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THE STRUCTURE OF REACHABLE SETS FOR AFFINE CONTROL SYSTEMS INDUCED BY GENERALIZED MARTINET SUB-LORENTZIAN METRICS

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Abstract. In this paper we investigate analytic affine control systems $\dot{q} = X + uY$, $u \in [a, b]$, where X, Y is an orthonormal frame for a generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type. We construct normal forms for such systems and, among other things, we study the connection between the presence of the singular trajectory starting at q_0 on the boundary of the reachable set from q_0 with the minimal number of analytic functions needed for describing the reachable set from q_0.

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1. Introduction

1.1. Preliminaries

In this paper we continue the study of non-contact sub-Lorentzian structures on $\mathbb{R}^3$ initiated in [9]. Additionally, we will apply the sub-Lorentzian geometry methods to the study of certain control affine systems, more precisely to their reachable sets. The main purpose of the present work is to establish the connection between the structure of the reachable set from q_0 and the geometric optimality of the singular trajectory starting at q_0 (a trajectory of a control system starting from a point q_0 is called geometrically optimal if it is entirely contained in the boundary of the reachable set from q_0 – *cf.* [1]). We also aim at providing an 'algorithm' for computing reachable sets in cases under consideration.

To start with, let us recall some basic facts and notions from the sub-Lorentzian geometry and affine control systems that will allow us to state main results of the paper. Let H be a smooth distribution of constant rank on a smooth manifold M. For a point $q \in M$ and a positive integer k let us denote by H_q^k the span of all vectors of the form

$$[X_1, [X_2, \ldots, [X_{i-1}, X_i] \ldots]](q),$$

where $X_1, \ldots, X_i$ are smooth local sections of H defined near q, $i \leq k$. We say that the distribution H is *bracket generating* if for every $q \in M$ there exists a positive integer $i = i(q)$ such that $H_q^i = T_qM$. By *a sub-Lorentzian structure* on a manifold M we mean a couple (H, g) where H is a smooth bracket generating distribution on M

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and g is a smooth Lorentzian metric on H (not every distribution admits Lorentzian metrics – see [7]). A triple (M, H, g), where M is a manifold and (H, g) is a sub-Lorentzian structure on M, is called a *sub-Lorentzian manifold*.

Fix a sub-Lorentzian manifold (M, H, g). A vector $v \in H_q$ is called *timelike* if $g(v, v) < 0$, is called *nonspacelike* if $g(v, v) \leq 0$, and is called *null* if $g(v, v) = 0$ and $v \neq 0$. Using the terminology from the Lorentzian geometry, a continuous timelike vector field on M is called a *time orientation* of (H, g). If such a field is given then we say that (M, H, g) is *time-oriented*. We will assume our (M, H, g) to be time-oriented by a vector field X. A nonspacelike $v \in H_q$ is said to be *future directed* if $g(v, X(q)) < 0$. To describe the geometry of (M, H, g) we will consider the so-called *horizontal* curves: a curve $\gamma : [\alpha, \beta] \longrightarrow M$ is called horizontal if it is absolutely continuous and $\dot{\gamma}(t) \in H_{\gamma(t)}$ a.e. on $[\alpha, \beta]$ (for technical reasons we also usually assume $\dot{\gamma}$ to be square integrable with respect to some Riemannian metric on M). A horizontal curve $\gamma : [\alpha, \beta] \longrightarrow M$ is *timelike* (resp. *timelike future directed, nonspacelike, nonspacelike future directed, null, null future directed*) if so its tangent $\dot{\gamma}(t)$ a.e. on $[\alpha, \beta]$.

From now on, unless otherwise stated, all vectors, vector fields and curves are supposed to be horizontal, *i.e.* tangent (resp. a.e. tangent) to H. We will also use the following abbreviations: *t.* for 'timelike', *nspc.* for 'nonspacelike', and *f.d.* for 'future directed'. Thus *e.g.* a t.f.d. curve is a horizontal curve which is timelike future directed.

Fix a point $q_0 \in M$ and a neighbourhood $U \subset M$ of q_0. By $J^+(q_0, U)$ (resp. $I^+(q_0, U)$, $N^+(q_0, U)$) we denote the *(future) nonspacelike* (resp. *timelike*, *null*) *reachable set (in U) from* q_0 which is defined to be the set of all points $q \in U$ that can be reached from q_0 by a nspc.f.d. (resp. t.f.d., null f.d.) curve contained in U. If U is a normal neighbourhood of q_0 (see the definition below) then it can be proved that $J^+(q_0, U)$ is closed with respect to U. Moreover, the three reachable sets have identical closures and interiors in U. In particular $\tilde{\partial} J^+(q_0, U) = \tilde{\partial} I^+(q_0, U) = \tilde{\partial} N^+(q_0, U)$; here and below, in cases where we have a fixed open set U, we will use $\tilde{\partial}$ to denote the boundary with respect to U.

In this paper we will also cope with affine control systems in $\mathbb{R}^3$ with a scalar input, *i.e.* with control systems of the form

$$\dot{q} = X + uY,\ u \in [a, b], \tag{1.1}$$

where X, Y are linearly independent vector fields defined on a neighbourhood U of a given point q_0, and $a, b \in \mathbb{R}$, $a < b$. For such systems we also consider three types of reachable sets. More precisely, we define $\mathcal{A}_{[a,b]}(q_0, U)$ (resp. $\mathcal{A}_{(a,b)}(q_0, U)$, $\mathcal{A}_{\{a,b\}}(q_0, U)$) to be the set of endpoints of all trajectories of (1.1) that start from q_0, are generated by measurable controls $u : [0, T] \longrightarrow [a, b]$ (resp. $u : [0, T] \longrightarrow (a, b)$, $u : [0, T] \longrightarrow \{a, b\}$), and are contained in U; here the final time T is not fixed and depends on a control u.

It was mentioned *e.g.* in [8, 9] that studying reachable sets for the affine control system $\dot{q} = X + uY$, $-1 \leq u \leq 1$, is equivalent to studying reachable sets for the suitable time-oriented sub-Lorentzian structure, namely the one determined by an orthonormal frame X, Y with a time orientation X. We recall this reasoning since it is crucial for further considerations.

Lemma 1.1. *Let (H, g) be a time-oriented sub-Lorentzian structure defined on an open set U. Suppose moreover that X, Y is an orthonormal frame for (H, g) with a time orientation X. Then nspc.f.d. curves contained in U are, up to a change of parameter, trajectories of the affine control system $\dot{q} = X + uY$, $|u| \leq 1$.*

Proof. Indeed, all trajectories of $\dot{q} = X + uY$, $|u| \leq 1$, are evidently nspc.f.d. Let, on the other hand, $\gamma : [0, T] \longrightarrow U$ be nspc.f.d. Then $\dot{\gamma}(t) = v_0(t)X + v_1(t)Y$, where $-v_0^2(t) + v_1^2(t) \leq 0$ and $v_0(t) > 0$ a.e. on $[0, T]$. Let $\beta(t) = \int_0^t v_0(\tau)\mathrm{d}\tau$. Clearly, $\beta : [0, T] \longrightarrow [0, T_1]$, $T_1 = \int_0^T v_0(\tau)\mathrm{d}\tau$, is an increasing function. Denote by $\alpha : [0, T_1] \longrightarrow [0, T]$ the inverse function, and let $\gamma_1 : [0, T_1] \longrightarrow U$ be defined by $\gamma_1(t) = \gamma(\alpha(t))$. Now $\dot{\gamma}_1(t) = X + \frac{v_1(\alpha(t))}{v_0(\alpha(t))}Y$, where $\left|\frac{v_1(\alpha(t))}{v_0(\alpha(t))}\right| \leq 1$ as required. □

Now we will extend this observation to general systems as in (1.1). More precisely, as we shall see in Lemma 1.2 below, the study of reachable sets for (1.1) is equivalent to the study of reachable sets for the sub-Lorentzian structure $(H, g^{a,b})$ defined on U by declaring the fields

$$\begin{aligned} Z^{a,b} &= X + \frac{1}{2}(b+a)Y \\ W^{a,b} &= \frac{1}{2}(b-a)Y \end{aligned} \tag{1.2}$$

to be an orthonormal basis for $(H, g^{a,b})$ with a time orientation $Z^{a,b}$. We will refer to the structure $(H, g^{a,b})$ as to *the sub-Lorentzian structure induced by the system* (1.1).

Lemma 1.2. *The reachable set $\mathcal{A}_{[a,b]}(q_0, U)$ (resp. $\mathcal{A}_{(a,b)}(q_0, U)$, $\mathcal{A}_{\{a,b\}}(q_0, U)$) coincides with the future non-spacelike (resp. timelike, null) reachable set from q_0 for the sub-Lorentzian structure $(H, g^{a,b})$ defined above.*

Proof. Suppose that $\gamma : [0, T] \longrightarrow U$ is nspc.f.d. with respect to $(H, g^{a,b})$. According to Lemma 1.1, after a change of parameterization $\dot{\gamma} = Z^{a,b} + u(t)W^{a,b} = X + \frac{1}{2}(b + a + (b-a)u(t))Y$, where $t \in [0, T_1]$ for suitable T_1 and $-1 \leq u(t) \leq 1$ a.e. It is seen that $a \leq \frac{1}{2}(b + a + (b-a)u(t)) \leq b$ a.e. on $[0, T_1]$, therefore γ is a trajectory of (1.1). Now suppose that $\gamma : [0, T] \longrightarrow U$ is a trajectory of (1.1), that is to say $\dot{\gamma} = X + u(t)Y$, $a \leq u(t) \leq b$ a.e. To show that γ is nspc.f.d. with respect to $(H, g^{a,b})$, it is enough to find $\tilde{u}(t)$, $-1 \leq \tilde{u}(t) \leq 1$ for $t \in [0, T]$, such that $X + u(t)Y = Z^{a,b} + \tilde{u}(t)W^{a,b}$. Such a $\tilde{u}(t)$ is given by $\tilde{u}(t) = \frac{2u(t)-a-b}{b-a}$. □

To simplify the computation of $\mathcal{A}_{[a,b]}(q_0, U)$ (resp. $\mathcal{A}_{(a,b)}(q_0, U)$, $\mathcal{A}_{\{a,b\}}(q_0, U)$), we will need a special type of neighbourhoods of a point q_0 which is a generalization of a concept of normal neighbourhoods in the sub-Lorentzian geometry (*cf.* [9]). So suppose that the system (1.1) is given on a bounded neighbourhood G of a point q_0. Let $f : G \longrightarrow \mathbb{R}$ be such a smooth function that $f(q_0) = 0$, $X(f)(q_0) = 1$, $Y(f)(q_0) = 0$. Setting $c = \max\{|a|, |b|\}$ we can suppose that, after shrinking G

$$\inf\{X(f)(q) : \ q \in G\} > c \sup\{|Y(f)(q))| : \ q \in G\}. \tag{1.3}$$

(1.3) of course implies that f increases along trajectories of (1.1). Indeed, if $\dot{\gamma} = X(\gamma(t)) + u(t)Y(\gamma(t))$, then $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = X(f)(\gamma(t)) + u(t)Y(f)(\gamma(t)) \geq X(f)(\gamma(t)) - c\,|Y(f)(\gamma(t))| > 0$. Now fix a sufficiently small $\delta > 0$ and set $U = G \cap \{f < \delta\}$. U has the following property: if $\gamma : [0, T] \longrightarrow G$ is a trajectory of (1.1) starting from $\gamma(0) \in U$ and there exists a $t \in (0, T)$ such that $\gamma(t) \in \partial U$ then one can find an $\varepsilon > 0$ for which $\gamma((t, t+\varepsilon)) \subset G \backslash \overline{U}$. The set U just constructed will be called a *normal neighbourhood of* q_0 for the system (1.1).

Now, by a normal neighbourhood of q_0 with respect to (H, g) we mean a normal neighbourhood of q_0 for a corresponding affine control system inducing (H, g) (note that such neighbourhoods can be used to prove theorems on reachable sets instead of normal neighbourhoods in the sub-Lorentzian sense, as defined in the previous papers by the author – *cf.* appendix in [9]).

One of the consequences of Lemma 1.2 is that the set $\mathcal{A}_{[a,b]}(q_0, U)$ is closed with respect to U, provided that U is a normal neighbourhood of q_0 (see [7]; note that this fact does not follow from standard theorems on closures of reachable sets that can be found *e.g.* in [5]). We also have $\mathcal{A}_{[a,b]}(q_0, U) = cl_U(int\mathcal{A}_{(a,b)}(q_0, U)) = cl_U(int\mathcal{A}_{\{a,b\}}(q_0, U))$, where cl_U stands for the closure with respect to U, and $int\mathcal{A}_{[a,b]}(q_0, U)) = int\mathcal{A}_{(a,b)}(q_0, U)) = int\mathcal{A}_{\{a,b\}}(q_0, U)$.

1.2. Statement of the results

In the paper [9] Martinet sub-Lorentzian structures of Hamiltonian type on $\mathbb{R}^3$ were studied. Let us recall that a rank 2 distribution H defined on an open set $U \subset \mathbb{R}^3$ is called a Martinet distribution if there exists a smooth hypersurface S in U (the so-called *Martinet surface for* H) with the following properties: (i) H is a contact structure on $U \backslash S$, (ii) $\dim(T_qS \cap H_q) = 1$ for every $q \in S$, (iii) $H_q^2 \subset H_q$ and $H_q^3 = T_q\mathbb{R}^3$ whenever $q \in S$. Let us also recall that the surface S is foliated by horizontal curves – trajectories of the nonsingular line field on S: $S \ni q \longrightarrow T_qS \cap H_q$, and that these curves are abnormal (see [12] for definition). Note that Martinet distributions are the simplest ones among all bracket generating non-contact distributions on $\mathbb{R}^3$; it can also be proved [14] that germs of Martinet distributions are stable. A sub-Lorentzian structure (H, g) is called *a Martinet sub-Lorentzian structure* if the mentioned line field on S is timelike.

In this moment, in order to be able to proceed further, we must state a definition of so-called Hamiltonian geodesics. To this end, let (M, H, g) be a sub-Lorentzian manifold, and let $X_0, \ldots, X_k$ be an orthonormal frame for (H, g) defined on an open set U, where X_0 is a time orientation. Let $\mathcal{H} : T^*U \longrightarrow \mathbb{R}$ be given as $\mathcal{H}(q,p) = -\frac{1}{2}\left\langle p, X_0(q)\right\rangle^2 + \frac{1}{2}\left\langle p, X_1(q)\right\rangle^2 + \ldots + \frac{1}{2}\left\langle p, X_k(q)\right\rangle^2$; $\mathcal{H}$ is called *the geodesic Hamiltonian* (note that $\mathcal{H}$ admits also a global definition – *cf.* [7]). Denote by Φ_t the (local) flow of $\overrightarrow{\mathcal{H}}$, the Hamiltonian vector field corresponding to $\mathcal{H}$. A curve $\gamma : [a, b] \longrightarrow U$ is said to be *a Hamiltonian geodesic* if it is of the form $\gamma(t) = \pi \circ \Phi_t(q_0, p_0)$, where $\pi : T^*U \longrightarrow \mathbb{R}$ is the canonical projection. It can be proved that Hamiltonian geodesics preserve their causal character (*i.e.* if a Hamiltonian geodesic is t.f.d. (resp. null f.d.) at a moment $t = t_0$, then it is t.f.d. (resp. null f.d.) for every t) and are locally length maximizing. Null f.d. Hamiltonian geodesics are also locally geometrically optimal (*cf.* [7]).

