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ON THE CONTINUITY OF DEGENERATE n-HARMONIC FUNCTIONS

Flavia Giannetti[1] and Antonia Passarelli di Napoli[1]

Abstract. We study the regularity of finite energy solutions to degenerate n-harmonic equations. The function $K(x)$, which measures the degeneracy, is assumed to be subexponentially integrable, *i.e.* it verifies the condition $\exp(P(K)) \in L^1_{\mathrm{loc}}$. The function $P(t)$ is increasing on $[0,\infty[$ and satisfies the divergence condition

$$\int_1^\infty \frac{P(t)}{t^2}\,\mathrm{d}t = \infty.$$

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1. Introduction

Let us consider the equation

$$\mathrm{div} A(x, Du) = \mathrm{div} f \tag{1.1}$$

in a bounded domain Ω of $\mathbb{R}^n$. We suppose that $A\colon \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ satisfies the following growth conditions

$$|A(x,\xi)| \le k(x)|\xi|^{n-1} \tag{1.2}$$

$$\langle A(x,\xi),\xi\rangle \ge \frac{1}{k(x)}|\xi|^n \tag{1.3}$$

for almost every $x\in\Omega$ and all $\xi\in\mathbb{R}^n$.

It is useful to observe that assumptions (1.2) and (1.3) are equivalent to

$$|\xi|^n + |A(x,\xi)|^{n'} \le K(x)\langle A(x,\xi),\xi\rangle, \tag{1.4}$$

where $n' = n/(n-1)$ and $K(x) = k(x)(k(x)^{n'}+1)$. This inequality is known as distortion inequality and the function $K(x)$, which measures the degeneracy of the equation, is called the distortion function.

We shall say that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a solution of (1.1) if $A(x,Du)$ and f are locally integrable in Ω and u satisfies the equation in the sense of distributions, that is,

$$\int_\Omega \langle A(x,Du) - f, D\varphi\rangle\,\mathrm{d}x = 0$$

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[1] Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università di Napoli "Federico II", via Cintia, 80126 Napoli, Italy. antonia.passarelli@unina.it; flavia.giannetti@unina.it

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for every $\varphi \in C_0^\infty(\Omega)$. A function u will be called a locally finite energy solution if in addition $\langle A(x, Du), Du\rangle$ is locally integrable in Ω.

If K is bounded, the equation is uniformly elliptic and finite energy solutions belong to $W^{1,n}_{\rm loc}(\Omega)$. In the case K unbounded, the condition to lie in $W^{1,n}_{\rm loc}(\Omega)$ is in general neither necessary nor sufficient for a solution to have finite energy.

Hence, in order to study the regularity properties of finite energy solutions to equation (1.1), the usual techniques cannot be used, since they provide test functions whose gradient is essentially proportional to the gradient Du of the solution, and *a priori* $A(x, Du)$ and Du are not in Hölder conjugate spaces.

However, many papers investigated the regularity properties of the solutions of degenerate p-harmonic equations under the assumption that the distortion function K is exponentially integrable, that is

$$\exp(\beta\, K) \in L^1_{\rm loc}(\Omega) \tag{1.5}$$

for some constant $\beta > 0$ (see for example [2,5,9,10,17]). More recently (see [7]), similar studies have been exploited under the more general assumption

$$\exp(P(K)) \in L^1_{\rm loc}(\Omega) \tag{1.6}$$

for a given Orlicz function $P(t)$ verifying the divergence condition

$$\int_1^\infty \frac{P(t)}{t^2}\,{\rm d}t = \infty. \tag{1.7}$$

It is worth pointing out that in all the above mentioned papers, the results hold true for solutions of degenerate p-harmonic type equations, for $1 < p < \infty$. In case $n-1 < p \le n$, it is well known that finite energy solutions of (1.1) with $f = 0$, under the assumption (1.5), are weakly monotone functions and therefore they are continuous in Ω except for a subset $\bar\Omega$ with $\mathcal{H}^{n-p}(\bar\Omega) = 0$ ([2,16]).

The aim of this paper is to study the continuity properties of finite energy solutions of (1.1), under the more general assumptions (1.6) and $f \neq 0$.

Our basic assumption on the function K will be

$$\Phi(K) \in L^1_{\rm loc}(\Omega) \tag{1.8}$$

for a given increasing function

$$\Phi\colon [0,\infty[\to [0,\infty]$$

verifying the following conditions:

$$\lim_{t\to 0}\frac{\Phi(t)}{t} = 0, \qquad \lim_{t\to\infty}\frac{\Phi(t)}{t} = \infty. \tag{1.9}$$

Remark 1.1. Since the distortion function $K(x) \ge 1$, the assumption $\Phi(K) \in L^1_{\rm loc}$ does not require Φ defined in $[0,1[$. Moreover, the local nature of our results makes the values $\Phi(t)$ relevant only for large t. However, it will be easier to work with function defined on the whole interval $[0,\infty[$; if Φ is only defined on $]a,\infty[$ for some $a > 0$, we extend it setting $\Phi(t) = 0$, $\forall t \in [0,a]$. Accordingly, even if the function $\Phi(t) = e^t$ does not verify the first condition in (1.9), we modify it by defining $\Phi(t) = 0$ for $t \in [0,e]$ and $\Phi(t) = e^t$ for $t > e$.

The function Φ may take the value ∞; in this case (1.6) means that K is locally essentially bounded in Ω.

If Φ is finite at every point, we impose further conditions on it. First, we assume that there exists $t_0 > 0$ such that

$$t \mapsto \frac{\Phi(t)}{t} \quad \text{is positive and increasing on } [t_0,\infty[. \tag{1.10}$$

In particular, the function

$$P(t) = \log \Phi(t) \tag{1.11}$$

is definite, finite and strictly increasing on $[t_0, \infty[$. We assume that it verifies the divergence condition:

$$\int_{t_0}^{\infty} \frac{P(t)}{t^2}\,\mathrm{d}t = \infty. \tag{1.12}$$

Hence, we include also the case of subexponentially integrable distortion. Typical examples are obtained choosing

$$P(t) = \frac{t}{\log(e+t)}, \qquad P(t) = \frac{t}{\log(e+t)\log\log(9+t)}\cdot$$

If we define the function

$$\mathscr{A}(t) = \exp\left[\int_{t_0}^{t} \frac{1}{\tau P^{-1}(\log\tau)}\mathrm{d}\tau\right], \tag{1.13}$$

our main result is the following:

Theorem 1.2. *Let u be a finite energy solution of equation (1.1). Assume that inequality (1.4) holds with a function K such that $\exp(P(K)) \in L^1_{\mathrm{loc}}(\Omega)$ and suppose that $K^{\frac{1}{n}}|f| \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > \frac{n}{n-1}$.*

If $\exp(P(t^{\frac{1-\vartheta}{\vartheta}}))$ is convex for some $\frac{n-1}{n} < \vartheta < 1$, then u is continuous in a subset Ω_0 of Ω with full measure. More precisely, there exist positive constants α, σ, c and a radius $R > 0$, such that

$$|u(x) - u(y)|^n \le c\,\frac{1}{\mathscr{A}^{\alpha}\left(\frac{1}{4^n|x-y|^n}\right)} + c\,|x-y|^{n\sigma} \tag{1.14}$$

for all Lebesgue points $x, y \in B_R \Subset \Omega$ of u.

We would like to note that, without assuming the divergence condition (1.12), no continuity can be expected even when the right hand side is zero and $A(x,\cdot)$ is the Beltrami operator of a mapping with finite distortion, as it has been shown in [11]. Our first step is to prove the local boundedness of the solutions, by using the classical truncation method due to Stampacchia, provided the right hand side f has a suitable degree of integrability. This first regularity result could be of interest by itself.

Once a weak local maximum principle has been proven, one could deduce the continuity arguing as in [13,16] and find that the modulus of continuity is logarithmic in a subset $\Omega_0 \subset \Omega$ with full measure. The point here is that we will establish the continuity of the solutions at every Lebesgue point and we will precise the modulus of continuity relating it to the degeneracy of the equation, trough the function $\mathscr{A}(t)$, defined at (1.13).

The proof of Theorem 1.2 strongly relies on an isoperimetric type inequality for the energy of the solution (see Prop. 3.1 below). We proved a similar inequality in the setting of div-curl couples with nonnegative scalar product (see [4,5]), but it cannot be used here since we deal with equations with right hand side different from zero.

Recall that (E, B) is a div-curl couple with nonnegative scalar product if

$$\mathrm{div} B = 0 \qquad\qquad \mathrm{curl} E = 0$$

in the sense of distributions and

$$\langle E, B\rangle \ge 0 \qquad \text{a.e.} \tag{1.15}$$

The assumption (1.15) is unavoidable to establish the result in [4,5], since the proof is based on an approximation argument. Note that to every weak solution of (1.1) it is possible to associate a div-curl couple, setting

$$B = A(x, Du) - f \qquad\qquad E = Du$$

but the presence of the right hand side $f \neq 0$ does not ensure that (1.15) holds.

In order to overcome this difficulty, we shall use an argument due to Lewis [15], based on a construction of test functions obtained truncating the maximal function of the gradient of the solution on its level sets, as in the pioneering paper by Acerbi and Fusco [1].

The isoperimetric type inequality allows us to establish a decay estimate for the energy of the solution, which will be the crucial step for the continuity result.

Observe that Theorem 1.2, in the particular case $f = 0$ and where $A(x, \cdot)$ is the Beltrami operator of a mapping with finite distortion, recovers the result in [12].

The following examples show, for different choices of the degree of degeneracy of the equation (1.1) and for large values of t, the explicit expression of the function $\mathscr{A}$ appearing in the modulus of continuity of the solution u. For more details, we refer to [6].

Example 1.3. If the distortion K is bounded, we define

$$\Phi(t) = \begin{cases} 0, & \text{for } 0 \le t \le \|K\|_\infty \\ \infty, & \text{for } t > \|K\|_\infty. \end{cases} \tag{1.16}$$

Then we find for $t \ge 1$

$$\mathscr{A}(t) = t^{1/\|K\|_\infty}.$$

Compare with [8].

