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APPROXIMATE CONTROLLABILITY BY BIRTH CONTROL FOR A NONLINEAR POPULATION DYNAMICS MODEL

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Abstract. In this paper we analyse an approximate controllability result for a nonlinear population dynamics model. In this model the birth term is nonlocal and describes the recruitment process in newborn individuals population, and the control acts on a small open set of the domain and corresponds to an elimination or a supply of newborn individuals. In our proof we use a unique continuation property for the solution of the heat equation and the Kakutani-Fan-Glicksberg fixed point theorem.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain where $N \geq 1$ is an integer, and consider the following system:

$$\begin{aligned}
\frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} - \Delta y + \mu y &= 0, \qquad \text{in } (0,T)\times(0,A)\times\Omega && (1.1)\\
y(t,a,\sigma) &= 0, \qquad \text{on } (0,T)\times(0,A)\times\partial\Omega && (1.2)\\
y(0,a,x) &= y_0(a,x), \qquad \text{in } (0,A)\times\Omega && (1.3)\\
y(t,0,x) &= v(t,x)1_\omega(x) + F_*(y), \quad \text{in } (0,T)\times\Omega && (1.4)
\end{aligned}$$

where, for a given function $F : (0,T)\times\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$, we denote by F_* the operator defined by

$$F_*(y)(t,x) := F\left(t, x, \int_0^A \beta(a) y(t,a,x)\mathrm{d}a\right).$$

In the above system $T > 0$, $A > 0$ are positive constants, $\partial\Omega$ is the boundary of Ω (for Dirichlet boundary conditions there is no need to assume any regularity, but for other boundary conditions appropriate boundary regularity must to be assumed). Here Δ is the Laplacian with respect to the spatial variable x and 1_ω is the

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characteristic function of a small subset $\omega \subset \Omega$, where the control v is assumed to act. The system (1.1)–(1.4) arises in population dynamics, where $y(t,a,x)$ denotes the distribution of individuals of age a at time t and location x, A is the maximal life expectancy; $\mu(a)$ and $\beta(a)$ are respectively the natural death rate and the natural birth rate of individuals of age a. The flux of individuals here has the form $-\nabla y(t,a,x)$, but actually a slightly more general flux can be treated with the same method, such as a flux defined by $-C(x)\nabla y(t,a,x)$, provided the resulting elliptic operator $y \mapsto -\mathrm{div}(C(x)\nabla y)$ possesses an appropriate unique continuation property, see below Remark 3.1. The boundary $\partial\Omega$ is assumed to be inhospitable (hence the boundary condition (1.2)). Finally, in (1.3), $y_0(a,x)$ is the initial distribution of individuals of age a at position x.

Let us introduce the so-called *net reproduction rate*

$$R := \int_0^A \beta(a) \exp\left(-\int_0^a \mu(s)\mathrm{d}s\right) \mathrm{d}a.$$

It is known that when $F(t,x,s) := s$ (see for instance Anita [2]), if $R < 1$ then

$$\lim_{t\to+\infty} \|y(t)\|_{L^2((0,A)\times\Omega)} = 0$$

while if $R > 1$ then

$$\lim_{t\to+\infty} \|y(t)\|_{L^2((0,A)\times\Omega)} = +\infty.$$

In this paper our aim is to analyse an approximate controllability result for the system above. More precisely in the sequel, for all $\varepsilon > 0$ small enough and all targets $h \in L^2((0,A)\times\Omega)$, we study the existence of a control $v \in L^2((0,T)\times\omega)$ such that the corresponding solution of the system (1.1)–(1.4) verifies

$$\|y(T,\cdot,\cdot) - h\| \le \varepsilon. \tag{1.5}$$

A first controllability result for a linear age and space structured population dynamics model was obtained by Ainseba and Langlais in [1], where it was shown that a certain set of profiles is approximately reachable at any given time T. As far as we may be aware of, a first work on the controllability with birth control was due to Barbu *et al.* in [4]. However in that work, the system did not involve diffusion terms and, as a consequence, one cannot use the method therein when the control acts on a small open subset ω of Ω. More precisely in [4], a null controllability result by birth control for a linear McKendrick model was proved by means of an internal controllability result.

In [10], the second author of the present paper studied an application of the approximate controllability property to data assimilation problems. The question addressed there is to determine whether one can use an approximate controllability result for recovering the initial data for a linear population dynamics model. The unique continuation result used there, is derived from a new Carleman inequality.

In this paper we address the question of the approximate controllability when the birth control acts on a small open set of the domain and in our proof we use a non standard unique continuation property. This unique continuation result is established using a classical unique continuation result for the heat equation (see for instance Lin [7]), and an approach developed in Kavian and de Teresa [5].

The remainder of this paper is organized as follows: in the next section we give the assumptions and state the main result; Section 3 is devoted to the study of an auxiliary linear equation, and in Section 4 we give the proof of our main result.

2. Assumptions and main results

In what follows we make the following assumptions:

$(\mathbf{H_1})$ $\mu \in L^1_{\mathrm{loc}}(0,A)$, $\mu \ge 0$ a.e. in $(0,A)$, and $\lim_{a\to A} \int_0^a \mu(s)\mathrm{d}s = +\infty$.

$(\mathbf{H_2})$ $\beta \in C^{0,1}([0,A])$, the space of Lipschitz functions on $[0,A]$, moreover $\beta \ge 0$ and there exists $A_0 < A_1$ such that $\mathrm{supp}(\beta) \subset [A_0,A_1] \subset]0,A[$.

($\mathbf{H_3}$) For $0 < a < A$, denoting $\pi(a) := \exp\left(-\int_0^a \mu(s)\mathrm{d}s\right)$ the survival likelihood, we assume that $\pi^{-1}y_0 \in L^2((0,A)\times\Omega)$.

($\mathbf{H_4}$) $F : (0,T)\times\Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ is a Caratheodory function such that a.e. in $(t,x)\in(0,T)\times\Omega$ the function $s \mapsto F(t,x,s)$ is in $C^1(\mathbb{R})$, verifies $F(t,x,0) = 0$ and moreover is globally Lipschitz: for some given $M_0 > 0$, a.e. in $(t,x)\in(0,T)\times\Omega$ and all $s_1, s_2 \in \mathbb{R}$ we have

$$|F(t,x,s_1) - F(t,x,s_2)| \leq M_0\, |s_1 - s_2|\,.$$

To such a function F, we associate an operator F_* defined as follows: for $y \in L^1((0,T)\times(0,A)\times\Omega)$ we set

$$F_*(y)(t,x) := F\left(t, x, \int_0^A \beta(a)y(t,a,x)\mathrm{d}a\right). \tag{2.1}$$

Remark 2.1. The assumptions (H_1) and (H_2) are classical assumptions in the modelisation of population dynamics. Indeed, since μ and β are supposed to be the natural rates of death and birth respectively, it is obvious that the conditions $\mu \geq 0$ a.e. in $(0,A)$ and $\beta \geq 0$ a.e. in $(0,A)$ are natural. On the other hand the condition

$$\lim_{a\to A}\int_0^a \mu(s)\mathrm{d}s = +\infty$$

means that the survival likelihood $\pi(a) = \exp\left(-\int_0^a \mu(s)\mathrm{d}s\right)$ tends to zero as a tends to A.

