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A SIMPLE PROOF OF THE CHARACTERIZATION OF FUNCTIONS OF LOW AVILES GIGA ENERGY ON A BALL *VIA* REGULARITY

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Abstract. The Aviles Giga functional is a well known second order functional that forms a model for blistering and in a certain regime liquid crystals, a related functional models thin magnetized films. Given Lipschitz domain $\Omega \subset \mathbb{R}^2$ the functional is $I_\epsilon(u) = \frac{1}{2}\int_\Omega \epsilon^{-1}\left|1-|Du|^2\right|^2 + \epsilon\left|D^2u\right|^2 \mathrm{d}z$ where u belongs to the subset of functions in $W_0^{2,2}(\Omega)$ whose gradient (in the sense of trace) satisfies $Du(x)\cdot\eta_x = 1$ where η_x is the inward pointing unit normal to $\partial\Omega$ at x. In [*Ann. Sc. Norm. Super. Pisa Cl. Sci.* **1** (2002) 187–202] Jabin *et al.* characterized a class of functions which includes all limits of sequences $u_n \in W_0^{2,2}(\Omega)$ with $I_{\epsilon_n}(u_n) \to 0$ as $\epsilon_n \to 0$. A corollary to their work is that if there exists such a sequence (u_n) for a bounded domain Ω, then Ω must be a ball and (up to change of sign) $u := \lim_{n\to\infty} u_n = \mathrm{dist}(\cdot, \partial\Omega)$. Recently [Lorent, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (submitted), `http://arxiv.org/abs/0902.0154v1`] we provided a quantitative generalization of this corollary over the space of convex domains using 'compensated compactness' inspired calculations of DeSimone *et al.* [*Proc. Soc. Edinb. Sect. A* **131** (2001) 833–844]. In this note we use methods of regularity theory and ODE to provide a sharper estimate and a much simpler proof for the case where $\Omega = B_1(0)$ without the requiring the trace condition on Du.

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1. Introduction

Let

$$I_\epsilon(u) := \int_\Omega \epsilon^{-1}\left|1-|Du|^2\right|^2 + \epsilon\left|D^2u\right|^2 \mathrm{d}z. \tag{1.1}$$

The functional I_ϵ forms a model for blistering and (in certain regimes) for a model for liquid crystals [6,17]. In addition there is a closely related functional modeling thin magnetic films [1,8–10,19]. For function $u \in W_0^{2,2}(\Omega)$ we refer to $I_\epsilon(u)$ as the *Aviles Giga energy* of u.

For an example of a candidate minimizer take the distance function from the boundary $\psi(x) := \mathrm{dist}(x, \partial\Omega)$ convolved by a standard convolution kernel ρ_ϵ with support of diameter ϵ. It has been conjectured that for convex domains Ω, the minimizers of I_ϵ have the structure suggested by this construction, *i.e.* they are in some quantitative sense close to the distance function from the boundary, Section 5.3 [4,13].

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The first progress on this conjecture was achieved by Jin and Kohn [17] whose showed that if I_ϵ is minimized over

$$\Lambda(\Omega) := \left\{ \begin{array}{ll} v \in W_0^{2,2}(\Omega) : & \frac{\partial v}{\partial \eta_z} = 1 \text{ where } \eta_z \text{ is the inwards} \\ & \text{pointing unit normal to } \partial\Omega \text{ at } z \end{array} \right\} \tag{1.2}$$

where Ω is taken to be an ellipse then as $\epsilon \to 0$ the energy of the minimizer of I_ϵ tends to the energy of $\psi * \rho_\epsilon$. Their method was to take arbitrary $u \in \Lambda(\Omega)$ and to construct vectors fields Σ_1, Σ_2 out of third order polynomials of the partial derivatives of u that have the property that the divergence of these vectors fields is bounded above by $I_\epsilon(u)$. Using the trace condition $\frac{\partial u}{\partial \eta} = 1$ and the fact that Ω is an ellipse the lower bound provided by the divergence of Σ_1, Σ_2 can be explicitly calculated and shown to be asymptotically sharp as $\epsilon \to 0$.

As has been discussed in [2,4,17] the functional I_ϵ minimized over $W_0^{2,2}(\Omega)$ has many features in common with the functional $J^p(v) = \int_{J_{Dv}} |Dv^+ - Dv^-|^p \,\mathrm{d}H^1$ for the case $p = 3$, when minimized over the space $Dv \in BV(\Omega)$ with $|Dv(x)| = 1$ a.e. x and $v = 0$ on $\partial\Omega$. Aviles and Giga [5] showed that if Ω is convex and polygonal then the distance function is the minimizer of J^1 over the subspace of piecewise affine functions satisfying these conditions. They conjectured the same is true for $p = 3$.

From a somewhat different direction a strong result has been proved [16] by Jabin *et al.* who characterized a class of functions which includes all limits of sequences $u_n \in W_0^{2,2}(\Omega)$ with $I_{\epsilon_n}(u_n) \to 0$ as $\epsilon_n \to 0$. A corollary to their work is that if there exists such a sequence (u_n) for a bounded domain Ω, then Ω must be a ball and (up to change of sign) $u := \lim_{n\to\infty} u_n = \mathrm{dist}(\cdot, \partial\Omega)$. In [18], a quantitative generalization of this corollary was achieved for the class of bounded convex domains, a corollary to the main result of [18] is the following.

Theorem 1.1 ([18]). *Let Ω be a convex set with diameter 2, C^2 boundary and curvature bounded above by $\epsilon^{-\frac{1}{2}}$. Let $\Lambda(\Omega)$ be defined by (1.2). There exists positive constants $C > 1$ and $\lambda < 1$ such that if u is a minimizer of I_ϵ over $\Lambda(\Omega)$, then*

$$\|u - \zeta\|_{W^{1,2}(\Omega)} \le C \left(\epsilon + \inf_y |\Omega \triangle B_1(y)| \right)^\lambda \tag{1.3}$$

where $\zeta(z) = \mathrm{dist}(z, \partial\Omega)$.

We take constant $\lambda = \frac{1}{2731}$ and thus the control represented by inequality (1.3) is far from optimal. Theorem 1.1 follows from Theorem 1 of [18] which is a characterization of domains Ω and functions u for which the Aviles energy is small, more specifically there exists a constant γ such that given $u \in \Lambda(\Omega)$ such that $I_\epsilon(u) = \beta$ then $|\Omega \triangle B_1(0)| \le c\beta^\gamma$ and $\int_{B_1(0)} \left| Du(z) + \frac{z}{|z|} \right|^2 \mathrm{d}z \le c\beta^\gamma$, here we can take $\gamma = 512^{-1}$. The proof of Theorem 1 of [18] is fairly involved, it relies heavily on the characterization of 'entropies' for the Aviles Giga energy that was achieved in [9] (see Lem. 3). While the calculations in [18] are elementary and self contained, they can appear quite unmotivated to those unfamiliar with the background of [9]. In addition the trace condition on the gradient in the definition of $\Lambda(\Omega)$ is used in an essential way.

The proof of Theorem 1 requires quite a careful construction of an upper bound of the Aviles Giga energy of a minimizer on a domain with smooth boundary that is 'close' to a ball, then the theorem follows by application of Theorem 1 [18]. The many steps required to complete the proof result in a gradual loss of control resulting in the constant $\lambda = \frac{1}{2731}$.

The propose of this note is twofold, firstly to provide a simple proof of a characterization of the minimizers of the Aviles Giga energy on a ball with a sharper estimate and secondly to prove the result without the trace condition on the gradient, specifically to characterize the minimizers over $W_0^{2,2}(B_1(0))$. Additionally we find it worthwhile to introduce new methods to study the characterization of minimizers of I_ϵ, the regularity theory and ODE approach of this note is quite different from previous methods of [5,16–18]. Our main theorem is:

Theorem 1.2. *Let u be a minimizer of I_ϵ over $W_0^{2,2}(B_1(0))$. Then there exists $\xi \in \{1, -1\}$*

$$\int_{B_1(0)} \left| Du(x) + \xi \frac{x}{|x|} \right|^2 \mathrm{d}x \le c\epsilon^{\frac{1}{6}} \left(\log\left(\epsilon^{-1}\right) \right)^{\frac{13}{6}}.$$

The desirability of a simpler proof with a better estimate has already been discussed, it is of interest to prove a characterization without a trace condition on the gradient due to the fact this is a strong assumption that is inappropriate for a number of physical models. More specifically the condition $Du(x) \cdot \eta = 1$ for $x \in \partial\Omega$ is not natural in the context of blistering, Gioia and Ortiz [13] proposed instead $Du(x) \cdot \eta_x = 0$. The original functional proposed by Aviles and Giga [4] to study liquid crystals also has this trace condition. In addition for the micro-magnetic analogue of functional I_ϵ there is nothing like a pointwise condition on the trace [8,10]. This micro-magnetic functional is given by $M_\epsilon(v) = \epsilon^{-1} \int_{\mathbb{R}^2} |H(\tilde{v})|^2 + \epsilon \int_\Omega |Dv|^2$ where H is the Hodge projection onto curl free vector fields and $\tilde{v}$ is the extension of v to 0 outside Ω, this functional is minimized over $W^{1,2}(\Omega : S^1)$. As mentioned, in the proof of Theorem 1 [18] the trace condition is used in an essential way, this is also true of the proof of Theorem 5.1 [17]. In order to achieve a characterization for less rigid functionals, methods need to be developed that do not use this trace condition. A related but different micro-magnetic functional E_ϵ was studied by Ignat and Otto [15]. They also achieved a characterization of minimizers E_ϵ showing that minimizers converge to Neel Walls, the focus of E_ϵ was to provide a two dimensional approximation of the micro-magnetic energy in the absence of an external field and crystal anisotropy.

The proof of Theorem 1.2 requires establishing the essentially folklore fact that critical points of the Aviles Giga energy have $W^{2,3}$ regularity and their gradients satisfy certain natural Caccioppoli inequalities. The much more subtle question of regularity of critical points of functional M_ϵ has been studied by Carbou [7] and Hardt and Kinderlehrer [14]. The non-local term in M_ϵ makes the Euler Lagrange equation harder to study and in some sense weaker regularity has been proved, it is not clear if the Caccioppoli inequalities needed for the proof presented in this note are available *via* the methods of [7]. Working with a three dimensional model, different methods are used in [14] and Caccioppoli inequalities are established off a discrete set[2].

