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FEEDBACK STABILIZATION OF A BOUNDARY LAYER EQUATION PART 1: HOMOGENEOUS STATE EQUATIONS*

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Abstract. We are interested in the feedback stabilization of a fluid flow over a flat plate, around a stationary solution, in the presence of perturbations. More precisely, we want to stabilize the laminar-to-turbulent transition location of a fluid flow over a flat plate. For that we study the Algebraic Riccati Equation (A.R.E.) of a control problem in which the state equation is a doubly degenerate linear parabolic equation. Because of the degenerate character of the state equation, the classical existence results in the literature of solutions to algebraic Riccati equations do not apply to this class of problems. Here taking advantage of the fact that the semigroup of the state equation is exponentially stable and that the observation operator is a Hilbert-Schmidt operator, we are able to prove the existence and uniqueness of solution to the A.R.E. satisfied by the kernel of the operator which associates the 'optimal adjoint state' with the 'optimal state'. In part 2 [Buchot and Raymond, *Appl. Math. Res. eXpress* (2010) doi:10.1093/amrx/abp007], we study problems in which the feedback law is determined by the solution to the A.R.E. and another nonhomogeneous term satisfying an evolution equation involving nonhomogeneous perturbations of the state equation, and a nonhomogeneous term in the cost functional.

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1. Introduction

We are interested in the feedback stabilization of a fluid flow over a flat plate, around a stationary solution, in the presence of perturbations. The control variable is a suction velocity through a small slot near the leading edge of the plate.

Keywords and phrases. Feedback control law, Crocco equation, degenerate parabolic equations, Riccati equation, boundary layer equations, unbounded control operator.

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In the stationary case, the fluid flow in the boundary layer may be described by the Prandtl equations, or similarly by the Crocco equations [14]:

$$\begin{cases} U_\infty^s \,\eta\, \dfrac{\partial w}{\partial \xi} - \nu w^2 \dfrac{\partial^2 w}{\partial \eta^2} = 0 & \text{in} \quad (0,L)\times(0,1), \\ \nu\left(w\dfrac{\partial w}{\partial \eta}\right)(\xi,0) = v_s\, w(\xi,0), \qquad \lim\limits_{\eta\to 1} w(\xi,\eta) = 0 & \text{for} \quad \xi\in(0,L), \\ w(0,\eta) = w_b(\eta) & \text{for} \quad \eta\in(0,1). \end{cases} \tag{1.1}$$

Here $(0,L)$ represents a part of the plate where the flow is laminar, $(0,1)$ is the *thickness* of the boundary layer in the Crocco variables, $(U_\infty^s, 0)$ is the velocity of the incident flow, w_b is the *velocity* profile in Crocco variables at $\xi = 0$, v_s is a suction velocity throughout the plate, the positive constant ν is the viscosity of the fluid. We set $\Omega = (0,L)\times(0,1)$. The transformation used to rewrite the Prandtl equations into the Crocco equation is

$$\xi = x, \qquad \eta = \frac{u_s(x,y)}{U_\infty^s}, \qquad w(\xi,\eta) = \frac{1}{U_\infty^s}\frac{\partial u_s}{\partial y}(x,y), \tag{1.2}$$

see [14], when (u_s, v_s) is the stationary solution of the Prandtl system, and $(x,y)\in(0,L)\times(0,\infty)$. Assuming that the regularity and compatibility conditions between w_b and v_s stated in [14], Theorem 3.3.2, are satisfied, the stationary equation (1.1) admits a unique solution w_s in the class of functions w satisfying

$$\begin{aligned} & w\in C_b(\Omega), \quad K_1|1-\eta| \le w(\xi,\eta) \le K_2|1-\eta|, \quad \left|\frac{\partial w}{\partial \xi}\right| \le K_3|1-\eta|, \\ & \frac{\partial w}{\partial \eta}\in L^\infty(\Omega), \qquad w\frac{\partial^2 w}{\partial \eta^2}\in L^\infty(\Omega), \qquad \frac{\partial w}{\partial \xi}\in L^\infty(\Omega), \end{aligned} \tag{1.3}$$

where K_1, K_2, and K_3 are positive constants. This class of solution will be called the class of '*asymptotic type solutions*' because they may correspond to an asymptotic profile of some solutions to the Prandtl equations when x tends to infinity (see [7], Sect. 6, where we give an explicit example of such solutions). Another class of solutions important for applications is the class of '*Blasius type solutions*' (the term comes from the fact that some solutions in that class can be obtained by solving the so-called Blasius differential equation) (see [7], Sect. 6, [14], p. 129).

We are interested in stabilizing a flow over a flat plate when the longitudinal incident velocity is of the form:

$$U_\infty(t) = U_\infty^s + u_\infty(t). \tag{1.4}$$

Using the Crocco transformation (see (1.2) and [14]) when the velocity of the external flow U_∞ is positive and only depends on t, the Prandtl system – describing the velocity field in the boundary layer over the flat plate – is transformed into a degenerate parabolic equation stated over $\Omega = (0,L)\times(0,1)$, called the Crocco equation [3,4], System 4.7, p. 85, [14], p. 174, written down below:

$$\begin{cases} \dfrac{\partial w}{\partial t} + U_\infty\,\eta\,\dfrac{\partial w}{\partial \xi} + \dfrac{U_\infty'}{U_\infty}(1-\eta)\dfrac{\partial w}{\partial \eta} \\ \qquad\qquad -\nu w^2\dfrac{\partial^2 w}{\partial \eta^2} + \dfrac{U_\infty'}{U_\infty} w = 0 & \text{in } \Omega\times(0,T), \\ w(\xi,\eta,0) = w_0(\xi,\eta) & \text{in } \Omega, \\ \left(\nu\, w\dfrac{\partial w}{\partial \eta}\right)(\xi,0,t) = (v_s + \mathbb{1}_\gamma u)\, w(\xi,0,t) - \dfrac{U_\infty'}{U_\infty}(t) & \text{for } (\xi,t)\in(0,L)\times(0,T), \\ \lim\limits_{\eta\to 1} w(\xi,\eta,t) = 0 & \text{for } (\xi,t)\in(0,L)\times(0,T), \\ w(0,\eta,t) = w_1(\eta,t) & \text{for } (\eta,t)\in(0,1)\times(0,T), \end{cases} \tag{1.5}$$

where $\mathbb{1}_\gamma$ is the characteristic function of the slot $\gamma = (x_0, x_1) \subset (0, L)$, u is a control variable and v_s is the function appearing in equation (1.1).

Due to the lack of existence result for the instationary Prandtl system when $U_\infty(t)$ is of the form (1.4) (or to the corresponding instationary Crocco equation – see [14] for some results corresponding to particular profiles, and the more recent results in [20]), we have chosen to describe the velocity field in the boundary layer by solving the Crocco equation linearized about the stationary solution w_s. Since the perturbation $u_\infty(t)$ and the control function u are supposed to be small with respect to U_∞^s, the linearized model is an accurate approximation of the nonlinear one. This assertion, which is not proved, is actually confirmed by numerical experiments [4,7]. The Crocco equation (1.5) linearized about w_s with a boundary control u is the degenerate parabolic equation:

$$\begin{cases} \dfrac{\partial z}{\partial t} = Az + f & (t,\xi,\eta) \in \ (0,\infty)\times\Omega, \\ z(0,\xi,\eta) = z_0(\xi,\eta) & (\xi,\eta) \in \ \Omega, \\ \sqrt{a}\, z(t,0,\eta) = \sqrt{a}\, z_b(t,\eta) & (t,\eta) \in (0,\infty)\times(0,1), \\ (bz)(t,\xi,1) = 0, \quad \dfrac{\partial z}{\partial \eta}(t,\xi,0) = (\mathbb{1}_\gamma u + g)(t,\xi) & (t,\xi) \in (0,\infty)\times(0,L), \end{cases} \tag{1.6}$$

where

$$\begin{aligned} &Az = -a(\eta)\frac{\partial z}{\partial \xi} + b(\xi,\eta)\frac{\partial^2 z}{\partial \eta^2} - c(\xi,\eta)z, \\ &f(t,\xi,\eta) = u_\infty(t)d(\xi,\eta) + \frac{u_\infty'(t)}{U_\infty^s}e(\xi,\eta), \qquad g(t,\xi) = -\frac{u_\infty'(t)}{\nu w_s(\xi,0)U_\infty^s}\cdot \end{aligned} \tag{1.7}$$

The coefficients a, b, c, d, e depend on the stationary solution w_s of the Crocco equation, and are defined by:

$$\begin{aligned} &a = U_\infty^s \eta, \quad b = \nu(w_s)^2, \quad c = -2w_s\frac{\partial^2 w_s}{\partial \eta^2}, \\ &d = -\eta\,\frac{\partial w_s}{\partial \xi}, \qquad e = -w_s - (1-\eta)\frac{\partial w_s}{\partial \eta}\cdot \end{aligned}$$

Assumptions on the coefficients a, b, c, d and e are not the same ones if w_s belongs to the class of *Blasius type solutions* or if it belongs to the class of *asymptotic type solutions*.

In this paper we only consider the class of *asymptotic type solutions* because we have studied equation (1.6) in [6] when w_s belongs to this class.

In the case of *Blasius type solutions* the so-called laminar-to-turbulent transition location – which is an important criterion in applications – is a nonlinear mapping depending on the state variable w and on U_∞. Its linearization about (w_s, U_∞^s) – called the linearized transition location – is of the form $\int_\Omega \psi(\xi,\eta)\, z(t,\xi,\eta)\, \mathrm{d}\xi\mathrm{d}\eta + c_0 u_\infty(t)$, where the function ψ belongs to $L^2(\Omega)$ and c_0 belongs to $\mathbb{R}$ (they can be determined numerically in a precise manner see [7], Sect. 6, Test 3).

Here, we consider observation operators of the more general form

$$Cz(t,\cdot) + y_d(t,\cdot) = \int_\Omega \phi(\cdot,\xi,\eta)\, z(t,\xi,\eta)\, \mathrm{d}\xi\mathrm{d}\eta + y_d(t,\cdot) \in L^2(\Omega), \tag{1.8}$$

where $\phi \in L^2(\Omega\times\Omega)$ and $y_\mathrm{d} \in L^2(0,\infty; L^2(\Omega))$ are given. Thus C is a Hilbert-Schmidt operator in $L^2(\Omega)$. (For the linearized laminar-to-turbulent transition location the function $\phi(x,y,\xi,\eta) = \psi(\xi,\eta)$ only depends on (ξ,η) and $y_d(t,\cdot) = c_0 u_\infty(t)$ only depends on t.) It is obvious that the identity in $L^2(\Omega)$ is not a Hilbert-Schmidt operator, however the identity operator from $L^2(\Omega)$ into $L^2(\Omega)$ equipped with a norm weaker than the usual one can also be written in the above form (see Prop. 2.1).

Our main objective is to determine a control u, in feedback form, in order that the observation $Cz(t)+y_d(t)$ decays to zero when t tends to infinity. For that we use the optimal control theory, and we consider the linear-quadratic control problem

$$(\mathcal{P}_{f,g,z_b,y_d,z_0}) \qquad \inf\Big\{J(z,u) \mid (z,u)\in L^2(0,\infty;Z)\times L^2(0,\infty;U),\ (z,u) \text{ satisfies } (1.6)\Big\},$$

where $Z=L^2(\Omega)$, $U=L^2(0,L)$, and

$$J(z,u)=\frac{1}{2}\int_0^\infty \|Cz(t)+y_d(t)\|_Z^2\,\mathrm{d}t+\frac{1}{2}\int_0^\infty \|u(t)\|_U^2\,\mathrm{d}t,$$

where $C\in\mathcal{L}(Z)$ is the Hilbert-Schmidt operator of kernel ϕ defined above.

First of all we would like to explain in which aspects problem $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ is a classical matter of the optimal control theory, and what are the questions that the existing results in the literature cannot answer.

In Section 2 we give a precise definition of solution to equation (1.6), and we prove that it can be rewritten in the form

$$z'=\mathcal{A}z+B(\mathbb{1}_\gamma u)+F,\qquad z(0)=z_0. \tag{1.9}$$

Moreover, the solution z to equation (1.9) belongs to $C_b([0,\infty);Z)\cap L^2(0,\infty;Z)$, the mapping $u\mapsto z$ is continuous from $L^2(0,\infty;U)$ into $C_b([0,\infty);Z)\cap L^2(0,\infty;Z)$, and the semigroup $(\mathrm{e}^{t\mathcal{A}})_{t\ge 0}$ is exponentially stable on Z. Thus it seems that we are in a very favorable position to characterize the optimal solution of $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ by means of a feedback law, and our control problem seems to enter into a classical setting.

Even if the analysis of the nonlinear model with the feedback law is not performed, let us explain why the results obtained for the LQ control problem $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ are quite new and interesting.

In Section 3, we are able to prove that $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ admits a unique solution (z,u), and that this solution is characterized by an optimality system of the form

$$\begin{cases} z'=\mathcal{A}z+B(\mathbb{1}_\gamma u)+F, & z(0)=z_0,\\ -p'=\mathcal{A}^*p+C^*(Cz+y_d), & p(\infty)=0,\\ u=-\mathbb{1}_\gamma B^*p. \end{cases} \tag{1.10}$$

We want to prove that there exists an operator $\Pi\in\mathcal{L}(Z)$ satisfying $\Pi=\Pi^*\ge 0$, and a function $r\in L^2(0,\infty;Z)$ such that

$$p(t)=\Pi z(t)+r(t).$$

The main objective of the present paper is to obtain an algebraic Riccati equation characterizing Π. The equation satisfied by r, which involves the nonhomogeneous terms f, g, z_b, and y_d is studied in Part 2 [7]. To find an equation satisfied by Π, we study problem $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ in the case when $f=0$, $g=0$, $z_b=0$ and $y_d=0$. Denoting this problem by $(\mathcal{P}_{z_0})$, we can easily show that

$$\inf(\mathcal{P}_{z_0})=\frac{1}{2}\big(\Pi z_0,z_0\big)_{L^2(\Omega)}.$$

Since $\mathcal{A}$ is a degenerate parabolic operator, we explain at the beginning of Section 5 why the existing results in the literature are not sufficient to obtain a Riccati equation characterizing Π in the domain of $\mathcal{A}$. To overcome this difficulty we look for Π in the form of a Hilbert-Schmidt operator in $L^2(\Omega)$, and we characterize the equation satisfied by its kernel π. The existence of a weak solution to the algebraic Riccati equation satisfied by π is studied in Section 5. In Section 6 we show that

$$\inf(P_{z_0})=\frac{1}{2}\int_{\Omega\times\Omega}\pi\, z_0\otimes z_0,$$

for all solution π to the algebraic Riccati equation. ($z_0 \otimes z_0$ denotes the function defined in $\Omega\times\Omega$ by $(x, y, \xi, \eta) \mapsto z_0(x, y)z_0(\xi, \eta)$.) Thus π is unique and it is the kernel of Π. The analysis in the nonhomogeneous case, that is when f, z_b, g and y_d are not necessarily zero, is performed in Part 2 [7]. Numerical results are also given in [7], showing the efficiency of the linear feedback law applied to the nonlinear Crocco equation in the presence of perturbations.

2. Assumptions and preliminary results

As in [6], we make the following assumptions on the coefficients a, b, and c.

(H_1) $a(\eta) = U^s_\infty\, \eta$ for $\eta \in [0, 1]$, and $b \in W^{1,\infty}(\Omega)$. There exist positive constants C_i, $i = 1$ to 4, such that

$$\begin{aligned} & C_1|1-\eta|^2 \le b(\xi,\eta) \le C_2|1-\eta|^2, \\ & \left|\frac{\partial b}{\partial \eta}(\xi,\eta)\right| \le C_3|1-\eta| \qquad \text{and} \qquad \left|\frac{\partial b}{\partial \xi}(\xi,\eta)\right| \le C_4|1-\eta|^2 \qquad \text{for all } (\xi,\eta)\in\Omega. \end{aligned} \tag{2.1}$$

(H_2) The function c belongs to $L^\infty(\Omega)$, and we denote by C_0 a positive constant such that

$$\|c\|_{L^\infty(\Omega)} \le C_0. \tag{2.2}$$

The nonhomogeneous terms f, g, z_b and the initial condition z_0 and the function ϕ satisfy

(H_3) $z_0 \in L^2(\Omega)$, $z_b \in L^2(0,\infty; L^2(0, 1))$ and $g \in L^2(0,\infty; L^2(0, L))$.

(H_4) $f \in L^2(0,\infty; L^2(\Omega))$, $\phi \in L^2(\Omega\times\Omega)$ and $y_{\rm d} \in L^2(0,\infty; L^2(\Omega))$.

Let us recall some notation introduced in [5,6]. Let $H^1(0, 1; d)$ be the closure of $C^\infty([0, 1])$ in the norm:

$$\|z\|_{H^1(0,1;d)} = \left(\int_0^1 |z|^2 + |1-\eta|^2\left|\frac{\partial z}{\partial\eta}\right|^2\,\mathrm{d}\eta\right)^{1/2}. \tag{2.3}$$

To take the Dirichlet boundary condition $bz(\xi, 1, t) = 0$ into account, we denote by $H^1_{\{1\}}(0, 1; d)$ the closure of $C^\infty_c([0, 1))$ in the norm $\|\cdot\|_{H^1(0,1;d)}$. According to Triebel [16], Theorem 2.9.2,

$$H^1(0, 1; d) = H^1_{\{1\}}(0, 1; d).$$

Let us set

$$\Gamma_0 = \big([0,L)\times\{0\}\big)\cup\big(\{0\}\times(0,1)\big), \qquad \Gamma_1 = \big(\{L\}\times(0,1)\big)\cup\big((0,L]\times\{1\}\big).$$

If the vectorfield $\left(az, -b\frac{\partial z}{\partial\eta}\right)$ belongs to $(L^2(\Omega))^2$, and its divergence belongs to $L^2(\Omega)$, the normal trace on the boundary Γ of the vectorfield $\left(az, -b\frac{\partial z}{\partial\eta}\right)$ belongs to $H^{-1/2}(\Gamma)$. We denote this normal trace by $T\left(az, -b\frac{\partial z}{\partial\eta}\right)$. Let us recall the definitions of some trace spaces (see [13] or [8], Chap. 7, Sect. 2, Rem. 1)

$$\begin{aligned} H^{1/2}_{00}(\Gamma_0) &= \Big\{\varphi\in L^2(\Gamma_0) \mid \exists\psi\in H^1(\Omega),\ \psi = 0 \text{ on } \Gamma_1 \text{ and } \psi=\varphi \text{ on } \Gamma_0\Big\}, \\ H^{1/2}_{00}(\Gamma_1) &= \Big\{\varphi\in L^2(\Gamma_1) \mid \exists\psi\in H^1(\Omega),\ \psi = 0 \text{ on } \Gamma_0 \text{ and } \psi=\varphi \text{ on } \Gamma_1\Big\}. \end{aligned}$$

We can define $T_0\left(az, -b\frac{\partial z}{\partial\eta}\right)$ as an element in $(H^{1/2}_{00}(\Gamma_0))'$ in the following way

$$\left\langle T_0\Big(az, -b\frac{\partial z}{\partial\eta}\Big), \varphi\right\rangle_{(H^{1/2}_{00}(\Gamma_0))', H^{1/2}_{00}(\Gamma_0)} = \left\langle T\Big(az, -b\frac{\partial z}{\partial\eta}\Big), \gamma_0\psi\right\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$

for all $\varphi \in H^{1/2}_{00}(\Gamma_0)$, where $\gamma_0 \in \mathcal{L}(H^1(\Omega), H^{1/2}(\Gamma))$ is the trace operator and $\psi \in H^1(\Omega)$ is a function such that $\psi = 0$ on Γ_1 and $\psi = \varphi$ on Γ_0.

Similarly, if the vectorfield $\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big)$ belongs to $(L^2(\Omega))^2$, and its divergence belongs to $L^2(\Omega)$, the normal trace on the boundary Γ of the vectorfield $\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big)$, denoted by $T\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big)$, belongs to $H^{-1/2}(\Gamma)$, and we can define $T_1\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big)$ by

$$\Big\langle T_1\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big), \varphi\Big\rangle_{(H^{1/2}_{00}(\Gamma_1))', H^{1/2}_{00}(\Gamma_1)} = \Big\langle T\Big(-az, -\frac{\partial}{\partial\eta}(bz)\Big), \gamma_0\psi\Big\rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$

for all $\varphi \in H^{1/2}_{00}(\Gamma_1)$, where $\psi \in H^1(\Omega)$ is a function such that $\psi = 0$ on Γ_0 and $\psi = \varphi$ on Γ_1.

The differential operators A and A^* are defined by

$$Az = -a\frac{\partial z}{\partial\xi} + b\frac{\partial^2 z}{\partial\eta^2} - cz, \qquad A^*p = a\frac{\partial p}{\partial\xi} + \frac{\partial^2(bp)}{\partial\eta^2} - cp.$$

The unbounded operators in $L^2(\Omega)$ associated with the above differential operators are given by:

$$\begin{aligned}
&D(\mathcal{A}) = \Big\{z \in L^2(0, L; H^1(0, 1; d)) \mid Az \in L^2(\Omega),\ T_0\Big(az, -b\frac{\partial z}{\partial\eta}\Big) = 0\Big\},\\
&\mathcal{A}z = Az \quad \text{for all } z \in D(\mathcal{A}),\\
&D(\mathcal{A}^*) = \Big\{p \in L^2(0, L; H^1(0, 1; d)) \mid A^*p \in L^2(\Omega),\ T_1\Big(-ap, -\frac{\partial}{\partial\eta}(bp)\Big) = 0\Big\},\\
&\mathcal{A}^*p = A^*p \quad \text{for all } p \in D(\mathcal{A}^*).
\end{aligned}$$

According to [6], Theorem 5.9, $(\mathcal{A}^*, D(\mathcal{A}^*))$ is the adjoint of $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous semigroup on $L^2(\Omega)$. As in [6], we also need to define the operators $(\mathcal{A}_k, D(\mathcal{A}_k))$ and $(\mathcal{A}^*_k, D(\mathcal{A}^*_k))$ by setting $D(\mathcal{A}_k) = D(\mathcal{A})$, $D(\mathcal{A}^*_k) = D(\mathcal{A}^*)$,

$$\mathcal{A}_k\zeta = \mathcal{A}\zeta - k\,a\,\zeta, \quad \text{for all } \zeta \in D(\mathcal{A}), \text{ and } \quad \mathcal{A}^*_k\zeta = \mathcal{A}^*\zeta - k\,a\,\zeta, \text{ for all } \zeta \in D(\mathcal{A}^*).$$

The interest of introducing the operator $(\mathcal{A}_k, D(\mathcal{A}_k))$ is explained right now. We can easily verify that a function $z \in L^2(0, T; L^2(\Omega))$ is a weak solution to

$$z' = \mathcal{A}z \quad \text{in } (0, T), \quad z(0) = z_0,$$

if and only if the function $\zeta = \mathrm{e}^{-k\xi}z$ is a weak solution to

$$\zeta' = \mathcal{A}_k\zeta \quad \text{in } (0, T), \quad \zeta(0) = \mathrm{e}^{-k\xi}z_0. \tag{2.4}$$

We are able to prove estimates for ζ that can be translated in estimates for z. Actually, we have proved in [6], Theorem 6.2, that, for all $z_0 \in L^2(\Omega)$, the weak solution $\zeta \in L^2(0, T; L^2(\Omega))$ to equation (2.4) obeys the following inequality

$$\begin{aligned}
&\frac{1}{2}\int_0^1\int_0^\xi |\zeta(x, \eta, t)|^2\,\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_0^1 a\,|\zeta(\xi, \eta, \tau)|^2\,\mathrm{d}\eta\mathrm{d}\tau\\
&\quad + \int_0^t\int_0^1\int_0^\xi \left(b\left|\frac{\partial\zeta}{\partial\eta}\right|^2 + \frac{\partial b}{\partial\eta}\frac{\partial\zeta}{\partial\eta}\zeta + (c + ka)|\zeta|^2\right)\mathrm{d}x\,\mathrm{d}\eta\,\mathrm{d}\tau \le \frac{1}{2}\int_0^1\int_0^\xi \mathrm{e}^{-2kx}\,|z_0(x, \eta)|^2\,\mathrm{d}x\,\mathrm{d}\eta,
\end{aligned} \tag{2.5}$$

for all $t \in (0,T)$ and all $\xi \in [0,L]$. Formally estimate (2.5) could be obtained by multiplying equation (2.4) by ζ and by making integrations in space and time. In that case we obtain an equality in (2.5) in place of an inequality. Due to the degenerate character of the operator $\mathcal{A}_k$ only an inequality has been proved in [6]. If we choose $k > 0$ big enough, due to Lemma 2.1 below, inequality (2.5) can provide estimates for ζ that can be translated in estimates for z. The existence of k, for which we can establish a coercivity condition, is established in [6], Lemma 3.1. Due to the crucial role of this coercivity condition, we state and we give a complete proof of this lemma below.

Lemma 2.1. *There exists $k > 0$ such that*

$$\int_0^1 \left(b(\xi,\cdot)\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2 + \frac{\partial b}{\partial \eta}(\xi,\cdot)\frac{\mathrm{d}z}{\mathrm{d}\eta} z + (-C_0 + ka)\, z^2 \right)\mathrm{d}\eta \geq \frac{C_1}{2}\|z\|^2_{H^1(0,1;d)} + \|z\|^2_{L^2(0,1)}, \tag{2.6}$$

for all $\xi \in [0,L]$, all z in $H^1(0,1;d)$.

Proof. **Step 1.** With the first inequality in (2.1) we can easily verify that

$$\alpha_1\|z\|^2_{H^1(0,1;d)} \leq \int_0^1 \left(|z|^2 + |b(\xi,\cdot)|\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2 \right)\mathrm{d}\eta \leq \alpha_2\|z\|^2_{H^1(0,1;d)}, \tag{2.7}$$

for all $\xi \in [0,L]$, and all $z \in H^1(0,1;d)$, with $\alpha_1 = \min(C_1, 1)$ and some $\alpha_2 > \alpha_1$.