The last condition on the birth rate β in (H_2) means, naturally, that the young and old individuals are not fertile. Under these hypotheses, it was proved in [8] that the system has a unique solution.

We also recall that if $u \in L^2\left((0,T)\times(0,A); H_0^1(\Omega)\right)$ verifies

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} \in L^2\left((0,T)\times(0,A); H^{-1}(\Omega)\right),$$

then one can define the trace $u(t_0,\cdot,\cdot) \in L^2\left((0,A)\times\Omega\right)$ at $t_0 \in [0,T]$, and analogously the trace $u(\cdot,a_0,\cdot) \in L^2\left((0,T)\times\Omega\right)$ at $a_0 \in [0,A]$ (see for instance Langlais [6], or Anita [2]).

Let us state now the main result of this paper.

Theorem 2.2. *Assume that the assumptions* (H_1)–(H_4) *are satisfied and that* $T > A$. *Then for all* $h \in L^2((0,A)\times\Omega)$ *and* $\varepsilon > 0$, *there exists* $v \in L^2\left((0,T)\times\omega\right)$ *such that the corresponding solution of the system* (1.1)–(1.4) *verifies* (1.5).

Remark 2.3. We introduce the following change of unknown functions:

$$\widehat{y}(t,a,x) := (\pi(a))^{-1}y(t,a,x), \tag{2.2}$$
$$\widehat{y}_0(a,x) := (\pi(a))^{-1}y_0(a,x), \tag{2.3}$$
$$\widehat{\beta}(a) := \pi(a)\beta(a). \tag{2.4}$$

Since $\pi(0) = 1$ and $\widehat{\beta}(a)\widehat{y}(t,a,x) = \beta(a)y(t,a,x)$, it follows that $\widehat{F}_*(\widehat{y})(t,x) = F_*(y)(t,x)$ on $(0,T)\times\Omega$ and therefore $\widehat{y}$ solves the problem:

$$\frac{\partial\widehat{y}}{\partial t} + \frac{\partial\widehat{y}}{\partial a} - \Delta\widehat{y} = 0 \quad \text{in } (0,T)\times(0,A)\times\Omega \tag{2.5}$$
$$\widehat{y}(t,a,\sigma) = 0 \quad \text{on } (0,T)\times(0,A)\times\partial\Omega \tag{2.6}$$
$$\widehat{y}(0,a,x) = \widehat{y}_0(a,x) \quad \text{in } (0,A)\times\Omega \tag{2.7}$$
$$\widehat{y}(t,0,x) = v(t,x)1_\omega(x) + \widehat{F}_*(\widehat{y}) \quad \text{in } (0,T)\times\Omega. \tag{2.8}$$

Therefore, the problem stated in Theorem 2.2 is reduced to find a control $v \in L^2((0,T)\times\omega)$ such that the solution of the system (2.5)–(2.8) with $\widehat{y}_0 \in L^2((0,A)\times\Omega)$ satisfies (1.5). On the other hand without loss of generality, we can assume that $\widehat{y}_0 \equiv 0$. Indeed, let φ be the solution of the free evolution equation, that is solution to the system

$$\begin{aligned}
\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial a}-\Delta\varphi &= 0 \quad \text{in } (0,T)\times(0,A)\times\Omega && (2.9)\\
\varphi(t,a,\sigma) &= 0 \quad \text{on } (0,T)\times(0,A)\times\partial\Omega && (2.10)\\
\varphi(0,a,x) &= \widehat{y}_0(a,x) \quad \text{in } (0,A)\times\Omega && (2.11)\\
\varphi(t,0,x) &= F_*(\varphi) \quad \text{in } (0,T)\times\Omega. && (2.12)
\end{aligned}$$

This system admits a unique solution, as one may see by an easy adaptation of the method used in Ouédraogo and Traoré [9], where the Neumann boundary condition case is studied. Then setting

$$\widetilde{y} := \widehat{y} - \varphi,$$

one checks that $\widetilde{y}$ solves the system:

$$\begin{aligned}
\frac{\partial\widetilde{y}}{\partial t}+\frac{\partial\widetilde{y}}{\partial a}-\Delta\widetilde{y} &= 0 \quad \text{in } (0,T)\times(0,A)\times\Omega && (2.13)\\
\widetilde{y}(t,a,\sigma) &= 0 \quad \text{on } (0,T)\times(0,A)\times\partial\Omega\\
\widetilde{y}(0,a,x) &= 0 \quad \text{in } (0,A)\times\Omega\\
\widetilde{y}(t,0,x) &= v(t,x)1_\omega(x)+G_*(\widetilde{y}) \text{ in } (0,T)\times\Omega && (2.14)
\end{aligned}$$

where one may check that the operators G and G_* defined by

$$\begin{aligned}
G_*(\widetilde{y}) &:= G\left(t,x,\int_0^A \widehat{\beta}(a)\widetilde{y}\mathrm{d}a\right)\\
&:= F\left(t,x,\int_0^A \widehat{\beta}(a)(\widetilde{y}+\varphi)\mathrm{d}a\right) - F\left(t,x,\int_0^A \widehat{\beta}(a)\varphi\mathrm{d}a\right),
\end{aligned}$$

satisfy clearly the same condition (H_4) as F, F_*. In this way one sees that the problem is reduced to finding v such that the solution of the system (2.13)–(2.14) satisfies

$$\|\widetilde{y}(T,\cdot,\cdot)-h_1\| \le \varepsilon,$$

where $h_1 := h - \varphi(T,\cdot,\cdot)$. Consequently, we may, and we will, consider the system (2.5)–(2.8) with $\widehat{y}_0 = y_0 = 0$ and write β instead of $\widehat{\beta}$, and y instead of $\widehat{y}$.

The next section is devoted to the study of the linear case.

3. Study of the linear system

We set

$$H(t,x,s) = \begin{cases} F'(t,x,0) & \text{if } s=0\\ \dfrac{F(t,x,s)}{s} & \text{if } s\neq 0. \end{cases} \tag{3.1}$$

Using (H_4) it follows that the function H is continuous and bounded on $(0,T)\times\Omega\times\mathbb{R}$; we shall denote by M the upper bound of the function H

$$M := \sup_{s\in\mathbb{R}} |s|^{-1}\|F(\cdot,\cdot,s)\|_{L^\infty((0,T)\times\Omega)}. \tag{3.2}$$

Let us consider for a given $Y^0 \in L^2((0,T)\times\Omega)$ the following system

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \Delta y &= 0 \quad \text{in } (0,T)\times(0,A)\times\Omega \\ y(t,a,\sigma) &= 0 \quad \text{on } (0,T)\times(0,A)\times\partial\Omega \\ y(0,a,x) &= 0 \quad \text{in } (0,A)\times\Omega \\ y(t,0,x) &= v(t,x)1_\omega + H(t,x,Y^0)Y \quad \text{in } (0,T)\times\Omega \end{cases} \tag{3.3}$$

where we set

$$Y(t,x) := \int_0^A \beta(a)y(t,a,x)\mathrm{d}a. \tag{3.4}$$

A solution of the system (2.5)–(2.8) with $y_0 = 0$ is obtained as a fixed point of the mapping $Y^0 \mapsto Y$.