Roughly speaking the main open problems related to the Aviles Giga functional are either: (A) conjectures on how the energy concentrates, specifically the Γ-convergence conjecture of [2] and related problems; or (B) conjectures about the minimizer of I_ϵ. It is know from [17] that for non-convex domains the minimizer does not need to be the distance function from the boundary (contrast this with the main theorem of [3] which showed that for a sequence $\epsilon_n \to 0$, the minimizer m_n of the micro-magnetics functional M_{ϵ_n} must converge to the rotated gradient of distance function for any connected open Lipschitz domain). However as mentioned for general convex domains the conjecture remains largely open, in [18] we developed methods that prove the conjecture for convex domains with low Aviles Giga energy, it is likely these methods could be used to prove the same result for general low energy domains with C^2 boundary. For domains with Aviles Giga energy of order $O(1)$ neither the methods of [18] or this note yield much. A very attractive open problem is to characterize the minimizers in the case where Ω is an ellipse, given the sharp lower bound provided by [17] in this case there seems to be much concrete information about this problem – yet it appears to be out of reach of current methods.

2. Proof sketch

Beyond the regularity issues mentioned in the introduction the proof reduces to essentially applying an ODE and using the Pythagorean theorem. In order to sketch the main strategy of the proof we will make a number of assumptions that we will later show are not needed.

We start by assuming for a moment that the cardinality of the set of critical points of Du is 1, *i.e.*

$$\mathrm{Card}\left(\{x \in B_1(0) : |Du(x)| = 0\}\right) = 1. \tag{2.1}$$

In addition let us temporarily assume we have the (in the sense of trace) boundary condition

$$Du(x) = -\frac{x}{|x|} \text{ for } x \in \partial B_1(0). \tag{2.2}$$

[2]It appears possible that the methods of [14] would establish the appropriate Caccioppoli inequalities everywhere in the interior if the arguments were carried through for the two dimensional model, if this is the case the strategy of this note would likely yield a characterization of minimizers of M_ϵ for where $\Omega = B_1(0)$.

So let $z_0 \in B_1(0)$ be the point for which $|Du(z_0)| = 0$. Take $y_0 = -z_0\mathbb{R} \cap \partial B_1(0)$ and let $X(0) = y_0$, $\frac{\mathrm{d}X}{\mathrm{d}t}(s) = Du(X(s))$. For $z \in \{X(s) : s \in [0,t]\}$ let t_z denote the tangent to this curve at z. Now for any $t > 0$

$$u(X(t)) = u(X(t)) - u(X(0)) = \int_{\{X(s):s\in[0,t]\}} Du(z)\cdot t_z \mathrm{d}H^1 z.$$

If we also assume

$$|Du(z)| \approx 1 \text{ for } z \in \{X(s) : s \in [0,t]\} \tag{2.3}$$

then we could conclude that

$$|u(X(t))| \approx H^1(X(s) : s \in [0,t]) \geq |X(t) - X(0)|\,.$$

Now by (2.2) we know that the path $X(t)$ has to run *into* $B_1(0)$ and can not escape this domain, so we must have $X(t) \to z_0$ as $t \to \infty$ we have $|u(z_0)| \geq |z_0 - X(0)| = |z_0| + 1$.

As will be established later in Lemma 3.3, $\inf_{v\in W_0^{2,2}(B_1(0))} I_\epsilon(v) \leq c\epsilon\log(\epsilon^{-1})$. Hence if u is a minimizer of I_ϵ,

$$\int_{B_1(0)} \left|1 - |Du|^2\right|^2 \mathrm{d}x \leq c\epsilon^2\log(\epsilon^{-1}) \tag{2.4}$$

so we know u 'is close to being' 1-Lipschitz and thus $|u(z_0)| \gtrapprox 1$, hence $|z_0| \approx 0$ and $|u(z_0)| \approx 1$. Again since u is close to 1-Lipschitz,

$$|u(x)| \approx 1 \text{ for any } x \in B_{\epsilon^{\frac{1}{4}}}(0). \tag{2.5}$$

Now for $y \in \partial B_1(0)$ let $\mathrm{e}^x(y) = \int_{[x,y]} \left|1 - |Du|^2\right| \mathrm{d}H^1$. Let $J_x(z) = \frac{z-x}{|z-x|}$, note that $|DJ_x(z)| \leq \frac{2}{|x-z|}$, so by the Co-area formula

$$\begin{aligned}
\int_{\partial B_1(0)} \mathrm{e}^x(y)\mathrm{d}H^1 y &= \int_{S^1}\int_{J_x^{-1}(\theta)} \left|1 - |Du(z)|^2\right| \mathrm{d}H^1 z \mathrm{d}H^1\theta \\
&= \int_{B_1(0)} \left|1 - |Du(z)|^2\right| |DJ_x(z)|\,\mathrm{d}z \\
&\leq c\int_{B_1(0)} \left|1 - |Du(z)|^2\right| |z-x|^{-1}\,\mathrm{d}z.
\end{aligned}$$

Now by Fubini and (2.4) we have

$$\int_{B_{\epsilon^{\frac{1}{4}}}(0)}\int_{B_1(0)} \left|1 - |Du(z)|^2\right| |z-x|^{-1}\,\mathrm{d}z\mathrm{d}x \leq c\epsilon^{\frac{5}{4}}\sqrt{\log(\epsilon^{-1})}$$

thus we can assume we chose $x \in B_{\epsilon^{\frac{1}{4}}}(0)$ such that $\int_{\partial B_1(0)} \mathrm{e}^x(y)\mathrm{d}H^1 y \leq c\epsilon^{\frac{3}{4}}\sqrt{\log(\epsilon^{-1})}$. Now

$$\begin{aligned}
\int_{[x,y]} \left|Du(z) + \frac{y-x}{|y-x|}\right|^2 \mathrm{d}H^1 z &= \int_{[x,y]} |Du(z)|^2 + 2Du(z)\cdot\frac{y-x}{|y-x|} + 1\mathrm{d}H^1 z \\
&\leq 2|x-y| - 2u(x) + \mathrm{e}^x(y) \\
&\overset{(2.5)}{\lessapprox} \mathrm{e}^x(y),
\end{aligned} \tag{2.6}$$

So

$$\begin{aligned}\int_{B_1(0)} \frac{\left|Du(z)+\frac{z-x}{|z-x|}\right|^2}{|z-x|}\mathrm{d}z &\leq c\int_{y\in\partial B_1(0)}\int_{[x,y]}\left|Du(z)+\frac{y-x}{|y-x|}\right|^2 \mathrm{d}H^1 z \mathrm{d}H^1 y\\ &\overset{(2.6)}{\leq} c\int_{y\in\partial B_1(0)} \mathrm{e}^x(y)\,\mathrm{d}H^1 y\\ &\leq c\epsilon^{\frac{3}{4}}\sqrt{\log\left(\epsilon^{-1}\right)}. \end{aligned} \tag{2.7}$$

As for 'most' $z\in B_1(0)$, $\left|\frac{z}{|z|}-\frac{z-x}{|z-x|}\right|\leq c\epsilon^{\frac{1}{8}}$ so we have $\int_{B_1(0)}\left|Du(z)+\frac{z}{|z|}\right|^2 \mathrm{d}z \overset{(2.7)}{\leq} c\epsilon^{\frac{1}{8}}$.

Now the big assumptions we made are (2.1), (2.3) and to a lesser extent (2.2). The main work of this note is to find substitutes for these assumptions.

What assumption (2.1) provides is the existence of a long integral path of the vector field Du which using assumption (2.3) we can show is close to a straight line. In order to find such a path, it is sufficient to show that the set of critical points of Du are merely low in number, using the energy upper bound and regularity of minimizers of I_ϵ that is what we will be able to do.

Now if we define $v(z)=u(\epsilon z)$ then v satisfies $\Delta^2 v+\mathrm{div}\left(\left(1-|Dv|^2\right)Dv\right)=0$ which is an Elliptic equation with right-hand side bounded in $H^{-1,p}(B_{\epsilon^{-1}}(0))$ for all $p>1$. Thus it is not hard to believe Dv is Holder so if $|Dv(z_0)|=0$ for some z_0 then there must be a constant c_0 such that $\sup\{|Dv(z)|: z\in B_{c_0}(z_0)\}\leq\frac{1}{2}$ so after rescaling we have that for every z_1 such that $|Du(z_1)|=0$ we have that $\sup\{|Du(z)|: z\in B_{c_1\epsilon}(z_0)\}\leq\frac{1}{2}$. Thus by (2.4) we have that we can have as most $c\log(\epsilon^{-1})$ critical points of I_ϵ that are spaced out by ϵ. So cutting $B_1(0)$ into $N=\left[\frac{4c\pi}{\log(\epsilon^{-1})}\right]$ equal angles slices which we denote by $T_1,T_2,\dots,T_N$ then at least half of them do not have any critical points of Du. So if T_1 is one of them, taking y_0 to be the center of the arc $T_1\cap\partial B_1(0)$ the ODE $X(0)=y_0$, $\frac{\mathrm{d}X}{\mathrm{d}t}(s)=Du(X(s))$ has to run until it hits ∂T_1.