Example 1.4. For $\Phi(t) = \exp(\beta\, t)$, with given $\beta > 0$, we find

$$\mathscr{A}(t) \sim \log t,$$

as $t \to \infty$.

Example 1.5. If $\Phi(t) = \exp(\beta\, t^\gamma)$, for given $\beta > 0$ and $\gamma > 1$, then

$$\mathscr{A}(t) \sim \exp \log^{1-1/\gamma} t.$$

Example 1.6. For

$$\Phi(t) = \exp\left[\frac{t}{\log(\mathcal{E} + t)}\right]$$

we have $\Psi(t) \sim t(\log t)^{-1}(\log\log t)^{-1}$, as $t \to \infty$, and

$$\mathscr{A}(t) \sim \log\log t.$$

In conclusion we underline that the results used to establish the main theorem hold true also in case of degenerate p-harmonic equations, for $1 < p < n$, even if they do not seem to be sufficient to derive the continuity. In virtue of further applications, in the Appendix, we will give precise statements and we will highlight only the proofs which significantly differ from the ones given for $p = n$.

2. Notation and preliminary results

We consider the class of Orlicz functions on $[0, \infty[$, *i.e.* increasing functions $\Phi\colon [0, \infty[\to [0, \infty]$ with $\Phi(0) = 0$. The conjugate to an Orlicz function Φ is defined by:

$$\Phi^*(s) = \sup_{t \ge 0}\{s\, t - \Phi(t)\}, \qquad s \ge 0. \tag{2.1}$$

It is well-known that Φ^* is a convex Orlicz function and the following Young inequality holds, for all t, $s \geq 0$:

$$t\,s \leq \Phi(t) + \Phi^*(s). \tag{2.2}$$

For basic properties of Orlicz functions, we refer to [14,18].

From now on Φ will denote an Orlicz function finite at every point, such that $\dfrac{\Phi(t)}{t}$ is positive and increasing on $[t_0, \infty[$, with $t_0 > 0$, and verifying

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0, \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty. \tag{2.3}$$

Note that we assume $\Phi(t)$ defined on $[t_0, \infty[$, for some $t_0 > 0$, since its values will be relevant only for sufficiently large values of the variable t.

In particular, the function

$$P(t) = \log \Phi(t)$$

is definite, finite and strictly increasing on $[t_0, \infty[$. In the whole paper we will assume that $P(t)$ verifies the divergence condition:

$$\int_{t_0}^{\infty} \frac{P(t)}{t^2}\,\mathrm{d}t = \infty \tag{2.4}$$

and that the inverse function P^{-1} satisfies the Δ_2-condition, *i.e.*, there exist constants $C_\Delta \geq 1$ and $s_0 > 0$ such that

$$P^{-1}(2s) \leq C_\Delta\, P^{-1}(s), \qquad s \geq s_0. \tag{2.5}$$

Note that, under assumptions (2.3), we have $\Phi^*(0) = 0$ and $0 < \Phi^*(s) < \infty$ for all $s > 0$. Hence Φ^* is strictly increasing and diverging at ∞. Therefore, we shall also consider the inverse function to Φ^*:

$$\Psi = (\Phi^*)^{-1}\colon [0, \infty[\to [0, \infty[,$$

which is concave, strictly increasing and verifying $\Psi(0) = 0$.
Moreover, the function Ψ still verifies the divergence condition (1.12).

In [6,7] a wide discussion on the functions defined above is given. In the sequel, we only list those properties we shall need for our arguments.

Lemma 2.1. *We have*

$$\lim_{t\to\infty} \frac{\Psi(t)}{t} = 0$$

and for $s \geq \Phi(t_0)$

$$\frac{s}{\Phi^{-1}(s)} \leq \Psi(s) \leq 2\,\frac{s}{\Phi^{-1}(s)}. \tag{2.6}$$

Moreover for every $\gamma > 1$*, there exists a constant* $C = C(\gamma, \Psi) > 0$ *such that*

$$\frac{s^\gamma}{\Psi(s^\gamma)} \leq C\,\frac{s}{\Psi(s)}, \qquad \forall s > 0 \tag{2.7}$$

and, for every $0 < \vartheta < 1$*, there exists a constant* $C = C(\vartheta, \Psi) > 0$ *such that*

$$s^\vartheta \leq C\,\Psi(s), \qquad \forall s > 1. \tag{2.8}$$

Remark 2.2. By using (2.6) and (2.8), it is easy to verify that for every $\theta > 1$, there exists a constant $C = C(\theta, \Phi) > 0$ such that $t^\theta \leq C\Phi(t)$, for every $t > 1$.

By means of inequality (1.4), the concavity of Ψ and the Young inequality (2.2), we deduce that

$$\Psi(|A(x,Du)|^{n'}+|Du|^n) \le \Psi(K\langle A(x,Du),Du\rangle) \le K\,\Psi(\langle A(x,Du),Du\rangle) \tag{2.9}$$
$$\le \Phi(K)+\langle A(x,Du),Du\rangle \tag{2.10}$$

and hence $\Psi(|A(x,Du)|^{n'}+|Du|^n)\in L^1_{\mathrm{loc}}(\Omega)$.

Note that the function $\mathscr{A}$ defined at (1.13) can be written, by means of Lemma 2.1, as

$$\mathscr{A}(t)=\exp\left[\int_1^t \frac{\Psi(\tau)}{\tau^2}\,\mathrm{d}\tau\right]. \tag{2.11}$$

Hence $\mathscr{A}$ is increasing and verifies that

$$\lim_{t\to\infty}\mathscr{A}(t)=\infty, \tag{2.12}$$

as follows by the divergence condition on Ψ. More precisely, since Φ is finite at every point, $\mathscr{A}$ increases at ∞ more slowly than any positive power of t.

3. An isoperimetric type inequality

The following isoperimetric type inequality, which is reminiscent of the result in [5], will be crucial for our aims. As already mentioned in the Introduction, the nonhomogeneity of equation (1.1) does not allow to use neither the inequality proven in [5] nor the same idea for the proof.

Proposition 3.1. Let u be a finite energy solution of equation (1.1). Assume that inequality (1.4) holds with a function K such that $\exp(P(K))\in L^1_{\mathrm{loc}}(\Omega)$ and suppose that $K^{\frac{1}{n-1}}|f|^{n'}\in L^1_{\mathrm{loc}}(\Omega)$, $n'=\frac{n}{n-1}$. Then, for every $x_0\in\Omega$,

$$\int_{B(x_0,r)}\langle A(x,Du),Du\rangle\,\mathrm{d}x \le c(n)r^{n+\frac{1-n}{\vartheta}}\left(\int_{\partial B(x_0,r)}|A(x,Du)-f|^{n'\vartheta}+|Du|^{n\vartheta}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{\vartheta}} + c(n)\int_{B(x_0,r)}K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x$$

for almost every radius $0<r<\mathrm{dist}(x_0,\partial\Omega)$ and for every $\frac{n-1}{n}<\vartheta<1$.

For the proof we shall use an argument due to Lewis [15], based on the following well-known approximation result by Lipschitz functions of Acerbi–Fusco [1].

Lemma 3.2. *Let $u\in W^{1,p}(\mathbb{R}^n)$, $p>1$. There exists a constant $C=C(n)$ such that, for every $t>0$, we can find a $C\,t$-Lipschitz function $v\colon\mathbb{R}^n\to\mathbb{R}$ which coincides with u a.e. on the set*

$$\{x\in\mathbb{R}^n : M|Dv|(x)\le t\}.$$

We denoted by M the Hardy–Littlewood maximal operator defined for every $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ as

$$Mf(x)=\sup_{x\in B\subset\mathbb{R}^n}\fint_B |f|$$

where B is a ball in $\mathbb{R}^n$. As usual $B(x_0,r)$ denotes the ball centered at x_0 of radius r, *i.e.*

$$B(x_0,r)=\{x\in\mathbb{R}^n : |x-x_0|<r\}.$$

We shall use the notation B_r and B when no confusion arises.

We shall need also next two lemmas which can be found in [7].

Lemma 3.3. *Let f, g, h be nonnegative functions on $\mathbb{R}^n$ and $p > 1$. If $f \le h^{p-1}$ and $g \le h$, then for all $t > 0$ we have*

$$t \int_{Mg>t} f \le Ct^{\vartheta} \int_{h>t/2} h^{p-\vartheta}, \tag{3.1}$$

where $\vartheta = \min\{p-1, 1\}$.

Lemma 3.4. *Let Ψ be an increasing nonnegative function on $[0,\infty[$ satisfying (2.7) and the divergence condition (1.12). If $\Psi(h^p) \in L^1(B)$ and $0 < \gamma < p$, then*

$$\liminf_{t\to\infty} t^{\gamma} \int_{h>t} h^{p-\gamma}\,\mathrm{d}x = 0. \tag{3.2}$$

In the proof of Proposition 3.1, we shall use test functions proportional to the solution. This is possible by virtue of next lemma, that could be of interest by itself.

Lemma 3.5. *Under the same assumptions of Proposition 3.1, we have*

$$\int_{\mathbb{R}^n} \langle A(x,Du), \eta Du\rangle\,\mathrm{d}x \le \int_{\mathbb{R}^n} |A(x,Du) - f||D\eta||u-c|\,\mathrm{d}x + \int_{\mathbb{R}^n} \eta|f||Du|\,\mathrm{d}x \tag{3.3}$$

for every $\eta \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let us fix a cut-off function $\eta \in C_0^\infty(B(x_0,r))$ and consider $\varphi = \eta(u-c)$, where the constant c will be explicited later. The function φ is extended to vanish outside $B(x_0,r)$.