Next we define the coefficient β_0 by

$$\beta_0(t,a,x) := H(t,x,Y^0)\beta(a), \tag{3.5}$$

and for $g \in L^2((0,A)\times\Omega)$ fixed we consider an adjoint system which reads:

$$-\frac{\partial p}{\partial t} - \frac{\partial p}{\partial a} - \Delta p = \beta_0(t,a,x)p(t,0,x) \quad \text{in } (0,T)\times(0,A)\times\Omega \tag{3.6}$$

$$p(t,a,\sigma) = 0 \quad \text{on } (0,T)\times(0,A)\times\partial\Omega \tag{3.7}$$

$$p(T,a,x) = g(a,x) \quad \text{in } (0,A)\times\Omega \tag{3.8}$$

$$p(t,A,x) = 0 \quad \text{in } (0,T)\times\Omega. \tag{3.9}$$

We recall that, using a fixed point method, and the arguments of Anita [2] or Ouédraogo and Traoré [9], one can show easily that the system (3.6)–(3.9), admits a unique solution.

Before proving our unique continuation result, which plays a crucial role in the proof of our main result, we state and prove the following elementary lemma.

Let us denote by $(\lambda_j,\varphi_j)_{j\geq 1}$ the eigenvalues and normalized eigenfunctions of $-\Delta$ on $H_0^1(\Omega)$, that is:

$$-\Delta\varphi_j = \lambda_j\varphi_j \quad \text{in } \Omega, \qquad \text{and } \varphi_j = 0 \quad \text{on } \partial\Omega.$$

It is known that $(\varphi_j)_{j\geq 1}$ is a Hilbert basis of $L^2(\Omega)$.

Remark 3.1. As one may see by a rapid inspection of the arguments we are using in this paper, instead of the elliptic operator $y \mapsto -\Delta y$ one can consider more general elliptic operators such as

$$\begin{gathered} y \mapsto -\mathrm{div}(C(\cdot)\nabla y), \\ \text{where } C \in \left(W^{1,\infty}(\Omega)\right)^{N\times N}, \quad C(x)^* = C(x), \\ \exists\,\alpha_* > 0, \quad \forall\,\xi\in\mathbb{R}^N,\ \forall\,x\in\Omega, \quad C(x)\xi\cdot\xi \geq \alpha_*|\xi|^2. \end{gathered}$$

Indeed for such an operator a unique continuation result is valid: if for a function $\varphi \in H_0^1(\Omega)$ one has $-\mathrm{div}(C\nabla\varphi) = \lambda\varphi$, and $\varphi \equiv 0$ in ω, then $\varphi \equiv 0$ in Ω. This can be used to show that Lemma 3.2 can be established for such an operator, where the φ_j's are the associated eigenfunctions. The remainder of the arguments of this paper are essentially unchanged (see also [5] where the relation between unique continuation results for elliptic operators and unique continuation results for solutions to the associated heat equation is extensively used).

Lemma 3.2. *Let $(c_j)_{j\geq 1}$ be a sequence of complex numbers such that for some $\tau > 0$ we have*

$$\sum_{j\geq 1} \mathrm{e}^{2\lambda_j \tau} |c_j|^2 < \infty.$$

Then the function

$$z := \sum_{j\geq 1} c_j \varphi_j \in L^2(\Omega)$$

is well defined and if $z \equiv 0$ on a nonempty open subset $\omega \subset \Omega$, then $z \equiv 0$ on Ω and $c_j = 0$ for all $j \geq 1$.

Proof. It is clear that the function

$$w_0 := \sum_{j\geq 1} \mathrm{e}^{\lambda_j \tau} c_j \varphi_j \in L^2(\Omega)$$

is well defined and that upon solving the linear heat equation

$$\frac{\partial w}{\partial s} - \Delta w = 0 \quad \text{in } (0,\infty)\times\Omega, \quad w(0,x) = w_0(x), \quad w(s,\cdot) = 0 \text{ on } (0,\infty)\times\partial\Omega,$$

the solution is given by

$$w(s,\cdot) = \sum_{j\geq 1} (w_0|\varphi_j)\mathrm{e}^{-\lambda_j s}\,\varphi_j = \sum_{j\geq 1} \mathrm{e}^{\lambda_j \tau}\, c_j\, \mathrm{e}^{-\lambda_j s}\,\varphi_j.$$

Observe that since $w(\tau,\cdot) = z(\cdot)$ vanishes identically on ω, according to the unique continuation principle for the heat equation (see for instance Lin [7], the main theorem of that paper) we conclude that $w(s,x) \equiv 0$ on $(0,\infty)\times\Omega$ and finally that $w_0 \equiv z \equiv 0$ on Ω, and $c_j = 0$ for all $j \geq 1$. □

Now, we prove the following unique continuation principle, which plays a crucial role in the proof of our main result.

Proposition 3.3. *Let $g \in L^2((0,A)\times\Omega)$, let the hypotheses* (H_1)*–*(H_4) *be satisfied, and set $\widehat{\beta}(a) := \pi(a)\beta(a)$ and $\beta_0(a,t,x) := H(t,x,Y^0(t,x))\widehat{\beta}(a)$ where H is given by* (3.1) *and $Y^0 \in L^2((0,T)\times\Omega)$ is fixed. If the solution $p \in L^2((0,T)\times(0,A);H_0^1(\Omega))$ of* (3.6)*–*(3.9) *verifies $p(t,0,x) = 0$ in $(0,T)\times\omega$ then $p \equiv 0$ in $(0,T)\times(0,A)\times\Omega$.*

Proof. It is rather more convenient to prove the result for the forward system, that is setting $z(t,a,x) := p(T-t,A-a,x)$, to prove that $z \equiv 0$ whenever $z(t,A,x) = 0$ in $(0,T)\times\omega$. Clearly z satisfies the system:

$$\left\{\begin{array}{rcll} \dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial a} - \Delta z &=& \beta_1(t,a,x)z(t,A,x) & \text{in } (0,T)\times(0,A)\times\Omega \\ z(t,a,\sigma) &=& 0 & \text{on } (0,T)\times(0,A)\times\partial\Omega \\ z(0,a,x) &=& k(a,x) & \text{in } (0,A)\times\Omega \\ z(t,0,x) &=& 0 & \text{in } (0,T)\times\Omega \end{array}\right. \tag{3.10}$$

where indeed we have set

$$\beta_1(t,a,x) := \beta_0(T-t,A-a,x), \qquad k(a,x) := g(A-a,x). \tag{3.11}$$

With the above notations, we may write z in the Hilbert basis $(\varphi_j)_j$

$$z(t,a,x) = \sum_{j=1}^{\infty} z_j(t,a)\varphi_j,$$

where z_j solves the linear hyperbolic system:

$$\begin{cases} \dfrac{\partial z_j}{\partial t} + \dfrac{\partial z_j}{\partial a} + \lambda_j z_j &= \gamma_j(t,a) \quad \text{in } (0,T)\times(0,A) \\ z_j(0,a) &= k_j(a) \quad \text{in } (0,A) \\ z_j(t,0) &= 0 \quad \text{in } (0,T)\,. \end{cases} \tag{3.12}$$

Here we have set

$$k_j(a) := \int_\Omega k(a,x)\varphi_j(x)\mathrm{d}x, \tag{3.13}$$

$$\gamma_j(t,a) := \int_\Omega \varphi_j(x)\beta_1(t,a,x)z(t,A,x)\mathrm{d}x. \tag{3.14}$$