Now the second main assumption we made is (2.3). Again since for minimizer u we know that $I_\epsilon(u)\leq c\epsilon\log(\epsilon^{-1})$, so

$$\int_{B_1(0)}\left|1-|Du|^2\right|\left|D^2u\right|\mathrm{d}x\leq c\epsilon\log\left(\epsilon^{-1}\right).$$

Take $v\in S^1$, for all but $c(\epsilon\log(\epsilon))^{\frac{1}{3}}$ lines L parallel to v we have that $\int_L\left|1-|Du|^2\right|\left|D^2u\right|\mathrm{d}H^1x\leq(\epsilon\log(\epsilon))^{\frac{2}{3}}$. Now on the line L if there is a point $z_1\in L$ with $\left|1-|Du(z_1)|^2\right|\geq 5(\epsilon\log(\epsilon^{-1}))^{\frac{1}{3}}$ then we must be able to find z_2,z_3 we have $\inf\left\{\left|1-|Du(y)|^2\right|: y\in[z_2,z_3]\right\}\geq 4(\epsilon\log(\epsilon^{-1}))^{\frac{1}{3}}$ and $\left|1-|Du(z_3)|^2\right|\geq 5(\epsilon\log(\epsilon^{-1}))^{\frac{1}{3}}$, $\left|1-|Du(z_2)|^2\right|\leq 4(\epsilon\log(\epsilon^{-1}))^{\frac{1}{3}}$ then

$$(\epsilon\log(\epsilon))^{\frac{2}{3}}\geq\int_{z_2}^{z_3}\left|1-|Du(y)|^2\right|\left|D^2u(y)\right|\mathrm{d}H^1y\geq 4\left(\epsilon\log\left(\epsilon^{-1}\right)\right)^{\frac{1}{3}}\int_{z_2}^{z_3}\left|D^2u(y)\right|\mathrm{d}H^1y\geq 4(\epsilon\log(\epsilon))^{\frac{2}{3}}$$

which is a contradiction. Thus for most lines L we know that $\sup\left\{\left|1-|Du(z)|^2\right|: y\in L\cap B_1(0)\right\}\leq 5(\epsilon\log(\epsilon))^{\frac{1}{3}}$. For vector $w\in\mathbb{R}^2$ define $\langle w\rangle:=\{\lambda w:\lambda\in\mathbb{R}\}$ and given subspace V let P_V denote the orthogonal projection onto V. For subset $S\subset\mathbb{R}^n$ let $|S|$ denote the Lebesgue n-measure of S. Now if we run an ODE $X(0)=y_0$, $\frac{\mathrm{d}X}{\mathrm{d}t}(s)=Du(X(s))$ between 0 and t then taking $v=\frac{X(t)-X(0)}{|X(t)-X(0)|}$ then we have a set $G\subset P_{\langle v\rangle}([X(0),X(t)])$ with $\left|P_{\langle v\rangle}([X(0),X(t)])\backslash G\right|\leq c(\epsilon\log(\epsilon^{-1}))^{\frac{1}{3}}$ and if $z\in\{X(s): s\in[0,t]\}\cap P^{-1}_{\langle v\rangle}(x)$ for some $x\in G$, then $\left||Du(z)|^2-1\right|\leq 5(\epsilon\log(\epsilon))^{\frac{1}{3}}$ thus the part of the path $\{X(s): s\in[0,t]\}$ that is in the set $P^{-1}_{\langle v\rangle}(G)$ is such that

$|Du(z)| \approx 1$. So the H^1 measure of the set of points $x \in \{X(s) : s \in [0,t]\}$ for which we can assume $|Du(x)| \approx 1$ is of measure as least $|X(0) - X(t)| - c(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}}$ and hence assumption (2.3) can in effect be justified. It is worth noting that the idea of following integral curves of the vector field given by Du (where u is the limit of a sequence of functions whose Aviles Giga energy tends to zero) was used by [16] and a similar idea later by [15].

Finally we also assumed (2.2), the only purpose of this assumption was to allow us to run an ODE starting from $y_0 \in \partial B_1(0)$ without it immediately trying to leave the domain. Recall y_0 was the point at the center of the arc $\partial T_1 \cap \partial B_1(0)$. If instead of starting at this point we started at $y_0 + c\frac{\eta_{y_0}}{(\log(\epsilon^{-1}))^2}$ then running the ODE forwards and backwards until both ends hit ∂T_1, then we will have a path of length (at least) $c(\log(\epsilon^{-1}))^{-2}$ which will be very close to a straight line, see Figure 1. Let $s < 0$, $r > 0$ be such that $X(s), X(e)$ are the endpoints of the path (where we assume without loss of generality $X(s)$ is closer to $\partial B_1(0)$ than $X(e)$). If we are able to show that $X(s) \in \partial T_1 \cap \partial B_1(0)$ then the argument can proceed very much as described in the paragraphs above. The only way this can fail is if the path is (close to) a line of length $c(\log(\epsilon^{-1}))^{-1}$ and runs, (roughly speaking) parallel to $\partial T_1 \cap \partial B_1(0)$. However as $|u(X(e)) - u(X(s))| \geq c(\log(\epsilon^{-1}))^{-1}$ this implies we must have $|u(X(e))| \geq c(\log(\epsilon^{-1}))^{-1}$, but since the path is close to 'parallel' to $\partial B_1(0) \cap \partial T_1$ we have $\text{dist}(X(e), \partial B_1(0)) \leq c \log(\epsilon^{-1})^{-2}$ which contradicts 1-Lipschitz type property as represented by inequality (2.4), thus we must have that $X(s) \in \partial T_1 \cap \partial B_1(0)$. By use of this argument assumption (2.2) can be avoided.

3. The E.L. equation

Note that if u is a critical point of I_ϵ it weakly satisfies the E.L. equation *i.e.*

$$\epsilon \Delta^2 u + \epsilon^{-1} \text{div}\left(\left(1 - |Du|^2 \right) Du \right) = 0. \tag{3.1}$$

Let $w \in W^{1,1}$ define $w_i := \frac{\partial w}{\partial x_i}$, similarly for $v \in W^{2,1}, s \in W^{3,1}$ define $v_{ij} := \frac{\partial^2 v}{\partial x_i \partial x_j}$ and $s_{ijk} := \frac{\partial^3 s}{\partial x_i \partial x_j \partial x_k}$.

Lemma 3.1. *Suppose $u \in W^{2,2}(\Omega)$ is a weak solution of (3.1). Define $\Omega_{\epsilon^{-1}} := \epsilon^{-1}\Omega$ and let $v : \Omega_{\epsilon^{-1}} \to \mathbb{R}$ be defined by $v(z) := u(\epsilon z)\epsilon^{-1}$, then v satisfies*

$$\Delta^2 v + \text{div}\left(\left(1 - |Dv|^2 \right) Dv \right) = 0 \tag{3.2}$$

weakly in $\Omega_{\epsilon^{-1}}$.

Proof. Follows directly from the definition of u. □

Lemma 3.2. *We will show that any $v \in W^{2,2}(\Omega_{\epsilon^{-1}})$ that satisfies (3.2) weakly in $\Omega_{\epsilon^{-1}}$ is such that for any $U \subset\subset \Omega_{\epsilon^{-1}}$, $v \in W^{3,2}(U)$ and v satisfies*

$$\int \sum_{i,j,p=1}^{2} v_{ijp}\phi_{ijp} + \left(\left(1 - |Dv|^2 \right) \cdot Dv \right)_p D\phi_p \, \mathrm{d}z = 0 \tag{3.3}$$

for any $\phi \in C_0^1(U)$.

Proof. Given set $S \subset \mathbb{R}^2$, let $d(x,S) = \inf\{|z - x| : z \in S\}$ and define $N_\delta(S) := \{x : d(x,S) < \delta\}$. □

Step 1. For $\delta > 0$ let $\Pi_\delta := \Omega_{\epsilon^{-1}} \backslash N_\delta(\partial \Omega_{\epsilon^{-1}})$. We will show that $D^2 v \in W^{1,2}(\Pi_{3\delta})$.

Proof of Step 1. Let $g(x) := Dv(x)\left(1 - |Dv(x)|^2\right)$ and $w := \Delta v$. Since $v \in W^{2,2}(\Omega_{\epsilon^{-1}})$, by Poincare's inequality (Thm. 2, Sect. 4.5.2 [12]) $Dv \in L^p(\Omega_{\epsilon^{-1}})$ for any $p < \infty$, hence $g \in L^q(\Omega_{\epsilon^{-1}})$ for any $q < \infty$. So

$$\int w \Delta \phi = \int g \cdot D\phi \text{ for any } \phi \in C_0^\infty(\Omega_{\epsilon^{-1}}).$$

Let $\rho \in C_0^\infty(B_1)$ be the standard convolution kernel and define $\rho_\sigma(z) = \rho\left(\frac{z}{\sigma}\right)\sigma^{-2}$. Given function $f \in W^{1,1}$ we denote the convolution of f and ρ_σ by $f * \rho_\sigma$. Let $\varphi \in (0, \delta)$ and define $w_\varphi := w * \rho_\varphi$ and $g_\varphi := g * \rho_\varphi$. Now for any $\phi \in C_0^\infty(\Omega_{\epsilon^{-1}})$, defining $\phi_\varphi = \phi * \rho_\varphi$ we have

$$\int w_\varphi \Delta\phi = \int w \Delta\phi_\varphi = \int g \cdot D\phi_\varphi = \int g_\varphi \cdot D\phi$$

which gives that $\Delta w_\varphi(z) = -\mathrm{div} g_\varphi(z)$ for any $z \in \Pi_\delta$. Let $\psi \in C_0^\infty(\Pi_\delta)$ with $\psi = 1$ on $\Pi_{2\delta}$ and $|D\psi| < c\delta^{-1}$ and $\left|D^2\psi\right| < c\delta^{-2}$. Define $s(x) = w_\varphi(x)\psi(x)$, so

$$\Delta s = -\mathrm{div} g_\varphi \psi + 2Dw_\varphi \cdot D\psi + w_\varphi \Delta\psi.$$

Now $\mathrm{div}(g_\varphi\psi) = \mathrm{div} g_\varphi \psi + g_\varphi \cdot D\psi$ and $2Dw_\varphi \cdot D\psi = \mathrm{div}(2w_\varphi D\psi) - 2w_\varphi \Delta\psi$ and thus

$$\Delta s = \mathrm{div}(-g_\varphi\psi + 2w_\varphi D\psi) + g_\varphi \cdot D\psi - w_\varphi \Delta\psi. \tag{3.4}$$

Let $X = Ds$, so by (3.4) we have that

$$\mathrm{curl}(X) = 0 \text{ and } \mathrm{div}(X + g_\varphi\psi - 2w_\varphi D\psi) = g_\varphi \cdot D\psi - w_\varphi \Delta\psi. \tag{3.5}$$

For any C^2 vector field V, let $H(V)$ denote the Hodge projection of V onto the subspace of curl free vector fields, *i.e.* $H(V) = -D\Delta^{-1}\mathrm{div}V$, so $H(V)$ satisfies $\mathrm{div}(H(V) + V) = 0$ and $\mathrm{curl}H(V) = 0$ on $\mathbb{R}^2$. So from (3.5) then we have

$$\mathrm{curl}(X - H(g_\varphi\psi - 2w_\varphi D\psi)) = 0 \text{ and } \mathrm{div}(X - H(g_\varphi\psi - 2w_\varphi D\psi)) = g_\varphi \cdot D\psi - w_\varphi \Delta\psi. \tag{3.6}$$

