ESAIM: COCV 17 (2011) 293–321
DOI: 10.1051/cocv/2010005

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

CUT LOCUS AND OPTIMAL SYNTHESIS IN THE SUB-RIEMANNIAN PROBLEM ON THE GROUP OF MOTIONS OF A PLANE*

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Abstract. The left-invariant sub-Riemannian problem on the group of motions (rototranslations) of a plane SE(2) is considered. In the previous works [Moiseev and Sachkov, *ESAIM: COCV*, DOI: 10.1051/cocv/2009004; Sachkov, *ESAIM: COCV*, DOI: 10.1051/cocv/2009031], extremal trajectories were defined, their local and global optimality were studied. In this paper the global structure of the exponential mapping is described. On this basis an explicit characterization of the cut locus and Maxwell set is obtained. The optimal synthesis is constructed.

Mathematics Subject Classification. 49J15, 93B29, 93C10, 53C17, 22E30.

Received May 28, 2009.
Published online March 24, 2010.

1. Introduction

This work completes the study of the left-invariant sub-Riemannian problem on the group of motions of a plane $\mathrm{SE}(2) = \mathbb{R}^2 \ltimes \mathrm{SO}(2)$ started in [7,10]. In visual geometric terms, this problem can be stated as follows: given two unit vectors $v_0 = (\cos\theta_0, \sin\theta_0)$, $v_1 = (\cos\theta_1, \sin\theta_1)$ attached respectively at two given points (x_0, y_0), (x_1, y_1) in the plane, one should find an optimal motion in the plane that transfers the vector v_0 to the vector v_1, see Figure 1. The vector can move forward or backward and rotate simultaneously. The required motion should be optimal in the sense of minimal length in the space (x, y, θ), where θ is the slope of the moving vector.

The corresponding optimal control problem reads as follows:

$$\dot{x} = u_1 \cos\theta, \quad \dot{y} = u_1 \sin\theta, \quad \dot{\theta} = u_2, \tag{1.1}$$

$$q = (x, y, \theta) \in M = \mathbb{R}^2_{x,y} \times S^1_\theta, \quad u = (u_1, u_2) \in \mathbb{R}^2, \tag{1.2}$$

$$q(0) = q_0 = (0,0,0), \qquad q(t_1) = q_1 = (x_1, y_1, \theta_1), \tag{1.3}$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, \mathrm{d}t \to \min, \tag{1.4}$$

Keywords and phrases. Optimal control, sub-Riemannian geometry, differential-geometric methods, left-invariant problem, group of motions of a plane, rototranslations, cut locus, optimal synthesis.

* *The author is partially supported by Russian Foundation for Basic Research, Project No. 09-01-00246-a, and by the Program of Presidium of Russian Academy of Sciences "Mathematical control theory".*

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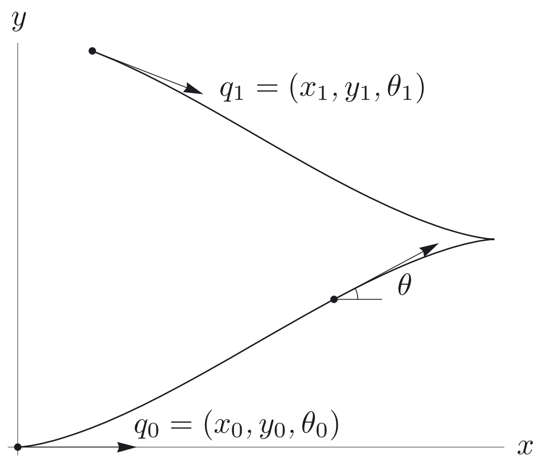

FIGURE 1. Problem statement.

or, equivalently,

$$J = \frac{1}{2}\int_0^{t_1} (u_1^2 + u_2^2)\,\mathrm{d}t \to \min. \tag{1.5}$$

This problem has important relations to vision [4,8,9], robotics [6], and diffusion equation on SE(2) [2].

Notice that before this work a global description of the cut locus and optimal synthesis was known for left-invariant sub-Riemannian problems on the following Lie groups only: the Heisenberg group (Vershik and Gershkovich [11]), and SO(3), SU(2) $\cong S^3$, SL(2) (Boscain and Rossi [3]).

First we recall the main results of the previous works [7,10]. In paper [7] the normal Hamiltonian system of Pontryagin Maximum Principle was written in a triangular form in appropriate coordinates on cotangent bundle T^*M, so that its vertical subsystem takes the form of mathematical pendulum:

$$\dot\gamma = c, \quad \dot c = -\sin\gamma, \qquad (\gamma, c) \in C \cong (2S^1_\gamma) \times \mathbb{R}_c, \tag{1.6}$$

$$\dot x = \sin\frac{\gamma}{2}\cos\theta, \quad \dot y = \sin\frac{\gamma}{2}\sin\theta, \quad \dot\theta = -\cos\frac{\gamma}{2}\cdot \tag{1.7}$$

The phase cylinder of pendulum (1.6) decomposes into invariant subsets according to values of the energy $E = c^2/2 - \cos\gamma$:

$$C = \bigcup_{i=1}^5 C_i,$$

$$C_1 = \{\lambda \in C \mid E \in (-1, 1)\}, \tag{1.8}$$

$$C_2 = \{\lambda \in C \mid E \in (1, +\infty)\}, \tag{1.9}$$

$$C_3 = \{\lambda \in C \mid E = 1,\ c \neq 0\}, \tag{1.10}$$

$$C_4 = \{\lambda \in C \mid E = -1\} = \{(\gamma, c) \in C \mid \gamma = 2\pi n,\ c = 0\}, \tag{1.11}$$

$$C_5 = \{\lambda \in C \mid E = 1,\ c = 0\} = \{(\gamma, c) \in C \mid \gamma = \pi + 2\pi n,\ c = 0\}. \tag{1.12}$$

In the subsets C_1, C_2, C_3 elliptic coordinates (φ, k) that rectify the flow of the pendulum were introduced: φ is the phase, and k a reparameterized energy of pendulum (1.6):

$$k = \sqrt{(E+1)/2} \text{ in } C_1 \cup C_3, \qquad k = \sqrt{2/(E+1)} \text{ in } C_2.$$