Now, a Martinet sub-Lorentzian structure is called *of Hamiltonian type* if, in addition, the abnormal curves are, up to a change of parameter, t.f.d. Hamiltonian geodesics.

We will generalize the notion of Martinet sub-Lorentzian structures of Hamiltonian type in the following way. Let k be a positive integer and let H be a bracket generating distribution defined on an open set $U \subset \mathbb{R}^3$. We will say that H satisfies the condition (M_k) if there exists a smooth hypersurface S in U such that

$$
(M_k)\quad
\begin{array}{l}
\text{(i) } H \text{ defines a contact structure on } U \backslash S;\\
\text{(ii) } \dim\left(T_qS \cap H_q\right) = 1 \text{ for every } q \in S;\\
\text{(iii) } H_q^l \subset H_q,\ 1 \leq l \leq k, \text{ and } H_q^{k+1} = T_q\mathbb{R}^3 \text{ whenever } q \in S.
\end{array}
$$

It is clear that the condition (M_2) is simply a definition of Martinet distributions. It is also clear that a (germ of a) distribution satisfying (M_k), $k \geq 3$, is not stable. It is seen *e.g.* by considering the following family of distributions: $H_\varepsilon = \ker\left(\mathrm{d}z - \left(y^2 - \varepsilon\right)(y\mathrm{d}x - x\mathrm{d}y)\right)$ for real ε: H_0 obeys the condition (M_3), while H_ε does not for $\varepsilon \neq 0$. Although not stable, such structures seem to be interesting due to the results concerning the structure of reachable sets proved below. The surface S appearing in the condition (M_k) will be called *the Martinet surface for* H.

Now, by a *(generalized) Martinet sub-Lorentzian structure of order* k we mean a sub-Lorentzian structure (H, g) defined on an open set $U \subset \mathbb{R}^3$ such that H satisfies the condition (M_k), $k \geq 2$, and g is such that the trajectories of the line field $S \ni q \longrightarrow T_qS \cap H_q$, *i.e.* the abnormal curves foliating S, are timelike. If, in addition, the abnormal curves are, up to a change of parameterization, t.f.d. Hamiltonian geodesics, then we say that (H, g) is of *Hamiltonian type*. By an analytic structure we mean a structure where all the data appearing in its definition, including the Martinet surface, are analytic.

The aim of this paper is to study the structure of reachable sets for affine control systems of the form

$$\dot{q} = X + uY,\ u \in [a, b],$$

where $(Span\{X, Y\}, g)$ is an analytic Martinet sub-Lorentzian structure of order k of Hamiltonian type, g is such that X, Y is an orthonormal basis for g, and X is a time orientation such that $X(q) \in T_qS$ for every $q \in S$, S being the Martinet surface for $Span\{X, Y\}$. In order to be able to compute reachable sets for the mentioned systems we must transform the fields X, Y into a more convenient form. This is done in the following theorem.

Theorem 1.3. *Let (H, g) be an analytic time-oriented generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type, defined on a neighbourhood U of the origin in $\mathbb{R}^3$. Then, supposing that U is sufficiently small, there exist analytic coordinates x, y, z on U such that (H, g) admits an orthonormal frame in the following normal form*

$$
\begin{aligned}
X &= \frac{\partial}{\partial x} + y\varphi\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + \frac{1}{2}y^k(1+\psi)\frac{\partial}{\partial z}\\
Y &= \frac{\partial}{\partial y} - x\varphi\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - \frac{1}{2}xy^{k-1}(1+\psi)\frac{\partial}{\partial z},
\end{aligned}
\tag{1.4}
$$

where $S = \{y = 0\}$ is the Martinet surface for H, X is a time orientation such that $X_{|S}$ is tangent to S, and φ, ψ are analytic functions on U satisfying $\psi(0, 0, z) = 0$.

Note that Theorem 1.3 contains normal forms for contact sub-Lorentzian structures from [6] (the case $k = 1$) as well as normal forms for Martinet sub-Lorentzian structures of Hamiltonian type (this is for $k = 2$) from [9] (compare it with normal forms for sub-Riemannian structures – see [2]).

Using normal forms (1.4) we then investigate the structure of reachable sets for the system (1.1), depending on the parity of k, and the signs of a and b. More precisely, we have two theorems.

Theorem 1.4. *Consider the system (1.1) defined on a normal neighbourhood U of the origin, where X and Y are given by the formula (1.4). Fix two numbers $a, b \in \mathbb{R}$, $a < b$. If (i) k is odd and a, b are arbitrary or (ii) k is even and a, b are either both nonpositive or both nonnegative then, provided that U is sufficiently small, there exist analytic functions $\eta_1^{a,b}$, $\eta_2^{a,b}$ on U such that*

$$\mathcal{A}_{[a,b]}(0, U) = \mathcal{A}_{\{a,b\}}(0, U) = A_1 \cup A_2,$$
$$\mathcal{A}_{(a,b)}(0, U) = int\,(A_1 \cup A_2)\,,$$

where
$$A_1 = \left\{(x, y, z) \in U :\ \eta_1^{a,b}(x, y, z) \leq 0\right\} \cap \{x \geq 0,\ z \geq 0\},$$
$$A_2 = \left\{(x, y, z) \in U :\ \eta_2^{a,b}(x, y, z) \leq 0\right\} \cap \{x \geq 0,\ z \leq 0\}.$$
In particular the three reachable sets are semi-analytic.

Theorem 1.5. *Consider the system (1.1) defined on a normal neighbourhood U of the origin, where X and Y are given by the formula (1.4) and the number k appearing there is even. Fix two numbers $a, b \in \mathbb{R}$, $a < 0 < b$, and suppose that U is sufficiently small. Then there exist analytic functions $\eta_1^{a,b}, \ldots, \eta_4^{a,b}$ on U, and a 2-dimensional semi-analytic set Σ with the property that $U \cap \{x \geq 0\} \setminus \Sigma$ has two connected components which we will denote by Σ^+, Σ^-, such that*

$$\mathcal{A}_{[a,b]}(0, U) = A_1 \cup \ldots \cup A_4,$$
$$\mathcal{A}_{(a,b)}(0, U) = int\,(A_1 \cup \ldots \cup A_4) \cup A_5,$$
$$\mathcal{A}_{\{a,b\}}(0, U) = int\,(A_1 \cup \ldots \cup A_4) \cup \tilde{\partial}\,(A_1 \cup A_2) \setminus A_5$$

where
$$A_1 = \left\{(x, y, z) \in U :\ \eta_1^{a,b}(x, y, z) \leq 0\right\} \cap \Sigma^+ \cap \{z \geq 0\},$$
$$A_2 = \left\{(x, y, z) \in U :\ \eta_2^{a,b}(x, y, z) \leq 0\right\} \cap \Sigma^- \cap \{z \geq 0\},$$
$$A_3 = \left\{(x, y, z) \in U :\ \eta_3^{a,b}(x, y, z) \leq 0\right\} \cap \{x \geq 0,\ y \geq 0\} \cap \{z \leq 0\},$$
$$A_4 = \left\{(x, y, z) \in U :\ \eta_4^{a,b}(x, y, z) \leq 0\right\} \cap \{x \geq 0,\ y \leq 0\} \cap \{z \leq 0\},$$
$$A_5 = \{(x, 0, 0) \in U :\ x \geq 0\}.$$
In particular, the three reachable sets are semi-analytic.

Obviously, the set A_5 is the set of points of the abnormal curve (with respect to the distribution H) starting from 0.

The presented results explain the connection between the presence on $\tilde{\partial}\mathcal{A}_{[a,b]}(q_0, U)$ of the abnormal curve starting from q_0 and generated by a control $u(t) \in (a, b)$, with the minimal number of analytic functions necessary for describing reachable sets from q_0. The formulation of Theorems 1.4, 1.5 (but not proofs) would simplify if we introduced the notion of *singular trajectories* (see [4]) for the system (1.1). In our situation, *i.e.* X, Y are given by (1.4), the singular trajectory is just the abnormal curve, provided that $0 \in (a, b)$. If $0 \notin (a, b)$ then the system (1.1) has no singular trajectories. Using this notion we can conclude that

Corollary 1.6. *Whenever the singular trajectory starting at q_0 lies on the boundary $\tilde{\partial}\mathcal{A}_{[a,b]}(q_0,U)$ then the minimal number of analytic functions needed to describe $\mathcal{A}_{[a,b]}(q_0,U)$ is four. On the other hand, if the singular trajectory starting at q_0 lies in the interior of $\mathcal{A}_{[a,b]}(q_0,U)$ or there are no singular trajectories, then two analytic functions suffice to describe $\mathcal{A}_{[a,b]}(q_0,U)$.*

Corollary 1.6 can be also reformulated in a more geometric language as follows.

Corollary 1.7. *Whenever the singular trajectory starting at q_0 lies on the boundary $\tilde{\partial}\mathcal{A}_{[a,b]}(q_0,U)$ then the minimal number of 2-dimensional strata in any analytic stratification of $\tilde{\partial}\mathcal{A}_{[a,b]}(q_0,U)$ is equal to four. On the other hand, if the singular trajectory starting at q_0 lies in the interior of $\mathcal{A}_{[a,b]}(q_0,U)$ or there are no singular trajectories, then there exists an analytic stratification of $\tilde{\partial}\mathcal{A}_{[a,b]}(q_0,U)$ containing two 2-dimensional strata.*

1.3. Organization of the paper

In Section 2 we compute reachable sets in the flat case. The proof of Theorem 1.3 is given in Section 3. In Section 4 we prove Theorems 1.4 and 1.5. In Section 5 we prove some facts concerning sub-Lorentzian structures $(H, g^{a,b})$, which are corollaries of Theorems 1.3–1.5. So we study the continuity of the sub-Lorentzian distance for the structures $(H, g^{a,b})$. We also compute the set reachable by Hamiltonian geodesics and give some results on conjugate and cut loci. Section 6 presents two more possible applications of normal forms (1.4). In Appendix A we construct other normal forms for generalized Martinet sub-Lorentzian structures of Hamiltonian type. In Appendix B we make some comments concerning sub-Lorentzian structures where the distribution is tangent to its Martinet surface in a single point.

2. Reachable sets in the flat case

In this section, by $(\hat{H}, \hat{g})$ we will denote the *flat generalized Martinet sub-Lorentzian structure of order k* which, by definition, is given by an orthonormal frame in the form

$$\begin{aligned}\hat{X} &= \frac{\partial}{\partial x} + \frac{1}{2}y^k\frac{\partial}{\partial z}\\ \hat{Y} &= \frac{\partial}{\partial y} - \frac{1}{2}xy^{k-1}\frac{\partial}{\partial z}\end{aligned}\tag{2.1}$$

with a time orientation $\hat{X}$ and $k > 2$. We see that (2.1) is obtained from (1.4) by setting φ and ψ to zero. It is also easy to see that the flat structure is of Hamiltonian type. As it was said in the introduction, we will compute reachable sets for the affine control system

$$\dot{q} = \hat{X} + u\hat{Y}, \quad u \in [a,b],\tag{2.2}$$

for arbitrarily fixed $a, b \in \mathbb{R}$, $a < b$. The corresponding reachable sets for (2.2) we will denote by $\hat{\mathcal{A}}_{[a,b]}(0) = \hat{\mathcal{A}}_{[a,b]}(0,\mathbb{R}^3)$, $\hat{\mathcal{A}}_{(a,b)}(0) = \hat{\mathcal{A}}_{(a,b)}(0,\mathbb{R}^3)$, $\hat{\mathcal{A}}_{\{a,b\}}(0) = \hat{\mathcal{A}}_{\{a,b\}}(0,\mathbb{R}^3)$. As we know from Lemma 1.2, the investigation of (2.2) and its reachable sets is equivalent to the study of the following time-oriented sub-Lorentzian structure $(\hat{H}, \hat{g}^{a,b})$ defined on $\mathbb{R}^3$ by declaring the fields

$$\hat{Z}^{a,b} = \hat{X} + \frac{1}{2}(b+a)\hat{Y}$$

$$\hat{W}^{a,b} = \frac{1}{2}(b-a)\hat{Y}$$

to be an orthonormal basis for $(\hat{H}, \hat{g}^{a,b})$ with a time orientation $\hat{Z}^{a,b}$.

To proceed further, let us define two hypersurfaces

$$\Gamma_1^{a,b} = \{(x, bx, z) : \ x, z \in \mathbb{R}\}$$

and

$$\Gamma_2^{a,b} = \{(x, ax, z) : \ x, z \in \mathbb{R}\}.$$

Looking at the system (2.2) it is seen that

$$\hat{\mathcal{A}}_{[a,b]}(0) \subset \{(x,y,z) : \ ax \leq y \leq bx\} \tag{2.3}$$

and

$$\hat{\mathcal{A}}_{(a,b)}(0) \subset \{(x,y,z) : \ ax < y < bx\}. \tag{2.4}$$

Below we will consider two cases, depending on the parity of k, but before we do it we will state one more definition. Let (M, H, g) be a sub-Lorentzian manifold. Let U be an open subset of M and suppose that $f : U \longrightarrow \mathbb{R}$ is a smooth function. By *the horizontal gradient of* f we mean a vector field denoted by $\nabla_H f$ and defined with the formula $v(f) = g(v, \nabla_H f(q))$ for every $q \in U$, $v \in H_q$. If $X_0, \ldots, X_k$ is an orthonormal basis for (H, g) defined on U with X_0 timelike, then it can be shown that $\nabla_H f = -X_0(f)X_0 + X_1(f)X_1 + \ldots + X_k(f)X_k$. It is clear that if $\nabla_H f$ is null f.d. and $\gamma : [a, b] \longrightarrow U$ is nspc.f.d. then the function $t \longrightarrow f(\gamma(t))$ is nonincreasing (see the previous papers by the author for more properties of $\nabla_H f$ and its possible applications).

2.1. The case $k = 2l + 1$

We start with the simpler case, *i.e.* for k being an odd positive integer.