Step 2. We set

$$\beta_k(\xi; z, z) = \int_0^1 \left(b(\xi,\cdot)\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2 + \frac{\partial b(\xi,\cdot)}{\partial \eta}\frac{\mathrm{d}z}{\mathrm{d}\eta} z + (-c + ka)|z|^2 \right)\mathrm{d}\eta.$$

Using (2.7) and inequality (2.1), we have

$$\begin{aligned}
\beta_k(\xi; z, z) &\geq \int_0^1 \left(b\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2 + \frac{\partial b}{\partial \eta}\frac{\mathrm{d}z}{\mathrm{d}\eta} z + (-C_0 + ka)|z|^2 \right)\mathrm{d}\eta \\
&\geq \int_0^1 \left(\frac{C_1}{2}|1-\eta|^2\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2 + \frac{\partial b}{\partial \eta}\frac{\mathrm{d}z}{\mathrm{d}\eta} z + \left(-C_0 + ka - \frac{1}{2}\right)|z|^2 \right)\mathrm{d}\eta + \frac{\alpha_1}{2}\|z\|^2_{H^1(0,1;d)}.
\end{aligned}$$

From inequality (2.1), and Young's inequality, it yields

$$\int_0^1 \frac{\partial b}{\partial \eta}\frac{\mathrm{d}z}{\mathrm{d}\eta} z\,\mathrm{d}\eta \geq -\frac{C_3\varepsilon}{2}\int_0^1 |1-\eta|^2\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2\mathrm{d}\eta - \frac{C_3}{2\varepsilon}\int_0^1 |z|^2\,\mathrm{d}\eta,$$

for all $\varepsilon > 0$. Consequently, $\beta_k(\xi;\cdot,\cdot)$ satisfies the estimate

$$\beta_k(\xi; z, z) \geq \frac{\alpha_1}{2}\|z\|^2_{H^1(0,1;d)} + \left(\frac{C_1}{2} - \frac{C_3\varepsilon}{2}\right)\int_0^1 |1-\eta|^2\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2\mathrm{d}\eta + \int_0^1 \left(-C_0 + ka - \frac{1}{2}(1 + \frac{C_3}{\varepsilon})\right)|z|^2\,\mathrm{d}\eta.$$

Now, we choose ε such that $\frac{C_1}{4} = \frac{C_1}{2} - \frac{C_3\varepsilon}{2} > 0$. We have

$$\beta_k(\xi; z, z) \geq \frac{\alpha_1}{2}\|z\|^2_{H^1(0,1;d)} + \frac{C_1}{4}\int_0^1 |1-\eta|^2\left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2\mathrm{d}\eta + \int_0^1 \left(-C_0 + ka - \frac{1}{2}(1 + \frac{C_3}{\varepsilon})\right)|z|^2\,\mathrm{d}\eta.$$

To establish the lemma, it is enough to prove that, there exists $k > 0$ such that

$$\widetilde{C}_1 \int_0^1 |1-\eta|^2 \left|\frac{\mathrm{d}z}{\mathrm{d}\eta}\right|^2 \mathrm{d}y + \int_0^1 \tilde{k}a\,|z|^2\,\mathrm{d}\eta \geq \|z\|^2_{L^2(0,1)},$$

with $\widetilde{C}_1 = C_1/(4\tilde{r}_0)$, $\tilde{k} = k/\tilde{r}_0$, $\tilde{c} = c/\tilde{r}_0$ and $\tilde{r}_0 = \frac{1}{2}\left(\frac{C_3}{\varepsilon}+1\right) + C_0 + 1$.

This can be shown by arguing by contradiction. We suppose that exists a sequence $(z_n)_n \subset H^1(0,1;d)$ that satisfies

$$\int_0^1 |z_n|^2\,\mathrm{d}y = 1 \quad \text{and} \quad \widetilde{C}_1 \int_0^1 |1-\eta|^2 \left|\frac{\mathrm{d}z_n}{\mathrm{d}\eta}\right|^2 \mathrm{d}\eta + n\int_0^1 a\,|z_n|^2\,\mathrm{d}\eta < 1. \tag{2.8}$$

Due to the second condition in (2.8), the sequence $(z_n)_n$ (or at least a subsequence) tends to 0 almost everywhere in $[0,1]$ and strongly in $L^2(\epsilon,1)$ for all $\epsilon > 0$. Since the imbedding from $H^1(0,1)$ in $L^2(0,1)$ is compact and since $((1-\eta)z_n)_n$ is bounded in $H^1(0,1)$, the sequence $\left((1-\eta)z_n\right)_n$ tends to 0 in $L^2(0,1)$. We know that the sequence $(z_n)_n$ converges to 0 in $L^2(1/2,1)$, and that the sequence $\left((1-\eta)z_n\right)_n$ converges to 0 in $L^2(0,1/2)$. Thus, the sequence $(z_n)_n$ converges to 0 in $L^2(0,1)$, which is in contradiction with the first condition in (2.8). □

Thanks to this lemma we can prove the following theorem.

Theorem 2.1. *The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous semigroup exponentially stable on $L^2(\Omega)$.*

Proof. The complete proof of this result is given in [6], Proof of Theorem 6.1. We only explain how the exponential stability of the semigroup $(\mathrm{e}^{\mathcal{A}t})_{t\geq 0}$, can be obtained. By using Lemma 2.1 and inequality (2.5), we can show that, for all $z_0 \in L^2(\Omega)$, the function $z(t) = \mathrm{e}^{\mathcal{A}t}z_0$ obeys

$$\|z\|_{L^2(0,\infty;L^2(\Omega))} \leq C\|z_0\|_{L^2(\Omega)}.$$

The exponential stability follows from Datko's Theorem (see *e.g.* [21], Thm. 3.1(i)). □

In the following we shall denote by $\omega > 0$ an exponent and $C(\omega) \geq 1$ a constant depending on ω such that

$$\|\mathrm{e}^{\mathcal{A}t}\|_{\mathcal{L}(L^2(\Omega))} \leq C(\omega)\,\mathrm{e}^{-\omega t} \quad \text{and} \quad \|\mathrm{e}^{\mathcal{A}^*t}\|_{\mathcal{L}(L^2(\Omega))} \leq C(\omega)\,\mathrm{e}^{-\omega t} \quad \text{for all } t > 0.$$

As in [6], it is useful to introduce a parameter k to obtain estimates of solutions of different equations related to the operator $\mathcal{A}$.

Now we show that there is a norm in $L^2(\Omega)$, weaker than the usual one, which is associated with a Hilbert-Schmidt operator. More precisely, we have the following:

Proposition 2.1. *For $1 \leq i < \infty$ and $1 \leq j < \infty$, let us set*

$$\psi_{i,j}(x,y) = \sqrt{\frac{2}{L}}\sin\left(\frac{i\pi x}{L}\right)\sqrt{2}\sin\left(j\pi y\right),$$

and

$$\phi_\alpha(x,y,\xi,\eta) = \sum_{i,j=1}^{\infty} \frac{1}{(i^{2\alpha}+j^{2\alpha})^{1/2}}\psi_{i,j}(x,y)\psi_{i,j}(\xi,\eta) \quad \textit{with } \alpha > 1.$$

Then ϕ_α belongs to $L^2(\Omega\times\Omega)$. Let $\mathcal{C}_\alpha$ be the Hilbert-Schmidt operator defined by

$$\mathcal{C}_\alpha z = \int_\Omega \phi_\alpha(\cdot,\xi,\eta)\,z(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta.$$

The mapping

$$z \longmapsto \|\mathcal{C}_\alpha z\|_{L^2(\Omega)} = \left(\sum_{i,j=1}^{\infty} \frac{1}{i^{2\alpha}+j^{2\alpha}} \left(\int_\Omega \psi_{i,j}\, z\right)^2\right)^{1/2},$$

is a norm in $L^2(\Omega)$ *weaker than the usual one.*

Proof. The family $(\psi_{i,j})_{1\le i,j\le\infty}$ is a Hilbertian basis of $L^2(\Omega)$, and the family $(\psi_{i,j}\otimes\psi_{i,j})_{1\le i,j\le\infty}$ is a Hilbertian basis of $L^2(\Omega\times\Omega)$. Thus it is easy to see that

$$\|\phi_\alpha\|^2_{L^2(\Omega\times\Omega)} = \sum_{i,j=1}^{\infty} \frac{1}{i^{2\alpha}+j^{2\alpha}} < \infty.$$

The end of proof is obvious. □

3. Control system

In this section, we want to prove that equation (1.6) can be rewritten as a control evolution equation of the form

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u) + F, \qquad z(0) = z_0. \tag{3.1}$$

In particular we want to define the operators $\mathcal{A}$ and B, and the function F.

3.1. Existence and uniqueness results for the state equation

To define solutions to equation (1.6) by the transposition method, we introduce the adjoint system:

$$-\,p' = \mathcal{A}^* p + \psi \quad \text{in } (0,\infty), \qquad p(\infty) = 0. \tag{3.2}$$

Due to Theorem 2.1, and with results in [6], we can prove the following theorem.

Theorem 3.1. *Let* $\psi\in L^2(0,\infty;L^2(\Omega))$. *The system* (3.2) *admits a unique weak solution* p *such that*

$$p\in C_b([0,\infty);L^2(\Omega))\cap L^2(0,\infty;L^2(0,L;H^1(0,1;d))),$$
$$\sqrt{a}p\in C_w([0,L];L^2(0,\infty;L^2(0,1))),$$

where $C_w([0,L];L^2(0,\infty;L^2(0,1)))$ *is the space of continuous functions from* $[0,L]$ *into* $L^2(0,\infty;L^2(0,1))$ *equipped with its weak topology and* $C_b([0,\infty);L^2(\Omega))$ *is the space of bounded and continuous functions from* $[0,\infty)$ *into* $L^2(\Omega)$. *It satisfies the estimate*

$$\|p\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\sqrt{a}p\|_{L^\infty(0,L;L^2(0,\infty;L^2(0,1)))} + \|p\|_{L^2(0,\infty;L^2(0,L;H^1(0,1;d)))} \le C\|\psi\|_{L^2(0,\infty;L^2(\Omega))}. \tag{3.3}$$

We define weak solutions to equation (1.6) by the transposition method.

Definition 3.1. A function $z\in L^2\left(0,\infty;L^2(\Omega)\right)$ is a weak solution to equation (1.6) if and only if we have

$$\begin{aligned}\int_Q z\psi\,\mathrm{d}\tau\mathrm{d}\xi\mathrm{d}\eta &= \int_Q fp\,\mathrm{d}\tau\mathrm{d}\xi\mathrm{d}\eta + \int_\Omega p(0,\xi,\eta)z_0(\xi,\eta)\,\mathrm{d}\xi\mathrm{d}\eta\\ &\quad - \int_0^\infty\int_0^L b(\xi,0)\,(g+\mathbb{1}_\gamma u)(\tau,\xi)\,p(\tau,\xi,0)\,\mathrm{d}\tau\mathrm{d}\xi + \int_0^\infty\int_0^1 a(\eta)z_b(\tau,\xi)p(\tau,0,\eta)\,\mathrm{d}\tau\mathrm{d}\eta,\end{aligned} \tag{3.4}$$

for all $\psi\in L^2\left(0,\infty;L^2(\Omega)\right)$, where p is the solution to equation (3.2), and $Q=\Omega\times(0,\infty)$.

In [6], Theorem 6.6, it is shown that if $z \in L^2\big(0,\infty;L^2(\Omega)\big)$ is a weak solution to equation (1.6), in the sense of semigroup theory, then it is also a solution in the sense of transposition, that is to say in the sense of Definition 3.1. By taking in (3.4) functions ψ of the form $\psi(t,\xi,\eta) = -\theta'(t)\zeta(\xi,\eta) - \theta(t)A^*\zeta(\xi,\eta)$, where $\zeta \in D(\mathcal{A}^*)$ and $\theta \in \mathcal{D}(\mathbb{R}^+)$, we recover the weak formulation of the definition in the sense of semigroup theory. The initial condition can also be recovered by choosing a particular sequence of functions ψ.

Theorem 3.2. *Let f be in $L^2(0,\infty;L^2(\Omega))$, $g \in L^2(0,\infty;L^2(0,L))$, $u \in L^2(0,\infty;L^2(0,L))$, $z_b \in L^2(0,\infty;L^2(0,1))$, and $z_0 \in L^2(\Omega)$, then equation (1.6) admits a unique weak solution $z \in L^2(0,\infty;\,L^2(\Omega))$. Moreover*

$$z \in L^2(0,\infty;L^2(0,L;H^1(0,1;d))) \cap C_b([0,\infty);L^2(\Omega)),$$

$$\sqrt{a}\,z \in C_w([0,L];L^2(0,\infty;L^2(0,1))),$$

and the solution obeys:

$$\begin{aligned}&\|z\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\sqrt{a}z\|_{L^\infty(0,L;L^2(0,\infty;L^2(0,1)))} + \|z\|_{L^2(0,\infty;L^2(0,L;H^1(0,1;d)))}\\ &\qquad \le C_5\Big(\|f\|_{L^2(Q)} + \|u\|_{L^2(0,\infty;L^2(0,L))} + \|g\|_{L^2(0,\infty;L^2(0,L))} + \|z_b\|_{L^2(0,\infty;L^2(0,1))} + \|z_0\|_{L^2(\Omega)}\Big).\end{aligned} \tag{3.5}$$

Proof. Theorem 3.2 is proved in [6], Theorem 6.6. Its proof relies on inequality (2.5), on Lemma 2.1, and on an approximation procedure (the boundary terms u, g and z_b are approximated by a sequence of distributed terms). □

3.2. **Dirichlet and Neumann operators**

Let v belong to $L^2(0,L)$ and $z_b \in L^2(0,1)$. We define the solution to the Neumann problem

$$Aw = 0 \text{ in } \Omega, \quad \sqrt{a}\,w(0,\cdot) = 0 \text{ in } (0,1), \quad (bw)(\cdot,1) = 0 \;\text{ and }\; \frac{\partial w}{\partial \eta}(\cdot,0) = v \quad \text{in } (0,L), \tag{3.6}$$

and to the Dirichlet problem

$$A\zeta = 0 \text{ in } \Omega, \quad \sqrt{a}\zeta(0,\cdot) = \sqrt{a}z_b \text{ in } (0,1), \quad (b\zeta)(\cdot,1) = 0 \;\text{ and }\; \frac{\partial \zeta}{\partial \eta}(\cdot,0) = 0 \quad \text{in } (0,L), \tag{3.7}$$

by the transposition method as follows.

Definition 3.2. A function $w \in L^2(\Omega)$ is a weak solution to equation (3.6) if and only if we have

$$\int_\Omega wA^*p \,\mathrm{d}\xi\mathrm{d}\eta = -\int_0^L b(\xi,0)v(\xi)p(\xi,0)\,\mathrm{d}\xi \qquad \text{for all } p \in D(\mathcal{A}^*). \tag{3.8}$$

Similarly, a function $\zeta \in L^2(\Omega)$ is a weak solution to equation (3.7) if and only if we have

$$\int_\Omega \zeta A^*p \,\mathrm{d}\xi\mathrm{d}\eta = -\int_0^1 a(\eta)z_b(\eta)p(0,\eta)\,\mathrm{d}\xi \qquad \text{for all } p \in D(\mathcal{A}^*). \tag{3.9}$$

Using the method in [6], Proof of Theorem 6.6, we can establish the following theorem.

Theorem 3.3. *Let $v \in L^2(0,L)$, then equation (3.6) admits a unique weak solution $w \in L^2(\Omega)$. Moreover*

$$w \in L^2(0,L;H^1(0,1;d)), \qquad \sqrt{a}w \in C_w([0,L];L^2(0,1)),$$

and

$$\|\sqrt{a}w\|_{L^\infty(0,L;L^2(0,1))} + \|w\|_{L^2(0,L;H^1(0,1;d))} \le C\|v\|_{L^2(0,L)}. \tag{3.10}$$

Let $z_b \in L^2(0,1)$, then equation (3.7) admits a unique weak solution $\zeta \in L^2(\Omega)$. Moreover

$$\zeta \in L^2(0,L;H^1(0,1;d)), \qquad \sqrt{a}\zeta \in C_w([0,L];L^2(0,1)),$$

and the solution obeys:

$$\|\sqrt{a}\zeta\|_{L^\infty(0,L;L^2(0,1))} + \|\zeta\|_{L^2(0,L;H^1(0,1;d))} \leq C\|z_b\|_{L^2(0,1)}. \tag{3.11}$$

Proof. We briefly give the proof of (3.10). The second statement can be proved in the same way. The uniqueness of solution to equation (3.6) is obvious. The only difficult point is the existence of a solution and estimate (3.10). We proceed by approximation. We set $v_n(\xi,\eta) = nv(\xi)\chi_n(\eta)$, where χ_n is the characteristic function of the interval $(0,\frac{1}{n})$. Let w_n be the solution to equation

$$\mathcal{A}w_n = b\,v_n. \tag{3.12}$$

It can be shown that $\zeta_n = \mathrm{e}^{-k\xi}w_n$ satisfies an inequality similar to (2.5). More precisely, we have

$$\frac{1}{2}\int_0^1 a\,\zeta_n(x,\eta)^2\,\mathrm{d}\eta + \int_0^1\int_0^x \left(b\left|\frac{\partial\zeta_n}{\partial\eta}\right|^2 + \frac{\partial b}{\partial\eta}\frac{\partial\zeta_n}{\partial\eta}\zeta_n + (c+ka)\zeta_n^2\right)\mathrm{d}\xi\,\mathrm{d}\eta \leq \int_0^1\int_0^x \mathrm{e}^{-k\xi}\,b\,v_n\,\zeta_n\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{3.13}$$

for all $x \in [0,L]$. With Lemma 2.1 and classical majorizations we arrive at

$$\|\sqrt{a}\zeta_n\|_{L^\infty(0,L;L^2(0,1))} + \|\zeta_n\|_{L^2(0,L;H^1(0,1;d))} \leq C\|v_n\|_{L^2(0,L)},$$

where the constant C is independent of n. Therefore, there exists a subsequence, still indexed by n to simplify the notation, such that

$$\begin{aligned} &\zeta_n \rightharpoonup w && \text{weakly in } L^2(0,L;H^1(0,1;d)),\\ &\sqrt{a}\zeta_n \rightharpoonup \sqrt{a}w && \text{weakly-star in } L^\infty(0,L;L^2(\Omega)), \end{aligned} \tag{3.14}$$

for some function $w \in L^\infty(0,L;L^2(0,1)) \cap L^2(0,L;H^1(0,1;d))$. By passing to the limit in the variational formulation satisfied by ζ_n, we can show that w is a weak solution to equation (3.6). □

3.3. Control system

We denote by N and D the operators defined by

$$Nv = w, \qquad Dz_b = \zeta$$

where w is the solution to equation (3.6), and ζ is the solution to equation (3.7).

Observe that N belongs to $\mathcal{L}(L^2(0,L), L^2(0,L;H^1(0,1;d)))$, and that D belongs to $\mathcal{L}(L^2(0,1), L^2(0,L;H^1(0,1;d)))$. Moreover according to Definition 3.2, we have

$$N^*A^*p = -b(\xi,0)p(\xi,0) \qquad \text{and} \qquad D^*A^*p = -a(\eta)p(0,\eta) \qquad \text{for all } p \in D(\mathcal{A}^*).$$

Thus N^*A^*p is the trace of $-bp$ on $(0,L)\times\{0\}$.

Using the extrapolation method the semigroup $(\mathrm{e}^{t\mathcal{A}})_{t\in\mathbb{R}^+}$ can be extended to $(D(\mathcal{A}^*))'$. Denoting the corresponding semigroup by $(\mathrm{e}^{t\widehat{\mathcal{A}}})_{t\in\mathbb{R}^+}$, the generator $(\widehat{\mathcal{A}}, D(\widehat{\mathcal{A}}))$ of this semigroup is an unbounded operator in $(D(\mathcal{A}^*))'$ with domain $D(\widehat{\mathcal{A}}) = Z$.

First assume that $g \in C_c^1(0,\infty,L^2(0,L))$, $u \in C_c^1(0,\infty;L^2(0,L))$, and $z_b \in C_c^1(0,\infty;L^2(0,1))$, and set

$$w(t) = N(\mathbb{1}_\gamma u(t) + g(t)), \qquad \zeta(t) = Dz_b(t).$$

Let z be the unique weak solution to equation (1.6), and set $Z = z - w - \zeta$. We can check that Z is the weak solution to the equation

$$Z' = \mathcal{A}Z - w' - \zeta' + f, \qquad Z(0) = z_0,$$

that is

$$Z(t) = \mathrm{e}^{t\mathcal{A}} z_0 + \int_0^t \mathrm{e}^{(t-\tau)\mathcal{A}} f(\tau)\mathrm{d}\tau - \int_0^t \mathrm{e}^{(t-\tau)\mathcal{A}} w'(\tau)\mathrm{d}\tau - \int_0^t \mathrm{e}^{(t-\tau)\mathcal{A}} \zeta'(\tau)\mathrm{d}\tau.$$

Making integration by parts, we can show that (see *e.g.* [2]) equation (1.6) can be rewritten in the form

$$z' = \widehat{\mathcal{A}}z + f + (-\widehat{\mathcal{A}})Ng + (-\widehat{\mathcal{A}})N(\mathbb{1}_\gamma u) + (-\widehat{\mathcal{A}})Dz_b, \qquad z(0) = z_0. \tag{3.15}$$

This equation is still meaningful if $g \in L^2(0,\infty; L^2(0,L))$, $u \in L^2(0,\infty; L^2(0,L))$, and $z_b \in L^2(0,\infty; L^2(0,1))$. We set

$$F = f + (-\widehat{\mathcal{A}})Ng + (-\widehat{\mathcal{A}})Dz_b \qquad \text{and} \qquad B = (-\widehat{\mathcal{A}})N, \tag{3.16}$$

and we obtain equation (3.1) if, by abuse of notation, we replace $\widehat{\mathcal{A}}$ by $\mathcal{A}$.

4. Optimal control

Let us recall the definition of

$$(\mathcal{P}_{f,g,z_b,y_d,z_0}) \qquad \inf\Big\{ J(z,u) \mid (z,u) \in L^2(0,\infty; Z) \times L^2(0,\infty; U),\ (z,u) \text{ satisfies } (4.2) \Big\},$$

where

$$J(z,u) = \frac{1}{2}\int_0^\infty \|Cz(t) + y_d(t)\|_Z^2 \,\mathrm{d}t + \frac{1}{2}\int_0^\infty \|u(t)\|_U^2 \,\mathrm{d}t, \tag{4.1}$$

with

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u) + F, \qquad z(0) = z_0, \tag{4.2}$$

and F is defined in (3.16). Let us recall that $Z = L^2(\Omega)$, $U = L^2(0,L)$, $C \in \mathcal{L}(Z)$, and $y_{\mathrm{d}} \in L^2(0,\infty; Z)$ are defined in the introduction. In the above setting $\|\cdot\|_Z$ and $\|\cdot\|_U$ denote respectively the norm in Z and in U, and the associated inner products will be denoted by $(\cdot,\cdot)_Z$ and $(\cdot,\cdot)_U$.

Theorem 4.1. *Assume that $(H_1) - (H_4)$ are fulfilled. Then problem $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ admits a unique solution $(\bar z, \bar u)$.*

Proof. The proof is classical. We briefly introduce the main ingredients for the convenience of the reader. Let us denote by $z(u)$ the solution to equation (4.2) corresponding to u. Due to Theorem 2.1, $J(z(0),0) < \infty$. Thus $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ admits minimizing sequences, and minimizing sequences are bounded in $L^2(0,\infty; U)$. Due to Theorem 3.2, if a sequence $(u_n)_n$ converges weakly in $L^2(0,\infty,U)$ to some u, then $(z(u_n))_n$ converges weakly in $L^2(0,\infty; L^2(0,L; H^1(0,1;d)))$ to $z(u)$. Thus, by standard arguments, if $(u_n)_n$ is a minimizing sequence, converging to u for the weak topology of $L^2(0,\infty; U)$, then

$$J(z(u),u) \le \liminf_{n\to\infty} J(z(u_n),u_n) = \inf(\mathcal{P}_{f,g,z_b,y_d,z_0}).$$

Thus, $(z(u),u)$ is a solution of $(\mathcal{P}_{f,g,z_b,y_d,z_0})$. The uniqueness follows from the strict convexity of the mapping $u \mapsto J(z(u),u)$. $\square$

Theorem 4.2. *If $(\bar z, \bar u)$ is the solution to $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ then*

$$\bar u(t) = \mathbb{1}_\gamma b\bar p|_{\gamma\times\{0\}} = -\mathbb{1}_\gamma B^*\bar p(t), \tag{4.3}$$

where $\bar p$ is the solution to equation (3.2) with

$$\psi = C^*(C\bar z + y_d).$$

Conversely if a pair $(z,p) \in \big(L^2(0,\infty;L^2(0,L;H^1(0,1;d)))\big)^2$ *obeys the system*

$$\begin{cases} z' = \mathcal{A}z + B(\mathbb{1}_\gamma b(\cdot,0)p(\cdot,0)) + F \quad \text{in } (0,\infty), \quad z(0)=z_0, \\ -p' = \mathcal{A}^*p + C^*(Cz+y_d) \quad \text{in } (0,\infty), \quad p(\infty)=0, \end{cases} \tag{4.4}$$

then the pair $\big(z, \mathbb{1}_\gamma bp|_{\gamma\times\{0\}}\big)$ *is the optimal solution to problem* $(\mathcal{P}_{f,g,z_b,y_d,z_0})$.