Let us denote by v the Ct-Lipschitz extension of φ to $\mathbb{R}^n$ given by Lemma 3.2. Recalling that $v = \varphi$ on the set $\{M|D\varphi| \le t\}$, we also have $v = \varphi$ on the set $\{MH \le t\}$, with $H = (|A(x,Du) - f|^{n'} + |D\varphi|^n)^{1/n}$. Using v as test function in the equation (1.1), we deduce

$$\begin{aligned}
\int_{\{MH\le t\}} \langle A(x,Du), D\varphi\rangle\,\mathrm{d}x &= -\int_{\{MH>t\}} \langle A(x,Du), Dv\rangle\,\mathrm{d}x \\
&\quad + \int_{\{MH\le t\}} \langle f, D\varphi\rangle\,\mathrm{d}x + \int_{\{MH>t\}} \langle f, Dv\rangle\,\mathrm{d}x \\
&\le Ct \int_{\{MH>t\}} |A(x,Du) - f|\,\mathrm{d}x + \int_{\{MH\le t\}} \langle f, D\varphi\rangle\,\mathrm{d}x.
\end{aligned}$$

Using the definition of φ and Lemma 3.3 with $p = n$, we infer that

$$\begin{aligned}
&\int_{\{MH\le t\}} \langle A(x,Du), \eta Du\rangle\,\mathrm{d}x \\
\le\ & Ct \int_{\{MH>t\}} |A(x,Du) - f|\,\mathrm{d}x + \int_{\{MH\le t\}} |A(x,Du) - f||D\eta||u-c|\,\mathrm{d}x \\
&+ \int_{\{MH\le t\}} \langle f, \eta Du\rangle\,\mathrm{d}x \le Ct \int_{\{H>\frac{t}{2}\}} |H|^{n-1}\,\mathrm{d}x \\
&+ \int_{\{MH\le t\}} |A(x,Du) - f||D\eta||u-c|\,\mathrm{d}x + \int_{\{MH\le t\}} \eta|f||Du|\,\mathrm{d}x.
\end{aligned}$$

Note that $|A(x,Du) - f||u-c| \in L^1_{\mathrm{loc}}(\Omega)$ since, by inequalities (2.8) and (2.9), we have $|A(x,Du) - f| \in L^q_{\mathrm{loc}}(\Omega)$ for every $q < n'$ and $|u-c| \in L^{s^*}_{\mathrm{loc}}(\Omega)$ for every $s < n$, where s^* denotes the Sobolev conjugate exponent of s. Moreover $|f||Du| \in L^1_{\mathrm{loc}}(\Omega)$, as one can easily see by using that $K^{\frac{1}{n-1}}|f|^{n'} \in L^1_{\mathrm{loc}}(\Omega)$ and u is a finite energy solution.

Taking the $\liminf$ as $t \to \infty$ and use Fatou's lemma in the left hand side and the Lebesgue dominated convergence theorem in the right hand side, with the aid of Lemma 3.4, we have the conclusion. □

Proof of Proposition 3.1. Let us define on $B(x_0,r)=B_r$ the function

$$\eta_\varepsilon(x)=\min\left\{1,\frac{r-|x|}{\varepsilon}\right\}.$$

Choosing $\eta(x)=\eta_\varepsilon(x)$ in (3.3), we get

$$\begin{aligned}\int_{B_{r-\varepsilon}}\langle A(x,Du),Du\rangle\,\mathrm{d}x \quad\le\quad & \left|\int_{B_r\setminus B_{r-\varepsilon}}\langle A(x,Du),Du\rangle\,\frac{r-|x|}{\varepsilon}\,\mathrm{d}x\right|\\ &+\int_{B_r\setminus B_{r-\varepsilon}}|A(x,Du)-f|\left|D\left(\frac{r-|x|}{\varepsilon}\right)\right|\,|u-c|\,\mathrm{d}x+\int_{B_{r-\varepsilon}}|f||Du|\,\mathrm{d}x\\ &+\int_{B_r\setminus B_{r-\varepsilon}}|f||Du|\frac{r-|x|}{\varepsilon}.\end{aligned}$$

If we observe that $\frac{r-|x|}{\varepsilon}<1$ in $B_r\setminus B_{r-\varepsilon}$ and that the second integral in the right hand side can be written as

$$\frac{1}{\varepsilon}\int_{r-\varepsilon}^{r}\left(\int_{\partial B_\rho}|A(x,Du)-f|\,|D(r-|x|)|\,|u-c|\,\mathrm{d}\mathcal{H}^{n-1}\right)\mathrm{d}\rho,$$

taking the limit as $\varepsilon\to 0$ we obtain

$$\int_{B_r}\langle A(x,Du),Du\rangle\,\mathrm{d}x\le\int_{\partial B_r}|A(x,Du)-f||u-c|\,\mathrm{d}\mathcal{H}^{n-1}+\int_{B_r}|f||Du|\,\mathrm{d}x.$$

Hence, by using the Sobolev inequality on spheres as formulated by Gehring in [3], we get for $\frac{n-1}{n}<\vartheta<1$

$$\begin{aligned}&\int_{B_r}\langle A(x,Du),Du\rangle\,\mathrm{d}x\\ &\le\left(\sup_{\partial B_r}u-\inf_{\partial B_r}u\right)\int_{\partial B_r}|A(x,Du)-f|\,\mathrm{d}\mathcal{H}^{n-1}+\int_{B_r}|f||Du|\,\mathrm{d}x\\ &\le c(n)r^{1+\frac{1-n}{n\vartheta}}\left(\int_{\partial B_r}|Du|^{n\vartheta}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{n\vartheta}}\left(\int_{\partial B_r}|A(x,Du)-f|\,\mathrm{d}\mathcal{H}^{n-1}\right)\\ &\quad+\int_{B_r}|f||Du|\,\mathrm{d}x\end{aligned}$$

where we have chosen $c=u_{\partial B_r}$. Hence by Hölder's and Young's inequalities we get

$$\begin{aligned}&\int_{B_r}\langle A(x,Du),Du\rangle\,\mathrm{d}x\\ &\le\; c(n)r^{n+\frac{1-n}{\vartheta}}\left(\int_{\partial B_r}(|A(x,Du)-f|^{n'\vartheta}+|Du|^{n\vartheta})\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{\vartheta}}\\ &\quad+\int_{B_r}|K^{\frac{1}{n}}f||K^{-\frac{1}{n}}Du|\,\mathrm{d}x\\ &\le\; c(n)r^{n+\frac{1-n}{\vartheta}}\left(\int_{\partial B_r}(|A(x,Du)-f|^{n'\vartheta}+|Du|^{n\vartheta})\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{\vartheta}}\\ &\quad+\frac{1}{2}\int_{B_r}\frac{1}{K}|Du|^n\,\mathrm{d}x+c\int_{B_r}K^{\frac{1}{n-1}}|f|^{n'}.\end{aligned}\tag{3.4}$$

Estimate (3.4) and inequality (1.4) give

$$\int_{B_r} \langle A(x,Du),Du\rangle\,\mathrm{d}x \le c(n) r^{n+\frac{1-n}{\vartheta}} \left(\int_{\partial B_r} (|A(x,Du)-f|^{n'\vartheta} + |Du|^{n\vartheta})\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{\vartheta}} + c\int_{B_r} K^{\frac{1}{n-1}}|f|^{n'}$$

i.e. the conclusion. □

4. The local boundedness

The proof of our main result deeply relies on the following weak maximum principle.

Proposition 4.1. Let u be a finite energy solution of equation (1.1). Assume that inequality (1.4) holds with a function K such that $\exp(P(K)) \in L^1_{\mathrm{loc}}(\Omega)$ and suppose that f is such that $K^{\frac{1}{n}}|f| \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > \frac{n}{n-1}$. Then u is a locally bounded function satisfying for every ball $B_r \Subset \Omega$

$$\sup_{B_r} u \le \sup_{\partial B_r} u + cr^\sigma \qquad \inf_{B_r} u \ge \inf_{\partial B_r} u - cr^\sigma,$$

where $\sigma = \sigma(n,q) > 0$.

The following well known lemma, due to Stampacchia, will be instrumental for the proof of proposition above.

Lemma 4.2 [19]**.** *Let $s_0 > 0$ and let $\zeta : (s_0,+\infty) \to [0,+\infty)$ be a decreasing function, such that for every $l > h > s_0$*

$$\zeta(l) \le \frac{c}{(l-h)^\alpha}\zeta^\beta(h)$$

where c,α are positive constants and $\beta > 1$. Then $\zeta(s_0+d) = 0$ where $d^\alpha = c2^{\frac{\alpha\beta}{\beta-1}}\zeta^{\beta-1}(s_0)$.

Proof of Proposition 4.1. Observe that inequalities (2.8) and (2.9) yield in particular that $|Du|^{n\theta} \in L^1_{\mathrm{loc}}(\Omega)$, for every $0 < \theta < 1$. Therefore, by the Sobolev imbedding theorem, u is locally bounded on all spheres well contained in Ω.

For a fixed ball $B_r \Subset \Omega$, let us denote by

$$M = \sup_{\partial B_r} u \qquad m = \inf_{\partial B_r} u.$$

For every $h \ge M$, let us define

$$T_h(u) = \begin{cases} -h & \text{if } u \le -h \\ u & \text{if } -h < u < h \\ h & \text{if } u \ge h \end{cases}$$

and consider the function $\varphi = u - T_h(u) \in W^{1,n\theta}_0(B_r)$. The function φ is extended to vanish outside B_r.

Let v be the Ct-Lipschitz extension of φ to $\mathbb{R}^n$ given by Lemma 3.2.

