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A HÖLDER INFINITY LAPLACIAN

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Abstract. In this paper we study the limit as $p \to \infty$ of minimizers of the fractional $W^{s,p}$-norms. In particular, we prove that the limit satisfies a non-local and non-linear equation. We also prove the existence and uniqueness of solutions of the equation. Furthermore, we prove the existence of solutions in general for the corresponding inhomogeneous equation. By making strong use of the barriers in this construction, we obtain some regularity results.

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1. Introduction and main result

1.1. Setting of the problem

Let Ω be a bounded open set in $\mathbb{R}^N$. Under suitable conditions, it is well-known that if u_p minimizes the integral

$$\int_\Omega |\nabla u|^p$$

then $u_p \to u$ as $p \to \infty$ where u solves the equation

$$\Delta_\infty u = \sum_{i,j=1,\ldots,N} u_{ij} u_i u_j = 0 \quad \text{on} \quad \Omega$$

with $u_i = \frac{\partial u}{\partial x_i}$ and $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, which is usually referred to as the infinity Laplace equation. See for instance [1,5] for discussions concerning this passage to the limit. Moreover, u is known to be a local minimizer of the Lipschitz norm, *i.e.*, a Lipschitz extension. A lot of the known results concerning infinity harmonic functions and Lipschitz extensions can be found in [3]. Some explicit Lipschitz extensions can be found in [13,17], and these are in general not infinity harmonic functions. Lipschitz extensions have been given a lot of attention recently, and as possible applications one has suggested for instance image interpolation (*cf.* [8]) and brain warping (*cf.* [14]).

Keywords and phrases. Lipschitz extensions, Hölder extensions, infinity Laplacian, non-local and non-linear equations, viscosity solutions.

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In the present paper, we address the following question:

What happens if we replace the space $W^{1,p}(\Omega)$ by $W^{s,p}(\Omega)$ with $s \in (0,1)$?

We study minimizers of the functional

$$\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} \mathrm{d}x\mathrm{d}y, \tag{1.1}$$

for $\alpha \in (0,1]$. We see that this is the $W^{s,p}$-norm for $s = \alpha - N/p$, and the form of the functional suggests that in the limit we should obtain a local minimizer of the α-Hölder semi-norm. The Euler-Lagrange equation of this functional is

$$\int_{\Omega} \left| \frac{u(x)-u(y)}{|x-y|^{\alpha}} \right|^{p-1} \frac{\operatorname{sgn}(u(x)-u(y))}{|x-y|^{\alpha}} \mathrm{d}y = 0. \tag{1.2}$$

Formally, one can see that, as $p \to \infty$, this should converge to the equation

$$Lu = 0 \quad \text{in} \quad \Omega \tag{1.3}$$

with the operator

$$(Lu)(x) = \sup_{y\in\overline{\Omega},\ y\neq x} \frac{u(y)-u(x)}{|y-x|^{\alpha}} + \inf_{y\in\overline{\Omega},\ y\neq x} \frac{u(y)-u(x)}{|y-x|^{\alpha}} \quad \text{for} \quad x \in \Omega,$$

that we call the Hölder infinity laplacian. In this paper, we study the Dirichlet problem

$$\begin{cases} Lu = f \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{1.4}$$

We obtain existence and some regularity results for this problem in general. In the case $f = 0$, we are also able to obtain uniqueness and an implicit representation formula of the solution. Moreover, we prove that the solution is an optimal Hölder extension, in the sense that the Hölder seminorm in Ω is always less than or equal to the one for the boundary data given on $\partial\Omega$.

At a first glance one might believe that for $\alpha = 1$, the Hölder infinity Laplace equation is equivalent to the infinity Laplace equation. This is not the case in general. Indeed, using (ii) in Theorem 1.5 one can quite easily see that the infinity harmonic function

$$u(x) = |x_1|^{\frac{4}{3}} - |x_2|^{\frac{4}{3}},$$

found by Aronsson (*cf.* [2]), is not a solution of (1.4) for $\alpha = 1$ and

$$\Omega = \{-2 \leq x \leq 2, -1 \leq y \leq 1\}.$$

Many of these results are also valid in the case when we replace Ω by $\mathbb{R}^n$ in the sense that we consider minimizers of

$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{\alpha p}} \mathrm{d}x\mathrm{d}y,$$

with prescribed values $u = g$ in $\mathbb{R}^n \setminus \Omega$ and under some appropriate growth condition on g at infinity. Then the limiting operator will instead be

$$\sup_{y\in\mathbb{R}^n,\ y\neq x} \frac{u(y)-u(x)}{|y-x|^{\alpha}} + \inf_{y\in\mathbb{R}^n,\ y\neq x} \frac{u(y)-u(x)}{|y-x|^{\alpha}} \text{ for } x \in \Omega.$$

If $\Omega \neq \mathbb{R}^n$ this operator does not coincide with the infinity Laplace operator. Indeed, the operator above will change if we change the values of g away from Ω, which is not the case for the infinity Laplace operator. Very recently, in [6], a closely related operator has been studied. There the authors consider a non-local "tug-of-war" game, which in the limit yields an operator also producing optimal Hölder extensions. Moreover, when a parameter is chosen correctly, this operator coincides with the infinity Laplace operator.

1.2. **Main results**

In all that follows, for $\alpha \in (0,1]$, we will denote the α-Hölder semi-norm of a function f defined on $A \subset \mathbb{R}^N$ by

$$[f]_{\alpha,\Omega} = \sup_{x,y\in A, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

We also recall the notation

$$C^{0,\alpha}(A) = \{f \in C(A), \|f\|_{L^\infty(A)} + [f]_{\alpha,A} < \infty\},$$

where $C(A)$ is the set of continuous function on A.

The first main result in this paper states that what we expect actually happens when we pass to the limit $p \to \infty$, as long as the integrals make sense.

Theorem 1.1 (limit equation as $p \to \infty$)**.** *Let $\alpha \in (0,1]$ and if $\alpha = 1$ assume $N \geq 2$. Consider a bounded Lipschitz domain Ω in $\mathbb{R}^N$, and boundary data $g \in C^{0,\alpha}(\partial\Omega)$. For any $p > 2N/\alpha$, there exists a unique minimizer u_p of* (1.1) *satisfying $u = g$ on $\partial\Omega$. Moreover, as $p \to \infty$, we have $u_p \to u_\infty$ uniformly in $\overline{\Omega}$ and $u_\infty \in C^{0,\alpha}(\overline{\Omega})$ is a viscosity solution of* (1.3)*.*

Remark 1.2. The reason why we haven't treated the case $\alpha = N = 1$ is simply that the Euler-Lagrange equation (1.2) is not well defined in a pointwise sense in this case.

Remark 1.3. If $\alpha = \alpha_p \to \alpha_\infty < 1$, the proof can easily be adapted to obtain a result similar to Theorem 1.1.

Remark 1.4. The reader might wonder why the assumption that Ω is a Lipschitz domain is necessary. The reason is that we at some point need to apply a fractional version of the Sobolev embedding, which, to the authors knowledge, is known only in the case when Ω is a bounded Lipschitz domain.

More generally we can consider the inhomogeneous Dirichlet problem

$$\begin{cases} Lu = f & \text{in} \quad \Omega, \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

for which the notion of viscosity solutions is given in Definition 4.1. Then, when $f = 0$, there exists a representation formula for u.

Theorem 1.5 (existence for general f, partial uniqueness)**.** *Let $\alpha \in (0,1]$, Ω be a bounded open set, $g \in C(\partial\Omega)$ and $f \in C(\Omega) \cap L^\infty(\Omega)$.*

(i) (Existence) *Then there exists a viscosity solution $u \in C(\overline{\Omega})$ of* (1.5)*.*

(ii) (Partial uniqueness) *Assume $f = 0$. Then the viscosity solution $u \in C(\overline{\Omega})$ of (1.5) is unique and is defined implicitly by the following:*

$$u(x) = \begin{cases} g(x) & \text{if} \quad x \in \partial\Omega \\ a \quad \text{with} \quad \ell_x(a) = 0 & \text{if} \quad x \in \Omega \end{cases}, \tag{1.6}$$

where

$$\ell_x(a) = \sup_{y\in\partial\Omega} \frac{g(y)-a}{|y-x|^\alpha} + \inf_{y\in\partial\Omega} \frac{g(y)-a}{|y-x|^\alpha}.$$

Remark 1.6. The solution defined by (1.6) is the same as the Lipschitz extension introduced by Oberman in [16] for the distance $d(x,y) = |x-y|^\alpha$.

Remark 1.7. It is not clear whether the uniqueness holds for general functions f or not. For the inhomogeneous infinity Laplace equation, the uniqueness is only known to hold if f does not change sign, see [12], where also a counterexample to the uniqueness for f changing sign is provided.

Finally we are also able to obtain the following regularity results, where we use the notation

$$\mathrm{diam}\ \Omega = \sup\{|x-y|, x, y \in \Omega\}.$$

Theorem 1.8 (regularity)**.** *Let $\alpha \in (0,1]$, Ω be a bounded open set, $g \in C(\partial\Omega)$, $f \in C(\Omega) \cap L^\infty(\Omega)$ and $u \in C(\overline{\Omega})$ a viscosity solution of* (1.5)*.*

(i) *For any $K \subset\subset \Omega$ and any $0 < \beta < \alpha$*

$$[u]_{\beta,K} \le C(\alpha, \beta, \|f\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \mathrm{diam}\ \Omega, \mathrm{dist}(K, \partial\Omega)).$$

(ii) *If $g \in C^{0,\beta}(\partial\Omega)$ for $0 < \beta < \alpha$ then*

$$[u]_{\beta,\Omega} \le C(\alpha, \beta, \|f\|_{L^\infty(\Omega)}, [g]_{\beta,\partial\Omega}, \mathrm{diam}\ \Omega).$$

(iii) *Assume that $f = 0$. Then for each ball $B \subset\subset \Omega$*

$$[u]_{1,B} \le C(\alpha, ||g||_{L^\infty(\partial\Omega)}, \mathrm{diam}\ \Omega, \mathrm{dist}(B, \partial\Omega)).$$

(iv) *If $f = 0$ and $g \in C^{0,\alpha}(\partial\Omega)$ then*

$$[u]_{\alpha,\Omega} = [g]_{\alpha,\partial\Omega}.$$

Remark 1.9. Part (iv) in Theorem 1.8 shows in particular that when $f = 0$, the solution is an optimal Hölder extension of g on Ω. This is also the limit solution given by Theorem 1.1.

Remark 1.10. The uniqueness and the optimal $C^{0,\alpha}$-regularity of the solution remain open for general functions f.

Remark 1.11. Parts of Theorem 1.5 remain true when the distance $|x-y|^\alpha$ is replaced by a more general distance of the type $d(x-y)$, see Section 12.2.

2. Organization of the paper

The structure of the paper is as follows: in Section 3 we try to make ourselves familiar with the operator L and study some continuity properties of L which later, in Section 4, motivates the introduction of the notion of viscosity solutions. In Section 5 we give a representation formula of the solution in the case $f = 0$. In Section 6 we prove Theorem 1.1. In Section 7 we prove a stability result, showing that certain limits of viscosity subsolutions are again viscosity subsolutions. In Section 8 we construct barriers, that we use later in Section 9, where we prove the existence of continuous solutions *via* Perron's method. In Section 10 we prove several regularity results of the solutions. In the end we also give the proof of Theorem 1.8. In Section 11 we prove a comparison principle in the case $f = 0$. Using this we can conclude the proof of Theorem 1.5. In Section 12.2 we mention some possible generalizations of the problem and also some open questions that can be of general interest.

3. Basic properties of L

Here we present some properties of the operator L, which is clearly not well defined for all functions. Define

$$(L^+u)(x) = \sup_{y\in\overline{\Omega}, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha}, \quad (L^-u)(x) = \inf_{y\in\overline{\Omega}, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha}\cdot$$

Lemma 3.1 (half relaxed limits for L^+ and L^-). *Consider a function $u : \overline{\Omega} \to \mathbb{R}$ and also a sequence of functions $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon : \overline{\Omega} \to \mathbb{R}$ such that*

$$|u_\varepsilon - u|_{L^\infty(\overline{\Omega})} \to 0 \quad \textit{as} \quad \varepsilon \to 0.$$

(i) *If u is upper semicontinuous, then*

$$\liminf_{\varepsilon\to 0}{}_*(L^+u_\varepsilon) \geq L^+u \quad \textit{on} \quad \Omega. \tag{3.1}$$

(ii) *If u is lower semicontinuous, then*

$$\limsup_{\varepsilon\to 0}{}^*(L^-u_\varepsilon) \leq L^-u \quad \textit{on} \quad \Omega. \tag{3.2}$$

Proof of Lemma 3.1. We give the proof of (3.1). The proof of (3.2) is similar. For any $x_0 \in \Omega$ and $r > 0$, let us set

$$(L_r^+u)(x_0) = \sup_{y\in\overline{\Omega}\backslash B_r(x_0)} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha}$$

where by definition, we have

$$(L^+u)(x_0) = \lim_{r\to 0}\,(L_r^+u)(x_0) = \sup_{r>0}\,(L_r^+u)(x_0).$$

Let us now consider a sequence $(x_\varepsilon)_\varepsilon$ of points of Ω such that $x_\varepsilon \to x_0$. For ε small enough, we have $|x_\varepsilon - x_0| < r/2$, and then

$$(L^+u_\varepsilon)(x_\varepsilon) \geq (L_{r/2}^+u_\varepsilon)(x_\varepsilon) = \sup_{y\in\overline{\Omega}\backslash B_{r/2}(x_\varepsilon)} \frac{u_\varepsilon(y)-u_\varepsilon(x_\varepsilon)}{|y-x_\varepsilon|^\alpha} \geq \sup_{y\in\overline{\Omega}\backslash B_r(x_0)} \frac{u_\varepsilon(y)-u_\varepsilon(x_\varepsilon)}{|y-x_\varepsilon|^\alpha}\cdot$$

Using that $-u$ is lower semicontinuous, we see that for any $y \in \overline{\Omega}\backslash B_r(x_0)$, we have

$$\liminf_{\varepsilon\to 0}(L^+u_\varepsilon)(x_\varepsilon) \geq \frac{u(y)-u(x_0)}{|y-x_0|^\alpha}\cdot$$

This implies

$$\liminf_{\varepsilon\to 0}(L^+u_\varepsilon)(x_\varepsilon) \geq \sup_{y\in\overline{\Omega}\backslash B_r(x_0)} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha}\cdot$$

Passing to the limit $r \to 0$, we deduce

$$\liminf_{\varepsilon\to 0}(L^+u_\varepsilon)(x_\varepsilon) \geq \sup_{y\in\overline{\Omega},\ y\neq x_0} \frac{u(y)-u(x_0)}{|y-x_0|^\alpha},$$

for any sequence of points x_ε converging to x_0. This shows (3.1).

