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NULL-CONTROL AND MEASURABLE SETS*

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Abstract. We prove the interior and boundary null-controllability of some parabolic evolutions with controls acting over measurable sets.

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1. Introduction

The control for evolution equations aims to drive the solution to a prescribed state starting from a certain initial condition. One acts on the equation through a source term, a so-called distributed control, or through a boundary condition. To achieve general results one wishes for the control to only act in part of the domain or its boundary and to have as much latitude as possible in the choice of the control region: location, size, shape.

Here, we focus on the heat equation in a smooth and bounded domain Ω in $\mathbb{R}^n$ for a time interval $(0,T)$, $T>0$ and for a distributed control f we consider

$$\begin{cases}\triangle u-\partial_t u=f(x,t)\chi_\omega(x), & \text{in } \Omega\times(0,T),\\ u=0, & \text{on } \partial\Omega\times[0,T],\\ u(0)=u_0, & \text{in } \Omega.\end{cases} \tag{1.1}$$

Here, $\omega\subset\Omega$ is an interior control region. The null controllability of this equation, *i.e.*, the existence for any u_0 in $L^2(\Omega)$ of a control f in $L^2(\omega\times(0,T))$ with

$$\|f\|_{L^2(\omega\times(0,T))}\le N\|u_0\|_{L^2(\Omega)}, \tag{1.2}$$

such that $u(T)=0$, was proved in [16] by means of local Carleman estimates for the elliptic operator $\triangle+\partial_y^2$ over $\Omega\times\mathbb{R}$. A second approach based on global Carleman estimates for the backward parabolic operator $\triangle+\partial_t$ [10], also led to the null controllability of the heat equation. The first approach has been used for the treatment

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of time-independent parabolic operators associated to self-adjoint elliptic operators, while the second allows to address time-dependent non-selfadjoint parabolic operators and semi-linear evolutions.

The method introduced in [16] was further extended to study thermoelasticity [17], thermoelastic plates [4] and semigroups generated by fractional orders of elliptic operators [20]. It has also been used to prove null controllability in the case of non smooth coefficients [5, 26]. The method of [16] has also be extended to treat some non-selfadjoint cases, e.g. non symmetric systems [15] and all 1-dimensional time-independent parabolic equations [2].

In the above works, the control region ω is always assumed to contain an open ball. Also, the cost of controllability (the smallest constant N found for the inequality (1.2)) depends on this fact. The reason for these is that the main technique used in the arguments, Carleman inequalities, requires to construct suitable Carleman weights: a role for functions which requires smoothness (at least C^2) and to have the extreme values in proper regions associated to the control region ω, the larger body Ω and possibly the value of $T > 0$. The construction of such functions seems to be not possible, when ω does not contain a ball.

Motivated by these facts Puel and Zuazua raised the question wether the null controllability of the heat equation is possible when the control region is a measurable set. A positive partial answer to this question was explained by the second author at the June 2008 meeting *Control of Physical Systems and Partial Differential Equations* held at the Institute Henri Poincaré. Here, we give a formal account of the results.

Theorem 1.1. *Let $n \geq 2$. Then, $\triangle - \partial_t$ is null-controllable at all positive times, with distributed controls acting over a measurable set $\omega \subset \Omega$ with positive Lebesgue measure, when*

$$\triangle = \nabla \cdot (\mathbf{A}(x)\nabla \cdot\,) + V(x),$$

is a self-adjoint elliptic operator, the coefficients matrix $\mathbf{A}$ is smooth in $\overline{\Omega}$, V is bounded in Ω and both are real-analytic in an open neighborhood of ω. The same holds when $n = 1$,

$$\triangle = \frac{1}{\rho(x)}\left[\partial_x\left(a(x)\partial_x\,\right) + b(x)\partial_x\, + c(x)\right]$$

and a, b, c and ρ are measurable functions in $\Omega = (0, 1)$.

In regard to boundary null controllability, *i.e.*, the existence for any u_0 in $L^2(\Omega)$ of a control h in $L^2(\gamma \times (0, T))$ with

$$\|h\|_{L^2(\gamma\times(0,T))} \leq N\|u_0\|_{L^2(\Omega)}, \tag{1.3}$$

such that the solution to

$$\begin{cases} \triangle u - \partial_t u = 0, & \text{in } \Omega \times (0, T), \\ u = h(x, t)\chi_\gamma(x), & \text{on } \partial\Omega \times [0, T], \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{1.4}$$

verifies $u(T) \equiv 0$, we have the following result.

Theorem 1.2. *Let $n \geq 2$. Then, $\triangle - \partial_t$ is null-controllable at all times $T > 0$ with boundary controls acting over a measurable set $\gamma \subset \partial\Omega$ with positive surface measure when*

$$\triangle = \nabla \cdot (\mathbf{A}(x)\nabla \cdot\,) + V(x)$$

is a self-adjoint elliptic operator, the coefficients matrix $\mathbf{A}$ is smooth in $\overline{\Omega}$, V is bounded in Ω and both are real-analytic in an open neighborhood of γ in $\overline{\Omega}$.

The results in Theorems 1.1 and 1.2 follow from a straightforward application of the linear construction of the control function for the systems (1.1) and (1.4) developed in [16] and the following observability inequality or propagation of smallness estimate established in [28] (see also [23, 24]).

Theorem 1.3. *Assume that $f : B_{2R} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is a real-analytic function verifying*

$$|\partial^\alpha f(x)| \le \frac{M|\alpha|!}{(\rho R)^{|\alpha|}}, \qquad \text{when } \; x \in B_{2R},\; \alpha \in \mathbb{N}^n, \tag{1.5}$$

for some $M > 0$, $0 < \rho \le 1$ and $E \subset B_{\frac{R}{2}}$ is a measurable set with positive measure. Then, there are positive constants $N = N(\rho, |E|/|B_R|)$ and $\theta = \theta(\rho, |E|/|B_R|)$ such that

$$\|f\|_{L^\infty(B_R)} \le N \left(\fint_E |f| \,\mathrm{d}x \right)^\theta M^{1-\theta}.$$

The experts will realize that the word *smooth* describing the global regularity of $\partial\Omega$, $\mathbf{A}$ and V (away from the measurable sets ω and γ respectively) in Theorems 1.1 and 1.2 can be relaxed to $\partial\Omega$ is C^2, $\mathbf{A}$ is Lipschitz in $\overline{\Omega}$ and V is bounded (See [10,16,17,25]).