The Hamiltonian system (1.6), (1.7) was integrated in Jacobi's functions [12]. The equation of pendulum (1.6) has a discrete group of symmetries $G = \{\mathrm{Id}, \varepsilon^1, \ldots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ generated by reflections in the axes of coordinates γ, c, and translations $(\gamma, c) \mapsto (\gamma + 2\pi, c)$. Reflections ε^i are symmetries of the exponential mapping

$$\mathrm{Exp} \,:\, N = C \times \mathbb{R}_+ \to M, \qquad \mathrm{Exp}(\lambda, t) = q_t.$$

The main result of work [7] is an upper bound on cut time

$$t_{\mathrm{cut}} = \sup\{t_1 > 0 \mid q_s \text{ is optimal for } s \in [0, t_1]\}$$

along extremal trajectories q_s. It is based on the fact that a sub-Riemannian geodesic cannot be optimal after a Maxwell point, *i.e.*, a point where two distinct geodesics of equal sub-Riemannian length meet one another. A natural idea is to look for Maxwell points corresponding to discrete symmetries of the exponential mapping. For each extremal trajectory $q_s = \mathrm{Exp}(\lambda, s)$, we described Maxwell times $t^n_{\varepsilon^i}(\lambda)$, $i = 1, \ldots, 7$, $n = 1, 2, \ldots$, corresponding to discrete symmetries ε^i. The following upper bound was proved in work [7]:

$$t_{\mathrm{cut}}(\lambda) \leq \mathbf{t}(\lambda), \qquad \lambda \in C, \tag{1.13}$$

where $\mathbf{t}(\lambda) = \min(t^1_{\varepsilon^i}(\lambda))$ is the first Maxwell time corresponding to the group of symmetries G. We recall the explicit definition of the function $\mathbf{t}(\lambda)$ below in equations (2.1)–(2.5).

In work [10], the local optimality of sub-Riemannian geodesics was completely characterized. Extremal trajectories corresponding to oscillating pendulum (*i.e.*, to $\lambda \in C_1$) do not have conjugate points, thus they are locally optimal forever. In the case of rotating pendulum ($\lambda \in C_2$) the first conjugate time is bounded from below and from above by the first Maxwell times $t^1_{\varepsilon^2}$ and $t^1_{\varepsilon^5}$ respectively. For critical values of energy of the pendulum, there are no conjugate points. As a consequence, the following bound was proved in Theorem 2.5 [10]:

$$\mathbf{t}(\lambda) \leq t_1^{\mathrm{conj}}(\lambda), \qquad \lambda \in C. \tag{1.14}$$

Also, in work [10] the global optimality of geodesics was studied. We constructed open dense domains in preimage and image of exponential mapping and proved that the exponential mapping transform these strata diffeomorphically. As a consequence, we showed that inequality (1.13) is in fact an equality.

In this work we obtain our further results for problem (1.1)–(1.5). We consider in detail the action of the exponential mapping at the boundary of the 3-dimensional diffeomorphic domains. This boundary is decomposed into smooth strata of dimension 2, 1, 0 so that restriction of exponential mapping to these strata is a diffeomorphism (Sect. 2). These results provide a detailed description of the global structure of the exponential mapping (Thm. 3.1). The optimal synthesis is constructed in Theorem 3.2.

In Theorems 3.4 and 3.5 we characterize the global structure of the Maxwell set (the set of points q_1 connected by more than one optimal trajectory with the initial point q_0), and the cut locus (the set of points where extremal trajectories lose optimality). For each point of the Maxwell set there are exactly two optimal trajectories. The cut locus has three connected components $\mathrm{Cut}^+_{\mathrm{loc}}$, $\mathrm{Cut}^-_{\mathrm{loc}}$, and $\mathrm{Cut}_{\mathrm{glob}}$. The initial point q_0 is contained in the closure of the local components $\mathrm{Cut}^{\pm}_{\mathrm{loc}}$, and is separated from the global component $\mathrm{Cut}_{\mathrm{glob}}$. The global component admits a simple description:

$$\mathrm{Cut}_{\mathrm{glob}} = \{q = (x, y, \theta) \in M \mid \theta = \pi\},$$

while the local components are subsets of the Moebius strip:

$$\mathrm{Cut}_{\mathrm{loc}}^{\pm} \subset \{q = (x, y, \theta) \in M \mid x \cos(\theta/2) + y \sin(\theta/2) = 0\}$$

defined by some inequalities, see Theorem 3.5. Embedding of the cut locus in the solid torus is shown at Figure 18.

In Section 4 we present explicit optimal solutions for special boundary conditions.

2. Structure of exponential mapping at the boundary of open strata

We recall some more definitions and notation introduced in the previous works [7,10]. The function $\mathbf{t} : C \to (0, +\infty]$ on the phase cylinder of pendulum $(2S^1_\gamma) \times \mathbb{R}_c = C = \cup_{i=1}^5 C_i$ that evaluates the cut time along sub-Riemannian geodesics $\mathrm{Exp}(\lambda, t)$, $t \in C$, is defined as follows:

$$\lambda \in C_1 \quad \Rightarrow \quad \mathbf{t}(\lambda) = 2K(k), \tag{2.1}$$

$$\lambda \in C_2 \quad \Rightarrow \quad \mathbf{t}(\lambda) = 2kp_1^1(k), \tag{2.2}$$

$$\lambda \in C_3 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty, \tag{2.3}$$

$$\lambda \in C_4 \quad \Rightarrow \quad \mathbf{t}(\lambda) = \pi, \tag{2.4}$$

$$\lambda \in C_5 \quad \Rightarrow \quad \mathbf{t}(\lambda) = +\infty, \tag{2.5}$$

where $p_1^1(k)$ is the first positive root of the equation $\operatorname{cn} p\,(\mathrm{E}(p) - p) - \operatorname{dn} p \operatorname{sn} p = 0$. Here and below we use Jacobi's functions cn, sn, dn, E, and the complete elliptic integral of the first kind K [12]. Further,

$$\begin{aligned}
&\widehat{M} = M \setminus \{q_0\},\\
&\widehat{N} = \{(\lambda, t) \in N \mid t \le \mathbf{t}(\lambda)\},\\
&N_i = C_i \times \mathbb{R}_+, \qquad i = 1, \dots, 5,\\
&\widetilde{N} = \{(\lambda, t) \in \cup_{i=1}^3 N_i \mid t < \mathbf{t}(\lambda), \ \operatorname{sn}\tau \operatorname{cn}\tau \neq 0\},\\
&N' = \{(\lambda, t) \in \cup_{i=1}^3 N_i \mid t = \mathbf{t}(\lambda) \text{ or } \operatorname{sn}\tau \operatorname{cn}\tau = 0\} \cup \widehat{N}_4 \cup N_5,\\
&\widehat{N}_4 = \widehat{N} \cap N_4,\\
&\widetilde{M} = \{q \in M \mid R_1(q) R_2(q) \sin\theta \neq 0\},\\
&M' = \{q \in M \mid R_1(q) R_2(q) \sin\theta = 0\},
\end{aligned}$$

where

$$R_1 = y \cos\frac{\theta}{2} - x \sin\frac{\theta}{2}, \qquad R_2 = x \cos\frac{\theta}{2} + y \sin\frac{\theta}{2}.$$