This ends the proof of the lemma. □

We then deduce immediately the following result.

Definition 3.2 (semicontinuous envelopes)**.** Consider a function $v : \overline{\Omega} \to \mathbb{R}$. Define

$$v^*(x) = \limsup_{y \to x} v(y)$$

and

$$v_*(x) = \liminf_{y \to x} v(y).$$

The functions v^* and v_* are called the upper and lower semicontinuous envelopes of v.

Definition 3.3 (semicontinuity)**.** We say that $v : \overline{\Omega} \to \mathbb{R}$ is upper semicontinuous (respectively lower semicontinuous) if $v^* = v$ (resp. $v_* = v$).

Corollary 3.4 (semicontinuity for L^+ and L^-)**.** *Consider a function $u : \overline{\Omega} \to \mathbb{R}$.*

(i) *If u is upper semicontinuous, then*

$$(L^+u)_* = L^+u \quad on \quad \Omega. \tag{3.3}$$

(ii) *If u is lower semicontinuous, then*

$$(L^-u)^* = L^-u \quad on \quad \Omega. \tag{3.4}$$

The following lemma motivates our choice of test functions when we later will define viscosity solutions.

Lemma 3.5 (continuity of $L^\pm\varphi$)**.** *Let $\varphi \in C^1(\Omega)$. Then $L^\pm\varphi \in C(\Omega)$.*

Proof of Lemma 3.5. We only do the proof for $L^+\varphi$, the result for $L^-\varphi$ following from the equality $L^-\varphi = -L^+(-\varphi)$. Take $x_0 \in \Omega$.

Case (i): $\alpha \in (0, 1)$

Then for δ small there exists a constant $C > 0$ such that

$$|\varphi(y) - \varphi(x)| \le C|y - x| \quad \text{for all} \quad x, y \in B_\delta(x_0) \subset \Omega.$$

We recall the definition for $r > 0$ of the operator for $x \in B_{\delta/2}(x_0)$

$$(L_r^+\varphi)(x) = \sup_{y \in \overline{\Omega} \setminus B_r(x)} \frac{\varphi(y) - \varphi(x)}{|y - x|^\alpha}\cdot$$

On one hand, by the continuity of φ, we see that $L_r^+\varphi$ is continuous on Ω. On the other hand, we have for $r < \delta/2$

$$|(L^+\varphi)(x) - (L_r^+\varphi)(x)| \le \sup_{y \in \Omega \cap B_r(x_0),\ y \ne x} \frac{|\varphi(y) - \varphi(x)|}{|y - x|^\alpha} \le Cr^{1-\alpha},$$

which shows that the family $L_r^+\varphi$ of functions converges uniformly to $L\varphi$ as $r \to 0$ on $B_{\delta/2}(x_0)$. This implies that $L^+\varphi$ is continuous.

Case (ii): $\alpha = 1$

Fix $\delta > 0$ such that $B_\delta(x_0) \subset \Omega$. Then there exists a modulus of continuity ω such that

$$|\nabla\varphi(y) - \nabla\varphi(x)| \le \omega(|y - x|) \quad \text{for all} \quad x, y \in B_\delta(x_0).$$

Using simply the formula for all $x, y \in B_\delta(x_0)$

$$\varphi(y) - \varphi(x) = \int_0^1 \mathrm{d}t\ \nabla\varphi(x + t(y - x)) \cdot (y - x),$$

we see that if furthermore $y \neq x$, then

$$\left| \frac{\varphi(y) - \varphi(x)}{|y - x|} - \nabla\varphi(x) \cdot \frac{y - x}{|y - x|} \right| \leq \omega(|y - x|). \tag{3.5}$$

In particular if $x \in \overline{B_{\delta/2}(x_0)}$, and $r \in (0, \delta/2)$, then $B_r(x) \subset B_\delta(x_0) \subset \Omega$ and

$$0 \leq \sup_{y \in \overline{\Omega} \cap B_r(x),\ y \neq x} \frac{\varphi(y) - \varphi(x)}{|y - x|} - |\nabla\varphi(x)| \leq \omega(r).$$

Remark that

$$(L^+\varphi)(x) = \max\left((L_r^+\varphi)(x), \quad \sup_{y \in \overline{\Omega} \cap B_r(x),\ y \neq x} \frac{\varphi(y) - \varphi(x)}{|y - x|} \right). \tag{3.6}$$

Now we have

$$|(L^+\varphi)(x) - (L^+\varphi)(x_0)| \leq |(L_r^+\varphi)(x) - (L_r^+\varphi)(x_0)| + ||\nabla\varphi(x)| - |\nabla\varphi(x_0)|| + 2\omega(r).$$

From the continuity of $L_r^+\varphi$ and $\nabla\varphi$, we deduce that

$$\limsup_{x \to x_0} |(L^+\varphi)(x) - (L^+\varphi)(x_0)| \leq 2\omega(r).$$

Choosing $r \to 0$, we deduce that

$$\limsup_{x \to x_0} |(L^+\varphi)(x) - (L^+\varphi)(x_0)| \leq 0,$$

and then $L^+\varphi$ is continuous at all points $x_0 \in \Omega$.

This ends the proof of the lemma. □

4. Notion of viscosity solutions

We have seen how L behaves when applied to sufficiently regular functions and we are now ready to introduce the notion of viscosity solutions. This notion follows the usual way of defining viscosity solutions. For a tour on the theory of viscosity solutions see [9]. For further reading on viscosity solutions of non-local operators, one can for instance consult [4].

Let

$$Lu = L^+u + L^-u$$

when it is well defined, which indeed is the case for $u \in C^1(\Omega)$. We wish to study

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega \end{cases} \tag{4.1}$$

with $f \in C(\Omega)$ and $g \in C(\partial\Omega)$.

Definition 4.1 (viscosity sub/super/solution)**.** Let $\alpha \in (0, 1]$ and $f \in C(\Omega)$.

We say that u is a subsolution (resp. supersolution) of (4.1) if u is an upper semicontinuous (resp. lower semicontinuous) function from $\overline{\Omega}$ to $\mathbb{R}$ such that

(i) $u \leq g$ (resp. $u \geq g$) on $\partial\Omega$

(ii) for any test function $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ satisfying

$$u \leq \varphi \quad \text{on} \quad \overline{\Omega} \text{ (resp. } u \geq \varphi)$$

and $u(x_0) = \varphi(x_0)$ for some $x_0 \in \Omega$, then

$$(L\varphi)(x_0) \geq f(x_0) \text{ (resp. } (L\varphi)(x_0) \leq f(x_0)).$$

A function $u : \overline{\Omega} \to \mathbb{R}$ is a viscosity solution of (4.1), if and only if u^* is a subsolution and u_* is a supersolution.

We will say that a function $u : \overline{\Omega} :\to \mathbb{R}$ is a solution (resp sub- or supersolution) of (4.1) in Ω if u only satisfies condition (ii) in Definition 4.1.

Remark 4.2. We see that this definition make sense intuitively, since if $u \in C^1(\Omega)$ and $\varphi \in C^1(\Omega) \cap \overline{\Omega}$ touches u from above at x_0, we would indeed have

$$(L\varphi)(x_0) \geq (Lu)(x_0).$$

5. A representation formula

In the homogeneous case, *i.e.*, when $f = 0$, one can obtain an implicit representation of the solution, as presented in the following lemma.

Lemma 5.1 (representation formula when $f = 0$)**.** *Let Ω be a bounded open set, $g \in C(\partial\Omega)$. Define for $x \in \Omega$ the non-increasing (in a) functions*

$$\ell_x^+(a) = \sup_{y \in \partial\Omega} \frac{g(y) - a}{|y - x|^\alpha}, \quad \ell_x^-(a) = \inf_{y \in \partial\Omega} \frac{g(y) - a}{|y - x|^\alpha}, \text{ and } \quad \ell_x(a) = \ell_x^+(a) + \ell_x^-(a).$$

Then the function u defined by

$$u(x) = \begin{cases} g(x) & \text{if} \quad x \in \partial\Omega \\ a \quad \text{with} \quad \ell_x(a) = 0 & \text{if} \quad x \in \Omega \end{cases}$$

is a solution of (4.1) which is continuous on $\overline{\Omega}$. Moreover, we have for all balls $B \subset\subset \Omega$, the estimate

$$[u]_{1,B} \leq C(\alpha, ||g||_{L^\infty(\partial\Omega)}, \text{diam } \Omega, \text{dist}(B, \partial\Omega)). \tag{5.1}$$

Before giving the proof of Lemma 5.1, we need the result below.

Lemma 5.2 ($|\cdot|^\alpha$ is a distance)**.** *For $\alpha \in (0, 1]$, the function $|\cdot|^\alpha$ is a distance,* i.e.,

$$|a + b|^\alpha \leq |a|^\alpha + |b|^\alpha.$$

Proof of Lemma 5.2. The lemma follows from the observation that the function $f(r) = r^\alpha$ for $r \geq 0$ is concave and non-decreasing. □

Proof of Lemma 5.1. We follow the ideas in [16]. From the definition of u, we deduce that

$$\inf_{\partial\Omega} g \leq u(x) \leq \sup_{\partial\Omega} g \quad \text{for all} \quad x \in \Omega$$

and then

$$L_x^- := \ell_x^-(u(x)) \leq 0 \leq \ell_x^+(u(x)) =: L_x^+ \quad \text{for all} \quad x \in \Omega.$$

Step 1: first estimate when $L^+_{x_1} \le L^+_{x_2}$

Let $x_1, x_2 \in \Omega$ and let $x_2^\pm \in \partial\Omega$ be such that

$$g(x_2^\pm) - u(x_2) = L^\pm_{x_2} |x_2^\pm - x_2|^\alpha.$$

Then

$$g(x_2^+) - u(x_1) \le L^+_{x_1} |x_2^+ - x_1|^\alpha.$$

This implies

$$\begin{aligned} u(x_2) - u(x_1) &\le L^+_{x_1} |x_2^+ - x_1|^\alpha - L^+_{x_2} |x_2^+ - x_2|^\alpha \\ &\le L^+_{x_1} \left(|x_2^+ - x_1|^\alpha - |x_2^+ - x_2|^\alpha \right) \\ &\le L^+_{x_1} |x_2 - x_1|^\alpha, \end{aligned}$$

where we have used Lemma 5.2 and $L^+_{x_1} \ge 0$.

Step 2: second estimate when $L^+_{x_1} \le L^+_{x_2}$

By the assumption on $L^+_{x_i}$ we have $L^-_{x_1} \ge L^-_{x_2}$. Then

$$g(x_2^-) - u(x_1) \ge L^-_{x_1} |x_2^- - x_1|^\alpha.$$

This implies

$$\begin{aligned} u(x_2) - u(x_1) &\ge L^-_{x_1} |x_2^- - x_1|^\alpha - L^-_{x_2} |x_2^- - x_2|^\alpha \\ &\ge L^-_{x_2} \left(|x_2^- - x_1|^\alpha - |x_2^- - x_2|^\alpha \right) \\ &\ge L^-_{x_2} |x_2 - x_1|^\alpha, \end{aligned}$$

where again we have used Lemma 5.2 and $L^-_{x_1} \le 0$. This implies that

$$u(x_1) - u(x_2) \le -L^-_{x_2} |x_2 - x_1|^\alpha = L^+_{x_2} |x_2 - x_1|^\alpha.$$

Step 3: estimate of L^+u

Adding the two steps above together, and interchanging the roles of x_1 and x_2 we have

$$\frac{u(x_2) - u(x_1)}{|x_2 - x_1|^\alpha} \le \begin{cases} L^+_{x_1} & \text{when } L^+_{x_1} \le L^+_{x_2}, \\ L^+_{x_1} & \text{when } L^+_{x_1} \ge L^+_{x_2}. \end{cases}$$

This implies $(L^+u)(x_1) = L^+_{x_1}$.

Step 4: estimate of L^-u

This can be done in a similar way as for L^+u.

Step 5: pointwise solution

Finally we get

$$(Lu)(x_1) = \ell_{x_1}(u(x_1)) = 0$$

which is true pointwise. In particular, this implies that u is a viscosity solution of the equation.

Step 6: local continuity estimate for u

Assume $b > a$ and take $a^{\pm}$ and $b^{\pm}$ such that

$$\ell_x^{\pm}(a) = \frac{g(a^{\pm}) - a}{|x - a^{\pm}|^{\alpha}},$$

and similarly for b. Then

$$\frac{b-a}{|x-b^+|^{\alpha}} \leq \frac{g(b^+) - a}{|x-b^+|^{\alpha}} - \frac{g(b^+) - b}{|x-b^+|^{\alpha}} \leq \ell_x^+(a) - \ell_x^+(b).$$

Hence,

$$\ell_x^+(a) - \ell_x^+(b) \geq \frac{b-a}{(\text{diam } \Omega)^{\alpha}}\cdot$$

After similar reasoning for ℓ_x^- one can conclude (using the fact that $\ell_x^{\pm}(a)$ is non-increasing in a)

$$\ell_x(a) - \ell_x(b) \geq \frac{2(b-a)}{(\text{diam } \Omega)^{\alpha}}\cdot \tag{5.2}$$

But for $x, y \in B \subset\subset \Omega$ we also have the inequality

$$\begin{aligned}|\ell_x(u(x)) - \ell_x(u(y))| &\leq |\ell_x(u(x)) - \ell_y(u(y))| + |\ell_y(u(y)) - \ell_x(u(y))| \\ &\leq C(\alpha, ||g||_{L^\infty(\partial\Omega)}, \text{dist}(B, \partial\Omega))|x-y|.\end{aligned}$$

Hence, with $b = \max(u(x), u(y))$ and $a = \min(u(x), u(y))$ in (5.2) we obtain

$$|u(x) - u(y)| \leq \frac{(\text{diam } \Omega)^{\alpha}}{2} C(\alpha, ||g||_{L^\infty(\partial\Omega)}, \text{dist}(B, \partial\Omega))|x-y|.$$

This implies (5.1).