Proof. Let $(\bar z,\bar u)$ be the optimal solution to problem $(\mathcal{P}_{f,g,z_b,y_d,z_0})$. Set $I(u) = J(z(u),u)$, where $z(u)$ is the solution to equation (4.2) corresponding to u. For every $v \in L^2(0,\infty;U)$ and $\lambda \in \mathbb{R}^*$, we denote by z_λ the solution to the equation (1.6) associated with $\bar u + \lambda v$. We have

$$I(\bar u+\lambda v) - I(\bar u) = \frac{1}{2}\int_0^\infty (C(z_\lambda - \bar z), C(z_\lambda+\bar z)+2y_d)_Z \,\mathrm{d}\tau + \frac{1}{2}\int_0^\infty \Big((2\lambda v,\bar u)_U + \lambda^2\|v(\tau)\|_U^2\Big)\,\mathrm{d}\tau. \tag{4.5}$$

The function $w = (z_\lambda - \bar z)/\lambda$ is the solution of equation

$$w' = \mathcal{A}w + B(\mathbb{1}_\gamma v) \quad \text{in } (0,\infty), \quad w(0)=0.$$

Due to Theorem 3.2, we have

$$\|w\|_{L^2(0,\infty;L^2(0,L;H^1(0,1;d)))} \le C\|v\|_{L^2(0,\infty;U)}.$$

Thus the sequence $(z_\lambda)_\lambda$ converges to $\bar z$ in $L^2(0,\infty;L^2(0,L;H^1(0,1;d)))$ when λ tends to zero. Dividing $I(\bar u + \lambda v) - I(\bar u)$ by λ and passing to the limit when λ tends to zero, we obtain

$$I'(\bar u)v = \int_0^\infty (Cw, C\bar z + y_d)_Z \,\mathrm{d}\tau + \int_0^\infty (v,\bar u)_U\,\mathrm{d}\tau.$$

With formula (3.4) in which z is replaced by w and p by the solution of equation (3.2) corresponding to $\psi = C^*(C\bar z + y_d)$, we have

$$\int_0^\infty (Cw, C\bar z+y_d)_Z\,\mathrm{d}\tau = -\int_0^\infty\int_\gamma b(\xi,0)v(\tau)\bar p(\tau,\xi,0)\,\mathrm{d}\xi\mathrm{d}\tau.$$

Hence

$$I'(\bar u)v = -\int_0^\infty\int_\gamma b(\xi,0)\bar p(\tau,\xi,0)v(\tau)\,\mathrm{d}\xi\mathrm{d}\tau + \int_0^\infty (\bar u(\tau), v(\tau))_U\,\mathrm{d}\tau.$$

Since $(\bar z,\bar u)$ is the solution to the problem $(\mathcal{P}_{f,g,z_b,y_d,z_0})$, we have $I'(\bar u)=0$ and $\bar u = \mathbb{1}_\gamma b\bar p|_{\gamma\times\{0\}} = -\mathbb{1}_\gamma B^*\bar p$.

Conversely, assume that $(z,p) \in \big(L^2(0,\infty;L^2(0,L;H^1(0,1;d)))\big)^2$ is the solution of system (4.4). Let us set

$$u(t,\xi) = \mathbb{1}_\gamma(\xi)b(\xi,0)p(t,\xi,0).$$

From previous calculations, it follows that

$$I'(\bar u) = 0.$$

Due to the convexity of the mapping I we deduce that $\bar u$ is the solution to problem $(\mathcal{P}_{f,g,z_b,y_d,z_0})$. $\square$

5. Riccati equation

In this section, we study problem $(\mathcal{P}_{f,g,z_b,y_d,z_0})$ in the case where $f = 0$, $z_b = 0$, $g = 0$ and $y_d = 0$. We denote it by $(\mathcal{P}_{z_0})$. In the previous section, we have proved that the solution (z, u) of $(\mathcal{P}_{z_0})$ is characterized by $u = -\mathbb{1}_\gamma B^*p$, where $(z, p) \in \left(L^2(0, \infty; L^2(0, L; H^1(0, 1; d)))\right)^2$ is the unique solution of system

$$\begin{cases} z' = \mathcal{A}z - B(\mathbb{1}_\gamma B^*p), \qquad z(0) = z_0, \\ -p' = \mathcal{A}^*p + C^*Cz, \qquad p(\infty) = 0. \end{cases} \tag{5.1}$$

Let us denote by Π the operator

$$\Pi \ : \ z_0 \longmapsto p(0). \tag{5.2}$$

This operator is well defined since p belongs to $C_b([0, \infty); L^2(\Omega))$ (it is sufficient to apply Thm. 3.2 to the adjoint equation).

5.1. **Failure of existing results**

Let us first explain why existing results in the literature do not permit to characterize Π as the weak solution to an algebraic Riccati with tests functions (in the definition of weak solutions) belonging to $D(\mathcal{A})$ (for existing results to algebraic Riccati equations, we refer to [9–12,15]). Using the dynamic programming principle, as in [11] it can be shown that the family of operators $(\mathcal{S}(t))_{t\in\mathbb{R}^+}$, defined by

$$\mathcal{S}(t)z_0 = z(t),$$

where $(z(t), p(t))_{t\in\mathbb{R}^+}$ is the solution of (5.1), is a strongly continuous semigroup exponentially stable on Z. Let us denote by $(\mathcal{A}_\Pi, D(\mathcal{A}_\Pi))$ its infinitesimal generator (formally $\mathcal{A}_\Pi = \mathcal{A} - B(\mathbb{1}_\gamma B^*\Pi)$). Let s belong to $(0, \infty)$. We denote by (z^s, p^s) the solution of the system

$$\begin{cases} \dfrac{\mathrm{d}z^s}{\mathrm{d}t} = \mathcal{A}z^s - B(\mathbb{1}_\gamma B^*p^s) \quad \text{in } (s, \infty), \qquad z^s(s) = z(s), \\ -\dfrac{\mathrm{d}p^s}{\mathrm{d}t} = \mathcal{A}^*p^s + C^*Cz^s \quad \text{in } (s, \infty), \qquad p^s(\infty) = 0. \end{cases} \tag{5.3}$$

It is clear that

$$p^s(s) = \Pi z^s(s).$$

Moreover, from the dynamic programming principle, it follows that $p^s(s) = p(s)$. Thus we have extended the identity (5.2) by showing that

$$p(t) = \Pi z(t) \qquad \text{for all } t \in [0, \infty).$$

Therefore we have proved that the optimal solution of $(\mathcal{P}_{z_0})$ obeys the feedback law

$$\bar{u}(t) = -\mathbb{1}_\gamma B^*\Pi z(t).$$

Moreover, with (5.3) we can show that

$$\inf(P_{z_0}) = \frac{1}{2}\big(p(0), z_0\big)_Z = \frac{1}{2}\big(\Pi z_0, z_0\big)_Z.$$

We can also show that Π obeys the following integral equation (see [11]):

$$\Pi = \int_0^\infty \mathrm{e}^{-\mathcal{A}^*t}C^*C\mathrm{e}^{\mathcal{A}_\Pi t}\,\mathrm{d}t. \tag{5.4}$$

However since Π is involved in the definition of the operator $\mathcal{A}_\Pi$, the above equation is not really useful for the computation of the operator Π.

Following [1], it can be shown that Π obeys the following formulation of the A.R.E.

$$\big(\mathcal{A}z,\Pi\zeta\big)_Z+\big(\Pi z,\mathcal{A}^*\zeta\big)_Z-\big(1\!\!1_\gamma B^*\Pi z,1\!\!1_\gamma B^*\Pi\zeta\big)_Z+\big(C^*Cz,\zeta\big)_Z=0,\quad \forall z,\,\zeta\in D(\mathcal{A}_\Pi). \tag{5.5}$$

Unfortunately the characterization of $D(\mathcal{A}_\Pi)$ is not obvious because it depends on Π which is precisely unknown, and in general this variational formulation is not satisfied for $z\in D(\mathcal{A})$, and it cannot be used to characterize the operator Π (see [17–19]).

Here taking advantage of the regularizing properties of the operator C, we look for Π in the form of a Hilbert-Schmidt operator, and we are able to study the partial differential equation satisfied by the kernel of the operator Π. We show that this partial differential equation admits a unique solution π in $L^2_s(\Omega\times\Omega)\cap L^2_+(\Omega\times\Omega)$ (see the definition of these spaces in Sect. 5.2). Showing in Section 6 that this unique solution π obeys

$$\inf(\mathcal{P}_{z_0})=\frac{1}{2}\int_{\Omega\times\Omega}\pi\, z_0\otimes z_0,$$

we can conclude that Π is a Hilbert-Schmidt operator and that π is the kernel of Π.

Since we want to characterize the operator $\Pi\in\mathcal{L}(L^2(\Omega))$ by a kernel $\pi\in L^2(\Omega\times\Omega)$, for notational simplicity we write $\Omega\times\Omega$ in the form $\Omega_X\times\Omega_\Xi$. The current point $(X,\Xi)\in\Omega_X\times\Omega_\Xi$ corresponds to $X=(x,y)\in\Omega_X$ and $\Xi=(\xi,\eta)\in\Omega_\Xi$. With this notation Π and π – if it exists in $L^2(\Omega_X\times\Omega_\Xi)$ – are related by the identity

$$\Pi z(X)=\int_\Omega \pi(X,\Xi)z(\Xi)\mathrm{d}\Xi. \tag{5.6}$$

Similarly, A^*_X (resp. A^*_Ξ) corresponds to the operator A^* written in X-variable (resp. in Ξ-variable), that is:

$$A^*_X p=a(y)\frac{\partial p}{\partial x}+\frac{\partial^2(b(x,y)p)}{\partial y^2}-c(x,y)\,p$$

(resp. $A^*_\Xi p=a(\eta)\frac{\partial p}{\partial\xi}+\frac{\partial^2(b(\xi,\eta)p)}{\partial\eta^2}-c(\xi,\eta)\,p$). To write the equation satisfied by π, let us introduce some new operators. Let us set $\mathcal{O}=\Omega_X\times\Omega_\Xi$. If $z\in L^2(\Omega)$ and $\zeta\in L^2(\Omega)$, we denote by $z\otimes\zeta$ the function belonging to $L^2(\mathcal{O})$ defined by

$$z\otimes\zeta\ :\ (X,\Xi)\longmapsto z(X)\zeta(\Xi).$$

We denote by $L^2_s(\mathcal{O})$ the space of functions $\pi\in L^2(\mathcal{O})$ satisfying:

$$\pi(X,\Xi)=\pi(\Xi,X)\qquad\text{for almost all } (X,\Xi)\in\Omega_X\times\Omega_\Xi.$$

We are going to see that

$$D(\mathcal{A}^*_{X,\Xi})=\left\{\varphi=\int_0^\infty \mathrm{e}^{t\mathcal{A}^*_X}\,\mathrm{e}^{t\mathcal{A}^*_\Xi}\,\psi\,\mathrm{d}t\mid\psi\in L^2(\mathcal{O})\right\},$$

is the domain of the infinitesimal generator of a strongly continuous exponentially stable semigroup on $L^2(\mathcal{O})$. We also set

$$D(\mathcal{A}^{s\ *}_{X,\Xi})=D(\mathcal{A}^*_{X,\Xi})\cap L^2_s(\mathcal{O}).$$

In Section 6, we show that the operator Π defined by (5.2) may be written in the form (5.6), where π is the unique solution to the algebraic Riccati equation

$$\pi\in D(\mathcal{A}^{s\ *}_{X,\Xi}),\quad \mathcal{A}^*_X\pi+\mathcal{A}^*_\Xi\pi-\int_\gamma|b(s,0)|^2\pi(s,0,\Xi)\pi(X,s,0)\,\mathrm{d}s\;+\mathbf{\Phi}=0, \tag{5.7}$$

and $\mathbf{\Phi}\in L^2_s(\mathcal{O})$ is the function defined by

$$\mathbf{\Phi}(X,\Xi)=\int_\Omega\phi(\cdot,X)\,\phi(\cdot,\Xi). \tag{5.8}$$

The function $\phi \in L^2(\mathcal{O})$ is the one defining the observation operator C (see (1.8)). Observe that by Cauchy-Schwarz inequality, we have

$$\|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})} \le \|\phi\|^2_{L^2(\mathcal{O})}.$$

The existence of at least one solution to equation (5.7) is established in Theorem 5.8. The uniqueness is proved in Theorem 6.2.

To study equation (5.7) we first study the differential Riccati equation

$$\begin{cases} \pi' = \mathcal{A}_X^*\pi + \mathcal{A}_\Xi^*\pi - \displaystyle\int_\gamma |b(s,0)|^2 \pi(t,s,0,\Xi)\pi(t,X,s,0)\,\mathrm{d}s + \mathbf{\Phi} \quad \text{in } (0,\infty), \\ \pi(0,\cdot) = \pi_0 \in L^2_s(\mathcal{O}). \end{cases} \tag{5.9}$$

Even if we prove that the solution of (5.7) is the limit when t tends to infinity of the solution to equation (5.9) when $\pi_0 = 0$, we need to study equation (5.9) with $\pi_0 \neq 0$ (see the proofs of Thm. 5.8 and Lem. 5.9).

5.2. Semigroup generated by $\mathcal{A}_X^* + \mathcal{A}_\Xi^*$

Lemma 5.1. *For every $z \in L^2(\Omega_X)$, and $\zeta \in L^2(\Omega_\Xi)$, we have*

$$\mathrm{e}^{t\mathcal{A}_X}\left(z \otimes \mathrm{e}^{t\mathcal{A}_\Xi}\zeta\right) = \mathrm{e}^{t\mathcal{A}_X} z \otimes \mathrm{e}^{t\mathcal{A}_\Xi}\zeta = \mathrm{e}^{t\mathcal{A}_\Xi}\left(\mathrm{e}^{t\mathcal{A}_X} z \otimes \zeta\right).$$

Proof. The result is a direct consequence of the definition of the tensor product. □

Lemma 5.2. *For all $t \ge 0$, $\tau \ge 0$, $\psi \in L^2(\mathcal{O})$, we have*

$$\mathrm{e}^{t\mathcal{A}_X^*}\,\mathrm{e}^{\tau\mathcal{A}_\Xi^*}\,\psi = \mathrm{e}^{\tau\mathcal{A}_\Xi^*}\,\mathrm{e}^{t\mathcal{A}_X^*}\,\psi.$$

Proof. The result can be deduced from Lemma 5.1 by using the density of $L^2(\Omega) \otimes L^2(\Omega)$ into $L^2(\mathcal{O})$. □

With Lemma 5.2 we can prove the following result.

Lemma 5.3. *For $t \ge 0$, let $S^*(t) \in \mathcal{L}(L^2(\mathcal{O}))$ be defined by*

$$S^*(t) \ : \ \psi \longmapsto \mathrm{e}^{t\mathcal{A}_X^*}\mathrm{e}^{t\mathcal{A}_\Xi^*}\psi.$$

The family $(S^*(t))_{t\ge 0}$ is a strongly continuous exponentially stable semigroup on $L^2(\mathcal{O})$.

Proof. We have $S^*(0) = I$. Since $\mathrm{e}^{t\mathcal{A}_\Xi^*}\,\mathrm{e}^{\tau\mathcal{A}_X^*} = \mathrm{e}^{\tau\mathcal{A}_X^*}\,\mathrm{e}^{t\mathcal{A}_\Xi^*}$, it is easy to show that $S^*(t)S^*(\tau) = S^*(t+\tau)$.

Let us show that the semigroup $(S^*(t))_{t\ge 0}$ is weakly continuous on $L^2(\mathcal{O})$. First we write:

$$\int_\mathcal{O} \psi\,\mathrm{e}^{t\mathcal{A}_X} z \otimes \mathrm{e}^{t\mathcal{A}_\Xi}\zeta - \int_\mathcal{O} \psi\, z \otimes \zeta = \int_{\Omega_\Xi}\int_{\Omega_X} \left((\mathrm{e}^{t\mathcal{A}_X} z - z)\psi(\cdot,\Xi)\right)\mathrm{e}^{t\mathcal{A}_\Xi}\zeta + \int_\mathcal{O} \psi\, z\left(\mathrm{e}^{t\mathcal{A}_\Xi}\zeta - \zeta\right).$$

We know that

$$\lim_{t\searrow 0}\int_\mathcal{O} \psi\, z\left(\mathrm{e}^{t\mathcal{A}_\Xi}\zeta - \zeta\right) = 0.$$

Moreover, for almost all $\Xi \in \Omega_\Xi$, we have

$$\lim_{t\searrow 0} \mathrm{e}^{t\mathcal{A}_\Xi}\zeta \int_{\Omega_X} (\mathrm{e}^{t\mathcal{A}_X} z - z)\psi(\cdot,\Xi)\,\mathrm{d}X = 0,$$

and

$$\left\|\mathrm{e}^{t\mathcal{A}_\Xi}\zeta \int_{\Omega_X} (\mathrm{e}^{t\mathcal{A}_X} z - z)\psi(\cdot,\Xi)\right\|_{L^2(\Omega_\Xi)} \le C\|z\|_{L^2(\Omega_X)}\|\zeta\|_{L^2(\Omega_\Xi)}\|\psi\|_{L^2(\mathcal{O})}.$$

Therefore with the dominated convergence theorem we have:

$$\lim_{t\searrow 0}\int_{\Omega_\Xi}\int_{\Omega_X}\left((\mathrm{e}^{t\mathcal{A}_X}z-z)\psi(\cdot,\Xi)\right)\mathrm{e}^{t\mathcal{A}_\Xi}\zeta=0.$$

Thus the semigroup $(S^*(t))_{t\geq 0}$ is weakly continuous on $L^2(\mathcal{O})$. It is also strongly measurable on $L^2(\mathcal{O})$. Thus it is also strongly continuous on $L^2(\mathcal{O})$. Let us show that it is exponentially stable. Using the exponential stability of the semigroups $(\mathrm{e}^{t\mathcal{A}_X^*})_{t\geq 0}$ and $(\mathrm{e}^{t\mathcal{A}_\Xi^*})_{t\geq 0}$, we can write

$$\|S^*(t)\psi\|_{L^2(\mathcal{O})}\leq C_\omega \mathrm{e}^{-\omega t}\|\mathrm{e}^{t\mathcal{A}_\Xi^*}\psi\|_{L^2(\mathcal{O})}\leq C_\omega^2\mathrm{e}^{-2\omega t}\|\psi\|_{L^2(\mathcal{O})}.$$

The proof is complete. □

Let us denote by $(\mathcal{A}_{X,\Xi}^*, D(\mathcal{A}_{X,\Xi}^*))$ the infinitesimal generator of $(S^*(t))_{t\geq 0}$ in $L^2(\mathcal{O})$. From the exponential stability of the semigroup $(S^*(t))_{t\geq 0}$, it follows that

$$(-\mathcal{A}_{X,\Xi}^*)^{-1}\psi=\int_0^\infty \mathrm{e}^{t\mathcal{A}_X^*}\,\mathrm{e}^{t\mathcal{A}_\Xi^*}\,\psi\,\mathrm{d}t\quad\text{and}\quad D(\mathcal{A}_{X,\Xi}^*)=\left\{\int_0^\infty \mathrm{e}^{t\mathcal{A}_X^*}\,\mathrm{e}^{t\mathcal{A}_\Xi^*}\,\psi\,\mathrm{d}t\mid \psi\in L^2(\mathcal{O})\right\}.$$

We cannot give a more precise characterization of $D(\mathcal{A}_{X,\Xi}^*)$. However, setting

$$H=L^2(\Omega_X;D(\mathcal{A}_\Xi^*))\cap L^2(\Omega_\Xi;D(\mathcal{A}_X^*)),$$

we can show that $H\subset D(\mathcal{A}_{X,\Xi}^*)$. Indeed if $\psi\in H$, we can write

$$\begin{aligned}
\lim_{t\searrow 0}\int_{\mathcal{O}}\frac{\mathrm{e}^{t\mathcal{A}_X^*}\mathrm{e}^{t\mathcal{A}_\Xi^*}\psi-\psi}{t}z\otimes\zeta &= \lim_{t\searrow 0}\int_{\mathcal{O}}\frac{\mathrm{e}^{t\mathcal{A}_\Xi^*}\psi-\psi}{t}z\otimes\zeta+\lim_{t\searrow 0}\int_{\mathcal{O}}\frac{\mathrm{e}^{t\mathcal{A}_X^*}\mathrm{e}^{t\mathcal{A}_\Xi^*}\psi-\mathrm{e}^{t\mathcal{A}_\Xi^*}\psi}{t}z\otimes\zeta\\
&= \lim_{t\searrow 0}\int_{\mathcal{O}}\psi\, z\,\frac{\mathrm{e}^{t\mathcal{A}_\Xi}\zeta-\zeta}{t}+\lim_{t\searrow 0}\int_{\mathcal{O}}\psi\,\mathrm{e}^{t\mathcal{A}_\Xi}\zeta\,\frac{\mathrm{e}^{t\mathcal{A}_X}z-z}{t}\\
&= \int_{\mathcal{O}}\psi\left(\mathcal{A}_X z\otimes\zeta+z\otimes\mathcal{A}_\Xi\zeta\right)=\int_{\mathcal{O}}\left(\mathcal{A}_X^*\psi+\mathcal{A}_\Xi^*\psi\right)z\otimes\zeta,
\end{aligned}$$

for all $z\in D(\mathcal{A})$ and all $\zeta\in D(\mathcal{A})$. By a density argument we deduce that

$$\lim_{t\searrow 0}\int_{\mathcal{O}}\frac{\mathrm{e}^{t\mathcal{A}_X^*}\mathrm{e}^{t\mathcal{A}_\Xi^*}\psi-\psi}{t}z\otimes\zeta=\int_{\mathcal{O}}\left(\mathcal{A}_X^*\psi+\mathcal{A}_\Xi^*\psi\right)z\otimes\zeta,$$

for all $z\in L^2(\Omega)$ and all $\zeta\in L^2(\Omega)$. Thus, if $\psi\in H$, $\mathcal{A}_\Xi^*\psi+\mathcal{A}_X^*\psi$ belongs to $L^2(\mathcal{O})$ and

$$\mathcal{A}_{X,\Xi}^*\psi=\mathcal{A}_X^*\psi+\mathcal{A}_\Xi^*\psi. \tag{5.10}$$

It is the reason why we shall often write $\mathcal{A}_X^*\psi+\mathcal{A}_\Xi^*\psi$ in place of $\mathcal{A}_{X,\Xi}^*\psi$, and $\mathrm{e}^{t(\mathcal{A}_X^*+\mathcal{A}_\Xi^*)}$ in place of $\mathrm{e}^{t\mathcal{A}_X^*}\,\mathrm{e}^{t\mathcal{A}_\Xi^*}$ or of $\mathrm{e}^{t\mathcal{A}_{X,\Xi}^*}$, even if it is an abuse of notation.

We also introduce the operators $\mathcal{A}_{k,X}^*$ and $\mathcal{A}_{k,\Xi}^*$ defined by $D(\mathcal{A}_{k,X}^*)=D(\mathcal{A}_X^*)$, $D(\mathcal{A}_{k,\Xi}^*)=D(\mathcal{A}_\Xi^*)$,

$$\mathcal{A}_{k,X}^*\zeta=\mathcal{A}_X^*\zeta-k\,a(y)\zeta\qquad\text{and}\qquad\mathcal{A}_{k,\Xi}^*\zeta=\mathcal{A}_\Xi^*\zeta-k\,a(\eta)\zeta,$$

where the parameter $k>0$ is the one in Lemma 2.1.

Theorem 5.1. *(i) The adjoint of the operator* $\mathrm{e}^{t\mathcal{A}_X^*}\,\mathrm{e}^{t\mathcal{A}_\Xi^*} \in \mathcal{L}(L^2(\mathcal{O}))$ *is the operator* $\mathrm{e}^{t\mathcal{A}_X}\,\mathrm{e}^{t\mathcal{A}_\Xi} \in \mathcal{L}(L^2(\mathcal{O}))$. *The family of operators* $(S(t))_{t\geq 0}$, *where* $S(t) = \mathrm{e}^{t\mathcal{A}_X}\,\mathrm{e}^{t\mathcal{A}_\Xi}$, *is the adjoint semigroup of* $(S^*(t))_{t\geq 0}$.
(ii) The infinitesimal generator of $(S(t))_{t\geq 0}$ *in* $L^2(\mathcal{O})$ *is* $(\mathcal{A}_{X,\Xi}, D(\mathcal{A}_{X,\Xi}))$, *the adjoint of* $(\mathcal{A}_{X,\Xi}^*, D(\mathcal{A}_{X,\Xi}^*))$.
(iii) The space $L^2(\Omega_X; D(\mathcal{A}_\Xi)) \cap L^2(\Omega_\Xi; D(\mathcal{A}_X))$ *is included in* $D(\mathcal{A}_{X,\Xi})$, *and*

$$\mathcal{A}_{X,\Xi}\psi = \mathcal{A}_\Xi\psi + \mathcal{A}_X\psi \qquad \text{if } \psi \in L^2(\Omega_X; D(\mathcal{A}_\Xi)) \cap L^2(\Omega_\Xi; D(\mathcal{A}_X)).$$

(iv) The family of operators $(S_k^*(t))_{t\geq 0}$, *where* $S_k^*(t) = \mathrm{e}^{t\mathcal{A}_{k,X}^*}\,\mathrm{e}^{t\mathcal{A}_{k,\Xi}^*}$ *is a strongly continuous exponentially stable semigroup on* $L^2(\mathcal{O})$. *Its infinitesimal generator* $(\mathcal{A}_{k,X,\Xi}^*, D(\mathcal{A}_{k,X,\Xi}^*))$ *satisfies* $H \subset D(\mathcal{A}_{k,X,\Xi}^*)$ *and*

$$\mathcal{A}_{k,X,\Xi}^*\psi = \mathcal{A}_{k,\Xi}^*\psi + \mathcal{A}_{k,X}^*\psi \qquad \text{if } \psi \in L^2(\Omega_X; D(\mathcal{A}_\Xi^*)) \cap L^2(\Omega_\Xi; D(\mathcal{A}_X^*)).$$

Proof. The first, the second and the fourth statements are obvious. The third one can be proved as above, when we have shown that $H \subset D(\mathcal{A}_{X,\Xi}^*)$. □

We make the same kind of abuse of notation as above: we shall often write $\mathcal{A}_\Xi\psi + \mathcal{A}_X\psi$ in place of $\mathcal{A}_{X,\Xi}\psi$, $\mathcal{A}_{k,\Xi}^*\psi + \mathcal{A}_{k,X}^*\psi$ in place of $\mathcal{A}_{k,X,\Xi}^*\psi$, $\mathrm{e}^{t(\mathcal{A}_X+\mathcal{A}_\Xi)}$ in place of $\mathrm{e}^{t\mathcal{A}_X}\,\mathrm{e}^{t\mathcal{A}_\Xi}$ or of $\mathrm{e}^{t\mathcal{A}_{X,\Xi}}$, and $\mathrm{e}^{t(\mathcal{A}_{k,X}^*+\mathcal{A}_{k,\Xi}^*)}$ in place of $\mathrm{e}^{t\mathcal{A}_{k,X}^*}\,\mathrm{e}^{t\mathcal{A}_{k,\Xi}^*}$ or of $\mathrm{e}^{t\mathcal{A}_{k,X,\Xi}^*}$.