Let $\eta \in C^\infty(\mathbb{R}^2)$ be such that

$$D\eta = X - H(g_\varphi\psi - 2w_\varphi D\psi), \tag{3.7}$$

so finally we have

$$\Delta\eta = g_\varphi \cdot D\psi - w_\varphi \Delta\psi. \tag{3.8}$$

Now recall $X = Ds$ where $s = w_\varphi\psi$. Thus $Ds = Dw_\varphi\psi + w_\varphi D\psi$ and thus for any $p \in [1, 2]$,

$$\begin{aligned}\|X\|_{L^p(\mathbb{R}^2)} &\le c\|Dw_\varphi\|_{L^p(\mathbb{R}^2)} + c\|w_\varphi\|_{L^p(\mathbb{R}^2)} \le c\|w * D\rho_\varphi\|_{L^p(\mathbb{R}^2)} + c\|w_\varphi\|_{L^p(\mathbb{R}^2)} \\ &\le c\varphi^{\frac{2-3p}{p}}\|D^2u\|_{L^2(\Omega_{\epsilon^{-1}})} \le c\varphi^{\frac{2-3p}{p}}. \end{aligned} \tag{3.9}$$

And by L^p boundedness of Hodge projection we know

$$\|H(g_\varphi\psi - 2w_\varphi D\psi)\|_{L^p(\mathbb{R}^2)} \le c\|g_\varphi\psi - 2w_\varphi D\psi\|_{L^p(\mathbb{R}^2)} \le c\|g_\varphi\|_{L^p(\Omega_{\epsilon^{-1}})} + c\|w_\varphi\|_{L^p(\Omega_{\epsilon^{-1}})} \le c. \tag{3.10}$$

Thus for $p = \frac{3}{2}$ we have $\|D\eta\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \overset{(3.10),(3.9),(3.7)}{\le} c\varphi^{-\frac{5}{3}}$. What we need to do is obtain an φ independent bound on $D\eta$, we will achieve this by use of (3.8). First note by Holder $g_\varphi \cdot D\psi - w_\varphi\Delta\psi \in L^{\frac{3}{2}}(\mathbb{R}^2)$ from (3.8) by standard L^p estimates on Riesz transforms (see Prop. 3, Sect. 1.3, Chap. 3 [20]) we know

$$\|D^2\eta\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \le c\|g_\varphi\|_{L^{\frac{3}{2}}(\Omega_{\epsilon^{-1}})} + c\|w_\varphi\|_{L^{\frac{3}{2}}(\Omega_{\epsilon^{-1}})} \le c. \tag{3.11}$$

So $D\eta \in W^{1,\frac{3}{2}}(\mathbb{R}^2)$ and thus by Sobolev embedding theorem (Thm. 1, Sect. 4.5.1 [12]) we have $\|D\eta\|_{L^6(\mathbb{R}^2)} \le c\|D^2\eta\|_{L^{\frac{3}{2}}(\mathbb{R}^2)} \overset{(3.11)}{\le} c$. As $\mathrm{Spt}X \subset \Pi_\delta \subset \Omega_{\epsilon^{-1}}$, $\|Ds\|_{L^2(\mathbb{R}^2)} = \|Ds\|_{L^2(\Omega_{\epsilon^{-1}})} \le c$ and using L^2 boundedness of the Hodge projection

$$\|Ds\|_{L^2(\mathbb{R}^2)} \overset{(3.7)}{\le} \|D\eta\|_{L^2(\Omega_{\epsilon^{-1}})} + \|H(g_\varphi\psi - 2w_\varphi D\psi)\|_{L^2(\Omega_{\epsilon^{-1}})} \le c. \tag{3.12}$$

Since $Ds = Dw_\varphi \psi + w_\varphi D\psi$, so $\|Dw_\varphi \psi\|_{L^2(\mathbb{R}^2)} \overset{(3.12)}{\leq} c + \|w_\varphi D\psi\|_{L^2(\mathbb{R}^2)}$. Now $w_\varphi = \triangle v_\varphi$ and so $\|w_\varphi D\psi\|_{L^2(\mathbb{R}^2)} \leq c\|D^2 v_\varphi\|_{L^2(\Pi_\delta)} \leq c$ for any $\varphi > 0$. Hence

$$\|Dw_\varphi\|_{L^2(\Pi_{2\delta})} < c \text{ for all } \varphi > 0. \tag{3.13}$$

Let $q \in C_0^\infty(\Pi_{2\delta})$ with $q \equiv 1$ on $\Pi_{3\delta}$. Let $z_\varphi = v_{\varphi,1} q$ so $\triangle z_\varphi = \triangle v_{\varphi,1} q + 2Dv_{\varphi,1} \cdot Dq + v_{\varphi,1} \triangle q$. Thus as $\triangle v_{\varphi,1} = w_{\varphi,1}$

$$\|\triangle z_\varphi\|_{L^2(\mathbb{R}^2)} \leq \|\triangle v_{\varphi,1} q\|_{L^2(\mathbb{R}^2)} + 2\|Dv_{\varphi,1} \cdot Dq\|_{L^2(\mathbb{R}^2)} + \|v_{\varphi,1} \triangle q\|_{L^2(\mathbb{R}^2)} \overset{(3.13)}{\leq} c.$$

Now as we have seen before by L^2 estimates on Riesz transforms, this implies $D^2 z_\varphi \in L^2(\mathbb{R}^2)$. As $D^2 z_\varphi = D^2 v_{\varphi,1} q + 2Dv_{\varphi,1} \otimes Dq + v_{\varphi,1} D^2 q$ we have that

$$\int_{\Pi_{3\delta}} \left|D^2 v_{\varphi,1}\right|^2 \mathrm{d}x \leq c \int_{\mathbb{R}^2} \left|D^2 z_\varphi\right|^2 \mathrm{d}x + c \int_{\mathbb{R}^2} \left|Dv_{\varphi,1}\right|^2 + c \int_{\mathbb{R}^2} \left|v_{\varphi,1}\right|^2 \mathrm{d}x \leq c \text{ for every } \varphi > 0. \tag{3.14}$$

Arguing in exactly the same way gives $\int_{\Pi_{3\delta}} \left|D^2 v_{\varphi,2}\right|^2 \mathrm{d}x \leq c$ for every $\varphi > 0$, thus

$$\int_{\Pi_{3\delta}} \left|D^3 v_\varphi\right|^2 \leq c \text{ for every } \varphi > 0.$$

Now for any $\varphi_n \to 0$, $D^2 v_{\varphi_n}$ is a bounded sequence in $W^{1,2}(\Pi_{3\delta})$, so for some subsequence k_n, $D^2 v_{\varphi_{k_n}} \rightharpoonup \zeta \in W^{1,2}(\Pi_{3\delta} : \mathbb{R}^{2\times 2})$. Clearly $\zeta = D^2 v$ for a.e. in $\Pi_{3\delta}$. Let $i, j, k \in \{1, 2\}$ and $\phi \in C_0^\infty(\Pi_3)$,

$$\begin{aligned} \int v_{,ij} \phi_{,k} &= \lim_{n\to\infty} \int v_{\varphi_{k_n},ij} \phi_{,k} \mathrm{d}x \\ &= \lim_{n\to\infty} \int -v_{\varphi_{k_n},ijk} \phi \mathrm{d}x \\ &= \int -\zeta_{ij,k} \phi \mathrm{d}x. \end{aligned}$$

Thus $v_{,ij} \in W^{1,2}(\Pi_{3\delta})$ for any $i, j \in \{1, 2\}$ and hence $D^2 v \in W^{1,2}(\Pi_{3\delta})$. □

Step 2. We will show that v satisfies (3.3).

Proof of Step 2. Take any arbitrary $\phi \in C^\infty(\Omega_{\epsilon^{-1}})$, letting $\psi^h(z) := \frac{\phi(z+he_p)-\phi(z)}{h}$ we know from (3.2)

$$\begin{aligned} &\int \sum_{i,j} v_{ij}(y)\, \phi_{ijp}(y) + \left(1 - |Dv(y)|^2\right) Dv(y)\, D\phi_p(y)\, \mathrm{d}y \\ &= \lim_{h\to 0} h^{-1} \int \sum_{i,j=1}^{2} v_{ij}(y)\, \psi^h_{ij}(y) + \left(1 + |Dv(y)|^2\right) Dv(y)\, D\psi^h(y)\, \mathrm{d}y \\ &= 0 \end{aligned} \tag{3.15}$$

thus integrating by parts

$$\int \sum_{i,j} v_{ijp} \phi_{ij} + \left(\left(1 - |Dv|^2\right) Dv\right)_p D\phi \mathrm{d}y = 0.$$

Repeating the argument gives us (3.3). □

Lemma 3.3. *Let $u \in W_0^{2,2}(B_1(0))$ be the minimizer of I_ϵ, then*

$$I_\epsilon(u) \leq c\epsilon \log(\epsilon^{-1}). \tag{3.16}$$

Proof. Let ρ be the standard rotationally symmetric convolution kernel with $\mathrm{Spt}\rho \subset B_2(0)$ and let $\rho_\epsilon(z) := \rho(\frac{z}{\epsilon})\epsilon^{-2}$. Let $w(x) = 1 - |x|$ and $w_\epsilon = w * \rho_\epsilon$. So if $y \in B_{4\epsilon}(0)$

$$\left|D^2 w_\epsilon(y)\right| \leq \left|\int (w(z)-1) D^2\rho_\epsilon(y-z)\mathrm{d}z\right| \leq c\epsilon^{-4}\int_{B_{6\epsilon}(0)} |w(z)-1|\,\mathrm{d}z \leq c\epsilon^{-1}. \tag{3.17}$$

Note $Dw(y) = -\frac{y}{|y|}$ and $D^2 w(y) = \frac{y\otimes y}{|y|^3} - |y|^{-1}\, Id$ so $\left|D^2 w(y)\right| \leq \frac{4}{|y|}$. So

$$\left|D^2 w_\epsilon(y)\right| \leq \left|\int D^2 w(z)\rho_\epsilon(y-z)\mathrm{d}z\right| \leq 4\int \frac{\rho_\epsilon(y-z)}{|z|}\mathrm{d}z \leq \frac{c}{|y|} \text{ for any } y \notin B_{4\epsilon}(0). \tag{3.18}$$

Thus

$$\int_{B_1(0)} \left|D^2 w_\epsilon\right|^2 \mathrm{d}y \leq \int_{B_{4\epsilon}(0)} \left|D^2 w_\epsilon\right|^2 \mathrm{d}y + \int_{B_1(0)\backslash B_{4\epsilon}(0)} \left|D^2 w_\epsilon\right|^2 \mathrm{d}y \overset{(3.17),(3.18)}{\leq} c + c\int_{4\epsilon}^1 r^{-1}\mathrm{d}r \leq c\log(\epsilon^{-1}).$$