Now, as it is customary in the study of such linear hyperbolic equations, integrating the transport equation (3.12) along characteristic lines, we obtain the explicit representation formula:

$$z_j(t,a) = \begin{cases} \mathrm{e}^{-\lambda_j t}k_j(a-t) + \displaystyle\int_0^t \mathrm{e}^{-\lambda_j s}\gamma_j(t-s,a-s)\mathrm{d}s, & \text{if } a \geq t \\ \displaystyle\int_0^a \mathrm{e}^{-\lambda_j s}\gamma_j(t-s,a-s)\mathrm{d}s, & \text{if } a < t. \end{cases} \tag{3.15}$$

Therefore we have that

$$z_j(t,A) = \begin{cases} \mathrm{e}^{-\lambda_j t}k_j(A-t) + \displaystyle\int_0^t \mathrm{e}^{-\lambda_j s}\gamma_j(t-s,A-s)\mathrm{d}s & t \leq A \\ \displaystyle\int_0^A \mathrm{e}^{-\lambda_j s}\gamma_j(t-s,A-s)\mathrm{d}s, & t > A. \end{cases} \tag{3.16}$$

We observe that since the support of β is contained in $[A_0, A_1]$ with $0 < A_1 < A_1 < A$ (*cf.* condition (H2)), we have that the support of $\beta_1(t,\cdot,x)$ is contained in $[A-A_1, A-A_0]$ and thus the support of $\gamma_j(t,\cdot)$ is contained in $[A-A_1, A-A_0]$ for all $j \geq 1$ and $t \in [0,T]$. Therefore for $0 < s < t < A_0$ we have $\gamma_j(t-s, A-s) = 0$ and

$$z_j(t,A) = \mathrm{e}^{-\lambda_j t}k_j(A-t) \qquad \text{for } \; 0 \leq t \leq A_0.$$

For almost all such fixed $t \in [0, A_0]$, setting

$$c_j := \mathrm{e}^{-\lambda_j t}k_j(A-t),$$

since $k(A-t,\cdot) \in L^2(\Omega)$, it follows that for $0 < \tau < t$ the sequence $(c_j)_{j\geq 1}$ satisfies the assumption of Lemma 3.2. Since

$$\sum_{j\geq 1} c_j\varphi_j = z(t,A,\cdot) \equiv 0 \quad \text{on } \; \omega,$$

thanks to the above mentioned lemma we conclude that $c_j = 0$ for all $j \geq 1$, and finally that

$$z(t,A,x) = 0 \qquad \text{for } \; (t,x) \in (0,A_0)\times\Omega. \tag{3.17}$$

This implies also that for all $j \geq 1$ we have $c_j = 0$, that is $k(A-t,x) \equiv 0$ for $(t,x) \in (0,A_0)\times\Omega$.

Next consider the case $A_0 \leq t \leq A$, and consider the coefficients

$$\alpha_j(t) := \int_0^t \mathrm{e}^{-\lambda_j s}\,\gamma_j(t-s,A-s)\mathrm{d}s.$$

Since $\gamma_j(t-s, A-s) = 0$ when $0 \leq s \leq A_0$, we may write

$$\alpha_j(t) := \int_{A_0}^{t} \mathrm{e}^{-\lambda_j s}\, \gamma_j(t-s, A-s)\mathrm{d}s \quad \text{for} \;\; A_0 \leq t \leq A.$$

Noting that for $t \in [A_0, A]$ we have

$$|\alpha_j(t)| \leq \mathrm{e}^{-\lambda_j A_0}\,(A - A_0)\,\|z(t, A, \cdot)\|_{L^2(\Omega)}\,,$$

upon setting

$$c_j := \mathrm{e}^{-\lambda_j t}\, k_j(A-t) + \alpha_j(t),$$

one sees that the conditions of Lemma 3.2 are satisfied for any $\tau \in (0, A_0)$ and again since $\sum_{j\geq 1} c_j\varphi_j = z(t, A, \cdot) \equiv 0$ on ω, we have that $z(t, A, x) \equiv 0$ on Ω. Thus we have shown that

$$z(t, A, x) = 0 \qquad \text{for} \;\; (t, x) \in (0, A) \times \Omega. \tag{3.18}$$

This in turn implies that $\gamma_j(t, A) = 0$ for all $j \geq 1$ and all $t \in (0, A)$, and also that $k(A - t, x) \equiv 0$ for $(t, x) \in (0, A) \times \Omega$.

For the case $A \leq t \leq T$, it is enough to apply the same observations to the coefficients

$$c_j := z_j(t, A) := \int_0^A \mathrm{e}^{-\lambda_j s}\, \gamma_j(t-s, A-s)\mathrm{d}s = \int_{A_0}^{A} \mathrm{e}^{-\lambda_j s}\, \gamma_j(t-s, A-s)\mathrm{d}s.$$

Indeed we have again $|z_j(t, A)| \leq \mathrm{e}^{-\lambda_j A_0}\,(A - A_0)\,\|z(t, A, \cdot)\|_{L^2(\Omega)}$ and Lemma 3.2 applies.

Finally, the above observations mean that

$$z(t, A, x) = 0 \quad \text{for} \;\; (t, x) \in (0, T) \times \Omega, \qquad k(a, x) = 0 \quad \text{for} \;\; (a, x) \in (0, A) \times \Omega.$$

Consequently z is solution of the homogeneous equation

$$\begin{cases} \dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial a} - \Delta z &= 0 \quad \text{in } (0, T) \times (0, A) \times \Omega \\ z(t, a, \sigma) &= 0 \quad \text{on } (0, T) \times (0, A) \times \partial\Omega \\ z(0, a, x) &= 0 \quad \text{in } (0, A) \times \Omega \\ z(t, 0, x) &= 0 \quad \text{in } (0, T) \times \Omega, \end{cases}$$

that is $z \equiv 0$ on $(0, T) \times (0, A) \times \Omega$, and the claim of the Proposition 3.3 is proved. □

Now let J be the functional defined on $L^2\left((0, A) \times \Omega\right)$ by:

$$J(g) = \frac{1}{2} \int_0^T \int_\omega |p(t, 0, x)|^2 \mathrm{d}t\mathrm{d}x + \varepsilon\|g\| - \int_0^A \int_\Omega h(a, x)g(a, x)\mathrm{d}a\mathrm{d}x \tag{3.19}$$

where for $g \in L^2((0, A) \times \Omega)$ given, p solves the adjoint system (3.6)–(3.9).

The following result is now classical.

Proposition 3.4. *The functional J is continuous, strictly convex and coercive. More precisely we have:*

$$\liminf_{\|g\|\to\infty} \frac{J(g)}{\|g\|} \geq \varepsilon, \tag{3.20}$$

where for convenience we set $\|g\| := \|g\|_{L^2((0,A)\times\Omega))}$. *In particular*

$$\lim_{\|g\|\to\infty} J(g) = +\infty$$

and J *achieves its minimum at a unique point* $\widehat{g} \in L^2((0,A)\times\Omega)$.