Step 7: $u \in C(\overline{\Omega})$

It remains thus to prove that u is continuous up to the boundary. Assume $x_n \to x_0 \in \partial\Omega$ and let

$$\ell_{x_n}^{\pm} = \frac{g(y_n^{\pm}) - u(x_n)}{|y_n^{\pm} - x_n|^{\alpha}},$$

for $y_n^{\pm} \in \partial\Omega$. Since $\pm\ell_{x_n}^{\pm} \geq 0$,

$$g(y_n^-) \leq u(x_n) \leq g(y_n^+). \tag{5.3}$$

We also know that

$$\ell_{x_n}(u(x_n)) = 0.$$

This implies that the limit of $\ell_{x_n}^+$ is finite if and only if the limit of $\ell_{x_n}^-$ is finite.

If they are both infinite then we must have $|y_n^{\pm} - x_n| \to 0$. Using this in (5.3) together with the continuity of g implies $u(x_n) \to g(x_0) = u(x_0)$.

If they are both finite then for some constant C

$$C \geq \limsup_n \ell_{x_n}^+ \geq \frac{g(x_0) - u(x_n)}{|x_0 - x_n|^{\alpha}} \geq \liminf_n \ell_{x_n}^- \geq -C.$$

This implies $u(x_n) \to u(x_0)$. This ends the proof of the lemma. □

6. The limit $p \to \infty$

As mentioned in the introduction we will work with the so called fractional Sobolev space $W^{s,p}(\Omega)$. This space is equipped with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\iint_{\Omega\times\Omega} \mathrm{d}x\mathrm{d}y \, \frac{|u(x)-u(y)|^p}{|x-y|^{sp+N}} \right)^{\frac{1}{p}}.$$

We recall the following result which can be found in [10], as Theorem 8.2.

Proposition 6.1 (Sobolev embedding)**.** *Let $u \in W^{s,p}(\Omega)$ for $s \in (0,1)$ and $s > N/p$ with Ω a bounded Lipschitz domain. Then with $\gamma = s - N/p$ we have*

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{s,p}(\Omega)}.$$

6.1. Proof of Theorem 1.1

A key result throughout this section is the following convexity inequality.

Lemma 6.2 (convexity inequality)**.** *For $p \geq 1$, there holds*

$$|\min(a,c) - \min(b,d)|^p + |\max(a,c) - \max(b,d)|^p \leq |a-b|^p + |c-d|^p.$$

For the sake of completeness we indicate a possible proof below. The idea is inspired by [15].

Proof of Lemma 6.2. The proof consists of, except in the obvious cases, observing that if $a > c$ and $b < d$ then there is θ such that

$$c - b = \theta(a-b) + (1-\theta)(c-d), \quad a - d = (1-\theta)(a-b) + \theta(c-d),$$

and using the convexity of the function $\phi(x) = |x|^p$. The case $a < c$ and $b > d$ can be treated in the exact same manner. □

The lemma below justifies the existence and uniqueness of minimizers for p large enough.

Lemma 6.3 (existence and uniqueness of a minimizer)**.** *Let $\alpha \in (0,1]$ and assume that Ω is a bounded Lipschitz domain. Consider $g \in C^{0,\alpha}(\partial\Omega)$ and define the set*

$$X_g = \left\{ u \in C(\overline{\Omega}), \quad u = g \quad on \quad \partial\Omega \right\}.$$

Define the minimization problem

$$I = \inf_{u \in X_g} \; E_p(u), \tag{6.1}$$

where

$$E_p(u) = \int_{\Omega\times\Omega} \mathrm{d}x\mathrm{d}y \left| \frac{u(x)-u(y)}{|x-y|^\alpha} \right|^p.$$

Then for any $p > 2N/\alpha$, problem (6.1) has a unique minimizer u_p. Moreover, for any function $\varphi \in C_c^\infty(\Omega)$, we have

$$\int_{\Omega\times\Omega} \mathrm{d}x\mathrm{d}y \, \left| \frac{u_p(y)-u_p(x)}{|y-x|^\alpha} \right|^{p-1} \left\{ \frac{\mathrm{sgn}\,(u_p(y)-u_p(x))}{|y-x|^\alpha} \right\} (\varphi(y) - \varphi(x)) = 0. \tag{6.2}$$

Proof of Lemma 6.3. We first remark that there is $h \in X_g$ such that $E_p(h) < \infty$ which shows that $I < \infty$. Indeed, we can take one of the extensions from [13,17]

$$h(x) = \sup_{y\in\partial\Omega} (g(y) - [g]_{\alpha,\partial\Omega}|x-y|^\alpha) \in C^{0,\alpha}(\overline{\Omega}).$$

Let us now consider a minimizing sequence $(u_n)_n$. We claim that we can assume $|u_n| \le \|g\|_{L^\infty(\partial\Omega)}$. Indeed, we have by Lemma 6.2

$$E_p(\max(u_n, \|g\|_{L^\infty(\partial\Omega)})) + E_p(\min(u_n, \|g\|_{L^\infty(\partial\Omega)})) \le E_p(u_n),$$

and also $\min(u_n, \|g\|_{L^\infty(\partial\Omega)}) \in X_g$. In the same way we can show that the energy decreases if we cut u_n from below at $-\|g\|_{L^\infty(\partial\Omega)}$. Hence, we can assume $|u_n| \le \|g\|_{L^\infty(\partial\Omega)}$.

In addition,

$$E_p(u_n) \le |\Omega|^2([h]_{\alpha,\Omega})^p \le C(\alpha, [g]_{\alpha,\partial\Omega}).$$

From Proposition 6.1, we deduce that

$$\|u_n\|_{C^{0,\gamma}(\overline{\Omega})} \le C\left((E_p(u_n))^{\frac{1}{p}} + |\Omega|\|g\|_{L^\infty(\partial\Omega)}\right) \le C(\alpha, [g]_{\alpha,\partial\Omega}, \|g\|_{L^\infty(\partial\Omega)})$$

for $\gamma = \alpha - \frac{2N}{p} > 0$. Therefore, up to the extraction of a subsequence, we deduce that u_n converges to a limit u_p in $C^{0,\beta}(\overline{\Omega})$ for $\beta < \gamma$. As a consequence we have $u_p \in X_g$. Since the integrand converges a.e. it follows by Fatou's Lemma that u_p is a minimizer. The uniqueness follows from the strict convexity of the functional and the fact that u_p satisfies the corresponding Euler-Lagrange equation follows by perturbing with a test function in a standard way. □

Now we will prove that minimizers are actually viscosity solutions, without knowing any regularity of the minimizer except continuity. For an example where a similar result is proved see [7].

Proposition 6.4 (minimizers are viscosity solutions)**.** *Let $p > 2\alpha/N$ and if $\alpha = 1$ let $N \ge 2$. Then the minimizer of E_p is a viscosity solution of the equation*

$$L_p u(x) = \int_\Omega \left|\frac{u(y)-u(x)}{|y-x|^\alpha}\right|^{p-1} \frac{\operatorname{sgn}(u(y)-u(x))}{|y-x|^\alpha} \mathrm{d}y = 0.$$

Proof of Proposition 6.4. Take u to be a minimizer of E_p. By Proposition 6.1, and the same arguments as in the proof of Lemma 6.3 we have $u \in C(\overline{\Omega})$. Now we need to prove that u satisfies the viscosity inequality. We prove that u is a subsolution.

Take $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ touching u from above at $x_0 \in \Omega$. Then we want to show that $L_p\varphi(x_0) \ge 0$. Let

$$\varphi^\varepsilon = \max(u, \varphi - \varepsilon)$$

and

$$\varphi_\varepsilon = \min(u, \varphi - \varepsilon).$$

Up to replacing $\varphi(x)$ by $\varphi(x) + \delta|x - x_0|^2$, we see that for ε small, we have $\varphi_\varepsilon = u$ and $\varphi^\varepsilon = \varphi - \varepsilon$ on $\partial\Omega$. Therefore $E_p(\varphi_\varepsilon) \ge E_p(u)$. Moreover, by Lemma 6.2

$$E_p(\varphi^\varepsilon) + E_p(\varphi_\varepsilon) \le E_p(u) + E_p(\varphi - \varepsilon) = E_p(u) + E_p(\varphi).$$

Consequently, $E_p(\varphi^\varepsilon) \le E_p(\varphi)$. The convexity of E_p then implies

$$E_p((1-t)\varphi + t\varphi^\varepsilon) \le (1-t)E_p(\varphi) + tE_p(\varphi^\varepsilon) \le E_p(\varphi).$$

Consider the convex function

$$f(t) = E_p(\varphi + t(\varphi^\varepsilon - \varphi)).$$

Then we have

$$\begin{aligned}
0 &\geq \frac{f(t) - f(0)}{t} \geq f'(0) \\
&= \iint_{\Omega\times\Omega} \left|\frac{\varphi(x) - \varphi(y)}{|x-y|^\alpha}\right|^{p-1} \left(\frac{\operatorname{sgn}(\varphi(x) - \varphi(y))}{|x-y|^\alpha}\right) ((\varphi^\varepsilon - \varphi + \varepsilon)(x) - (\varphi^\varepsilon - \varphi + \varepsilon)(y))\,\mathrm{d}y\mathrm{d}x \\
&= 2\int_\Omega (\varphi^\varepsilon - \varphi + \varepsilon)(x) \left(\int_\Omega \left|\frac{\varphi(x) - \varphi(y)}{|x-y|^\alpha}\right|^{p-1} \left(\frac{\operatorname{sgn}(\varphi(x) - \varphi(y))}{|x-y|^\alpha}\right) \mathrm{d}y\right) \mathrm{d}x \\
&= 2\int_\Omega (\varphi^\varepsilon - \varphi + \varepsilon)(x)(-L_p\varphi)(x)\mathrm{d}x.
\end{aligned}$$

Now we argue by contradiction. If $L_p\varphi(x_0) < 0$, then by continuity, which holds under our assumptions, because of Lebesgue's dominated convergence theorem, there is a small ball $B_r(x_0)$ such that $L_p\varphi < 0$ in $B_r(x_0)$. Moreover, when ε is small then $\operatorname{supp}(\varphi^\varepsilon - \varphi + \varepsilon) \subset B_r(x_0)$. We also observe that $\varphi^\varepsilon \geq \varphi - \varepsilon$ and in particular $(\varphi^\varepsilon - \varphi + \varepsilon)(x_0) = \varepsilon$. Hence, from the continuity of u, we see that there is a ball $B_\delta(x_0) \subset B_r(x_0)$ such that $\varphi^\varepsilon - \varphi + \varepsilon > 0$ in $B_\delta(x_0)$. Therefore,

$$\begin{aligned}
0 \geq \int_\Omega (\varphi^\varepsilon - \varphi + \varepsilon)(x)(-L_p\varphi)(x)\mathrm{d}x &= \int_{B_r(x_0)} (\varphi^\varepsilon - \varphi + \varepsilon)(x)(-L_p\varphi)(x)\mathrm{d}x \\
&\geq \int_{B_\delta(x_0)} (\varphi^\varepsilon - \varphi + \varepsilon)(x)(-L_p\varphi)(x)\mathrm{d}x > 0,
\end{aligned}$$

which is a contradiction.

In the same way it can be proved that u is a viscosity supersolution. □

To prove Theorem 1.1 we need the following technical result, whose proof is given in Section 6.2.

Lemma 6.5 (convergence of the L^p-norms)**.** *For $\varphi \in C^1(\Omega)$ let*

$$f_p(y) = \frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^\alpha}$$

and

$$f(y) = \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha},$$

where $x_p \to x_0 \in \Omega$ as $p \to \infty$. If $\alpha = 1$ assume in addition $N \geq 2$. Then

$$\lim_{p\to\infty} \left\| \frac{f_p^+(y)}{|y - x_p|^{\frac{\alpha}{p}}} \right\|_{L^p(\Omega)} = \left\| f^+ \right\|_{L^\infty(\Omega)},$$

where $f_p^\pm = \max(\pm f_p, 0)$. The same also holds for f_p^-.

Now we are ready to pass to the limit in the equation.

Proof of Theorem 1.1. Since

$$E_p(u_p) \leq |\Omega|^2 [h]^p_{\alpha,\Omega},$$

we have with $q = 2N/\alpha + \delta$ for $\delta > 0$ that

$$E_q(u_p) \leq E_p^{\frac{q}{p}} |\Omega|^{\frac{2(p-q)}{p}} \leq [h]_{\alpha,\Omega}^q |\Omega|^2.$$

By the same arguments as in the proof of Lemma 6.3 we can prove that $|u_p| \leq \|g\|_{L^\infty(\Omega)}$. Therefore, by Proposition 6.1, u_p is uniformly bounded in $C^{0,\gamma}(\overline{\Omega})$ with $\gamma = \alpha - 2N/q > 0$. Hence, for a subsequence, again labelled u_p, we have $u_p \to u$ in $C(\overline{\Omega})$.