Since $L_s^2(\mathcal{O})$ is a closed subspace in $L^2(\mathcal{O})$, we can show that $\mathcal{A}_{X,\Xi}^{s\ *}$, the restriction of $\mathcal{A}_{X,\Xi}^*$ to $L_s^2(\mathcal{O})$, is an unbounded operator in $L_s^2(\mathcal{O})$ whose domain is defined by $D(\mathcal{A}_{X,\Xi}^{s\ *}) = D(\mathcal{A}_{X,\Xi}^*) \cap L_s^2(\mathcal{O})$.

Theorem 5.2. *The operator* $(\mathcal{A}_{X,\Xi}^{s\ *}, D(\mathcal{A}_{X,\Xi}^{s\ *}))$ *is the infinitesimal generator of an exponentially stable semigroup on* $L_s^2(\mathcal{O})$.

We denote by $L_+^2(\mathcal{O})$ the cone in $L_s^2(\mathcal{O})$ of functions π satisfying:

$$\int_{\mathcal{O}} \pi\, z \otimes z \geq 0 \qquad \text{for all } z \in L^2(\Omega).$$

Let us notice that if $f \in L^2(\Omega)$ and $f \geq 0$, then $f \otimes f$ belongs to $L_+^2(\mathcal{O})$. If $\pi_1 \in L_s^2(\mathcal{O})$ and $\pi_2 \in L_s^2(\mathcal{O})$, we shall write $\pi_1 \geq \pi_2$ if

$$\int_{\mathcal{O}} (\pi_1 - \pi_2)\, z \otimes z \geq 0 \qquad \text{for all } z \in L^2(\Omega).$$

We are going to prove that the optimal pair $(\bar{u}, \bar{z})$ obeys the feedback law

$$\bar{u}(t) = 1\!\!1_\gamma b(s,0) \int_\Omega \pi(s,0,\Xi)\bar{z}(t,\Xi)\,\mathrm{d}\Xi, \tag{5.11}$$

where π is solution to the algebraic Riccati equation (5.7).

5.3. Lyapunov equation

To prove the existence of a solution to system (5.9), we study the following differential Lyapunov equation:

$$\pi' = \mathcal{A}_X^*\pi + \mathcal{A}_\Xi^*\pi + \psi(t,X,\Xi) \quad \text{in } (0,\infty), \qquad \pi(0,\cdot) = \pi_0. \tag{5.12}$$

Weak solutions to equation (5.12) are defined as weak solutions for evolution equations.

Theorem 5.3. *Let ψ be in $L^1_{\rm loc}([0,\infty);L^2_s(\mathcal{O}))$ and $\pi_0\in L^2_s(\mathcal{O})$. The system (5.12) admits a unique weak solution π in $L^1_{\rm loc}([0,\infty);L^2_s(\mathcal{O}))$ defined by*

$$\pi(t)=\mathrm{e}^{t(\mathcal{A}^*_X+\mathcal{A}^*_\Xi)}\pi_0+\int_0^t \mathrm{e}^{(t-\tau)(\mathcal{A}^*_X+\mathcal{A}^*_\Xi)}\psi(\tau)\,\mathrm{d}\tau.$$

(i) If ψ belongs to $L^1(0,\infty;L^2_s(\mathcal{O}))$, then

$$\|\pi\|_{L^1(0,\infty;L^2_s(\mathcal{O}))}+\|\pi\|_{L^\infty(0,\infty;L^2_s(\mathcal{O}))}\le C\big(\|\pi_0\|_{L^2_s(\mathcal{O})}+\|\psi\|_{L^1(0,\infty;L^2_s(\mathcal{O}))}\big).$$

(ii) If ψ belongs to $L^\infty(0,\infty;L^2_s(\mathcal{O}))$, then

$$\|\pi\|_{L^\infty(0,\infty;L^2_s(\mathcal{O}))}\le C\big(\|\pi_0\|_{L^2_s(\mathcal{O})}+\|\psi\|_{L^\infty(0,\infty;L^2_s(\mathcal{O}))}\big).$$

(iii) If in addition π_0 belongs to $L^2_+(\mathcal{O})$ and $\psi\in L^1_{\rm loc}([0,\infty);L^2_+(\mathcal{O}))$, then π in $L^1_{\rm loc}([0,\infty);L^2_+(\mathcal{O}))$.

Proof. The first statement follows from Theorem 5.2. Assertions (i) and (ii) follows from Young inequality for convolutions, and from the exponential stability of the semigroup $(\mathrm{e}^{t(\mathcal{A}^*_X+\mathcal{A}^*_\Xi)})_{t\ge0}$ on $L^2_s(\mathcal{O})$. To prove the third assertion, we observe that

$$\int_{\mathcal{O}}\left(\mathrm{e}^{t(\mathcal{A}^*_X+\mathcal{A}^*_\Xi)}\pi_0\right)\,z\otimes z=\int_{\mathcal{O}}\pi_0\,\left(\mathrm{e}^{t(\mathcal{A}_X+\mathcal{A}_\Xi)}z\otimes z\right)=\int_{\mathcal{O}}\pi_0\,\mathrm{e}^{t\mathcal{A}}z\otimes\mathrm{e}^{t\mathcal{A}}z\ge0.$$

The same kind of calculation can be made for the term

$$\int_{\mathcal{O}}\left(\int_0^t \mathrm{e}^{(t-\tau)(\mathcal{A}^*_X+\mathcal{A}^*_\Xi)}\psi(\tau)\,\mathrm{d}\tau\right)\,z\otimes z.$$

The proof is complete. □

Let $k>0$ be the constant in Lemma 2.1, then π is a weak solution of equation (5.12) if and only if the function

$$\hat\pi(t,X,\Xi)=\mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\pi(t,X,\Xi)\tag{5.13}$$

is the solution of equation

$$\hat\pi'=\mathcal{A}^*_{k,X}\hat\pi+\mathcal{A}^*_{k,\Xi}\hat\pi+\mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\psi(t,X,\Xi)\quad\text{in }(0,\infty),\qquad \hat\pi(0,\cdot)=\mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\pi_0.\tag{5.14}$$

Lemma 5.4. *If $\psi(t,\cdot)=z(t,\cdot)\otimes\zeta(t,\cdot)$, with $z\in L^2(0,T;D(\mathcal{A}^*_X))$, $\zeta\in L^2(0,T;D(\mathcal{A}^*_\Xi))$, and $\pi_0=z_0\otimes\zeta_0$, with $z_0\in D(\mathcal{A}^*)$ and $\zeta_0\in D(\mathcal{A}^*)$, then the solution $\hat\pi$ of equation (5.14) belongs to $W^{1,1}(0,T;L^2(\mathcal{O}))\cap L^\infty(0,T;L^2(\Omega_\Xi;D(\mathcal{A}^*_X)))\cap L^\infty(0,T;L^2(\Omega_X;D(\mathcal{A}^*_\Xi)))$.*

Proof. We have

$$\hat\pi(t)=\mathrm{e}^{t\mathcal{A}^*_{k,X}}\mathrm{e}^{-kx}z_0\otimes\mathrm{e}^{t\mathcal{A}^*_{k,\Xi}}\mathrm{e}^{-k\xi}\zeta_0+\int_0^t\mathrm{e}^{(t-\tau)\mathcal{A}^*_{k,X}}\mathrm{e}^{-kx}z(\tau)\otimes\mathrm{e}^{(t-\tau)\mathcal{A}^*_{k,\Xi}}\mathrm{e}^{-k\xi}\zeta(\tau)\,\mathrm{d}\tau,$$

which gives

$$\mathcal{A}^*_{k,X}\hat\pi(t)=\mathrm{e}^{t\mathcal{A}^*_{k,X}}\mathcal{A}^*_{k,X}\mathrm{e}^{-kx}z_0\otimes\mathrm{e}^{t\mathcal{A}^*_{k,\Xi}}\mathrm{e}^{-k\xi}\zeta_0+\int_0^t\mathrm{e}^{(t-\tau)\mathcal{A}^*_{k,X}}\mathcal{A}^*_{k,X}\mathrm{e}^{-kx}z(\tau)\otimes\mathrm{e}^{(t-\tau)\mathcal{A}^*_{k,\Xi}}\mathrm{e}^{-k\xi}\zeta(\tau)\,\mathrm{d}\tau,$$

and

$$\mathcal{A}^*_{k,\Xi}\hat\pi(t)=\mathrm{e}^{t\mathcal{A}^*_{k,X}}\mathrm{e}^{-kx}z_0\otimes\mathrm{e}^{t\mathcal{A}^*_{k,\Xi}}\mathcal{A}^*_{k,\Xi}\mathrm{e}^{-k\xi}\zeta_0+\int_0^t\mathrm{e}^{(t-\tau)\mathcal{A}^*_{k,X}}\mathrm{e}^{-kx}z(\tau)\otimes\mathrm{e}^{(t-\tau)\mathcal{A}^*_{k,\Xi}}\mathcal{A}^*_{k,\Xi}\mathrm{e}^{-k\xi}\zeta(\tau)\,\mathrm{d}\tau.$$

Thus $\hat{\pi} \in L^\infty(0,T;L^2(\Omega_\Xi;D(\mathcal{A}_X^*))) \cap L^\infty(0,T;L^2(\Omega_X;D(\mathcal{A}_\Xi^*)))$. Due to (5.10), we have

$$\hat{\pi}' = \mathcal{A}_{k,X,\Xi}^*\hat{\pi} + \psi = \mathcal{A}_{k,X}^*\hat{\pi} + \mathcal{A}_{k,\Xi}^*\hat{\pi} + \psi \in L^1(0,T;L^2(\mathcal{O})).$$

Since $\mathcal{A}_{k,X}^*\hat{\pi} \in L^2(0,T;L^2(\mathcal{O}))$, $\mathcal{A}_{k,\Xi}^*\hat{\pi} \in L^2(0,T;L^2(\mathcal{O}))$, and $\psi \in L^1(0,T;L^2(\mathcal{O}))$, we have $\hat{\pi} \in W^{1,1}(0,T;L^2(\mathcal{O}))$ and the proof is complete. □

Theorem 5.4. *The weak solution π of system (5.12) satisfies the estimate*

$$2\|\pi(t)\|^2_{L^2_s(\mathcal{O})} + \|\pi\|^2_{L^2(0,t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \le C_6\left(\left|\int_0^t\int_{\mathcal{O}} \pi\,\psi\,\mathrm{d}X\mathrm{d}\Xi\mathrm{d}\tau\right| + \|\pi_0\|^2_{L^2_s(\mathcal{O})}\right), \tag{5.15}$$

for all $t \in [0,\infty)$ (for some $C_6 > 0$).

Observe that estimate (5.15) is more precise than estimate (i) in Theorem 5.3. It is needed in the proof of Theorem 5.5.

Proof. Let $k > 0$ be the parameter in Lemma 2.1. Let π be the solution of system (5.12). First assume that $\psi(t) = z(t) \otimes \zeta(t)$, with $z \in L^2(0,T;D(\mathcal{A}_X^*))$, $\zeta \in L^2(0,T;D(\mathcal{A}_\Xi^*))$, and $\pi_0 = z_0 \otimes \zeta_0$, with $z_0 \in D(\mathcal{A}_X^*)$ and $\zeta_0 \in D(\mathcal{A}_\Xi^*)$. Let us set $\hat{\pi}(t,X,\Xi) = \mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\pi(t,X,\Xi)$. It is clear that $\hat{\pi}$ is the solution of system (5.14). We can apply Lemma 5.4, and we can rewrite equation (5.14) in the form

$$\hat{\pi}' = \mathcal{A}_{k,X}^*\hat{\pi} + \Psi, \qquad \hat{\pi}(0) = \mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\pi_0 = \hat{\pi}_0,$$

with $\Psi = \mathcal{A}_{k,\Xi}^*\hat{\pi} + \mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\psi$. This equation is considered as an evolution equation in $L^2(\Omega_X)$, the variable Ξ being considered as a parameter. Thus applying [6], Theorem 6.2, we can write:

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{O}} \hat{\pi}(t)^2\,\mathrm{d}X\,\mathrm{d}\Xi - \frac{1}{2}\int_{\mathcal{O}} \hat{\pi}_0^2\,\mathrm{d}X\,\mathrm{d}\Xi + \frac{1}{2}\int_0^t\int_{\Omega_\Xi}\int_0^1 a(y)\,\hat{\pi}(\tau,L,y,\Xi)^2\,\mathrm{d}y\,\mathrm{d}\Xi\,\mathrm{d}\tau&\\ + \int_0^t\int_{\mathcal{O}}\left(b(X)\left|\frac{\partial\hat{\pi}}{\partial y}\right|^2 + \frac{\partial b}{\partial y}\frac{\partial\hat{\pi}}{\partial y}\hat{\pi} + (c+ka)(X)\hat{\pi}^2\right)\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}\tau&\\ \le \int_0^t\int_{\mathcal{O}} \Psi\,\hat{\pi}\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}\tau \le \int_0^t\int_{\mathcal{O}} \mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\psi\,\hat{\pi}\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}\tau,&\end{aligned} \tag{5.16}$$

for all $t > 0$. Since $\mathcal{A}_{k,\Xi}^*$ is dissipative (see [6]) and $\hat{\pi} \in L^\infty(0,T;L^2(\Omega_X;D(\mathcal{A}_\Xi^*)))$, we have

$$\int_0^t\int_{\mathcal{O}} \mathcal{A}_{k,\Xi}^*\hat{\pi}\,\hat{\pi}\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}\tau \le 0.$$

This explains the last inequality in (5.16). In a similar way, we can prove that $\hat{\pi}$ satisfies the inequality

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{O}} \hat{\pi}(t)^2\,\mathrm{d}X\,\mathrm{d}\Xi - \frac{1}{2}\int_{\mathcal{O}} \hat{\pi}_0^2\,\mathrm{d}X\,\mathrm{d}\Xi + \frac{1}{2}\int_0^t\int_{\Omega_X}\int_0^1 a(\eta)\,\hat{\pi}(\tau,X,L,\eta)^2\,\mathrm{d}\eta\,\mathrm{d}X\,\mathrm{d}\tau&\\ + \int_0^t\int_{\mathcal{O}}\left(b(\Xi)\left|\frac{\partial\hat{\pi}}{\partial \eta}\right|^2 + \frac{\partial b}{\partial \eta}\frac{\partial\hat{\pi}}{\partial \eta}\hat{\pi} + (c+ka)(\Xi)\hat{\pi}^2\right)\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}\tau \le \int_0^t\int_{\mathcal{O}} \mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\psi\,\hat{\pi}\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}\tau,&\end{aligned} \tag{5.17}$$

for all $t > 0$. Thus, we have

$$
\begin{aligned}
&\int_{\mathcal{O}} \hat{\pi}(t)^2 \,\mathrm{d}X \,\mathrm{d}\Xi - \int_{\mathcal{O}} \hat{\pi}_0^2 \,\mathrm{d}X \,\mathrm{d}\Xi + \frac{1}{2}\int_0^t \int_{\Omega_\Xi} \int_0^1 a(y)\, \hat{\pi}(\tau, L, y, \Xi)^2 \,\mathrm{d}y \,\mathrm{d}\Xi \,\mathrm{d}\tau \\
&\quad + \frac{1}{2}\int_0^t \int_{\Omega_X} \int_0^1 a(\eta)\, \hat{\pi}(\tau, X, L, \eta)^2 \,\mathrm{d}\eta \,\mathrm{d}X \,\mathrm{d}\tau \\
&\quad + \int_0^t \int_{\mathcal{O}} \left(b(X) \left|\frac{\partial \hat{\pi}}{\partial y}\right|^2 + \frac{\partial b}{\partial y}\frac{\partial \hat{\pi}}{\partial y}\hat{\pi} + (c+ka)(X)\hat{\pi}^2 \right) \mathrm{d}X \,\mathrm{d}\Xi \,\mathrm{d}\tau \\
&\quad + \int_0^t \int_{\mathcal{O}} \left(b(\Xi) \left|\frac{\partial \hat{\pi}}{\partial \eta}\right|^2 + \frac{\partial b}{\partial \eta}\frac{\partial \hat{\pi}}{\partial \eta}\hat{\pi} + (c+ka)(\Xi)\hat{\pi}^2 \right) \mathrm{d}X \,\mathrm{d}\Xi \,\mathrm{d}\tau \\
&\quad \le 2\int_0^t \int_{\mathcal{O}} \mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\psi\, \hat{\pi} \,\mathrm{d}X \,\mathrm{d}\Xi \,\mathrm{d}\tau,
\end{aligned}
\tag{5.18}
$$

for all $t > 0$. With Lemma 2.1, we obtain

$$
\begin{aligned}
&\|\hat{\pi}(t)\|^2_{L^2_s(\mathcal{O})} - \|\hat{\pi}_0\|^2_{L^2_s(\mathcal{O})} + \frac{C_1}{2}\|\hat{\pi}\|^2_{L^2(0,t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \\
&\quad + \frac{C_1}{2}\|\hat{\pi}\|^2_{L^2(0,t;L^2(\Omega_\Xi;L^2(0,L;H^1(0,1;d))))} \le 2\int_0^t \int_{\mathcal{O}} \mathrm{e}^{-kx}\mathrm{e}^{-k\xi}\psi\, \hat{\pi} \,\mathrm{d}X \,\mathrm{d}\Xi \,\mathrm{d}\tau,
\end{aligned}
\tag{5.19}
$$

for all $t > 0$. By a density argument, we can show that this inequality also holds if $\psi(t) = z(t) \otimes \zeta(t)$, with $z \in L^2(0, T; L^2(\Omega))$, $\zeta \in L^2(0, T; L^2(\Omega))$, and $\pi_0 = z_0 \otimes \zeta_0$, with $z_0 \in L^2(\Omega)$ and $\zeta_0 \in L^2(\Omega)$. Finally, still with a density argument we can establish inequality (5.19) for all $\psi \in L^1(0, T; L^2_s(\mathcal{O}))$ and all $\pi_0 \in L^2_s(\mathcal{O})$. The theorem clearly follows from (5.19) and (5.13). $\square$

5.4. **Differential Riccati equation**

Now, we define weak solutions to equation (5.9).

Definition 5.1. A function $\pi \in L^2(0, T; L^2_s(\mathcal{O})) \cap L^2(0, T; L^2(\Omega_X; L^2(0, L; H^1(0, 1; d))))$ is a weak solution to equation (5.9) if it is a weak solution of system (5.12) in $(0, T)$ with

$$
\psi(t, X, \Xi) = -\int_\gamma |b(s, 0)|^2 \pi(t, s, 0, \Xi)\pi(t, X, s, 0)\,\mathrm{d}s + \mathbf{\Phi}(X, \Xi),
$$

where $\mathbf{\Phi}$ is defined in (5.8).

Theorem 5.5. *Let π_0 be in $L^2_s(\mathcal{O})$. There exists $\bar{t} > 0$, depending on $\|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})}$ and $\|\pi_0\|_{L^2_s(\mathcal{O})}$, such that system (5.9) admits a unique weak solution π that belongs to the space*

$$
L^2(0, \bar{t}; L^2(\Omega_X; L^2(0, L; H^1(0, 1; d)))) \cap C([0, \bar{t}]; L^2_s(\mathcal{O})).
$$

Proof. Let $M > 0$ be a constant such that $\|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})} \le M$ and $\|\pi_0\|_{L^2_s(\mathcal{O})} \le M^2/(2C_6)^{1/2}$. Let $\bar{t}$ be the constant defined by

$$
\max\left(9M^4 C_\gamma^2 C_I^2 \|b\|^2_\infty |\bar{t}|^{\frac{2-\varepsilon}{4-\varepsilon}} + C_6 \bar{t} M^2, \frac{3+3\sqrt{2}}{\sqrt{2}} C_6 C_I^2 C_\gamma^2 M^2 \,\|b\|^2_\infty |\bar{t}|^{\frac{2-\varepsilon}{4-\varepsilon}} \right) = \min\left(\frac{2\sqrt{2}-1}{1+\sqrt{2}} M^2, \frac{1}{2} \right),
$$

where C_I and C_γ are the constants appearing in (5.21) and (5.22). Let us set

$$E_M = \Big\{\pi \in C([0,\bar t]; L^2_s(\mathcal{O})) \cap L^2(0,\bar t; L^2(\Omega_X; L^2(0,L; H^1(0,1;d)))),\\ \|\pi\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))} + \|\pi\|_{L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \le 3M^2\Big\}.$$

Equipped with the metric corresponding to the norm:

$$\|\cdot\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))} + \|\cdot\|_{L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))},$$

E_M is a complete metric space. Let v be in E_M, then

$$\psi(t,X,\Xi) = -\int_\gamma |b(s,0)|^2 v(t,s,0,\Xi) v(t,X,s,0)\,\mathrm{d}s + \mathbf{\Phi}(X,\Xi)$$

belongs to $L^1(0,\bar t; L^2_s(\mathcal{O}))$. Due to Theorem 5.3, the equation

$$\begin{cases} \pi' = \mathcal{A}_X^*\pi + \mathcal{A}_\Xi^*\pi - \displaystyle\int_\gamma |b(s,0)|^2 v(t,X,s,0)\, v(t,s,0,\Xi)\,\mathrm{d}s + \mathbf{\Phi}(X,\Xi) \quad \text{in } (0,T), \\ \pi(0,\cdot) = \pi_0, \end{cases} \tag{5.20}$$

admits a unique weak solution π_v in $L^\infty(0,\bar t; L^2_s(\mathcal{O}))$. Due to Theorem 5.4, this solution also belongs to $L^2(0,\bar t; L^2(\Omega_X; L^2(0,L; H^1(0,1;d))))$. Let us show that the mapping $\Psi : v \mapsto \pi_v$ is a contraction in E_M. The proof is divided into two steps.

Step 1. Let us show that Ψ is a mapping from E_M into E_M. With Theorem 5.4, we can write

$$2\|\pi_v(t)\|^2_{L^2_s(\mathcal{O})} + \|\pi_v\|^2_{L^2(0,t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \le C_6 \left|\int_0^t \int_{\mathcal{O}} \pi_v(\tau,X,\Xi)\mathbf{\Phi}(X,\Xi)\,\mathrm{d}X\mathrm{d}\Xi\,\mathrm{d}\tau\right| \\ + C_6\|\pi_0\|^2_{L^2_s(\mathcal{O})} + C_6 \left|\int_0^t \int_{\mathcal{O}} \pi_v \left[\int_\gamma |b(s,0)|^2 v(\tau,X,s,0) v(\tau,s,0,\Xi)\,\mathrm{d}s\right] \mathrm{d}X\mathrm{d}\Xi\,\mathrm{d}\tau\right|,$$

for all $t \in [0,\bar t]$. With Hölder's inequality, and due to assumptions on $\mathbf{\Phi}$ and π_0, we have $C_6\|\pi_0\|^2_{L^2_s(\mathcal{O})} \le M^4/2$ and

$$C_6 \left|\int_0^t \int_{\mathcal{O}} \pi_v(\tau,X,\Xi)\mathbf{\Phi}(X,\Xi)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}\tau\right| \le C_6\bar t\|\pi_v\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))}\|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})} \\ \le C_6\bar t\|\pi_v\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))} M^2 \le \frac{1}{2}C_6^2\bar t^2 M^4 + \frac{1}{2}\|\pi_v\|^2_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))}.$$

With a trace theorem we have

$$\int_\gamma |b(s,0)|^2 |v(\tau,X,s,0)|\,|v(\tau,s,0,\Xi)|\,\mathrm{d}s \le C_\gamma^2\|b\|^2_\infty \|v(\tau,\cdot,\Xi)\|_{L^2(0,L;H^{1/2+\varepsilon'}(0,1;d))}\|v(\tau,X,\cdot)\|_{L^2(0,L;H^{1/2+\varepsilon'}(0,1;d))}, \tag{5.21}$$

for all $\varepsilon' > 0$. (The constant C_γ depends on $\varepsilon' > 0$.) Thus we can write

$$\left\|\int_\gamma |b(s,0)|^2 |v(\cdot,\cdot,s,0,\cdot)|\,|v(\cdot,s,0,\cdot)|\,\mathrm{d}s\right\|_{L^1(0,\bar t;L^2_s(\mathcal{O}))} \le C_\gamma^2\|b\|^2_\infty\|v\|^2_{L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^{1/2+\varepsilon'}(0,1;d))))}.$$

With the interpolation identity

$$\big[L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d)))),L^\infty(0,\bar t;L^2(\Omega_X;L^2(0,L;L^2(0,1))))\big]_{(2-\varepsilon)/(4-\varepsilon)}$$
$$= L^{4-\varepsilon}(0,\bar t;L^2(\Omega_X;L^2(0,L;H^{1/2+\varepsilon/(8-2\varepsilon)}(0,1;d)))),\qquad 0<\varepsilon<1,$$

we have

$$\|\cdot\|_{L^{4-\varepsilon}(0,\bar t;L^2(\Omega_X;L^2(0,L;H^{1/2+\varepsilon/(8-2\varepsilon)}(0,1;d))))}\le C_I\|\cdot\|^{2/(4-\varepsilon)}_{L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))}\|\cdot\|^{(2-\varepsilon)/(4-\varepsilon)}_{L^\infty(0,\bar t;L^2(\Omega_X;L^2(0,L;L^2(0,1))))}. \tag{5.22}$$

Setting $\varepsilon'=\varepsilon/(8-2\varepsilon)$, from Hölder's inequality it follows that

$$\|v\|_{L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^{1/2+\varepsilon'}(0,1;d))))}\le|\bar t|^{\frac{2-\varepsilon}{2(4-\varepsilon)}}\|v\|_{L^{4-\varepsilon}(0,\bar t;L^2(\Omega_X;L^2(0,L;H^{1/2+\varepsilon/(8-2\varepsilon)}(0,1;d))))}.$$

Thus, we obtain

$$\left\|\int_\gamma|b(s,0)|^2|v(\cdot,\cdot,s,0,\cdot)|\,|v(\cdot,s,0,\cdot)|\,\mathrm{d}s\right\|_{L^1(0,\bar t;L^2_s(\mathcal{O}))}\le C_\gamma^2\|b\|^2_\infty C_I^2|\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}}\|v\|^{4/(4-\varepsilon)}_{L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))}$$
$$\times\|v\|^{(4-2\varepsilon)/(4-\varepsilon)}_{L^\infty(0,\bar t;L^2(\Omega_X;L^2(0,L;L^2(0,1))))}\le 9M^4\,C_\gamma^2\|b\|^2_\infty C_I^2|\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}}.$$

From the previous inequality, it yields

$$C_6\left|\int_0^t\int_{\mathcal{O}}\pi_v\left[\int_\gamma|b(s,0)|^2v(\tau,X,s,0)v(\tau,s,0,\Xi)\,\mathrm{d}s\right]\mathrm{d}X\mathrm{d}\Xi\,\mathrm{d}\tau\right|$$
$$\le C_6\left\|\int_\gamma|b(s,0)|^2v(s,0,\cdot)v(s,0,\cdot)\,\mathrm{d}s\right\|_{L^1(0,\bar t;L^2_s(\mathcal{O}))}\|\pi_v\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))}$$
$$\le\ 9M^4\,C_\gamma^2\|b\|^2_\infty C_I^2C_6|\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}}\|\pi_v\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))}\le\ \frac12 81M^8\,C_\gamma^4\|b\|^4_\infty C_I^4C_6^2|\bar t|^{\frac{4-2\varepsilon}{4-\varepsilon}}+\frac12\|\pi_v\|^2_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))}. \tag{5.23}$$

Collecting together the previous estimates we arrive at

$$2\|\pi_v(t)\|^2_{L^2_s(\mathcal{O})}+\|\pi_v\|^2_{L^2(0,t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))}\le\frac12C_6^2\bar t^2M^4+\frac{M^4}{2}$$
$$+\frac12 81M^8\,C_\gamma^4\|b\|^4_\infty C_I^4C_6^2|\bar t|^{\frac{4-2\varepsilon}{4-\varepsilon}}+\|\pi_v\|^2_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))}.$$

Therefore we have

$$\|\pi\|^2_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))}\le\left(\frac12C_6^2\bar t^2M^4+\frac{M^4}{2}+\frac12 81M^8\,C_\gamma^4\|b\|^4_\infty C_I^4C_6^2|\bar t|^{\frac{4-2\varepsilon}{4-\varepsilon}}\right),$$

$$\|\pi\|^2_{L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))}\le\left(C_6^2\bar t^2M^4+M^4+81M^8\,C_\gamma^4\|b\|^4_\infty C_I^4C_6^2|\bar t|^{\frac{4-2\varepsilon}{4-\varepsilon}}\right),$$

and

$$\|\pi\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))} + \|\pi\|_{L^2(0,\bar t;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \le \frac{1+\sqrt{2}}{\sqrt{2}}\left(C_6^2\bar t^2 M^4 + M^4 + 81M^8\, C_\gamma^4\|b\|_\infty^4 C_I^4 C_6^2|\bar t|^{\frac{4-2\varepsilon}{4-\varepsilon}}\right)^{1/2}$$

$$\le \frac{1+\sqrt{2}}{\sqrt{2}}\left(C_6\bar t M^2 + M^2 + 9M^4\, C_\gamma^2\|b\|_\infty^2 C_I^2 C_6|\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}}\right) \le 3M^2,$$

provided that $\bar t$ obey the condition:

$$\left(9M^4 C_\gamma^2 C_I^2\|b\|_\infty^2|\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}} + C_6\bar t M^2\right) \le \frac{2\sqrt{2}-1}{1+\sqrt{2}}M^2.$$

Thus we have proved that π_v belongs to E_M.