Along with the coordinates (k, φ, t), we use in the domains N_1, N_2, N_3 also the coordinates (k, p, τ):

$$\begin{aligned}
&(\lambda, t) \in N_1 \cup N_3 \quad \Rightarrow \quad \tau = (2\varphi + t)/2, \qquad p = t/2,\\
&(\lambda, t) \in N_2 \quad \Rightarrow \quad \tau = (2\varphi + t)/(2k), \qquad p = t/(2k).
\end{aligned}$$

In work [10] we proved that $\mathrm{Exp} : \widetilde{N} \to \widetilde{M}$ is a diffeomorphism, and that $\mathrm{Exp}(N') \subset M'$. In this section we describe the action of the exponential mapping

$$\mathrm{Exp} : N' = \widehat{N} \setminus \widetilde{N} \to M' = \widehat{M} \setminus \widetilde{M}.$$

2.1. Decomposition of the set N'

Consider the following subsets of the set N':

$$N_{\text{cut}} = \{(\lambda, t) \in N \mid t = \mathbf{t}(\lambda)\}, \tag{2.6}$$
$$N_{\text{conj}} = \{(\lambda, t) \in N_2 \mid t = \mathbf{t}(\lambda),\ \operatorname{sn}\tau = 0\}, \tag{2.7}$$
$$N_{\text{Max}} = N_{\text{cut}} \setminus N_{\text{conj}}, \qquad N_{\text{rest}} = N' \setminus N_{\text{cut}}.$$

The meaning of the subscripts in N_{cut}, N_{conj}, N_{Max}, and N_{rest} is the following: we will show that $\operatorname{Exp}(N_{\text{cut}})$ is the cut locus, $\operatorname{Exp}(N_{\text{Max}})$ is the first Maxwell set, $\operatorname{Exp}(N_{\text{conj}})$ is the intersection of the cut locus with the conjugate locus (caustic), and $\operatorname{Exp}(N_{\text{rest}})$ has no special meaning in this problem (so it contains all the rest strata), see Theorem 3.4. We have the following decompositions:

$$\widehat{N} = \widetilde{N} \sqcup N', \qquad N' = N_{\text{cut}} \sqcup N_{\text{rest}}, \qquad N_{\text{cut}} = N_{\text{Max}} \sqcup N_{\text{conj}}, \tag{2.8}$$

here and below we denote by $\sqcup$ the union of disjoint sets.

In order to study the structure of the exponential mapping at the set N', we need a further decomposition into subsets N'_i, $i = 1, \dots, 58$, defined by Table 1.

Images of the projections

$$N'_i \cap \{t < \mathbf{t}(\lambda),\ \operatorname{sn}\tau\ \operatorname{cn}\tau = 0\} \to \{p = 0\}, \qquad (k, \tau, p) \mapsto (k, \tau, 0),$$
$$N'_i \cap \{t = \mathbf{t}(\lambda)\} \to \{p = 0\}, \qquad (k, \tau, p) \mapsto (k, \tau, 0),$$

are shown respectively in Figures 2 and 3.

Table 1 provides a definition of the sets N'_i, *e.g.*, the second column of this table means that

$$N'_1 = \{(\lambda, t) \in N \mid \lambda \in C_1^0,\ \tau \in (0, K),\ p = K,\ k \in (0, 1)\}.$$

Here we use the following decomposition of the sets C_i into connected components:

$$C_1 = \cup_{i=0}^1 C_1^i, \qquad C_1^i = \{(\gamma, c) \in C_1 \mid \operatorname{sgn}(\cos(\gamma/2)) = (-1)^i\}, \quad i = 0, 1,$$
$$C_2 = C_2^+ \cup C_2^-, \qquad C_2^\pm = \{(\gamma, c) \in C_2 \mid \operatorname{sgn} c = \pm 1\},$$
$$C_3 = \cup_{i=0}^1 (C_3^{i+} \cup C_3^{i-}),$$
$$C_3^{i\pm} = \{(\gamma, c) \in C_3 \mid \operatorname{sgn}(\cos(\gamma/2)) = (-1)^i,\ \operatorname{sgn} c = \pm 1\}, \quad i = 0, 1,$$
$$C_4 = \cup_{i=0}^1 C_4^i, \qquad C_4^i = \{(\gamma, c) \in C \mid \gamma = 2\pi i,\ c = 0\}, \quad i = 0, 1,$$
$$C_5 = \cup_{i=0}^1 C_5^i, \qquad C_5^i = \{(\gamma, c) \in C \mid \gamma = \pi + 2\pi i,\ c = 0\}, \quad i = 0, 1.$$

Introduce the following index sets for numeration of the subsets N'_i:

$$I = \{1, \dots, 58\}, \qquad C = \{1, \dots, 34\}, \qquad J = \{26, 28, 30, 32\}, \tag{2.9}$$
$$R = \{35, \dots, 58\}, \qquad X = C \setminus J. \tag{2.10}$$

Notice that $I = C \sqcup R$, $J \subset C$.

Lemma 2.1.

(1) *We have* $N'_i \cap N'_j = \emptyset$ *for any distinct* $i, j \in I$.

TABLE 1. Definition of sets N_i'.