Proposition 2.1. *Consider the affine control system* (2.2) *defined on* $\mathbb{R}^3$*. Suppose that* $a < b$ *and* $a, b \neq 0$*. Then*

$$\hat{\mathcal{A}}_{[a,b]}(0) = \hat{\mathcal{A}}_{\{a,b\}}(0) = \hat{A}_1 \cup \hat{A}_2 \tag{2.5}$$

and

$$\hat{\mathcal{A}}_{(a,b)}(0) = int(\hat{A}_1 \cup \hat{A}_2), \tag{2.6}$$

where

$$\hat{A}_1 = \left\{(x,y,z) \in \mathbb{R}^3 : \ z - \frac{1}{2ka}(y - ax)\left(y^k - \frac{b^k}{(b-a)^k}(y-ax)^k\right) \leq 0\right\} \cap \{x \geq 0, \ z \geq 0\},$$

$$\hat{A}_2 = \left\{(x,y,z) \in \mathbb{R}^3 : \ -z - \frac{1}{2kb}(bx - y)\left(y^k - \frac{a^k}{(b-a)^k}(bx-y)^k\right) \leq 0\right\} \cap \{x \geq 0, \ z \leq 0\}.$$

To motivate the construction below, let us notice that the z-coordinate of

$$\hat{Z}^{a,b} - \hat{W}^{a,b} = \hat{X} + a\hat{Y} = \frac{\partial}{\partial x} + a\frac{\partial}{\partial y} + \frac{1}{2}y^{k-1}(y - ax)\frac{\partial}{\partial z} \tag{2.7}$$

is positive, while the the z-coordinate of

$$\hat{Z}^{a,b} + \hat{W}^{a,b} = \hat{X} + b\hat{Y} = \frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + \frac{1}{2}y^{k-1}(y - bx)\frac{\partial}{\partial z} \tag{2.8}$$

is negative, everything in the sector $\{ax < y < bx\}$. It means that the trajectories of $\hat{Z}^{a,b} - \hat{W}^{a,b}$ (resp. $\hat{Z}^{a,b} + \hat{W}^{a,b}$) starting from $\{y = bx, \ z = 0\}$ (resp. from $\{y = ax, \ z = 0\}$) enter the half-space $\{z > 0\}$ (resp. $\{z < 0\}$). Having this in mind, and following [8], we consider two Cauchy problems:

$$(\hat{X} + a\hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_1^{a,b}}(x, bx, z) = z, \tag{2.9}$$

and

$$(\hat{X} + b\hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_2^{a,b}}(x, ax, z) = -z. \tag{2.10}$$

After simple computations we find that

$$\hat{\eta}_1^{a,b}(x,y,z) = z - \frac{1}{2ka}(y-ax)\left(y^k - \frac{b^k}{(b-a)^k}(y-ax)^k\right) \tag{2.11}$$

is the solution to (2.9), and

$$\hat{\eta}_2^{a,b}(x,y,z) = -z - \frac{1}{2kb}(bx-y)\left(y^k - \frac{a^k}{(b-a)^k}(bx-y)^k\right) \tag{2.12}$$

is the solution to (2.10). We will compute the horizontal gradient $\nabla_{\hat{H}^{a,b}}\hat{\eta}_i^{a,b}$ of the function $\hat{\eta}_i^{a,b}$, $i=1,2$, with respect to the structure $(\hat{H},\hat{g}^{a,b})$. Making use of (2.9) and (2.10) we have

$$\nabla_{\hat{H}^{a,b}}\hat{\eta}_1^{a,b} = -\hat{Z}^{a,b}(\hat{\eta}_1^{a,b})(\hat{Z}^{a,b} - \hat{W}^{a,b}) \tag{2.13}$$

and

$$\nabla_{\hat{H}^{a,b}}\hat{\eta}_2^{a,b} = -\hat{Z}^{a,b}(\hat{\eta}_1^{a,b})(\hat{Z}^{a,b} + \hat{W}^{a,b}). \tag{2.14}$$

Now

$$\hat{Z}^{a,b}(\hat{\eta}_1^{a,b}) = \frac{k+1}{4k}\,(y-bx)\left(y^{k-1} + y^{k-2}\frac{b\,(ax-y)}{a-b} + \ldots + y\left(\frac{b\,(ax-y)}{a-b}\right)^{k-2} + \left(\frac{b\,(ax-y)}{a-b}\right)^{k-1}\right),$$

where the first and the third term is positive (recall that $k=2l+1$) and the second term is negative in the sector $\{ax<y<bx\}$. Therefore $\hat{Z}^{a,b}(\hat{\eta}_1^{a,b})<0$, and $\nabla_{\hat{H}^{a,b}}\hat{\eta}_1^{a,b}$ is future directed in the mentioned sector. In the similar way we obtain

$$\hat{Z}^{a,b}(\hat{\eta}_2^{a,b}) = -\frac{k+1}{4k}\,(y-ax)\left(y^{k-1} + y^{k-2}\frac{a\,(bx-y)}{b-a} + \ldots + y\left(\frac{a\,(bx-y)}{b-a}\right)^{k-2} + \left(\frac{a\,(bx-y)}{b-a}\right)^{k-1}\right)$$

proving that $\nabla_{\hat{H}^{a,b}}\hat{\eta}_2^{a,b}$ is future directed in the sector $\{ax<y<bx\}$.

Now, by (2.3), (2.4), the reasoning is the same as in *e.g.* [9]. Since for every curve $\gamma:[0,T]\longrightarrow\mathbb{R}^3$ which is nspc.f.d. (resp. t.f.d.) with respect the structure $(\hat{H},\hat{g}^{a,b})$, the function $t\longrightarrow\hat{\eta}_i^{a,b}(\gamma(t))$ is nonincreasing (resp. decreasing), $i=1,2$, it follows that $\mathcal{A}_{[a,b]}(0)\subset\hat{A}_1\cup\hat{A}_2$ and $\mathcal{A}_{(a,b)}(0)\subset int(\hat{A}_1\cup\hat{A}_2)$. To prove the reverse inclusion, take a point $q\in\hat{A}_1\cup\hat{A}_2$ and send the trajectory of the field $-\hat{Z}^{a,b}$ from q. Sooner or later we will reach the boundary $\partial(\hat{A}_1\cup\hat{A}_2)$ which, by our construction, is made up of null f.d. curves starting from the origin. To be more precise $\partial(\hat{A}_1\cup\hat{A}_2)\cap\{z>0\}$ is made up of trajectories of $\hat{Z}^{a,b}-\hat{W}^{a,b}$ that start from $\{y=bx,\, z=0\}$, and $\partial(\hat{A}_1\cup\hat{A}_2)\cap\{z<0\}$ is made up of trajectories of $\hat{Z}^{a,b}+\hat{W}^{a,b}$ that start from $\{y=ax,\, z=0\}$. Thus $\mathcal{A}_{[a,b]}(0)=\hat{A}_1\cup\hat{A}_2$ and the other claims in (2.5), (2.6) follow now from properties of reachable sets – see [7].

It remains to consider two cases: $a=0<b$ and $a<b=0$.

Proposition 2.2. *Under the same assumptions as in Proposition 2.1, except that $a=0<b$, the formulas*

$$\hat{\mathcal{A}}_{[0,b]}(0) = \hat{\mathcal{A}}_{\{0,b\}}(0) = \hat{A}_1\cup\hat{A}_2$$
$$\hat{\mathcal{A}}_{(0,b)}(0) = int(\hat{A}_1\cup\hat{A}_2)$$

hold true, where

$$\hat{A}_1 = \left\{(x,y,z)\in\mathbb{R}^3:\ z-\frac{1}{2b}y^k(bx-y)\le 0\right\}\cap\{x\ge 0,\, z\ge 0\},$$
$$\hat{A}_2 = \left\{(x,y,z)\in\mathbb{R}^3:\ -z-\frac{1}{2kb}y^k(bx-y)\le 0\right\}\cap\{x\ge 0,\, z\le 0\}.$$

Proposition 2.3. *Under the same assumptions as in Proposition 2.1, except that $a < b = 0$, the formulas*

$$\hat{\mathcal{A}}_{[a,0]}(0) = \hat{\mathcal{A}}_{\{a,0\}}(0) = \hat{A}_1 \cup \hat{A}_2$$
$$\hat{\mathcal{A}}_{(a,0)}(0) = int(\hat{A}_1 \cup \hat{A}_2)$$

hold true, where

$$\hat{A}_1 = \left\{(x,y,z) \in \mathbb{R}^3 : \ z + \frac{1}{2ka} y^k (ax - y) \leq 0\right\} \cap \{x \geq 0,\ z \geq 0\},$$
$$\hat{A}_2 = \left\{(x,y,z) \in \mathbb{R}^3 : \ -z + \frac{1}{2a} y^k (ax - y) \leq 0\right\} \cap \{x \geq 0,\ z \leq 0\}.$$

To prove Propositions 2.2, 2.3 we obviously use the same method as in the proof of Proposition 2.1. Take for instance Proposition 2.2. Noting that $\hat{Z}^{0,b} - \hat{W}^{0,b} = \hat{X}$, $\hat{Z}^{0,b} + \hat{W}^{0,b} = \hat{X} + b\hat{Y}$, we see that the z-coordinate of $\hat{Z}^{0,b} - \hat{W}^{0,b}$ is positive, while the z-coordinate of $\hat{Z}^{0,b} + \hat{W}^{0,b}$ is negative, both facts taking place in $\{0 < y < bx\}$. Hence we consider two Cauchy problems:

$$\hat{X}(\eta) = 0, \quad \eta_{|\Gamma_1^{0,b}}(x, bx, z) = z \tag{2.15}$$

and

$$(\hat{X} + b\hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_2^{0,b}}(x, 0, z) = -z. \tag{2.16}$$

The solution to (2.15) is

$$\hat{\eta}_1^{0,b}(x,y,z) = z - \frac{1}{2b} y^k (bx - y),$$

and the solution to (2.16) is

$$\hat{\eta}_2^{0,b}(x,y,z) = -z - \frac{1}{2kb} y^k (bx - y).$$

Evidently

$$\nabla_{\hat{H}^{0,b}} \hat{\eta}_1 = -\hat{Z}^{0,b}(\hat{\eta}_1^{0,b})(\hat{Z}^{0,b} - \hat{W}^{0,b}),$$

and in the sector $\{0 < y < bx\}$

$$\hat{Z}^{0,b}(\hat{\eta}_1^{0,b}) = -\frac{1}{4}\,(k+1)\,y^{k-1}\,(bx - y) < 0$$

meaning that $\nabla_{\hat{H}^{0,b}} \hat{\eta}_1^{0,b}$ is null f.d. in this sector. Similarly

$$\nabla_{\hat{H}^{0,b}} \hat{\eta}_2^{0,b} = -\hat{Z}^{0,b}(\hat{\eta}_1^{0,b})(\hat{Z}^{0,b} + \hat{W}^{0,b})$$

with

$$\hat{Z}^{0,b}(\hat{\eta}_2^{0,b}) = -\frac{k+1}{4k} y^k$$

again meaning that $\nabla_{\hat{H}^{0,b}} \hat{\eta}_2^{0,b}$ is null f.d. in the sector $\{0 < y < bx\}$. Having in mind (2.3), (2.4), we continue exactly as in the proof of the previous proposition.

Applying once more the above procedure, this time to the case $a < b = 0$, we construct the functions

$$\hat{\eta}_1^{a,0}(x,y,z) = z + \frac{1}{2ka} y^k (ax - y),$$
$$\hat{\eta}_2^{a,0}(x,y,z) = -z + \frac{1}{2a} y^k (ax - y).$$

appearing in the hypothesis of Proposition 2.3.

Corollary 2.4. *The reachable sets for the structure $(\hat{H}, \hat{g})$ with k being an odd positive integer are as follows:*

$$J^+(0) = N^+(0) = \{\eta_1 \le 0,\, x \ge 0,\, z \ge 0\} \cup \{\eta_2 \le 0,\, x \ge 0,\, z \le 0\}$$

$$I^+(0) = int J^+(0) = \{\eta_1 < 0,\, x \ge 0,\, z \ge 0\} \cup \{\eta_2 < 0,\, x \ge 0,\, z \le 0\}\,,$$

where

$$\hat{\eta}_1(x,y,z) = z + \frac{1}{2k}(x+y)\left(y^k - \frac{(x+y)^k}{2^k}\right),$$

$$\hat{\eta}_2(x,y,z) = -z - \frac{1}{2k}(x-y)\left(y^k + \frac{(x-y)^k}{2^k}\right).$$

2.2. The case $k = 2l$

This case is a little more complicated, since the structure of reachable sets will essentially depend on whether 0 belongs to (a,b) or does not.

First of all let us consider the case $0 < a < b$.

Proposition 2.5. *Let $0 < a < b$. Then the reachable sets for (2.2) have the form*

$$\hat{\mathcal{A}}_{[a,b]}(0) = \hat{\mathcal{A}}_{\{a,b\}}(0) = \hat{A}_1 \cup \hat{A}_2 \tag{2.17}$$

and

$$\hat{\mathcal{A}}_{(a,b)}(0) = int(\hat{A}_1 \cup \hat{A}_2), \tag{2.18}$$

where

$$\hat{A}_1 = \left\{(x,y,z) \in \mathbb{R}^3 :\; z - \frac{1}{2ka}(y-ax)\left(y^k - \frac{b^k}{(b-a)^k}(y-ax)^k\right) \le 0\right\} \cap \{x \ge 0,\, z \ge 0\},$$

$$\hat{A}_2 = \left\{(x,y,z) \in \mathbb{R}^3 :\; -z - \frac{1}{2kb}(bx-y)\left(y^k - \frac{a^k}{(b-a)^k}(bx-y)^k\right) \le 0\right\} \cap \{x \ge 0,\, z \le 0\}.$$

Proof. By (2.7), (2.8) we see that as in the case of Proposition 2.1 the z-coordinate of the field $\hat{Z}^{a,b} - \hat{W}^{a,b} = \hat{X} + a\hat{Y}$ is positive, while the z-coordinate of the field $\hat{Z}^{a,b} + \hat{W}^{a,b} = \hat{X} + b\hat{Y}$ is negative, everything in the sector $\{ax < y < bx\}$. It terminates the proof since evidently the same formulas as in Proposition 2.1 hold true. □

Now consider the case $a < b < 0$.