Now $\left\{x \in \mathbb{R}^2 : w_\epsilon(x) = 0\right\}$ is a circle of radius $h \approx 1$ so defining $v(x) = w_\epsilon\left(\frac{x}{h}\right)h$, $v \in W_0^{2,2}(B_1(0))$ and $\int_{B_1(0)} \left|D^2 v\right|^2 \mathrm{d}x \leq c\log(\epsilon^{-1})$. Now if $x \notin B_{4\epsilon}(0)$, $|Dw_\epsilon(x) - Dw(x)| = \left|\int (Dw(z) - Dw(x))\rho_\epsilon(x-z)\mathrm{d}z\right| \leq \frac{c\epsilon}{|x|}$. So $\left||Dw_\epsilon(x)|^2 - 1\right|^2 \leq c\,||Dw_\epsilon(x)| - 1|^2 \leq \frac{c\epsilon^2}{|x|^2}$. Thus

$$\begin{aligned}
\int_{B_1(0)} \left|1 - |Dw_\epsilon(x)|^2\right|^2 \mathrm{d}x &\leq c\epsilon^2 + \int_{B_1(0)\backslash B_{4\epsilon}(0)} \left|1 - |Dw_\epsilon(x)|^2\right|^2 \mathrm{d}x \\
&\leq c\epsilon^2 + \int_{4\epsilon}^1 \frac{\epsilon^2}{r}\mathrm{d}r \\
&\leq c\log(\epsilon^{-1})\epsilon^2
\end{aligned}$$

and this establishes (3.16). □

Lemma 3.4. *Let $u \in W_0^{2,2}(B_1(0))$ be a minimizer of I_ϵ. Let $\mathcal{C}_1$ be a some small positive constant to be chosen later. Define $A(x,\alpha,\beta) := B_\beta(x)\backslash\overline{B_\alpha(x)}$. We divide $B_1(0)$ into $N = \left[\mathcal{C}_1^{-2}\log(\epsilon^{-1})\right]$ slices of equal angle, denote their closure by $T_1, T_2, \ldots, T_N$. There must exists a set $\Pi \subset \{1,2,\ldots,N\}$ with* Card $(\Pi) \geq \frac{N}{2}$ *such that if $i \in \Pi$*

$$\begin{aligned}
\inf\left\{|Du(z)| : z \in T_i \cap A(0, c\log(\epsilon^{-1})\epsilon, 1-2\epsilon)\right\} > \frac{1}{2} \text{ and}\\
\sup\left\{|Du(z)| : z \in T_i \cap A(0, c\log(\epsilon^{-1})\epsilon, 1-2\epsilon)\right\} < 2.
\end{aligned} \tag{3.19}$$

Proof of Lemma 3.4. Define $v(z) = u(\epsilon z)\epsilon^{-1}$. Let $S_i = \epsilon^{-1}T_i$ for $i = 1,2,\ldots,N$. For $i \in \{2,3,\ldots,N-1\}$ define

$$\widetilde{S}_i = S_{i-1} \cup S_i \cup S_{i+1} \text{ and let } \widetilde{S}_1 = S_{N-1} \cup S_1 \cup S_2,\ \widetilde{S}_N = S_{N-1} \cup S_N \cup S_1.$$

Define

$$G_0 := \left\{i \in \{1,2,\ldots,N\} : \int_{\widetilde{S}_i} \left|1 - |Dv|^2\right|^2 + \left|D^2 v\right|^2 \mathrm{d}z \leq \mathcal{C}_1\right\}. \tag{3.20}$$

Note that by (3.16) of Lemma 3.3 we know $\int_{B_{\epsilon^{-1}}(0)} \left|1 - |Dv|^2\right|^2 + \left|D^2 v\right|^2 \mathrm{d}x \leq c\log(\epsilon^{-1})$, so $\mathcal{C}_1(N - \mathrm{Card}\,(G_0)) \leq c\log(\epsilon^{-1})$, thus (assuming we chose $\mathcal{C}_1$ small enough) $\frac{\mathcal{C}_1^{-2}}{2}\log(\epsilon^{-1}) \leq \mathrm{Card}\,(G_0)$.

Step 1. Let $i \in G_0$, we will show that for any $y_0 \in \widetilde{S}_i$ such that $B_2(y_0) \subset \widetilde{S}_i$ and $\psi \in C_0^\infty(B_2(y_0))$ such that $\psi \equiv 1$ on $B_1(y_0)$ we have

$$\int \left|D^3 v\right|^2 \psi^6 \mathrm{d}z \leq c. \tag{3.21}$$

Proof of Step 1. Let $Y = (4\pi)^{-1} \int_{B_2(y_0)} Dv$, $T = (4\pi)^{-1} \int_{B_2(y_0)} v$ and we define $\tilde{v}(z) = v(z) - Y \cdot (z - y_0) - T$. Let $\phi := \tilde{v}\psi^6$. So $\phi_p = \tilde{v}_p \psi^6 + 6\tilde{v}\psi^5\psi_p$ and

$$\phi_{pi} = v_{pi}\psi^6 + 6\tilde{v}_p\psi^5\psi_i + 6\tilde{v}_i\psi^5\psi_p + 6\tilde{v}\left(\psi^5\psi_p\right)_i. \tag{3.22}$$

$$\begin{aligned}\phi_{pij} \quad = \quad & v_{pij}\psi^6 + 6v_{pi}\psi^5\psi_j + 6v_{pj}\psi^5\psi_i + 6\tilde{v}_p\left(\psi^5\psi_i\right)_j \\ & + 6v_{ij}\psi^5\psi_p + 6\tilde{v}_i\left(\psi^5\psi_p\right)_j + 6\tilde{v}_j\left(\psi^5\psi_p\right)_i + 6\tilde{v}\left(\psi^5\psi_p\right)_{ij}.\end{aligned} \tag{3.23}$$

By the fact that $B_2(y_0) \subset \widetilde{S}_i$ we know $\int_{B_2(y_0)} \left|D^2 v\right|^2 \leq \mathcal{C}_1$, by Poincare's inequality this implies $\|D\tilde{v}\|_{L^2(B_2(y_0))} \leq c$ and $\|\tilde{v}\|_{L^2(B_2(y_0))} \leq c$. So from (3.23)

$$\begin{aligned}\left|\int v_{ijp}\phi_{ijp} - \int (v_{ijp})^2 \psi^6\right| & \overset{(3.23)}{\leq} c\|v_{ijp}\psi^3\|_{L^2}\left(\|D^2 v\|_{L^2(B_2(y_0))} + \|D\tilde{v}\|_{L^2(B_2(y_0))} + \|\tilde{v}\|_{L^2(B_2(y_0))}\right) \\ & \leq c\|D^3 v\psi^3\|_{L^2}.\end{aligned} \tag{3.24}$$

Now

$$\begin{aligned}\left|\int \left(\left(1 - |Dv|^2\right) Dv\right)_p \cdot D\phi_p \,\mathrm{d}z\right| & = \left|\int \left(\left(1 - |Dv|^2\right) Dv\right) \cdot D\phi_{pp} \mathrm{d}z\right| \\ & \leq \left|\int \left(\left(1 - |Dv|^2\right) Dv\right) \cdot \left(D\phi_{pp} - Dv_{pp}\psi^6\right) \mathrm{d}z\right| \\ & \quad + \left|\int \left(\left(1 - |Dv|^2\right) Dv\right) \cdot Dv_{pp}\psi^6 \mathrm{d}z\right| \\ & \overset{(3.23)}{\leq} c\left\|\left(1 - |Dv|^2\right) Dv\right\|_{L^2(B_2(y_0))}\|D^2 v\|_{L^2(B_2(y_0))} \\ & \quad + \|D^3 v\psi^3\|_{L^2}\left\|\left(1 - |Dv|^2\right) Dv\psi^3\right\|_{L^2} \\ & \overset{(3.20)}{\leq} c\left(1 + \|D^3 v\psi^3\|_{L^2(B_2(y_0))}\right).\end{aligned} \tag{3.25}$$

Recalling the fact that by Lemma 3.2, v satisfies (3.3) we have

$$\begin{aligned}\left|\int \sum_{i,j,p=1}^{2} (v_{ijp})^2 \psi^6 \mathrm{d}z\right| & \overset{(3.3)}{=} \left|\int \sum_{i,j,p=1}^{2} (v_{ijp})^2 \psi^6 - v_{ijp}\phi_{ijp} - \int \left(\left(1 - |Dv|^2\right) Dv\right)_p \cdot D\phi_p \mathrm{d}z\right| \\ & \overset{(3.24),(3.25)}{\leq} c\|D^3 v\psi^3\|_{L^2} + c.\end{aligned}$$

And this establishes (3.21). □

Proof of Lemma 3.4 *completed.* By Theorem 2, Section 5.6 [11]

$$\|D^2 v\|_{L^4(B_2(y_0))} \leq \|D^2 v\|_{W^{1,2}(B_2(y_0))} \leq c + \|D^3 v\|_{L^2(B_2(y_0))} \overset{(3.3)}{\leq} c.$$

By Sobolev embedding this implies Dv is $\frac{1}{2}$-Holder in $B_1(y_0)$.