N_i'	N_1'	N_2'	N_3'	N_4'	N_5'	N_6'	N_7'	N_8'
λ	C_1^0	C_1^0	C_1^0	C_1^0	C_1^1	C_1^1	C_1^1	C_1^1
τ	$(0,K)$	$(K,2K)$	$(2K,3K)$	$(3K,4K)$	$(0,K)$	$(K,2K)$	$(2K,3K)$	$(3K,4K)$
p	K	K	K	K	K	K	K	K

N_i'	N_9'	N_{10}'	N_{11}'	N_{12}'	N_{13}'	N_{14}'	N_{15}'	N_{16}'
λ	C_2^+	C_2^+	C_2^+	C_2^+	C_2^-	C_2^-	C_2^-	C_2^-
τ	$(3K,4K)$	$(0,K)$	$(K,2K)$	$(2K,3K)$	$(-3K,-2K)$	$(-2K,-K)$	$(-K,0)$	$(0,K)$
p	p_1^1	p_1^1	p_1^1	p_1^1	p_1^1	p_1^1	p_1^1	p_1^1

N_i'	N_{17}'	N_{18}'	N_{19}'	N_{20}'	N_{21}'	N_{22}'	N_{23}'	N_{24}'	N_{25}'	N_{26}'	N_{27}'	N_{28}'	N_{29}'	N_{30}'
λ	C_1^0	C_1^0	C_1^0	C_1^0	C_1^1	C_1^1	C_1^1	C_1^1	C_2^+	C_2^+	C_2^+	C_2^+	C_2^-	C_2^-
τ	0	K	$2K$	$3K$	0	K	$2K$	$3K$	$3K$	0	K	$2K$	K	$-2K$
p	K	K	K	K	K	K	K	K	p_1^1	p_1^1	p_1^1	p_1^1	p_1^1	p_1^1

N_i'	N_{31}'	N_{32}'	N_{35}'	N_{36}'	N_{37}'	N_{38}'	N_{39}'	N_{40}'	N_{41}'	N_{42}'
λ	C_2^-	C_2^-	C_1^0	C_1^0	C_1^0	C_1^0	C_1^1	C_1^1	C_1^1	C_1^1
τ	$-K$	0	0	K	$2K$	$3K$	0	K	$2K$	$3K$
p	p_1^1	p_1^1	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$	$(0,K)$

N_i'	N_{47}'	N_{48}'	N_{49}'	N_{50}'	N_{51}'	N_{52}'	N_{53}'	N_{54}'
λ	C_3^{0+}	C_3^{0-}	C_3^{1+}	C_3^{1-}	C_2^+	C_2^+	C_2^+	C_2^+
τ	0	0	0	0	$3K$	0	K	$2K$
p	$(0,+\infty)$	$(0,+\infty)$	$(0,+\infty)$	$(0,+\infty)$	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$

N_i'	N_{55}'	N_{56}'	N_{57}'	N_{58}'
λ	C_2^-	C_2^-	C_2^-	C_2^-
τ	K	$-2K$	$-K$	0
p	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$	$(0,p_1^1)$

N_i'	N_{33}'	N_{34}'	N_{43}'	N_{44}'	N_{45}'	N_{46}'
λ	C_4^0	C_4^1	C_4^0	C_4^1	C_5^0	C_5^1
t	π	π	$(0,\pi)$	$(0,\pi)$	$(0,+\infty)$	$(0,+\infty)$

(2) *There are the following decompositions of subsets of the set N':*

$$N_{\text{cut}} = \cup_{i\in C} N_i', \qquad N_{\text{conj}} = \cup_{i\in J} N_i', \qquad N_{\text{rest}} = \cup_{i\in R} N_i',$$

thus

$$N_{\text{Max}} = \cup_{i\in X} N_i', \tag{2.11}$$

$$N' = \sqcup_{i\in I} N_i'. \tag{2.12}$$

Proof. Both statements (1) and (2) follow directly from Table 1, definitions of the sets N', N_{Max}, N_{cut}, N_{conj}, N_{rest}, and decompositions (2.8). □

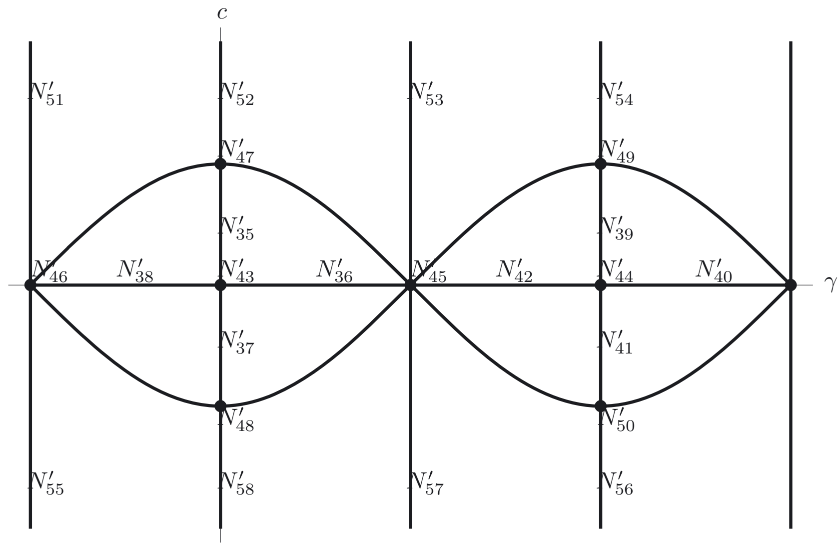

FIGURE 2. $N'_i \cap \{t < \mathbf{t}(\lambda),\ \operatorname{sn}\tau\ \operatorname{cn}\tau\ = 0\}$.

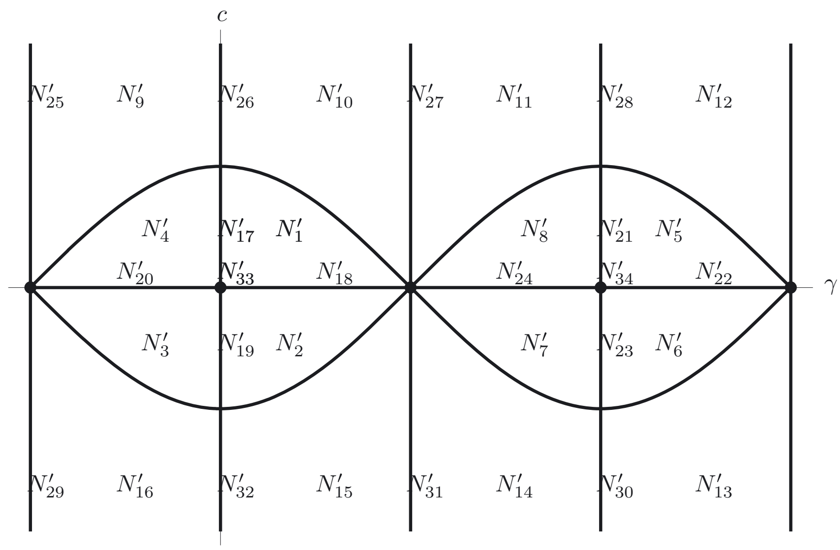

FIGURE 3. $N'_i \cap \{t = \mathbf{t}(\lambda)\}$.