Proof. The proof is similar to the one given in Traoré [10] and follows the arguments used by Zuazua in [12]. First observe that, as we mentioned earlier, the operator

$$g \mapsto \left(p, \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a}\right)$$

from $L^2((0,A)\times\Omega))$ into

$$L^2\left((0,T)\times(0,A); H_0^1(\Omega)\right) \times L^2\left((0,T)\times(0,A); H^{-1}(\Omega)\right)$$

is continuous. Therefore the trace $p(\cdot,0,\cdot)$ is well-defined (see for instance Langlais [6] or Anita [2]) and depends continuously on g, so that, the mapping $g \mapsto p(t,0,x)1_\omega$, considered from $L^2\left((0,A)\times\Omega)\right)$ into the space $L^2\left((0,T)\times\omega\right)$, is linear and continuous, and therefore the functional J is continuous. Let us now prove (3.20). Consider a sequence $(g_n)_n$ in $L^2\left((0,A)\times\Omega\right)$ such that $\|g_n\| \to \infty$ as $n\to\infty$. We note

$$\widehat{g}_n = \frac{g_n}{\|g_n\|},$$

and $\widehat{p}_n$ the associated solution of (3.6)–(3.9), with $g := \widehat{g}_n$. Then,

$$\frac{J(g_n)}{\|g_n\|} = \frac{\|g_n\|}{2}\int_0^T\int_\omega |\widehat{p}_n(t,0,x)|^2 \mathrm{d}t\mathrm{d}x + \varepsilon - \int_{(0,A)\times\Omega} h_1\widehat{g}_n \mathrm{d}a\mathrm{d}x. \tag{3.21}$$

Using the fact that

$$\int_0^T\int_\omega |\widehat{p}_n(t,0,x)|^2 \mathrm{d}t\mathrm{d}x \geq 0, \tag{3.22}$$

we obtain either

$$\liminf_{n\to\infty}\int_0^T\int_\omega |\widehat{p}_n(t,0,x)|^2 \mathrm{d}t\mathrm{d}x > 0 \tag{3.23}$$

or

$$\liminf_{n\to\infty}\int_0^T\int_\omega |\widehat{p}_n(t,0,x)|^2 \mathrm{d}t\mathrm{d}x = 0. \tag{3.24}$$

In the first case, we get obviously

$$\liminf_{n\to\infty}\frac{J(g_n)}{\|g_n\|} = +\infty, \tag{3.25}$$

and this yields (3.20). In the second case, we extract a subsequence still denoted $(\widehat{g}_n)_n$ such that

$$\begin{gathered}
\int_0^T\int_\omega |\widehat{p}_n(t,0,x)|^2 \mathrm{d}t\mathrm{d}x \to 0 \qquad \text{as } \ n\to+\infty,\\
\widehat{g}_n \rightharpoonup \widehat{g} \qquad \text{weakly in } \ L^2\left((0,A)\times\Omega\right),\\
\widehat{p}_n \rightharpoonup \widehat{p} \qquad \text{weakly in } \ L^2\left((0,T)\times(0,A), H_0^1(\Omega)\right),\\
\frac{\partial\widehat{p}_n}{\partial t} + \frac{\partial\widehat{p}_n}{\partial a} \rightharpoonup \frac{\partial\widehat{p}}{\partial t} + \frac{\partial\widehat{p}}{\partial a} \quad \text{weakly in } \ L^2\left((0,T)\times(0,A), H^{-1}(\Omega)\right).
\end{gathered}$$

Therefore, we get that $\widehat{p}$ is solution to (3.6)–(3.9) and verifies

$$\widehat{p}(t,0,x)=0 \quad \text{a.e.} \quad (0,T)\times\omega.$$

Using now the unique continuation result of Proposition 3.3, we conclude that

$$\widehat{p}\equiv 0 \qquad \text{a.e. in} \quad (0,T)\times(0,A)\times\Omega.$$

From this we infer that $\widehat{p}^0\equiv 0$ a.e. in $(0,A)\times\Omega$, and in particular that $\widehat{g}\equiv 0$, that is $\widehat{g}_n\rightharpoonup 0$, which in turn implies that

$$\int_0^A\int_\Omega h(a,x)\widehat{g}_n(a,x)\mathrm{d}a\mathrm{d}x\to 0,$$

and finally equality (3.21) yields (3.20). Finally, due to the presence of the quadratic term in J and the unique continuation result, we note that J is strictly convex, and therefore the minimum of J is achieved at a unique point. □

Proposition 3.5. *Let $\widehat{g}$ be the unique minimizer of J on $L^2((0,A)\times\Omega)$. There exists a control $v\in L^2\left((0,T)\times\omega\right)$ such that the corresponding solution y of (3.3) verifies (1.5). More precisely, if $\varepsilon\geq\|h\|$, one can take $v:=0$, while if $\varepsilon<\|h\|$, one may take $v:=\widehat{p}(t,0,x)1_\omega$ where $\widehat{p}$ is the solution of equation (3.6)–(3.9) with $g:=\widehat{g}_\varepsilon$, the unique minimizer of J given by Proposition 3.4. Moreover, there exists $R>0$ independent of Y^0 such that*

$$\|v\|=\|\widehat{p}(\cdot,0,\cdot)1_\omega\|\leq R\|h\|. \tag{3.26}$$

Proof. First consider the case $\varepsilon\geq\|h\|$. Clearly, taking $v:=0$ one sees that $y\equiv 0$ and so we have (1.5).

Next consider the case $\varepsilon<\|h\|$. We know that J has a unique minimizer denoted by $\widehat{g}_\varepsilon$, and using the fact that $J(0)=0$ and $\|h\|>\varepsilon$, we infer that there exists $g\in L^2\left((0,A)\times\Omega\right)$ such that $J(g)<0$. This implies that $\widehat{g}_\varepsilon\neq 0$. It follows that J is differentiable at $\widehat{g}$ and we have:

$$J'(\widehat{g}_\varepsilon)g=\int_{(0,T)\times\omega}\widehat{p}p(t,0,x)\mathrm{d}x\mathrm{d}t+\frac{\varepsilon}{\|\widehat{g}_\varepsilon\|}\int_{(0,A)\times\Omega}\widehat{g}_\varepsilon g\mathrm{d}x\mathrm{d}a-\int_{(0,A)\times\Omega}h(a,x)g(a,x)\mathrm{d}a\mathrm{d}x=0.$$

This gives

$$\int_0^T\int_\omega\widehat{p}p(t,0,x)\mathrm{d}x\mathrm{d}t=-\frac{\varepsilon}{\|\widehat{g}_\varepsilon\|}\int_0^A\int_\Omega\widehat{g}_\varepsilon g\mathrm{d}a\mathrm{d}x+\int_0^A\int_\Omega h(a,x)g(a,x)\mathrm{d}a\mathrm{d}x. \tag{3.27}$$

Now consider $\widehat{y}$ the solution of the system

$$\begin{cases}\dfrac{\partial\widehat{y}}{\partial t}+\dfrac{\partial\widehat{y}}{\partial a}-\Delta\widehat{y} &=& 0 \quad \text{in } (0,T)\times(0,A)\times\Omega\\ \widehat{y}(t,a,\sigma) &=& 0 \quad \text{on } (0,T)\times(0,A)\times\partial\Omega\\ \widehat{y}(0,a,x) &=& 0 \quad \text{in } (0,A)\times\Omega\\ \widehat{y}(t,0,x) &=& \widehat{p}(t,x)1_\omega+H(t,x,Y^0)\widehat{Y} \quad \text{in } (0,T)\times\Omega,\end{cases} \tag{3.28}$$

where

$$\widehat{Y}(t,x):=\int_0^A\widehat{\beta}(a)\widehat{y}(t,a,x)\mathrm{d}a.$$

Upon multiplying the first equation in the above system by p and integrating over $(0,T)\times(0,A)\times\Omega$, after some integration by parts and using the fact that

$$\beta_0(t,a,x)=H(t,x,Y^0)\widehat{\beta}(a),$$

we obtain:

$$\int_0^T \int_\omega \widehat{p}p(t,0,x)\mathrm{d}x\mathrm{d}t = \int_0^A \int_\Omega \widehat{y}(T,a,x)g(a,x)\mathrm{d}x\mathrm{d}a.$$