To simplify the exposition and to show the strength of Theorem 1.3 we give the proof of Theorems 1.1 and 1.2 under the assumptions that $\partial\Omega$, $\mathbf{A}$ and V are *globally real analytic*. We do it because it makes more clear how the construction algorithm of the control function in [16] and Theorem 1.3 can also be applied to prove the *interior* null-controllability (Thm. 1.1) for other parabolic evolutions whose corresponding observability or spectral inequalities (suitable Carleman inequalities) are otherwise unknown. Examples of these parabolic evolutions are the ones associated to selfadjoint elliptic systems with unknowns $\mathbf{u} = (u^1, \dots, u^m)$,

$$L_\alpha \mathbf{u} = \partial_i (a_{ij}^{\alpha\beta}(x) \partial_j u^\beta), \qquad \alpha = 1, \dots, m$$

with $a_{ij}^{\alpha\beta} = a_{ji}^{\beta\alpha}$, for $\alpha, \beta = 1, \dots, m$, $i, j = 1, \dots, n$, and with coefficients matrices verifying for some $\delta > 0$ the strong ellipticity condition,

$$\sum_{i,j,\alpha,\beta} a_{ij}^{\alpha\beta}(x) \xi_i^\alpha \xi_j^\beta \ge \delta \sum_{i,\alpha} |\xi_i^\alpha|^2, \qquad \text{when } \boldsymbol{\xi} \in \mathbb{R}^{nm}, x \in \mathbb{R}^n,$$

or the more general Legendre-Hadamard condition

$$\sum_{i,j,\alpha,\beta} a_{ij}^{\alpha\beta}(x) \xi_i \xi_j \eta^\alpha \eta^\beta \ge \delta |\xi|^2 |\boldsymbol{\eta}|^2, \qquad \text{when } \xi \in \mathbb{R}^n, \boldsymbol{\eta} \in \mathbb{R}^m, x \in \mathbb{R}^n. \tag{1.6}$$

We recall that the Lamé system of elasticity

$$\nabla \cdot \left(\mu(x) \left(\nabla \mathbf{u} + \nabla \mathbf{u}^t \right) \right) + \nabla \left(\lambda(x) \nabla \cdot \mathbf{u} \right),$$

with $\mu \ge \delta$, $\mu + \lambda \ge 0$ in $\mathbb{R}^n$, $m = n$ and $a_{ij}^{\alpha\beta} = \mu(\delta_{\alpha\beta}\delta_{ij} + \delta_{i\beta}\delta_{j\alpha}) + \lambda \delta_{j\beta}\delta_{\alpha i}$, are examples of systems verifying (1.6). Here, $a_{ij}^{\alpha\beta}$, μ and λ can either be constants or real analytic functions on $\overline{\Omega}$.

It also makes clear that under such hypothesis one may replace the Carleman inequalities used in the literature to prove the observability or propagation of smallness inequalities necessary in the process of applying the construction algorithm in [16,17] by a finite number of successive applications of Theorem 1.3. Of course, it has the drawback that it requires more smoothness on the operators and on the boundary of Ω but on the contrary, with Theorem 1.3 and the construction methods in [16,17] one can handle the null-controllability with distributed controls of other parabolic evolutions, which as far as the authors know were unknown: like the second order evolutions explained above or for higher order evolutions as

$$\partial_t u + (-1)^m \triangle^m u, \; m = 2, \dots,$$

with Dirichlet boundary conditions on $\partial\Omega$, *i.e.*, $u = \nabla u = \cdots = \nabla^{m-1} u = 0$.

In Section 2 we give first the proofs of Theorems 1.1 and 1.2 as it is explained above. We then show how to extend Theorem 1.1 to the evolutions (2.25) and (2.31) in Remark 2.2, while the problems we find to extend Theorem 1.2 to these evolutions are explained in Remark 2.3 for the simpler case of parabolic systems. In Remark 2.4 we give the proof of Theorem 1.1 when $\partial\Omega$, $\mathbf{A}$ and V are globally smooth and $\mathbf{A}$ and V are real analytic in a neighborhood of the measurable set ω. An outline of a proof of Theorem 1.2, when $\partial\Omega$, $\mathbf{A}$ and V are real analytic near γ and smooth elsewhere appears in Remark 2.4.

For the sake of completeness, we include a proof of Theorem 1.3 in Section 3. It is built with ideas taken from [18, 23, 28].

2. Proof of Theorems 1.1 and 1.2

We begin by setting up the formal hypothesis: first we assume there is $0 < \delta \le 1$ such that

$$\delta|\xi|^2 \le \mathbf{A}(x)\xi\cdot\xi \le \delta^{-1}|\xi|^2, \quad \text{for all } x \in \Omega, \quad \xi \in \mathbb{R}^n,$$

Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 2$, with a real analytic boundary and $\mathbf{A}$, V are real analytic in $\overline{\Omega}$, *i.e.*, there are $r > 0$ and $0 < \delta \le 1$ such that

$$|\partial^\alpha \mathbf{A}(x)| + |\partial^\alpha V(x)| \le |\alpha|!\delta^{-|\alpha|-1}, \quad \text{when } x \in \overline{\Omega},\ \alpha \in \mathbb{N}^n,$$

and for each $x \in \partial\Omega$, there are a new coordinate system (where $x = 0$) and a real analytic function $\varphi : B'_r \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ verifying

$$\begin{aligned} &\varphi(0') = 0,\ |\partial^\alpha\varphi(x')| \le |\alpha|!\delta^{-|\alpha|-1}, \quad \text{when } x' \in B'_r,\ \alpha \in \mathbb{N}^{n-1},\\ &B_r \cap \Omega = B_r \cap \{(x', x_n) : x' \in B'_r,\ x_n > \varphi(x')\},\\ &B_r \cap \partial\Omega = B_r \cap \{(x', x_n) : x' \in B'_r,\ x_n = \varphi(x')\}. \end{aligned} \tag{2.1}$$

For $\rho > 0$, we set

$$\Omega_\rho = \{x \in \Omega : d(x, \partial\Omega) \le \rho\}, \quad \Omega^\rho = \{x \in \Omega : d(x, \partial\Omega) \ge \rho\},$$
$$\Omega(\rho) = \{x \in \mathbb{R}^n : d(x, \Omega) \le \rho\}$$

and $|E|$ denotes the Lebesgue or surface measure of a measurable set E.

Proof of Theorem 1.1. We may assume that the eigenvalues with zero Dirichlet condition for $\triangle = \nabla\cdot(\mathbf{A}(x)\nabla\cdot) + V(x)$ on Ω are all positive

$$0 < \omega_1^2 < \omega_2^2 \le \omega_3^2 \le \dots \le \omega_j^2 \le \dots$$

and $\{e_j\}$ denotes the sequence of $L^2(\Omega)$-normalized eigenfunctions,

$$\begin{cases} \triangle e_j + \omega_j^2 e_j = 0, & \text{in } \Omega,\\ e_j = 0, & \text{in } \partial\Omega. \end{cases}$$

When $\omega \subset \Omega$ is measurable with positive Lebesgue measure, the method in [16] shows that one can find and $L^2(\omega \times (0,T))$ control function f verifying (1.2) for the system (1.1), provided there is $N = N(|\omega|, \Omega, r, \delta)$ such that the inequality

$$\sum_{\omega_j \le \mu} a_j^2 + b_j^2 \le \mathrm{e}^{N\mu} \int\int_{\omega\times[\frac14,\frac34]} \left| \sum_{\omega_j\le\mu} \left(a_j \mathrm{e}^{\omega_j y} + b_j \mathrm{e}^{-\omega_j y}\right) e_j \right|^2 \mathrm{d}x\mathrm{d}y, \tag{2.2}$$

holds for $\mu \geq \omega_1$ and all sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$ Set then

$$u(x,y) = \sum_{\omega_j \leq \mu} \left(a_j \mathrm{e}^{\omega_j y} + b_j \mathrm{e}^{-\omega_j y} \right) e_j. \tag{2.3}$$

It satisfies

$$\begin{cases} \triangle u + \partial_y^2 u = 0, & \text{in } \Omega \times \mathbb{R}, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}, \end{cases} \tag{2.4}$$

and u is real analytic in $\overline{\Omega} \times \mathbb{R}$. Moreover, given (x_0, y_0) in $\overline{\Omega} \times \mathbb{R}$ and $R \leq 1$, there are $N = N(r, \delta)$ and $\rho = \rho(r, \delta)$, $0 < \rho \leq 1$, such that

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(B_R(x_0,y_0) \cap \Omega \times \mathbb{R})} \leq \frac{N(|\alpha| + \beta)!}{(R\rho)^{|\alpha|+\beta}} \left(\fint_{B_{2R}(x_0,y_0) \cap \Omega \times \mathbb{R}} |u|^2 \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}, \tag{2.5}$$

when $\alpha \in \mathbb{N}^n$ and $\beta \geq 1$. For the later see [21], Chapter 5, [13], Chapter 3.