2.2. Exponential mapping of the sets N'_{35}, N'_{47}, N'_{26}, N'_{52}

2.2.1. *Exponential mapping of the set* N'_{35}

In order to describe the image $\operatorname{Exp}(N'_{35})$, we will need the following function:

$$R_1^2(\theta) = 2(\operatorname{artanh}(\sin(\theta/2)) - \sin(\theta/2)), \qquad \theta \in [0, \pi). \tag{2.13}$$

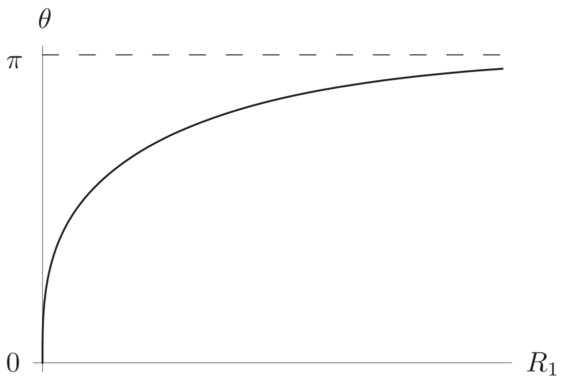

FIGURE 4. Plot of $R_1 = R_1^2(\theta)$.

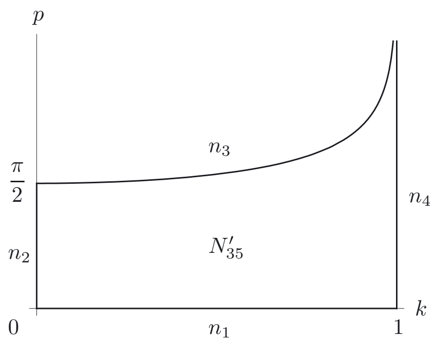

FIGURE 5. Domain N'_{35}.

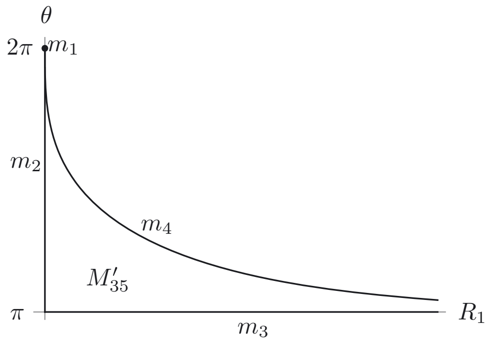

FIGURE 6. Domain M'_{35}.

It is obvious that $R_1^2 \in C^\infty[0,\pi)$, $R_1^2(0) = 0$, $R_1^2(\theta) > 0$ for $\theta \in (0,\pi)$, $\lim_{\theta\to\pi-0} R_1^2(\theta) = +\infty$, and

$$\frac{\mathrm{d}\,R_1^2}{\mathrm{d}\,\theta}(0) = 0. \tag{2.14}$$

A plot of the function $R_1^2(\theta)$ is given at Figure 4.

Define the following subset of the set M', see Figures 6 and 14:

$$M'_{35} = \{q \in M \mid \theta \in (\pi, 2\pi),\ R_1 \in (0, R_1^2(2\pi - \theta)),\ R_2 = 0\}.$$

Lemma 2.2. *The mapping* Exp : $N'_{35} \to M'_{35}$ *is a diffeomorphism of 2-dimensional manifolds.*

To be more precise, we state that $\mathrm{Exp}(N'_{35}) = M'_{35}$ and $\mathrm{Exp}|_{N'_{35}}$ is a diffeomorphism of the manifold N'_{35} onto the manifold M'_{35}. Below we will write such statements briefly as in Lemma 2.2.

Proof. Formulas (5.2)–(5.6) [7] imply that in the domain N'_{35} we have the following:

$$\sin(\theta/2) = \operatorname{sn} p\,, \qquad \cos(\theta/2) = -\operatorname{cn} p\,, \tag{2.15}$$

$$R_1 = 2(p - \mathrm{E}(p))/k, \qquad R_2 = 0. \tag{2.16}$$

By Theorem 2.5 [10] (see (1.14)), the restriction $\mathrm{Exp}|_{N'_{35}}$ is nondegenerate. Thus the set $\mathrm{Exp}(N'_{35})$ is an open connected domain in the 2-dimensional manifold

$$S = \{q \in M \mid \theta \in (\pi, 2\pi),\ R_1 > 0,\ R_2 = 0\}.$$

On the other hand, the set N'_{35} is an open connected simply connected domain in the 2-dimensional manifold

$$T = \{\nu = (\lambda, t) \in N_1 \mid \tau = 0,\ p \in (0, K),\ k \in (0, 1)\}.$$

In the topology of T, we have

$$\begin{aligned}
&\partial N'_{35} = \cup_{i=1}^{4} n_i,\\
&n_1 = \{\nu \in N_1 \mid \tau = 0,\ p = 0,\ k \in [0, 1]\},\\
&n_2 = \{\nu \in N_1 \mid \tau = 0,\ p \in [0, \pi/2],\ k = 0\},\\
&n_3 = \{\nu \in N_1 \mid \tau = 0,\ p = K(k),\ k \in [0, 1)\},\\
&n_4 = \{\nu \in N_1 \mid \tau = 0,\ p \in [0, +\infty),\ k = 1]\},
\end{aligned}$$

see Figure 5.

It follows from formulas (2.15) and (2.16) that

$$\begin{aligned}
&\mathrm{Exp}(n_1) = m_1 = \{q \in M \mid \theta = 2\pi,\ R_1 = 0,\ R_2 = 0\},\\
&\mathrm{Exp}(n_2) = m_2 = \{q \in M \mid \theta \in [\pi, 2\pi],\ R_1 = 0,\ R_2 = 0\},\\
&\mathrm{Exp}(n_3) = m_3 = \{q \in M \mid \theta = \pi,\ R_1 > 0,\ R_2 = 0\},\\
&\mathrm{Exp}(n_4) = m_4 = \{q \in M \mid \theta \in [\pi, 2\pi],\ R_1 = R_1^2(2\pi - \theta),\ R_2 = 0\},
\end{aligned}$$

moreover, $\partial M'_{35} = \cup_{i=1}^{4} m_i$, see Figure 6.