Consider a test function $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} u \leq \varphi, \\ u(x_0) = \varphi(x_0) \quad \text{for some} \quad x_0 \in \Omega, \end{cases}$$

and assume towards a contradiction that

$$(L\varphi)(x_0) < 0. \tag{6.3}$$

Up to replacing φ by $\varphi + \delta|x - x_0|^2$ for δ small enough, we can furthermore assume that x_0 is a point of strict maximum of $u - \varphi$. Then

$$\sup_{\overline{\Omega}} (u_p - \varphi) = (u_p - \varphi)(x_p) = M_p$$

with

$$x_p \to x_0, \quad M_p \to 0.$$

This shows that

$$\begin{cases} u_p \leq \varphi_p := M_p + \varphi, \\ u_p(x_p) = \varphi_p(x_p). \end{cases}$$

By Proposition 6.4, u_p is a viscosity solution, therefore

$$0 \leq (L_p\varphi_p)(x_p) = (L_p\varphi)(x_p).$$

We recall that

$$0 \leq (L_p\varphi)(x_p) = 2\int_\Omega \mathrm{d}y \left|\frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^\alpha}\right|^{p-1} \left\{\frac{\operatorname{sgn}\,(\varphi(y) - \varphi(x_p))}{|y - x_p|^\alpha}\right\}.$$

Written in another way we have

$$\left\|\left(\frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^{\alpha + \frac{\alpha}{p-1}}}\right)^+\right\|_{L^{p-1}(\Omega)} \geq \left\|\left(\frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^{\alpha + \frac{\alpha}{p-1}}}\right)^-\right\|_{L^{p-1}(\Omega)}.$$

Lemma 6.5 now implies that we can pass to the limit in this inequality. Hence, we obtain

$$\sup_{y\in\Omega}\left(\max\left(\frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha}, 0\right)\right) + \inf_{y\in\Omega}\left(\min\left(\frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha}, 0\right)\right) \geq 0.$$

Since φ is C^1 at x_0 it is clear that $\pm(L^\pm\varphi)(x_0) \geq 0$. Combined with the last inequality this implies,

$$(L\varphi)(x_0) \geq 0,$$

which contradicts (6.3). In the same way it can be proved that u is a supersolution.

By (ii) in Theorem 1.5 the solution u is unique, so the whole sequence converges to the solution. Moreover, by (iv) in Theorem 1.8 we have

$$[u]_{\alpha,\overline{\Omega}} = [g]_{\alpha,\partial\Omega}.$$

This ends the proof of the theorem. □

6.2. **Proof of Lemma 6.5**

In order to prove Lemma 6.5 we first need the following result.

Lemma 6.6. *For $\varphi \in C^1(\Omega)$ let*

$$f_p(y) = \frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^\alpha}$$

and

$$f(y) = \frac{\varphi(y) - \varphi(x_0)}{|y - x_0|^\alpha},$$

where $x_p \to x_0 \in \Omega$ as $p \to \infty$. In addition, let

$$f_p^\pm = \max(\pm f_p, 0)$$

and assume

$$\sup_\Omega f^+ > 0.$$

Then for any

$$0 < t < \sup_\Omega f^+$$

there is a $p_0 < \infty$ and $c > 0$ such that

$$\big|\{f_p^+ > t\}\big| > c,$$

for all $p \geq p_0$. The same also holds for f_p^-.

Proof of Lemma 6.6. For $\alpha < 1$ this is obvious since f_p will be uniformly continuous and then also f_p^+. Therefore we treat only the case $\alpha = 1$. By arguments identical to those in the proof of Lemma 3.5 one can prove that

$$\sup_\Omega f_p^+ \to \sup_\Omega f^+. \tag{6.4}$$

Since $t < \sup_\Omega f^+$, there is a sequence z_p such that $f_p^+(z_p) > t + \varepsilon$ for ε small enough. We split the proof into two cases.

Case 1: $z_p \to x_0$.

By Taylor expansion we have $t + \varepsilon/2 \leq |\nabla\varphi(x_0)|$ for p large enough. We also have for all y

$$f_p(y) \geq \nabla\varphi(x_0) \cdot \frac{y - x_p}{|y - x_p|} - o_{|y-x_p|}(1) - o_{|x_p - x_0|}(1).$$

Therefore, if we choose p large enough and y such that $o_{|y-x_p|}(1) + o_{|x_p-x_0|}(1) < \varepsilon/4$ and

$$\nabla\varphi(x_0) \cdot \frac{y - x_p}{|y - x_p|} \geq |\nabla\varphi(x_0)| - \varepsilon/4$$

then $f_p(y) > t$. Clearly, this set of y:s has positive measure, independently of p, as long as p is large enough.

Case 2: $z_p \to z \neq x_0$.

In this case, for p large enough, there is a δ such that f_p^+ is uniformly continuous in $B_\delta(z_p)$, uniformly also in p. Consequently there is δ', independent of p, such that $f_p^+ > t$ in $B_{\delta'}(z_p)$. □

Proof of Lemma 6.5.

Case 1: $\sup_\Omega f^+ > 0$.

Take

$$0 < t < \sup_\Omega f^+$$

and let

$$A(t,p) = \{f_p^+ > t\}.$$

By Lemma 6.6, for $p > p_0$, $|A(t,p)| > c > 0$ with c independent of p. Therefore

$$\int_\Omega \frac{(f_p^+)^p}{|y - x_p|^\alpha} \geq t^p \int_{A(t,p)} \frac{1}{|y - x_p|^\alpha} \geq \frac{c}{(\mathrm{diam}\ (\Omega))^\alpha} t^p.$$

This implies

$$\left\| \frac{f_p^+}{|y - x_p|^{\frac{\alpha}{p}}} \right\|_{L^p(\Omega)} \geq t \left(\frac{c}{(\mathrm{diam}\ (\Omega))^\alpha} \right)^{\frac{1}{p}} \to t.$$

For the other side of the inequality we have

$$\int_\Omega \frac{(f_p^+)^p}{|y - x_p|^\alpha} \leq \sup_\Omega (f_p^+)^p \int_\Omega \frac{1}{|y - x_p|^\alpha} \leq C \sup_\Omega (f_p^+)^p.$$

Thus

$$\left\| \frac{f_p^+}{|y - x_p|^{\frac{\alpha}{p}}} \right\|_{L^p(\Omega)} \leq C^{\frac{1}{p}} \sup_\Omega f_p^+ \to \sup_\Omega f^+, \tag{6.5}$$

where we have used (6.4) for the convergence. All together we have

$$t \leq \liminf_{p \to \infty} \left\| \frac{f_p^+}{|y - x_p|^{\frac{\alpha}{p}}} \right\|_{L^p(\Omega)} \leq \limsup_{p \to \infty} \left\| \frac{f_p^+}{|y - x_p|^{\frac{\alpha}{p}}} \right\|_{L^p(\Omega)} \leq \sup_\Omega f^+,$$

for all

$$0 < t < \sup_\Omega f^+.$$

This implies the desired result.

Case 2: $\sup_\Omega f^+ = 0$.

Then (6.5) implies the result. □

7. Limits of viscosity solutions

In this section we prove the result that says that limits of subsolutions are again subsolutions.

Proposition 7.1 (stability of subsolutions)**.**

(i) *Consider a family $(F_\varepsilon)_\varepsilon$ of sets F_ε of subsolutions of (4.1) in Ω and define for any $x_0 \in \overline{\Omega}$*

$$\overline{u}(x_0) = \limsup_{\varepsilon \to 0,\ x_\varepsilon \to x_0,\ u_\varepsilon \in F_\varepsilon} u_\varepsilon(x_\varepsilon),$$

which we assume to be bounded from above. Then $\overline{u}$ is a subsolution of (4.1) in Ω.

(ii) *Moreover, in the special case where the sets $F_\varepsilon = F$ are independent of ε, then we have*

$$\overline{u} = \overline{v}^* \quad \textit{with} \quad \overline{v}(x) = \sup_{u \in F} u(x) \quad \textit{for all} \quad x \in \overline{\Omega}.$$

In fact, we will only be using the second statement of this proposition, but we give the full result since it can be of general interest.

To prove the proposition, we will need the following:

Lemma 7.2 (perturbation by a small parabola)**.** *Let $\varphi \in C^1(\Omega)$ and define for some $x_0 \in \Omega$ and $\delta \in \mathbb{R}$*

$$\overline{\varphi}(x) = \varphi(x) + \delta(x - x_0)^2.$$

Then, with the notation $R = \operatorname{diam} \Omega$ we have

$$|(L\overline{\varphi})(x) - (L\varphi)(x)| \le 4|\delta|R^{2-\alpha} \quad \textit{for every} \quad x \in \Omega.$$

Proof of Lemma 7.2. Consider points $y, x \in \Omega \setminus \{x_0\}$. We deduce

$$\left| \frac{\delta(y - x_0)^2 - \delta(x - x_0)^2}{|y - x|^\alpha} \right| = |\delta| \left| \frac{y - x}{|y - x|^\alpha} \cdot (y + x - 2x_0) \right| \le 2|\delta|R^{2-\alpha}.$$

This implies for $x \in \Omega \setminus \{x_0\}$

$$|(L^\pm\overline{\varphi})(x) - (L^\pm\varphi)(x)| \le 2|\delta|R^{2-\alpha}$$

and then

$$|(L\overline{\varphi})(x) - (L\varphi)(x)| \le 4|\delta|R^{2-\alpha}.$$

This ends the proof of the lemma. □

Proof of Proposition 7.1.

Preliminary: $\overline{u}$ is upper semicontinuous

Consider a sequence $(x_\varepsilon)_\varepsilon$ such that $x_\varepsilon \to x_0$ as $\varepsilon \to 0$ and

$$\overline{u}^*(x_0) = \lim_{\varepsilon \to 0} \overline{u}(x_\varepsilon).$$

In particular, for any $\delta > 0$, there exists a point x_δ such that

$$\overline{u}^*(x_0) - \delta \le \overline{u}(x_\delta) \quad \text{and} \quad |x_\delta - x_0| \le \delta.$$

By the definition of $\overline{u}$, there exist a sequence y_ε and a function $u_\varepsilon \in F$ such that for $\varepsilon = \varepsilon_\delta < \delta$ we have

$$\overline{u}(x_\delta) - \delta \le u_{\varepsilon_\delta}(y_{\varepsilon_\delta}) \quad \text{and} \quad |x_\delta - y_{\varepsilon_\delta}| \le \delta.$$

Therefore

$$\overline{u}^*(x_0) - 2\delta \le u_{\varepsilon_\delta}(y_{\varepsilon_\delta}), \quad |y_{\varepsilon_\delta} - x_0| \le 2\delta \quad \text{and} \quad \varepsilon_\delta < \delta.$$

Since this is true for any $\delta > 0$, this shows that

$$\overline{u}^*(x_0) \le \overline{u}(x_0)$$

and then $\overline{u} = \overline{u}^*$.

Part I: proof that $\overline{u}$ is a subsolution

We argue by contradiction and assume that there exists $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\overline{u} \le \varphi \quad \text{on} \quad \overline{\Omega}$$

with $\overline{u}(x_0) = \varphi(x_0)$ and $(L\varphi)(x_0) < f(x_0)$ for some $x_0 \in \Omega$.

Step I.1: reducing the problem to a point of strict maximum

Let us set for $\delta > 0$

$$\overline{\varphi}(x) = \varphi(x) + \delta(x - x_0)^2$$

such that x_0 is a point of strict maximum of $\overline{u} - \overline{\varphi}$. From Lemma 7.2 we deduce that

$$(L\overline{\varphi})(x_0) \leq (L\varphi)(x_0) + 4\delta R^{2-\alpha} < f(x_0) \tag{7.1}$$

if δ is chosen small enough.

Step I.2: coming back to the ε-problem

Let us choose a sequence ε with x_ε and u_ε such that

$$\overline{u}(x_0) = \lim_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon).$$

Then let us set

$$M_\varepsilon := \sup_{x \in \overline{\Omega}} (u_\varepsilon - \overline{\varphi})(x) = (u_\varepsilon - \overline{\varphi})(y_\varepsilon) \quad \text{with} \quad y_\varepsilon \in \overline{\Omega}.$$

Because x_0 is a point of strict maximum of $\overline{u} - \overline{\varphi}$, it is classical to realize that $M_\varepsilon \to 0$ and $y_\varepsilon \to x_0$. Let us set

$$\overline{\varphi}_\varepsilon(x) = M_\varepsilon + \overline{\varphi}.$$

Then we have

$$u_\varepsilon \leq \overline{\varphi}_\varepsilon$$

and

$$u_\varepsilon(y_\varepsilon) = \overline{\varphi}_\varepsilon(y_\varepsilon) \tag{7.2}$$

which implies $(L\overline{\varphi})(y_\varepsilon) \geq f(y_\varepsilon)$, where we have used the fact that $L\overline{\varphi}_\varepsilon = L\overline{\varphi}$.

Therefore, by letting $y_\varepsilon \to x_0$ we can conclude that $(L\overline{\varphi})(x_0) \geq f(x_0)$. A contradiction to (7.1).

Part II: proof that $\overline{u} = \overline{v}^*$ when $F_\varepsilon = F$

Step II.1: $\overline{u} \geq \overline{v}^*$

By definition we have

$$\overline{u}(x_0) = \limsup_{\varepsilon \to 0,\ x_\varepsilon \to x_0,\ u_\varepsilon \in F} u_\varepsilon(x_\varepsilon).$$

Setting $x_\varepsilon = x_0$, we see in particular that $\overline{u} \geq \overline{v}$, and then $\overline{u}^* \geq \overline{v}^*$. Using the fact that $\overline{u} = \overline{u}^*$, we deduce that

$$\overline{u} \geq \overline{v}^*.$$

Step II.2: $\overline{u} \leq \overline{v}^*$

Let us fix $x_0 \in \overline{\Omega}$ and sequences $(x_\varepsilon)_\varepsilon$, $(u_\varepsilon)_\varepsilon$ such that

$$\overline{u}(x_0) = \lim_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) \quad \text{and} \quad x_\varepsilon \to x_0.$$

In particular, for any $\delta > 0$, there exist ε_δ such that

$$\overline{u}(x_0) - \delta \leq u_{\varepsilon_\delta}(x_{\varepsilon_\delta}) \leq \overline{v}(x_{\varepsilon_\delta}) \quad \text{and} \quad |x_{\varepsilon_\delta} - x_0| \leq \delta.$$

This implies that $\overline{v}^*(x_0) \geq \overline{u}(x_0)$, *i.e.*

$$\overline{u} \leq \overline{v}^*.$$

Step II.3: conclusion

We conclude that $\overline{u} = \overline{v}^*$.

This ends the proof of the proposition. □

8. Barriers

In order the prove the existence of solutions we need barriers, *i.e.*, sub- and supersolutions. This section is devoted to the construction of barriers.