From this, and equality (3.27) we infer that for all $g \in L^2((0,A)\times\Omega)$

$$\int_0^A \int_\Omega \left(\widehat{y}(T,a,x) - h + \frac{\varepsilon}{\|\widehat{g}_\varepsilon\|}\widehat{g}_\varepsilon\right) g(a,x)\mathrm{d}x\mathrm{d}a = 0.$$

This means that

$$\widehat{y}(T,a,x) - h = -\frac{\varepsilon}{\|\widehat{g}_\varepsilon\|}\widehat{g}_\varepsilon,$$

an equality which implies (1.5).

Let us prove now (3.26). It suffices to prove that the minimizer $\widehat{g}$ is uniformly bounded with respect to the function Y^0. Taking $g = \widehat{g}_\varepsilon$ in (3.27) we get

$$\int_0^T \int_\omega |\widehat{p}(t,0,x)|^2\mathrm{d}x\mathrm{d}t + \frac{\varepsilon}{\|\widehat{g}_\varepsilon\|}\int_0^A \int_\Omega |\widehat{g}_\varepsilon|^2\mathrm{d}x\mathrm{d}a = \int_0^A\int_\Omega h(a,x)\widehat{g}_\varepsilon\mathrm{d}a\mathrm{d}x.$$

This gives

$$J(\widehat{g}) = -\frac{1}{2}\int_0^A\int_0^T\int_\omega |\widehat{p}(t,0,x)|^2\mathrm{d}x\mathrm{d}t. \tag{3.29}$$

Now, we argue by contradiction. Indeed let $\left(Y_n^0\right) \subset L^2\left((0,T)\times\Omega\right)$ be a sequence such that the sequence of minimizers $\widehat{g}_n$ of J_n verifies

$$\lim_{n\to+\infty}\|\widehat{g}_n\| = \infty,$$

with J_n being

$$J_n(g) := \frac{1}{2}\int_0^T\int_\omega |p_n(t,0,x)|^2\mathrm{d}t\mathrm{d}x + \varepsilon\|g\| - \int_0^A\int_\Omega h(a,x)g(a,x)\mathrm{d}a\mathrm{d}x$$

where p_n solves (3.6)–(3.9) with $\beta_0^n := H(Y_n^0)\beta$ instead of β_0. We set $\widetilde{g}_n := \widehat{g}_n/\|\widehat{g}_n\|$ and $\widetilde{p}_n := \widehat{p}_n/\|\widehat{g}_n\|$. Then we get:

$$\frac{J_n(\widehat{g}_n)}{\|\widehat{g_n}\|} = \frac{1}{2}\|\widehat{g}_n\|\int_0^T\int_\omega |\widetilde{p}_n(t,0,x)|^2\mathrm{d}t\mathrm{d}x + \varepsilon - \int_0^A\int_\Omega h(a,x)\widetilde{g}_n(a,x)\mathrm{d}a\mathrm{d}x.$$

Since $J_n(\widehat{g}_n) < 0$, using the Cauchy-Schwarz inequality it follows that:

$$\frac{1}{2}\|\widehat{g_n}\|\int_0^T\int_\omega |\widetilde{p}_n(t,0,x)|^2\mathrm{d}t\mathrm{d}x + \varepsilon \le \|h\|.$$

Now, since $\|\widehat{g_n}\| \to +\infty$, this implies

$$\lim_{n\to\infty}\int_0^T\int_\omega |\widetilde{p}_n(t,0,x)|^2\mathrm{d}t\mathrm{d}x = 0.$$

On the one hand as the sequence $(\widetilde{g}_n)_n$ is bounded, we infer that there exists subsequences still denoted $\widetilde{p}_n, \beta_n, \widetilde{g}_n$ that converge weakly respectively to $\widetilde{p}, \beta, \widetilde{g}$. On the other hand $(\widetilde{p}, \beta, \widetilde{g})$ verifies (3.6)–(3.9), and using the fact that $\widetilde{p}(t,0,x) = 0$ in $(0,T)\times\Omega$ we deduce from the Proposition 3.3 that $\widetilde{p} \equiv 0$. Consequently the sequence $(\widetilde{g}_n)_n$ converges weakly to zero in $L^2\left((0,T)\times\Omega\right)$, and from (3.29) and the fact that $\widehat{p}_n = \|\widehat{g}_n\|\widetilde{p}_n$ we get that:

$$\frac{J(\widehat{g}_n)}{\|\widehat{g}_n\|} = -\frac{1}{2}\|\widehat{g}_n\|\int_0^T\int_\omega |\widetilde{p}_n(t,0,x)|^2\mathrm{d}x\mathrm{d}t \le 0. \tag{3.30}$$

On the other hand by the very definition of the functional J we have:

$$\frac{J(\widehat{g}_n)}{\|\widehat{g}_n\|} = \frac{1}{2}\|\widehat{g}_n\| \int_0^T \int_\omega |\widetilde{p}_n(t,0,x)|^2 \mathrm{d}x\mathrm{d}t + \varepsilon - \int_0^A \int_\Omega h(a,x)\widetilde{g}_n \mathrm{d}a\mathrm{d}x.$$

Since $\widetilde{g}_n$ converges weakly to zero, it follows that

$$\liminf_{n\to+\infty} \frac{J(\widehat{g}_n)}{\|\widehat{g}_n\|} \geq \varepsilon.$$

We conclude by observing that this is in contradiction with (3.30).

Therefore this contradiction shows that the sequence $(\widehat{g}_n)_n$ is bounded, that is the minimizer $\widehat{g}$ is uniformly bounded with respect to the function Y^0. This means that there exists $R_0 > 0$ such that for all $Y^0 \in L^2\left((0,T)\times\Omega\right)$ we have $\|\widehat{g}\| \leq R_0$. Since $J(\widehat{g}) < 0$, then the Cauchy-Schwarz inequality implies

$$\frac{1}{2}\int_0^A \int_0^T \int_\omega |\widehat{p}(t,0,x)|^2 \mathrm{d}x\mathrm{d}t \leq R_0\|h\|$$

and the proof is complete. □

4. Proof of the main result

We examine only the case $\|h\| > \varepsilon$, since if $\|h\| \leq \varepsilon$ one may take $v = 0$ in order to get (1.5).