Proposition 2.6. *Let $a < b < 0$. Then the reachable sets for (2.2) have the form*

$$\hat{\mathcal{A}}_{[a,b]}(0) = \hat{\mathcal{A}}_{\{a,b\}}(0) = \hat{A}_1 \cup \hat{A}_2 \tag{2.19}$$

and

$$\hat{\mathcal{A}}_{(a,b)}(0) = int(\hat{A}_1 \cup \hat{A}_2), \tag{2.20}$$

where

$$\hat{A}_1 = \left\{(x,y,z) \in \mathbb{R}^3 :\; -z + \frac{1}{2ka}(y-ax)\left(y^k - \frac{b^k}{(b-a)^k}(y-ax)^k\right) \le 0\right\} \cap \{x \ge 0,\, z \le 0\},$$

$$\hat{A}_2 = \left\{(x,y,z) \in \mathbb{R}^3 :\; z + \frac{1}{2kb}(bx-y)\left(y^k - \frac{a^k}{(b-a)^k}(bx-y)^k\right) \le 0\right\} \cap \{x \ge 0,\, z \ge 0\}.$$

This time, by (2.7) and (2.8), we conclude that the z -coordinate of $\hat{Z}^{a,b} - \hat{W}^{a,b} = \hat{X} + a\hat{Y}$ is negative, while the z-coordinate of $\hat{Z}^{a,b} + \hat{W}^{a,b} = \hat{X} + b\hat{Y}$ is positive, everything in the sector $\{ax < y < bx\}$. It suggests considering the following Cauchy problems:

$$(\hat{X} + a\hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_1^{a,b}}(x, bx, z) = -z \tag{2.21}$$

and

$$(\hat{X}+b\hat{Y})(\eta)=0, \quad \eta_{|\Gamma_2^{a,b}}(x,ax,z)=z. \tag{2.22}$$

An easy computation shows that

$$\hat{\eta}_1^{a,b}(x,y,z)=-z+\frac{1}{2ka}(y-ax)\left(y^k-\frac{b^k}{(b-a)^k}(y-ax)^k\right)$$

is the solution to (2.21), and

$$\hat{\eta}_2^{a,b}(x,y,z)=z+\frac{1}{2kb}(bx-y)\left(y^k-\frac{a^k}{(b-a)^k}(bx-y)^k\right)$$

is the solution to (2.22). Next we find that

$$\hat{Z}^{a,b}(\hat{\eta}_1^{a,b})=-\frac{k+1}{4k\,(a-b)}\,(y-bx)\left((a-b)y+b(ax-y)\right)G_1,$$

with

$$G_1=y^{k-2}+y^{k-4}\left(\frac{b\,(ax-y)}{a-b}\right)^2+\ldots+y^2\left(\frac{b\,(ax-y)}{a-b}\right)^{k-4}+\left(\frac{b\,(ax-y)}{a-b}\right)^{k-2}. \tag{2.23}$$

Similarly

$$\hat{Z}^{a,b}(\hat{\eta}_2^{a,b})=-\frac{k+1}{4k\,(a-b)}\,(ax-y)\left((a-b)\,y-a\,(bx-y)\right)G_2$$

with

$$G_2=y^{k-2}+y^{k-4}\left(\frac{(bx-y)\,a}{a-b}\right)^2+\ldots+y^2\left(\frac{(bx-y)\,a}{a-b}\right)^{k-4}+\left(\frac{(bx-y)\,a}{a-b}\right)^{k-2}. \tag{2.24}$$

By (2.13), (2.14) and (2.3), (2.4) we see that $\nabla_{\hat{H}^{a,b}}\hat{\eta}_1^{a,b}$ and $\nabla_{\hat{H}^{a,b}}\hat{\eta}_2^{a,b}$ are null f.d. in $\{ax<y<bx\}$, so the procedure used in the previous subsection ends the proof of Proposition 2.6.

Similar results, *i.e.* two functions describing reachable sets but with different formulas, hold in two further cases, namely $0=a<b$ and $a<b=0$.

Proposition 2.7. *Let $0=a<b$. Then the reachable sets for (2.2) have the form*

$$\hat{\mathcal{A}}_{[0,b]}(0)=\hat{\mathcal{A}}_{\{0,b\}}(0)=\hat{A}_1\cup\hat{A}_2 \tag{2.25}$$

and

$$\hat{\mathcal{A}}_{(0,b)}(0)=int(\hat{A}_1\cup\hat{A}_2), \tag{2.26}$$

where

$$\hat{A}_1=\left\{(x,y,z)\in\mathbb{R}^3:\ z-\frac{1}{2b}y^k(bx-y)\le 0\right\}\cap\{x\ge 0,\ z\ge 0\},$$

$$\hat{A}_2=\left\{(x,y,z)\in\mathbb{R}^3:\ -z-\frac{1}{2kb}y^k(bx-y)\le 0\right\}\cap\{x\ge 0,\ z\le 0\}.$$

Proof. The z-coordinates of the fields $\hat{Z}^{0,b}-\hat{W}^{0,b}=\hat{X}$, $\hat{Z}^{0,b}+\hat{W}^{0,b}=\hat{X}+b\hat{Y}$ are exactly the same as in the case of Proposition 2.2. □

Proposition 2.8. *Let $a<b=0$. Then the reachable sets for (2.2) have the form*

$$\hat{\mathcal{A}}_{[a,0]}(0)=\hat{\mathcal{A}}_{\{a,0\}}(0)=\hat{A}_1\cup$$
$$\hat{\mathcal{A}}_{(a,0)}(0)=int(\hat{A}_1\cup\hat{A}_2),$$

where

$$\hat{A}_1 = \left\{(x,y,z) \in \mathbb{R}^3 : \ -z - \frac{1}{2ka} y^k (ax - y) \le 0\right\} \cap \{x \ge 0,\ z \le 0\},$$

$$\hat{A}_2 = \left\{(x,y,z) \in \mathbb{R}^3 : \ z - \frac{1}{2a} y^k (ax - y) \le 0\right\} \cap \{x \ge 0,\ z \ge 0\}.$$

Proof. Looking at the signs of z-coordinates of the fields $\hat{Z}^{a,0} - \hat{W}^{a,0} = \hat{X} + a\hat{Y}$, $\hat{Z}^{a,0} + \hat{W}^{a,0} = \hat{X}$ in the sector $\{ax < y < 0\}$, we deduce that the appropriate Cauchy problems to consider are

$$(\hat{X} + a\hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_1^{a,0}}(x,0,z) = -z$$

with the solution equal to

$$\hat{\eta}_1^{a,0}(x,y,z) = -z - \frac{1}{2ka} y^k (ax - y),$$

and

$$\hat{X}(\eta) = 0, \quad \eta_{|\Gamma_1^{a,0}}(x,ax,z) = z$$

with the solution equal to

$$\hat{\eta}_2^{a,0}(x,y,z) = z - \frac{1}{2a} y^k (ax - y).$$

It remains to check whether $\nabla_{\hat{H}^{a,0}} \hat{\eta}_i^{a,0}$'s are suitably directed. By (2.13), (2.14) it suffices to ensure that

$$\hat{Z}^{a,0}(\hat{\eta}_1^{a,0}) = -\frac{1+k}{4k} y^k < 0,$$

and

$$\hat{Z}^{a,0}(\hat{\eta}_2^{a,0}) = -\frac{k+1}{4} y^{k-1} (ax - y) < 0,$$

both holding in $\{ax < y < 0\}$. $\square$

The last case to consider is the case $a < 0 < b$. This time, as we are about to see, we need four functions to describe the reachable sets.

Proposition 2.9. *Let $a < 0 < b$. Then the reachable sets for (2.2) have the form*

$$\begin{aligned}
\hat{\mathcal{A}}_{[a,b]}(0) &= \hat{A}_1 \cup \ldots \cup \hat{A}_4 \\
\hat{\mathcal{A}}_{(a,b)}(0) &= int(\hat{A}_1 \cup \ldots \cup \hat{A}_4) \cup A_5 \\
\hat{\mathcal{A}}_{\{a,b\}}(0) &= int(\hat{A}_1 \cup \ldots \cup \hat{A}_4) \cup \partial(\hat{A}_1 \cup \hat{A}_2) \backslash A_5,
\end{aligned}$$

where

$$\hat{A}_1 = \left\{(x,y,z) \in \mathbb{R}^3 : \ z - \frac{1}{2ka}(y - ax)\left(y^k - \frac{b^k}{(b-a)^k}(y-ax)^k\right) \le 0\right\} \cap \{y \ge 0\} \cap \{x \ge 0,\ z \ge 0\},$$

$$\hat{A}_2 = \left\{(x,y,z) \in \mathbb{R}^3 : \ z + \frac{1}{2kb}(bx - y)\left(y^k - \frac{a^k}{(b-a)^k}(bx-y)^k\right) \le 0\right\} \cap \{y \le 0\} \cap \{x \ge 0,\ z \ge 0\},$$

$$\hat{A}_3 = \left\{(x,y,z) \in \mathbb{R}^3 : \ -z - \frac{1}{2kb} y^k (bx - y) \le 0\right\} \cap \{y \ge 0\} \cap \{x \ge 0,\ z \le 0\},$$

$$\hat{A}_4 = \left\{(x,y,z) \in \mathbb{R}^3 : \ -z - \frac{1}{2ka} y^k (ax - y) \le 0\right\} \cap \{y \le 0\} \cap \{x \ge 0,\ z \le 0\},$$

where A_5 is the abnormal curve starting from 0.

At the beginning let us notice that $\gamma(t) = (t, 0, 0)$, *i.e.* the abnormal curve for the distribution $\hat{H}$ starting from zero and at the same time the trajectory of (2.2) corresponding to the control $u(t) = 0$, lies on the boundary $\partial\hat{\mathcal{A}}_{[a,b]}(0)$. Indeed, every trajectory of (2.2) is a horizontal curve with respect to $\hat{H}$. Suppose that $\eta = (\eta_1, \eta_2, \eta_3) : [\alpha, \beta] \longrightarrow \mathbb{R}^3$ is a trajectory of (2.2) such that $\eta(\alpha) = \gamma(t_1)$, $\eta(\beta) = \gamma(t_2)$. The equation for horizontal curves is $\dot{z} = \frac{1}{2}y^{k-1}(y\dot{x} - x\dot{y})$. In particular

$$0 = \eta_3(\alpha) + \int_\alpha^\beta \frac{1}{2}\eta_2^{k-1}(\eta_2\dot{\eta}_1 - \eta_1\dot{\eta}_2)\mathrm{d}t = \int_{(\eta_1,\eta_2)} \frac{1}{2}y^{k-1}(y\mathrm{d}x - x\mathrm{d}y)$$

(note that $\dot{\eta}_1 > 0$ since η is f.d. for $(\hat{H}, \hat{g}^{a,b})$). Since $d\left(\frac{1}{2}y^{k-1}(y\mathrm{d}x - x\mathrm{d}y)\right) = -\frac{1}{2}(k+1)y^{k-1}\mathrm{d}x \wedge \mathrm{d}y$ and $k-1$ is odd, it follows by Stokes' theorem that $\eta_2(t)$ vanishes on $[\alpha, \beta]$. But then $\eta_3(t) = 0$ on $[\alpha, \beta]$ and, consequently, η is a reparameterization of γ. Using the facts recalled in Section 1.1, γ is contained in $\partial\mathcal{A}_{[a,b]}(0)$.

Next let us notice that in the sector $\{ax < y < bx\}$ there hold the following relations. The z-coordinate of the field $\hat{Z}^{a,b} - \hat{W}^{a,b} = \hat{X} + a\hat{Y}$ in a neighbourhood of $\{y = bx,\ z = 0,\ x \geq 0\}$, and the z-coordinate of $\hat{Z}^{a,b} + \hat{W}^{a,b} = \hat{X} + b\hat{Y}$ in a neighbourhood of $\{y = ax,\ z = 0,\ x \geq 0\}$ are both positive. At the same time they are both negative in a neighbourhood of $\{y = 0,\ z = 0,\ x \geq 0\}$.

Summing all these facts up we conclude that this case is similar to the Martinet case considered in [9]. As indicated there, we construct four functions $\hat{\eta}_1^{a,b}, \ldots, \hat{\eta}_4^{a,b}$.

$\hat{\eta}_1^{a,b}$ is the solution to the Cauchy problem

$$(\hat{X} + a\hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_1^{a,b}}(x, bx, z) = z,$$

$\hat{\eta}_2^{a,b}$ is the solution to the Cauchy problem

$$(\hat{X} + b\hat{Y})(\eta) = 0, \quad \eta_{|\Gamma_2^{a,b}}(x, ax, z) = z,$$

$\hat{\eta}_3^{a,b}$ is the solution to

$$(\hat{X} + b\hat{Y})(\eta) = 0, \quad \eta_{|S}(x, 0, z) = -z,$$

and finally $\hat{\eta}_4^{a,b}$ is the solution to

$$(\hat{X} + a\hat{Y})(\eta) = 0, \quad \eta_{|S}(x, 0, z) = -z,$$

where as above S is the Martinet surface. After calculations we have

$$\begin{aligned}
\hat{\eta}_1^{a,b}(x,y,z) &= z - \frac{1}{2ka}(y - ax)\left(y^k - \frac{b^k}{(b-a)^k}(y-ax)^k\right) \\
\hat{\eta}_2^{a,b}(x,y,z) &= z + \frac{1}{2kb}(bx - y)\left(y^k - \frac{a^k}{(b-a)^k}(bx-y)^k\right) \\
\hat{\eta}_3^{a,b}(x,y,z) &= -z - \frac{1}{2kb}y^k(bx - y) \\
\hat{\eta}_4^{a,b}(x,y,z) &= -z - \frac{1}{2ka}y^k(ax - y).
\end{aligned}$$

Of course, for $k = 2$ and $a = -1$, $b = 1$ we obtain the same formulas as in [9]. In order to be able to apply the reasoning used in [9] we must compute $\nabla_{\hat{H}^{a,b}}\hat{\eta}_i^{a,b}$, *i.e.* again by (2.13), (2.14) to compute $\hat{Z}^{a,b}(\hat{\eta}_i^{a,b})$, $i = 1, 2, 3, 4$. They are as follows:

$$\hat{Z}^{a,b}(\hat{\eta}_1^{a,b}) = \frac{k+1}{4k\,(a-b)}\,(y - bx)\,((a-b)\,y + b\,(ax - y))\,G_1 \tag{2.27}$$

with G_1 defined in (2.23),

$$\hat{Z}^{a,b}(\hat{\eta}_2^{a,b}) = -\frac{k+1}{4k\,(a-b)}\,(ax-y)\,((a-b)\,y - a\,(bx-y))\,G_2 \tag{2.28}$$

with G_2 defined as in (2.24),

$$\hat{Z}^{a,b}(\hat{\eta}_3^{a,b}) = \frac{k+1}{4kb}\,(a-b)\,y^k, \tag{2.29}$$

and finally

$$\hat{Z}^{a,b}(\hat{\eta}_4^{a,b}) = -\frac{k+1}{4ka}\,(a-b)\,y^k. \tag{2.30}$$

Lemma 2.10. *$\hat{Z}^{a,b}(\hat{\eta}_1^{a,b}) < 0$ on $\left\{\frac{ab}{2b-a}x < y < bx\right\}$, $\hat{Z}^{a,b}(\hat{\eta}_2^{a,b}) < 0$ on $\left\{ax < y < \frac{|a|b}{b-2a}x\right\}$, $\hat{Z}^{a,b}(\hat{\eta}_3^{a,b}) < 0$ and $\hat{Z}^{a,b}(\hat{\eta}_4^{a,b}) < 0$ on $\{y \neq 0\}$.*

Proof. It is easily seen using the expressions presented above. $\square$

Note that $0 < \frac{|a|b}{2b-a} < \frac{|a|b}{2b} = \frac{1}{2}|a|$, and $\frac{|a|b}{b-2a} < \frac{|a|b}{2|a|} = \frac{1}{2}b$.