As in the proof of Proposition 3.1, we recall that $v = \varphi$ on the set $\{MH \le t\}$, with $H = (|A(x,Du)-f|^{n'} + |Dv|^n)^{1/n}$. Hence, using v as test function in the equation (1.1), we deduce

$$\begin{aligned}\int_{\{MH\le t\}} \langle A(x,Du),D\varphi\rangle\,\mathrm{d}x &= -\int_{\{MH>t\}} \langle A(x,Du),Dv\rangle\,\mathrm{d}x \\ &\quad + \int_{\{MH\le t\}} \langle f,D\varphi\rangle\,\mathrm{d}x + \int_{\{MH>t\}} \langle f,Dv\rangle\,\mathrm{d}x \\ &\le Ct\int_{\{MH>t\}} |A(x,Du)-f|\,\mathrm{d}x + \int_{\{MH\le t\}} \langle f,D\varphi\rangle\,\mathrm{d}x.\end{aligned}$$

Using the definition of φ and Lemma 3.3, we get

$$\begin{aligned}
&\int_{\{MH\le t\}} \langle A(x,Du), D(u-T_h(u))\rangle \,\mathrm{d}x\\
&\le Ct\int_{\{MH>t\}} |A(x,Du)-f|\,\mathrm{d}x\\
&\quad+\int_{\{MH\le t\}} \langle f, D(u-T_h(u))\rangle\,\mathrm{d}x\\
&\le Ct\int_{\{H>\frac{t}{2}\}} |H|^{n-1}\,\mathrm{d}x\\
&\quad+\int_{\{MH\le t\}} |f||D(u-T_h(u))|\,\mathrm{d}x.
\end{aligned}$$

Taking the lim inf as $t\to\infty$ and use Fatou's lemma in the left hand side and the Lebesgue dominated convergence theorem in the right hand side, with the aid of Lemma 3.4, we have

$$\int_{B_r} \langle A(x,Du), D(u-T_h(u))\rangle\,\mathrm{d}x \le \int_{B_r} |f||D(u-T_h(u))|\,\mathrm{d}x.$$

It follows that

$$\int_{\{|u|\ge h\}} \langle A(x,Du), Du\rangle \mathrm{d}x \le \int_{\{|u|\ge h\}} |f||Du| \tag{4.1}$$

just observing that $u=T_h(u)$ in $\{x\in B_r : |u|<h\}$.

Let s be an exponent such that $\frac{1}{n}+\frac{1}{q}+\frac{1}{s}=1$. By means of Hölder's inequality and assumption (1.4), we estimate the right hand side of (4.1) as follows

$$\begin{aligned}
\int_{\{|u|\ge h\}} |f||Du| &= \int_{\{|u|\ge h\}} K^{\frac{1}{n}}\frac{1}{K^{\frac{1}{n}}}|f||Du|\,\mathrm{d}x\\
&\le \left(\int_{\{|u|\ge h\}} \frac{1}{K}|Du|^n\mathrm{d}x\right)^{\frac{1}{n}} \left(\int_{\{|u|\ge h\}} (K^{\frac{1}{n}}|f|)^q\mathrm{d}x\right)^{\frac{1}{q}} |\{|u|\ge h\}|^{\frac{1}{s}}\\
&\le \left(\int_{\{|u|\ge h\}} \langle A(x,Du),Du\rangle\mathrm{d}x\right)^{\frac{1}{n}} \left(\int_{\{|u|\ge h\}} (K^{\frac{1}{n}}|f|)^q\mathrm{d}x\right)^{\frac{1}{q}} |\{|u|\ge h\}|^{\frac{1}{s}}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\int_{\{|u|\ge h\}} \langle A(x,Du),Du\rangle\mathrm{d}x\\
&\le \left(\int_{\{|u|\ge h\}} \langle A(x,Du),Du\rangle\mathrm{d}x\right)^{\frac{1}{n}} \left(\int_{\{|u|\ge h\}} (K^{\frac{1}{n}}|f|)^q\mathrm{d}x\right)^{\frac{1}{q}} |\{|u|\ge h\}|^{\frac{1}{s}}
\end{aligned}$$

and hence

$$\int_{\{|u|\ge h\}} \langle A(x,Du),Du\rangle\mathrm{d}x \le \left(\int_{\{|u|\ge h\}} (K^{\frac{1}{n}}|f|)^q\mathrm{d}x\right)^{\frac{n}{q(n-1)}} |\{|u|\ge h\}|^{\frac{n}{s(n-1)}}. \tag{4.2}$$

By using again inequality (1.4) and Hölder's inequality, since by assumption and Remark 2.2, $K \in L^p$ for every $p > 1$, we easily get

$$\int_{\{|u|\geq h\}} |Du|^{n\theta} \mathrm{d}x \leq c \left(\int_{\{|u|\geq h\}} \langle A(x, Du), Du \rangle \mathrm{d}x \right)^{\theta}. \tag{4.3}$$

Combining (4.2) with (4.3) and recalling that $(K^{\frac{1}{n}}|f|)^q \in L^1_{\mathrm{loc}}$, we obtain

$$\int_{\{|u|\geq h\}} |Du|^{n\theta} \mathrm{d}x \leq c\, |\{|u| \geq h\}|^{\frac{n\theta}{s(n-1)}}$$

and therefore, by the Sobolev imbedding theorem, it follows that

$$\int_{\{|u|\geq h\}} (|u|-h)^{\frac{n\theta}{1-\theta}} \mathrm{d}x \leq \left(\int_{\{|u|\geq h\}} |Du|^{n\theta} \mathrm{d}x \right)^{\frac{1}{1-\theta}} \leq c |\{|u| \geq h\}|^{\frac{n}{s(n-1)} \frac{\theta}{1-\theta}}.$$

Moreover, since for $l > h$ the inclusion $\{|u| \geq l\} \subset \{|u| \geq h\}$ holds true, we have

$$\int_{\{|u|\geq h\}} (|u|-h)^{\frac{n\theta}{1-\theta}} \mathrm{d}x \geq \int_{\{|u|\geq l\}} (|u|-h)^{\frac{n\theta}{1-\theta}} \mathrm{d}x \geq (l-h)^{\frac{n\theta}{1-\theta}} |\{|u| \geq l\}|.$$

Hence, combining the last two inequalities, we get

$$|\{|u| \geq l\}| \leq \frac{c}{(l-h)^{\frac{n\theta}{1-\theta}}} |\{|u| \geq h\}|^{\frac{n}{s(n-1)} \frac{\theta}{1-\theta}}. \tag{4.4}$$

Note now that the exponent $\dfrac{n}{s(n-1)} \dfrac{\theta}{1-\theta}$ in the right hand side of (4.4) is strictly greater than 1 if θ can be chosen such that

$$\frac{1}{\theta} < 1 + \frac{n}{n-1} \frac{1}{s} = 1 + 1 - \frac{n}{q(n-1)}.$$

Such a choice of θ is possible since, by assumption, $q > \frac{n}{n-1}$. Therefore, we can apply Lemma 4.2 to the function $\zeta(h) = |\{|u| \geq h\}|$ to deduce that

$$\sup_{B(x_0,r)} u \leq M + cr^{\sigma},$$

for some positive exponent σ. Similar arguments give

$$\inf_{B(x_0,r)} u \geq m - cr^{\sigma}$$

which concludes the proof. $\square$

5. The main result

This section is devoted to the proof of Theorem 1.2. In Theorem 5.1 we shall prove a decay estimate for the energy integral that will be fundamental for our aims. In the whole section we shall use the notation

$$\tilde{K}_t = \int_{B_t} \exp(P(K))\, \mathrm{d}x,$$

for a ball $B_t \Subset \Omega$.

Theorem 5.1. *Let u be a finite energy solution of equation (1.1). Assume that inequality (1.4) holds with a function K such that $\exp(P(K)) \in L^1_{\text{loc}}(\Omega)$ and suppose that $K^{\frac{1}{n-1}}|f|^{n'} \in L^1_{\text{loc}}(\Omega)$.*

If $\exp\left(P\left(t^{\frac{1-\vartheta}{\vartheta}}\right)\right)$ is convex for some $\frac{n-1}{n} < \vartheta < 1$, then there exists a positive constant $\alpha = \alpha(n, ||\exp(P(K))||_{L^1(B)})$ such that, for every ball $B_R \subset B \Subset \Omega$, we have

$$\int_{B_r} \langle A(x,Du), Du\rangle \,\mathrm{d}x \le \frac{c(n)}{\mathscr{A}^\alpha\left(\frac{1}{2^n r^n}\right)} \left[\int_{B_R} \langle A(x,Du), Du\rangle \,\mathrm{d}x + \int_{B_R} K^{\frac{1}{n-1}}|f|^{n'}\,\mathrm{d}x\right], \tag{5.1}$$

whenever $0 < r < R/2$.

Proof. Let us denote by

$$H = |A(x,Du) - f|^{n'} + |Du|^n.$$

Applying Proposition 3.1, we have

$$\int_{B_s} \langle A(x,Du), Du\rangle \,\mathrm{d}x$$

$$\le c(n)\left[s^{n+\frac{1-n}{\vartheta}}\left(\int_{\partial B_s} H^\vartheta \,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{\vartheta}} + \int_{B_s} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right] \tag{5.2}$$

for almost every $0 < s < R$. For every $i \in \mathbf{N}$, let us denote now by Δ_i the interval $\left(\frac{R}{2^i}, \frac{R}{2^{(i-1)}}\right)$ and by A_i the annulus $B_{\frac{R}{2^{(i-1)}}} \setminus B_{\frac{R}{2^i}}$. Using Fubini's theorem one can easily check that the set

$$E_i = \left\{t \in \Delta_i : \int_{\partial B_t} H^\vartheta \,\mathrm{d}\mathcal{H}^{n-1} \le \frac{2}{|\Delta_i|}\int_{A_i} H^\vartheta \,\mathrm{d}x\right\}$$

has positive measure. Choosing $r \in E_i$ so that inequality (5.2) holds, we obtain the estimate

$$\begin{aligned}
&\int_{B_{\frac{R}{2^i}}} \langle A(x,Du), Du\rangle \,\mathrm{d}x \le \int_{B_r} \langle A(x,Du), Du\rangle \,\mathrm{d}x \\
&\le c(n)\left[r^{n+\frac{1-n}{\vartheta}}\left(\int_{\partial B_r} H^\vartheta \,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1}{\vartheta}} + \int_{B_r} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right] \\
&\le c(n)\left[r^{n+\frac{1-n}{\vartheta}}\left(\frac{2}{|\Delta_i|}\right)^{\frac{1}{\vartheta}}\left(\int_{A_i} H^\vartheta \,\mathrm{d}x\right)^{\frac{1}{\vartheta}} + \int_{B_r} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right] \\
&\le c(n) r^{n+\frac{1-n}{\vartheta}}\left(\frac{1}{|\Delta_i|}\right)^{\frac{1}{\vartheta}}\left(\int_{A_i}\left(|A(x,Du)|^{n'} + |Du|^n\right)^\vartheta \mathrm{d}x\right)^{\frac{1}{\vartheta}} \\
&\quad + c(n)\left[r^{n+\frac{1-n}{\vartheta}}\left(\frac{1}{|\Delta_i|}\right)^{\frac{1}{\vartheta}}\left(\int_{A_i} |f|^{n'\vartheta}\mathrm{d}x\right)^{\frac{1}{\vartheta}} + \int_{B_r} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right].
\end{aligned} \tag{5.3}$$