Let us denote by $X := L^2\left((0,T)\times\Omega\right)$. Recall that for $Y^0 \in X$ the linear equation (3.3) has a unique solution y and that we have defined Y in (3.4) as being the integral of y over $(0,A)$. We shall denote by $K(Y^0)$ the set defined by

$$K(Y^0) := \left\{Y \text{ as in (3.4)};\ (y,v) \text{ as in (3.3)},\ (1.5),\ v \in L^2((0,T)\times\omega)\right\}. \tag{4.1}$$

The goal is now to prove that the multivalued mapping K has a fixed point, that is that there exists Y such that $Y \in K(Y)$. This will be a consequence of the following version of the Kakutani fixed point theorem, due to Ky Fan and I. Glicksberg (see *e.g.* Aubin [3], and Zeidler's book [11], Sect. 77.8; in particular in Chap. 77 of the latter reference a very nice account of various forms of fixed point theorems are given). So we shall prove:

Theorem 4.1 (Kakutani-Fan-Glicksberg fixed point theorem)**.** *Let X be a reflexive Banach space and $K : X \longrightarrow 2^X$ a multivalued mapping which satisfies the following conditions:*

(1) *For all $Y^0 \in X$ the set $K(Y^0)$ is a nonempty convex closed subset of X.*
(2) *There exists a compact set X_c such that $K(X_c) \subset X_c$.*
(3) *For all $Z \in X$ the mapping*

$$Y^0 \mapsto \sup_{P\in K(Y^0)} \int_0^T \int_\Omega Z(t,x)P(t,x)\mathrm{d}x\mathrm{d}t$$

is upper semicontinuous.

Then the mapping K has at least one fixed point, that is there exists $Y \in X$ such that $Y \in K(Y)$.

We begin by proving the first property of Theorem 4.1.

Lemma 4.2 (Property (1) of Thm. 4.1)**.** *With the above notations, for all $Y^0 \in X$ the set $K(Y^0)$ is a non empty convex closed subset of X.*

Proof. Let $Y^0 \in X$. Using the result of the previous section we infer that $K(Y^0)$ is nonempty, and it is clear that the mapping $Y^0 \mapsto y$ being affine, thanks to its very definition, the set $K(Y^0)$ is convex. In order to prove that $K(Y^0)$ is closed, let $(Y^n)_{n\geq 1}$ be a sequence of $K(Y^0)$ converging strongly to Y. We have to check that $Y \in K(Y^0)$. Clearly, thanks to Proposition 3.5, for all n there exists a pair (y^n, v^n) that solves (3.3) such that

$$(y^n, v^n) \text{ satisfies (1.5)}, \quad Y^n(t,x) = \int_0^A \beta(a) y^n(t,a,x)\mathrm{d}a, \quad \|v^n\| \leq R\|h\|.$$

Multiplying (3.3) by y^n and integrating by parts, we get

$$\|\nabla y^n\|^2 \leq \|v^n\|^2 + \|Y^n\|^2. \tag{4.2}$$

Therefore the sequence $(y^n)_n$ is bounded in $L^2\left((0,T)\times(0,A)\times\Omega\right)$. Consequently, we can extract subsequences also indexed by n such that

$$\begin{aligned} y^n &\rightharpoonup y \ \text{ in } \ L^2\left((0,T)\times(0,A)\times\Omega\right)\\ v^n &\rightharpoonup v \ \text{ in } \ L^2\left((0,T)\times\omega\right)\\ Y^n &\rightharpoonup \int_0^A \beta(a)y(t,a,x)\mathrm{d}a \ \text{ in } \ L^2\left((0,T)\times\Omega\right). \end{aligned}$$

Hence, we conclude that the pair (y, v) solves (3.3) and verifies (3.26) as well as condition (1.5). This means that $Y \in K(Y^0)$, and therefore $K(Y^0)$ is closed. □

The following lemma is straightforward:

Lemma 4.3. *Let M be given by (3.2) and $R > 0$ fixed. There exists a constant $C > 0$ such that if $\widetilde{H} \in L^\infty((0,T)\times\Omega)$ with $\|\widetilde{H}\|_\infty \leq M$, and $v \in L^2((0,T)\times\omega)$ is such that $\|v\|_{L^2} \leq R$, a function y such that*

$$y \in L^2((0,T)\times(0,A); H_0^1(\Omega)), \quad \frac{\partial y}{\partial t} + \frac{\partial y}{\partial a} \in L^2((0,T)\times((0,A)); H^{-1}(\Omega))$$

and satisfying the equation

$$\begin{cases} \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial a} - \Delta y &= 0 \qquad \text{in } (0,T)\times(0,A)\times\Omega\\ y(t,a,\sigma) &= 0 \qquad \text{on } (0,T)\times(0,A)\times\partial\Omega\\ y(0,a,x) &= 0 \qquad \text{in } (0,A)\times\Omega\\ y(t,0,x) &= v(t,x)1_\omega + \widetilde{H}(t,x)\displaystyle\int_0^A \beta(a)y(t,a,x)\mathrm{d}a \quad \text{in } (0,T)\times\Omega, \end{cases} \tag{4.3}$$

then one has

$$Y(t,x) := \int_0^A \beta(a)y(t,a,x)\mathrm{d}a \quad \text{satisfies} \ \ \|Y\|_{L^2((0,T)\times\Omega)} \leq C. \tag{4.4}$$

Next, in order to prove the second property needed in the application of Kakutani-Fan-Glicksberg fixed point theorem, we show the following lemma.

Lemma 4.4 (Property (2) of Thm. 4.1)**.** *There exists a compact set X_c such that $K(X_c) \subset X_c$.*

Proof. Denote by X_c the set of functions $Y := \int_0^A \beta y \mathrm{d}a$, where y is such that there exist $v \in L^2\left((0,T)\times\omega\right)$ verifying (3.26) and $\widetilde{H} \in L^\infty\left((0,T)\times\Omega\right)$ such that $\|\widetilde{H}^n\|_\infty \leq M$ and moreover, the pair (y, v) solves the system (4.3). Note that by construction, for any $Y^0 \in L^2((0,T)\times\Omega)$ we have $K(Y^0) \subset X_c$, since $\|H(Y^0)\|_\infty \leq M$.

Therefore, since by Lemma 4.3 we may infer that X_c is bounded, we need only to prove that X_c is relatively compact.

Let $(y^n)_n$ be a sequence such that $Y^n := \int_0^A \beta y^n da$ is a sequence belonging to X_c: we have to show that $(Y^n)_n$ contains a convergent subsequence. We begin by noting that there exists v^n and $\widetilde{H}^n$ such that if $\|\widetilde{H}^n\|_\infty \leq M$, while v^n verifies (3.26) and y^n satisfies the system:

$$\begin{cases} \dfrac{\partial y^n}{\partial t} + \dfrac{\partial y^n}{\partial a} - \Delta y^n &= 0 \quad \text{in } \ (0,T)\times(0,A)\times\Omega \\ y^n(t,a,\sigma) &= 0 \quad \text{on } \ (0,T)\times(0,A)\times\partial\Omega \\ y^n(0,a,x) &= 0 \quad \text{in } \ (0,A)\times\Omega \\ y^n(t,0,x) &= v^n(t,x)1_\omega + \widetilde{H}^n Y^n \quad \text{in } \ (0,T)\times\Omega. \end{cases} \tag{4.5}$$

Multiplying (4.5) by β, integrating this on $(0,A)$ and using the assumptions on β we obtain that Y^n solves the system:

$$\begin{cases} \dfrac{\partial Y^n}{\partial t} - \Delta Y^n &= \displaystyle\int_0^A y^n(t,a,x)\beta'(a)\mathrm{d}a \quad \text{in } \ (0,T)\times\Omega \\ Y^n(t,\sigma) &= 0 \quad \text{on } \ (0,T)\times\partial\Omega \\ Y^n(0,x) &= 0 \quad \text{in } \ \Omega. \end{cases} \tag{4.6}$$