Lemma 8.1 (fundamental supersolutions)**.** *Consider a bounded open set* Ω *such that* $0 \in \partial\Omega$*. We also choose* $R > 0$ *such that*

$$\overline{\Omega} \subset \overline{B_R(0)}. \tag{8.1}$$

Then for $\alpha \in (0,1]$*, the function*

$$\Psi(x) := |x|^\alpha$$

satisfies

$$0 \geq -\delta(x) \geq (L\Psi)(x) \quad \textit{for} \quad x \in \Omega$$

where when $\alpha \in (0,1)$*, we can choose*

$$-\delta(x) := -1 + \frac{\rho^\alpha - 1}{(\rho-1)^\alpha} < 0 \quad \textit{with} \quad \rho = R/|x|.$$

Proof of Lemma 8.1. We simply estimate $(L\Psi)(x)$ for every $x \in \Omega$. We first remark that

$$(L^-\Psi)(x) \leq \liminf_{y \to 0} \frac{|y|^\alpha - |x|^\alpha}{|y-x|^\alpha} = -1. \tag{8.2}$$

On the other hand we have with $e = x/|x|$

$$\begin{aligned}
(L^+\Psi)(x) &= \sup_{y \in \overline{\Omega},\ y \neq x} \frac{|y|^\alpha - |x|^\alpha}{|y-x|^\alpha} \\
&= \sup_{z \in \overline{\Omega}/|x|,\ z \neq e} \frac{|z|^\alpha - 1}{|z-e|^\alpha} \\
&\leq \sup_{z \in \mathbb{R}^N,\ 1<|z|\leq R/|x|} \frac{|z|^\alpha - 1}{|z-e|^\alpha} \\
&= \sup_{1<r\leq R/|x|} \frac{r^\alpha - 1}{(r-1)^\alpha},
\end{aligned}$$

where we have used for the last line the fact that $|z-e| \geq ||z|-|e||$. Now we set

$$g(r) := \frac{r^\alpha - 1}{(r-1)^\alpha}$$

and compute

$$\begin{aligned}
g'(r) &= \frac{\alpha r^{\alpha-1}(r-1)^\alpha - (r^\alpha - 1)\alpha(r-1)^{\alpha-1}}{(r-1)^{2\alpha}} \\
&= \frac{\alpha(r-1)^{\alpha-1}}{(r-1)^{2\alpha}} \left\{ r^{\alpha-1}(r-1) - (r^\alpha - 1) \right\} \\
&= \frac{\alpha(r-1)^{\alpha-1}}{(r-1)^{2\alpha}} \left\{ 1 - r^{\alpha-1} \right\}.
\end{aligned}$$

In particular for $r > 1$, we get $g'(r) \geq 0$ and moreover

$$g'(r) > 0 \text{ for } r > 1 \text{ if } \alpha \in (0,1). \tag{8.3}$$

This implies that

$$(L^+\Psi)(x) \leq g(R/|x|)$$

where $g(R/|x|) \leq g(\infty) = 1$ and moreover $g(R/|x|) < 1$ if $\alpha \in (0,1)$. Joint to (8.2), this proves the lemma. □

Lemma 8.2 (fundamental strict supersolutions for $\alpha = 1$)**.** *Let $\alpha = 1$. Consider a bounded open set Ω such that $0 \in \partial\Omega$. For $\varepsilon > 0$ we set*

$$\Psi_\varepsilon(x) := |x| - \varepsilon|x|^2$$

Then we have

$$0 > -\varepsilon|x| \geq (L\Psi_\varepsilon)(x) \quad \text{for all} \quad x \in \Omega.$$

Proof of Lemma 8.2. We proceed as earlier. We have

$$(L^-\Psi_\varepsilon)(x) \leq \liminf_{y\to 0} \frac{|y| - \varepsilon|y|^2 - (|x| - \varepsilon|x|^2)}{|y-x|} = -1 + \varepsilon|x|. \tag{8.4}$$

On the other hand we have with $e = x/|x|$

$$\begin{aligned}
(L^+\Psi_\varepsilon)(x) &= \sup_{y\in\overline{\Omega},\ y\neq x} \frac{|y| - \varepsilon|y|^2 - (|x| - \varepsilon|x|^2)}{|y-x|} \\
&= \sup_{z\in\overline{\Omega}/|x|,\ z\neq e} \frac{|z| - \bar{\varepsilon}|z|^2 - (1-\bar{\varepsilon})}{|z-e|} \\
&\leq \sup_{z\in\mathbb{R}^N,\ 1<|z|} \frac{\left(|z| - \bar{\varepsilon}|z|^2 - (1-\bar{\varepsilon})\right)^+}{||z| - |e||} \\
&\leq \sup_{1<r} \frac{r - 1 - \bar{\varepsilon}(r^2-1)}{r-1} \\
&= \sup_{1<r} \quad 1 - \bar{\varepsilon}(r+1) \\
&= 1 - 2\bar{\varepsilon},
\end{aligned}$$

where in the second line we have set

$$\bar{\varepsilon} = \varepsilon|x|$$

and where in the third line, we have used the fact that $|z-e| \geq ||z| - |e||$. Joint to (8.4), this shows that

$$(L^+\Psi_\varepsilon)(x) + (L^-\Psi_\varepsilon)(x) \leq -\varepsilon|x| < 0$$

which ends the proof of the lemma. □

We see that the strict sub- or supersolutions we have constructed above are not uniformly strict as we approach the origin $x = 0$. However, if we demand less regularity, it is possible to construct strict sub- and supersolutions that remain strict when approaching the origin. These sub- and supersolutions will be useful later.

Lemma 8.3 (less regular strict subsolutions/supersolutions)**.** *Consider a bounded open set* Ω *such that* $0 \in \partial\Omega$*. For* $0 < \beta < \alpha \in (0, 1]$*, the function*

$$\Psi(x) := |x|^{\beta}$$

satisfies

$$-\delta(x) \geq (L\Psi)(x) \quad \textit{for} \quad x \in \Omega$$

where

$$\delta(x) = C(\alpha, \beta)|x|^{\beta-\alpha} > 0.$$

Proof of Lemma 8.3. We proceed with the same computations as in Lemma 8.1 and obtain

$$(L^{-}\Psi)(x) \leq -|x|^{\beta-\alpha},$$

and

$$(L^{+}\Psi)(x) \leq |x|^{\beta-\alpha} \sup_{1<r} \frac{r^{\beta}-1}{(r-1)^{\alpha}}\cdot$$

Now let

$$h(r) = \frac{r^{\beta}-1}{(r-1)^{\alpha}}\cdot$$

Clearly, $g \to 0$ when $r \to \infty$. So for R large enough, $r \geq R$ implies $g(r) < 1/2$.

Case 1: $\alpha \in (0, 1)$

When $r \leq R$ we have

$$h(r) \leq \frac{r^{\alpha}-1}{(r-1)^{\alpha}} = g(r) \leq g(R) < 1,$$

where we have used (8.3). Therefore,

$$(L^{+}\Psi)(x) < \max(h(R), 1/2)|x|^{\beta-\alpha} < |x|^{\beta-\alpha}.$$

Case 2: $\alpha = 1$

We have $h(r) \to h(1) = \beta$ as $r \searrow 1$. Moreover, $h(r) < 1$ for $r > 1$. Therefore,

$$\sup_{1<r<R} h(r) = C_0 < 1.$$

This implies

$$(L^{+}\Psi)(x) < \max(C_0, 1/2)|x|^{\beta-\alpha} < |x|^{\beta-\alpha}.$$

Hence finally, in both cases

$$(L\Psi)(x) \leq -C(\alpha, \beta)|x|^{\beta-\alpha}.$$

□

Lemma 8.4 (natural subsolutions/supersolutions with boundary conditions)**.** *Let* $f \in C(\Omega) \cap L^{\infty}(\Omega)$ *and* $g \in C(\partial\Omega)$*. For* $\beta \in (0, \alpha)$*,* $x_0 \in \mathbb{R}^N$ *and* $a, b \in \mathbb{R}$*, we define*

$$u_{x_0,a,b}(x) = a + b|x - x_0|^{\beta}.$$

Furthermore, let

$$S^{\pm} = \left\{ \begin{array}{l} u_{x_0,a,b} \quad \textit{for} \quad (x_0, a, \pm b) \in (\partial\Omega) \times \mathbb{R} \times (0, \infty) \\ \quad \textit{such that} \\ \pm u_{x_0,a,\pm b} \geq \pm g \quad \textit{on} \quad \partial\Omega \\ \pm L u_{x_0,a,\pm b} \leq \pm f \quad \textit{in} \quad \Omega \end{array} \right\}$$

and define for all $x \in \overline{\Omega}$

$$\underline{v}(x) = \inf_{u \in S^{+}} u(x), \quad \overline{v}(x) = \sup_{u \in S^{-}} u(x).$$

Then $\underline{v} \in C(\overline{\Omega})$ *is a supersolution and* $\overline{v} \in C(\overline{\Omega})$ *is a subsolution of* (4.1).

Moreover, we have

$$\overline{v} \leq \underline{v} \quad on \quad \overline{\Omega} \tag{8.5}$$

and

$$\overline{v} = g = \underline{v} \quad on \quad \partial\Omega. \tag{8.6}$$

Proof of Lemma 8.4. Let us show that $\overline{v}$ is a continuous subsolution satisfying (8.6), the proof being similar to show that $\underline{v}$ is a supersolution.

Step 1: $\overline{v}^*$ is a subsolution

From Lemma 8.3, we first deduce that for $x_0 \in \partial\Omega$ if a and b are chosen properly, then $u_{x_0,a,b} \in S^-$. This shows that $S^- \neq \emptyset$. On the other hand if $u_{x_0,a,b} \in S^-$ then

$$a \leq g(x_0) \leq \sup_{\partial\Omega} g,$$

which implies that for all $u \in S^-$ we have

$$u \leq \sup_{\partial\Omega} g.$$

Therefore, applying the stability result (Prop. 7.1), and setting $F_\varepsilon = S^-$, we know that

$$\overline{u}(x) = \limsup_{\varepsilon\to 0,\ x_\varepsilon \to x,\ u_\varepsilon \in S^-} u_\varepsilon(x_\varepsilon) \tag{8.7}$$

is a viscosity subsolution. Moreover we have $\overline{v}^* = \overline{u}$.

Step 2: $\overline{v} \geq g$ on $\partial\Omega$

For any $x_0 \in \partial\Omega$, and any $\delta > 0$, we see (using the continuity of g) that there exists $b_\delta > 0$ large enough such that with $a_\delta = g(x_0) - \delta$ there holds

$$u_{x_0,a_\delta,-b_\delta} \leq g \quad \text{on} \quad \partial\Omega.$$

Therefore

$$\overline{v}(x_0) \geq g(x_0) - \delta.$$

Since this true for any $\delta > 0$, this implies that

$$\overline{v}(x_0) \geq g(x_0)$$

and then

$$\overline{v} \geq g \quad \text{on} \quad \partial\Omega. \tag{8.8}$$

Step 3: $\overline{v}_* = \overline{v}$ on $\overline{\Omega}$

Let $x_0 \in \overline{\Omega}$ and take a sequence of functions $(u_\delta)_\delta$ with $u_\delta \in S^-$ such that

$$\overline{v}(x_0) = \lim_{\delta\to 0} u_\delta(x_0).$$

Now consider a sequence $(x_\varepsilon)_\varepsilon$ of points $x_\varepsilon \in \overline{\Omega}$ such that

$$\overline{v}_*(x_0) = \lim_{\varepsilon\to 0} \overline{v}(x_\varepsilon).$$

Then we have

$$\overline{v}(x_\varepsilon) \geq u_\delta(x_\varepsilon)$$

which implies

$$\overline{v}_*(x_0) \geq u_\delta(x_0).$$

Taking now the limit $\delta \to 0$, we get

$$\overline{v}_*(x_0) \geq \overline{v}(x_0)$$

and then

$$\overline{v}_* = \overline{v}.$$

Step 4: $\overline{v}^* = \overline{v}$ on $\overline{\Omega}$

From (8.7), we deduce that for any $x_0 \in \overline{\Omega}$, there exist a sequence $(y_\varepsilon)_\varepsilon$ of points $y_\varepsilon \in \overline{\Omega}$ such that $y_\varepsilon \to x_0$ and a sequence $(u_\varepsilon)_\varepsilon$ of functions $u_\varepsilon \in S^-$ such that

$$\overline{v}^*(x_0) = \lim_{\varepsilon \to 0} u_\varepsilon(y_\varepsilon). \tag{8.9}$$

We write

$$u_\varepsilon(x) = u_{x_\varepsilon, a_\varepsilon, -b_\varepsilon}(x) = a_\varepsilon - b_\varepsilon |x - x_\varepsilon|^\beta$$

with $a_\varepsilon \in \mathbb{R}, b_\varepsilon \in (0, \infty), x_\varepsilon \in \partial\Omega$.

Case $b_\varepsilon \to \infty$

Since $a_\varepsilon \leq \sup_\Omega g$ and $\overline{v}^*(x_0) \leq \sup_\Omega g$ we deduce that $|y_\varepsilon - x_\varepsilon| \to 0$ which shows that

$$x_\varepsilon \to x_0 \quad \text{and} \quad x_0 \in \partial\Omega.$$

On the other hand we have $u_\varepsilon(x_\varepsilon) \leq g(x_\varepsilon)$ which means

$$a_\varepsilon \leq g(x_\varepsilon).$$

Therefore

$$u_\varepsilon(y_\varepsilon) \leq a_\varepsilon \leq g(x_\varepsilon).$$

Passing to the limit as ε goes to zero, and using the continuity of g, we deduce from (8.9) that

$$\overline{v}^*(x_0) \leq g(x_0) \leq \overline{v}(x_0)$$

where we have used (8.8) for the last inequality. This shows in that case that

$$\overline{v}^*(x_0) \leq \overline{v}(x_0).$$

Case b_ε bounded

Because of (8.9), we see that a_ε is bounded. Then up to extraction of a subsequence, we can assume the following:

$$\begin{cases} a_\varepsilon \to a \in \mathbb{R}, \\ b_\varepsilon \to b \in [0, \infty), \\ x_\varepsilon \to \bar{x}_0. \end{cases}$$

Therefore, with

$$u_0 = u_{\bar{x}_0, a, -b}$$

we get

$$\overline{v}^*(x_0) = u_0(x_0) \leq \overline{v}(x_0)$$

and then we conclude in every case that

$$\overline{v}^* = \overline{v} \quad \text{on} \quad \overline{\Omega}.$$

Step 5: intermediate conclusion

From the previous steps, we deduce that $\overline{v} \in C(\overline{\Omega})$ is a subsolution.