It follows that $\nabla_{\hat{H}^{a,b}}\hat{\eta}_1^{a,b}$ is null f.d. on $\{0 \leq 0 < bx\}$, $\nabla_{\hat{H}^{a,b}}\hat{\eta}_2^{a,b}$ is null f.d. on $\{ax < y \leq 0\}$, $\nabla_{\hat{H}^{a,b}}\hat{\eta}_3^{a,b}$ and $\nabla_{\hat{H}^{a,b}}\hat{\eta}_4^{a,b}$ are null f.d. on $\{y \neq 0\}$. Let us also notice, and this will be important later in Section 4, that $\hat{\eta}_1^{a,b} \leq \hat{\eta}_2^{a,b}$ on $\{0 \leq y \leq bx\}$, *i.e.*

$$\left\{\hat{\eta}_2^{a,b} \leq 0\right\} \cap \{0 \leq y \leq bx\} \subset \left\{\hat{\eta}_1^{a,b} \leq 0\right\} \cap \{0 \leq y \leq bx\}, \tag{2.31}$$

and $\hat{\eta}_2^{a,b} \leq \hat{\eta}_1^{a,b}$ on $\{ax \leq y \leq 0\}$, *i.e.*

$$\left\{\hat{\eta}_1^{a,b} \leq 0\right\} \cap \{ax \leq y \leq 0\} \subset \left\{\hat{\eta}_2^{a,b} \leq 0\right\} \cap \{ax \leq y \leq 0\}. \tag{2.32}$$

Indeed, take (2.31) for instance. Notice that by Lemma 2.10 $(\hat{Z}^{a,b} - \hat{W}^{a,b})(\hat{\eta}_2^{a,b} - \hat{\eta}_1^{a,b}) = (\hat{Z}^{a,b} - \hat{W}^{a,b})(\hat{\eta}_2^{a,b}) > 0$ on $\left\{\frac{|a|b}{b-2a}x < y < bx\right\}$ and $(\hat{Z}^{a,b} - \hat{W}^{a,b})(\hat{\eta}_2^{a,b} - \hat{\eta}_1^{a,b}) < 0$ on $\left\{0 < y < \frac{|a|b}{b-2a}x\right\}$. It means that $\hat{\eta}_2^{a,b}$ increases along the trajectories of $\hat{Z}^{a,b} - \hat{W}^{a,b}$ on $\left\{\frac{|a|b}{b-2a}x < y < bx\right\}$, while $\hat{\eta}_1^{a,b}$ remains constant, that is to say $\hat{\eta}_2^{a,b} - \hat{\eta}_1^{a,b} > 0$ on $\left\{\frac{|a|b}{b-2a}x < y < bx\right\}$. Then $\hat{\eta}_2^{a,b} - \hat{\eta}_1^{a,b}$ starts to decrease and attains zero on $\{y = 0\}$.

Now the reasoning similar to that used in [9] ends the proof of Proposition 2.9.

Corollary 2.11. *The reachable sets for the structure $(\hat{H}, \hat{g})$ with k being an even positive integer are as follows:*

$$J^+(0) = \hat{A}_1 \cup \ldots \cup \hat{A}_4$$

$$I^+(0) = int(\hat{A}_1 \cup \ldots \cup \hat{A}_4) \cup A_5$$

$$N^+(0) = int(\hat{A}_1 \cup \ldots \cup \hat{A}_4) \cup \partial(\hat{A}_1 \cup \hat{A}_2) \backslash A_5,$$

where

$$\hat{A}_1 = \left\{(x,y,z) \in \mathbb{R}^3 :\ z + \frac{1}{2k}(y+x)\left(y^k - \frac{1}{2^k}(x+y)^k\right) \leq 0\right\} \cap \{y \geq 0\} \cap \{x \geq 0,\ z \geq 0\},$$

$$\hat{A}_2 = \left\{(x,y,z) \in \mathbb{R}^3 :\ z + \frac{1}{2k}(x-y)\left(y^k - \frac{1}{2^k}(x-y)^k\right) \leq 0\right\} \cap \{y \leq 0\} \cap \{x \geq 0,\ z \geq 0\},$$

$$\hat{A}_3 = \left\{(x,y,z) \in \mathbb{R}^3 :\ -z - \frac{1}{2k}y^k(x-y) \leq 0\right\} \cap \{y \geq 0\} \cap \{x \geq 0,\ z \leq 0\},$$

$$\hat{A}_4 = \left\{(x,y,z) \in \mathbb{R}^3 :\ -z - \frac{1}{2k}y^k(x+y) \leq 0\right\} \cap \{y \leq 0\} \cap \{x \geq 0,\ z \leq 0\},$$

and A_5 is the same as in the hypothesis of Proposition 2.8.

Remark 2.12. Being motivated by the sub-Lorentzian geometry we considered only reachable sets $\hat{\mathcal{A}}_{[a,b]}(0)$, $\hat{\mathcal{A}}_{(a,b)}(0)$, $\hat{\mathcal{A}}_{\{a,b\}}(0)$, but of course one can also consider *e.g.* sets $\hat{\mathcal{A}}_{[a,b)}(0)$, $\hat{\mathcal{A}}_{(a,b]}(0)$. Clearly, in the cases considered in Propositions 2.1–2.8 we have

$$\hat{\mathcal{A}}_{[a,b)}(0) = \hat{\mathcal{A}}_{(a,b)}(0) \cup \{y = ax,\, z = 0,\, x \geq 0\}$$
$$\hat{\mathcal{A}}_{(a,b]}(0) = \hat{\mathcal{A}}_{(a,b)}(0) \cup \{y = bx,\, z = 0,\, x \geq 0\},$$

while in the case of Proposition 2.9 the formulas are

$$\hat{\mathcal{A}}_{[a,b)}(0) = \hat{\mathcal{A}}_{(a,b)}(0) \cup \left\{\hat{\eta}_4^{a,b} = 0,\, z \leq 0,\, -x \leq y \leq 0,\, x \geq 0\right\}$$
$$\hat{\mathcal{A}}_{(a,b]}(0) = \hat{\mathcal{A}}_{(a,b)}(0) \cup \left\{\hat{\eta}_3^{a,b} = 0,\, z \leq 0,\, 0 \leq y \leq x,\, x \geq 0\right\}.$$

3. Normal forms

In this section we prove Theorem 1.1. We begin with a simple lemma. Let Y, Z be vector fields on a manifold. We will use the standard notation $ad^kY \cdot Z$ which is defined as follows: $ad^0Y \cdot Z = Z$, and $ad^{n+1}Y \cdot Z = [Y, ad^nY \cdot Z]$ for $n = 1, 2, \ldots$ Next, if Y is a vector field regarded as an operator acting on smooth functions then we will write $Y^l = Y \circ \ldots \circ Y$ (l terms); by Y^0 we will mean the identity operator.

Lemma 3.1. *For any vector fields Y, Z we have*

$$ad^nY \cdot Z = \sum_{l=0}^{n} (-1)^l \binom{n}{l} Y^{n-l} \circ Z \circ Y^l \tag{3.1}$$

as an operator acting on smooth functions. In particular

$$ad^nY \cdot [X, Y] = \sum_{l=0}^{n} (-1)^l \binom{n}{l} Y^{n-l} \circ X \circ Y^{l+1} - \sum_{l=0}^{n} (-1)^l \binom{n}{l} Y^{n-l+1} \circ X \circ Y^l. \tag{3.2}$$

Proof. (3.1) is proved by induction with respect to n. (3.2) is then an easy consequence of (3.1). □

Now we will prove a proposition which gives a key argument in proving theorem 1.1.

Proposition 3.2. *Let k be a positive integer, $k \geq 2$, and let (H, g) be an analytic time-oriented generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type, defined on a neighbourhood U of the origin in $\mathbb{R}^3$. Then, provided that U is sufficiently small, there are coordinates x, y, z on U in which (H, g) admits an orthonormal frame in the form*

$$X = \frac{\partial}{\partial x} - yB\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - y^k A \frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y} + xB\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + xy^{k-1} A \frac{\partial}{\partial z}, \tag{3.3}$$

where A, B are analytic functions on U, X is a time orientation, $S = \{y = 0\}$ is the Martinet surface for H, and $X_{|S}$ is tangent to S.

Proof. The proposition clearly holds for $k = 2$ – see [9]. As the induction hypothesis let us assume that a structure of order k can be transformed to the form (3.3). Now let (H, g) be a Martinet structure of order $k + 1$ of Hamiltonian type defined on a neighbourhood U of the origin. By induction hypothesis there are coordinates

x, y, z on U (U sufficiently small) such that (H, g) admits an orthonormal frame X, Y in the form (3.3) with the time orientation X.

We will make use of the satisfaction of condition (M_{k+1}). In order to do it, since $H_{|S} = \ker dz$, we must examine the z-coordinate of successive Lie brackets of X and Y. The z-coordinate of $[X, Y] = adX \cdot Y$ is

$$(adX \cdot Y)(z) = X(xy^{k-1}A) + Y(y^k A) = O(y^{k-1}).$$

Next

$$(adX \cdot [X, Y])(z) = X \circ X(xy^{k-1}A) + 2X \circ Y(y^k A) - Y \circ X(y^k A) = O(y^{k-1})$$

and

$$(adY \cdot [X, Y])(z) = 2Y \circ X(xy^{k-1}A) + Y \circ Y(y^k A) - X \circ Y(xy^{k-1}A) = O(y^{k-2}).$$

Thus we see that the action of adX does not reduce the order with respect to y (unless it produces zero), while adY reduces this order by one. It follows that only $\left(ad^{k-1}Y \cdot [X, Y]\right)(z)_{|S} = 0$ can give a non-trivial condition. Rewriting (3.2) we are led to a sequence of equations

$$\begin{aligned} ad^{k-1}Y \cdot [X, Y] &= \sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} Y^{k-1-l} \circ X \circ Y^{l+1} - \sum_{l=0}^{k-1} (-1)^l \binom{k-1}{l} Y^{k-l} \circ X \circ Y^l \\ &= (-1)^{k-1} X \circ Y^k - Y^k \circ X + \sum_{l=0}^{k-2} (-1)^l \left[\binom{k-1}{l} + \binom{k-1}{l+1}\right] Y^{k-1-l} \circ X \circ Y^{l+1} \\ &= (-1)^{k-1} X \circ Y^k - Y^k \circ X + \sum_{l=0}^{k-2} (-1)^l \binom{k}{l+1} Y^{k-1-l} \circ X \circ Y^{l+1}. \end{aligned}$$

We will compute all summands appearing in $\left(ad^{k-1}Y \cdot [X, Y]\right)(z)$. Let us start with $Y(z) = xy^{k-1}A$ which leads to the general formula

$$Y^r(z) = (k-1)\ldots(k-r+1)x(1+x^2B)^{r-1}y^{k-r}A + O(y^{k-r+1}).$$

Further

$$\begin{aligned} X \circ Y^r(z) = (k-1)\ldots(k-r+1)y^{k-r}\Bigg[&\frac{\partial}{\partial x}\left(x(1+x^2B)^{r-1}\right)A + x(1+x^2B)^{r-1}\frac{\partial A}{\partial x} \\ &- (k-r)x^2(1+x^2B)^{r-1}AB\Bigg] + O(y^{k-r+1}). \end{aligned} \tag{3.4}$$

Setting

$$F_r = (k-1)\ldots(k-r+1)\left[\frac{\partial}{\partial x}\left(x(1+x^2B)^{r-1}\right)A + x(1+x^2B)^{r-1}\frac{\partial A}{\partial x} - (k-r)x^2(1+x^2B)^{r-1}AB\right],$$

for $r > 1$, and

$$F_1 = A + x\frac{\partial A}{\partial x} - (k-1)x^2AB,$$

(3.4) reduces to $X \circ Y^r(z) = y^{k-r}F_r + O(y^{k-r+1})$ which for $r = k$ gives

$$X \circ Y^k(z) = F_k + O(y). \tag{3.5}$$

Next we get

$$Y^s \circ X \circ Y^r(z) = (k-r)\ldots(k-r-s+1)(1+x^2B)^s y^{k-r-s}F_r + O(y^{k-r-s+1})$$

which finally gives

$$Y^{k-1-l} \circ X \circ Y^{l+1}(z) = (k-l-1)!(1+x^2B)^{k-1-l} F_{l+1} + O(y). \tag{3.6}$$

At the end we need to know $Y^k \circ X(z) = -Y^k(y^k A)$. After computations we obtain

$$Y^k(y^k A) = -k!(1+x^2B)^k A + O(y). \tag{3.7}$$

Suppose that A does not vanish identically on S. Then there exists a z such that $A(x,0,z) = a_m(z)x^m + O(x^{m+1})$ as $x \longrightarrow 0$, where $a_m(z) \neq 0$. Now on $S = \{y = 0\}$

$$F_1 = A + x\frac{\partial A}{\partial x} + O(x^{m+2}) \tag{3.8}$$

and

$$F_l = (k-1)\dots(k-l+1)\left(A + x\frac{\partial A}{\partial x}\right) + O(x^{m+2}), \tag{3.9}$$

where A and $\frac{\partial A}{\partial x}$ should be evaluated at $(x,0,z)$. Using (3.6), (3.9) we get

$$\sum_{l=0}^{k-2}(-1)^l \binom{k}{l+1} Y^{k-1-l} \circ X \circ Y^{l+1}(z)_{|\{y=0\}} = (k-1)!(1+(-1)^k)\left(A + x\frac{\partial A}{\partial x}\right) + O(x^{m+2})$$

(again A and $\frac{\partial A}{\partial x}$ are to be taken at $(x,0,z)$). Next (3.7) gives $Y^k \circ X(z)_{|S} = -k!A + O(x^{m+2})$ and (3.5) gives $X \circ Y^k(z)_{|S} = (k-1)!(A + x\frac{\partial A}{\partial x}) + O(x^{m+2})$. Taking all what we have said together we finally obtain that

$$\left(ad^{k-1}Y \cdot [X,Y]\right)(z)_{|S} = (-1)^{k-1}(k-1)!\left(A + x\frac{\partial A}{\partial x}\right) + k!A + (k-1)!(1+(-1)^k)\left(A + x\frac{\partial A}{\partial x}\right) + O(x^{m+2}). \tag{3.10}$$

Thus $\left(ad^{k-1}Y \cdot [X,Y]\right)(z)_{|S} = 0$ is equivalent to

$$\left((-1)^{k-1} + k + 1 + (-1)^k\right)A + \left((-1)^{k-1} + 1 + (-1)^k\right)x\frac{\partial A}{\partial x} + O(x^{m+2}) = 0$$

which in turn gives

$$(k+m+1)\, a_m(z) = 0$$

contradicting the assumption that $a_m(z) \neq 0$. It follows that $A_{|S} = 0$ identically, therefore A may be replaced in (3.3) by yA, for some other analytic function A, leading to normal forms

$$X = \frac{\partial}{\partial x} - yB\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - y^{k+1}A\frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y} + xB\left(y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + xy^k A\frac{\partial}{\partial z}\cdot$$

The proof of the proposition is over. □

Suppose that (H,g) is a generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type. By Proposition 3.2 we can assume that it is already transformed to the form (3.3). To obtain (1.4) we notice that, using considerations from the proof of the previous proposition, we must have

$$Span\left\{X(q), Y(q), \left(ad^{k-1}Y \cdot [X,Y]\right)(q)\right\} = T_q\mathbb{R}^3$$

for every $q \in S$. By (3.10) it is enough to have

$$(k-1)!\left((-1)^{k-1} + k + 1 + (-1)^k\right)A(0,0,z) = (k-1)!\,(k+1)\,A(0,0,z) \neq 0.$$

To finish the proof of Theorem 1.3 we normalize the z-axis as in [9], *i.e.* we consider the change of coordinates $x \longrightarrow x$, $y \longrightarrow y$, $z \longrightarrow \alpha(z)z$ with α satisfying the equation

$$\alpha(z) + z\frac{\mathrm{d}\alpha}{\mathrm{d}z} = \frac{1}{2A(0,0,z)}.$$

In this way, in the new coordinates, we have $A(0,0,z) = \frac{1}{2}$ for every $(0,0,z) \in U$. Now it suffices to define $\varphi = -B$, $\psi = 2A - 1$.