Inserting inequality (1.4) in (5.3) and using Hölder's inequality, we find that

$$\begin{aligned}
\int_{B_{\frac{R}{2^i}}} \langle A(x,Du), Du\rangle \,\mathrm{d}x &\le c(n) r^{n+\frac{1-n}{\vartheta}}\left(\frac{1}{|\Delta_i|}\right)^{\frac{1}{\vartheta}}\left(\int_{A_i} K^{\frac{\vartheta}{1-\vartheta}}\,\mathrm{d}x\right)^{\frac{1-\vartheta}{\vartheta}}\int_{A_i} \langle A(x,Du), Du\rangle \,\mathrm{d}x \\
&+ c(n)\left[\int_{B_r} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x + r^{n+\frac{1-n}{\vartheta}}\left(\frac{1}{|\Delta_i|}\right)^{\frac{1}{\vartheta}}|A_i|^{\frac{1-\vartheta}{\vartheta}}\int_{B_r}|f|^{n'}\mathrm{d}x\right].
\end{aligned}$$

At this point, Jensen's inequality applied to the convex function $\Phi(t^{\frac{1-\vartheta}{\vartheta}}) = \exp(P(t^{\frac{1-\vartheta}{\vartheta}}))$, gives

$$\begin{aligned}
&\int_{B_{\frac{R}{2^i}}} \langle A(x,Du),Du\rangle\,\mathrm{d}x \\
\le\ & c(n) r^{n+\frac{1-n}{\vartheta}} \left(\frac{1}{|\Delta_i|}\right)^{\frac{1}{\vartheta}} |A_i|^{\frac{1-\vartheta}{\vartheta}} \left[\Phi^{-1}\left(\fint_{A_i} \Phi(K)\,\mathrm{d}x\right)\right] \int_{A_i} \langle A(x,Du),Du\rangle\,\mathrm{d}x \\
&+ c(n)\left[\int_{B_r} K^{\frac{1}{n-1}} |f|^{n'}\,\mathrm{d}x + r^{n+\frac{1-n}{\vartheta}} \left(\frac{1}{|\Delta_i|}\right)^{\frac{1}{\vartheta}} |A_i|^{\frac{1-\vartheta}{\vartheta}} \int_{B_r} |f|^{n'}\,\mathrm{d}x\right] \\
\le\ & c(n)\left[\Phi^{-1}\left(\frac{1}{|A_i|}\int_{B_R} \Phi(K)\,\mathrm{d}x\right)\right] \int_{A_i} \langle A(x,Du),Du\rangle\,\mathrm{d}x \\
&+ c(n)\int_{B_{\frac{R}{2^{i-1}}}} K^{\frac{1}{n-1}} |f|^{n'}\mathrm{d}x
\end{aligned}$$

where we used that $K(x) \ge 1$ and that, since

$$|A_i| = C(n)\frac{(2^n-1)R^n}{2^{ni}}, \qquad |\Delta_i| = \frac{R}{2^i}, \qquad r < \frac{R}{2^{i-1}}$$

we have

$$r^{n+\frac{1-n}{\vartheta}} \left(\frac{1}{|\Delta_i|}\right)^{\frac{1}{\vartheta}} |A_i|^{\frac{1-\vartheta}{\vartheta}} \le c(n).$$

Hence

$$\begin{aligned}
&\int_{B_{\frac{R}{2^i}}} \langle A(x,Du),Du\rangle\,\mathrm{d}x \\
&\le c(n)\frac{1}{|\Delta_i|}\frac{R}{2^i}\Phi^{-1}\left(\frac{\tilde{K}_R}{|A_i|}\right) \int_{A_i} \langle A(x,Du),Du\rangle\,\mathrm{d}x + c(n)\int_{B_{\frac{R}{2^{i-1}}}} K^{\frac{1}{n-1}} |f|^{n'}\mathrm{d}x \\
&\le c(n)\frac{1}{|\Delta_i|}\frac{R}{2^i}\Phi^{-1}\left(\frac{C(n)2^{ni}\tilde{K}_R}{R^n}\right) \int_{A_i} \langle A(x,Du),Du\rangle\,\mathrm{d}x \\
&\quad + c(n)\frac{1}{|\Delta_i|}\frac{R}{2^i}\int_{B_{\frac{R}{2^{i-1}}}} K^{\frac{1}{n-1}} |f|^{n'}\mathrm{d}x.
\end{aligned} \tag{5.4}$$

Since $K \ge 1$ and Φ is increasing, we get that $\tilde{K}_R \ge \Phi(1)R^n$. Hence

$$\frac{1}{\Phi^{-1}\left(\frac{C(n)2^{ni}\tilde{K}_R}{R^n}\right)} \le \frac{1}{\Phi^{-1}\left(\frac{C(n)2^{n}\tilde{K}_R}{R^n}\right)} \le C(n). \tag{5.5}$$

Therefore, from estimate (5.4) it follows

$$\int_{B_{\frac{R}{2^i}}} \langle A(x,Du),Du\rangle\,\mathrm{d}x \le \frac{c(n)}{|\Delta_i|}\frac{R}{2^i}\Phi^{-1}\left(\frac{C(n)2^{ni}\tilde{K}_R}{R^n}\right)\cdot\left[\int_{A_i}\langle A(x,Du),Du\rangle\,\mathrm{d}x + \int_{B_{\frac{R}{2^{i-1}}}} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right]. \tag{5.6}$$

Now, for $t \in \Delta_i$, we set

$$v_i(t) = \int_{B_{\frac{R}{2^i}}} \langle A(x, Du), Du\rangle \,\mathrm{d}x + \frac{t - R2^{-i}}{|\Delta_i|} \left[\int_{A_i} \langle A(x, Du), Du\rangle \,\mathrm{d}x + \int_{B_{\frac{R}{2^{i-1}}}} K^{\frac{1}{n-1}} |f|^{n'} \,\mathrm{d}x \right]$$

and

$$v(t) = v_1(t)\chi_{\left[\frac{R}{2}, R\right]}(t) + \sum_{i=2}^{\infty} v_i(t)\chi_{\left[\frac{R}{2^i}, \frac{R}{2^{(i-1)}}\right)}(t).$$

Estimate (5.6) implies that

$$v_i(t) \le \left[c(n)\frac{R}{2^i}\Phi^{-1}\left(\frac{C(n)2^{ni}\tilde{K}_R}{R^n}\right) + t - \frac{R}{2^i}\right] \frac{1}{|\Delta_i|} \cdot \left[\int_{A_i} \langle A(x, Du), Du\rangle \,\mathrm{d}x + \int_{B_{\frac{R}{2^{i-1}}}} K^{\frac{1}{n-1}} |f|^{n'} \,\mathrm{d}x \right].$$

Since

$$v_i'(t) = \frac{1}{|\Delta_i|} \left[\int_{A_i} \langle A(x, Du), Du\rangle \,\mathrm{d}x + \int_{B_{\frac{R}{2^{i-1}}}} K^{\frac{1}{n-1}} |f|^{n'} \,\mathrm{d}x \right]$$

we obtain

$$v_i(t) \le \left[c(n)\frac{R}{2^i}\Phi^{-1}\left(\frac{C(n)2^{ni}\tilde{K}_R}{R^n}\right) + t - \frac{R}{2^i}\right] v_i'(t) \tag{5.7}$$

for all $t \in \Delta_i$. Since $t - \frac{R}{2^i} \le \frac{R}{2^{(i-1)}} - \frac{R}{2^i} = \frac{R}{2^i}$, from (5.7) we deduce

$$v_i(t) \le \frac{R}{2^i}\Phi^{-1}\left(\frac{C(n)2^{ni}\tilde{K}_R}{R^n}\right) \left[c(n) + \frac{1}{\Phi^{-1}\left(\frac{C(n)\tilde{K}_R 2^{ni}}{R^n}\right)}\right] v_i'(t). \tag{5.8}$$

Inserting (5.5) in (5.8), it follows that

$$v_i(t) \le C(n)t\Phi^{-1}\left(\frac{C(n)\tilde{K}_R}{t^n}\right) v_i'(t)$$

and hence summing on i and observing that $v(t)$ is a piecewise affine function, we get

$$v(t) \le C(n)t\Phi^{-1}\left(\frac{C(n)\tilde{K}_R}{t^n}\right) v'(t). \tag{5.9}$$

In order to short the notation, we denote by $\gamma = C(n)\tilde{K}_R$ and we rewrite (5.9) as follows

$$v(t) \le C(n)t\Phi^{-1}\left(\frac{\gamma}{t^n}\right) v'(t)$$

and therefore

$$\frac{v'(t)}{v(t)} \ge \frac{C(n)}{t\Phi^{-1}\left(\frac{\gamma}{t^n}\right)}.$$

Now using that $\Phi^{-1}(s) \approx \frac{s}{\Psi(s)}$ by Lemma 2.1, we easily obtain for every $\rho < R$

$$\int_\rho^R \frac{v'(t)}{v(t)}\mathrm{d}t \ge C(n)\int_\rho^R \frac{\mathrm{d}t}{t\Phi^{-1}\left(\frac{\gamma}{t^n}\right)} \ge C(n)\int_\rho^R \frac{1}{t}\frac{\Psi(\gamma/t^n)}{\gamma/t^n}\mathrm{d}t.$$