Since $\int_{B_1(y_0)} \left|1-|Dv|^2\right|^2 \mathrm{d}z \leq \mathcal{C}_1$. Let $L = \left\{z \in B_1(y_0) : \left|1-|Dv|^2\right|^2 \leq \sqrt{\mathcal{C}_1}\right\}$ so we have $|B_1(y_0) \backslash L| \leq \sqrt{\mathcal{C}_1}$. So $B_{4\mathcal{C}_1^{\frac{1}{4}}}(y_0) \cap L \neq \emptyset$ so we can pick $z_1 \in B_{4\mathcal{C}_1^{\frac{1}{4}}}(y_0) \cap L$. Since Dv is $\frac{1}{2}$ Holder

$$\begin{aligned} ||Dv(y_0)|-1| &\leq |Dv(y_0) - Dv(z_1)| + \mathcal{C}_1^{\frac{1}{4}} \\ &\leq c\,|y_0 - z_1|^{\frac{1}{2}} + \mathcal{C}_1^{\frac{1}{4}} \\ &\leq c\mathcal{C}_1^{\frac{1}{8}}, \end{aligned}$$

assuming we chose $\mathcal{C}_1$ small enough this implies $|Dv(y_0)| \in (\frac{1}{2}, 2)$. Since y_0 is an arbitrary point in $\widetilde{S}_i \backslash N_2(\partial \widetilde{S}_i)$ and $Du(\epsilon y_0) = Dv(y_0)$ this implies (3.19). □

Lemma 3.5. *Let $u \in W^{2,2}(B_1(0))$. Suppose*

$$\int_{B_1(0)} \left|1-|Du|^2\right| \left|D^2 u\right| \mathrm{d}z \leq \beta \tag{3.26}$$

and

$$\int_{B_1(0)} \left|1-|Du|^2\right| \mathrm{d}z \leq \beta. \tag{3.27}$$

We will show that for any $w \in S^1$ we can find a set $G_w \subset P_{w^\perp}(B_1(0))$ with

$$|P_{w^\perp}(B_1(0)) \backslash G_w| \leq \beta^{\frac{1}{3}} \tag{3.28}$$

and for any $x \in G_w$ we have

$$\sup\left\{||Du(z)|-1| : z \in P_{w^\perp}^{-1}(x) \cap B_1(0)\right\} \leq 5\beta^{\frac{1}{3}}. \tag{3.29}$$

Proof. Let

$$B_w := \left\{x \in P_{w^\perp}(B_1(0)) : \int_{P_{w^\perp}^{-1}(x) \cap B_1(0)} \left|1-|Du|^2\right| \left|D^2 u\right| + \left|1-|Du|^2\right| \mathrm{d}z \leq \beta^{\frac{2}{3}}\right\}.$$

By Chebyshev's inequality we have $|P_{w^\perp}(B_1(0)) \backslash B_w| \leq 2\beta^{\frac{1}{3}}$. For any $x \in P_{w^\perp}(B_{1-\beta^{\frac{2}{3}}}(0))$ we know $\left|P_{w^\perp}^{-1}(x) \cap B_1(0)\right| \geq \beta^{\frac{1}{3}}$ and so if in addition $x \in B_w$ we have that there must exists $z_x \in P_{w^\perp}^{-1}(x) \cap B_1(0)$ such that $|1-|Du(z_x)|| \leq \beta^{\frac{1}{3}}$.

Suppose $x \in B_w \cap P_{w^\perp}(B_{1-\beta^{\frac{2}{3}}}(0))$ and for some $y_x \in P_{w^\perp}^{-1}(x) \cap B_1(0)$ we have $|1-|Du(y_x)|| \geq 5\beta^{\frac{1}{3}}$. Then as we can assume without loss of generality that Du is continuous on $P_{w^\perp}^{-1}(x) \cap B_1(0)$ and so there must exists $a_x, b_x \in P_{w^\perp}^{-1}(x) \cap B_1(0)$ such that $||Du(a_x)| - |Du(b_x)|| \geq \beta^{\frac{1}{3}}$ and $\inf\{|Du(x)| : x \in [a_x, b_x]\} \geq 1 + 4\beta^{\frac{1}{3}}$. However by the fundamental theorem of calculus

$$4\beta^{\frac{1}{3}}\,||Du(a_x)| - |Du(b_x)|| \leq \int_{a_x}^{b_x} |1-|Du||\left|D^2 u\right| \leq \beta^{\frac{2}{3}}$$

which is a contradiction. Thus taking $G_w := B_w \cap P_{w^\perp}(B_{1-\beta^{\frac{1}{3}}}(0))$ completes the proof of the lemma. □

Lemma 3.6. *Suppose $\tilde{u}$ is a C^2 function that satisfies (3.26), (3.27) and $\Lambda \subset B_1(0)$ is convex with the property that* $\inf\{|D\tilde{u}(x)| : x \in \Lambda\} > \frac{1}{3}$ *and* $\sup\{|D\tilde{u}(x)| : x \in \Lambda\} < 3$.

Given function $X : \mathbb{R} \to \mathbb{R}^2$ that solves $X(0) = x$ and $\dot{X}(s) = D\tilde{u}(X(s))$, suppose $s_1 < 0 < s_2$ are such that $X(s) \in \Lambda$ for any $s \in [s_1, s_2]$ then

$$\tilde{u}(X(s_2)) - \tilde{u}(X(s_1)) \geq (1 - \beta^{\frac{1}{3}}) |X(s_2) - X(s_1)| - c\beta^{\frac{1}{3}}. \tag{3.30}$$

And if in addition $X(s_1), X(s_2) \notin B_r(x)$ for some $B_r(x) \subset \Omega$, then

$$\{X(s) : s \in [s_1, s_2]\} \subset N_{c\frac{\beta^{\frac{1}{6}}}{\sqrt{r}}}([X(s_1), X(s_2)]). \tag{3.31}$$

Proof. Let $w \in S^1$ be orthogonal to $X(s_2) - X(s_1)$. Let G_w be the set satisfying (3.28) and (3.29) from Lemma 3.5. Let $P = \{X(t) : t \in [s_1, s_2]\}$ and $\Gamma = P \cap P_{w^\perp}^{-1}(G_w)$. So $H^1(\Gamma) \geq |P_{w^\perp}([X(s_1), X(s_1)]) \cap G_w| \geq |X(s_2) - X(s_1)| - \beta^{\frac{1}{3}}$ and so

$$\begin{aligned} \tilde{u}(X(s_2)) - \tilde{u}(X(s_1)) &= \int_P D\tilde{u}(z) \cdot t_z \mathrm{d}H^1 z \\ &\geq (1 - c\beta^{\frac{1}{3}}) H^1(\Gamma) + \frac{1}{3} H^1(P \backslash \Gamma) \\ &\geq (1 - c\beta^{\frac{1}{3}}) |X(s_2) - X(s_1)| + \frac{1}{3} H^1(P \backslash \Gamma) - c\beta^{\frac{1}{3}} \end{aligned} \tag{3.32}$$

which establishes (3.30). Now

$$\begin{aligned} \tilde{u}(X(s_2)) - \tilde{u}(X(s_1)) &\leq \int_{[X(s_1), X(s_2)]} |D\tilde{u}(z)| \, \mathrm{d}H^1 z \\ &\leq (1 + c\beta^{\frac{1}{3}}) |P_{v^\perp}([X(s_2), X(s_1)] \cap G_w)| + 3 |P_{v^\perp}([X(s_2), X(s_1)] \backslash G_w)| \\ &\leq |X(s_2) - X(s_1)| + c\beta^{\frac{1}{3}} \end{aligned} \tag{3.33}$$

now putting (3.32) and (3.33) together we have $H^1(P \backslash \Gamma) \leq c\beta^{\frac{1}{3}}$. Now this and the second inequality of (3.32) and inequality (3.33) imply that

$$|X(s_2) - X(s_1)| - c\beta^{\frac{1}{3}} \geq H^1(P). \tag{3.34}$$

If $X(s_1), X(s_2) \notin B_r(x)$ then as $X(0) = x \in P$ and as P is connected we know $H^1(P) \geq |X(s_1) - X(0)| + |X(s_2) - X(0)| \geq 2r$ which by (3.34) implies $|X(s_1) - X(s_2)| \geq r$ and so $|X(s_1) - X(s_2)| (1 + \frac{c\beta^{\frac{1}{3}}}{r}) \geq H^1(P)$. Now letting t_z denote the tangent to the curve P at point z we have

$$\begin{aligned} \int_P \left| t_z - \frac{X(s_2) - X(s_1)}{|X(s_2) - X(s_1)|} \right|^2 \mathrm{d}H^1 z &= \int_P 2 - 2t_z \cdot \left(\frac{X(s_2) - X(s_1)}{|X(s_2) - X(s_1)|} \right) \mathrm{d}H^1 z \\ &= 2H^1(P) - 2 |X(s_2) - X(s_1)| \\ &\leq \frac{c\beta^{\frac{1}{3}}}{r}. \end{aligned}$$

By Holder's inequality and the fundamental theorem of calculus this immediately implies (3.31). □

Lemma 3.7. *Suppose u is a minimizer of I_ϵ over $W_0^{2,2}(B_1(0))$. There exists $r \approx \epsilon^{\frac{1}{6}} (\log(\epsilon^{-1}))^{\frac{13}{6}}$ and $\xi \in \{1, -1\}$ such that*

$$\inf \{\xi u(z) : z \in B_r(0)\} \geq 1 - c\epsilon^{\frac{1}{6}} (\log(\epsilon^{-1}))^{\frac{13}{6}} \tag{3.35}$$

Proof. First recall that by Lemma 3.3, (3.16) we know that $I_\epsilon(u) \leq c\epsilon \log(\epsilon^{-1})$. Let $T_1, T_2, \dots, T_N$ be as defined in Lemma 3.4. By Lemma 3.4 there exists $i \in \{1, 2, \dots, N\}$ such that T_i satisfies (3.19). □

By Lemma 3.2 we know $u \in W^{3,2}(B_{1-2\epsilon}(0))$. Now by approximation of Sobolev functions (see Thm. 3, Sect. 5.33 [11]), for any small $\tau > 0$ we can find $\tilde{u} \in C^\infty(B_{1-2\epsilon}(0))$ such that

$$\|\tilde{u} - u\|_{W^{3,2}(B_{1-2\epsilon}(0))} < \tau. \tag{3.36}$$

Since

$$\int_{B_1(0)} \left|1 - |Du|^2\right|^2 \mathrm{d}x \leq c\epsilon^2 \log(\epsilon^{-1}) \tag{3.37}$$

and

$$\int_{B_1(0)} \left|1 - |Du|^2\right| \left|D^2 u\right| \mathrm{d}x \leq c\epsilon \log(\epsilon^{-1}). \tag{3.38}$$

By Sobolev embedding we have that u is $\frac{1}{2}$-Holder and thus

$$\sup\{|u(z)| : z \in \partial B_{1-2\epsilon}(0)\} \leq c\sqrt{\epsilon}. \tag{3.39}$$

Now assuming τ is small enough, as by Sobolev embedding $D\tilde{u}$ is Holder continuous, $\tilde{u}$ must satisfy $\sup\{|\tilde{u}(z)| : z \in \partial B_{1-2\epsilon}(0)\} \leq c\sqrt{\epsilon}$ and

$$\begin{aligned} \inf\left\{|D\tilde{u}\,(z)| : z \in A(0, c\log(\epsilon^{-1})\epsilon, 1-2\epsilon) \cap T_i\right\} > \frac{1}{3} \text{ and} \\ \sup\left\{|D\tilde{u}\,(z)| : z \in A(0, c\log(\epsilon^{-1})\epsilon, 1-2\epsilon) \cap T_i\right\} < 3. \end{aligned} \tag{3.40}$$

It is also clear that for small enough τ, $\tilde{u}$ satisfies $I_\epsilon(\tilde{u}) \leq c\epsilon\log(\epsilon^{-1})$.