Step 6: proof of $\overline{v} \leq \underline{v}$ on $\overline{\Omega}$

Step 6.1: $u^- \leq u^+$

Let us consider $u^+ = u_{x_0^+,a^+,b^+} \in S^+$ and $u^- = u_{x_0^-,a^-,-b^-} \in S^-$. By assumption we have

$$u^- \leq g \leq u^+ \quad \text{on} \quad \partial\Omega. \tag{8.10}$$

We want to show that

$$u^- \leq u^+ \quad \text{on} \quad \overline{\Omega}. \tag{8.11}$$

Let us proceed by contradiction. If this is false, then we have

$$0 < \sup_{x\in\overline{\Omega}} (u^- - u^+) = (u^- - u^+)(y_0) \quad \text{for some point} \quad y_0 \in \Omega \tag{8.12}$$

and then

$$\nabla(u^- - u^+)(y_0) = 0,$$

i.e., for $x = y_0$

$$\frac{(x - x_0^+)}{|x - x_0^+|}\, b^+ |x - x_0^+|^{\beta-1} + \frac{(x - x_0^-)}{|x - x_0^-|}\, b^- |x - x_0^-|^{\beta-1} = 0.$$

This implies that $y_0 \in [x_0^-, x_0^+]$, because $b^\pm > 0$. Let us call $I = (z^-, z^+)$ the connected component of $[x_0^-, x_0^+] \cap \Omega$ containing y_0. In particular since $\beta \in (0,1)$, $u^- - u^+$ is strictly convex on I and reaches it maximum at the interior point $y_0 \in I$. This gives immediately a contradiction.

Step 6.2: conclusion

From (8.12), we deduce that for any $x \in \overline{\Omega}$

$$\overline{v}(x) = \sup_{u^-\in S^-} u^-(x) \leq u^+(x)$$

and then

$$\overline{v}(x) \leq \inf_{u^+\in S^+} u^+(x) = \underline{v}(x).$$

Therefore

$$\overline{v} \leq \underline{v} \text{ on } \overline{\Omega}. \tag{8.13}$$

Step 7: proof of $\overline{v} = g = \underline{v}$ on $\partial\Omega$

Similarly to (8.8), we show that

$$\underline{v} \leq g \text{ on } \partial\Omega.$$

Therefore from (8.13), we deduce that

$$g \leq \overline{v} \leq \underline{v} \leq g \quad \text{on} \quad \partial\Omega$$

and then

$$\overline{v} = g = \underline{v} \quad \text{on} \quad \partial\Omega.$$

This ends the proof of the lemma. □

9. Perron's method

In this section we construct the solutions applying the Perron's method.

Theorem 9.1 (existence by Perron's method)**.** *Let $u^- \in C(\overline{\Omega})$ be a subsolution (resp. $u^+ \in C(\overline{\Omega})$ be a supersolution) of (4.1) with continuous boundary data g, satisfying*

$$\begin{cases} u^- \le u^+ & \text{on} \quad \overline{\Omega}, \\ u^- = g = u^+ & \text{on} \quad \partial\Omega. \end{cases}$$

Define

$$\mathcal{S} = \left\{ u \quad \text{subsolution of (4.1) such that} \quad u^- \le u \le u^+ \quad \text{on} \quad \overline{\Omega} \right\}$$

and for all $x_0 \in \overline{\Omega}$

$$\overline{u}(x_0) = \limsup_{\varepsilon \to 0, x_\varepsilon \to x_0, w_\varepsilon \in \mathcal{S}} w_\varepsilon(x_\varepsilon).$$

Then $\overline{u}$ is upper semicontinuous on $\overline{\Omega}$ and $\overline{u}$ is a viscosity solution of (4.1) in Ω. Moreover, $\overline{u}$ satisfies

$$u^- \le \overline{u} \le u^+ \quad \text{on} \quad \overline{\Omega}. \tag{9.1}$$

Remark 9.2. From Lemma 8.4, we can set $u^- = \overline{u}$ and $u^+ = \underline{u}$ and then Theorem 9.1 provides the existence of a solution.

Proof of Theorem 9.1.

Step 1: construction of the maximal subsolution on $\overline{\Omega}$

By assumption we have $\mathcal{S} \neq \emptyset$, because $u^- \in \mathcal{S}$. Applying the stability property of subsolutions (Prop. 7.1), we deduce that $\overline{u}$ is a subsolution on $\overline{\Omega}$. Finally, by construction, we get (9.1).

Step 2: $\overline{u}_*$ is a supersolution on Ω

Let us proceed by contradiction and assume that $\overline{u}_*$ is not a supersolution on Ω. Then there exists a test function $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ and a point $x_0 \in \Omega$ such that

$$\begin{cases} \overline{u}_* \ge \varphi & \text{on} \quad \overline{\Omega}, \\ \overline{u}_*(x_0) = \varphi(x_0). \end{cases} \tag{9.2}$$

and $\overline{u}_*$ is not a supersolution at the point x_0, *i.e.*

$$(L\varphi)(x_0) = \theta + f(x_0) > f(x_0). \tag{9.3}$$

Step 2.1: $\overline{u}_*(x_0) < u^+(x_0)$

We already know that $\overline{u} \le u^+$ on $\overline{\Omega}$, and then

$$\varphi \le \overline{u}_* \le u^+ \quad \text{on} \quad \Omega.$$

If $\overline{u}_*(x_0) = u^+(x_0)$ and $x_0 \in \Omega$, then φ is a test function for u^+ which is then in contradiction with the supersolution property of u^+ at x_0. Therefore we have

$$\overline{u}_*(x_0) < u^+(x_0). \tag{9.4}$$

Step 2.2: preliminary

Similarly to what was done in Step 1 of the proof of Proposition 7.1, we can set for $\delta > 0$

$$\varphi_\delta(x) = \varphi(x) - \delta |x - x_0|^2.$$

From the result on perturbations by a small parabola (Lem. 7.2), we deduce that for $\delta > 0$ small enough, the exists a radius $\rho > 0$ such that

$$(L\varphi_\delta) \geq \theta/2 + f > f \quad \text{on} \quad B_\rho(x_0) \subset \Omega. \tag{9.5}$$

In particular, we see that x_0 is a point of strict minimum of $\overline{u}_* - \varphi_\delta$. We set for $\eta \geq 0$

$$u_\eta(x) = \max(\overline{u}(x), \eta + \varphi_\delta(x)).$$

Let us consider a point $y_0 \in \Omega$ and a test function $\psi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} u_\eta \leq \psi \quad \text{on} \quad \overline{\Omega}, \\ \\ u_\eta(y_0) = \psi(y_0). \end{cases}$$

Step 2.3: u_η is a subsolution on $\{u_\eta = \overline{u}\}$

Let us assume that $y_0 \in \{u_\eta = \overline{u}\}$. Because $u_\eta \geq \overline{u}$, we deduce that ψ is also a test function for $\overline{u}$ at y_0 and then u_η satisfies the subsolution property at y_0 with the test function ψ.

Step 2.4: u_η is a subsolution on $\{u_\eta > \overline{u}\} \cap \Omega$

When $\eta > 0$, let us choose $r > 0$ such that

$$\eta = \delta r^2. \tag{9.6}$$

This implies that

$$\eta + \varphi_\delta(x) \leq \varphi(x) \leq \overline{u}_*(x) \leq \overline{u}(x) \quad \text{if} \quad x \notin B_r(x_0) \cap \overline{\Omega}$$

and then

$$\{u_\eta > \overline{u}\} \subset B_r(x_0) \subset B_\rho(x_0) \subset \Omega,$$

if we choose η small enough such that r given by (9.6) satisfies

$$r \leq \rho. \tag{9.7}$$

Assume that $y_0 \in \{u_\eta > \overline{u}\}$. Because $u_\eta \geq \eta + \varphi_\delta$, we deduce that ψ is also a test function for $\eta + \varphi_\delta$ at y_0 and then

$$(L\psi)(y_0) \geq (L(\eta + \varphi_\delta))(y_0) = (L\varphi_\delta)(y_0).$$

From (9.5) and for the choice (9.7), we see that

$$(L\varphi_\delta)(y_0) \geq \theta/2 + f(y_0) > f(y_0).$$

This shows that u_η is a subsolution at y_0.

Step 2.5: conclusion

Therefore u_η is a subsolution on $\overline{\Omega}$. On one hand we deduce from (9.4) that

$$\eta + \varphi_\delta \leq u^+ \quad \text{on} \quad \overline{\Omega}$$

for $\eta > 0$ small enough, and then

$$u^- \leq u_\eta \leq u^+ \quad \text{on} \quad \overline{\Omega}.$$

This shows that $u_\eta \in \mathcal{S}$ for $\eta > 0$ small enough, and then $u_\eta \leq \overline{u}$. On the other hand, by definition of $\overline{u}_*$ there is a sequence of points $x_\varepsilon \to x_0$ such that

$$\overline{u}_*(x_0) = \lim_{\varepsilon \to 0} \overline{u}(x_\varepsilon) \geq \lim_{\varepsilon \to 0} u_\eta(x_\varepsilon) \geq \lim_{\varepsilon \to 0} \eta + \varphi(x_\varepsilon) = \eta + \varphi(x_0) = \eta + \overline{u}_*(x_0),$$

which is a contradiction. We finally conclude that $\overline{u}_*$ is a supersolution on Ω, and then $\overline{u}$ is a viscosity solution on Ω.

This ends the proof of the theorem. □

10. Regularity properties

10.1. Continuity of subsolutions

First out is the result that all subsolutions are actually continuous.

Proposition 10.1 (viscosity subsolutions are continuous)**.** *Let $f \in C(\Omega) \cap L^\infty(\Omega)$. If u is a subsolution of (4.1) then $u \in C(\Omega)$.*

Proof of Proposition 10.1. The proof is divided into several steps.

Step 1: discontinuity at x_0

We proceed by contradiction and assume that there exists a point $x_0 \in \Omega$ and a sequence $(x_\varepsilon)_\varepsilon$ such that for some $\delta > 0$

$$u(x_\varepsilon) \le u(x_0) - 3\delta \quad \text{and} \quad x_\varepsilon \to x_0.$$

Because u is upper semicontinuous, for each point x_ε, there exists $r_\varepsilon > 0$ such that

$$u \le u(x_\varepsilon) + \delta \le u(x_0) - 2\delta \quad \text{on} \quad B_{r_\varepsilon}(x_\varepsilon). \tag{10.1}$$

Step 2: construction of a first test function φ

Because u is upper semicontinuous, for any $\eta > 0$, there exists $\rho_\eta \in (0,1)$ such that

$$u < u(x_0) + \eta \quad \text{on} \quad B_{2\rho_\eta}(x_0) \subset \Omega.$$

Consider a test function $\varphi \in C^1(\overline{\Omega})$ satisfying

$$\begin{cases} \varphi = u(x_0) + \eta \quad \text{on} \quad B_{\rho_\eta}(x_0) \subset\subset \Omega, \\ \varphi \ge u \quad \text{on} \quad \overline{\Omega}. \end{cases}$$

Step 3: the first perturbed test function

Let us now consider a function ψ satisfying

$$\begin{cases} \psi \in C^1(\mathbb{R}), \\ 0 \le \psi(-z) = \psi(z) \le 2, \\ \psi = 0 \quad \text{on} \quad \mathbb{R}\backslash[-1,1], \\ \psi(1/2) = 1, \\ \psi' < 0 \quad \text{on} \quad (0,1), \\ \psi(0) = 2. \end{cases}$$

Put

$$\Psi(x) = \psi(|x|) \quad \text{and} \quad M = \sup_{x \in \mathbb{R}^N} |\nabla \Psi(x)|,$$

and define for $\lambda > 0$

$$\Psi_{x_\varepsilon}^\lambda = \Psi((x - x_\varepsilon)/\lambda).$$

Choosing the sequence r_ε such that $r_\varepsilon \to 0$ as $\varepsilon \to 0$, we know that for ε small enough we have

$$B_{r_\varepsilon}(x_\varepsilon) \subset B_{\rho_\eta/4}(x_0). \tag{10.2}$$

We then define

$$u_\varepsilon^\lambda = u(x_0) + \eta - \eta\Psi_{x_\varepsilon}^\lambda$$

and we set

$$\lambda_\varepsilon = \sup A_\varepsilon \quad \text{with} \quad A_\varepsilon := \left\{\lambda > 0, \quad u < u_\varepsilon^\lambda \quad \text{on} \quad B_{\rho_\eta}(x_0)\right\}.$$

From (10.1), we deduce that if $\lambda \in (0, r_\varepsilon]$, then $\lambda \in A_\varepsilon$ if $\eta < \delta$. Moreover for

$$\bar\lambda_\varepsilon = 2|x_\varepsilon - x_0|$$

we have $u_\varepsilon^{\bar\lambda_\varepsilon}(x_0) = u(x_0)$ and therefore $\bar\lambda_\varepsilon \notin A_\varepsilon$. Moreover we have $B_{\bar\lambda_\varepsilon}(x_\varepsilon) \subset B_{3|x_\varepsilon - x_0|}(x_0) \subset B_{3\rho_\eta/4}(x_0)$ because of (10.2). Therefore for any $0 < \lambda \le \bar\lambda_\varepsilon$, we have

$$u_\varepsilon^\lambda = u(x_0) + \eta \quad \text{in a neighborhood of} \quad \partial B_{\rho_\eta}(x_0). \tag{10.3}$$