The boundedness of $(v^n)_n$ and $(Y^n)_n$ implies that $(\int_0^A y^n\beta'(a)\mathrm{d}a)_n$ is bounded in $L^2((0,T)\times\Omega)$ and that $(\nabla Y^n)_n$ is bounded in $(L^2((0,T)\times\Omega))^N$. Consequently, $(\partial Y^n/\partial t)_n$ is bounded in $L^2((0,T);H^{-1}(\Omega))$ and $(Y^n)_n$ is bounded in $L^2((0,T);H_0^1(\Omega))$. Therefore, using a classical result due to Aubin and Lions, we infer that $(Y^n)_n$ is relatively compact in $L^2((0,T)\times\Omega)$. Finally we can conclude that there exist subsequences still indexed by n such that:

$$\begin{gathered} y^n \rightharpoonup y \quad \text{in } \ L^2\left((0,T)\times(0,A);H_0^1(\Omega)\right) \\ Y^n \to Y = \int_0^A \beta y \mathrm{d}a \quad \text{in } \ X \ \text{ and a.e. on } \ (0,T)\times\Omega \\ v^n \rightharpoonup v \quad \text{in } \ L^2\left((0,T)\times\omega\right) \\ \widetilde{H}^n \to \widetilde{H} \quad \text{in } L^\infty\left((0,T)\times\Omega\right) \text{weak} *. \end{gathered}$$

Therefore we infer that

$$\widetilde{H}^n Y^n \rightharpoonup \widetilde{H} Y \quad \text{in } \ L^1\left((0,T)\times\Omega\right).$$

Consequently, we may observe that the pair (y,v) solves (4.3) and (3.26), and this shows that X_c is relatively compact in X, thus Lemma 4.4 is proved. (Actually this shows that the set $K(Y^0)$ is compact for all $Y^0 \in X$.) $\square$

It remains now to prove the following lemma which states the third property needed in the application of Kakutani-Fan-Glicksberg fixed point theorem.

Lemma 4.5 (Property (3) of Thm. 4.1)**.** *Let $Z \in X$ be fixed and denote by $Y \mapsto \widehat{K}(Y)$ the mapping defined by*

$$\widehat{K}(Y) := \sup_{P\in K(Y)} \int_0^T \int_\Omega Z(t,x)P(t,x)\mathrm{d}x\mathrm{d}t.$$

Then $\widehat{K}$ is upper semicontinuous.

Proof. Consider a sequence (Y^n) that converges strongly to Y in X. We must prove that

$$\limsup_{n\to+\infty} \widehat{K}(Y^n) \leq \widehat{K}(Y).$$

Since each set $K(Y^n)$ is compact, we infer that there exists $P^n \in K(Y^n)$ such that

$$\widehat{K}(Y^n) = \int_0^A \int_\Omega Z(t,x) P^n(t,x) \mathrm{d}x \mathrm{d}t.$$

Note that, at the cost of extracting a subsequence, without loss of generality (thanks to a result which is the partial converse of Lebesgue's dominated convergence theorem) we may assume that there exists $Y_* \in L^2((0,T)\times\Omega)$ such that

$$Y^n(t,x) \to Y(t,x) \quad \text{and} \quad |Y^n(t,x)| \leq Y_*(t,x) \quad \text{a.e. on} \quad (0,T)\times\Omega.$$

Since H is continuous and bounded, this implies in particular that $H(Y^n) \to H(Y)$ almost everywhere and in $L^r((0,T)\times\Omega)$ for all $r < \infty$. Now, as $P^n \in K(Y^n)$, we know that there exists a pair (z^n, v^n) that solves:

$$\left\{ \begin{array}{rcl} \dfrac{\partial z^n}{\partial t} + \dfrac{\partial z^n}{\partial a} - \Delta z^n & = & 0 \quad \text{in} \quad (0,T)\times(0,A)\times\Omega \\ z^n(t,a,\sigma) & = & 0 \quad \text{on} \quad (0,T)\times(0,A)\times\partial\Omega \\ z^n(0,a,x) & = & 0 \quad \text{in} \quad (0,A)\times\Omega \\ z^n(t,0,x) & = & v^n(t,x)1_\omega + H(Y^n)\displaystyle\int_0^A \beta z^n \mathrm{d}a \quad \text{in} \quad (0,T)\times\Omega, \end{array} \right. \tag{4.7}$$

and the pair (z^n, v^n) verifies also (3.26) and (1.5), while $P^n = \int_0^A \beta(a) z^n \mathrm{d}a$. Therefore, since X_c is relatively compact and $P^n \in X_c$, up to extraction of subsequences, we have that:

$$\begin{gathered} z^n \rightharpoonup z \quad \text{in} \quad L^2\left((0,T)\times(0,A); H_0^1(\Omega)\right) \\ v^n \rightharpoonup v \quad \text{in} \quad L^2\left((0,T)\times\omega\right) \\ Y^n \rightharpoonup Y \quad \text{a.e. and in} \quad L^2\left((0,T)\times\Omega\right) \\ H(Y^n) \to H(Y) \quad \text{a.e. and in} \quad L^r((0,T)\times\Omega) \\ P^n \to P \quad \text{a.e. and in} \quad X \\ H(Y^n)P^n \to H(Y)P \quad \text{a.e. and in} \quad L^2\left((0,T)\times\Omega\right). \end{gathered}$$

Therefore we may infer that the pair (z, v) also verifies equation (4.7) with $H(Y)$ instead of $H(Y^n)$ and verifies also (3.26) and (1.5). Consequently we have $P = \int_0^A \beta z \mathrm{d}a \in K(Y)$ and

$$\begin{aligned} \limsup_{n\to+\infty} \widehat{K}(Y^n) &= \lim_{n\to+\infty} \int_0^A \int_\Omega Z(t,x) P^n(t,x) \mathrm{d}x \mathrm{d}t \\ &= \int_0^A \int_\Omega Z(t,x) P(t,x) \mathrm{d}x \mathrm{d}t. \end{aligned}$$

Finally, since $P \in K(Y)$, one deduces that

$$\limsup_{n\to+\infty} \widehat{K}(Y^n) \leq \widehat{K}(Y),$$

and this ends the proof of the lemma. $\square$

5. Conclusion

In Lemmas 4.2, 4.4 and 4.5 we have shown that the mapping K defined by (4.1) satisfies the conditions required by Kakutani-Fan-Glicksberg Theorem 4.1: it follows that K has at least one fixed point, that is there exists

$$(y, v) \in L^2((0,T) \times (0,A); H_0^1(\Omega)) \times L^2((0,T) \times \omega)$$

satisfying (3.3) and (1.5) such that $Y := \int_0^A \beta(a) y(\cdot, a, \cdot) \mathrm{d}a \in K(Y)$. This means precisely that there exists a pair (y, v) such that y solves the system (3.3), with $y(t, 0, x) = v 1_\omega + F(t, x, y)$ and verifies also the condition (1.5), that is $\|y(T, ., .) - h_1\| \leq \varepsilon$. The proof of our main result is over. □

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