Observe that if $\gamma < 1$ the concavity of Ψ implies that

$$C(n)\int_\rho^R \frac{1}{t}\frac{\Psi(\gamma/t^n)}{\gamma/t^n}\mathrm{d}t \ge C(n)\int_\rho^R \frac{1}{t}\frac{\Psi(\frac{1}{t^n})}{\frac{1}{t^n}}\mathrm{d}t$$

otherwise

$$C(n)\int_\rho^R \frac{1}{t}\frac{\Psi(\gamma/t^n)}{\gamma/t^n}\mathrm{d}t \ge \frac{C(n)}{\tilde K_R}\int_\rho^R \frac{1}{t}\frac{\Psi(\frac{1}{t^n})}{\frac{1}{t^n}}\mathrm{d}t$$

thank to the monotonicity of Ψ. Therefore, using that $\tilde K_R \le ||\Phi(K)||_{L^1(B)}$, we get

$$\int_\rho^R \frac{v'(t)}{v(t)}\mathrm{d}t \ge C(n, ||\Phi(K)||_{L^1(B)})\int_\rho^R \frac{1}{t}\frac{\Psi(\frac{1}{t^n})}{\frac{1}{t^n}}\mathrm{d}t.$$

The change of variable $s = \frac{1}{t^n}$ in the last integral yields

$$\int_\rho^R \frac{v'(t)}{v(t)}\mathrm{d}t \ge C(n, ||\Phi(K)||_{L^1(B)})\int_{\frac{1}{R^n}}^{\frac{1}{\rho^n}} \frac{\Psi(s)}{s^2}\mathrm{d}s$$

and therefore

$$\log\frac{v(R)}{v(\rho)} \ge C(n, ||\Phi(K)||_{L^1(B)})\int_{\frac{1}{R^n}}^{\frac{1}{\rho^n}} \frac{\Psi(s)}{s^2}\mathrm{d}s$$

which implies

$$v(\rho) \le \left(\exp\left[C(n, ||\Phi(K)||_{L^1(B)})\int_{\frac{1}{R^n}}^{\frac{1}{\rho^n}} \frac{\Psi(s)}{s^2}\mathrm{d}s\right]\right)^{-1} v(R).$$

By using (2.11) and setting $\alpha = C(n, ||\Phi(K)||_{L^1(B)})$, we obtain

$$v(\rho) \le \frac{v(R)}{\mathscr{A}^\alpha\left(\frac{1}{\rho^n}\right)}\cdot \tag{5.10}$$

Since $\rho < R$, there exists $j \in$ such that $\rho \in [R2^{-j}, R2^{-j+1})$. Then, by the definition of the function $v(t)$, from (5.10) it follows

$$\begin{aligned}&\int_{B_{\frac{R}{2^j}}} \langle A(x,Du),Du\rangle\,\mathrm{d}x + \frac{\rho - R2^{-j}}{|\Delta_j|}\left[\int_{A_j}\langle A(x,Du),Du\rangle\,\mathrm{d}x + \int_{B_{\frac{R}{2^{j-1}}}} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right]\\ \le\ & \frac{1}{\mathscr{A}^\alpha\left(\frac{1}{\rho^n}\right)}\left[\int_{B_R}\langle A(x,Du),Du\rangle\,\mathrm{d}x + \int_{B_R} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right]\end{aligned}$$

which obviously implies

$$\int_{B_{\frac{R}{2^j}}} \langle A(x,Du),Du\rangle\,\mathrm{d}x \le \frac{1}{\mathscr{A}^\alpha\left(\frac{1}{\rho^n}\right)}\left[\int_{B_R}\langle A(x,Du),Du\rangle\,\mathrm{d}x + \int_{B_R} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right].$$

At this point, choosing $r = \frac{\rho}{2} < \frac{R}{2}$, we get

$$\int_{B_r} \langle A(x,Du),Du\rangle\,\mathrm{d}x \le \frac{1}{\mathscr{A}^\alpha\left(\frac{1}{2^n r^n}\right)}\left[\int_{B_R}\langle A(x,Du),Du\rangle\,\mathrm{d}x + \int_{B_R} K^{\frac{1}{n-1}}|f|^{n'}\mathrm{d}x\right]$$

which concludes the proof. $\square$

Now, we are ready to embark in the core of the proof of our main result.

Proof of Theorem 1.2. Let us divide the proof into two steps.

Step 1. Let us fix a ball B_R, which, without loss of generality, we may suppose centered at the origin. We shall use the notation B_r in place of $B(0,r)$.

For every $\bar{r} < R$, fix Lebesgue points $x, y \in B_{\frac{\bar{r}}{4}}$. There exists a ball $B(a,r) \subset B_{\bar{r}}$, such that $x, y \in B(a,r)$. In fact, it's enough to choose $a = \frac{x+y}{2}$ and $\frac{|x-y|}{2} < r < \frac{\bar{r}}{2}$.

The following obvious inequalities hold for every $t \in (r, \bar{r})$

$$|u(x) - u(y)| \leq \sup_{B(a,r)} u - \inf_{B(a,r)} u \leq \sup_{B(a,t)} u - \inf_{B(a,t)} u.$$

Combining Proposition 4.1 with the Sobolev inequality on spheres, we have

$$\begin{aligned} |u(x) - u(y)| &\leq \sup_{B(a,t)} u - \inf_{B(a,t)} u \leq \sup_{\partial B(a,t)} u - \inf_{\partial B(a,t)} u + ct^{\sigma} \\ &\leq ct \left(\fint_{\partial B(a,t)} |Du(x)|^{n\vartheta} \mathrm{d}\mathcal{H}^{n-1} \right)^{\frac{1}{n\vartheta}} + ct^{\sigma} \\ &= ct^{1+\frac{1-n}{n\vartheta}} \left(\int_{\partial B(a,t)} |Du(x)|^{n\vartheta} \mathrm{d}\mathcal{H}^{n-1} \right)^{\frac{1}{n\vartheta}} + ct^{\sigma} \end{aligned}$$

for almost every $t \in (r, \bar{r})$ and for $\frac{n-1}{n} < \vartheta < 1$. Therefore

$$\begin{aligned} |u(x) - u(y)|^n &\leq ct^{n+\frac{1-n}{\vartheta}} \left(\int_{\partial B(a,t)} |Du(x)|^{n\vartheta} \mathrm{d}\mathcal{H}^{n-1} \right)^{\frac{1}{\vartheta}} + ct^{n\sigma} \\ &\leq ct^{n+\frac{1-n}{\vartheta}} \left(\int_{\partial B(a,t)} (K\langle A(x,Du), Du\rangle)^{\vartheta} \mathrm{d}\mathcal{H}^{n-1} \right)^{\frac{1}{\vartheta}} + ct^{n\sigma} \end{aligned}$$

where we used inequality (1.4). Applying Hölder's inequality with exponents $\frac{1}{\vartheta}$ and $\frac{1}{1-\vartheta}$ we get

$$|u(x) - u(y)|^n \leq ct^{n+\frac{1-n}{\vartheta}} \left(\int_{\partial B(a,t)} \langle A(x,Du), Du\rangle \mathrm{d}\mathcal{H}^{n-1} \right) \cdot \left(\int_{\partial B(a,t)} K^{\frac{\vartheta}{1-\vartheta}} \mathrm{d}\mathcal{H}^{n-1} \right)^{\frac{1-\vartheta}{\vartheta}} + ct^{n\sigma}. \tag{5.11}$$

Moreover, recalling that $\Phi(K) = \exp(P(K))$, let us define the set $E_i = E_i^1 \cap E_i^2$, with

$$E_i^1 = \left\{ t \in (2^{i-1}r, 2^i r) : \int_{\partial B(a,t)} \langle A(x,Du), Du\rangle \mathrm{d}\mathcal{H}^{n-1} \leq \frac{12}{2^i r} \int_{A_i} \langle A(x,Du), Du\rangle \mathrm{d}x \right\}$$

and

$$E_i^2 = \left\{ t \in (2^{i-1}r, 2^i r) : \int_{\partial B(a,t)} \Phi(K) \mathrm{d}\mathcal{H}^{n-1} \leq \frac{12}{2^i r} \int_{A_i} \Phi(K) \mathrm{d}x \right\}$$

for every $i \in \mathbb{N} \cap [1, \log_2 \frac{\bar{r}}{r}] = I$ and where A_i denotes the annulus $B(a, 2^i r) \setminus B(a, 2^{i-1} r)$. Since $2r < \bar{r}$ we have that $I \neq \emptyset$. Observe that by Fubini's theorem we have

$$|\mathcal{C}(E_i^1)| \leq \frac{2^i r}{12} \qquad \text{and} \qquad |\mathcal{C}(E_i^2)| \leq \frac{2^i r}{12}$$

and hence

$$|E_i| \geq \frac{2^i r}{3} > \frac{2^{i-1} r}{2}.$$

For $t \in E_i$, estimate (5.11) combined with Jensen's inequality yields

$$\begin{aligned} |u(x)-u(y)|^n &\leq ct^{n+\frac{1-n}{\vartheta}}\left(\frac{12}{2^i r}\right)\left(\int_{A_i}\langle A(x,Du),Du\rangle \mathrm{d}x\right)\cdot\left(\fint_{\partial B(a,t)} K^{\frac{\vartheta}{1-\vartheta}}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{1-\vartheta}{\vartheta}} t^{\frac{(n-1)(1-\vartheta)}{\vartheta}} + ct^{n\sigma} \\ &\leq ct\Phi^{-1}\left(\frac{12\tilde{K}_{\bar r}}{t^{n-1}2^i r}\right)\left(\frac{1}{2^i r}\right)\left(\int_{A_i}\langle A(x,Du),Du\rangle \mathrm{d}x\right) \\ &\quad + ct^{n\sigma} \\ &\leq ct\Phi^{-1}\left(\frac{12\tilde{K}_{\bar r}}{t^{n}}\right)\left(\frac{1}{2^i r}\right)\left(\int_{A_i}\langle A(x,Du),Du\rangle \mathrm{d}x\right) \\ &\quad + ct^{n\sigma} \end{aligned}$$

where, in the last line, we used the monotonicity of the function Φ^{-1}. It follows that

$$\frac{|u(x)-u(y)|^n}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar r}}{t^n}\right)} \leq \frac{c}{2^i r}\int_{A_i}\langle A(x,Du),Du\rangle \mathrm{d}x + \frac{ct^{n\sigma}}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar r}}{t^n}\right)}.$$