The orthonormality of $\{e_j\}$ in Ω and (2.5) with $R = 1$ imply that

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(\Omega \times [-5,5])} \leq \mathrm{e}^{N\mu} (|\alpha| + \beta)! \rho^{-|\alpha|-\beta} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \right)^{\frac{1}{2}}, \quad \text{for } \alpha \in \mathbb{N}^n, \quad \beta \geq 0, \tag{2.6}$$

and there is $C > 0$ such that replacing the constants N and ρ in (2.6) by CN and ρ/C respectively, u has a real analytic extension to $\Omega(\rho) \times [-4, 4]$ satisfying

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(\Omega(\rho) \times [-4,4])} \leq M(|\alpha| + \beta)!(2\rho)^{-|\alpha|-\beta}, \quad \text{for } \alpha \in \mathbb{N}^n, \quad \beta \geq 0, \tag{2.7}$$

with

$$M = \mathrm{e}^{N\mu} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2 \right)^{\frac{1}{2}}. \tag{2.8}$$

For (x_0, y_0) in $\Omega \times [0, 1]$ with $d(x_0, \partial\Omega) = \rho$, we have $B_{2\rho}(x_0, y_0) \subset \Omega(\rho) \times [-4, 4]$, and if we apply Theorem 1.3 to the real analytic extension of u in $B_{2\rho}(x_0, y_0)$ with $E = B_{\frac{\rho}{4}}(x_0, y_0)$, (2.7) implies that there are new constants $N = N(\rho) > 0$ and $0 < \theta = \theta(\rho) < 1$ such that

$$\|u\|_{L^\infty(B_\rho(x_0,y_0))} \leq N \|u\|_{L^\infty(B_{\frac{\rho}{4}}(x_0,y_0))}^\theta M^{1-\theta}.$$

The later implies

$$\|u\|_{L^\infty(\Omega_\rho \times [0,1])} \leq N \|u\|_{L^\infty(\Omega^{\frac{3\rho}{4}} \times [-\rho, 1+\rho])}^\theta M^{1-\theta}, \tag{2.9}$$

so that from (2.7) with $\alpha = \beta = 0$ and (2.9)

$$\|u\|_{L^\infty(\Omega \times [0,1])} \leq N \|u\|_{L^\infty(\Omega^{\frac{3\rho}{4}} \times [-1,2])}^\theta M^{1-\theta}. \tag{2.10}$$

Thus, from Theorem 3 and without Carleman inequalities it is possible to bound except for the factor M all the information on the size of u over $\Omega \times [0, 1]$ by information on the size of u over $\Omega^{\frac{3\rho}{4}} \times [-1, 2]$, a region located in the interior and away from the boundary of $\Omega \times \mathbb{R}$.

After choosing ρ smaller and replacing ω by a smaller measurable set if it is necessary, we can assume that $\omega \subset B_{\frac{\rho}{4}}(0)$, $|\omega|/|B_{\frac{\rho}{4}}(0)| \geq \frac{1}{2}$ and $B_{2\rho}(0) \subset \Omega^{\frac{3\rho}{4}}$, so that $E = \omega \times (\frac{2-\rho}{4}, \frac{2+\rho}{4}) \subset B_{\frac{\rho}{2}}(0, \frac{1}{2})$ has positive Lebesgue measure inside $B_{\frac{\rho}{2}}(0, \frac{1}{2})$ and $B_{2\rho}(0, \frac{1}{2}) \subset \Omega \times [0, 1]$. Then, a second application of Theorem 1.3 and (2.7) gives

$$\|u\|_{L^\infty(B_\rho(0,\frac{1}{2}))} \leq N \|u\|_{L^2(\omega \times [\frac{1}{4}, \frac{3}{4}])}^\theta M^{1-\theta}. \tag{2.11}$$

Also, (2.7) and Theorem 1.3 with $E = B_{\frac{R}{4}}(x_2, y_2)$ imply that

$$\|u\|_{L^\infty(B_R(x_1,y_1))} \le N\|u\|^{\theta}_{L^\infty(B_{\frac{R}{4}}(x_2,y_2))} M^{1-\theta},$$

whenever $B_{2R}(x_1, y_1) \subset \Omega \times [-4, 4]$, (x_2, y_2) is in $B_{\frac{R}{4}}(x_1, y_1)$ and $0 < R \le 1$. In particular,

$$\|u\|_{L^\infty(B_R(x_1,y_1))} \le N\|u\|^{\theta}_{L^\infty(B_R(x_2,y_2))} M^{1-\theta}, \tag{2.12}$$

when $B_{2R}(x_1, y_1) \subset \Omega \times [-4, 4]$, (x_2, y_2) is in $B_{\frac{R}{4}}(x_1, y_1)$ and $0 < R \le 1$.

Because ρ is now fixed and $\overline{\Omega}$ is compact there are R in $(0, 1)$ and $k \ge 2$, which depend on ρ and the geometry of Ω, such that for any (x_0, y_0) in $\Omega^{\frac{3\rho}{4}} \times [-1, 2]$ there are k points $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)$ in $\Omega^{\frac{3\rho}{4}} \times [-1, 2]$ with

$$B_{2R}(x_i, y_i) \subset \Omega \times [-4, 4],\ (x_{i+1}, y_{i+1}) \in B_{\frac{R}{4}}(x_i, y_i), \quad \text{for } i = 0, \dots, k-1$$

and $(x_k, y_k) = (0, \frac{1}{2})$. The later and (2.12) show that

$$\|u\|_{L^\infty(B_R(x_i,y_i))} \le N\|u\|^{\theta}_{L^\infty(B_R(x_{i+1},y_{i+1}))} M^{1-\theta}, \quad \text{for } i = 0, \dots, k-1, \tag{2.13}$$

while the iteration of (2.13) gives

$$\|u\|_{L^\infty(\Omega^{\frac{3\rho}{4}} \times[-1,2])} \le N^k \|u\|^{\theta^k}_{L^\infty(B_\rho(0,\frac{1}{2}))} M^{1-\theta^k}. \tag{2.14}$$

Combining then (2.10), (2.14) and (2.11), one finds that

$$\|u\|_{L^\infty(\Omega\times[0,1])} \le N^{k+2} \|u\|^{\theta^{k+2}}_{L^2(\omega\times[\frac{1}{4},\frac{3}{4}])} M^{1-\theta^{k+2}}.$$

Thus,

$$\|u\|_{L^\infty(\Omega\times[0,1])} \le N \|u\|^{\theta}_{L^2(\omega\times[\frac{1}{4},\frac{3}{4}])} M^{1-\theta}, \tag{2.15}$$

for some new N and $0 < \theta < 1$, which depend on ρ, $|\omega|$ and the geometry of Ω but not on u.