Step 2. Let π_1 and π_2 be two solutions to system (5.20) respectively associated with $v_1 \in E_M$ and $v_2 \in E_M$. The function $(\pi_1 - \pi_2)$ is the solution of

$$\pi_1' - \pi_2' = \mathcal{A}_X^*(\pi_1-\pi_2) + \mathcal{A}_\Xi^*(\pi_1-\pi_2) + \psi \quad \text{in } (0,\bar t), \quad (\pi_1-\pi_2)(0) = 0, \tag{5.24}$$

where

$$\begin{aligned}\psi(t,X,\Xi) \;&=\; -\int_\gamma |b(s,0)|^2 v_1(t,s,0,\Xi)\big(v_1(t,X,s,0) - v_2(t,X,s,0)\big)\,\mathrm{d}s\\ &\quad + \int_\gamma |b(s,0)|^2\big(v_2(t,s,0,\Xi) - v_1(t,s,0,\Xi)\big)v_2(t,X,s,0)\,\mathrm{d}s.\end{aligned}$$

With the same estimates as in Step 1, we obtain

$$\begin{aligned}&\left\|\int_\gamma b^2 v_1\,(v_1-v_2)\,\mathrm{d}s\right\|_{L^1(0,\bar t;L^2(\mathcal{O}))} + \left\|\int_\gamma b^2(v_2-v_1)v_2\,\mathrm{d}s\right\|_{L^1(0,\bar t;L^2(\mathcal{O}))}\\ &\quad\le 3C_I^2C_\gamma^2M^2|\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}}\|b\|_\infty^2\big(\|v_1-v_2\|_{L^\infty(0,\bar t;L^2(\mathcal{O}))} + \|v_1-v_2\|_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))}\big).\end{aligned}$$

With Cauchy-Schwarz inequality and with Theorem 5.4, we get

$$\begin{aligned}&2\|(\pi_1-\pi_2)(t)\|^2_{L^2_s(\mathcal{O})} + \|\pi_1-\pi_2\|^2_{L^2(0,t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))} \le \frac{1}{4}9C_6^2C_I^4C_\gamma^4M^4\|b\|_\infty^4|\bar t|^{\frac{4-2\varepsilon}{4-\varepsilon}}\big(\|v_1-v_2\|_{L^\infty(0,\bar t;L^2(\mathcal{O}))}\\ &\qquad + \|v_1-v_2\|_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))}\big)^2 + \|\pi_1-\pi_2\|^2_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))},\end{aligned}$$

for all $t\in[0,\bar t]$. Thus, we have

$$\begin{aligned}\|\pi_1-\pi_2\|^2_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))} \;&\le\; \frac{1}{4}9C_6^2C_I^4C_\gamma^4M^4\|b\|_\infty^4|\bar t|^{\frac{4-2\varepsilon}{4-\varepsilon}}\big(\|v_1-v_2\|_{L^\infty(0,\bar t;L^2(\mathcal{O}))}\\ &\quad + \|v_1-v_2\|_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))}\big)^2,\\ \|\pi_1-\pi_2\|^2_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))} \;&\le\; \frac{1}{2}9C_6^2C_I^4C_\gamma^4M^4\|b\|_\infty^4|\bar t|^{\frac{4-2\varepsilon}{4-\varepsilon}}\big(\|v_1-v_2\|_{L^\infty(0,\bar t;L^2(\mathcal{O}))}\\ &\quad + \|v_1-v_2\|_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))}\big)^2,\end{aligned}$$

and

$$\|\pi_1 - \pi_2\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))} + \|\pi_1 - \pi_2\|_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))}$$
$$\leq \frac{3+3\sqrt{2}}{2} C_6 C_I^2 C_\gamma^2 M^2 \|b\|_\infty^2 |\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}} \left(\|v_1 - v_2\|_{L^\infty(0,\bar t;L^2(\mathcal{O}))} + \|v_1 - v_2\|_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))}\right).$$

By definition of $\bar t$, we have

$$\frac{3+3\sqrt{2}}{2} C_6 C_I^2 C_\gamma^2 M^2 \, \|b\|_\infty^2 |\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}} \leq \frac{3+3\sqrt{2}}{\sqrt{2}} C_6 C_I^2 C_\gamma^2 M^2 \, \|b\|_\infty^2 |\bar t|^{\frac{2-\varepsilon}{4-\varepsilon}} \leq \frac{1}{2},$$

therefore, it yields

$$\begin{aligned}\|\pi_1 - \pi_2\|_{L^\infty(0,\bar t;L^2_s(\mathcal{O}))} + \|\pi_1 - \pi_2\|_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))} \quad \leq \quad & \frac{1}{2}\big(\|v_1 - v_2\|_{L^\infty(0,\bar t;L^2(\mathcal{O}))} \\ & + \|v_1 - v_2\|_{L^2(0,\bar t;L^2(\Omega_X\times(0,L);H^1(0,1;d)))}\big).\end{aligned}$$

Thus the mapping $\Psi : v \mapsto \pi_v$ is a contraction in the complete metric space E_M, and equation (5.24) admits a unique weak solution π in E_M. □

Theorem 5.6. *In addition to assumptions in Theorem 5.5 we assume that π_0 in $L^2_+(\mathcal{O})$. Then the solution π of equation (5.9) belongs to $C([0,\bar t]; L^2_+(\mathcal{O}))$.*

To prove this theorem, we have to establish different lemmas.

Lemma 5.5. *Let τ be in $[0,\bar t)$, and $u \in C^1([\tau,\bar t]; L^2(0,L))$. There exists a sequence $(f_n)_n$ in $C^1([\tau,\bar t]; L^2(\Omega))$ such that*

$$\left|\int_\tau^{\bar t}\int_\Omega b\, f_n\, \varphi - \int_\tau^{\bar t}\int_\gamma b\, u\, \varphi\right| \leq \frac{C}{n^{1/2}} \|\varphi\|_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))} \|u\|_{L^2(\tau,\bar t;L^2(\gamma))},$$

for all $\varphi \in L^2(\tau,\bar t; L^2(0,L;H^1(0,1;d)))$.

Proof. Let $\theta \in C^2_c([0,1))$ be such that $0 \leq \theta$ and $\int_0^1 \theta(y)\mathrm{d}y = 1$. Let us set

$$f_n(t,x,y) = n\theta(ny)u(t,x)\mathbb{1}_\gamma(x).$$

For $n \geq 2$, we have

$$\begin{aligned}\left|\int_\tau^{\bar t}\int_\Omega b\, f_n\, \varphi - \int_\tau^{\bar t}\int_\gamma b(\cdot,0)\, u\, \varphi\right| &= \left|\int_\tau^{\bar t}\int_\gamma \left(u\int_0^1 n\theta(ny)\big(b(x,y)\varphi(t,x,y) - b(\cdot,0)\varphi(t,x,0)\big)\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \left|\int_\tau^{\bar t}\int_\gamma \left(|u|\int_0^{\frac{1}{n}} n\theta(ny)\left(\int_0^y \left|\frac{\partial(b\,\varphi)}{\partial y}(t,x,\zeta)\right|\mathrm{d}\zeta\right)\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \left|\int_\tau^{\bar t}\int_\gamma \left(|u|\int_0^1 \theta(\eta)\left(\int_0^{\frac{\eta}{n}} \left|\frac{\partial(b\,\varphi)}{\partial y}(t,x,\zeta)\right|\mathrm{d}\zeta\right)\mathrm{d}\eta\right)\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \left|\int_\tau^{\bar t}\int_\gamma \left(|u|\int_0^1 \left|\frac{\eta}{n}\right|^{\frac12}\theta(\eta)\left(\int_0^{\frac12} \left|\frac{\partial(b\,\varphi)}{\partial y}(t,x,\zeta)\right|^2\mathrm{d}\zeta\right)^{\frac12}\mathrm{d}\eta\right)\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq \frac{C}{n^{1/2}}\|\varphi\|_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))}\|u\|_{L^2(\tau,\bar t;L^2(\gamma))}. \qquad \square\end{aligned}$$

Lemma 5.6. *Let ψ be in $C([\tau,\bar t]; D(\mathcal{A}^{s\,*}_{X,\Xi}))$, $\pi_0 \in D(\mathcal{A}^{s\,*}_{X,\Xi})$, and π be the solution of*

$$-\pi' = \mathcal{A}_X^*\pi + \mathcal{A}_\Xi^*\pi + \psi \quad \textit{in } (\tau,\bar t), \quad \pi(\bar t) = \pi_0,$$

where $\tau \in [0,\bar t)$. Let u be in $L^2(\tau,\bar t;U)$, $z_0 \in L^2(\Omega)$, and z be the solution to equation

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u) \quad \textit{in } (\tau,\bar t), \qquad z(\tau) = z_0. \tag{5.25}$$

Then π and z obeys the following identity:

$$\begin{aligned}\int_\tau^{\bar t}\int_{\mathcal{O}} \big(\pi'(t,X,\Xi) + \mathcal{A}^*_{X,\Xi}\pi(t,X,\Xi)\big)\; z(t)\otimes z(t) &= \int_{\mathcal{O}} \pi_0\, z(\bar t)\otimes z(\bar t) - \int_{\mathcal{O}} \pi(\tau)\, z_0\otimes z_0 \\ &+ 2\int_\tau^{\bar t}\int_\gamma b(s,0)u(t,s)\int_\Omega \pi(t,s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{5.26}$$

Proof. We first prove the identity when u belongs to $C^1([\tau,\bar t];L^2(0,L))$. Let $(f_n)_n$ be the sequence in $C^1([\tau,\bar t];L^2(\Omega))$ defined in Lemma 5.5, and $(z_{0,n})_n$ be a sequence in $D(\mathcal{A})$ converging to z_0 in $L^2(\Omega)$. Let us denote by z_n the solution to

$$z' = \mathcal{A}z - bf_n, \qquad z(0) = z_{0,n}.$$

As in Lemma 6.2 we can show that the sequence $(z_n)_n$ is bounded in $L^\infty(\tau,\bar t;L^2(\Omega))$ and in $L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))$, the sequence $(\sqrt{a}z_n)_n$ is bounded in $L^\infty(0,L;L^2(\tau,\bar t;L^2(0,1)))$, and all the sequence $(z_n)_n$ converges to the solution z of equation (5.25) for the weak-star topology of $L^\infty(\tau,\bar t;L^2(\Omega))$ and the weak topology of $L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))$. Moreover, z belongs to $C([\tau,\bar t];L^2(\Omega))$, we can show that, for every $t \in (\tau,\bar t]$, $(z_n(t))_n$ converges to $z(t)$ for the weak topology of $L^2(\Omega)$. Since bf_n belongs to $C^1([\tau,\bar t];L^2(\Omega))$, we have $z_n \in C([\tau,\bar t]; D(\mathcal{A})) \cap C^1([\tau,\bar t]; L^2(\Omega))$, and $\pi \in C([\tau,\bar t]; D(\mathcal{A}^*_{X,\Xi})) \cap C^1([\tau,\bar t]; L^2_s(\mathcal{O}))$, we can write

$$\begin{aligned}
\int_\tau^{\bar t}\int_{\mathcal{O}} \big(\pi'(t,X,\Xi) + \mathcal{A}^*_{X,\Xi}\pi(t,X,\Xi)\big)\, z_n(t)\otimes z_n(t) &= \int_\tau^{\bar t}\int_{\mathcal{O}} \pi'(t)\, z_n(t)\otimes z_n(t) + \int_\tau^{\bar t}\int_{\mathcal{O}} \pi(t)\,\mathcal{A}_X z_n(t)\otimes z_n(t)\\
&\quad + \int_\tau^{\bar t}\int_{\mathcal{O}} \pi(t)\, z_n(t)\otimes \mathcal{A}_\Xi z_n(t)\\
&= \int_\tau^{\bar t}\int_{\mathcal{O}} \pi'(t,X,\Xi)\, z_n(t)\otimes z_n(t) + \int_\tau^{\bar t}\int_{\mathcal{O}} \pi(t,X,\Xi)\, z_n'(t)\otimes z_n(t)\\
&\quad + \int_\tau^{\bar t}\int_{\mathcal{O}} \pi(t,X,\Xi)\, z_n(t)\otimes z_n'(t)\\
&\quad + \int_\tau^{\bar t}\int_{\mathcal{O}} bf_n(t,X)\,\pi(t,X,\Xi) z_n(t,\Xi) + \int_\tau^{\bar t}\int_{\mathcal{O}} bf_n(t,\Xi)\,\pi(t,X,\Xi) z_n(t,X)\\
&= \int_{\mathcal{O}} \pi_0\, z_n(\bar t)\otimes z_n(\bar t) - \int_{\mathcal{O}} \pi(\tau)\, z_{0,n}\otimes z_{0,n} + 2\int_\tau^{\bar t}\int_{\mathcal{O}} bf_n(t,X)\,\pi(t,X,\Xi) z_n(t,\Xi).
\end{aligned}$$

Let us pass to the limit when n tends to infinity in the above identity. For every $t \in (\tau,\bar t]$, $(z_n(t))_n$ converges $z(t)$ for the weak topology of $L^2(\Omega)$. Thus

$$\lim_{n\to\infty}\int_{\mathcal{O}} \big(z_n(t)\otimes z_n(t)\big)\big(\varphi\otimes\zeta\big) = \int_{\mathcal{O}} \big(z(t)\otimes z(t)\big)\big(\varphi\otimes\zeta\big),$$

for all $\varphi \in L^2(\Omega_X)$, and all $\zeta \in L^2(\Omega_\Xi)$. Since $L^2(\Omega_X)\otimes L^2(\Omega_\Xi)$ is dense in $L^2(\mathcal{O})$, we obtain

$$\lim_{n\to\infty}\int_{\mathcal{O}} \big(z_n(t)\otimes z_n(t)\big)\,\varphi = \int_{\mathcal{O}} \big(z(t)\otimes z(t)\big)\,\varphi$$

for all $\varphi \in L^2(\mathcal{O})$. In particular we have

$$\lim_{n\to\infty}\int_{\mathcal{O}} \big(\pi'(t) + \mathcal{A}^*_{X,\Xi}\pi(t)\big)\, z_n(t)\otimes z_n(t) = \int_{\mathcal{O}} \big(\pi'(t) + \mathcal{A}^*_{X,\Xi}\pi(t)\big)\, z(t)\otimes z(t)$$

for almost all $t \in (\tau,\bar t)$. Moreover

$$\left|\int_{\mathcal{O}} \big(\pi'(t) + \mathcal{A}^*_{X,\Xi}\pi(t)\big)\, z_n(t)\otimes z_n(t)\right| \le \|\psi(t,\cdot)\|_{L^2_s(\mathcal{O})}\|z_n\|^2_{L^\infty(\tau,\bar t;L^2(\Omega))} \le C\|\psi(t,\cdot)\|_{L^2_s(\mathcal{O})}.$$

With the dominated convergence theorem we can write

$$\lim_{n\to\infty}\int_\tau^{\bar t}\left(\int_{\mathcal{O}} \big(\pi'(t) + \mathcal{A}^*_{X,\Xi}\pi(t)\big)\, z_n(t)\otimes z_n(t)\right)\mathrm{d}t = \int_\tau^{\bar t}\left(\int_{\mathcal{O}} \big(\pi'(t) + \mathcal{A}^*_{X,\Xi}\pi(t)\big)\, z(t)\otimes z(t)\right)\mathrm{d}t.$$

From Lemma 5.5 it follows that

$$\left| \int_\tau^{\bar t} \int_\Omega b f_n(t,X) \int_\Omega \pi(t,X,\Xi) z_n(t,\Xi) \mathrm{d}\Xi\, \mathrm{d}X \mathrm{d}t - \int_\tau^{\bar t} \int_\gamma b(s,0) u(t,s) \int_\Omega \pi(t,s,0,\Xi) z_n(t,\Xi) \mathrm{d}\Xi\, \mathrm{d}s \mathrm{d}t \right|$$

$$\leq \frac{C}{n^{1/2}} \|u\|_{L^2(\tau,\bar t;L^2(\gamma))} \left\| \int_\Omega \pi(\cdot,\cdot,\Xi) z_n(\cdot,\Xi) \mathrm{d}\Xi \right\|_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))}$$

$$\leq \frac{C}{n^{1/2}} \|u\|_{L^2(\tau,\bar t;L^2(\gamma))} \|\pi\|_{L^2(\tau,\bar t;L^2(\Omega_\Xi;L^2(0,L;H^1(0,1;d))))} \|z_n\|_{L^\infty(\tau,\bar t;L^2(\Omega_\Xi))}.$$

Therefore identity (5.26) is established when u belongs to $C^1([\tau,\bar t]; L^2(0,L))$. When u belongs to $L^2(\tau,\bar t; L^2(0,L))$ we recover identity (5.26) by a density argument. □

Lemma 5.7. *Let π be the solution to equation*

$$\begin{cases} -\pi' = \mathcal{A}_X^* \pi + \mathcal{A}_\Xi^* \pi - \displaystyle\int_\gamma |b(s,0)|^2 \pi(t,s,0,\Xi) \pi(t,X,s,0)\, \mathrm{d}s + \mathbf{\Phi} \quad \textit{in } (\tau,\bar t), \\ \pi(\bar t) = \pi_0, \end{cases} \tag{5.27}$$

where $\tau \in [0,\bar t)$ and $\pi_0 \in L^2_s(\mathcal{O})$. For all $u \in L^2(0,\infty;U)$, $z_0 \in L^2(\Omega)$, we have

$$\begin{aligned} &\frac12 \int_\tau^{\bar t} \int_{\mathcal{O}} \mathbf{\Phi}(X,\Xi) z(t,X)\, z(t,\Xi)\, \mathrm{d}X \mathrm{d}\Xi\, \mathrm{d}t + \frac12 \int_\tau^{\bar t} \int_\gamma |u|^2 \mathrm{d}s\, \mathrm{d}t + \frac12 \int_{\mathcal{O}} \pi_0\, z(\bar t) \otimes z(\bar t) \\ &\quad = \frac12 \int_{\mathcal{O}} \pi(\tau)\, z_0 \otimes z_0 + \frac12 \int_\tau^{\bar t} \int_\gamma \left| u(t,s) - b(s,0) \int_\Omega \pi(t,s,0,\Xi)\, z(t,\Xi) \right|^2 \mathrm{d}s\, \mathrm{d}t, \end{aligned} \tag{5.28}$$

where z is the solution to equation (5.25).

Proof. Let $\hat\pi$ be the solution to equation (5.9). Setting $\pi(t) = \hat\pi(\bar t - t)$, we can verify that π is the solution to equation (5.27). Let $(\psi_\ell)_\ell$ be a sequence in $C([\tau,\bar t]; D(\mathcal{A}^{s\,*}_{X,\Xi}))$, converging to $-\int_\gamma |b(s,0)|^2 \pi(t,s,0,\Xi)\pi(t,X,s,0)\mathrm{d}s + \mathbf{\Phi}$ in $L^2(\tau,\bar t; L^2_s(\mathcal{O}))$, and $(\pi_{0,\ell})_\ell$ be a sequence in $D(\mathcal{A}^{s\,*}_{X,\Xi})$, converging to π_0 in $L^2_s(\mathcal{O})$. Let π_ℓ be the solution to

$$-\pi_\ell' = \mathcal{A}_X^* \pi_\ell + \mathcal{A}_\Xi^* \pi_\ell + \psi_\ell \quad \text{in } (\tau,\bar t), \quad \pi_\ell(\bar t) = \pi_{0,\ell}. \tag{5.29}$$

With Lemma 5.6 applied to π_ℓ, we can write

$$\int_{\mathcal{O}} \pi_{0,\ell}\, z(\bar t) \otimes z(\bar t) - \int_{\mathcal{O}} \pi_\ell(\tau)\, z_0 \otimes z_0 + 2 \int_\tau^{\bar t} \int_\gamma b(s,0) u(t,s) \int_\Omega \pi_\ell(s,0,\Xi) z(t,\Xi)\, \mathrm{d}\Xi\, \mathrm{d}s\, \mathrm{d}t$$

$$= \int_\tau^{\bar t} \int_{\mathcal{O}} \left(\pi_\ell'(t,X,\Xi) + \mathcal{A}^*_{X,\Xi} \pi_\ell(t,X,\Xi) \right) z(t) \otimes z(t)\, \mathrm{d}X \mathrm{d}\Xi\, \mathrm{d}t$$

$$= -\int_\tau^{\bar t} \int_{\mathcal{O}} \psi_\ell(t,X,\Xi)\, z(t) \otimes z(t)\, \mathrm{d}X \mathrm{d}\Xi\, \mathrm{d}t.$$

By passing to the limit when ℓ tends to infinity, we obtain:

$$\int_{\mathcal{O}} \pi_0\, z(\bar{t}) \otimes z(\bar{t}) - \int_{\mathcal{O}} \pi(\tau)\, z_0 \otimes z_0 + 2 \int_\tau^{\bar{t}} \int_\gamma b(s,0) u(t,s) \int_\Omega \pi(s,0,\Xi) z(t,\Xi)\, \mathrm{d}\Xi\, \mathrm{d}s\, \mathrm{d}t$$
$$= \int_\tau^{\bar{t}} \int_\gamma \left| b(s,0) \int_\Omega \pi(s,0,X) z(t,X) \mathrm{d}X \right|^2 \mathrm{d}s\, \mathrm{d}t - \int_\tau^{\bar{t}} \int_{\mathcal{O}} \mathbf{\Phi}(X,\Xi) z(t,X)\, z(t,\Xi)\, \mathrm{d}X \mathrm{d}\Xi\, \mathrm{d}t.$$

Thus we have

$$\frac{1}{2} \int_\tau^{\bar{t}} \int_{\mathcal{O}} \mathbf{\Phi}(X,\Xi) z(t,X)\, z(t,\Xi)\, \mathrm{d}X \mathrm{d}\Xi\, \mathrm{d}t + \frac{1}{2} \int_\tau^{\bar{t}} \int_\gamma |u|^2 \mathrm{d}s\, \mathrm{d}t + \frac{1}{2} \int_{\mathcal{O}} \pi_0\, z(\bar{t}) \otimes z(\bar{t})\, \mathrm{d}X \mathrm{d}\Xi$$
$$= \frac{1}{2} \int_{\mathcal{O}} \pi(\tau)\, z_0 \otimes z_0 + \int_\tau^{\bar{t}} \int_\gamma |u(t,s)|^2\, \mathrm{d}s\, \mathrm{d}t - \int_\tau^{\bar{t}} \int_\gamma b(s,0) u(t,s) \int_\Omega \pi(s,0,\Xi) z(t,\Xi)\, \mathrm{d}\Xi\, \mathrm{d}s\, \mathrm{d}t$$
$$+ \frac{1}{2} \int_\tau^{\bar{t}} \int_\gamma \left| b(s,0) \int_\Omega \pi(s,0,X) z(t,X) \mathrm{d}X \right|^2 \mathrm{d}s\, \mathrm{d}t$$
$$= \frac{1}{2} \int_{\mathcal{O}} \pi(\tau)\, z_0 \otimes z_0 + \frac{1}{2} \int_\tau^{\bar{t}} \int_\gamma \left| u(t,s) - b(s,0) \int_\Omega \pi(t,s,0,\Xi)\, z(t,\Xi) \right|^2 \mathrm{d}s\, \mathrm{d}t.$$

The proof is complete. □

Let π be the solution of equation (5.27), and consider the evolution equation

$$z' = \mathcal{A}z - B(\mathbb{1}_\gamma B^* \Pi z) \quad \text{in } (\tau, \bar{t}), \qquad z(\tau) = z_0, \tag{5.30}$$

where

$$B^* \Pi z(s,t) = -b(s,0) \int_\Omega \pi(t,s,0,\Xi) z(t,\Xi) \mathrm{d}\Xi \quad \text{for } s \in (0,L),\ t \in (\tau, \bar{t}).$$

Weak solutions to equation (5.30) are defined as weak solutions to equation

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u) \quad \text{in } (\tau, \bar{t}), \qquad z(\tau) = z_0, \tag{5.31}$$

when $u = -B^* \Pi z$. This is meaningful because if $z \in L^2(\tau, \bar{t}; L^2(\Omega))$, then $\mathbb{1}_\gamma B^* \Pi z \in L^2(\tau, \bar{t}; L^2(0,L))$.