At the end let us note that it would be good to know if φ and ψ are invariants of the sub-Lorentzian structure. This is however not clear to the author.

4. Reachable sets in the general case

In this section we will prove Theorems 1.4 and 1.5. The proof is based in Propositions 2.1–2.9 and on normal forms constructed in the previous section. Here and below we assume that X and Y are given in the normal form (1.4), $k > 2$, while $\hat{X}$ and $\hat{Y}$ keep the same meaning as in (2.1).

Consider the system $\dot{q} = X + uY$, $u \in [a,b]$, on a normal neighbourhood U of the origin. Throughout the proof we will suppose that U is as small as is needed for our purposes. By (H,g) we denote the time-oriented sub-Lorentzian structure given on U by declaring X, Y to be an orthonormal frame with a time orientation X; $(H, g^{a,b})$, $Z^{a,b}$ and $W^{a,b}$ are given by (1.2). The horizontal gradient of a function f with respect to the structure $(H, g^{a,b})$ will be denoted by $\nabla_{H^{a,b}} f$. We slightly correct the definition of hypersurfaces $\mathit{\Gamma}_i^{a,b}$, $i = 1, 2$:

$$\mathit{\Gamma}_1^{a,b} = \{(x, bx, z) : \ x, z \in \mathbb{R}\} \cap U$$
$$\mathit{\Gamma}_2^{a,b} = \{(x, ax, z) : \ x, z \in \mathbb{R}\} \cap U.$$

Note that (2.3) and (2.4) carry over to the general case with a small modification:

$$\mathcal{A}_{[a,b]}(0,U) \subset \{(x,y,z) : \ ax \le y \le bx\} \cap U, \tag{4.1}$$
$$\mathcal{A}_{(a,b)}(0,U) \subset \{(x,y,z) : \ ax < y < bx\} \cap U. \tag{4.2}$$

4.1. The case $k = 2l + 1$ and $a < b$ arbitrary or $k = 2l$ and either $0 \le a < b$ or $a < b \le 0$

We will state the exact proof only for the case where k is odd and $a < 0 < b$. All other cases contained in the hypothesis of Theorem 1.4 can be treated in the same manner, the only difference being the formulas of functions appearing there.

The procedure we are going to use is almost the same as in the proof of Proposition 2.1 (the only difference is a local character of considerations here. So, for instance, the operator ∂ should be replaced by $\tilde{\partial}$). Since, like in the flat case, the z-coordinate of $Z^{a,b} - W^{a,b} = X + aY$ is positive, and the z-coordinate of $Z^{a,b} + W^{a,b} = X + bY$ is negative, both in the region $\{ax < y < bx\} \cap U$, we construct two analytic functions $\eta_i^{a,b}$, $i = 1, 2$ as follows. $\eta_1^{a,b}$ is the solution to the Cauchy problem

$$(Z^{a,b} - W^{a,b})(\eta) = (X + aY)(\eta) = 0, \quad \eta_{|\mathit{\Gamma}_1^{a,b}}(x, bx, z) = z$$

while $\eta_2^{a,b}$ is the solution to the Cauchy problem

$$(Z^{a,b} + W^{a,b})(\eta) = (X + bY)(\eta) = 0, \quad \eta_{|\mathit{\Gamma}_2^{a,b}}(x, ax, z) = -z.$$

Let us write $X = \hat{X} + X_1$, $\hat{Y} = Y + Y_1$. Setting $\eta_i^{a,b} = \hat{\eta}_i^{a,b} + R_i$, $i = 1, 2$, we see that finding *e.g.* $\eta_1^{a,b}$ is equivalent to solving the following Cauchy problem

$$(X + aY)(R_1) = -(X_1 + aY_1)(\hat{\eta}_1^{a,b}), \quad R_{1|\mathit{\Gamma}_1^{a,b}}(x, bx, z) = 0.$$

We deduce that $R_1 = O(r^{k+2})$ where $r = \sqrt{x^2+y^2+z^2}$, and similarly $R_2 = O(r^{k+2})$. Since $\hat{\eta}_i^{a,b} = \pm z + O(r^{k+1})$, $\eta_i^{a,b}$ can be viewed as a perturbation of $\hat{\eta}_i^{a,b}$, $i=1,2$.

Next, let us notice that

$$(X+cY)_{|y=cx} = \frac{\partial}{\partial x} + c\frac{\partial}{\partial y} \tag{4.3}$$

for any real number c. Now $\nabla_{H^{a,b}}\eta_1^{a,b} = -Z^{a,b}(\eta_1^{a,b})(Z^{a,b}-W^{a,b})$ where

$$Z^{a,b}(\eta_1^{a,b}) = \frac{1}{2}(Z^{a,b}-W^{a,b}+Z^{a,b}+W^{a,b})(\eta_1^{a,b}) = \frac{1}{2}(Z^{a,b}+W^{a,b})(\eta_1^{a,b}) = \frac{1}{2}(X+bY)(\eta_1^{a,b}).$$

$X+bY_{|\varGamma_1^{a,b}}$ is tangent to $\varGamma_1^{a,b}$ by (4.3), and the trajectories of $X+bY_{|\varGamma_1^{a,b}}$ are, geometrically, lines $\varGamma_1^{a,b}\cap\{z=\text{const.}\}$. Therefore, by definition of $\eta_1^{a,b}$, $Z^{a,b}(\eta_1^{a,b})_{|\varGamma_1^{a,b}} = 0$ which means that $\nabla_{H^{a,b}}\eta_1^{a,b}$ is divisible by $y-bx$. Now

$$Z^{a,b}(\eta_1^{a,b}) = Z^{a,b}(\hat{\eta}_1^{a,b}+R_1) = \hat{Z}^{a,b}(\hat{\eta}_1^{a,b}) + Z_1^{a,b}(\hat{\eta}_1^{a,b}) + Z^{a,b}(R_1), \tag{4.4}$$

where $Z_1^{a,b} = X_1 + \frac{1}{2}(b+a)Y_1$, $Z_1^{a,b}(\hat{\eta}_1^{a,b}) + Z^{a,b}(R_1) = O(r^{k+1})$ and $\hat{Z}^{a,b}(\hat{\eta}_1^{a,b})$ is known from (2.11). In this way we are led to

$$Z^{a,b}(\eta_1^{a,b}) = \frac{k+1}{4k}(y-bx)\left(y^{k-1} + y^{k-2}\frac{b\,(ax-y)}{a-b} + \ldots + y\left(\frac{b\,(ax-y)}{a-b}\right)^{k-2} + \left(\frac{b\,(ax-y)}{a-b}\right)^{k-1} + O(r^k)\right).$$

This proves that $\nabla_{H^{a,b}}\eta_1^{a,b}$ is null f.d. with respect to $(H,g^{a,b})$ in the sector $\{ax<y<bx\}\cap U$. In the same way we make sure that $\nabla_{H^{a,b}}\eta_2^{a,b} = -Z^{a,b}(\eta_2^{a,b})(Z^{a,b}+W^{a,b})$ is divisible by $y-ax$ with

$$Z^{a,b}(\eta_2^{a,b}) = -\frac{k+1}{4k}(y-ax)\left(y^{k-1} + y^{k-2}\frac{a\,(bx-y)}{b-a} + \ldots + y\left(\frac{a\,(bx-y)}{b-a}\right)^{k-2} + \left(\frac{a\,(bx-y)}{b-a}\right)^{k-1} + O(r^k)\right),$$

showing that $\nabla_{H^{a,b}}\eta_2^{a,b}$ is null f.d. in $\{ax<y<bx\}\cap U$. Now we continue as in the proof of Proposition 2.1.

4.2. The case $k=2l$ and $a<0<b$

To begin with let us note that, similarly as in the flat case described in Section 2.2, there hold two facts. First of all, the abnormal curve $t \longrightarrow (t,0,0)$ starting at the origin is geometrically optimal - this follows from considerations below. Secondly, the z-coordinate of $X+bY$ and that of $X+aY$ have exactly the same signs as z-coordinates of $\hat{X}+b\hat{Y}$ and $\hat{X}+a\hat{Y}$ in the corresponding areas intersected with U. Therefore we proceed as in [9], *i.e.* we construct four analytic functions $\eta_1^{a,b},\ldots,\eta_4^{a,b}$ as follows. $\eta_1^{a,b}$ is the solution to the Cauchy problem

$$(X+aY)(\eta) = 0, \quad \eta_{|\varGamma_1^{a,b}}(x,bx,z) = z,$$

$\eta_2^{a,b}$ is the solution to the Cauchy problem

$$(X+bY)(\eta) = 0, \quad \eta_{|\varGamma_2^{a,b}}(x,ax,z) = z,$$

$\eta_3^{a,b}$ is the solution to the Cauchy problem

$$(X+bY)(\eta) = 0, \quad \eta_{|S}(x,0,z) = -z, \tag{4.5}$$

and finally $\eta_4^{a,b}$ is the solution to the Cauchy problem

$$(X+aY)(\eta) = 0, \quad \eta_{|S}(x,0,z) = -z.$$

As in Section 4.1, $\eta_i^{a,b} = \hat{\eta}_i^{a,b} + R_i$, where $R_i = O(r^{k+2})$, so $\eta_i^{a,b}$ can be regarded as a perturbation of $\hat{\eta}_i^{a,b}$, $i = 1, \ldots, 4$. Similarly as above we prove that $\nabla_{H^{a,b}}\eta_1^{a,b}$ (resp. $\nabla_{H^{a,b}}\eta_2^{a,b}$) is divisible by $y - bx$ (resp. by $y - ax$). More precisely, making use of (2.27), (2.28) (*cf.* also (4.4)) we obtain

$$Z^{a,b}(\eta_1^{a,b}) = \frac{k+1}{4k\,(a-b)}\,(y - bx)\left[((a-b)\,y + b\,(ax - y))\,G_1 + O(r^k)\right] \tag{4.6}$$

with G_1 as in (2.23), and

$$Z^{a,b}(\eta_2^{a,b}) = -\frac{k+1}{4k\,(a-b)}\,(ax - y)\left[((a-b)\,y - a\,(bx - y))\,G_2 + O(r^k)\right] \tag{4.7}$$

with G_2 defined in (2.24).

Next we will show that $\nabla_{H^{a,b}}\eta_i^{a,b}$, $i = 3, 4$, is divisible by y^2. In order to do it, first we prove

Lemma 4.1. *There exist analytic functions ζ_3, ζ_4 on U such that $\eta_3^{a,b}(x,y,z) = -z + y^k\zeta_3$ and $\eta_4^{a,b}(x,y,z) = -z + y^k\zeta_4$ on U.*

Proof. We will state the proof for $\eta_3^{a,b}$ only. Clearly, by (4.5), $\eta_3^{a,b} = -z + y\xi$ with an analytic function ξ on U. Suppose that $\eta_3^{a,b}$ can be written in the form $\eta_3^{a,b} = -z + y^l\xi$ where $l < k$ and ξ is analytic on U. Again by (4.5), $(X + bY)(-z + y^l\xi) = \left(xy\varphi + b(1 - x^2\varphi)\right)\left(ly^{l-1}\xi + y^l\frac{\partial\xi}{\partial y}\right) + O(y^l)$ vanishes identically on U. Taking this into account we get

$$\frac{\partial^{l-1}}{\partial y^{l-1}}(X + bY)(-z + y^l\xi) = l!b(1 - x^2\varphi)\xi + O(y) = 0.$$

Since $1 - x^2\varphi$ does not vanish on U, we are led to $\xi|_{y=0} = 0$. Thus ξ may be replaced by $y\xi$ with some other analytic ξ on U, and we obtain $\eta_3^{a,b} = -z + y^{l+1}\xi$. The induction principle yields the existence of an analytic function ζ_3 such that $\eta_3^{a,b} = -z + y^k\zeta_3$ on U. □

Using (2.29), (2.30) we easily come to

$$Z^{a,b}(\hat{\eta}_3^{a,b}) = \frac{k+1}{4kb}\,(a-b)\,y^2\left(y^{k-2} + O(r^{k-1})\right), \tag{4.8}$$

$$Z^{a,b}(\hat{\eta}_4^{a,b}) = -\frac{k+1}{4ka}\,(a-b)\,y^2\left(y^{k-2} + O(r^{k-1})\right), \tag{4.9}$$

and indeed $\nabla_{H^{a,b}}\eta_3^{a,b}$ and $\nabla_{H^{a,b}}\eta_4^{a,b}$ are divisible by y^2.