Integrating the obtained estimate over the set E_i with respect to t, we get

$$\begin{aligned} &|u(x)-u(y)|^n \int_{E_i}\frac{1}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar r}}{t^n}\right)}\,\mathrm{d}t \\ \leq\ & c\int_{A_i}\langle A(x,Du),Du\rangle \mathrm{d}x + c\int_{2^{i-1}r}^{2^i r}\frac{t^{n\sigma}}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar r}}{t^n}\right)}\,\mathrm{d}t. \end{aligned} \tag{5.12}$$

In order to deal with the second integral in the right hand side of (5.12), let us recall that, for s sufficiently large, we have $\frac{1}{\Phi^{-1}(s)} \leq c\frac{\Psi(s)}{s}$ and $\Psi(s) \leq cs$ (see Lem. 2.1). Combining these two properties we have

$$\frac{1}{\Phi^{-1}(s)} \leq c$$

for s sufficiently large. Hence

$$\int_{2^{i-1}r}^{2^i r}\frac{t^{n\sigma}}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar r}}{t^n}\right)}\,\mathrm{d}t \leq c\int_{2^{i-1}r}^{2^i r} t^{n\sigma-1}\,\mathrm{d}t = c\int_{2^{i-1}r}^{2^i r} t^{n\sigma-1}\,\mathrm{d}t. \tag{5.13}$$

Inserting (5.13) in (5.12), we get

$$\begin{aligned} &|u(x)-u(y)|^n \int_{E_i}\frac{1}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar r}}{t^n}\right)}\,\mathrm{d}t \\ \leq\ & c\int_{A_i}\langle A(x,Du),Du\rangle \mathrm{d}x + c\int_{2^{i-1}r}^{2^i r} t^{n\sigma-1}\,\mathrm{d}t. \end{aligned} \tag{5.14}$$

Since the function $t \to \frac{1}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar{r}}}{t^n}\right)}$ is decreasing for t sufficiently small and $|E_i| > \frac{2^{i-1}r}{2}$, one can easily check that

$$\int_{E_i} \frac{1}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar{r}}}{t^n}\right)}\,\mathrm{d}t \geq \int_{\frac{3}{4}2^i r}^{2^i r} \frac{1}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar{r}}}{t^n}\right)}\,\mathrm{d}t.$$

Inserting previous estimate in (5.14), we obtain

$$\begin{aligned} &|u(x)-u(y)|^n \int_{\frac{3}{4}2^i r}^{2^i r} \frac{1}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar{r}}}{t^n}\right)}\,\mathrm{d}t \\ \leq\ & c\int_{A_i} \langle A(x,Du),Du\rangle \mathrm{d}x + c\int_{2^{i-1}r}^{2^i r} t^{n\sigma-1}\,\mathrm{d}t. \end{aligned} \tag{5.15}$$

Summing estimates (5.15) with respect to i in the set I, we can conclude that

$$\begin{aligned} &|u(x)-u(y)|^n \int_{\frac{3}{2}r}^{\bar{r}} \frac{1}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar{r}}}{t^n}\right)}\,\mathrm{d}t \\ \leq\ & c\int_{B_{\bar{r}}} \langle A(x,Du),Du\rangle \mathrm{d}x + c\int_{r}^{\bar{r}} t^{n\sigma-1}\,\mathrm{d}t. \end{aligned} \tag{5.16}$$

Now, with the change of variable $s = \frac{12\tilde{K}_{\bar{r}}}{t^n}$, we estimate the integral in the left hand side of (5.16) as follows

$$\begin{aligned} \int_{\frac{3}{2}r}^{\bar{r}} \frac{1}{t\Phi^{-1}\left(\frac{12\tilde{K}_{\bar{r}}}{t^n}\right)}\,\mathrm{d}t &= \int_{\frac{12\tilde{K}_{\bar{r}}}{\bar{r}^n}}^{\frac{2^{n+2}\tilde{K}_{\bar{r}}}{3^{n-1}r^n}} \frac{1}{s\Phi^{-1}(s)}\,\mathrm{d}s \\ \geq \frac{1}{2}\int_{\frac{12\tilde{K}_{\bar{r}}}{\bar{r}^n}}^{\frac{2^{n+2}\tilde{K}_{\bar{r}}}{3^{n-1}r^n}} \frac{\Psi(s)}{s^2}\,\mathrm{d}s &\geq \frac{1}{2}\int_{c\Phi(1)}^{\frac{2^{n+2}\tilde{K}}{3^{n-1}r^n}} \frac{\Psi(s)}{s^2}\,\mathrm{d}s \end{aligned}$$

where we used Lemma 2.1 and the fact that $\tilde{K}_{\bar{r}} \geq \Phi(1)\bar{r}^n$. On the other hand, the divergence condition of Ψ implies that for every $M>0$ there exists a radius R_M such that

$$\int_{c\Phi(1)}^{\frac{2^{n+2}\tilde{K}}{3^{n-1}r^n}} \frac{\Psi(s)}{s^2} \geq M$$

for every $r < R_M$. Finally we get

$$|u(x)-u(y)|^n \leq c(M)\int_{B_{\bar{r}}} \langle A(x,Du),Du\rangle \mathrm{d}x + c\int_r^{\bar{r}} t^{n\sigma-1}\,\mathrm{d}t \tag{5.17}$$

for every $x,y \in B_{\frac{\bar{r}}{4}}$ and for every $r < \bar{r} < R_M$.

Step 2. Let us choose in Step 1 $R = R_M$, $\bar{r} = 2|x-y|$ and $r = \frac{2}{3}|x-y|$. Hence for every $x,y \in B_{\frac{\bar{r}}{4}}$ by (5.17), we infer that

$$|u(x)-u(y)|^n \leq c(n,M)\int_{B_{2|x-y|}} \langle A(x,Du),Du\rangle \mathrm{d}x + c(n,M)|x-y|^{n\sigma}. \tag{5.18}$$

As $|x-y| < \frac{R_M}{2}$, the decay estimate in Theorem 5.1 holds with $2|x-y|$ in place of r.

Combining (5.18) with the decay estimate in (5.1), we obtain

$$|u(x)-u(y)|^n \leq c(n,M)\left(\frac{1}{\mathscr{A}^\alpha\left(\frac{1}{4^n|x-y|^n}\right)}+|x-y|^{n\sigma}\right)$$

which is the conclusion. □

A. Appendix

This section is devoted to degenerate p-harmonic equations. More precisely, we shall consider equation (1.1) with the operator $A\colon \Omega\times\mathbb{R}^n\to\mathbb{R}^n$ satisfying the following growth conditions

$$|A(x,\xi)| \leq k(x)|\xi|^{p-1} \tag{A.1}$$

$$\langle A(x,\xi),\xi\rangle \geq \frac{1}{k(x)}|\xi|^p \tag{A.2}$$

for almost every $x\in\Omega$, all $\xi\in\mathbb{R}^n$ and $1<p<n$.

As for $p=n$, we have that assumptions (A.1) and (A.2) are equivalent to the following inequality

$$|\xi|^p+|A(x,\xi)|^{p'} \leq K(x)\langle A(x,\xi),\xi\rangle, \tag{A.3}$$

where $K(x)\geq 1$ and $p'=p/(p-1)$.

The solutions of (1.1) are locally bounded, provided they satisfy a suitable bounded boundary condition. Namely, we have:

Proposition A.1. Let us fix a ball $B_R\Subset\Omega$ and $u_0\in L^\infty(\partial B_R)$. Let u be a finite energy solution of the following Dirichlet problem

$$\begin{cases}\operatorname{div} A(x,Du)=\operatorname{div} f & \text{in } B_R\\[2ex] u=u_0 & \text{on } \partial B_R.\end{cases}$$

Assume that inequality (A.3) holds with a function K such that $\exp(P(K))\in L^1_{\mathrm{loc}}(\Omega)$ and suppose $K^{\frac{1}{p}}|f|\in L^q_{\mathrm{loc}}(\Omega)$ for some $q>\frac{n}{p-1}$. Then u is a locally bounded function satisfying

$$\sup_{B_R} u \leq u_0+cr^\sigma \qquad\qquad \inf_{B_R} u\geq u_0-cr^\sigma,$$

where $\sigma=\sigma(n,p,q)>0$.

For the proof it suffices to argue as in the proof of Proposition 4.1.

The isoperimetric type inequality of Proposition 3.1 reads as:

Proposition A.2. Let u be a finite energy solution of equation (1.1). Assume that inequality (A.3) holds with a function K such that $K^{\frac{1}{p-1}}|f|^{p'}\in L^1_{\mathrm{loc}}(\Omega)$ and $\exp(P(K))\in L^1_{\mathrm{loc}}(\Omega)$. Then

$$\int_{B(x_0,r)}\langle A(x,Du),Du\rangle\,\mathrm{d}x$$

$$\leq c(n,p)\left[\left(\int_{\partial B(x_0,r)}(|A(x,Du)-f|^{p'}+|Du|^p)^{\frac{n-1}{n}}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}}+\int_{B(x_0,r)}K^{\frac{1}{p-1}}|f|^{p'}\mathrm{d}x\right]$$

for every ball $B(x_0,r)\Subset\Omega$.