The fact that the inequality

$$\mathrm{e}^{-\mu}\left(\frac{\sinh \omega_1}{\omega_1} - 1\right)\left(a^2 + b^2\right) \le \int_0^1 \left(a\mathrm{e}^{\omega y} + b\mathrm{e}^{-\omega y}\right)^2 \mathrm{d}y,$$

holds, when $\omega_1 \le \omega \le \mu$ and $a, b \in \mathbb{R}$ and the orthonormality of $\{e_j\}$ in $L^2(\Omega)$ show that

$$\left(\sum_{\omega_j \le \mu} a_j^2 + b_j^2\right)^{\frac{1}{2}} \le \mathrm{e}^{N\mu}\|u\|_{L^2(\Omega\times[0,1])} \le \mathrm{e}^{N\mu}|\Omega|^{\frac{1}{2}}\|u\|_{L^\infty(\Omega\times[0,1])}, \tag{2.16}$$

and (2.2) follows from (2.16), (2.15) and the definition of M in (2.8).

It is shown in [2] that the null-controllability of the system (1.1) over $\Omega = (0, 1)$ with

$$\triangle = \frac{1}{\rho(x)}\left[\partial_x\left(a(x)\partial_x\right) + b(x)\partial_x + c(x)\right],$$

$$\delta \le a(x),\ \rho(x) \le \delta^{-1}, \qquad |b(x)| + |c(x)| \le \delta^{-1}, \quad \text{a.e. in } [0, 1], \tag{2.17}$$

is equivalent to the null-controllability of the system

$$\begin{cases} \partial_x^2 z - \rho(x)\partial_t z = f\chi_\omega, & 0 < x < 1\ ,\ 0 < t < T,\\ z(0,t) = z(1,t) = 0, & 0 \le t \le T,\\ z(x,0) = z_0, & 0 \le x \le 1, \end{cases} \tag{2.18}$$

where ρ is a new function verifying (2.17) and ω a new measurable set in $[0,1]$ with positive measure. The later follows from the bilipschitz change of variables used in [2]. Let then, $0 < \omega_1^2 < \omega_2^2 \le \omega_3^2 \le \dots \le \omega_j^2 \le \dots$ and $\{e_j\}$ denote the sequences of eigenvalues and $L^2(\Omega)$-normalized eigenfunctions [29], Chapter VII, verifying

$$\begin{cases} e_j'' + \rho(x)\omega_j^2 e_j = 0, & 0 < x < 1,\\ e_j(0) = e_j(1) = 0. \end{cases}$$

From [16] it suffices to show that (2.2) holds in order to find an interior null-control f for (2.18) verifying (1.2). Extend then e_j and ρ to $[-1,1]$ by odd and even reflections respectively, and to all $\mathbb{R}$ as periodic functions of period 2. The extended e_j is in $C^{1,1}(\mathbb{R})$ and verifies $e_j'' + \rho(x)\omega_j e_j = 0$ in $\mathbb{R}$, $j = 1, 2 \dots$. As before, let u be defined by (2.3), it verifies

$$\partial_x^2 u + \partial_y\left(\rho(x)\partial_y u\right) = 0, \quad \text{in } \mathbb{R}^2.$$

By Chebyshev's inequality and defining E as

$$\left(\omega \times \left[\frac{1}{4}, \frac{3}{4}\right]\right) \setminus E = \left\{(x,y) \in \omega \times \left[\frac{1}{4}, \frac{3}{4}\right] : |u(x,y)|/2 > \fint_{\omega\times[\frac{1}{4},\frac{3}{4}]} |u| \,\mathrm{d}x\mathrm{d}y\right\},$$

we have

$$|E| \ge \frac{1}{2}\left|\omega \times \left[\frac{1}{4}, \frac{3}{4}\right]\right| \quad \text{and} \quad \|u\|_{L^\infty(E)} \le 2 \fint_{\omega\times[\frac{1}{4},\frac{3}{4}]} |u| \,\mathrm{d}x\mathrm{d}y. \tag{2.19}$$

Set then $u_\epsilon(x,y) = u(\frac{x}{\epsilon}, \frac{y}{\epsilon})$, it verifies

$$\partial_x^2 u_\epsilon + \partial_y\left(\rho(x/\epsilon)\partial_y u_\epsilon\right) = 0, \quad \text{in } \mathbb{R}^2$$

and let v_ϵ be the *stream* function of u_ϵ, *i.e.*, the solution to

$$\begin{cases} \partial_x v_\epsilon = -\rho(\frac{x}{\epsilon})\partial_y u_\epsilon,\\ \partial_y v_\epsilon = \partial_x u_\epsilon,\\ v_\epsilon(0,0) = 0. \end{cases}$$

Then, $f = u_\epsilon + iv_\epsilon$ is $(1/\delta)$-quasiregular, *i.e.*,

$$f \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2), \quad \partial_{\overline{z}} f = \nu(z)\partial_z f, \quad |\nu(z)| \le \frac{1-\delta}{1+\delta}, \quad z \in \mathbb{C},$$

and by the Ahlfors-Bers representation theorem [1] (see [6] or [7]) all the $(1/\delta)$-quasiregular mappings f in B_4 can be written as

$$f = F \circ \Psi,$$

where $F = U + iV$ is holomorphic in B_4 and $\Psi : B_4 \longrightarrow B_4$ is a $(1/\delta)$-quasiconformal mapping, *i.e.*, a $(1/\delta)$-quasiregular homeomorphism from B_4 onto B_4 verifying

$$\begin{gathered} \partial_{\overline{z}}\Psi = \nu(z)\partial_z\Psi, \quad \Psi(0) = 0, \quad \Psi(4) = 4,\\ N^{-1}|z_1 - z_2|^{\frac{1}{\alpha}} \le |\Psi(z_1) - \Psi(z_2)| \le N|z_1 - z_2|^{\alpha}, \quad \text{when } z_1, z_2 \in B_4, \end{gathered} \tag{2.20}$$

for some $0 < \alpha < 1$ and $N \geq 1$ depending on δ. Now, $\epsilon E \subset B_{2\epsilon}$, and from (2.20), $\Psi(\epsilon E) \subset B_{N(2\epsilon)^\alpha}$. Choose then ϵ so that $N(2\epsilon)^\alpha = \frac{1}{2}$. Thus, $\Psi(\epsilon E) \subset B_{\frac{1}{2}}$, $u_\epsilon = U \circ \Psi$,

$$\|U\|_{L^\infty(B_4)} = \|u\|_{L^\infty\left(B_{\frac{4}{\epsilon}}\right)}$$

while the L^∞-interior estimates for subsolutions of elliptic equations [12], Section 8.6, the periodicity and orthogonality of the eigenfunctions e_j in $L^2([0,1], \rho(x)\,\mathrm{d}x)$, gives

$$\|u\|_{L^\infty(B_{\frac{4}{\epsilon}})} \lesssim \|u\|_{L^2(B_{\frac{6}{\epsilon}})} \lesssim \mathrm{e}^{6\mu/\epsilon} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2\right)^{\frac{1}{2}}.$$

Thus, U is harmonic in B_4,

$$\|U\|_{L^\infty(B_4)} \leq \mathrm{e}^{N\mu} \left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2\right)^{\frac{1}{2}}$$

and from (2.19)

$$\|U\|_{L^\infty(\Psi(\epsilon E))} \leq 2 \fint_{\omega \times \left[\frac{1}{4}, \frac{3}{4}\right]} |u| \,\mathrm{d}x\mathrm{d}y. \tag{2.21}$$

All together, U verifies the conditions in Theorem 1.3 in B_2 with $R = 1$ and the universal constant $0 < \rho \leq 1$ associated to the quantitative property of analyticity over B_2 for bounded harmonic functions on B_4 [9], Chapter 2. From (2.21), (2.2) holds provided we can find a lower bound for the Lebesgue measure of $\Psi(\epsilon E) \subset B_{\frac{1}{2}}$. The lower bound follows from (2.19) and the following rescaled version of [3], Theorem 1:

Let $\Psi : B_4 \longrightarrow B_4$ be a $(1/\delta)$-quasiconformal mapping with $\Psi(0) = 0$ and $E \subset B_4$ be a measurable set. Then, there is $N = N(\delta)$ such that

$$|E|^{\frac{1}{\delta}}/N \leq |\Psi(E)| \leq N|E|^\delta.$$

□

Remark 2.1. Theorem 1.3 also implies the version of (2.2) appearing in [17]. For if Ω, $\mathbf{A}$ and V are as above and

$$u(x,y) = \sum_{\omega_j \leq \mu} a_j \mathrm{e}^{\omega_j y} e_j(x),$$

u verifies (2.5) and

$$\|\partial_x^\alpha u(\,.\,,0)\|_{L^\infty(\Omega)} \leq M|\alpha|!(2\rho)^{-|\alpha|}, \quad \text{for } \alpha \in \mathbb{N}^n, \quad \text{with } M = \mathrm{e}^{N\mu} \left(\sum_{\omega_j \leq \mu} a_j^2\right)^{\frac{1}{2}}.$$

Thus, $u(\,.\,,0)$ has an analytic extension to a ρ-neighborhood of $\overline{\Omega}$ and after a finite number of applications of Theorem 1.3 and a suitable covering argument we get

$$\|u(\,.\,,0)\|_{L^2(\Omega)} \leq N\|u(\,.\,,0)\|_{L^2(\omega)}^\theta M^{1-\theta}.$$

In particular,

$$\sum_{\omega_j \leq \mu} a_j^2 \leq \mathrm{e}^{N\mu} \int_\omega \left|\sum_{\omega_j \leq \mu} a_j e_j\right|^2 \mathrm{d}x, \quad \text{with } N = N(|\omega|, \Omega, r, \delta).$$

Proof of Theorem 1.2. Let u be defined by (2.3). From [16], one can find a boundary control h verifying (1.3) provided there is $N = N(|\gamma|, \Omega, r, \delta)$ such that the inequality

$$\sum_{\omega_j \le \mu} a_j^2 + b_j^2 \le \mathrm{e}^{N\mu} \int\int_{\gamma \times [\frac{1}{4}, \frac{3}{4}]} \left| \sum_{\omega_j \le \mu} \left(a_j \mathrm{e}^{\omega_j y} + b_j \mathrm{e}^{-\omega_j y}\right) \frac{\partial e_j}{\partial \nu} \right|^2 \mathrm{d}\sigma \mathrm{d}y, \tag{2.22}$$

holds for $\mu \ge \omega_1$ and all sequences $a_1, a_2, \dots$ and $b_1, b_2, \dots$. Here, ν, σ and $\frac{\partial}{\partial \nu}$ denote respectively the exterior unit normal vector to Ω, the surface measure on $\partial\Omega$ and the conormal derivative for $\partial_y^2 + \triangle$ on $\partial\Omega \times \mathbb{R}$, $\frac{\partial e}{\partial \nu} = A\nabla e \cdot \nu$. We may also assume that $0 \in \partial\Omega$, $\gamma \subset B_{\frac{\rho}{2}} \cap \partial\Omega$, where $B_{2\rho} \cap \partial\Omega$ is the region above the graph of a real analytic function, $\varphi : B'_\rho \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, as in (2.1). From [16], Section 3(2), there is N such that

$$\left(\sum_{\omega_j \le \mu} a_j^2 + b_j^2\right)^{\frac{1}{2}} \le \mathrm{e}^{N\mu} \left\| \frac{\partial u}{\partial \nu} \right\|_{L^\infty(\partial\Omega \times [-1,2])}, \tag{2.23}$$

with

$$\frac{\partial u}{\partial \nu} = \sum_{\omega_j \le \mu} \left(a_j \mathrm{e}^{\omega_j y} + b_j \mathrm{e}^{-\omega_j y}\right) \frac{\partial e_j}{\partial \nu} \cdot$$

From (2.6) and (2.1), there are $N = N(r, \delta)$ and $\rho = \rho(r, \delta)$ such that $h(x', y) = \frac{\partial u}{\partial n}(x', \varphi(x'), y)$ verifies

$$\|\partial_{x'}^\alpha \partial_y^\beta h\|_{L^\infty(B'_{2\rho} \times [-4,4])} \le M\,(|\alpha| + \beta)!(2\rho)^{-|\alpha| - \beta}, \quad \text{for } \alpha \in \mathbb{N}^{n-1}, \quad \beta \in \mathbb{N},$$

with M as in (2.8) and provided that $B_{2\rho} \cap \partial\Omega$ is a coordinate chart of $\partial\Omega$ as in (2.1). This fact, a suitable covering argument of $\partial\Omega$ and the successive iteration of a finite number of applications of the *three-spheres type inequalities* associated to the obvious extension of Theorem 1.3 for real analytic functions defined over real analytic hypersurfaces in $\mathbb{R}^{n+1}$ (See [14], pp. 67–69 for a quantitative version of the real analyticity of composite functions) imply that there are $N = N(|\gamma|, r, \delta)$ and $\theta = \theta(|\gamma|, r, \delta)$ such that

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^\infty(\partial\Omega \times [-1,2])} \le N \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\gamma \times [\frac{1}{4}, \frac{3}{4}])}^\theta M^{1-\theta}. \tag{2.24}$$

Finally, (2.22) follows from (2.23) and (2.24). □

Remark 2.2. The extension of Theorem 1.1 to the parabolic system

$$\begin{cases} \partial_i(a_{ij}^{\alpha\beta} \partial_j e_k^\beta) - \partial_t u^\alpha = f^\alpha(x,t)\chi_\omega(x), & \text{in } \Omega \times (0,T),\ \alpha = 1, \dots, m, \\ \mathbf{u} = 0, & \text{on } \partial\Omega \times [0,T], \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases} \tag{2.25}$$

with $\partial\Omega$ as in (2.1), $a_{ij}^{\alpha\beta}$ verifying (1.6) and

$$|\partial^\gamma a_{ij}^{\alpha\beta}(x)| \le |\gamma|!\delta^{-|\gamma|-1}, \quad \text{when } x \in \overline{\Omega}, \quad \gamma \in \mathbb{N}^n, \tag{2.26}$$

for some $0 < \delta \le 1$ is now obvious: the symmetry, coerciveness and compactness of the operator $L^2(\Omega)^m \longrightarrow W_0^{1,2}(\Omega)^m$, mapping $\mathbf{f} = (f^1, \dots, f^m)$ into the unique solution $\mathbf{u} = (u^1, \dots, u^m)$ to

$$\begin{cases} \partial_i(a_{ij}^{\alpha\beta} \partial_j u^\beta) - \Lambda u^\alpha = f^\alpha, & \text{in } \Omega,\ \alpha = 1, \dots, m, \\ \mathbf{u} = 0, & \text{in } \partial\Omega \end{cases}$$

where $\varLambda > 0$ is sufficiently large [11], Proposition 2.1, gives the existence of a complete system $\{\mathbf{e}_k\}$ in $L^2(\varOmega)^m$, $\mathbf{e}_k = (e_k^1, \dots, e_k^m)$, of eigenfunctions verifying

$$\begin{cases} \partial_i(a_{ij}^{\alpha\beta}\partial_j e_k^\beta) + \omega_k^2 e_k^\alpha = 0, & \text{in } \varOmega,\ \alpha = 1, \dots, m, \\ \mathbf{e}_k = 0, & \text{in } \partial\varOmega \end{cases}$$

with eigenvalues $0 \le \omega_1 \le \dots \omega_k \le \dots$ and $\lim_{k\to+\infty}\omega_k = +\infty$. By separation of variables, the Green's matrix for the system (2.25) over $\varOmega \times \mathbb{R}$ is the $m \times m$ matrix

$$\boldsymbol{\varGamma}(x, y, t - s) = \sum_{k=1}^{+\infty} \mathrm{e}^{-\omega_k^2(t-s)} \mathbf{e}_k(x) \otimes \mathbf{e}_k(y). \tag{2.27}$$