Taking (4.6)–(4.9) together and applying Lemma 2.10 we conclude that there exists a sufficiently small $\varepsilon > 0$ with $\frac{ab}{2b-a} + \varepsilon < 0$ and $\frac{|a|b}{b-2a} - \varepsilon > 0$, such that $\nabla_{H^{a,b}}\eta_1^{a,b}$ is null f.d. on $\left\{\left(\frac{ab}{2b-a} + \varepsilon\right)x < y < bx\right\} \cap U$, $\nabla_{H^{a,b}}\eta_2^{a,b}$ is null f.d. on $\left\{ax < y < \left(\frac{|a|b}{b-2a} - \varepsilon\right)x\right\} \cap U$, while $\nabla_{H^{a,b}}\eta_3^{a,b}$ and $\nabla_{H^{a,b}}\eta_4^{a,b}$ are null f.d. on $\{y \neq 0\} \cap U$. Further, by (2.31) and (2.32), maybe for a smaller $\varepsilon > 0$, we have $\eta_1^{a,b} < \eta_2^{a,b}$, *i.e.* $\left\{\eta_2^{a,b} \leq 0\right\} \subset \left\{\eta_1^{a,b} \leq 0\right\}$ on $\{\varepsilon x < y < bx\} \cap U$, and $\eta_1^{a,b} > \eta_2^{a,b}$, *i.e.* $\left\{\eta_1^{a,b} \leq 0\right\} \subset \left\{\eta_2^{a,b} \leq 0\right\}$ on $\{ax < y < -\varepsilon x\} \cap U$.

Consider a semi-analytic set

$$Z = \left\{\eta_1^{a,b} = \eta_2^{a,b} = 0\right\} \cap \{-\varepsilon x \leq y \leq \varepsilon x\} \cap U$$

(see [13] for definition and properties of semi-analytic sets). Clearly, $\dim Z = 1$ which can be seen by observing that $\nabla_{H^{a,b}}\eta_1^{a,b}$ and $\nabla_{H^{a,b}}\eta_2^{a,b}$ are linearly independent away from zero on $\{-\varepsilon x \leq y \leq \varepsilon x\}$. Also Z is made up of a single analytic curve entering zero which follows from $(Z^{a,b} - W^{a,b})(\hat{\eta}_2^{a,b} - \hat{\eta}_1^{a,b}) = (Z^{a,b} - W^{a,b} + Z^{a,b} + W^{a,b})(\hat{\eta}_2^{a,b}) = 2Z^{a,b}(\hat{\eta}_2^{a,b}) < 0$ on $\{-\varepsilon x \leq y \leq \varepsilon x\}$.

Finally let us define a semi-analytic set Σ by the formula $\Sigma = \rho^{-1}(\rho(Z)) \cap U$, where $\rho : \mathbb{R}^3 \ni (x, y, z) \longrightarrow (x, y) \in \mathbb{R}^2$ is the projection onto the (x, y)-plane. Now we can apply exactly the same reasoning as in [9] to finish the proof of Theorem 1.5. We will not repeat arguments from [9] here, however, for further use, we will explicitly indicate geometrically optimal curves initiating at zero. These are up to reparameterization (i) concatenations of a segment of the trajectory of $X + bY$ starting from zero with a segment of a trajectory of $X + aY$, (ii) concatenations of a segment of the trajectory of $X + aY$ starting from zero with a segment of a trajectory of $X + bY$, (iii) concatenations of a segment of the trajectory of X starting from zero with a segment of a trajectory of $X + bY$, (iv) concatenations of a segment of the trajectory of X starting from zero with a segment of a trajectory of $X + aY$.

Remark 4.2. Remark that similarly as in the process of constructing nilpotent approximations (*cf. e.g.* [3]), we could introduce, for structures of order k, weights for coordinates in the following way: $weight(x) = weight(y) = 1$, $weight(z) = k+1$. Then our functions $\hat{\eta}_i^{a,b}$ become homogeneous polynomials of order $k+1$, and the structure given by $\hat{X}, \hat{Y}$ as in (2.1) is the nilpotent approximation of the structure defined by X, Y, where X, Y are as in (1.4).

Remark 4.3. Let us remark that a key object to study, when computing reachable sets for the system $\dot{q} = X+uY$, $u \in [a, b]$, is the horizontal gradient with respect to the corresponding sub-Lorentzian structure $(H, g^{a,b})$. Rewriting this gradient in terms of our affine control system, we obtain the field which we could denote by ∇_{aff}, call it *the affine gradient*, and which is defined as

$$\nabla_{aff}(f) = -(X(f) + \frac{1}{2}(a + b)Y(f))X - \left(\frac{1}{2}(a + b)X(f) + abY(f)\right)Y \tag{4.10}$$

for a smooth function f. The reachable sets $\mathcal{A}_{[a,b]}(0, U)$ (resp. $\mathcal{A}_{(a,b)}(0, U)$, $\mathcal{A}_{\{a,b\}}(0, U)$) are described by such functions f for which $\nabla_{aff}(f)$ is parallel either to $X + aY$ or to $X + bY$.

Remark 4.4. Suppose we are given the system (1.1) where X, Y define a generalized Martinet sub-Lorentzian structure of Hamiltonian type, and X is tangent to the Martinet surface. Having proved Theorems 1.3–1.5, we do not have to transform our system to normal forms (1.4) in order to compute its reachable sets from a point q_0 lying on the Martinet surface (if $[a, b] = [-1, 1]$ we also do not have to assume that X is tangent to the Martinet surface). We can proceed as follows. We consider only the case where k is even and $a < 0 < b$. All other remaining cases are treated similarly as the contact case – see [8]. Let γ_1, γ_2 be the two null f.d. Hamiltonian geodesics with respect to the structure $(H, g^{a,b})$ starting at q_0. Let moreover S denote the Martinet surface and γ be the abnormal curve starting from q_0. Further we chose two analytic hypersurfaces M_1, M_2 such that M_i contains γ_i and is transverse to γ_j, $i \neq j$, $i, j = 1, 2$. Next we will chose three analytic functions $\zeta_i \in C^\omega(M_i)$, $i = 1, 2$, $\zeta \in C^\omega(S)$. First of all we impose the condition $\zeta_i^{-1}(0) = \gamma_i$, $i = 1, 2$, $\zeta^{-1}(0) = \gamma$. Now we will determine the signs of these functions. Consider the trajectories of $X + aY$ (resp. of $-X - aY$) staring from γ_1, and then project them onto M_1; we assume that the obtained curves enter the region $\{\zeta_1 > 0\}$ (resp. the region $\{\zeta_1 < 0\}$). Note that this definition makes sense, provided that the mentioned trajectories are considered in a sufficiently small neighbourhood of M_1. Later consider the trajectories of $X + bY$ (resp. of $-X - bY$) staring from γ_2, and then project them onto M_2; ζ_2 is chosen in such a way that the projected curves enter the region $\{\zeta_2 > 0\}$ (resp. $\{\zeta_2 < 0\}$). Again this makes sense if the considered trajectories stay in a sufficiently small neighbourhood of M_2. Finally, project onto S sufficiently short pieces of the trajectories of $X + aY$ (resp. $-X - aY$) starting from γ and assume that the projected curves enter the region $\{\zeta > 0\}$ (resp. $\{\zeta < 0\}$). Note that in the latter case we could also use the field $X + bY$, resulting in the same function ζ. Now we build four functions describing the reachable set: η_1 is the solution to the Cauchy problem $(X + aY)(\eta) = 0$, $\eta_{|M_1} = \zeta_1$, η_2 is the solution to the problem $(X + bY)(\eta) = 0$, $\eta_{|M_2} = \zeta_2$, η_3 is the solution to the problem $(X + bY)(\eta) = 0$, $\eta_{|S} = \zeta$, and η_4 is the solution to $(X + aY)(\eta) = 0$, $\eta_{|S} = \zeta$. Of course, during the whole process we chose all the data M_i, ζ_i, ζ in the simplest possible way to make the calculations the easiest.

Remark 4.5. Note that even when functions η_i's cannot be obtained explicitly, knowing the fields X, Y and using computer methods, we can find as many terms in the Taylor expansions of η_i as we need.

5. Some further results

In this section we state and prove some corollaries of Theorems 1.3–1.5.

5.1. Image under the exponential mapping

Let (M, H, g) be a sub-Lorentzian manifold. Recall that by $\mathcal{H}$ we agreed to denote the geodesic Hamiltonian induced by the structure (H, g). Let Φ_t stand for the (local) flow of $\overrightarrow{\mathcal{H}}$. Fix a point $q_0 \in M$ and consider the set $\mathcal{D}_{q_0} = \left\{\lambda \in T^*_{q_0}M : \quad t \longrightarrow \Phi_t(\lambda) \text{ is defined on } [0,1]\right\}$. $\mathcal{D}_{q_0}$ is nonempty and open. We define *the exponential mapping* with the pole at q_0:

$$\exp_{q_0} : \mathcal{D}_{q_0} \longrightarrow \mathbb{R}, \quad \exp_{q_0}(\lambda) = \pi \circ \Phi_1(\lambda).$$

Obviously, $\exp_{q_0}$ is smooth (analytic) whenever (M, H, g) is smooth (analytic).

Let (H, g) be a generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type defined on a neighbourhood U of a point q_0. Fix two numbers $a, b,\ a < b$. Now, by $\exp_{q_0}$ we will denote the exponential mapping with the pole at q_0 for the structure $(H, g^{a,b})$ defined in Section 4; by $\mathcal{D}_{q_0}$ we denote its domain Further, we shall write $\mathcal{H}$ for the geodesic Hamiltonian associated with $(H, g^{a,b})$. Using the similar reasoning based on the Pontriagin maximum principle as in [9] we can prove the following proposition.

Proposition 5.1. *Let U be a sufficiently small normal neighbourhood of a point q_0 belonging to the Martinet surface. Then*

$$\exp_{q_0}\left(\left\{\lambda \in \mathcal{D}_{q_0} :\ \mathcal{H}(\lambda) < 0,\ \left\langle \lambda, Z^{a,b}(q_0)\right\rangle < 0\right\}\right) \cap U = \mathcal{A}_{(a,b)}(q_0, U),$$

$$\exp_{q_0}\left(\left\{\lambda \in \mathcal{D}_{q_0} :\ \mathcal{H}(\lambda) \leq 0,\ \left\langle \lambda, Z^{a,b}(q_0)\right\rangle < 0\right\}\right) \cap U = \mathcal{A}_{(a,b)}(q_0, U) \cup \sigma_1 \cup \sigma_2,$$

where σ_1, σ_2 are the two null f.d. Hamiltonian geodesics initiating at q_0.

It follows that the set of all points that are not accessible along Hamiltonian maximizers is equal to $\tilde{\partial}\mathcal{A}_{(a,b)}(0, U) \setminus (\sigma_1 \cup \sigma_2)$ in case $k = 2l + 1$ and $a < b$ arbitrary or $k = 2l$ and either $0 \leq a < b$ or $a < b \leq 0$, and to $\tilde{\partial}\mathcal{A}_{(a,b)}(0, U) \setminus (\sigma_1 \cup \sigma_2 \cup \gamma)$ in case $k = 2l$ and $a < 0 < b$, γ being the abnormal curve for H starting from q_0.

Thus the timelike reachable set for generalized Martinet sub-Lorentzian structures is accessible by Hamiltonian geodesics, and it would be interesting to know if (locally) the uniqueness of geodesics holds. So far the uniqueness of geodesics has been established in the Heisenberg case – see [11].

5.2. Continuity of the sub-Lorentzian distance

Let $(H, g^{a,b})$, q_0, U and γ keep the same meaning as in the previous subsection. Recall [7] that $f[U] : U \longrightarrow \mathbb{R}$, *the sub-Lorentzian distance function from q_0 relative to U* for the structure $(H, g^{a,b})$, is defined by formula

$$f[U](q) = \begin{cases} \sup\left\{L(\gamma) : \gamma \in \Omega^{nspc}_{q_0,q}(U)\right\} : & q \in J^+(q_0, U) \\ 0 : & q \notin J^+(q_0, U) \end{cases},$$

where $q \in U$, $\Omega^{nspc}_{q_0,q}(U)$ stands for the set of all nspc.f.d. with respect to $(H, g^{a,b})$ curves in U joining q_0 to q, and $L(\cdot)$ is *the sub-Lorentzian length* which for a curve $\gamma : [a, b] \longrightarrow M$ is defined to be

$$L(\gamma) = \int_a^b |g(\dot\gamma(t), \dot\gamma(t))|^{1/2}\, \mathrm{d}t.$$

Using [7] we easily obtain

Proposition 5.2. *Let $f[U]$ be the sub-Lorentzian distance from a point q_0 belonging to the Martinet surface. Suppose that $a < 0 < b$. If $k = 2l+1$ then $f[U]$ is continuous at points of γ, and if $k = 2l$, $f[U]$ is not continuous at points of γ.*

Thus $f[U]$ is not continuous at points of the singular trajectory starting at q_0 if this trajectory is geometrically optimal.

5.3. Conjugate locus

Let (M, H, g) be a sub-Lorentzian manifold. A point q is said to be *conjugate to a point* q_0 if there exists a covector $\lambda \in T^*_{q_0}M$ such that $\exp_{q_0}(\lambda) = q$ and $d_\lambda \exp_{q_0}$ is not of full rank. In such a situation we say that q *is conjugate to* q_0 *along a geodesic* $\gamma(t) = \exp_{q_0}(t\lambda)$. *The future null* (resp. *future timelike*) *conjugate locus of a point* q_0 is defined to be the set of all points conjugate to q_0 along null f.d. (resp. t.f.d) Hamiltonian geodesics. We will denote it by $Conj^{null}_{q_0}$ (resp. $Conj^t_{q_0}$). Similarly as in [9] we prove that

Proposition 5.3. *Let (M, H, g) be an analytic time-oriented generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type. Then the future null conjugate locus of a point q_0 for the structure $(H, g^{a,b})$ is equal to the union of the two null f.d. Hamiltonian geodesics starting at q_0.*

Let us remark that, similarly as in [9], the two null f.d. Hamiltonian geodesics starting from q_0 are unique maximizers, and the same time they are entirely contained in $Conj^{null}_{q_0}$. Also, when q_0 belongs to the Martinet surface, k is even, and $a < 0 < b$, the abnormal curve starting from q_0 is the unique maximizer which is entirely contained in $Conj^t_{q_0}$.

5.4. Future null cut locus

Let (M, H, g) be a time-oriented sub-Lorentzian manifold. Recall a definition of $Cut^{null}_{q_0}(M)$, *the (future) null cut locus* of a given point q_0. So, by definition, a point $q \in M$ belongs to $Cut^{null}_{q_0}(M)$ if there exists a null f.d. geodesic $\gamma : [0, T] \longrightarrow M$ with the following properties: $\gamma(0) = q_0$, $\gamma(t_1) = q$ where $0 < t_1 < T$, $\gamma_{|[0,t_1]}$ is a length maximizer and $\gamma_{|[0,t_1+\varepsilon]}$ is not a length maximizer for any $\varepsilon > 0$, $t_1 + \varepsilon \leq T$.

Now let (H, g) be an analytic generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type, defined by (1.4) on a neighbourhood U of the origin in $\mathbb{R}^3$. Take two numbers $a < b$ and suppose that U is a normal neighbourhood for $(H, g^{a,b})$. If $Cut^{null}_0(U)$ stands for the future null cut locus of zero for $(U, H, g^{a,b})$ then.