Now we show that $\mathrm{Exp}(N'_{35}) \subset M'_{35}$ and $\mathrm{Exp} : N'_{35} \to M'_{35}$ is a diffeomorphism.

(a) We show that $\mathrm{Exp}(N'_{35}) \cap M'_{35} \neq \emptyset$. Formulas (2.15) and (2.16) give the following asymptotics as $k \to 0$:

$$\theta = 2\pi - 2p + o(1), \qquad R_1 = k(p/2 - (\sin 2p)/4) + o(k).$$

There exists (p, k) close to $(\pi/2, 0)$ such that the corresponding point (θ, R_1) is arbitrarily close to $(0, 0)$, with $\theta > 0$, $R_1 > 0$. Thus there exists $\nu \in N'_{35}$ such that $\mathrm{Exp}(\nu) \in M'_{35}$.

(b) We show that $\mathrm{Exp}(N'_{35}) \neq S$. Formulas (2.15) and (2.16) yield the following chain:

$$\theta \to 2\pi - 0 \quad \Rightarrow \quad \mathrm{sn}\, p \to 0 \quad \Rightarrow \quad p \to 0 \quad \Rightarrow \quad R_1 \to 0.$$

Thus there exists $q \in S \setminus \mathrm{Exp}(N'_{35})$.

(c) We prove that $\mathrm{Exp}(N'_{35}) \subset M'_{35}$. By contradiction, suppose that there exists a point $q_1 \in \mathrm{Exp}(N'_{35}) \setminus M'_{35}$. Since the mapping $\mathrm{Exp}|_{N'_{35}}$ is nondegenerate, we can choose this point such that $q_1 \in \mathrm{Exp}(N'_{35}) \setminus \mathrm{cl}(M'_{35})$.

Choose any point $q_2 \in S \setminus \mathrm{cl}(M'_{35})$. Connect the points q_1, q_2 by a continuous curve in S, and find at this curve a point $q_3 \in S \setminus \mathrm{Exp}(N'_{35})$, $q_3 \notin \mathrm{cl}(M'_{35})$ such that there exists a converging sequence $q^n \to q_3$, $q^n = \mathrm{Exp}(\nu^n) \in \mathrm{Exp}(N'_{35})$. Further, there exist a subsequence $\nu^{n_i} \in N'_{35}$ converging to a finite or infinite limit. If $\nu^{n_i} \to \bar{\nu} \in N'_{35}$, then $q_3 = \mathrm{Exp}(\bar{\nu}) \in \mathrm{int}\,\mathrm{Exp}(N'_{35})$ by nondegeneracy of $\mathrm{Exp}|_{N'_{35}}$, a contradiction. If $\nu^{n_i} \to \bar{\nu} \in \partial N'_{35}$, then

$$q_3 = \mathrm{Exp}(\bar{\nu}) \in \mathrm{Exp}(\partial N'_{35}) = \partial M'_{35} \subset \mathrm{cl}(M'_{35}),$$

a contradiction. Finally, if $\nu^{n_i} \to \infty$, then at this sequence $k^{n_i} \to 1-0$, $p^{n_i} \to \infty$, thus $R_1(q^{n_i}) \to \infty$, a contradiction.

Consequently, $\operatorname{Exp}(N'_{35}) \subset M'_{35}$.

(d) The mapping $\operatorname{Exp} : N'_{35} \to M'_{35}$ is a diffeomorphism since $\operatorname{Exp}|_{N'_{35}}$ is nondegenerate and proper, and N'_{35}, M'_{35} are connected and simply connected. □

2.2.2. *Exponential mapping of the set* N'_{47}

Define the following subset of M', see Figure 14:

$$M'_{47} = \{q \in M \mid \theta \in (\pi, 2\pi),\ R_1 = R_1^2(2\pi - \theta),\ R_2 = 0\}.$$

Lemma 2.3. *The mapping* Exp : $N'_{47} \to M'_{47}$ *is a diffeomorphism of 1-dimensional manifolds.*

Proof. We pass to the limit $k \to 1-0$ in formulas (2.15) and (2.16) and obtain for $\nu \in N'_{47}$:

$$\sin(\theta/2) = \tanh p, \quad \cos(\theta/2) = -1/\cosh p, \quad R_1 = 2(p - \tanh p), \quad R_2 = 0.$$

This coordinate representation shows that $\operatorname{Exp} : N'_{47} \to M'_{47}$ is a diffeomorphism. □

2.2.3. *Exponential mapping of the set* N'_{26}

Before the study of $\operatorname{Exp}|_{N'_{52}}$, postponed till the next subsection, we need to consider the set N'_{26} contained in the boundary of N'_{52}. In order to parameterize regularly the image $\operatorname{Exp}(N'_{26})$, we introduce the necessary functions.

Recall that the function $p = p_1^1(k)$, $k \in [0,1)$, is the first positive root of the function $f_1(p) = \operatorname{cn} p\,(\operatorname{E}(p) - p) - \operatorname{dn} p \operatorname{sn} p$, see equation (5.12) and Corollary 5.1 [7]. Define the function

$$v_1^1(k) = \operatorname{am}(p_1^1(k), k), \qquad k \in [0,1). \tag{2.17}$$

Lemma 2.4.

(1) *The number* $v = v_1^1(k)$ *is the first positive root of the function*

$$h_1(v,k) = E(v,k) - F(v,k) - \sqrt{1 - k^2 \sin^2 v}\ \tan v, \qquad k \in [0,1).$$

(2) $v_1^1 \in C^\infty[0,1)$.
(3) $v_1^1(k) \in (\pi/2, \pi)$ *for* $k \in (0,1)$*; moreover,* $v_1^1(0) = \pi$.
(4) *The function* $v_1^1(k)$ *is strictly decreasing at the segment* $k \in [0,1)$.
(5) $\lim_{v \to 1-0} v_1^1(k) = \pi/2$*, thus setting* $v_1^1(1) = \pi/2$*, we obtain* $v_1^1 \in C[0,1]$.
(6) $v_1^1(k) = \pi - (\pi/2)k^2 + o(k^2)$*,* $k \to +0$.

Proof. (1) follows from (2.17) since $p = p_1^1$ is the first positive root of the function $f_1(p)$.