Moreover, the interior and boundary regularity for the elliptic system $\partial_y^2 + \partial_i(a_{ij}^{\alpha\beta}\partial_j\ )$ in $\varOmega \times \mathbb{R}$, shows that (2.5) holds for $\mathbf{u}$ as in (2.3) but with $\mathbf{e}_k$ replacing e_k ([11, 22], Chap. II). These and [16] suffice to find a control function $\mathbf{f}$ for the system (2.25) verifying (1.2). Furthermore, if you wish to get bounds on the regularity of the control function $\mathbf{f}$, [16] shows that suffices to know that

$$\|e_k\|_{H^s(\varOmega)} \le N_s(1 + \omega_k)^s, \quad \text{when } k, s \ge 0$$

and

$$\#\{k \ge 1 : 0 \le \omega_k \le \mu\} \le N\mu^n, \quad \text{when } \mu \ge 1, \tag{2.28}$$

for some N which does not depend on $\mu \ge 1$. The first holds from elliptic regularity and (2.26), while (2.28) follows from the Gaussian estimates verified by $\boldsymbol{\varGamma}$; *i.e.*, there are N and κ [8], Corollary 4.14, such that

$$|\boldsymbol{\varGamma}(x, y, t)| \le N(1 \wedge t)^{-\frac{n}{2}} \mathrm{e}^{\varLambda t - \kappa|x-y|^2/t}, \quad \text{for } x, y \in \mathbb{R}^n \text{ and } t > 0. \tag{2.29}$$

To verify the last claim, observe that (2.27), (2.29) and the orthonormality of the eigenfunctions $\mathbf{e}_k$ give

$$\int_\varOmega \int_\varOmega |\boldsymbol{\varGamma}(x, y, t)|^2 \,\mathrm{d}x\mathrm{d}y = \sum_{k\ge1} \mathrm{e}^{-2\omega_k^2 t} \le N\mathrm{e}^{2\varLambda t}|\varOmega|t^{-\frac{n}{2}}, \tag{2.30}$$

and it suffices to choose $t = 1/\mu^2$ in (2.30) to get (2.28). In particular, the methods in [16] also generate a control function $\mathbf{f}$ in $C_0^\infty((0,T), C^\infty(\overline{\varOmega}))$.

The null-controllability of the system

$$\begin{cases} \partial_t u + (-1)^m \triangle^m u = f(x,t)\chi\omega, & \text{in } \varOmega \times (0,T], \\ u = \nabla u = \dots = \nabla^{m-1} u = 0, & \text{in } \partial\varOmega \times (0,T), \\ u(0) = u_0, & \text{in } \varOmega, \end{cases} \tag{2.31}$$

$m \ge 2$, is better managed with the approach in [17]: if $\{e_j\}$ and $0 \le \omega_1^{2m} \le \dots \le \omega_k^{2m} \le \dots$ are the eigenvectors and eigenvalues for $\triangle^m$ in $W_0^{m,2}(\varOmega)$,

$$\begin{cases} (-1)^m \triangle^m e_j - \omega_j^{2m} e_j = 0, & \text{in } \varOmega, \\ e_j = \nabla e_j = \dots = \nabla^{m-1} e_j = 0, & \text{in } \partial\varOmega, \end{cases}$$

define

$$u(x, y) = \sum_{w_j^m \le \mu} a_j X_j(y) e_j(x), \quad \text{with} \quad X_j(y) = \begin{cases} \mathrm{e}^{\omega_j y}, & \text{for } m \text{ odd}, \\ \mathrm{e}^{\omega_j \mathrm{e}^{\frac{\pi i}{2m}} y}, & \text{for } m \text{ even}. \end{cases}$$

It verifies

$$\begin{cases} \partial_y^{2m} u + \triangle^m u = 0, & \text{in } \Omega \times \mathbb{R}, \\ u = \nabla u = \cdots = \nabla^{m-1} u = 0, & \text{in } \partial\Omega \times \mathbb{R}. \end{cases}$$

Again, u verifies (2.5) [22] and from Theorem 1.3 applied to $u(\,.\,,0)$ as in Remark 2.1,

$$\sum_{\omega_j^m \le \mu} a_j^2 \le \mathrm{e}^{N\mu^{\frac{1}{m}}} \int_\omega \Big| \sum_{\omega_j^m \le \mu} a_j e_j \Big|^2 \mathrm{d}x, \quad \text{with } N = N(|\omega|, \Omega, r, \delta, m).$$

From [17], the last inequality suffices to find a control f in $L^2(\omega \times (0,T))$ for (2.31).

Remark 2.3. The proof of Theorem 1.2 for the scalar case is based on the bound (2.23). In the literature, it is is obtained from (2.16) and the interpolation inequality below which has been proved with Carleman inequalities: there are N and θ such that the inequality

$$\|u\|_{L^2(\Omega\times[0,1])} \le N \left\| \frac{\partial u}{\partial \nu} \right\|^\theta_{L^2(\partial\Omega\times[-1,2])} \|u\|^{1-\theta}_{L^2(\Omega\times[-3,3])},$$

holds when u verifies (2.4). However, the authors are not aware wether the interpolation inequality

$$\|\mathbf{u}\|_{L^2(\Omega\times[0,1])} \le N \left\| \frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}} \right\|^\theta_{L^2(\partial\Omega\times[-1,2])} \|\mathbf{u}\|^{1-\theta}_{L^2(\Omega\times[-3,3])},$$

with

$$\left(\frac{\partial \mathbf{u}}{\partial \boldsymbol{\nu}}\right)^\alpha = a_{ij}^{\alpha\beta} \partial_j u^\beta \nu_i, \quad \alpha = 1, \ldots, m,$$

holds for solutions $\mathbf{u}$ to the corresponding analog elliptic system and lateral boundary conditions:

$$\begin{cases} \partial_i (a_{ij}^{\alpha\beta} \partial_j u^\beta) + \partial_y^2 u^\alpha = 0, & \text{in } \Omega \times \mathbb{R},\ \alpha = 1, \ldots, m, \\ \mathbf{u} = 0, & \text{in } \partial\Omega \times \mathbb{R}. \end{cases}$$

The later explains why we can not extend the boundary null-controllability results in Theorem 1.2 to general parabolic systems with analytic coefficients and also analytic lateral boundaries. In spite of that, the inverse mapping theorem shows that there is $\rho > 0$ such that the mapping

$$\partial\Omega \times (0,\rho) \longrightarrow U_\rho, \quad (Q,t) \rightsquigarrow Q + t\nu(Q),$$

is an analytic diffeomorphism onto $U_\rho = \{x \in \mathbb{R}^n \setminus \overline{\Omega} : 0 < d(x, \partial\Omega) < \rho\}$ and the null-controllability for parabolic systems with analytic coefficients over $\overline{\Omega} \cup U_\rho$ and with controls acting over $\omega = U_{\frac{3\rho}{4}} \backslash U_{\frac{\rho}{2}}$ has been stablished in Remark 2.2. From these, standard arguments show that *at least* the parabolic system

$$\begin{cases} \partial_i (a_{ij}^{\alpha\beta} \partial_j e_k^\beta) - \partial_t u^\alpha = 0, & \text{in } \Omega \times (0,T),\ \alpha = 1, \ldots, m, \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \times [0,T], \\ \mathbf{u}(0) = \mathbf{u}_0, & \text{in } \Omega, \end{cases}$$

can be null-controlled with controls $\mathbf{g}$ acting over the *full lateral boundary* of Ω.