Thus, there exists $\lambda_\varepsilon \in (r_\varepsilon, \bar\lambda_\varepsilon]$ and y_ε such that

$$\begin{cases} u \le u_\varepsilon^{\lambda_\varepsilon} \quad \text{on} \quad B_{\rho_\eta}(x_0), \\ u = u_\varepsilon^{\lambda_\varepsilon} \quad \text{at} \quad y_\varepsilon \in B_{\lambda_\varepsilon}(x_\varepsilon) \subset\subset B_{\rho_\eta}(x_0), \end{cases} \tag{10.4}$$

and due to (10.1) we can see that $y_\varepsilon \notin B_{r_\varepsilon}(x_\varepsilon)$. See Figure 1 for a possible situation. We now define

$$\varphi_\varepsilon(x) = \begin{cases} u_\varepsilon^{\lambda_\varepsilon} \ge u(x_0) - \eta & \text{if} \quad x \in B_{\rho_\eta}(x_0), \\ \varphi & \text{if} \quad x \in \overline{\Omega} \backslash B_{\rho_\eta}(x_0). \end{cases}$$

This can also be written as

$$\varphi_\varepsilon = \varphi - \eta\Psi_{x_\varepsilon}^{\lambda_\varepsilon}.$$

Because of (10.3), we see that $\varphi_\varepsilon \in C^1(\overline{\Omega})$ and satisfies

$$|\nabla\varphi_\varepsilon| \le M\eta/\lambda_\varepsilon \quad \text{on} \quad B_{\lambda_\varepsilon}(x_\varepsilon) \subset\subset B_{\rho_\eta}(x_0)$$

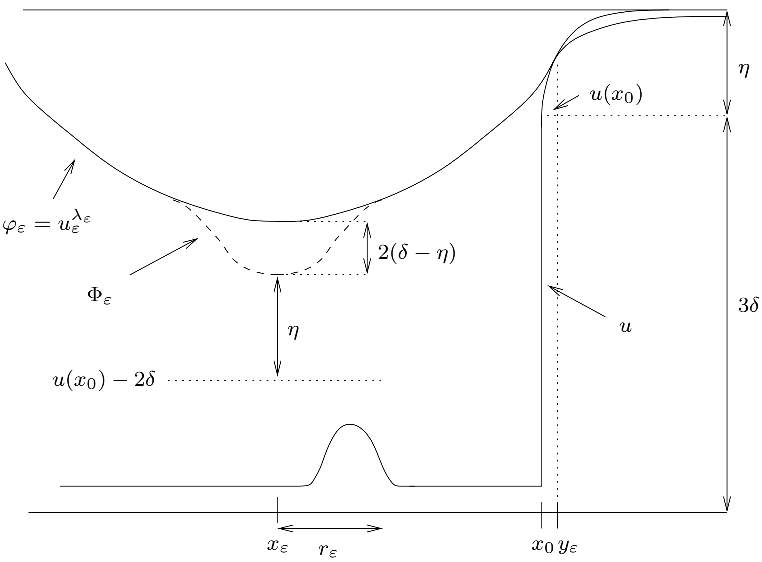

FIGURE 1. One possible situation of the chosen test functions.

and in particular we have with $\lambda_\varepsilon \le \bar\lambda_\varepsilon = 2|x_\varepsilon - x_0| < 1/2$ for ε small enough

$$\begin{aligned}(L^+\varphi_\varepsilon)(y_\varepsilon) &\le \max\left(\sup_{y\in\overline{\Omega}\cap B_{\lambda_\varepsilon}(x_\varepsilon),\ y\ne y_\varepsilon} \frac{\varphi_\varepsilon(y)-\varphi_\varepsilon(y_\varepsilon)}{|y-y_\varepsilon|^\alpha},\quad \sup_{y\in\overline{\Omega}\setminus B_{\lambda_\varepsilon}(x_\varepsilon)} \frac{\varphi_\varepsilon(y)-\varphi_\varepsilon(y_\varepsilon)}{|y-y_\varepsilon|^\alpha}\right)\\
&\le \max\left(\sup_{y\in\overline{\Omega}\cap B_{\lambda_\varepsilon}(x_\varepsilon),\ y\ne y_\varepsilon} \frac{-\eta\Psi_{x_\varepsilon}^{\lambda_\varepsilon}(y)+\eta\Psi_{x_\varepsilon}^{\lambda_\varepsilon}(y_\varepsilon)}{|y-y_\varepsilon|^\alpha},\quad \sup_{y\in\overline{\Omega}\setminus B_{\lambda_\varepsilon}(x_\varepsilon)} \frac{\varphi(y)-\varphi(y_\varepsilon)+\eta\Psi_{x_\varepsilon}^{\lambda_\varepsilon}(y_\varepsilon)}{|y-y_\varepsilon|^\alpha}\right)\\
&\le \max\left(\frac{M\eta}{(\lambda_\varepsilon)^\alpha},\quad (L^+\varphi)(y_\varepsilon)+\frac{\eta\Psi_{x_\varepsilon}^{\lambda_\varepsilon}(y_\varepsilon)}{(\lambda_\varepsilon)^\alpha}\right)\\
&\le \max\left(\frac{2M\eta}{(\lambda_\varepsilon)^\alpha},\quad (L^+\varphi)(y_\varepsilon)+\frac{2\eta}{(\lambda_\varepsilon)^\alpha}\right)\\
&\le \sup_{x\in\overline{B_{\rho_\eta}(x_0)}} (L^+\varphi)(x) + \frac{2\bar M\eta}{(\lambda_\varepsilon)^\alpha} \quad \text{with} \quad \bar M = \max(M,1).\end{aligned}$$

Step 4: the second perturbed test function

Define (with $\delta > \eta$)

$$\Phi_\varepsilon = \varphi_\varepsilon - (\delta-\eta)\Psi_{x_\varepsilon}^{r_\varepsilon} \le \varphi_\varepsilon$$

which by (10.1) and (10.4) satisfies

$$\begin{cases} \Phi_\varepsilon(y_\varepsilon) = u(y_\varepsilon) = \varphi_\varepsilon(y_\varepsilon), \\ u \le \Phi_\varepsilon. \end{cases}$$

Therefore

$$(L^+\Phi_\varepsilon)(y_\varepsilon) \le (L^+\varphi_\varepsilon)(y_\varepsilon) \le \sup_{B_1(x_0)} (L^+\varphi) + \frac{2\bar{M}\eta}{(\lambda_\varepsilon)^\alpha}\cdot$$

Step 5: estimate of $L^-\Phi_\varepsilon$

We have

$$(L^-\Phi_\varepsilon)(y_\varepsilon) \le \frac{\Phi_\varepsilon(x_\varepsilon) - \Phi_\varepsilon(y_\varepsilon)}{|x_\varepsilon - y_\varepsilon|^\alpha}\cdot$$

Using the fact that $\Phi_\varepsilon(x_\varepsilon) = u(x_0) + \eta - 2\delta$, $\Phi_\varepsilon(y_\varepsilon) = \varphi_\varepsilon(y_\varepsilon) \ge u(x_0) - \eta$ and (10.4), we get

$$(L^-\Phi_\varepsilon)(y_\varepsilon) \le -\frac{2(\delta - \eta)}{(\lambda_\varepsilon)^\alpha}\cdot$$

Step 6: conclusion

For the choice $\eta < \delta/(\bar{M}+1)$ and using the fact that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$, we see that

$$(L\Phi_\varepsilon)(y_\varepsilon) = (L^-\Phi_\varepsilon)(y_\varepsilon) + (L^+\Phi_\varepsilon)(y_\varepsilon) < f(y_\varepsilon)$$

for ε small enough. This is in contradiction with the fact that u is a subsolution.

This ends the proof of the proposition. □

As a consequence we obtain the continuity of the solutions constructed by Perron's method.

Corollary 10.2. *The solutions constructed in Theorem 9.1 are continuous on $\overline{\Omega}$.*

Proof of Corollary 10.2. By the previous proposition, any subsolution and thus any solution is continuous inside Ω. By construction, since the solutions is trapped between u^- and u^+, the solution is then also continuous up the boundary. □

10.2. Uniform regularity

First we present a comparison result for certain sub- and supersolutions in "domains minus a point".

Lemma 10.3 (comparison)**.** *Let $x_0 \in \Omega$ and assume that u is upper semicontinuous and that in the viscosity sense there holds*

$$\begin{cases} Lu \ge f & \text{in } \Omega, \\ Lv < f & \text{in } \Omega \setminus \{x_0\}, \end{cases}$$

with the boundary condition

$$u \le v \text{ on } \partial\Omega \cup \{x_0\}.$$

If $v \in C^1(\Omega \setminus \{x_0\}) \cap C(\overline{\Omega})$, then $u \le v$ in $\overline{\Omega}$.

Proof of Lemma 10.3. We argue by contradiction. If the assertion does not hold, then there is a point $y \in \Omega \setminus \{x_0\}$ so that $u - v$ attains a positive maximum at y. If $v \in C^1(\Omega)$, v will essentially be a test function for u which gives a contradiction. If we assume only $v \in C^1(\Omega \setminus \{x_0\})$ the result can be obtained by approximation. □

We remark that due to this result combined with Lemma 8.3, we can compare solutions to "Hölder cones" of the type $C|x|^\beta$ for $\beta < \alpha < 1$. Furthermore, if we are dealing with the homogeneous equations, we can take $\beta = \alpha$ due to Lemma 8.1 (and for $\alpha = 1$ we had the special construction in Lem. 8.2).

Proposition 10.4 (bound in L^∞)**.** *Let $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C(\partial\Omega)$ and u be a viscosity solution of (4.1). Then there is $C(\alpha, \|g\|_{L^\infty(\partial\Omega)}, \|f\|_{L^\infty(\Omega)}, \text{diam } \Omega)$ such that*

$$\|u\|_{L^\infty(\Omega)} \le C.$$

Proof of Proposition 10.4. Fix $\beta \in (0,\alpha)$ and $x_0 \in \partial\Omega$, and let

$$v(x) = a + b|x - x_0|^\beta,$$

where

$$a > \|g\|_{L^\infty(\partial\Omega)},$$

and b is chosen so that

$$Lv < -\|f\|_{L^\infty(\Omega)}.$$

This is possible if $\beta < \alpha$ due to Lemma 8.3, choosing b such that

$$-bC(\alpha,\beta)(\text{diam }\Omega)^{\beta-\alpha} < -\|f\|_{L^\infty(\Omega)}.$$

Then we are in the situation of Lemma 10.3 which implies $u^* \le v$ in Ω. Similarly we can obtain a bound from below. □

Proposition 10.5 (partial regularity of solutions to the inhomogeneous equations)**.** *Let $f \in C(\Omega) \cap L^\infty(\Omega)$ and u be a continuous viscosity solution of (4.1). Then for all $0 < \beta < \alpha$, for all compact sets $K \subset\subset \Omega$ and with $d = \text{dist}(K, \partial\Omega)$ we have*

$$[u]_{\beta,K} \le \max\left(\frac{2\|u\|_{L^\infty(\Omega)}}{d^\beta}, \frac{\|f\|_{L^\infty(\Omega)}(\text{diam }\Omega)^{\alpha-\beta}}{C(\alpha,\beta)}\right),$$

where $C(\alpha,\beta)$ is defined in Lemma 8.3. If moreover, $g \in C^{0,\beta}(\partial\Omega)$. Then

$$[u]_{\beta,\Omega} \le \max\left(\|g\|_{C^{0,\beta}(\partial\Omega)}, \frac{\|f\|_{L^\infty(\Omega)}(\text{diam }\Omega)^{\alpha-\beta}}{C(\alpha,\beta)}\right).$$

Proof of Proposition 10.5. For the first part, take $x_0 \in K$ and

$$v(x) = u(x_0) + \frac{C\|u\|_{L^\infty(\Omega)}}{d^\beta}|x - x_0|^\beta$$

with $C \ge 2$ and so that $Lv < f$ in $\Omega \setminus \{x_0\}$. This C can be chosen uniformly with respect to the point $x_0 \in K$. It is sufficient to assume

$$\frac{C\|u\|_{L^\infty(\Omega)}}{d^\beta}C(\alpha,\beta)(\text{diam }\Omega)^{\beta-\alpha} \ge \|f\|_{L^\infty(\Omega)}.$$

Then for $x \in \partial\Omega$ we have

$$u(x) - v(x) = u(x) - u(x_0) - \frac{C\|u\|_{L^\infty(\Omega)}}{d^\beta}|x - x_0|^\beta \le 2\|u\|_{L^\infty(\Omega)} - \frac{C\|u\|_{L^\infty(\Omega)}}{d^\beta}|x - x_0|^\beta \le 0.$$

Hence, by Lemma 10.3, $u \le v$ everywhere. Similarly we can obtain a bound from below of $u(x) - u(x_0)$. This concludes the first part.

For the second part, let $x_0 \in \partial\Omega$ and

$$v(x) = u(x_0) - C|x - x_0|^\beta.$$

Clearly, $v(x) \le u(x)$ for any $x \in \partial\Omega$ and $Lv > f$ when C is large enough (since $g \in C^{0,\alpha}$ and due to Lem. 8.3). Indeed, choose C such that

$$C \ge [g]_{\beta,\partial\Omega} \quad \text{and} \quad CC(\alpha,\beta)(\text{diam }\Omega)^{\beta-\alpha} \ge \|f\|_{L^\infty(\Omega)}. \tag{10.5}$$

Thus, Lemma 10.3 implies $v(x) \leq u(x)$ for all $x \in \Omega$. Written differently, we have for any $x_0 \in \partial\Omega$ and $x \in \Omega$,

$$u(x_0) \leq u(x) + C|x - x_0|^\beta = w(x_0).$$

Thus, Lemma 10.3 applied with w implies (becase of (10.5))

$$u(y) \leq u(x) + C|x - y|^\beta,$$

for any $x, y \in \Omega$. Applying the same arguments to $-u$, implies a similar bound from below of $u(y) - u(x)$, and thus the proof is finished. □

Proof of Theorem 1.8. Part (i) follows from Lemma 5.1, part (ii) follows from Proposition 10.5 and part (iii) follows from Proposition 10.5.