(2) follows since $p_1^1 \in C^\infty[0,1)$ by Lemma 5.3 [7].

(3) follows since $p_1^1 \in (K, 2K)$ and $p_1^1(0) = \pi$, see Corollary 5.1 [7].

(4) We have for $v \in (\pi/2, \pi]$:

$$\frac{\partial h_1}{\partial v} = -\sqrt{1 - k^2 \sin^2 v}/\cos^2 v < 0,$$
$$\frac{\partial h_1}{\partial k} = -\frac{k}{1-k^2}(E(v,k) - \sqrt{1 - k^2 \sin^2 v} \tan v) < 0.$$

Thus $\dfrac{d v_1^1}{d k} = -\dfrac{\partial h_1/\partial k}{\partial h_1/\partial v} < 0$ for $k \in [0,1)$.

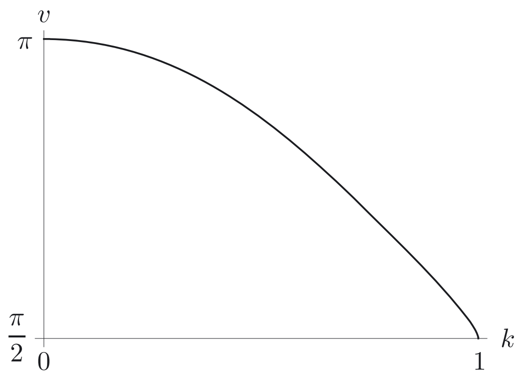

FIGURE 7. Plot of $v = v_1^1(k)$.

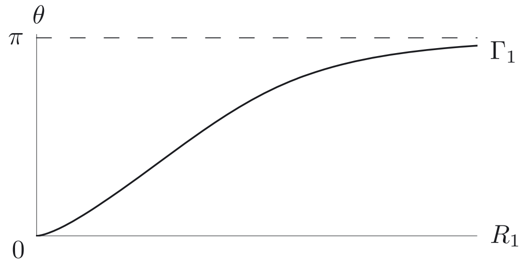

FIGURE 8. The curve Γ_1.

(5) Monotonicity and boundedness of $v_1^1(k)$ imply that there exists a limit $\lim_{k\to 1-0} v_1^1(k) = \bar{v} \in [\pi/2, \pi)$. If $\bar{v} \in (\pi/2, \pi)$, then as $k \to 1-0$

$$h_1(v_1^1(k), k) \to \int_0^{\bar{v}} (|\cos t| - 1/|\cos t|)\, \mathrm{d}t - \sqrt{1-\sin^2 \bar{v}} \tan \bar{v} = \infty,$$

which contradicts the identity $h_1(v_1^1(k), k) \equiv 0$, $k \in [0, 1)$. Thus $\bar{v} = \pi/2$.

(6) As $(k, v) \to (0, \pi)$, we have $h_1(v, k) = v - \pi + (\pi/2)k^2 + o(k^2 + (v-\pi)^2)$, thus $v_1^1(k) = \pi - (\pi/2)k^2 + o(k^2)$, $k \to +0$. □

A plot of the function $v_1^1(k)$ is given in Figure 7.

Define the curve $\Gamma_1 \subset S = \{q \in M \mid \theta \in (0, \pi),\ R_1 > 0,\ R_2 = 0\}$ given parametrically as follows:

$$\theta = 2 \arcsin(k \sin v_1^1(k)), \tag{2.18}$$

$$R_1 = 2(F(v_1^1(k), k) - E(v_1^1(k), k)), \qquad k \in [0, 1), \tag{2.19}$$

see Figure 8.

Lemma 2.5.

(1) *The function $k \sin v_1^1(k)$ is strictly increasing as $k \in [0, 1]$, thus the function $\theta = \theta(k)$, $k \in [0, 1]$, determined by (2.18) has an inverse function $k = k_1^1(\theta)$, $\theta \in [0, \pi]$.*

(2) $k_1^1 \in C[0, \pi] \cap C^\infty[0, \pi)$.

(3) *The function $k_1^1(\theta)$ is strictly increasing as $\theta \in [0, \pi]$.*

(4) *The curve Γ_1 is a graph of the function*

$$R_1 = R_1^1(\theta), \qquad \theta \in [0, \pi],$$
$$R_1^1(\theta) = 2(F(v_1^1(k), k) - E(v_1^1(k), k)), \qquad k = k_1^1(\theta). \tag{2.20}$$

(5) $R_1^1 \in C[0, \pi] \cap C^\infty(0, \pi)$.

(6) $R_1^1(\theta) = \sqrt[3]{\pi}/2\ \theta^{2/3} + o(\theta^{2/3})$, $\theta \to +0$.

Proof. (1) As $k \in [0, 1]$, we have:

$$v_1^1(k)\ \downarrow, \qquad v_1^1(k) \in [\pi/2, \pi],$$
$$\sin v_1^1(k)\ \uparrow, \qquad k \sin v_1^1(k)\ \uparrow, \qquad 2\arcsin(k v_1^1(k))\ \uparrow.$$

(2) follows from items (2) and (5) of Lemma 2.4.
(3) follows from item (1) of this lemma.
(4) follows from (2.18) and (2.19).
(5) follows from item (2) of this lemma.
(6) As $k \to +0$, we have

$$v_1^1(k) = \pi - (\pi/2)k^2 + o(k^2), \qquad \sin v_1^1(k) = (\pi/2)k^2 + o(k^2),$$

and for the functions (2.18), (2.19)

$$\theta = \pi k^3 + o(k^3), \qquad R_1 = (\pi/2)k^2 + o(k^2).$$

Thus as $\theta \to +0$, we have

$$k_1^1(\theta) = \sqrt[3]{\theta/\pi} + o(\sqrt[3]{\theta}), \qquad R_1^1(\theta) = \sqrt[3]{\pi}/2\ \theta^{2/3} + o(\theta^{2/3}).$$ □

Define the following subset of M', see Figure 14:

$$M'_{26} = \{q \in M \mid \theta \in (\pi, 2\pi),\ R_1 = R_1^1(2\pi - \theta),\ R_2 = 0\}.$$

Lemma 2.6. *The mapping* Exp : $N'_{26} \to M'_{26}$ *is a diffeomorphism of 1-dimensional manifolds.*