Remark 2.4. To prove Theorem 1.1 when $\partial\Omega$, $\mathbf{A}$ and V are globally smooth and $\mathbf{A}$ and V are only real analytic in a neighborhood of ω, one may assume that

$$\omega \subset B_{\frac{R}{4}},\ |\omega|/|B_R| \geq 1/2, \qquad B_{4R} \subset \Omega \tag{2.32}$$

$$|\partial^\alpha \mathbf{A}(x)| + |\partial^\alpha V(x)| \leq |\alpha|!\delta^{-|\alpha|-1}, \qquad \text{when} \quad x \in B_{4R},\ \alpha \in \mathbb{N}^n, \tag{2.33}$$

for some fixed $0 < R, \delta \leq 1$ and the goal is to show that (2.2) holds. The Carleman and interpolation inequalities stablished in [16], Section 3(1). for solutions to elliptic operators with smooth coefficients and for $\partial\Omega$ smooth show that there is $N = N(\mathbf{A}, V, \Omega, R)$ such that the inequalities

$$\left(\sum_{\omega_j \leq \mu} a_j^2 + b_j^2\right)^{\frac{1}{2}} \leq \mathrm{e}^{N\mu} \|u\|_{L^\infty(B_R(0,\frac{1}{2}))}, \tag{2.34}$$

hold when $\mu \geq \omega_1$, for all sequences $a_1, a_2, \ldots,\ b_1, b_2, \ldots$ and for u as in (2.3). Because u verifies

$$\Delta u + \partial_y^2 u = 0, \qquad \text{in } B_{4R}\left(0, \frac{1}{2}\right)$$

and (2.33) holds, u is a local solution to an elliptic equation with local real analytic coefficients and there are $N = N(\delta)$ and $\rho = \rho(\delta)$, $0 < \rho \leq 1$, such that

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(B_{2R}(0,\frac{1}{2}))} \leq \frac{N(|\alpha| + \beta)!}{(R\rho)^{|\alpha|+\beta}} \left(\fint_{B_{4R}(0,\frac{1}{2})} |u|^2 \,\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}, \tag{2.35}$$

when $\alpha \in \mathbb{N}^n$ and $\beta \geq 1$. The later follows from the corresponding result for $R = 1$ and rescaling ([21], Chap. 5, [13], Chap. 3). The orthonormality of $\{e_j\}$ in Ω and (2.35) show that

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(B_{2R}(0,\frac{1}{2}))} \leq M(|\alpha| + \beta)!(R\rho)^{-|\alpha|-\beta}, \qquad \text{for } \alpha \in \mathbb{N}^n, \quad \beta \geq 0, \tag{2.36}$$

with M as in (2.8). Finally, Theorem 1.3 in $B_{2R}(0, \frac{1}{2})$ with

$$E = \omega \times \left(\frac{2-R}{4}, \frac{2+R}{4}\right) \subset B_{\frac{R}{2}}\left(0, \frac{1}{2}\right),$$

(2.36) and (2.32) show that

$$\|u\|_{L^\infty(B_R(0,\frac{1}{2}))} \leq N\|u\|^\theta_{L^2(\omega\times[\frac{1}{4},\frac{3}{4}])} M^{1-\theta}, \tag{2.37}$$

and (2.2) follows from (2.34), (2.37) and (2.8).

To prove Theorem 1.2 with $\partial\Omega$, $\mathbf{A}$ and V real analytic near γ and smooth elsewhere, one may assume that 0 is in $\partial\Omega$ and that there are $0 < R, \delta \leq 1$ and $\varphi : B'_R \subset \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that

$$\gamma \subset B_{\frac{R}{4}} \cap \partial\Omega,\ |\gamma|/|B_R \cap \partial\Omega| \geq \frac{1}{2}, \tag{2.38}$$

$$\begin{aligned}
&\varphi(0') = 0,\ |\partial^\alpha \varphi(x')| \leq |\alpha|!\delta^{-|\alpha|-1}, \qquad \text{when } x' \in B'_{4R}, \quad \alpha \in \mathbb{N}^{n-1},\\
&|\partial^\alpha \mathbf{A}(x)| + |\partial^\alpha V(x)| \leq |\alpha|!\delta^{-|\alpha|-1}, \qquad \text{when } x \in B_{4R} \cap \overline{\Omega}, \quad \alpha \in \mathbb{N}^n,\\
&B_{4R} \cap \Omega = B_{4R} \cap \{(x', x_n) : x' \in B'_{4R},\ x_n > \varphi(x')\},\\
&B_{4R} \cap \partial\Omega = B_{4R} \cap \{(x', x_n) : x' \in B'_{4R},\ x_n = \varphi(x')\}
\end{aligned} \tag{2.39}$$

and the goal is to show that (2.22) holds. The Carleman and interpolation inequalities stablished in [16], Section 3(2) for solutions to elliptic operators with smooth coefficients and with $\partial\Omega$ smooth show that there is $N = N(\mathbf{A}, V, \Omega, R)$ such that the inequalities

$$\left(\sum_{\omega_j \le \mu} a_j^2 + b_j^2\right)^{\frac{1}{2}} \le \mathrm{e}^{N\mu} \left\| \frac{\partial u}{\partial \nu} \right\|_{L^\infty(B_R(0,\frac{1}{2})\cap\partial\Omega\times\mathbb{R})}, \tag{2.40}$$

hold when $\mu \ge \omega_1$, for all sequences $a_1, a_2, \dots, b_1, b_2, \dots$ and for u as in (2.3). Because u verifies

$$\begin{cases} \Delta u + \partial_y^2 u = 0, & \text{in } B_{4R}\left(0, \frac{1}{2}\right) \cap \Omega \times \mathbb{R}, \\ u = 0, & \text{in } B_{4R}\left(0, \frac{1}{2}\right) \cap \partial\Omega \times \mathbb{R}, \end{cases}$$

and (2.39) holds there are $N = N(\delta)$ and $\rho = \rho(\delta)$, $0 < \rho \le 1$, such that

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(B_{2R}(0,\frac{1}{2})\cap\Omega\times\mathbb{R})} \le \frac{N(|\alpha| + \beta)!}{(R\rho)^{|\alpha|+\beta}} \left(\fint_{B_{4R}(0,\frac{1}{2})\cap\Omega\times\mathbb{R}} |u|^2 \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{2}}, \tag{2.41}$$

when $\alpha \in \mathbb{N}^n$ and $\beta \ge 1$. The later follows from the corresponding result for $R = 1$ and rescaling ([21], Chap. 5, [13], Chap. 3). The orthonormality of $\{e_j\}$ in Ω and (2.41) show that

$$\|\partial_x^\alpha \partial_y^\beta u\|_{L^\infty(B_{2R}(0,\frac{1}{2})\cap\Omega\times\mathbb{R})} \le M(|\alpha| + \beta)!(R\rho)^{-|\alpha|-\beta}, \quad \text{for } \alpha \in \mathbb{N}^n, \quad \beta \ge 0, \tag{2.42}$$

with M as in (2.8). Finally, the obvious extension of Theorem 1.3 for real analytic functions over $B_{2R}(0, \frac{1}{2}) \cap \partial\Omega \times \mathbb{R}$ [14], pages 67–69, with

$$E = \gamma \times \left(\frac{2-R}{4}, \frac{2+R}{4}\right) \subset B_{\frac{R}{2}}\left(0, \frac{1}{2}\right) \cap \partial\Omega \times \mathbb{R},$$

(2.42) and (2.38) show that

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^\infty(B_R(0,\frac{1}{2})\cap\partial\Omega\times\mathbb{R})} \le N \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\gamma\times[\frac{1}{4},\frac{3}{4}])}^\theta M^{1-\theta}, \tag{2.43}$$

and (2.22) follows from (2.40), (2.43) and (2.8).