Proof. We argue exactly as in the proof of Proposition 3.1 until we arrive at

$$\int_{B_r}\langle A(x,Du),Du\rangle\,\mathrm{d}x \le \int_{\partial B_r}|A(x,Du)-f||u-c|\,\mathrm{d}\mathcal{H}^{n-1}+\int_{B_r}|f||Du|\,\mathrm{d}x.$$

Hence, by using Hölder's and Poincaré's inequalities we get

$$\begin{aligned}
&\int_{B_r}\langle A(x,Du),Du\rangle\,\mathrm{d}x\\
&\le \left(\int_{\partial B_r}|A(x,Du)-f|^{p'\frac{n-1}{n}}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n}{(n-1)p'}}\left(\int_{\partial B_r}|u-c|^{p\frac{n-1}{n-p}}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n-p}{p(n-1)}}+\int_{B_r}|f||Du|\,\mathrm{d}x\\
&\le c(n,p)\left(\int_{\partial B_r}|A(x,Du)-f|^{p'\frac{n-1}{n}}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n}{(n-1)p'}}\left(\int_{\partial B_r}|Du|^{p\frac{n-1}{n}}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n}{(n-1)p}}+\int_{B_r}|f||Du|\,\mathrm{d}x
\end{aligned}$$

where we have chosen $c=u_{\partial B_r}$. Hence

$$\begin{aligned}
\int_{B_r}\langle A(x,Du),Du\rangle\,\mathrm{d}x &\le c(n,p)\left(\int_{\partial B_r}(|A(x,Du)-f|^{p'\frac{n-1}{n}}+|Du|^{p\frac{n-1}{n}})\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}}\\
&\quad+\int_{B_r}|K^{\frac1p}f||K^{-\frac1p}Du|\,\mathrm{d}x\\
&\le c(n,p)\left(\int_{\partial B_r}(|A(x,Du)-f|^{p'\frac{n-1}{n}}+|Du|^{p\frac{n-1}{n}})\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}}\\
&\quad+\frac12\int_{B_r}\frac1K|Du|^p\,\mathrm{d}x+c\int_{B_r}K^{\frac{1}{p-1}}|f|^{p'}. \qquad \text{(A.4)}
\end{aligned}$$

Estimate (A.4) and inequality (A.3) give

$$\int_{B_r}\langle A(x,Du),Du\rangle\,\mathrm{d}x\le c(n,p)\left[\left(\int_{\partial B_r}(|A(x,Du)-f|^{p'}+|Du|^p)^{\frac{n-1}{n}}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}}+\int_{B_r}K^{\frac{1}{p-1}}|f|^{p'}\right]$$

i.e. the conclusion. □

Finally, we will show that a decay estimate, similar to the one proven in Theorem 5.1, is still valid.

Theorem A.3. *Let u be a finite energy solution of equation (1.1). Assume that inequality (A.3) holds with a function K such that $K^{\frac{1}{p-1}}|f|^{p'}\in L^1_{\mathrm{loc}}(\Omega)$ and $\exp(P(K))\in L^1_{\mathrm{loc}}(\Omega)$. If $\exp(P(t^{\frac{1}{n-1}}))$ is convex, then there exists a positive constant $\alpha=\alpha(n,||\exp(P(K))||_{L^1(B)})$ such that for every ball $B_R\subset B\Subset\Omega$, we have*

$$\int_{B_r}\langle A(x,Du),Du\rangle\,\mathrm{d}x\le\frac{c(n)}{\mathscr{A}^{\alpha}\left(\frac{1}{r^n}\right)}\left[\int_{B_R}\langle A(x,Du),Du\rangle\,\mathrm{d}x+\int_{B_R}K^{\frac{1}{p-1}}|f|^{p'}\mathrm{d}x\right],$$

whenever $0<r<R/2$.

Proof. Let us denote by

$$H=|A(x,Du)-f|^{p'}+|Du|^p.$$

Applying Proposition A.2, we have

$$\int_{B_s}\langle A(x,Du),Du\rangle\,\mathrm{d}x\le c(n,p)\left[\left(\int_{\partial B_s}H^{\frac{n-1}{n}}\,\mathrm{d}\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}}+\int_{B_s}K^{\frac{1}{p-1}}|f|^{p'}\mathrm{d}x\right] \qquad \text{(A.5)}$$

for almost every $0 < s < R$. Let us denote now by Δ_i the interval $\left(\frac{R}{2^i}, \frac{R}{2^{(i-1)}}\right)$ and by A_i the annulus $B_{\frac{R}{2^{(i-1)}}} \setminus B_{\frac{R}{2^i}}$, for every $i \in \mathbf{N}$. Using Fubini's theorem one can easily check that the set

$$E_i = \left\{ t \in \Delta_i : \int_{\partial B_t} H^{\frac{n-1}{n}} \, \mathrm{d}\mathcal{H}^{n-1} \le \frac{2}{|\Delta_i|} \int_{A_i} H^{\frac{n-1}{n}} \, \mathrm{d}x \right\}$$

has positive measure. Choosing $r \in E_i$ so that inequality (A.5) holds, we obtain the estimate

$$\begin{aligned}
&\int_{B_{\frac{R}{2^i}}} \langle A(x, Du), Du \rangle \, \mathrm{d}x \le \int_{B_r} \langle A(x, Du), Du \rangle \, \mathrm{d}x \\
&\le c(n,p) \left[\left(\int_{\partial B_r} H^{\frac{n-1}{n}} \, \mathrm{d}\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} + \int_{B_r} K^{\frac{1}{p-1}} |f|^{p'} \mathrm{d}x \right] \\
&\le c(n,p) \left[\left(\frac{2}{|\Delta_i|} \right)^{\frac{n}{n-1}} \left(\int_{A_i} H^{\frac{n-1}{n}} \, \mathrm{d}x \right)^{\frac{n}{n-1}} + \int_{B_r} K^{\frac{1}{p-1}} |f|^{p'} \mathrm{d}x \right] \\
&\le c(n,p) \left(\frac{1}{|\Delta_i|} \right)^{\frac{n}{n-1}} \left(\int_{A_i} |A(x, Du)|^{p'} + (|Du|^p)^{\frac{n-1}{n}} \, \mathrm{d}x \right)^{\frac{n}{n-1}} \\
&\quad + c(n,p) \left[\left(\frac{1}{|\Delta_i|} \right)^{\frac{n}{n-1}} \left(\int_{A_i} |f|^{p' \frac{n-1}{n}} \mathrm{d}x \right)^{\frac{n}{n-1}} + \int_{B_r} K^{\frac{1}{p-1}} |f|^{p'} \mathrm{d}x \right].
\end{aligned} \tag{A.6}$$

Inserting inequality (A.3) in (A.6) and using Hölder's inequality, we find that

$$\begin{aligned}
&\int_{B_{\frac{R}{2^i}}} \langle A(x, Du), Du \rangle \, \mathrm{d}x \le c(n,p) \left(\frac{1}{|\Delta_i|} \right)^{\frac{n}{n-1}} \left(\int_{A_i} K^{n-1} \, \mathrm{d}x \right)^{\frac{1}{n-1}} \int_{A_i} \langle A(x, Du), Du \rangle \, \mathrm{d}x \\
&+ c(n,p) \left[\int_{B_r} K^{\frac{1}{p-1}} |f|^{p'} \mathrm{d}x + \int_{B_r} |f|^{p'} \, \mathrm{d}x \right].
\end{aligned}$$

At this point, Jensen's inequality applied to the convex function $\Phi(t^{\frac{1}{n-1}}) = \exp(P(t^{\frac{1}{n-1}}))$, gives

$$\begin{aligned}
&\int_{B_{\frac{R}{2^i}}} \langle A(x, Du), Du \rangle \, \mathrm{d}x \\
&\le c(n,p) \left(\frac{1}{|\Delta_i|} \right)^{\frac{n}{n-1}} |A_i|^{\frac{1}{n-1}} \left[\Phi^{-1} \left(\fint_{A_i} \Phi(K) \, \mathrm{d}x \right) \right] \int_{A_i} \langle A(x, Du), Du \rangle \, \mathrm{d}x \\
&\quad + c(n,p) \left[\int_{B_r} K^{\frac{1}{p-1}} |f|^{p'} \mathrm{d}x + \int_{B_r} |f|^{p'} \, \mathrm{d}x \right] \\
&\le c(n,p) \left(\frac{1}{|\Delta_i|} \right)^{\frac{n}{n-1}} |A_i|^{\frac{1}{n-1}} \left[\Phi^{-1} \left(\frac{1}{|A_i|} \int_{B_R} \Phi(K) \, \mathrm{d}x \right) \right] \int_{A_i} \langle A(x, Du), Du \rangle \, \mathrm{d}x \\
&\quad + c_0(n,p) \int_{B_R} K^{\frac{1}{p-1}} |f|^{p'} \mathrm{d}x
\end{aligned}$$

where we have used that $K(x) \ge 1$ and that

$$|A_i| = C(n) \frac{(2^n - 1) R^n}{2^{ni}} \qquad\qquad |\Delta_i| = \frac{R}{2^i}.$$

Hence, for $\tilde{K}_R = \int_{B_R} \exp(P(K))\,\mathrm{d}x$, we have

$$\int_{B_{\frac{R}{2^i}}} \langle A(x,Du),Du\rangle\,\mathrm{d}x \tag{A.7}$$

$$\leq c(n,p)\frac{1}{|\Delta_i|}\frac{R}{2^i}\Phi^{-1}\left(\frac{C(n)2^{ni}\tilde{K}_R}{R^n}\right)\int_{A_i}\langle A(x,Du),Du\rangle\,\mathrm{d}x \tag{A.8}$$

$$+\,c_0(n,p)\int_{B_R} K^{\frac{1}{p-1}}|f|^{p'}\mathrm{d}x \tag{A.9}$$

that is analogous to the estimate (5.4). From now on the proof goes as in the proof of Proposition 5.1 setting

$$v_i(t) = \int_{B_{\frac{R}{2^i}}} \langle A(x,Du),Du\rangle\,\mathrm{d}x + \frac{t-R2^{-i}}{|\Delta_i|}\int_{A_i}\langle A(x,Du),Du\rangle\,\mathrm{d}x + \varphi(t)\int_{B_R} K^{\frac{1}{p-1}}|f|^{p'}\mathrm{d}x. \qquad \square$$

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