}_1,$$

and

$$J(u_1(\cdot), u_2(\cdot)) = \inf_{v_1(\cdot)\in\mathcal{U}_1} J(v_1(\cdot), u_2(\cdot)).$$

(c) If both (a) and (b) are true, then $(u_1(\cdot), u_2(\cdot))$ *is a saddle point which implies*

$$\sup_{v_2(\cdot)\in\mathcal{U}_2}\Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot), v_2(\cdot)\big)\Big) = J(u_1(\cdot), u_2(\cdot)) = \inf_{v_1(\cdot)\in\mathcal{U}_1}\Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot), v_2(\cdot)\big)\Big).$$

Proof.
(a) Using the similar proceeding shown in proving Theorem 2.3, we can obtain the following:

$$J(u_1(\cdot), v_2(\cdot)) \le J(u_1(\cdot), u_2(\cdot)), \qquad \text{for all } v_2(\cdot) \in \mathcal{U}_2.$$

Furthermore,

$$\sup_{v_2(\cdot)\in\mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)) \le J(u_1(\cdot), u_2(\cdot)).$$

Since $u_2(\cdot) \in \mathcal{U}_2$, we have

$$\sup_{v_2(\cdot)\in\mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)) = J(u_1(\cdot), u_2(\cdot)).$$

(b) This proof is a counterpart of (a), and consequently we omit the proof for simplicity.
(c) If both (a) and (b) are true, then

$$J(u_1(\cdot), v_2(\cdot)) \le J(u_1(\cdot), u_2(\cdot)) \le J(v_1(\cdot), u_2(\cdot)),$$

for all $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, *i.e.* $(u_1(\cdot), u_2(\cdot))$ is a saddle point.

In the following, we have

$$J(u_1(\cdot), u_2(\cdot)) \le \inf_{v_1(\cdot)\in\mathcal{U}_1} J(v_1(\cdot), u_2(\cdot)) \le \inf_{v_1(\cdot)\in\mathcal{U}_1} \Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot)v_2(\cdot)\big)\Big),$$

and

$$J(u_1(\cdot), u_2(\cdot)) \ge \sup_{v_2(\cdot)\in\mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)) \ge \sup_{v_2(\cdot)\in\mathcal{U}_2} \Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot), v_2(\cdot)\big)\Big),$$

which imply that

$$\sup_{v_2(\cdot)\in\mathcal{U}_2} \Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot), v_2(\cdot)\big)\Big) \le J(u_1(\cdot), u_2(\cdot)) \le \inf_{v_1(\cdot)\in\mathcal{U}_1} \Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot), v_2(\cdot)\big)\Big). \tag{3.2}$$

On the other hand, we derive

$$J(u_1(\cdot), u_2(\cdot)) \le \inf_{v_1(\cdot)\in\mathcal{U}_1} J(v_1(\cdot), u_2(\cdot)) \le \sup_{v_2(\cdot)\in\mathcal{U}_2} \Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot), v_2(\cdot)\big)\Big)$$

and

$$J(u_1(\cdot), u_2(\cdot)) \ge \sup_{v_2(\cdot)\in\mathcal{U}_2} J(u_1(\cdot), v_2(\cdot)) \ge \inf_{v_1(\cdot)\in\mathcal{U}_1} \Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot), v_2(\cdot)\big)\Big),$$

which show that

$$\sup_{v_2(\cdot)\in\mathcal{U}_2} \Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot), v_2(\cdot)\big)\Big) \ge J(u_1(\cdot), u_2(\cdot)) \ge \inf_{v_1(\cdot)\in\mathcal{U}_1} \Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot), v_2(\cdot)\big)\Big). \tag{3.3}$$

Combining (3.2) and (3.3), we obtain

$$\sup_{v_2(\cdot)\in\mathcal{U}_2} \Big(\inf_{v_1(\cdot)\in\mathcal{U}_1} J\big(v_1(\cdot), v_2(\cdot)\big)\Big) = J(u_1(\cdot), u_2(\cdot)) = \inf_{v_1(\cdot)\in\mathcal{U}_1} \Big(\sup_{v_2(\cdot)\in\mathcal{U}_2} J\big(v_1(\cdot), v_2(\cdot)\big)\Big).$$ □

Remark 3.4. Similar to the results in Section 2, we can also give the corresponding corollaries for maximum principle and verification theorem for a saddle point of full information zero-sum differential games. For simplicity, we omit them here.

4. Conclusion

We investigate a new stochastic differential game problem of FBDSDEs. Compared with the previous literature, our game systems are initial coupled FBDSDEs and are under the framework of partial information. We established a maximum principle and a verification theorem for an equilibrium point of nonzero-sum differential games and a saddle point of zero-sum differential games. We also gave an LQ nonzero-sum game to specify how to apply the theoretical results to find an explicit expression of the equilibrium point.

The subject issue studied in this paper possesses fine generality. Firstly, the FBDSDE game system covers many systems as its particular case. For example, if we drop the terms on backward Itô's integral or forward equation or both them, then the FBDSDE can be reduced to FBSDE or BDSDE or BSDE. Secondly, if we suppose that $\mathcal{E}_t = \mathcal{F}_t$ for all $t \in [0, T]$, all the results are reduced to the case of full information. Finally, if the present zero-sum stochastic differential game has only one player, the game problem is reduced to some related optimal control. Particularly, our results are a partial extension to optimal control of partial information FBSDEs [17], BSDEs [6] and full information BDSDEs [5], and to differential games of full information BSDEs [15] and partial information BSDEs [16]. In our game systems, the forward equations are coupled with the backward equations at initial time, rather than terminal time. In this regard, this paper offers new results on the initial time' issue, which previous terminal time' studies had not addressed.

Finally, since many optimization and game problems in finance and economics can be associated with forward-backward stochastic systems, the outcomes of this paper bear much relevance in these areas.

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