3. Proof of Theorem 1.3

First we recall Hadamard's three-circle theorem [19] and prove two Lemmas before the proof of Theorem 1.3.

Theorem 3.1. *Let F be a holomorphic function of a complex variable in the ball B_{r_2}. Then, the following is valid for $0 < r_1 \le r \le r_2$,*

$$\|F\|_{L^\infty(B_r)} \le \|F\|_{L^\infty(B_{r_1})}^\theta \|F\|_{L^\infty(B_{r_2})}^{1-\theta}, \qquad \theta = \frac{\log \frac{r_2}{r}}{\log \frac{r_2}{r_1}}.$$

Lemma 3.2. *Let f be holomorphic in B_1, $|f(z)| \le 1$ in B_1 and E be a measurable set in $\left[-\frac{1}{5}, \frac{1}{5}\right]$. Then, there are $N = N(|E|)$ and $\gamma = \gamma(|E|)$ such that*

$$\|f\|_{L^\infty(B_{\frac{1}{2}})} \le N \|f\|_{L^\infty(E)}^\gamma.$$

Proof. For $n \geq 1$, there are $n+1$ points with $-\frac{1}{5} \leq x_0 < x_1 < \cdots < x_n \leq \frac{1}{5}$, with $x_i \in \overline{E}$, $i = 0, \dots, n$ and $x_i - x_{i-1} \geq \frac{|E|}{n+1}$, $i = 1, \dots, n$. For example, $x_0 = \inf E$, $x_i = \inf \left(E \cap [x_{i-1} + \frac{|E|}{n+1}, \frac{1}{5}] \right)$. Let

$$P_n(z) = \sum_{i=0}^{n} f(x_i) \, \frac{\prod_{j \neq i}(z - x_j)}{\prod_{j \neq i}(x_i - x_j)}.$$

Then,

$$|P_n(z)| \leq \|f\|_{L^\infty(E)} |E|^{-n} \sum_{i=0}^{n} \frac{(n+1)^n}{i!(n-i)!} \leq \|f\|_{L^\infty(E)} \left(\frac{3}{|E|} \right)^n, \quad \text{for } |z| \leq \frac{1}{2}.$$

By Cauchy's formula,

$$|f(z) - P_n(z)| = \left| \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(\xi)(z-x_0)\dots(z-x_n)}{(\xi - z)(\xi - x_0)\dots(\xi - x_n)} \, \mathrm{d}\xi \right| \leq 2 \left(\frac{7}{8} \right)^n, \quad \text{for } |z| \leq \frac{1}{2}.$$

The last two inequalities give

$$\|f\|_{L^\infty(B_{\frac{1}{2}})} \leq \|f\|_{L^\infty(E)} \left(\frac{3}{|E|} \right)^n + 2 \left(\frac{7}{8} \right)^n, \quad \text{for all } n \geq 1, \tag{3.1}$$

and the minimization in the n-variable of the right hand side of (3.1) implies Lemma 3.2. □

Lemma 3.3. *Let f be analytic in $[0,1]$, E be a measurable set in $[0,1]$ and assume there are positive constants M and ρ such that*

$$|f^{(k)}(x)| \leq M k! (2\rho)^{-k}, \quad \text{for } k \geq 0, \quad x \in [0,1]. \tag{3.2}$$

Then, there are $N = N(\rho, |E|)$ and $\gamma = \gamma(\rho, |E|)$ such that

$$\|f\|_{L^\infty([0,1])} \leq N \|f\|^{\gamma}_{L^\infty(E)} M^{1-\gamma}.$$

Proof. (3.2) implies that f has a holomorphic extension to $D_\rho = \cup_{0 \leq x \leq 1} B(x, \rho)$, with $|f| \leq 2M$ in D_ρ. Write $[0,1]$ as a disjoint union of $\frac{5}{2\rho}$ non-overlapping closed intervals of length $\frac{2\rho}{5}$. Among them there is at least one, $I = [x_0 - \frac{\delta}{5}, x_0 + \frac{\delta}{5}]$, such that $|E \cap I| \geq \frac{2\delta |E|}{5}$. Then, $g(z) = f(x_0 + \delta z)/2M$ is holomorphic in B_1, $E_{x_0,\rho} = \rho^{-1}(E \cap I - x_0)$ is measurable in $[-\frac{1}{5}, \frac{1}{5}]$ with measure bounded from below by $\frac{2|E|}{5}$, $\|g\|_{L^\infty(E_{x_0,\rho})} \leq \|f\|_{L^\infty(E)}$ and applying Lemma 3.2 to g, we have

$$\|f\|_{L^\infty(B_{\frac{\rho}{2}}(x_0))} \leq N \|f\|^{\gamma}_{L^\infty(E)} M^{1-\gamma}, \tag{3.3}$$

for some $0 \leq x_0 \leq 1$. By making successive iterations of Hadamard's three-circle theorem (a finite number which depends only ρ) with a suitable chain of three-circles contained in D_ρ of radius comparable to ρ and with centers at points in $[0,1]$, while recalling that on the largest ball $|f| \leq 2M$, we get that

$$\|f\|_{L^\infty(0,1)} \leq N \|f\|^{\theta}_{L^\infty(B_{\frac{\rho}{2}}(x_0))} M^{1-\theta}, \quad \theta = \theta(\rho). \tag{3.4}$$

Finally, Lemma 3.3 follows from (3.4) and (3.3). □

Proof of Theorem 1.3. We may assume $R = 1$. Let $x \in B_{\frac{1}{2}}$. Using spherical coordinates centered at x,

$$|E| \leq \int_{S^{n-1}} |\{t \in [0,1] : x + tz \in E\}| \, \mathrm{d}z,$$

and there is at least one $z \in S^{n-1}$ with $|\{t \in [0,1] : x + tz \in E\}| \geq |E|/(2\omega_n)$, with ω_n the surface measure on S^{n-1}. Set $\varphi(t) = f(x+tz)$. From (1.5), φ satisfies (3.2), $\|\varphi\|_{L^\infty(E_z)} \leq \|f\|_{L^\infty(E)}$ and Lemma 3.3 gives

$$\|f\|_{L^\infty(B_{\frac{1}{2}})} \leq N \|f\|_{L^\infty(E)}^{\gamma} M^{1-\gamma}. \tag{3.5}$$

Finally, setting

$$\widetilde{E} = \left\{ x \in E : |f(x)|/2 \leq \fint_E |f| \,\mathrm{d}x \right\}.$$

Chebyshev's inequality shows that

$$|\widetilde{E}| \geq |E|/2 \ , \qquad \|f\|_{L^\infty(\widetilde{E})} \leq 2 \fint_E |f| \,\mathrm{d}x,$$

and Theorem 1.3 follows after replacing E by $\widetilde{E}$ in (3.5). □

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