Lemma 5.8. *Equation (5.30) admits a unique weak solution in $L^\infty(\tau, \bar{t}; L^2(\Omega))$. Moreover this solution also belongs to $L^2(\tau, \bar{t}; L^2(0,L; H^1(0,1;d)))$.*

Proof. We first show that equation (5.30) admits a unique weak solution in $L^\infty(\tau, \hat{t}; L^2(\Omega))$, for some $\hat{t} > \tau$, by using a fixed point argument. We need an estimate of the solution z of equation (5.31) in the case when $u \in L^{2-\varepsilon'}(\tau, \bar{t}; L^2(0,L))$ for some $\varepsilon' > 0$.

Step 1. *Estimate for the solution to equation (5.31).* We use the technique in [6], Proof of Theorem 6.6, and an approximation process. Set $f_n(t,x,y) = n \mathbb{1}_{(0,\frac{1}{n})}(y) u(t,x) \mathbb{1}_\gamma$, where $\mathbb{1}_{(0,\frac{1}{n})}$ is the characteristic function of $(0, \frac{1}{n})$. Let us denote by z_n the solution to

$$z' = \mathcal{A}z - b\, f_n, \qquad z(\tau) = z_0,$$

and $\zeta_n = \mathrm{e}^{-kx} z_n$ be the solution to

$$\zeta' = \mathcal{A}_k \zeta - \mathrm{e}^{-kx} b\, f_n, \qquad \zeta(\tau) = \mathrm{e}^{-kx} z_0.$$

From [6], Inequality 6.4, it follows that

$$\frac{1}{2}\int_0^1\int_0^x \zeta_n(\xi,y,t)^2\,\mathrm{d}\xi\,\mathrm{d}y - \frac{1}{2}\int_0^1\int_0^x \mathrm{e}^{-k\xi} z_0(\xi,y)^2\,\mathrm{d}\xi\,\mathrm{d}y + \frac{1}{2}\int_\tau^t\int_0^1 a\,\zeta_n(x,y,\theta)^2\,\mathrm{d}y\,\mathrm{d}\theta$$
$$+\int_\tau^t\int_0^1\int_0^x\left(b\left|\frac{\partial\zeta_n}{\partial y}\right|^2+\frac{\partial b}{\partial y}\frac{\partial\zeta_n}{\partial y}\zeta_n+(c+ka)\zeta_n^2\right)\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\theta \le \int_\tau^t\int_0^1\int_0^x \mathrm{e}^{-k\xi} b\,f_n\,\zeta_n\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\theta, \quad (5.32)$$

for all $t\in(\tau,\bar t)$ and all $x\in[0,L]$. We have

$$\begin{aligned}\left|\int_\tau^t\int_0^1\int_0^x \mathrm{e}^{-k\xi} b\,f_n\,\zeta_n\,\mathrm{d}\xi\,\mathrm{d}y\,\mathrm{d}\theta\right| &= \left|\int_\tau^t\int_0^x \mathrm{e}^{-k\xi}u(x,\theta)n\int_0^{1/n} b\,\zeta_n\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}\theta\right|\\
&\le \|u\|_{L^{\frac{4-\varepsilon}{3-\varepsilon}}(\tau,\bar t;L^2(0,L))}\|b\,\zeta_n\|_{L^{4-\varepsilon}(\tau,\bar t;L^2(0,L;L^\infty(0,1/2)))}\\
&\le C\|u\|_{L^{\frac{4-\varepsilon}{3-\varepsilon}}(\tau,\bar t;L^2(0,L))}\|b\,\zeta_n\|_{L^{4-\varepsilon}(\tau,\bar t;L^2(0,L;H^{\frac12+\frac{\varepsilon}{8-2\varepsilon}}(0,1;d)))}\\
&\le C\|u\|_{L^{\frac{4-\varepsilon}{3-\varepsilon}}(\tau,\bar t;L^2(0,L))}\|b\,\zeta_n\|^{2/(4-\varepsilon)}_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))}\|b\,\zeta_n\|^{(2-\varepsilon)/(4-\varepsilon)}_{L^\infty(\tau,\bar t;L^2(0,L;L^2(0,1)))}\\
&\le \frac{C^2}{\alpha}\|u\|^2_{L^{\frac{4-\varepsilon}{3-\varepsilon}}(\tau,\bar t;L^2(0,L))}+\frac{\alpha}{2}\|\zeta_n\|^2_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))}\\
&\quad+\frac{\alpha}{2}\|\zeta_n\|^2_{L^\infty(\tau,\bar t;L^2(0,L;L^2(0,1)))},\end{aligned}$$

for all $\alpha>0$ and $0<\varepsilon<1$. With (5.32) and with Lemma 2.1, we obtain

$$\frac{1}{2}\|\zeta_n\|^2_{L^\infty(\tau,\bar t;L^2(\Omega))}+\frac{1}{2}\|\sqrt{a}\zeta_n\|_{L^\infty(0,L;L^2(\tau,\bar t;L^2(0,1)))}+\frac{C_1}{2}\|\zeta_n\|_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))}\le\frac{3C^2}{\alpha}\|u\|^2_{L^{\frac{4-\varepsilon}{3-\varepsilon}}(\tau,\bar t;L^2(0,L))}$$
$$+\frac{3\alpha}{2}\|\zeta_n\|^2_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))}+\frac{3\alpha}{2}\|\zeta_n\|^2_{L^\infty(\tau,\bar t;L^2(0,L;L^2(0,1)))}+\frac{3}{2}\int_0^1\int_0^x \mathrm{e}^{-k\xi}z_0(\xi,y)^2\mathrm{d}\xi\mathrm{d}y.$$

Thus, choosing α suitablely, we prove that there exists a constant $C>0$ such that

$$\|\zeta_n\|_{L^\infty(\tau,\bar t;L^2(\Omega))}+\|\sqrt{a}\zeta_n\|_{L^\infty(0,L;L^2(\tau,\bar t;L^2(0,1)))}+\|\zeta_n\|_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))}\le C\Big(\|u\|_{L^{\frac{4-\varepsilon}{3-\varepsilon}}(\tau,\bar t;L^2(0,L))}+\|z_0\|_{L^2(\Omega)}\Big).$$

By passing to the limit when n tends to infinity, we recover the same estimate for ζ, and next for z. Thus we have

$$\|z\|_{L^\infty(\tau,\bar t;L^2(\Omega))}+\|\sqrt{a}z\|_{L^\infty(0,L;L^2(\tau,\bar t;L^2(0,1)))}+\|z\|_{L^2(\tau,\bar t;L^2(0,L;H^1(0,1;d)))}\le C_7\Big(\|u\|_{L^{2-\varepsilon'}(\tau,\bar t;L^2(0,L))}+\|z_0\|_{L^2(\Omega)}\Big), \quad (5.33)$$

for some $\varepsilon'>0$, and where C_7 is independent of τ and $\bar t$.

Step 2. *Existence of solution to equation* (5.30). If v belongs to $L^\infty(\tau,\bar t;L^2(\Omega))$, then from calculations in the proof of Theorem 5.5 it follows that

$$\|B^*\Pi v\|_{L^{2-\varepsilon'}(\tau,\bar t;L^2(0,L))}\le C_8|\bar t-\tau|^{\frac{\varepsilon'}{2-\varepsilon'}}\|v\|_{L^\infty(\tau,\bar t;L^2(\Omega))}, \quad (5.34)$$

for some constant C_8 depending on $\|\phi\|_{L^2(\Omega)}$, but independent of τ and $\bar{t}$. We choose $\hat{t} > 0$ such that $C_7\, C_8 |\hat{t} - \tau|^{\frac{\varepsilon'}{2-\varepsilon'}} \leq 1/2$. Let v be in $L^\infty(\tau, \hat{t}; L^2(\Omega))$ and $z_v \in L^\infty(\tau, \hat{t}; L^2(\Omega))$ be the solution to

$$z' = \mathcal{A}z - B(\mathbb{1}_\gamma B^* \Pi v) \quad \text{in } (\tau, \hat{t}), \qquad z(0) = z_0.$$

Let us denote by Ψ the mapping $v \mapsto z_v$. Let v_1 and v_2 be in $L^\infty(\tau, \hat{t}; L^2(\Omega))$. With (5.33) and (5.34) we have

$$\|z_{v_1} - z_{v_2}\|_{L^\infty(\tau,\hat{t};L^2(\Omega))} \leq C_7\, C_8 |\hat{t} - \tau|^{\frac{\varepsilon'}{2-\varepsilon'}} \|v_1 - v_2\|_{L^\infty(\tau,\bar{t};L^2(\Omega))}.$$

Since $C_7 C_8 |\hat{t} - \tau|^{\frac{\varepsilon'}{2-\varepsilon'}} \leq 1/2$, Ψ is a contraction in $L^\infty(\tau, \hat{t}; L^2(\Omega))$. Thus equation (5.30) admits a unique solution $z \in L^\infty(\tau, \hat{t}; L^2(\Omega))$. If $z \in L^\infty(\tau, \hat{t}; L^2(\Omega))$, with (5.33) and (5.34) it follows that z belongs to $L^2(\tau, \hat{t}; L^2(0, L; H^1(0, 1; d)))$. We can repeat the fixed point argument on $(\tau, 2\hat{t} - \tau)$ in the following way. Let us set

$$E = \Big\{ v \in L^\infty(\tau, 2\hat{t} - \tau; L^2(\Omega)) \mid v|_{(\tau,\hat{t})} = z \Big\},$$

where z is the solution of (5.30) in $(\tau, \hat{t})$. Step by step, we prove that equation (5.30) admits a unique solution in $L^\infty(\tau, \bar{t}; L^2(\Omega))$. Observe that $\mathbb{1}_\gamma B^* \pi z$ belongs not only to $L^{2-\varepsilon'}(\tau, \bar{t}; L^2(0, L))$, but also to $L^2(\tau, \bar{t}; L^2(0, L))$. □

Proof of Theorem 5.6. Let π be the solution to equation (5.9). Let us show that $\pi \geq 0$. Let us set $\hat{\pi}(t) = \pi(\bar{t} - t)$. We verify that $\hat{\pi}$ is the solution to equation (5.27). Denote by $\hat{\Pi}$ the operator whose kernel is $\hat{\pi}$. Let z be the solution to equation (5.30). We can apply Lemma 5.7 to z with $u(t) = -\mathbb{1}_\gamma B^* \hat{\Pi} z(t) = \mathbb{1}_\gamma\, b(s, 0) \int_{\Omega_X} \hat{\pi}(X, s, 0) z(X, t)\, \mathrm{d}X$, and we get

$$\frac{1}{2} \int_{\mathcal{O}} \hat{\pi}(\tau)\, z_0 \otimes z_0 - \frac{1}{2} \int_{\mathcal{O}} \pi_0\, z(\bar{t}) \otimes z(\bar{t}) = \frac{1}{2} \int_\tau^{\bar{t}} \int_{\mathcal{O}} \mathbf{\Phi}(X, \Xi) z(t, X)\, z(t, \Xi)\, \mathrm{d}X \mathrm{d}\Xi\, \mathrm{d}t + \frac{1}{2} \int_\tau^{\bar{t}} \int_\gamma |B^* \hat{\pi}\, z|^2 \mathrm{d}s\, \mathrm{d}t.$$

Since $\pi_0 \in L^2_+(\mathcal{O})$ we have

$$\int_{\mathcal{O}} \pi(\bar{t} - \tau)\, z_0 \otimes z_0 = \int_{\mathcal{O}} \hat{\pi}(\tau)\, z_0 \otimes z_0 \geq \int_{\mathcal{O}} \pi_0\, z(\bar{t}) \otimes z(\bar{t}) \geq 0,$$

for all $\tau \in [0, \bar{t})$. The proof is complete. □

Theorem 5.7. *The solution π to equation* (5.9) *exists over the time interval* $(0, \infty)$ *and satisfies*

$$\|\pi\|_{L^\infty(0,\infty;L^2_s(\mathcal{O}))} \leq C \left(\|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})} + \|\pi_0\|_{L^2_s(\mathcal{O})} \right).$$

Moreover, there exist two constants C_9 and C_{10}, independent of $T > 0$, such that

$$\|\pi\|^2_{L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \leq C_9\, T \left(\|\mathbf{\Phi}\|^3_{L^2_s(\mathcal{O})} + \|\pi_0\|^3_{L^2_s(\mathcal{O})} + \|\mathbf{\Phi}\|^{\frac{8-3\varepsilon}{2-\varepsilon}}_{L^2_s(\mathcal{O})} + \|\pi_0\|^{\frac{8-3\varepsilon}{2-\varepsilon}}_{L^2_s(\mathcal{O})} \right) + C_{10} \|\pi_0\|^2_{L^2_s(\mathcal{O})} \tag{5.35}$$

for all $T > 0$ and all $\varepsilon > 0$. (C_9 depends on $\varepsilon > 0$.)

Proof. We argue by contradiction, we suppose that there exists a maximal solution which is not a global one. Let $[0, T_{\max}[$ be the maximal interval such that, for all $\bar{t} \in [0, T_{\max}[$ equation (5.9) admits a solution π in $L^\infty(0, \bar{t}; L^2_s(\mathcal{O})) \cap L^2(0, \bar{t}; L^2(\Omega_X; L^2(0, L; H^1(0, 1; d))))$ and

$$\lim_{\bar{t} \to T_{\max}} \left(\|\pi\|_{L^\infty(0,\bar{t};L^2_s(\mathcal{O}))} + \|\pi\|_{L^2(0,\bar{t};L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \right) = \infty. \tag{5.36}$$

Let π_ℓ be the solution to the Lyapunov equation (5.12) corresponding to

$$\psi(t,X,\Xi)=\mathbf{\Phi}(X,\Xi).$$

We can verify that $\pi_\ell-\pi$ is the solution to Lyapunov equation (5.12) corresponding to

$$\psi(t,X,\Xi)=\int_\gamma |b(s,0)|^2\pi(t,s,0,\Xi)\pi(t,X,s,0)\,\mathrm{d}s\ \geq 0.$$

From assertion (iii) in Theorem 5.3 it follows that $\pi_\ell(t)\geq\pi(t)$ for all $t\in[0,T_{\max}[$. We have

$$\begin{aligned}\|\pi\|_{L^2(\Omega_X;L^2(\Omega_\Xi))} &= \sup\left\{\int_{\Omega_X}\sup\left\{\int_{\Omega_\Xi}\pi\,\zeta\mathrm{d}\Xi\mid\|\zeta\|_{L^2(\Omega_\Xi)}=1\right\}z\,\mathrm{d}X\mid\|z\|_{L^2(\Omega_X)}=1\right\}\\ &= \sup\left\{\int_{\mathcal{O}}\pi\,z\otimes\zeta\,\mathrm{d}X\mathrm{d}\Xi\mid\|\zeta\|_{L^2(\Omega_\Xi)}=1,\ \|z\|_{L^2(\Omega_\Xi)}=1\right\},\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\mathcal{O}}\pi\,z\otimes\zeta\right| &\leq \frac{1}{4}\int_{\mathcal{O}}\pi\,(z+\zeta)\otimes(z+\zeta)+\frac{1}{4}\int_{\mathcal{O}}\pi\,(z-\zeta)\otimes(z-\zeta)\\ &\leq \frac{1}{4}\int_{\mathcal{O}}\pi_\ell\,(z+\zeta)\otimes(z+\zeta)+\frac{1}{4}\int_{\mathcal{O}}\pi_\ell\,(z-\zeta)\otimes(z-\zeta)\\ &\leq \frac{3}{2}\|\pi_\ell\|_{L^\infty(0,\infty;L^2_s(\mathcal{O}))}\left(\|z\|^2_{L^2(\Omega_X)}+\|\zeta\|^2_{L^2(\Omega_\Xi)}\right).\end{aligned}$$

Thus

$$\|\pi\|_{L^\infty(0,T_{\max};L^2_s(\mathcal{O}))}\leq C\|\pi_\ell\|_{L^\infty(0,\infty;L^2_s(\mathcal{O}))}\leq C\left(\|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})}+\|\pi_0\|_{L^2_s(\mathcal{O})}\right). \tag{5.37}$$

Therefore we have

$$\|\pi\|_{L^\infty(0,T_{\max};L^2_s(\mathcal{O}))}<\infty \tag{5.38}$$

and

$$\lim_{\bar{t}\to T_{\max}}\|\pi\|_{L^2(0,\bar{t};L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))}=\infty.$$

Now, as in the proof of Theorem 5.5, we can write

$$\begin{aligned}2\|\pi(T)\|^2_{L^2_s(\mathcal{O})}+\|\pi\|^2_{L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} &\leq C_6\,T\,\|\pi\|^2_{L^\infty(0,T;L^2_s(\mathcal{O}))}\|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})}+C_6\|\pi_0\|^2_{L^2_s(\mathcal{O})}\\ &+C_6\,\|\pi\|^2_{L^\infty(0,T;L^2_s(\mathcal{O}))}\left[C_\gamma^2C_I^2\,\|b\|^2_\infty|T|^{\frac{2-\varepsilon}{4-\varepsilon}}\,\|\pi\|^{\frac{4-2\varepsilon}{4-\varepsilon}}_{L^\infty(0,\infty;L^2_s(\mathcal{O}))}\|\pi\|^{\frac{4}{4-\varepsilon}}_{L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))}\right],\end{aligned}$$

for all $0 < T < T_{\max}$. With Young's inequality and with (5.37), we obtain

$$\begin{aligned}
\|\pi\|^2_{L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} &\le C_6\, T\, \|\pi\|^2_{L^\infty(0,T;L^2_s(\mathcal{O}))} \|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})} + C_6 \|\pi_0\|^2_{L^2_s(\mathcal{O})} \\
&\quad + \Big[C_6\, C_\gamma^2 C_I^2\, \|b\|^2_\infty |T|^{\frac{2-\varepsilon}{4-\varepsilon}} \|\pi\|^{\frac{8-3\varepsilon}{4-\varepsilon}}_{L^\infty(0,\infty;L^2_s(\mathcal{O}))} \|\pi\|^{\frac{4}{4-\varepsilon}}_{L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \Big] \\
&\le C_6\, T\, \|\pi\|^2_{L^\infty(0,T;L^2_s(\mathcal{O}))} \|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})} + C_6 \|\pi_0\|^2_{L^2_s(\mathcal{O})} \\
&\quad + \frac{4(4-\varepsilon)}{(4-\varepsilon)^2} \Big[C_6\, C_\gamma^2 C_I^2\, \|b\|^2_\infty |T|^{\frac{2-\varepsilon}{4-\varepsilon}} \|\pi\|^{\frac{8-3\varepsilon}{4-\varepsilon}}_{L^\infty(0,\infty;L^2_s(\mathcal{O}))} \Big]^{\frac{4-\varepsilon}{2-\varepsilon}} \\
&\quad + \frac{1}{2} \|\pi\|^2_{L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))}.
\end{aligned}$$

Thus with (5.37) we obtain

$$\|\pi\|^2_{L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \le C_9\, T \left(\|\mathbf{\Phi}\|^3_{L^2_s(\mathcal{O})} + \|\pi_0\|^3_{L^2_s(\mathcal{O})} + \|\mathbf{\Phi}\|^{\frac{8-3\varepsilon}{2-\varepsilon}}_{L^2_s(\mathcal{O})} + \|\pi_0\|^{\frac{8-3\varepsilon}{2-\varepsilon}}_{L^2_s(\mathcal{O})} \right) + C_{10} \|\pi_0\|^2_{L^2_s(\mathcal{O})}.$$

By passing to the limit when T tends to $T_{\max}$, we obtain a contradiction with (5.36). Thus we obtain the existence of solution for all $T > 0$, and the estimates in the theorem are already proved. □

5.5. Algebraic Riccati equation

By studying the asymptotic behaviour of the solution to the differential Riccati equation (5.9), we prove the existence of a solution to the algebraic Riccati equation (5.9). Let ψ be in $L^2_s(\mathcal{O})$, the solution to equation

$$\pi \in D(\mathcal{A}^{s\,*}_{X,\Xi}), \quad \mathcal{A}^*_X \pi + \mathcal{A}^*_\Xi \pi + \psi = 0, \tag{5.39}$$

is explicitly defined by

$$\pi = \int_0^\infty \mathrm{e}^{t\mathcal{A}^*_X} \mathrm{e}^{t\mathcal{A}^*_\Xi} \psi \, \mathrm{d}t.$$

Moreover to give a meaning to the nonlinear term in the Riccati equation (5.7), we have to look for solutions π such that the trace of π on $\gamma \times \{0\} \times \Omega_\Xi$ and on $\Omega_X \times \gamma \times \{0\}$ are well defined. Thus it is natural to define solutions to equation (5.7) as follows.

Definition 5.2. A function $\pi \in D(\mathcal{A}^{s\,*}_{X,\Xi}) \cap L^2(\Omega_X; L^2(0, L; H^1(0, 1; d)))$ is a weak solution to equation (5.7) if it is solution of equation (5.39) with

$$\psi(X, \Xi) = -\int_\gamma |b(s, 0)|^2 \pi(s, 0, \Xi) \pi(X, s, 0)\, \mathrm{d}s + \mathbf{\Phi}(X, \Xi).$$

Remark 5.1. Observe that if $\pi \in D(\mathcal{A}^{s\,*}_{X,\Xi}) \cap L^2(\Omega_X; L^2(0, L; H^1(0, 1; d)))$, then $\pi \in L^2(\Omega_\Xi; L^2(0, 1; H^1(0, 1; d))$. Moreover, if $\pi \in L^2(\Omega_X; L^2(0, L; H^1(0, 1; d))) \cap L^2(\Omega_\Xi; L^2(0, L; H^1(0, 1; d)))$, then the term $\int_\gamma |b(s, 0)|^2 \pi(s, 0, \Xi) \pi(X, s, 0)\, \mathrm{d}s$ belongs to $L^2_s(\mathcal{O})$. Thus Definition 5.2 is meaningful.

Lemma 5.9. *Let $(\pi_{0,n})_n$ be a sequence in $L^2_s(\mathcal{O})$ and let $\pi_{0,\infty}$ belong to $L^2_s(\mathcal{O})$. We assume that, for all n, $m \ge n$, $\pi_{0,n} \le \pi_{0,m} \le \pi_{0,\infty}$ and that, for all $\zeta \in L^2(\Omega)$, $(\int_{\Omega_X} \pi_{0,n}\zeta)_n$ converges to $\int_{\Omega_X} \pi_0 \zeta$ in $L^2(\Omega_\Xi)$.*

Let π_n (respectively π_∞) be the solution to equation (5.9) corresponding to the initial condition $\pi_{0,n}$ (respectively $\pi_{0,\infty}$). Then, for all $T > 0$ and all $z_0 \in L^2(\Omega)$, the sequence $(\int_{\mathcal{O}} \pi_n(T) z_0 \otimes z_0)_n$ converges to $\int_{\mathcal{O}} \pi(T) z_0 \otimes z_0$.

Let us notice that if $(\int_{\Omega_X} \pi_{0,n}\zeta)_n$ converges to $\int_{\Omega_X} \pi_0 \zeta$ in $L^2(\Omega_\Xi)$, then $(\int_{\Omega_\Xi} \pi_{0,n}\zeta)_n$ converges to $\int_{\Omega_\Xi} \pi_0 \zeta$ in $L^2(\Omega_X)$ because $\pi_{0,n}$ and π_0 belong to $L^2_s(\mathcal{O})$.