Step 1. Let ϑ denote the center point of $\partial B_{1-2\epsilon}(0) \cap T_i$ define $\varsigma = 2(1 - \cos(\frac{\pi}{N}))$, so $\varsigma \approx \frac{\mathcal{C}_1^4 \pi^2}{(\log(\epsilon^{-1}))^2}$. Let $\varrho = (1-\varsigma)\vartheta$. For any set A let $\mathrm{conv}(A)$ denote the convex hull of A. Note that (see Fig. 1)

$$\mathrm{dist}\,(\varrho, \mathrm{conv}(\partial B_{1-2\epsilon}(0) \cap T_i)) > \frac{\varsigma}{2}. \tag{3.41}$$

Let $X : \mathbb{R} \to \mathbb{R}^2$ be the solution of $X(0) = \varrho$ and $\dot{X}(s) = D\tilde{u}(X(s))$. Let $\mathcal{T}_i := T_i \cap A(0, c\log(\epsilon^{-1})\epsilon, 1-2\epsilon)$. Let $t_2 > 0$ be the smallest number such that $X(t_2) \in \partial\mathcal{T}_i$ and let $t_1 < 0$ be the largest number so that $X(t_1) \in \partial\mathcal{T}_i$. Let $s \in \{t_1, t_2\}$ be such that

$$d(X(s), \partial B_{1-2\epsilon}(0)) = \min\{d(X(t_1), \partial B_{1-2\epsilon}(0)), d(X(t_2), \partial B_{1-2\epsilon}(0))\}. \tag{3.42}$$

Let $e \in \{t_1, t_2\} \setminus \{s\}$. See Figure 1.

We will show $X(s) \in \partial B_{1-2\epsilon}(0) \cap B_{\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}/2}(\vartheta)$ and $X(e) \in \partial\mathcal{T}_i \backslash \partial B_{1-2\epsilon}(0)$.

Proof of Step 1. We claim

$$\cos^{-1}\left(\frac{X(s) - X(e)}{|X(s) - X(e)|} \cdot \frac{\vartheta}{|\vartheta|}\right) \leq \frac{\pi}{2} - \frac{1}{129}. \tag{3.43}$$

Let $\psi = \cos^{-1}\left(\frac{X(s)-X(e)}{|X(s)-X(e)|} \cdot \frac{\vartheta}{|\vartheta|}\right)$. Suppose (3.43) not true, *i.e.* $\psi \geq \frac{\pi}{2} - \frac{1}{129}$. Since $X(s), X(e) \notin B_\varsigma(\vartheta)$ and by (3.36)–(3.38) $\tilde{u}$ satisfies (3.26), (3.27) for $\beta = \epsilon\log(\epsilon^{-1})$ so applying Lemma 3.6 we have that by (3.31)

$$\varrho \in N_{c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{7}{6}}}([X(s), X(e)]), \tag{3.44}$$

i.e. points $\varrho, X(s_2), X(s_1)$ are *roughly* (with error $c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{7}{6}}$) aligned, so by (3.41) we must have

$$X(e) \in \partial\mathcal{T}_i \backslash \partial B_{1-2\epsilon}(0)$$

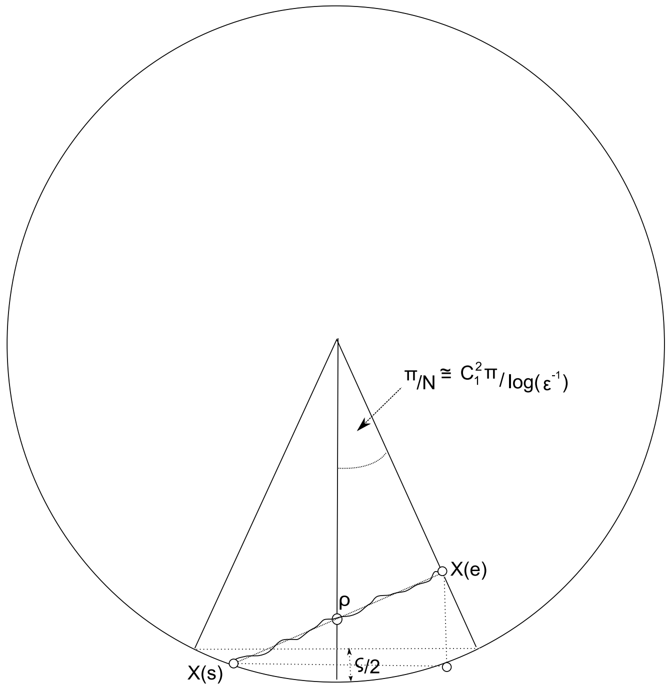

FIGURE 1. An integral path of the vector field $D\tilde{u}$ going through ρ.

and in particular $|X(e)-X(s)| > \frac{\mathcal{C}_1^2}{2}(\log(\epsilon^{-1}))^{-1}$. Note also by (3.42) and by (3.44) we have that

$$d(X(s), \partial B_{1-2\epsilon}(0)) \leq c(\log(\epsilon^{-1}))^{-2}. \tag{3.45}$$

Thus by (3.30)

$$|\tilde{u}(X(e)) - \tilde{u}(X(s))| \geq \frac{\mathcal{C}_1^2}{3}(\log(\epsilon^{-1}))^{-1}. \tag{3.46}$$

Since $\tilde{u}$ is 3-Lipschitz and $d(X(s), \partial B_{1-2\epsilon}(0)) \leq 2\varsigma$ we have $|\tilde{u}(X(s))| \leq 6\varsigma \leq \frac{c}{(\log(\epsilon^{-1}))^2}$. Thus by (3.46) we have

$$|\tilde{u}(X(e))| \geq \frac{\mathcal{C}_1^2}{4}(\log(\epsilon^{-1}))^{-1}. \tag{3.47}$$

Now let L be the line parallel to $[X(s), X(e)]$ that passes through ϱ, by (3.31) we can pick $\nu \in L \cap B_{\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{7}{6}}}(X(s))$ and let $\mu = (X(e) + \langle\vartheta\rangle) \cap (\nu + \vartheta^{\perp})$. Note that by trigonometry

$$d(\mu, \partial B_{1-2\epsilon}(0)) \leq d(\nu, \partial B_{1-2\epsilon}(0)) + c(\log(\epsilon^{-1}))^{-2}. \tag{3.48}$$

And so

$$d(\mu, \partial B_{1-2\epsilon}(0)) \leq d(X(s), \partial B_{1-2\epsilon}(0)) + c(\log(\epsilon^{-1}))^{-2} \overset{(3.45)}{\leq} c(\log(\epsilon^{-1}))^{-2}. \tag{3.49}$$

Recall we have assumed by contradiction that $\psi \geq \frac{\pi}{2} - \frac{1}{129}$. By (3.44) $X(s)$, ϱ, $X(e)$ are with error $(\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1})))^{\frac{7}{6}}$ aligned and by (3.42) $X(s)$ is closer (or equally close) to $\partial B_{1-2\epsilon}(0)$ than $X(e)$, so $X(s) \cdot \frac{\vartheta}{|\vartheta|} > X(e) \cdot \frac{\vartheta}{|\vartheta|} - c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{7}{6}}$, hence $\psi \leq \frac{\pi}{2} + \frac{1}{129}$. We will denote a triangle with corners at a, b, c by $T(a,b,c)$. Consider the right angle triangle $T(\nu, X(e), \mu)$. Now let $\tilde{\psi}$ denote the angle of the corner of the triangle

$T(\nu, X(e), \mu)$ at $X(e)$. By construction as $|\nu - X(s)| < \epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{7}{6}}$ so $\left|\psi - \tilde{\psi}\right| \leq \epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}} \leq \frac{1}{128} - \frac{1}{129}$, thus $\tilde{\psi} \in \left[\frac{\pi}{2} - \frac{1}{128}, \frac{\pi}{2} + \frac{1}{128}\right]$. Thus

$$\frac{127}{128}|\nu - X(e)| \leq |\nu - X(e)|\sin(\tilde{\psi}) \leq |\mu - \nu| \leq 2\pi\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}.$$

So

$$|\nu - X(e)| \leq 8\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}. \tag{3.50}$$

Thus

$$\begin{aligned} |X(e) - \mu| &\leq \cos(\tilde{\psi})\,|\nu - X(e)| \\ &\overset{(3.50)}{\leq} 8\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}\cos\left(\frac{\pi}{2} - \frac{1}{128}\right) \\ &\leq \frac{\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}}{16}. \end{aligned} \tag{3.51}$$

Hence

$$\begin{aligned} d(X(e), \partial B_{1-2\epsilon}(0)) &\overset{(3.51)}{\leq} d(\mu, \partial B_{1-2\epsilon}(0)) + \frac{\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}}{16} \\ &\overset{(3.49)}{\leq} \frac{\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}}{16} + c(\log(\epsilon^{-1}))^{-2}. \end{aligned}$$

Thus $|\tilde{u}(X(e))| \leq \frac{3\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}}{16} + c\left(\log(\epsilon^{-1})\right)^2$ which is a contradicts (3.47). So (3.43) is established.