Proposition 5.4. *Suppose that k, a, b are as in the hypothesis of Theorem 1.4. Then $Cut^{null}_0(U) = \{0\}$.*

Proof. The situation is exactly the same as in the contact case – *cf.* [8]. Each null f.d. maximizer starting from 0 is a piecewise smooth curve with at most one corner point, where smooth pieces are segments null f.d. Hamiltonian geodesics. Such curves do not cease to be maximizing. □

Proposition 5.5. *Suppose that k, a, b satisfy the hypothesis of Theorem 1.5. Then $Cut^{null}_0(U) = \tilde{\partial}\mathcal{A}_{[a,b]}(0, U) \cap \{z > 0\} \cap \Sigma$.*

Proof. See the proof of the analogous result in [9]. Note that all geometrically optimal curves are listed in Section 4 just before Remark 4.2. □

Let us remark that in all cases the set $Cut^{null}_0(U)$ is semi-analytic. Let us also notice that $Cut^{null}_0(U) \cap Conj^{null}_0 = \varnothing$.

6. More applications of normal forms

Suppose that X, Y is an orthonormal basis, with X being a time orientation, for a generalized Martinet sub-Lorentzian structure (H, g) of order k, defined on an open set $U \subset \mathbb{R}^3$. Note that we do not assume that X is tangent to S. Let us see what we can say about reachable sets for the system

$$\dot{q} = X + uY,\ u \in [a, b]. \tag{6.1}$$

Using a Lorentz transformation we can find functions α, β, analytic on U, such that the field $\alpha(q)X(q)+\beta(q)Y(q)$ is tangent to S whenever $q \in S$. Assuming that

$$\tilde{X} = X + \beta(q)/\alpha(q)Y, \quad \tilde{Y} = Y$$

is an orthonormal basis with a time orientation equal to $\tilde{X}$ we have defined a generalized Martinet sub-Lorentzian structure on U which we will denote by $(H, \tilde{g})$ (by the definition of Lorentz transformations $\alpha \neq 0$). Now (6.1) can be rewritten in the form

$$\dot{q} = \tilde{X} - (u - \beta(q)/\alpha(q))\,\tilde{Y}.$$

Provided that $(H, \tilde{g})$ is of Hamiltonian type, we can transform $\tilde{X}$ and $\tilde{Y}$ to normal form (1.4) which can, in many cases, simplify the study of (6.1).

Suppose now that (H, g) is a sub-Lorentzian structure such that H satisfies the condition (M_k) and g is such that the field of directions $S \ni q \longrightarrow T_qS \cap H_q$ is null f.d. Let X, Y be an orthonormal basis for (H, g) with a time orientation X. We are interested in the reachable set for the system (6.1). Multiplying Y by -1 if necessary, we can assume that $X - Y$ restricted to S is tangent to S. Define a sub-Lorentzian structure $(H, \tilde{g})$ by assuming that the two fields

$$\tilde{X} = X - Y, \quad \tilde{Y} = Y$$

form an orthonormal basis for $(H, \tilde{g})$ with $\tilde{X}$ being a time orientation. We see that $(H, \tilde{g})$ is a generalized Martinet sub-Lorentzian structure of order k, and if it is of Hamiltonian type we can transform $\tilde{X}$, $\tilde{Y}$ to normal forms (1.4). Now (6.1) takes the form

$$\dot{q} = \tilde{X} + \tilde{u}\tilde{Y},\ \tilde{u} \in [a+1, b+1],$$

and its reachable set is known by above considerations.

Appendix A. Some other normal forms

Normal forms centered at a point q_0, as those constructed in Theorem 1.3, are adjusted to problems which are 'measured' by the distance from q_0. Similarly as in [9] we can construct normal forms which may be more suitable when considering problems involving, in some sense, the distance to the Martinet surface. This is done in the following

Proposition A.1. *Let (H, g) be a smooth time-oriented generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type, defined on a neighbourhood U of the origin in $\mathbb{R}^3$. Denote by S the Martinet surface for H. Then, provided that U is sufficiently small, there exist coordinates x, y, z on U in which $S = \{y = 0\}$, and (H, g) admits an orthonormal frame in the following normal form*

$$\begin{aligned} X &= (1 + ya)\frac{\partial}{\partial x} + y^k(1 + b)\frac{\partial}{\partial z} \\ Y &= \frac{\partial}{\partial y} \end{aligned} \tag{A.1}$$

with a time orientation X, where $a, b \in C^\infty(U)$, $b(0, 0, z) = 0$.

The proof is based on the following:

Lemma A.2. *Let (H, g) be a generalized Martinet sub-Lorentzian structure of order k of Hamiltonian type, $k \geq 2$, defined on a neighbourhood U of the origin in $\mathbb{R}^3$. Then, provided that U is sufficiently small, there exist coordinates x, y, z on U in which (H, g) admits an orthonormal frame in the form*

$$X = (1 + ya)\frac{\partial}{\partial x} + y^k b\frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y}, \tag{A.2}$$

where a, b are smooth functions on U, X is a time orientation, $S = \{y = 0\}$ is the Martinet surface for H, and $X_{|S}$ is tangent to S.

Proof. The proof is by induction and is quite similar to that of Proposition 3.2. The case $k = 2$ is treated in [9], equation (5.4). As the induction hypothesis we assume that any structure of order k can be transformed to (A.2). Now suppose we are given a structure (H, g) of order $k+1$, defined in a neighbourhood U of the origin. By induction hypothesis, possibly after shrinking U, we can find coordinates x, y, z on U in which (H, g) has an orthonormal frame X, Y in the form (A.2).

We will make use of the condition (M_{k+1}). In view of $H_{|S} = \ker \mathrm{d}z$, we must compute z-coordinates of successive Lie brackets of the fields X and Y. To this end notice that

$$[X, Y]\,(z) = y^{k-1}\left(kb + y\frac{\partial b}{\partial y}\right) = O(y^{k-1}),$$

$$(adX \cdot [X, Y])\,(z) = O(y^{k-1}),$$

while

$$(adY \cdot [X, Y])\,(z) = -\left(ad^2 Y \cdot X\right)(z) = O(y^{k-2})$$

as $y \longrightarrow 0$. Consequently, only computation of $\left(ad^k Y.X\right)(z)_{|S}$ can gives any nontrivial conditions. Clearly

$$\left(ad^l Y.X\right)(z) = k(k-1)\ldots(k-l+1)y^{k-l}b + O(y^{k-l+1}),$$

from which

$$\left(ad^k Y \cdot X\right)(z) = k!b + O(y). \tag{A.3}$$

By assumption $\left(ad^k Y \cdot X\right)(z)_{|y=0} = b_{|y=0}$ vanishes identically, thus b in (A.2) can be replaced by yb for some other smooth function b, therefore we are led to

$$X = (1 + ya)\frac{\partial}{\partial x} + y^{k+1}b\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}. \qquad \square$$

To end the proof of Proposition A.1 suppose we are given a generalized Martinet sub-Lorentzian structure (H, g) of order k of Hamiltonian type. By Lemma A.2 we can assume that it is given by an orthonormal frame X, Y as in (A.2). In view of (A.3), in order to have

$$Span\left\{X(q), Y(q), ad^k Y \cdot X(q)\right\} = T_q\mathbb{R}^3$$

for $q = (x, 0, z) \in S$, it is necessary that $b(x, 0, z) \neq 0$. Then we can achieve $b(0, 0, z) = 1$ by renormalizing the z-axis as in Section 3. This permits us to replace b in (A.2) by $1 + b$ for yet some other smooth b satisfying $b(0, 0, z) = 0$. The proof is over.

Appendix B. Possible generalizations

When one looks at the results presented in this paper, there appear at least two possible directions for further studies. One is to consider structures with non-smooth Martinet surfaces, and this is the objective of the next paper by the author. The other is to keep smoothness of the Martinet surface but drop the assumption (ii) in the definition of condition (M_k) in Section 1. We will make some comments on this latter case, since the situation when a given distribution is tangent to its Martinet surface in isolated points appears generically (*cf.* [10] where the authors give complete classification and list of normal forms of generic singularities of rank 2 distributions on 3-manifolds). So let H be a distribution defined on a neighbourhood of the origin in $\mathbb{R}^3$, and let M be its Martinet surface. Suppose that $T_0M = H_0$. According to [10], the field of directions $M \ni q \longrightarrow T_qM \cap H_q$ generically has only two types of singularities at 0: a focus and a saddle. We will present two examples describing the structure of reachable sets in the two indicated cases.

Example B.1. Let $H = \ker\omega$, where $\omega = \mathrm{d}z - \frac{1}{2}(z - x^2 - y^2)(y\mathrm{d}x - x\mathrm{d}y)$. We define a time-oriented Lorentzian metric g on H by supposing the frame

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y(z - x^2 - y^2)\frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x(z - x^2 - y^2)\frac{\partial}{\partial z}$$

to be orthonormal with X being a time orientation. Since $[X,Y] = \left(-z + 2x^2 + 2y^2\right)\frac{\partial}{\partial z}$, the Martinet surface in our case is equal to $M = \left\{z = 2x^2 + 2y^2\right\}$. Obviously $TM = Span\left\{\frac{\partial}{\partial x} + 4x\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + 4y\frac{\partial}{\partial z}\right\}$, hence $H_0 = T_0M$, while $H_q \cap T_qM$ is 1-dimensional for every $q \in M$, $q \neq 0$. It is easy to verify that, under suitable parameterization, the abnormal curves are trajectories of the field

$$\left(8y + x\left(2x^2 + 2y^2\right)\right)X - \left(8x - y\left(2x^2 + 2y^2\right)\right)Y \tag{B.1}$$

considered as a vector field on M. Writing equations for the abnormal curves in polar coordinates:

$$\dot{r} = 2r^3, \quad \dot{\theta} = -8$$

we see that they are spirals that enter zero for $t \longrightarrow -\infty$. It follows that 0 is a focus for (B.1) and, consequently, the system $\dot{q} = X + uY$, $|u| \leq 1$, has no singular trajectories starting from zero. As in Sections 2 and 4 we will compute null f.d. curves starting from Γ_1 and Γ_2, respectively. So the trajectories of $X - Y$ initiating at $\Gamma_1 \cap \{z = 0\}$ are of the form $\gamma(t) = (x_0 + t, x_0 - t, z(t))$ with

$$z(t) = -\frac{1}{x_0^2}\left(2x_0^4 + 4\right)\mathrm{e}^{x_0 t} + \frac{1}{x_0^2}\left(2x_0^4 + 2x_0^2t^2 + 4x_0t + 4\right) = -2x_0^3t + O\left(t^2\right).$$

Similarly, the trajectories of $X + Y$ initiating at $\Gamma_2 \cap \{z = 0\}$ are of the form $(x_0 + t, -x_0 + t, z(t))$ where this time

$$z(t) = -\frac{1}{x_0^2}\left(2x_0^4 + 4\right)\mathrm{e}^{-x_0 t} + \frac{1}{x_0^2}\left(2x_0^4 + 2x_0^2t^2 - 4x_0t + 4\right) = 2x_0^3t + O\left(t^2\right).$$

Repeating arguments from Section 2 we make sure that $J^+(0,U)$ is described by two analytic functions, where U is supposed to be sufficiently small normal neighbourhood of 0.

Example B.2. Now, let $H = \ker\omega$, where $\omega = \mathrm{d}z - \frac{1}{2}(z - xy)(y\mathrm{d}x - x\mathrm{d}y)$. A time-oriented Lorentzian metric g on H will be defined by supposing the frame

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y(z - xy)\frac{\partial}{\partial z}$$
$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x(z - xy)\frac{\partial}{\partial z}$$

to be orthonormal with respect to g with a time orientation X. This time $M = \{z = 2xy\}$ and $TM = Span\left\{\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z}\right\}$. It follows that $H_0 = T_0M$ and $\dim(H_q \cap T_qM) = 1$ for every $q \in M$, $q \neq 0$. The abnormal curves, under suitable parameterization, are trajectories of the field

$$x\,(4 + xy)\,X - y\,(4 - xy)\,Y \tag{B.2}$$

restricted to M. It is seen that the origin is a saddle for (B.2), and there is a unique t.f.d. abnormal curve starting from the origin, namely the x-axis. As in the previous sections we find null f.d. curves. So, the trajectories of $X - Y$ starting from $\Gamma_1 \cap \{z = 0\}$ are computed to be $\gamma(t) = (x_0 + t, x_0 - t, z(t))$ where

$$z\,(t) = -\frac{1}{x_0^2}\left(x_0^4 - 2\right)\mathrm{e}^{x_0 t} + \frac{1}{x_0^2}\left(x_0^4 - t^2x_0^2 - 2x_0t - 2\right) = -x_0^3t + O\left(t^2\right);$$

the trajectories of $X + Y$ starting from $\Gamma_2 \cap \{z = 0\}$ are computed to be $\gamma(t) = (x_0 + t, x_0 - t, z(t))$ with

$$z\,(t) = \frac{1}{x_0^2}\left(x_0^4 - 2\right)\mathrm{e}^{-x_0 t} + \frac{1}{x_0^2}\left(-x_0^4 + t^2x_0^2 - 2x_0t + 2\right) = -x_0^3t + O\left(t^2\right).$$

On the other hand, the trajectories of $X+Y$ initiating at $\{y = 0\}\cap\{z = 0\}$ are of the form $\gamma(t) = (x_0+t, t, z(t))$ where

$$z\,(t) = \frac{1}{x_0^2}\left(2x_0^2 - 8\right)\mathrm{e}^{-\frac{1}{2}x_0 t} + \frac{1}{x_0^2}\left(x_0^3t - 2x_0^2 + t^2x_0^2 - 4x_0t + 8\right) = \frac{1}{4}x_0^2t^2 + O\left(t^3\right),$$

and the trajectories of $X - Y$ initiating at $\{y = 0\} \cap \{z = 0\}$ are of the form $\gamma(t) = (x_0 + t, -t, z(t))$ with

$$z\,(t) = \frac{1}{x_0^2}\left(2x_0^2 + 8\right)\mathrm{e}^{\frac{1}{2}x_0 t} + \frac{1}{x_0^2}\left(-x_0^3t - 2x_0^2 - t^2x_0^2 - 4x_0t - 8\right) = \frac{1}{4}x_0^2t^2 + O\left(t^3\right).$$

Using similar reasoning as in Section 2 we deduce that $J^+(0, U)$ (U being a sufficiently small normal neighbourhood of the origin) is described by four analytic functions.

Thus, apparently, similar interrelation between geometric optimality of singular trajectories and the number of analytic functions needed for describing reachable sets also holds after dropping the condition (ii) in the generic case.

In order to obtain some general theorems, we need normal forms convenient for computations, and it seems to be a hard task. It is already not easy without a metric, as it can be again seen from [10]. The ideas used in Section 3 of the present paper are of course not applicable, since they are based on transversality of the distribution to its Martinet surface. Therefore, to generalize the results from Examples B.1, B.2 probably quite different reasoning should be used.

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