Proof. Let π be the solution to (5.27) with $(\tau,\bar t)=(0,T)$ and $\pi(T)=\pi_0$, and let π_n be the solution to (5.27) in $(\tau,\bar t)=(0,T)$ corresponding to the terminal condition $\pi_n(T)=\pi_{0,n}$. To prove the lemma it is sufficient to establish that

$$\lim_{n\to\infty}\int_{\mathcal{O}}\pi_n(0)\,z_0\otimes z_0=\int_{\mathcal{O}}\pi(0)\,z_0\otimes z_0.$$

Let us introduce the control problem

$$(Q^T_{0,z_0})\qquad \inf\Big\{I^T_0(z,u)\mid (z,u)\in L^2(0,T;Z)\times L^2(0,T;U),\ (z,u)\text{ satisfies (5.40)}\Big\},$$

where

$$I^T_0(z,u)=\frac12\int_0^T\int_{\mathcal{O}}\mathbf{\Phi}(X,\Xi)z(\tau,X)\,z(\tau,\Xi)\,\mathrm{d}X\mathrm{d}\Xi\,\mathrm{d}\tau+\frac12\int_0^T\int_\gamma|u|^2+\frac12\int_{\mathcal{O}}\pi_0 z(T)\otimes z(T),$$

and

$$z'=\mathcal{A}z+B(\mathbb{1}_\gamma u),\qquad z(0)=z_0, \tag{5.40}$$

and let us consider the family of control problems

$$(Q^T_{0,n,z_0})\qquad \inf\Big\{I^T_{0,n}(z,u)\mid (z,u)\in L^2(0,T;Z)\times L^2(0,T;U),\ (z,u)\text{ satisfies (5.40)}\Big\},$$

where

$$I^T_{0,n}(z,u)=\frac12\int_0^T\int_{\mathcal{O}}\mathbf{\Phi}(X,\Xi)z(\tau,X)\,z(\tau,\Xi)\,\mathrm{d}X\mathrm{d}\Xi\,\mathrm{d}\tau+\frac12\int_0^T\int_\gamma|u|^2+\frac12\int_{\mathcal{O}}\pi_{0,n} z(T)\otimes z(T).$$

Let us denote by $\varphi(T,z_0)$ the value function of (Q^T_{0,z_0}) and by (z,u) its optimal pair. Similarly, we denote by $\varphi_n(T,z_0)$ the value function of (Q^T_{0,n,z_0}) and by (z_n,u_n) its optimal pair. From Lemma 5.7, it follows that (z,u) and (z_n,u_n) obey the feedback formulas

$$u(t,s)=b(s,0)\int_{\Omega_\Xi}\pi(t,s,0,\Xi)\,z(t,\Xi)\,\mathrm{d}\Xi\quad\text{and}\quad u_n(t,s)=b(s,0)\int_{\Omega_\Xi}\pi_n(t,s,0,\Xi)\,z_n(t,\Xi)\,\mathrm{d}\Xi,$$

and the value functions satisfy

$$\varphi(T,z_0)=\frac12\int_{\mathcal{O}}\pi(0)\,z_0\otimes z_0\quad\text{and}\quad\varphi_n(T,z_0)=\frac12\int_{\mathcal{O}}\pi_n(0)\,z_0\otimes z_0.$$

We are going to show that $(u_n)_n$ converges to u in $L^2(0,T;U)$. First, since we have

$$I^T_{0,n}(z_n,u_n)\le I^T_{0,n}(z,u),$$

we notice that the sequence $(u_n)_n$ is bounded in $L^2(0,T;U)$ and that, from any subsequence, we can extract another subsequence, still indexed by n to simplify the notation, weakly converging in $L^2(0,T;U)$ to some $\bar u$. Let us denote by $\bar z$ the solution to (5.40) corresponding to $\bar u$. We can easily see that $(z_n)_n$ converges to $\bar z$ for the weak topology in $L^2(0,T;Z)$ and that $z_n(T)$ converges to $\bar z(T)$ for the weak topology of Z. Thus, by passing to the inferior limit when n tends to infinity, we obtain

$$I^T_{0,n_0}(\bar z,\bar u)\le\liminf_{n\to\infty}I^T_{0,n_0}(z_n,u_n)\le\liminf_{n\to\infty}I^T_{0,n}(z_n,u_n)\le\lim_{n\to\infty}I^T_{0,n}(z,u)=I^T_0(z,u),$$

where $n_0 \in \mathbb{N}$ is given fixed (here we have used that $\pi_{0,n_0} \leq \pi_{0,n}$ when $n_0 \leq n$). Next by passing to the limit when n_0 tends to infinity, we obtain

$$I_0^T(\bar{z}, \bar{u}) = \lim_{n\to\infty} I_{0,n_0}^T(\bar{z}, \bar{u}) \leq I_0^T(z, u).$$

Thus $I_0^T(\bar{z}, \bar{u}) = I_0^T(z, u)$, $\bar{u} = u$, $\bar{z} = z$, $(u_n)_n$ converges to u in $L^2(0, T; U)$ and $(z_n)_n$ converges to z in $C([0, T]; Z)$. Therefore

$$\lim_{n\to\infty} \varphi_n(T, z_0) = \lim_{n\to\infty} \frac{1}{2} \int_{\mathcal{O}} \pi_n(0)\, z_0 \otimes z_0 = \varphi(T, z_0) = \frac{1}{2} \int_{\mathcal{O}} \pi(0)\, z_0 \otimes z_0.$$

The proof is complete. □

Theorem 5.8. *The algebraic Riccati equation* (5.7) *admits at least one solution* π *in the sense of Definition* 5.2*, and it satisfies:*

$$\|\pi\|^2_{L^2(\Omega_X; L^2(0,L;H^1(0,1;d)))} \leq C \left(\|\mathbf{\Phi}\|^2_{L^2_s(\mathcal{O})} + \|\mathbf{\Phi}\|^3_{L^2_s(\mathcal{O})} + \|\mathbf{\Phi}\|^{\frac{8-3\varepsilon}{2-\varepsilon}}_{L^2_s(\mathcal{O})} \right). \tag{5.41}$$

Proof. **Step 1.** Let π be the solution to equation (5.9) corresponding to $\pi_0 = 0$, and π_ε be the solution to equation (5.9) corresponding to $\pi_\varepsilon(0) = \pi(\varepsilon)$, $\varepsilon > 0$. For all $t > 0$ and $z_0 \in L^2(\Omega)$, let us introduce the control problem

$$(\mathcal{P}^t_{0,z_0}) \qquad \inf\left\{ J_0^t(z, u) \mid (z, u) \in L^2(0, t; Z) \times L^2(0, t; U),\ (z, u) \text{ satisfies } (5.42) \right\},$$

where

$$J_0^t(z, u) = \frac{1}{2} \int_0^t \int_{\mathcal{O}} \mathbf{\Phi}(X, \Xi) z(\tau, X)\, z(\tau, \Xi)\, \mathrm{d}X \mathrm{d}\Xi\, \mathrm{d}\tau + \frac{1}{2} \int_0^t \int_\gamma |u|^2,$$

and

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u), \qquad z(0) = z_0. \tag{5.42}$$

Let us denote by $\varphi(t, z_0)$ the value function of $(\mathcal{P}^t_{0,z_0})$. From Lemma 5.7 it follows that

$$\varphi(t, z_0) = \frac{1}{2} \int_{\mathcal{O}} \pi(t)\, z_0 \otimes z_0.$$

Since $\varphi(t + \varepsilon, z_0) \geq \varphi(t, z_0)$, we have

$$\int_{\mathcal{O}} \pi(t + \varepsilon)\, z_0 \otimes z_0 = \int_{\mathcal{O}} \pi_\varepsilon(t)\, z_0 \otimes z_0 \geq \int_{\mathcal{O}} \pi(t)\, z_0 \otimes z_0.$$

Thus the mapping $t \to \int_{\mathcal{O}} \pi(t)\, z_0 \otimes z_0$ is nondecreasing. We denote by $\Pi(t) \in \mathcal{L}(L^2(\Omega))$ the operator defined by:

$$(\Pi(t)z)(X) = \int_\Omega \pi(t, X, \Xi) z(\Xi)\, \mathrm{d}\Xi.$$

Since $\|\pi\|_{L^\infty(0,\infty;L^2_s(\mathcal{O}))} < \infty$, and

$$\big(\Pi(t)z, \zeta\big)_{L^2(\Omega)} = \frac{1}{4}\big(\Pi(t)(z + \zeta), (z + \zeta)\big)_{L^2(\Omega)} - \frac{1}{4}\big(\Pi(t)(z - \zeta), (z - \zeta)\big)_{L^2(\Omega)}, \tag{5.43}$$

we have

$$\sup_{t\geq 0} \left| \big(\Pi(t)z, \zeta\big)_{L^2(\Omega)} \right| < \infty,$$

for all $t \geq 0$, $z \in L^2(\Omega)$, and all $\zeta \in L^2(\Omega)$. Applying the Banach-Steinhaus Theorem, we deduce that $\sup_{t\geq 0} \big\|\big(\Pi(t)z,\cdot\big)_{L^2}\big\|_{\mathcal{L}(L^2(\Omega))} < \infty$. Applying another time the Banach-Steinhaus Theorem, we obtain $\sup_{t\geq 0} \big\|\big(\Pi(t)\cdot,\cdot\big)_{L^2}\big\|_{\mathcal{L}(L^2(\Omega)\times L^2(\Omega))} < \infty$. Therefore there exists $\Pi_{\min} \in \mathcal{L}(L^2(\Omega))$ such that

$$\lim_{t\to\infty} \big(\Pi(t)z,\zeta\big)_{L^2(\Omega)} = \big(\Pi_{\min}z,\zeta\big)_{L^2(\Omega)}. \tag{5.44}$$

Since $\Pi(t) = \Pi^*(t) \geq 0$, it follows that $\Pi_{\min} = \Pi^*_{\min} \geq 0$. Let us notice that $\|(\Pi_{\min} - \Pi(t))^{1/2}\|_{\mathcal{L}(L^2(\Omega))}$ is bounded uniformly with respect to $t \in \mathbb{R}^+$, thus we have

$$\|(\Pi_{\min} - \Pi(t))\zeta\|_{L^2(\Omega)} \leq C\|(\Pi_{\min} - \Pi(t))^{1/2}\zeta\|_{L^2(\Omega)},$$

and with (5.44) we deduce

$$\begin{aligned}&\lim_{t\to\infty} \|(\Pi_{\min} - \Pi(t))\zeta\|_{L^2(\Omega)}\\ &\leq C\lim_{t\to\infty} \|(\Pi_{\min} - \Pi(t))^{1/2}\zeta\|_{L^2(\Omega)} = \lim_{t\to\infty} \big((\Pi_{\min} - \Pi(t))\zeta,\zeta\big)_{L^2(\Omega)} = 0.\end{aligned} \tag{5.45}$$

Besides the sequence $(\pi(n))_n$ is bounded in $L^2_+(\mathcal{O})$. Without loss of generality, we can suppose that $(\pi(n))_n$ converges to some $\pi_{\min} \in L^2_+(\mathcal{O})$ weakly in $L^2_s(\mathcal{O})$. Thus we also have

$$\lim_{n\to\infty} \int_{\mathcal{O}} \pi(n)z\otimes\zeta = \int_{\mathcal{O}} \pi_{\min}z\otimes\zeta.$$

By uniqueness of the limit, we have

$$\int_{\mathcal{O}} \pi_{\min}z\otimes\zeta = \big(\Pi_{\min}z,\zeta\big)_{L^2(\Omega)}.$$

From (5.45) it follows that

$$\lim_{n\to\infty} \|(\pi_{\min} - \pi(n))\zeta\|_{L^2(\Omega)} = 0.$$

Therefore the assumptions of Lemma 5.9 are satisfied by the sequence $(\pi(n))_n$ and the limit $\pi_{\min}$.

Step 2. We show that $\pi_{\min}$ is solution to the algebraic Riccati equation (5.7). Let $\hat{\pi}$ be the solution to (5.9) corresponding to $\pi_0 = \pi_{\min}$. Let $\bar{\pi}$ be the solution to (5.9) corresponding to $\pi_0 = 0$, and $\bar{\pi}_n$ the solution to (5.9) corresponding to $\pi_0 = \bar{\pi}(n)$. By using the dynamic programming principle, we have

$$\bar{\pi}_n(t) = \bar{\pi}(t+n), \quad t > 0.$$

Due to the first step, we have

$$\lim_{n\to\infty} \int_{\mathcal{O}} \bar{\pi}(n)z\otimes z = \lim_{n\to\infty} \int_{\mathcal{O}} \bar{\pi}_n(0)z\otimes z = \int_{\mathcal{O}} \pi_{\min}z\otimes z,$$

for all $z \in L^2(\Omega)$. Due to Lemma 5.9, we can write

$$\int_{\mathcal{O}} \hat{\pi}(t)z\otimes z = \lim_{n\to\infty} \int_{\mathcal{O}} \bar{\pi}_n(t)z\otimes z.$$

Therefore

$$\int_{\mathcal{O}} \hat{\pi}(t)z\otimes z = \lim_{n\to\infty} \int_{\mathcal{O}} \bar{\pi}_n(t)z\otimes z = \lim_{n\to\infty} \int_{\mathcal{O}} \bar{\pi}(t+n)z\otimes z = \int_{\mathcal{O}} \pi_{\min}z\otimes z$$

for all $t > 0$ and all $z \in L^2(\Omega)$. Thus, $\hat\pi$ is constant and equal to $\pi_{\min}$. This implies that $\pi_{\min} \in L^2(\Omega_X; L^2(0,L;H^1(0,1;d)))$, and that

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{O}} \hat\pi(t) z\otimes z \\ &= \big(\mathcal{A}_X z, \hat\pi(t)z\big) + \big(\hat\pi(t)z, \mathcal{A}_\Xi z\big) - \int_\gamma\int_{\mathcal{O}} (b\hat\pi(t)z)\otimes(b\hat\pi(t)z)\,\mathrm{d}s + \int_{\mathcal{O}} \mathbf{\Phi}(X,\Xi) z(X)\, z(\Xi)\,\mathrm{d}X\mathrm{d}\Xi \\ &= \big(\mathcal{A}_X z, \pi_{\min} z\big) + \big(\pi_{\min} z, \mathcal{A}_\Xi z\big) - \int_\gamma\int_{\mathcal{O}} (b\pi_{\min} z)\otimes(b\pi_{\min} z)\,\mathrm{d}s + \int_{\mathcal{O}} \mathbf{\Phi}(X,\Xi) z(X)\, z(\Xi)\,\mathrm{d}X\mathrm{d}\Xi. \end{aligned}$$

Consequently, $\pi_{\min}$ is a solution to the algebraic Riccati equation (5.7).

Let us prove estimate (5.41). With estimate (5.35) for $\hat\pi$ and the fact that $\hat\pi$ is constant with respect to t, we have

$$\begin{aligned} T\|\pi\|^2_{L^2(\Omega_X;L^2(0,L;H^1(0,1;d)))} &= \|\pi\|^2_{L^2(0,T;L^2(\Omega_X;L^2(0,L;H^1(0,1;d))))} \\ &\le C\,T\left(\|\mathbf{\Phi}\|^3_{L^2_s(\mathcal{O})} + \|\pi_{\min}\|^3_{L^2_s(\mathcal{O})} + \|\mathbf{\Phi}\|^{\frac{8-3\varepsilon}{2-\varepsilon}}_{L^2_s(\mathcal{O})} + \|\pi_{\min}\|^{\frac{8-3\varepsilon}{2-\varepsilon}}_{L^2_s(\mathcal{O})}\right) + C\|\pi_{\min}\|^2_{L^2_s(\mathcal{O})}. \end{aligned}$$

Choosing $T=1$ and using $\|\pi_{\min}\|_{L^2_s(\mathcal{O})} \le C\|\mathbf{\Phi}\|_{L^2_s(\mathcal{O})}$, the proof is complete. □

6. Feedback control law

The main objective of this section is to prove that the algebraic Riccati equation (5.7) admits a unique solution π and that $(\bar z, \bar u)$, the optimal solution to $(\mathcal{P}_{z_0})$, obeys the feedback formula

$$\bar u(s,\tau) = \mathbb{1}_\gamma(s)\, b(s,0)\left(\int_\Omega \pi(s,0,\Xi)\bar z(\tau,\Xi)\,\mathrm{d}\Xi\right), \quad s\in(0,L), \quad \tau\in\mathbb{R}^+.$$

To prove this result we first show that if π is a solution to equation (5.7), and if Π is the Hilbert-Schmidt operator of kernel π, then the equation

$$z' = \mathcal{A}z - B(\mathbb{1}_\gamma B^*\Pi z) \quad \text{in } (0,T), \qquad z(0) = z_0,$$

admits a unique solution (Thm. 6.1). Next we show that if

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u), \qquad z(0) = z_0,$$

then we have (see Lem. 6.4):

$$J(z,u) = \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0\otimes z_0 + \int_0^\infty \int_\gamma \left| u(\tau,s) - b(s,0)\int_{\Omega_\Xi} \pi(s,0,\Xi)\, z(\tau,\Xi)\right|^2 \mathrm{d}s\,\mathrm{d}\tau.$$

Combining these results we prove that any solution π to the algebraic Riccati equation (5.7) obeys

$$\frac{1}{2}\int_{\mathcal{O}} \pi\, z_0 \otimes z_0 \;= \inf(\mathcal{P}_{z_0}).$$

The uniqueness follows.

To establish such results we have to justify some integration by parts. We do it by using a regularization argument which is developed in the two following lemmas.

Lemma 6.1. *Let u belong to $C_c^1([0,\infty); L^2(0,L))$. There exists a sequence $(f_n)_n$ in $C_c^1([0,\infty); L^2(\Omega))$ such that*

$$\left|\int_0^t \int_\Omega b\, f_n\, \varphi - \int_0^t \int_\gamma b\, u\, \varphi\right| \le \frac{C}{n^{1/2}} \|\varphi\|_{L^2(0,t;L^2(0,L;H^1(0,1;d)))} \|u\|_{L^2(0,t;L^2(0,L))}\,,$$

for all $t > 0$ and all $\varphi \in L^2(0,\infty; L^2(0,L; H^1(0,1;d)))$.

Proof. The proof is similar to that of Lemma 5.5, where C is independent of t. □

Remark 6.1. If we identify $B(\mathbb{1}_\gamma u)$ with the functional defined in $L^2(0,\infty; L^2(0,L; H^1(0,1;d)))$ by

$$\varphi \longmapsto \int_0^\infty \int_\gamma b(s,0)\, u(t,s)\, \varphi(t,s,0)\, \mathrm{d}s\, \mathrm{d}t,$$

the sequence $(b\, f_n)_n$ can be considered as an approximation of $B(\mathbb{1}_\gamma u) \in L^2(0,\infty; L^2(0,L; (H^1(0,1;d))'))$.

Lemma 6.2. *Let u be in $C_c^1([0,\infty); L^2(0,L))$, $(f_n)_n$ be the sequence in $C_c^1([0,\infty); L^2(\Omega))$ defined in Lemma 6.1, z be the solution to equation*

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u), \qquad z(0) = z_0,$$

and z_n be the solution to equation

$$z' = \mathcal{A}z - b\, f_n, \qquad z(0) = z_0.$$

Then $(z_n)_n$ converges to z for the weak topology of $L^2(0,\infty; L^2(0,L; H^1(0,1;d)))$ and for the weak-star topology of $L^\infty(0,\infty; L^2(\Omega))$.

Proof. Let $k > 0$ be the parameter defined in Lemma 2.1. We set $\zeta = \mathrm{e}^{-kx} z$ and $\zeta_n = \mathrm{e}^{-kx} z_n$. To prove the lemma it is sufficient to show that $(\zeta_n)_n$ converges to ζ for the weak topology of $L^2(0,\infty; L^2(0,L; H^1(0,1;d)))$ and the weak-star topology of $L^\infty(0,\infty; L^2(\Omega))$. The functions ζ and ζ_n are respectively the solutions to

$$\zeta' = \mathcal{A}_k \zeta + B(\mathbb{1}_\gamma\, \mathrm{e}^{-kx} u), \qquad \zeta(0) = \mathrm{e}^{-kx} z_0,$$

and

$$\zeta_n' = \mathcal{A}_k \zeta - \mathrm{e}^{-kx} b\, f_n, \qquad \zeta_n(0) = \mathrm{e}^{-kx} z_0.$$

With [6], Theorem 6.2, we can write

$$\frac{1}{2}\int_\Omega |\zeta_n(t)|^2 + \frac{1}{2}\int_0^t \int_0^1 a\, \zeta_n(L,y,\tau)^2\, \mathrm{d}y\, \mathrm{d}\tau - \frac{1}{2}\int_\Omega |\mathrm{e}^{-kx} z_0|^2 + \int_0^t \int_0^1 \int_0^L \left(b \left|\frac{\partial \zeta_n}{\partial y}\right|^2 + \frac{\partial b}{\partial y}\frac{\partial \zeta_n}{\partial y}\zeta_n \right.$$
$$\left. + (c + ka)\zeta_n^2 \right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\tau \le -\int_0^t \int_0^1 \int_0^L \mathrm{e}^{-kx} b\, f_n\, \zeta_n\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}\tau.$$

From Lemma 6.1, it follows that

$$\left|\int_0^t \int_\Omega \mathrm{e}^{-kx} b\, f_n\, \zeta_n\right| \le \|u\|_{L^2(0,t;L^2(0,L))} \|b\zeta_n\|_{L^2(0,t;L^2(0,L;H^1(0,1;d)))} \left(1 + \frac{C}{n^{1/2}}\right)\cdot$$

Combining the two previous inequalities, with Lemma 2.1, we obtain:

$$\frac{1}{2}\int_\Omega |\zeta_n(t)|^2 + \frac{1}{2}\int_0^t\int_0^1 a\,\zeta_n(L,y,\tau)^2\,\mathrm{d}y\,\mathrm{d}\tau - \frac{1}{2}\int_\Omega |\mathrm{e}^{-kx}z_0|^2 + \frac{C_1}{2}\|\zeta_n\|^2_{L^2(0,t;L^2(0,L;H^1(0,1;d)))}$$
$$\leq \frac{1}{2\varepsilon}\left(1+\frac{C}{n^{1/2}}\right)^2 \|u\|^2_{L^2(0,t;L^2(0,L))} + \frac{\varepsilon}{2}\|b\zeta_n\|^2_{L^2(0,t;L^2(0,L;H^1(0,1;d)))},$$

for all $\varepsilon > 0$. Thus, we can choose $\varepsilon > 0$ to obtain:

$$\|\zeta_n\|^2_{L^\infty(0,\infty;L^2(\Omega))} + \int_0^\infty\int_0^1 a\,\zeta_n(L,y,\tau)^2\,\mathrm{d}y\,\mathrm{d}\tau + \|\zeta_n\|^2_{L^2(0,\infty;L^2(0,L;H^1(0,1;d)))}$$
$$\leq C\left(\|u\|^2_{L^2(0,\infty;L^2(0,L))} + \int_\Omega |\mathrm{e}^{-kx}z_0|^2\right).$$

The sequence $(\zeta_n)_n$ being bounded in $L^2(0,\infty;L^2(0,L;H^1(0,1;d)))$ and in $L^\infty(0,\infty;L^2(\Omega))$, we can easily prove that $(\zeta_n)_n$ converges to ζ for the weak topology of $L^2(0,\infty;L^2(0,L;H^1(0,1;d)))$ and the weak-star topology of $L^\infty(0,\infty;L^2(\Omega))$. □

Lemma 6.3. *Let π be a solution to the Riccati equation (5.7), $u\in L^2(0,\infty;L^2(0,L))$, $z_0\in L^2(\Omega)$, and z be the solution to equation*

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u), \qquad z(0) = z_0.$$

Then z satisfies the following identity:

$$\int_0^\infty\int_{\mathcal{O}} ((\mathcal{A}_X^* + \mathcal{A}_\Xi^*)\pi)\; z(t)\otimes z(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t = -\int_{\mathcal{O}} \pi\, z_0\otimes z_0\,\mathrm{d}X\,\mathrm{d}\Xi$$
$$+2\int_0^\infty\int_\gamma b(s,0)u(t,s)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi\,\mathrm{d}s\,\mathrm{d}t. \quad (6.1)$$

Proof. We first prove the identity when u belong to $C_c^1([0,\infty);L^2(0,L))$. Let $(f_n)_n$ be the sequence in $C_c^1([0,\infty);L^2(\Omega))$ defined in Lemma 6.1, and $(z_{0,n})_n$ be a sequence in $D(\mathcal{A})$ converging to z_0 in $L^2(\Omega)$. Let us denote by z_n the solution to

$$z' = \mathcal{A}z - b\,f_n, \qquad z(0) = z_{0,n}.$$

Since $z_n\in C([0,\infty);D(\mathcal{A}))\cap C^1([0,\infty);L^2(\Omega))$, we can write

$$\int_0^T\int_{\mathcal{O}} ((\mathcal{A}_X^* + \mathcal{A}_\Xi^*)\pi)\; z_n(t)\otimes z_n(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t = \int_0^T\int_{\mathcal{O}} \pi\,\mathcal{A}_X z_n(t)\otimes z_n(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t$$
$$+\int_0^T\int_{\mathcal{O}} \pi\, z_n(t)\otimes \mathcal{A}_\Xi z_n(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t$$
$$= \int_0^T\int_{\mathcal{O}} \pi\, z_n'(t)\otimes z_n(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t + \int_0^T\int_{\mathcal{O}} \pi\, z_n(t)\otimes z_n'(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t$$
$$+\int_0^T\int_{\mathcal{O}} b\,f_n(t,X)\,\pi(X,\Xi)z_n(t,\Xi)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t + \int_0^T\int_{\mathcal{O}} b\,f_n(t,\Xi)\,\pi(X,\Xi)z_n(t,X)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t$$
$$= \int_{\mathcal{O}} \pi\, z_n(T)\otimes z_n(T)\,\mathrm{d}X\,\mathrm{d}\Xi - \int_{\mathcal{O}} \pi\, z_{0,n}(\Xi)\,z_{0,n}(X)\,\mathrm{d}X\,\mathrm{d}\Xi + 2\int_0^T\int_{\mathcal{O}} b\,f_n(t,X)\,\pi(X,\Xi)z_n(t,\Xi)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t.$$

We first pass to the limit when n tends to infinity. As in the proof of Lemma 5.6, we can show that

$$\lim_{n\to\infty}\int_0^T\int_{\mathcal{O}}((\mathcal{A}_X^*+\mathcal{A}_\Xi^*)\pi)\ z_n(t)\otimes z_n(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t=\int_0^T\int_{\mathcal{O}}((\mathcal{A}_X^*+\mathcal{A}_\Xi^*)\pi)\ z(t)\otimes z(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t.$$

Due to Lemma 6.2, $(z_n)_n$ is bounded in $L^2(0,\infty;L^2(0,L;H^1(0,1;d)))$, and $(z_n)_n$ converges to z weakly in $L^2(0,\infty;L^2(0,L;H^1(0,1;d)))$. Moreover, with Lemma 6.1, we have

$$\left|\int_0^T\int_{\Omega_X} b\,f_n\int_{\Omega_\Xi}\pi(X,\Xi)z_n(t,\Xi)\,\mathrm{d}\Xi\,\mathrm{d}X\,\mathrm{d}t-\int_0^T\int_\gamma b\,u\int_{\Omega_\Xi}\pi(s,0,\Xi)z_n(t,\Xi)\,\mathrm{d}\Xi\,\mathrm{d}s\,\mathrm{d}t\right|$$

$$\leq\frac{C}{n^{1/2}}\left\|\int_{\Omega_\Xi}\pi(\cdot,\Xi)z_n(\cdot,\Xi)\,\mathrm{d}\Xi\right\|_{L^2(0,T;L^2(0,L;H^1(0,1;d)))}\|u\|_{L^2(0,T;L^2(0,L))}\,.$$