Let $\omega = L \cap (\vartheta + \vartheta^{\perp})$. Consider the right angle triangle $T(\omega, \varrho, \vartheta)$. By trigonometry we know that $|\omega - \vartheta|\tan\left(\frac{\pi}{2} - \psi\right) = \varsigma$ which implies $|\omega - \vartheta| \leq 258\varsigma$, hence $X(s) \in \partial B_{1-2\epsilon}(0) \cap B_{\frac{\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}}{2}}(\vartheta)$. As we know already $X(e) \in \partial\mathcal{T}_i \backslash B_{1-2\epsilon}(0)$ this completes the proof of Step 1. $\square$

Step 2. We will show

$$\left|\cos^{-1}\left(\frac{X(s)}{|X(s)|} \cdot \frac{(X(s) - X(e))}{|X(s) - X(e)|}\right)\right| \leq c\epsilon^{\frac{1}{6}}\log(\epsilon^{-1})^{\frac{7}{6}}. \tag{3.52}$$

Proof of Step 2. Let $\theta = \cos^{-1}\left(\frac{X(s)}{|X(s)|} \cdot \frac{(X(s)-X(e))}{|X(s)-X(e)|}\right)$. Let

$$\kappa = (X(s) + (X(s))^{\perp}) \cap (X(e) + \mathbb{R}X(s))\,.$$

Note that the points $X(s), X(e), \kappa$ forms the corners of a right-angle triangle where the angle at the point $X(e)$ is θ. Since $\kappa \notin \mathcal{T}_i$ and as $\mathcal{T}_i$ is convex, $[\kappa, X(e)]$ intersects $\partial\mathcal{T}_i$ at one point only, so let $\zeta = (\kappa, X(e)) \cap \partial\mathcal{T}_i$. We claim that $\zeta \in \partial B_{1-2\epsilon}(0)$. To see this suppose it is not true, then the line segment $[\kappa, X(e)]$ must cross one of the flat sides of $\partial\mathcal{T}_i$. Recall the angle at 0 of the 'pie slice' $\mathcal{T}_i$ is $\frac{2\pi}{N}$. So the angle between ϑ and either of the sides of $\partial\mathcal{T}_i$ is $\frac{\pi}{N}$. However the line segment $[\kappa, X(e)]$ is parallel to the line segment $[0, X(s)]$ so $\cos^{-1}\left(\frac{\vartheta}{|\vartheta|} \cdot \frac{\kappa - X(e)}{\kappa - X(e)}\right) < \frac{\pi}{N}$. Now in order for $[\kappa, X(e)]$ to cross the flat sides of $\partial\mathcal{T}_i$ without first intersecting $\partial B_{1-2\epsilon}(0)$ it has to make a larger angle with ϑ than the flat sides of $\partial\mathcal{T}_i$ so this a contradiction. Thus the claim is established and we have $\cos(\theta)\,|X(s) - X(e)| \geq |X(e) - \zeta|$.

Now since $X(s) \in \partial B_{1-2\epsilon}(0)$ so $|\tilde{u}(X(s))| \leq c\sqrt{\epsilon}$ and thus

$$\begin{aligned} \tilde{u}(X(e)) &\overset{(3.30)}{\geq} (1 - c(\epsilon\log(\epsilon^{-1}))^{\frac{1}{3}})\,|X(e) - X(s)| - c(\epsilon\log(\epsilon^{-1}))^{\frac{1}{3}} \\ &\geq \frac{|X(e) - \zeta|}{\cos\theta} - c(\epsilon\log(\epsilon^{-1}))^{\frac{1}{3}}. \end{aligned} \tag{3.53}$$

By Lemma 3.5 there exists a line segment $\Gamma \subset \mathcal{T}_i$ parallel to $[X(e), \zeta]$ whose end points are within $(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}}$ of $X(e), \zeta$ and for which $\sup\{||D\tilde{u}(z)| - 1| : z \in \Gamma\} \leq c(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}}$. Let a, b be the end points of Γ, so by the fundamental theorem of calculus, $|\tilde{u}(a) - \tilde{u}(b)| \leq (1 + c(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}})\, |a - b|$. Since $\tilde{u}$ is Lipschitz on $\mathcal{T}_i$ and $|\tilde{u}(\zeta)| \leq c\sqrt{\epsilon}$ we have that $|\tilde{u}(X(e))| \leq (1 + c(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}})\, |X(e) - \zeta|$, thus putting this together with (3.53) we have

$$|X(e) - \zeta| \geq \frac{|X(e) - \zeta|}{(1 + c(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}}) \cos\theta} - c(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}}. \tag{3.54}$$

Recall $B_\varsigma(\varrho) \subset \mathcal{T}_i$ and as we know $X(s)$ is closer to $\partial B_{1-2\epsilon}(0)$ than $X(e)$, so by (3.44) we have that $|X(e) - \zeta| \geq \frac{\varsigma}{2}$, so by (3.54) we have $\cos(\theta) \geq 1 - c\epsilon^{\frac{1}{3}}(\log(\epsilon^{-1}))^{\frac{7}{3}}$ which implies $|\theta| \leq c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{7}{6}}$ and this completes the proof of Step 2. $\square$

Proof of Lemma 3.7 completed. By Step 1 we know $X(s) \in B_{\frac{\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}}{2}}(\vartheta)$, so the angle between the line segment $[X(s), 0]$ and the sides of $\partial\mathcal{T}_i$ is at least $\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}/4$. So if we consider the triangle $T(0, X(s), X(e))$. Let η be the angle of the triangle at corner 0, so $\eta \geq \frac{\mathcal{C}_1^2(\log(\epsilon^{-1}))^{-1}}{4}$. Recall the angle at corner $X(s)$ is θ and by (3.52) $\theta \leq c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{7}{6}}$. So by the law of sins, $\frac{|X(e)|}{\sin\theta} = \frac{|X(e) - X(s)|}{\sin\eta}$. So

$$|X(e)| \leq \frac{2\sin\theta}{\sin\eta} \leq c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}}. \tag{3.55}$$

Now as noted previously, (3.39) and (3.36), $|\tilde{u}(X(s))| \leq c\sqrt{\epsilon}$. So by (3.30) we have that

$$\begin{aligned} |\tilde{u}(X(e))| &\geq (1 - (\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}})\, |X(e) - X(s)| - c(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}} \\ &\geq (1 - (\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}})\mathrm{d}(X(e), \partial B_{1-2\epsilon}(0)) - c(\epsilon \log(\epsilon^{-1}))^{\frac{1}{3}} \\ &\geq 1 - c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}}. \end{aligned} \tag{3.56}$$

So we must have $r \in (|X(e)| + \frac{1}{2}\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}}, |X(e)| + c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}})$ such that

$$\int_{\partial B_r(0)} \left|1 - |D\tilde{u}|^2\right| \mathrm{d}H^1 z \overset{(3.37),(3.36)}{\leq} c\epsilon^{\frac{5}{6}}(\log(\epsilon^{-1}))^{-\frac{10}{6}}.$$

By the fundamental theorem of calculus we have that

$$|\tilde{u}(x) - \tilde{u}(y)| \leq c\epsilon^{\frac{5}{6}}(\log(\epsilon^{-1}))^{-\frac{10}{6}} \text{ for all } x, y \in \partial B_r(0). \tag{3.57}$$

Let $\xi = \frac{\tilde{u}(X(e))}{|\tilde{u}(X(e))|}$. Pick $z \in \partial B_r(0) \cap \mathcal{T}_i$, since $\tilde{u}$ is Lipschitz on $\mathcal{T}_i$ we know

$$|\tilde{u}(z) - \tilde{u}(X(e))| \leq c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}}. \tag{3.58}$$

Thus for any $x \in \partial B_r(0)$

$$\xi\tilde{u}(x) \overset{(3.58)(3.57)}{\geq} \xi\tilde{u}(X(e)) - c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}} \overset{(3.56)}{\geq} 1 - c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}}, \tag{3.59}$$

together with (3.36) (using the fact that (3.36) implies $\|\tilde{u} - u\|_{L^\infty(B_{1-2\epsilon}(0))} \leq c\epsilon$) this completes the proof of Lemma 3.7. $\square$

Proof of Theorem 1.2 completed. Let $r \approx \epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}}$, $\xi \in \{-1,1\}$ be the numbers that satisfy (3.35) from Lemma 3.7. Let $A(x) = \frac{x}{|x|}$ note $|DA(x)| \leq \frac{c}{|x|}$. Note by Fubini

$$\begin{aligned}\int_{B_r(0)} \int_{B_1(0)} \left|1 - |Du(z)|^2\right| |DA(x-z)| \,\mathrm{d}z\mathrm{d}x \\ &= \int_{B_1(0)} \left(\int_{B_r(0)} |DA(x-z)| \,\mathrm{d}x\right) \left(1 - |Du(z)|^2\right) \mathrm{d}z \\ &\leq c\epsilon\sqrt{\log(\epsilon^{-1})}.\end{aligned} \tag{3.60}$$

So there must exist a set $G \subset B_r(0)$ with $|G| \geq \epsilon^{\frac{1}{3}}(\log(\epsilon^{-1}))^{\frac{13}{3}}$ such that if $x \in G$ we have

$$\int_{B_1(0)} \left|1 - |Du(z)|^2\right| |DA(x-z)| \,\mathrm{d}z \leq c\epsilon^{\frac{1}{3}}. \tag{3.61}$$

For $\theta \in S^1$, $y \in \mathbb{R}^2$ define $l_\theta^y := y + \mathbb{R}_+\theta$. Pick $x \in G$, by the Co-area formula

$$\int_{\psi \in S^1} \int_{l_\psi^x} \left|1 - |Du(z)|^2\right| \mathrm{d}H^1 z \mathrm{d}H^1\psi \leq c\epsilon^{\frac{1}{3}}.$$

For each $\psi \in S^1$ let $x_\psi = \partial B_r(0) \cap l_\psi^x$, $y_\psi = \partial B_1(0) \cap l_\psi^x$ and $e_\psi = \int_{l_\psi^x} \left|1 - |Du(z)|^2\right| \mathrm{d}H^1 z$. So

$$\begin{aligned}\int_{[x_\psi, y_\psi]} |Du(z) + \xi\psi|^2 \,\mathrm{d}H^1 z &= \int_{[x_\psi, y_\psi]} |Du(z)|^2 + 2\xi Du(z)\cdot\psi + 1 \mathrm{d}H^1 z \\ &\leq 2|y_\psi - x_\psi| - 2\xi u(x_\psi) + ce_\psi \\ &\overset{(3.35)}{\leq} c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}} + ce_\psi.\end{aligned} \tag{3.62}$$

Thus

$$\begin{aligned}\int_{B_1(0)\setminus B_r(x)} \left|Du(z) + \xi\frac{z}{|z|}\right|^2 \mathrm{d}z &\leq \int_{B_1(0)\setminus B_r(x)} \left|Du(z) + \xi\frac{z}{|z|}\right|^2 |DA(x-z)| \,\mathrm{d}z \\ &\leq \int_{S^1} \int_{[x_\psi, y_\psi]} |Du(z) + \xi\psi| \,\mathrm{d}H^1 z\mathrm{d}H^1\psi \\ &\overset{(3.62)}{\leq} c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}} + c\int_{S^1} e_\psi \mathrm{d}H^1\psi \\ &\leq c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{B_1(0)} \left|Du(z) + \xi\frac{z}{|z|}\right|^2 \mathrm{d}z &\leq \int_{B_r(0)} \left|Du(z) + \xi\frac{z}{|z|}\right|^2 \mathrm{d}z + c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}} \\ &\leq c\int_{B_r(0)} |1 - ||Du(z)| - 1||^2 \,\mathrm{d}z + c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}} \\ &\leq c\epsilon^{\frac{1}{6}}(\log(\epsilon^{-1}))^{\frac{13}{6}}.\end{aligned}$$

□

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