For part (iv), the result follows from the exact same arguments as in the proof of Proposition 10.5 with $\beta = \alpha$ and $C = [g]_{\alpha,\partial\Omega}$, using Lemmas 8.1 and 8.2. The reason why we can do this for the α-barriers is simply that we do not need to compare with solutions having big or small operators L, since we are dealing with the homogeneous case.

Alternatively, one can apply the estimate in Proposition 10.5, taking $f = 0$ and letting $\beta \to \alpha$. □

Remark 10.6. As remarked by Luis Silvestre, when $f = 0$, we obtain an optimal Hölder extension of g, for all exponents $\beta < \alpha$, and this holds also true for Δ_∞.

In fact, following the proof of Proposition 10.5, one realizes that something similar holds for a general operator A (non-local or local) under quite mild assumptions on A, if we can find a strict supersolution (away from the origin) v regular enough to be admissible as a test function such that

$$v(x) = v(|x|), v(0) = 0, v \geq 0 \text{ and } |g(x) - g(x_0)| \leq C_v v(x - x_0).$$

Then if $Au = 0$ in Ω and $u = g$ on $\partial\Omega$ there holds for all $x, x_0 \in \Omega$

$$|u(x) - u(x_0)| \leq C_v v(x - x_0).$$

11. Uniqueness

Finally we prove a uniqueness result under the same assumptions as in Lemma 5.1. The idea is to compare sub- and supersolutions to the solution given by the representation formula in Lemma 5.1, which then yields in the uniqueness.

Lemma 11.1 (convolution and Lipschitz with respect to the distance $|\cdot|^\alpha$)**.** *For* $\alpha \in (0, 1]$ *assume that*

$$\frac{u(y) - u(x)}{|y - x|^\alpha} \leq L \quad \textit{for all} \quad y, x \in B_r(0).$$

In addition, let ρ_ε *be a mollifier* ($\int \rho_\varepsilon = 1$ *and* $\rho_\varepsilon \geq 0$) *with support in* $B_\varepsilon(0)$. *Then* $u_\varepsilon = \rho_\varepsilon \star u$ *satisfies when* $\varepsilon < r$

$$\frac{u_\varepsilon(y) - u_\varepsilon(x)}{|y - x|^\alpha} \leq L \quad \textit{for all} \quad y, x \in B_{r-\varepsilon}(0).$$

Proof of Lemma 11.1. For all $x, y \in B_{r-\varepsilon}(0)$, we have with $y = x + h$

$$\begin{aligned} u_\varepsilon(x+h) - u_\varepsilon(x) &= (\rho_\varepsilon \star u)(x+h) - (\rho_\varepsilon \star u)(x) \\ &= \int \mathrm{d}z \rho_\varepsilon(z) \{u(x+h-z) - u(x-z)\} \\ &\leq \int \mathrm{d}z \rho_\varepsilon(z) L|h|^\alpha \\ &= L|h|^\alpha. \end{aligned}$$

This shows exactly the expected result. □

Proposition 11.2 (comparison when $f = 0$)**.** *Under the hypotheses of Lemma 5.1, take u to be the therein implicitly defined solution and v a subsolution (resp. a supersolution) of (4.1). Then $u \geq v$ (resp. $u \leq v$).*

Proof of Proposition 11.2. We give the proof for the case when v is a subsolution, the proof being similar when v is a supersolution.

Step 1: preliminaries

We first observe that we can apply steps 1–4 of the proof of Lemma 5.1 to obtain

$$(L^\pm u)(x) = \ell_x^\pm(u(x)). \tag{11.1}$$

Assume that

$$M = \sup_{\overline{\Omega}} (v - u) > 0$$

and consider the set

$$K_0 = \left\{x \in \overline{\Omega},\ v(x) - u(x) = M\right\}.$$

Because $u \in C(\overline{\Omega})$ and v is upper semi continuous, we see that the compact set K_0 satisfies

$$K_0 \subset\subset \Omega.$$

For some fixed $\delta > 0$ small enough, let us consider a compact δ-neighborhood K_δ^+ of K_0 satisfying

$$K_0 \subset\subset K_\delta^+ \subset\subset \Omega$$

and a δ-neighborhood Ω_δ of Ω. We first extend u on Ω_δ by a continuous function still denoted by u. Since u is continuous on $\overline{\Omega}$ this can be done thanks to a theorem of Lebesgue, found in [11]. In fact there is also an explicit extension

$$u_{\text{ext}}(x) = \inf_{y \in \overline{\Omega}} (\omega(x-y) + u(y)),$$

if ω, the modulus of continuity of u on $\overline{\Omega}$, is assumed to be continuous. If ω is a distance, then u_{ext} is C^ω-continuous, otherwise it might have a slighty worse modulus of continuity.

Then consider a mollifier $\rho_\varepsilon(x)$ and set

$$u_\varepsilon = \rho_\varepsilon \star u$$

and

$$M_\varepsilon = \sup_{\overline{\Omega}} (v - u_\varepsilon) \geq M/2 > 0$$

where the bound from below holds for ε small enough. Moreover we also have

$$K_\varepsilon := \left\{x \in \overline{\Omega},\ v(x) - u_\varepsilon(x) = M_\varepsilon\right\} \subset\subset K_\delta^+ \subset\subset \Omega$$

for ε small enough. We then deduce that

$$\begin{cases} v \le M_\varepsilon + u_\varepsilon =: \varphi_\varepsilon, \\ v = \varphi_\varepsilon \quad \text{on} \quad K_\varepsilon. \end{cases}$$

On the other hand, by the upper semi continuity of v, there exists a neighborhood V of $\partial\Omega$ in $\overline{\Omega}$ such that for ε small enough

$$v \le u_\varepsilon + M/8 \quad \text{on} \quad V \subset \overline{\Omega}\backslash K_\delta^+.$$

Let $\psi \in C^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} \psi = 1 \quad \text{on} \quad \partial\Omega, \\ \psi = 0 \quad \text{on} \quad \Omega\backslash V, \\ 0 \le \psi \le 1 \quad \text{on} \quad \overline{\Omega}. \end{cases}$$

Then set

$$\tilde{\varphi}_\varepsilon = \varphi_\varepsilon - \frac{M}{4}\psi$$

which satisfies

$$\begin{cases} v \le \tilde{\varphi}_\varepsilon, \\ v = \tilde{\varphi}_\varepsilon \quad \text{on} \quad K_\varepsilon. \end{cases}$$

Therefore for any $x_\varepsilon \in K_\varepsilon$, $\tilde{\varphi}_\varepsilon$ is a test function for v at x_ε, and then

$$0 \le (L\tilde{\varphi}_\varepsilon)(x_\varepsilon).$$

Step 2: limit for L^-

Up to extraction of a subsequence, we have $x_\varepsilon \to x_0 \in \Omega$. Moreover u_ε converges to u uniformly on $\overline{\Omega}$, and then $M_\varepsilon \to M$. From Lemma 3.1 (ii), we deduce with $\tilde{\varphi}_0 = M + u - \dfrac{M}{4}\psi$ that

$$\begin{aligned} \limsup_{\varepsilon\to 0}\, (L^-\tilde{\varphi}_\varepsilon)(x_\varepsilon) &\le (L^-\tilde{\varphi}_0)(x_0) \\ &\le \left(L^-\left(u - \frac{M}{4}\psi\right)\right)(x_0) \\ &\le \inf_{y\in\partial\Omega} \frac{g(y) - M/4 - u(x_0)}{|y - x_0|^\alpha} \\ &\le (L^-u)(x_0) - \delta_0 \quad \text{with} \quad \delta_0 = \frac{M}{4\sup_{y\in\partial\Omega}|y - x_0|^\alpha}. \end{aligned} \tag{11.2}$$

Step 3: limit for L^+

We have

$$(L^+\tilde{\varphi}_\varepsilon)(x_\varepsilon) \le (L^+\varphi_\varepsilon)(x_\varepsilon) = (L^+u_\varepsilon)(x_\varepsilon). \tag{11.3}$$

For any $x \in \Omega$ let us set

$$L_x^+ = \sup_{y\in\partial\Omega} \frac{g(y) - u(x)}{|y - x|^\alpha}.$$

From the continuity of u, we deduce that the map $x \mapsto L_x^+$ is continuous on Ω. In particular for any $\eta > 0$, there exists $r > 0$ such that

$$|L_x^+ - L_{x_0}^+| \le \eta \quad \text{for all} \quad x \in B_r(x_0) \subset\subset \Omega.$$

We also recall that due to (11.1), for all $x \in \Omega$ we have

$$u(y) - u(x) \leq L_x^+ |y - x|^\alpha \quad \text{for all} \quad y \in \overline{\Omega}$$

and then for all $x \in B_r(x_0)$

$$u(y) - u(x) \leq (L_{x_0}^+ + \eta)|y - x|^\alpha \quad \text{for all} \quad y \in \overline{\Omega}.$$

Up to choosing δ small enough, we can always assume that the extension u on Ω_δ satisfies for all $x \in B_r(x_0)$

$$u(y) - u(x) \leq a|y - x|^\alpha \quad \text{for all} \quad y \in \Omega_\delta \quad \text{with} \quad a = (L_{x_0}^+ + 2\eta).$$

Lemma 11.1 implies for ε small enough that

$$u_\varepsilon(y) - u_\varepsilon(x) \leq a|y - x|^\alpha \quad \text{on} \quad B_{r/2}(x_0).$$

Now, choose ε small enough such that $|x_\varepsilon - x_0| < r/4$. Then we have

$$(L^+ u_\varepsilon)(x_\varepsilon) \leq \max\left(a, \sup_{y \in \overline{\Omega} \setminus B_{r/4}(x_\varepsilon)} \frac{u_\varepsilon(y) - u_\varepsilon(x_\varepsilon)}{|y - x_\varepsilon|^\alpha}\right).$$

Therefore we deduce from the uniform convergence of u_ε towards u that

$$\limsup_{\varepsilon \to 0} (L^+ u_\varepsilon)(x_\varepsilon) \leq \max\left(a, \sup_{y \in \overline{\Omega} \setminus B_{r/4}(x_0)} \frac{u(y) - u(x_0)}{|y - x_0|^\alpha}\right) = a = L_{x_0}^+ + 2\eta.$$

Since this is true for any $\eta > 0$, we obtain

$$\limsup_{\varepsilon \to 0} (L^+ \tilde{\varphi}_\varepsilon)(x_\varepsilon) \leq \limsup_{\varepsilon \to 0} (L^+ u_\varepsilon)(x_\varepsilon) \leq L_{x_0}^+ = (L^+ u)(x_0). \tag{11.4}$$

Step 4: conclusion

From (11.2)–(11.4), we deduce that

$$\limsup_{\varepsilon \to 0} (L\tilde{\varphi}_\varepsilon)(x_\varepsilon) \leq (Lu)(x_0) - \delta_0 = 0 - \delta_0 \quad \text{with} \quad \delta_0 > 0.$$

This is in contradiction with the property $Lu = 0$ satisfied by u pointwisely.

This ends the proof. □

Remark 11.3. In the proof above, the essential key is the fact that the supremum and the infimum in $L^\pm u$ are attained on $\partial\Omega$ for the solution given by the representation formula in Lemma 5.1.

Proof of Theorem 1.5. Part (i) follows from Theorem 9.1, Remark 9.2 and Corollary 10.2, while part (ii) is an immediate consequence of Proposition 11.2. □

12. Generalizations

12.1. Replacing Ω with $\mathbb{R}^n$

We remark here that we can replace Ω by $\mathbb{R}^n$ and instead consider the problem

$$\begin{cases} L_{\mathbb{R}^n} u = f \text{ in } \Omega \\ u = g \text{ in } \mathbb{R}^n \setminus \Omega \end{cases}$$

where

$$L_{\mathbb{R}^n} u = \sup_{y\in\mathbb{R}^n,\ y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} + \inf_{y\in\mathbb{R}^n,\ y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha},$$

and g grows at most like $|x|^\beta$ with $\beta < \alpha$ at infinity. For this problem, the corresponding of Theorem 1.5 and Theorem 1.8 will also hold true with $\partial\Omega$ replaced by $\mathbb{R}^n \setminus \Omega$. The crucial result is Lemma 8.3, which allows us to compare with functions of the type $|x|^\beta$.

12.2. More general moduli of continuity

Many of the results in this paper can be generalized when we replace $|x-y|^\alpha$ with some other modulus of continuity.

Consider a function $\omega : \mathbb{R}^N \to [0,\infty)$ such that

$$\begin{cases} \omega(x) > 0 = \omega(0) \quad \text{for all} \quad x \in \mathbb{R}^N \setminus \{0\}, \\ \omega(x+y) \leq \omega(x) + \omega(y) \quad \text{for all} \quad x, y \in \mathbb{R}^N. \end{cases}$$

Define for $x \in \Omega$

$$(L_\omega u)(x) = \sup_{y\in\overline{\Omega},\ y\neq x} \frac{u(y)-u(x)}{\omega(y-x)} + \inf_{y\in\overline{\Omega},\ y\neq x} \frac{u(y)-u(x)}{\omega(y-x)}.$$

For this case, in [16] a representation formula (Lem. 5.1) is found when $f = 0$, and also the analogue of (iv) in Theorem 1.8 for the solution given by the representation formula, with the $C^{0,\alpha}$-regularity replaced by C^ω-regularity.

It seems plausible that one can, following the ideas of the present paper, extend the following results to hold for the operators L_ω:

- The existence *via* Perron's method (Thm. 9.1), when f has compact support.
- The comparison (Prop. 11.2), again under the assumption in Lemma 5.1.

13. Open questions

Some questions that remain unanswered in this paper that could be interesting to study in the future are listed below.

- The uniqueness for general functions f.
- Is the $C^{0,\alpha}$-regularity valid for general f, disprove or prove?
- What happens if we instead consider higher order operators of the form

$$Lu(x) = \sup_{\overline{\Omega}\setminus\{x\}} \frac{u(y)-u(x)-\nabla u(x)\cdot(x-y)}{|x-y|^{1+\alpha}} + \inf_{\overline{\Omega}\setminus\{x\}} \frac{u(y)-u(x)-\nabla u(x)\cdot(x-y)}{|x-y|^{1+\alpha}},$$

  with $\alpha \in [0,1]$. Will this yield in $C^{1,\alpha}$-extensions?

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