Since $\|\int_{\Omega_\Xi}\pi(\cdot,\Xi)z_n(\cdot,\Xi)\|_{L^2(0,T;L^2(0,L;H^1(0,1;d)))}$ is bounded, passing to the limit when n tends to infinity, we obtain

$$\begin{aligned}\int_0^T\int_{\mathcal{O}}((\mathcal{A}_X^*+\mathcal{A}_\Xi^*)\pi)\ z(t)\otimes z(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t=&\int_{\mathcal{O}}\pi\,z(T)\otimes z(T)-\int_{\mathcal{O}}\pi\,z_0\otimes z_0\\&-2\int_0^T\int_\gamma b(s,0)u(t,s)\int_\Omega\pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi\,\mathrm{d}s\,\mathrm{d}t,\end{aligned}$$

when u belongs to $C_c^1([0,\infty);L^2(0,L))$. Since $u\in C_c^1([0,\infty);L^2(0,L))$, due to the exponential stability on $L^2(\Omega)$ of the semigroup $(\mathrm{e}^{\mathcal{A}t})_{t\geq0}$, it follows that

$$\lim_{T\to\infty}\int_{\mathcal{O}}\pi\,z(T)\otimes z(T)=0.$$

Passing to the limit when T tends to infinity, we finally obtain

$$\int_0^\infty\int_{\mathcal{O}}((\mathcal{A}_X^*+\mathcal{A}_\Xi^*)\pi)\,z\otimes z\mathrm{d}X\,\mathrm{d}\Xi=-\int_{\mathcal{O}}\pi z_0(\Xi)z_0(X)\mathrm{d}X\mathrm{d}\Xi+2\int_0^\infty\int_\gamma b(s,0)u(t,s)\int_\Omega\pi(s,0,\Xi)z(t,\Xi)\mathrm{d}\Xi\mathrm{d}s\mathrm{d}t, \tag{6.2}$$

when u belongs to $C_c^1([0,\infty);L^2(0,L))$. Let us now consider the case where $u\in L^2(0,\infty;L^2(0,L))$. Since $C_c^1([0,\infty);L^2(0,L))$ is dense in $L^2(0,\infty;L^2(0,L))$, there exists a sequence $(u_n)_n$ in $C_c^1([0,\infty);L^2(0,L))$ converging to u in $L^2(0,\infty;L^2(0,L))$. The solution z_n of equation

$$z_n'=\mathcal{A}z_n+B(\mathbb{1}_\gamma u_n),\qquad z_n(0)=z_0,$$

converges to z in $L^2(0,\infty;L^2(0,L;H^1(0,1;d)))$. Thus we can write the identity (6.1) for z_n, and we establish (6.1) for z by passing to the limit when n tends to infinity. □

Lemma 6.4. *Let π be a solution to the system (5.7), $u\in L^2(0,\infty;U)$, $z_0\in L^2(\Omega)$, and z be the solution to equation*

$$z'=\mathcal{A}z+B(\mathbb{1}_\gamma u),\qquad z(0)=z_0.$$

Then the cost function satisfies

$$J(z,u)=\frac{1}{2}\int_{\mathcal{O}}\pi\,z_0\otimes z_0+\int_0^\infty\int_\gamma\left|u(\tau,s)-b(s,0)\int_\Omega\pi(s,0,\Xi)\,z(\tau,\Xi)\,\mathrm{d}\Xi\right|^2\mathrm{d}s\,\mathrm{d}\tau. \tag{6.3}$$

Proof. With Lemma 6.3 and equation (5.7), we can write

$$-\int_{\mathcal{O}} \pi\, z_0 \otimes z_0 + 2\int_0^\infty \int_\gamma b(s,0)u(t,s) \int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi\,\mathrm{d}s\,\mathrm{d}t \;= \int_0^\infty \int_{\mathcal{O}} ((\mathcal{A}_X^* + \mathcal{A}_\Xi^*)\pi)\; z(t)\otimes z(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t$$

$$= \int_0^\infty \int_\gamma \left| b(s,0) \int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi \right|^2 \mathrm{d}s\,\mathrm{d}t - \int_0^\infty \int_{\mathcal{O}} \mathbf{\Phi}(X,\Xi) z(t,X)\, z(t,\Xi)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t.$$

Thus we have

$$\begin{aligned}
J(z,u) &= \frac{1}{2}\int_0^\infty \int_{\mathcal{O}} \mathbf{\Phi}(X,\Xi)z(t,X)\,z(t,\Xi)\,\mathrm{d}X\mathrm{d}\Xi\,\mathrm{d}t + \frac{1}{2}\int_0^\infty \int_\gamma |u|^2 \mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0\otimes z_0 + \frac{1}{2}\int_0^\infty \int_\gamma |u|^2\mathrm{d}s\,\mathrm{d}t - \int_0^\infty \int_\gamma b(s,0)u(t,s)\left(\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi\right)\mathrm{d}s\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^\infty \int_\gamma \left| b(s,0)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi\right|^2 \mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0\otimes z_0 + \frac{1}{2}\int_0^\infty \int_\gamma \left| u(t,s) - b(s,0)\int_\Omega \pi(s,0,X)z(t,\Xi)\,\mathrm{d}\Xi\right|^2 \mathrm{d}s\,\mathrm{d}t.
\end{aligned}$$

The proof is complete. □

For a given solution π to equation (5.7), we consider the evolution equation

$$z' = \mathcal{A}z - B(\mathbb{1}_\gamma B^*\Pi z) \quad \text{in } (0,\infty), \qquad z(0) = z_0, \tag{6.4}$$

where

$$B^*\Pi z(t,s) = -b(s,0)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi, \quad s\in(0,L),\ t\in(0,\infty).$$

Weak solutions to equation (6.4) are defined as weak solutions to equation

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u) \quad \text{in } (0,T), \qquad z(0) = z_0, \tag{6.5}$$

with $u = -B^*\Pi z$. This is meaningful because if $z \in L^2(0,T;L^2(\Omega))$, then $B^*\Pi z \in L^2(0,T;L^2(0,L))$.

Lemma 6.5. *For a given solution π to equation (5.7), equation (6.4) admits a unique weak solution in $L^\infty(0,T;L^2(\Omega))$. Moreover this solution also belongs to $L^2(0,T;L^2(0,L;H^1(0,1;d)))$.*

Proof. We first show, by using a fixed point argument, that equation (6.4) admits a unique weak solution in $L^\infty(0,\bar{t};L^2(\Omega))$, for some $0<\bar{t}\le T$. In (3.5), it is stated that the weak solution z of equation (6.5) obeys

$$\|z\|_{L^\infty(0,T;L^2(\Omega))} + \|\sqrt{a}z\|_{L^\infty(0,L;L^2(0,T;L^2(0,1)))} + \|z\|_{L^2(0,T;L^2(0,L;H^1(0,1;d)))} \le C_5\Big(\|u\|_{L^2(0,T;L^2(0,L))} + \|z_0\|_{L^2(\Omega)}\Big), \tag{6.6}$$

where C_5 is independent of T. If v belongs to $L^\infty(0,T;L^2(\Omega))$, then from Theorem 5.8 it follows that

$$\|\mathbb{1}_\gamma B^*\Pi v\|_{L^2(0,T;L^2(0,L))} \le C_{11}T^{1/2}\|v\|_{L^\infty(0,T;L^2(\Omega))}, \tag{6.7}$$

for some constant C_{11} depending on $\|\phi\|_{L^2(\Omega)}$, but independent of T. We choose $\bar{t} > 0$ such that $C_5\, C_{11}|\bar{t}|^{1/2} \leq 1/2$. Let v be in $L^\infty(0,\bar{t}; L^2(\Omega))$ and $z_v \in L^\infty(0,\bar{t}; L^2(\Omega))$ be the solution to

$$z' = \mathcal{A}z - B(\mathbb{1}_\gamma B^*\Pi v) \quad \text{in } (0,\bar{t}), \qquad z(0) = z_0.$$

Let us denote Ψ the mapping $v \mapsto z_v$. Let v_1 and v_2 be in $L^\infty(0,\bar{t}; L^2(\Omega))$. With (6.6) and (6.7) we have

$$\|z_{v_1} - z_{v_2}\|_{L^\infty(0,\bar{t};L^2(\Omega))} \leq C_5\, C_{11}|\bar{t}|^{1/2}\|v_1 - v_2\|_{L^\infty(0,\bar{t};L^2(\Omega))}.$$

Since $C_5 C_{11}|\bar{t}|^{1/2} \leq 1/2$, Ψ is a contraction in $L^\infty(0,\bar{t}; L^2(\Omega))$. Thus equation (6.4) admits a unique solution in $L^\infty(0,\bar{t}; L^2(\Omega))$. If $v \in L^2(0,\bar{t}; L^2(\Omega))$, with (6.6) and (6.7) it follows that z belongs to $L^2(0,T; L^2(0,L; H^1(0,1;d)))$. We can repeat the fixed point argument on $(\bar{t}, 2\bar{t})$ in the following way. Let us set

$$E = \Big\{v \in L^\infty(0,2\bar{t}; L^2(\Omega)) \mid v|_{(0,\bar{t})} = z\Big\},$$

where z is the solution of (6.4) in $(0,\bar{t})$. Step by step, we prove that for all $T > 0$ equation (6.4) admits a unique solution in $L^\infty(0,T; L^2(\Omega))$ for all $T > 0$. $\square$

Theorem 6.1. *For a given solution π to equation (5.7), equation (6.4) admits a unique weak solution in $C_b([0,\infty); L^2(\Omega))$. Moreover this solution also belongs to $L^2(0,\infty; L^2(0,L; H^1(0,1;d)))$ and*

$$\|z\|_{L^\infty(0,\infty;L^2(\Omega))} + \|\sqrt{a}z\|_{L^\infty(0,L;L^2(0,\infty;L^2(0,1)))} + \|z\|_{L^2(0,\infty;L^2(0,L;H^1(0,1;d)))} \leq C_6\|z_0\|_{L^2(\Omega)}. \tag{6.8}$$

Proof. Let u be in $L^2(0,\infty; L^2(0,L))$, $z_0 \in L^2(\Omega)$, and z be the solution to equation

$$z' = \mathcal{A}z + B(\mathbb{1}_\gamma u), \qquad z(0) = z_0.$$

As in the proof of Lemma 6.3, we can show that

$$\begin{aligned}\int_0^T \int_{\mathcal{O}} ((\mathcal{A}_X^* + \mathcal{A}_\Xi^*)\pi)\; z(t)\otimes z(t)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t \;=\; &\int_{\mathcal{O}} \pi\, z(T)\otimes z(T) - \int_{\mathcal{O}} \pi\, z_0\otimes z_0\\ &- 2\int_0^T \int_\gamma b(s,0)\,u(t,s)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

Next, as in the proof of Lemma 6.4, we can establish the identity

$$\begin{aligned}&\frac{1}{2}\int_0^T \int_{\mathcal{O}} \mathbf{\Phi}(X,\Xi)z(t,X)\,z(t,\Xi)\,\mathrm{d}X\,\mathrm{d}\Xi\,\mathrm{d}t + \frac{1}{2}\int_0^T \int_\gamma |u|^2\mathrm{d}s\,\mathrm{d}t + \frac{1}{2}\int_{\mathcal{O}} \pi\, z(T)\otimes z(T)\\ &\qquad = \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0\otimes z_0 + \frac{1}{2}\int_0^T \int_\gamma \left|u(t,s) - b(s,0)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi\right|^2 \mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

In particular, if $u(t,s) = b(s,0)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\,\mathrm{d}\Xi$, we obtain

$$\int_0^T \int_\gamma \left|b(s,0)\int_\Omega \pi(s,0,X)z(t,X)\,\mathrm{d}X\right|^2 \mathrm{d}s\,\mathrm{d}t \leq \int_{\mathcal{O}} \pi\, z_0\otimes z_0.$$

This means that the solution to equation (6.4) is such that the mapping $(t,s) \mapsto \mathbb{1}_\gamma b(s,0)\int_\Omega \pi(s,0,\Xi)z(t,\Xi)\mathrm{d}\Xi$ belongs to $L^2(0,\infty; L^2(0,L))$. Estimate (6.8) follows from (6.6) for $T = \infty$. $\square$

Theorem 6.2. *The algebraic Riccati equation (5.7) admits a unique solution.*

Proof. Let $(\bar{z}, \bar{u})$ be the solution to problem $(\mathcal{P}_{z_0})$. Let π be a solution to equation (5.7), and let z be the solution to equation (6.4) corresponding to π. From Theorem 6.1 we deduce that $-\mathbb{1}_\gamma B^*\pi z$ is an admissible control. Due to Lemma 6.4 we have:

$$J(z,u) = \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0 \otimes z_0,$$

and

$$J(\bar{z},\bar{u}) = \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0 \otimes z_0 + \int_0^\infty \int_\gamma \left| \bar{u}(\tau,s) - b(s,0)\int_{\Omega_\Xi} \pi(s,0,\Xi)\, \bar{z}(\tau,\Xi)\, \mathrm{d}\Xi \right|^2 \mathrm{d}s\, \mathrm{d}\tau.$$

Thus

$$J(z,u) = J(\bar{z},\bar{u}) = \frac{1}{2}\int_{\mathcal{O}} \pi\, z_0 \otimes z_0,$$

and

$$\bar{u}(\tau,s) = b(s,0)\int_{\Omega_\Xi} \pi(s,0,\Xi)\, \bar{z}(\tau,\Xi)\, \mathrm{d}\Xi.$$

Henceforth, there is a unique operator π such that

$$\frac{1}{2}\int_{\mathcal{O}} \pi\, z_0 \otimes z_0 \ = \inf(\mathcal{P}_{z_0}),$$

for all $z_0 \in L^2(\Omega)$. The proof is complete. □

Theorem 6.3. *Let $(\bar{z}, \bar{u})$ be the optimal solution to problem $(\mathcal{P}_{z_0})$. The optimal control $\bar{u}$ obeys the feedback formula*

$$\bar{u}(\tau,s) = \mathbb{1}_\gamma(s)\, b(s,0)\left(\int_\Omega \pi(s,0,\Xi)\bar{z}(\tau,\Xi)\, \mathrm{d}\Xi\right) \quad s \in (0,L), \quad \tau \in (0,\infty), \tag{6.9}$$

where π is the solution to the algebraic Riccati equation (5.7). The optimal cost is given by

$$J(\bar{z},\bar{u}) = \frac{1}{2}\int_{\mathcal{O}} \pi z_0 \otimes z_0.$$

Proof. Theorem 6.3 is a direct consequence of Theorem 6.1 and Lemma 6.4. □

We finish this section by introducing the infinitesimal generator of the semigroup associated with the optimal solution of problem (P). For every $z_0 \in L^2(\Omega)$, let us denote by z_{z_0} the solution to equation (6.4). According to Theorem 6.1, the family of operators

$$\Big(z_0 \longmapsto z_{z_0}(t)\Big)_{t\geq 0}$$

is an exponentially stable semigroup on $L^2(\Omega)$. The exponential stability follows from (6.8) and from Datko's Theorem [21], Theorem 3.1(i), Part IV. Let us denote it by $(\mathrm{e}^{tA_\pi})_{t\geq 0}$ and by $(A_\pi, D(A_\pi))$ its infinitesimal generator. Since $(\mathrm{e}^{tA_\pi})_{t\geq 0}$ is an exponentially stable semigroup on $L^2(\Omega)$, the domain $D(A_\pi)$ is defined by

$$D(A_\pi) = \left\{ \int_0^\infty \mathrm{e}^{\tau A_\pi}\, \psi\, \mathrm{d}\tau \mid \psi \in L^2(\Omega) \right\}.$$

Moreover,

$$z \in D(\mathcal{A}_\pi) \quad \text{and} \quad \mathcal{A}_\pi z = \psi,$$

if and only if

$$z = -\int_0^\infty \mathrm{e}^{\tau A_\pi}\, \psi\, \mathrm{d}\tau.$$

We are now going to give another characterization of $D(A_\pi)$.

Theorem 6.4. *A function $z \in L^2(\Omega)$ belongs to $D(\mathcal{A}_\pi)$ if and only if z is the solution to the variational problem*

$$\begin{aligned}
&z \in L^2(0,L;H^1(0,1;d)),\\
&Az, \text{ calculated in the sense of distributions in } \Omega, \text{ belongs to } L^2(\Omega),\\
&Az = \psi \quad \text{in } \Omega, \qquad T_0\Big(az, -b\frac{\partial z}{\partial y}\Big) = -\mathbb{1}_\gamma(s)\, b(s,0)^2 \int_\Omega \pi(s,0,\Xi)\, z(\Xi)\,\mathrm{d}\Xi.
\end{aligned} \tag{6.10}$$

Proof. Let $z \in D(\mathcal{A}_\pi)$ be the unique solution to the equation $\mathcal{A}_\pi z = \psi$, that is to say

$$z = -\int_0^\infty \mathrm{e}^{\tau A_\pi}\,\psi\,\mathrm{d}\tau.$$

Thus z is the limit in $L^2(\Omega)$, when t tends to infinity, of the function $\zeta(t)$ defined by

$$\zeta(t) = -\int_0^t \mathrm{e}^{\tau A_\pi}\psi\,\mathrm{d}\tau = -\int_0^t \mathrm{e}^{(t-s)A_\pi}\psi\,\mathrm{d}s.$$

Observe that ζ is the solution to the equation

$$\zeta' = A\zeta - B(\mathbb{1}_\gamma B^*\Pi\zeta) - \psi, \qquad \zeta(0) = 0.$$

Therefore ζ obeys the following boundary condition

$$T_0\Big(a\zeta(t), -b\frac{\partial\zeta(t)}{\partial y}\Big) = -\mathbb{1}_\gamma\, b(s,0)^2 \int_\Omega \pi(s,0,\Xi)\,\zeta(t,\Xi)\,\mathrm{d}\Xi.$$

We can pass to the limit when t tends to infinity in the above identity, and we obtain the same one for z.

To prove that Az, calculated in the sense of distributions in Ω, is equal to ψ, we notice that, for all $\varphi \in \mathcal{D}(\Omega)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \zeta(t)\varphi = \int_\Omega \zeta(t)A^*\varphi - \int_\Omega \psi\varphi.$$

Thus the mapping

$$t \longmapsto \int_\Omega \zeta(t)\varphi$$

belongs to $C^1([0,\infty))$, it admits a limit and together with its derivative when t tends to infinity. Thus the limit of $\int_\Omega \zeta(t)A^*\varphi - \int_\Omega \psi\varphi$, when t tends to infinity, is equal to zero, *i.e.*:

$$\int_\Omega zA^*\varphi - \int_\Omega \psi\varphi = 0 \qquad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

This means that $Az = \psi$ in $\mathcal{D}'(\Omega)$.

Now we want to show that $z \in L^2(0,L;H^1(0,1;d))$. Observe that

$$\|z\|_{L^2(\Omega)} \le C\|\psi\|_{L^2(\Omega)},$$

and that ζ belongs to $L^2_{\mathrm{loc}}([0,\infty);L^2(0,L;H^1(0,1;d)))$. Thus we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \zeta(t)\varphi = &\int_\Omega \Big(a\,\zeta(t)\frac{\partial\varphi}{\partial x} - b\frac{\partial\zeta(t)}{\partial y}\frac{\partial\varphi}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial\zeta(t)}{\partial y}\varphi - c\zeta(t)\varphi\Big)\,\mathrm{d}x\,\mathrm{d}y\\
&- \int_\Omega \psi\varphi + \int_\gamma \varphi(s,0)\int_\Omega \pi(s,0,X)\zeta(t,X)\,\mathrm{d}X\,\mathrm{d}s,
\end{aligned}$$

for all $\varphi \in F$, where

$$F = \Big\{\varphi \in L^2(0, L; H^1(0, 1; d)) \cap H^1(0, L; L^2(0, 1)) \mid \varphi(L, \cdot) = 0\Big\}.$$

As previously we can show that, if $\varphi \in F$, the mapping

$$t \longmapsto \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \zeta(t)\varphi$$

tends to zero when t tends to infinity. Thus z is also the solution of the variational equation

$$\int_\Omega \left(a\, z \frac{\partial \varphi}{\partial x} - b \frac{\partial z}{\partial y}\frac{\partial \varphi}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial z}{\partial y}\varphi - cz\varphi \right) \mathrm{d}x\,\mathrm{d}y - \int_\Omega \psi\varphi + \int_\gamma \varphi(s,0) \int_\Omega \pi(s,0,\Xi) z(\Xi)\,\mathrm{d}\Xi\,\mathrm{d}s = 0,$$

for all $\varphi \in F$. With the estimate of z in $L^2(\Omega)$, and with the estimates obtained in [6] we can show that

$$\|z\|_{L^2(0,L;H^1(0,1;d))} \le C\|\psi\|_{L^2(\Omega)}.$$

Let us give a short explanation. Setting $Z = \mathrm{e}^{-kx} z$, with $k > 0$, we can show that Z is the solution of the variational equation

$$\int_\Omega \left(a\, Z \frac{\partial \varphi}{\partial x} - b \frac{\partial z}{\partial y}\frac{\partial \varphi}{\partial y} - \frac{\partial b}{\partial y}\frac{\partial Z}{\partial y}\varphi - (c + ka)Z\varphi \right) \mathrm{d}x\,\mathrm{d}y - \int_\Omega \mathrm{e}^{-kx}\psi\varphi + \int_\gamma \phi(s,0)\mathrm{e}^{-ks} g(s)\,\mathrm{d}s = 0,$$

for all $\varphi \in F$, where

$$g(s) = \int_\Omega \pi(s,0,X) z(\Xi)\,\mathrm{d}\Xi.$$

We can verify that

$$\|g\|_{L^2(0,L)} \le C\|z\|_{L^2(\Omega)}.$$

Next using the techniques in [6], the following estimate can be shown

$$\|Z\|_{L^2(0,L;H^1(0,1;d))} \le C\|g\|_{L^2(0,L)},$$

from which we can deduce the corresponding estimate for z.

Conversely, if z is a solution to the variational problem (6.10), with the results in [6], Section 5, we can show that z is the limit in $L^2(\Omega)$, when t tends to infinity of the function ζ introduced above. The proof is complete. □

References

[1] V. Barbu, I. Lasiecka and R. Triggiani, Extended algebraic Riccati equations in the abstract hyperbolic case. *Nonlinear Anal.* **40** (2000) 105–129.

[2] A. Bensoussan, G. Da. Prato, M.C. Delfour and S.K. Mitter, *Representation and Control of Infinite Dimensional Systems*, *Systems & Control: Fondations & Applications* **2**. Boston, Birkhäuser (1993).

[3] J.-M. Buchot, Stabilization of the laminar turbulent transition location, in *Proceedings MTNS 2000*, El Jaï Ed. (2000).

[4] J.-M. Buchot, *Stabilisation et contrôle optimal des équations de Prandtl.* Ph.D. Thesis, École supérieure d'Aéronautique et de l'Espace, Toulouse (2002).

[5] J.-M. Buchot and J.-P. Raymond, A linearized model for boundary layer equations, in *International Series of Numerical Mathematics* **139**, Birkhäuser (2001) 31–42.

[6] J.-M. Buchot and J.-P. Raymond, A linearized Crocco equation. *J. Math. Fluid Mech.* **8** (2006) 510–541.

[7] J.-M. Buchot and J.-P. Raymond, Feedback stabilization of a boundary layer equation – Part 2: Nonhomogeneous state equation and numerical experiments. *Appl. Math. Res. eXpress* (2010) doi:10.1093/amrx/abp007.

[8] R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique* **4**. Masson, Paris (1988).

[9] F. Flandoli, Algebric Riccati Equations arising in boundary control problems. *SIAM J. Control Optim.* **25** (1987) 612–636.

[10] F. Flandoli, I. Lasiecka and R. Triggiani, Algebraic Riccati Equations with non-smoothing observation arising in hyperbolic and Euler-Bernoulli boundary control problems. *Ann. Math. Pura Appl.* **153** (1988) 307–382.

[11] I. Lasiecka and R. Triggiani, *Control theory for partial differential equations I, Abstract parabolic systems*. Cambridge University Press, Cambridge (2000).

[12] I. Lasiecka and R. Triggiani, Optimal Control and Algebraic Riccati Equations under Singular Estimates for $e^{At}B$ in the Abscence of Analycity, Part I: The stable case, in *Lecture Notes in Pure in Applied Mathematics* **225**, Marcel Dekker (2002) 193–219.

[13] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes*. Dunod, Paris (1968).

[14] O.A. Oleinik and V.N. Samokhin, *Mathematical Models in Boundary Layer Theory, Applied Mathematics and Mathematical Computation* **15**. Chapman & Hall/CRC, Boca Raton, London, New York (1999).

[15] A.J. Pritchard and D. Salamon, The linear quadratic control of problem for infinite dimensional systems with unbounded input and output operators. *SIAM J. Control Optim.* **25** (1987) 121–144.

[16] H. Triebel, *Interpolation theory, Functions spaces, Differential operators*. North Holland (1978).

[17] R. Triggiani, An optimal control problem with unbounded control operator and unbounded observation operator where Algebraic Riccati Equation is satisfied as a Lyapunov equation. *Appl. Math. Letters* **10** (1997) 95–102.

[18] R. Triggiani, The Algebraic Riccati Equation with unbounded control operator: The abstract hyperbolic case revisited. *Contemporary mathematics* **209** (1997) 315–338.

[19] G. Weiss and H. Zwart, An example in LQ optimal control. *Syst. Control Lett.* **33** (1998) 339–349.

[20] Z. Xin and L. Zhang, On the global existence of solutions to the Prandtl's system. *Adv. Math.* **181** (2004) 88–133.

[21] J. Zabczyck, *Mathematical Control Theory*. Birkhäuser (1995).