Proof. For $\nu \in N'_{26}$ we obtain from formulas (5.7)–(5.12) [7]:

$$\begin{aligned}
&\sin(\theta/2) = k \,\mathrm{sn}\, p_1^1(k) = k \sin v_1^1(k),\\
&\cos(\theta/2) = -\,\mathrm{dn}\, p_1^1(k) = -\sqrt{1 - k^2 \sin^2 v_1^1(k)},\\
&R_1 = 2(p_1^1(k) - \mathrm{E}(p_1^1(k))) = 2(F(v_1^1(k), k) - E(v_1^1(k), k)),\\
&R_2 = 0.
\end{aligned}$$

Thus $\mathrm{Exp}(N'_{26}) = M'_{26}$. Moreover, the mapping Exp : $N'_{26} \to M'_{26}$ decomposes into the chain

$$\begin{aligned}
&N'_{26} \overset{(*)}{\to} \Gamma_1 \overset{(**)}{\to} M'_{26},\\
&(*)\ :\ k \mapsto (\theta = 2\arcsin(k \sin v_1^1(k)),\ R_1 = 2(F(v_1^1(k), k) - E(v_1^1(k), k)),\ R_2 = 0),\\
&(**)\ :\ (\theta, R_1, R_2) \mapsto (2\pi - \theta, R_1, R_2).
\end{aligned}$$

The mapping $(*)$ is a diffeomorphism by Lemma 2.5. Thus Exp : $N'_{26} \to M'_{26}$ is a diffeomorphism. □

2.2.4. *Exponential mapping of the set* N'_{52}

Lemma 2.7.

(1) *The functions* $R_1^1(\theta)$, $R_1^2(\theta)$ *defined in* (2.20) *and* (2.13) *satisfy the inequality*

$$R_1^2(\theta) < R_1^1(\theta), \qquad \theta \in (0, \pi).$$

(2) *The mapping* Exp : $N'_{52} \to M'_{52}$ *is a diffeomorphism of 2-dimensional manifolds, where*

$$M'_{52} = \{q \in M \mid \theta \in (\pi, 2\pi),\ R_1 \in (R_1^2(2\pi - \theta), R_1^1(2\pi - \theta)),\ R_2 = 0\},$$

see Figure 14.

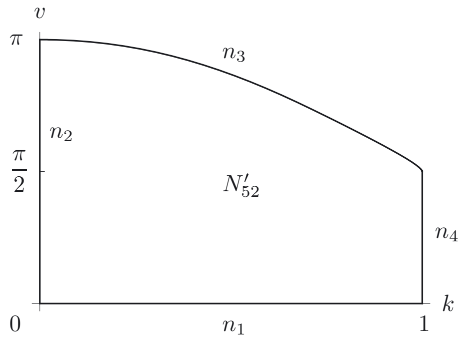

FIGURE 9. Domain N'_{52}.

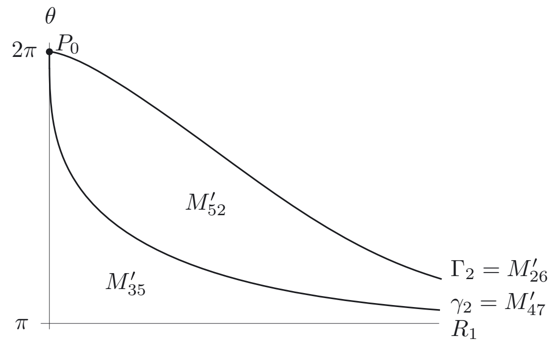

FIGURE 10. Curves γ_2 and Γ_2.

Proof. We have

$$N'_{52} = \{\nu \in N_2^+ \mid \tau = 0,\ v \in (0, v_1^1(k)),\ k \in (0,1)\},$$

where $v = \operatorname{am}(p,k)$. Thus

$$N'_{52} \subset T = \{\nu \in N_2^+ \mid \tau = 0,\ v \in [0,\pi],\ k \in [0,1]\},$$

and in the 2-dimensional topology of T

$$\begin{aligned}
&\partial N'_{52} = \cup_{i=1}^4 n_i,\\
&n_1 = \{\nu \in N_2^+ \mid \tau = 0,\ v = 0,\ k \in [0,1]\},\\
&n_2 = \{\nu \in N_2^+ \mid \tau = 0,\ v \in [0,\pi],\ k = 0\},\\
&n_3 = \{\nu \in N_2^+ \mid \tau = 0,\ v = v_1^1(k),\ k \in [0,1]\},\\
&n_4 = \{\nu \in N_2^+ \mid \tau = 0,\ v \in [0,\pi/2],\ k = 1\},
\end{aligned}$$

see Figure 9.

By formulas (5.7)–(5.12) [7], the exponential mapping in the domain N'_{52} reads as follows:

$$\begin{aligned}
&\sin(\theta/2) = k \sin v, && \cos(\theta/2) = -\sqrt{1 - k^2 \sin^2 v},\\
&R_1 = 2(F(v,k) - E(v,k)), && R_2 = 0.
\end{aligned}$$

Thus

$$\begin{aligned}
&\operatorname{Exp}(N'_{52}) \subset S = \{q \in M \mid \theta \in (\pi, 2\pi),\ R_1 > 0,\ R_2 = 0\},\\
&\operatorname{Exp}(n_1) = \operatorname{Exp}(n_2) = P_0 = \{q \in M \mid \theta = 2\pi,\ R_1 = 0,\ R_2 = 0\},\\
&\operatorname{Exp}(n_3) = \overline{M'_{26}} = \overline{\Gamma_2}, \qquad \Gamma_2 := M'_{26},\\
&\operatorname{Exp}(n_4) = \overline{M'_{47}} = \overline{\gamma_2}, \qquad \gamma_2 := M'_{47}.
\end{aligned}$$

By Theorem 2.5 [10] (see (1.14)), the mapping $\operatorname{Exp}|_{N'_{52}}$ is nondegenerate, thus $\operatorname{Exp}(N'_{52})$ is an open connected domain in S, with $\partial \operatorname{Exp}(N'_{52}) \subset \operatorname{Exp}(\partial N'_{52}) = \overline{\Gamma_2} \cup \overline{\gamma_2}$.

The curves $\overline{\Gamma_2}$ and $\overline{\gamma_2}$ intersect one another at the point P_0. We show that they have no other intersection points. By contradiction, assume that the curves $\overline{\Gamma_2}$ and $\overline{\gamma_2}$ have intersection points distinct from P_0, then the domain $\operatorname{Exp}(N'_{52})$ is bounded by finite arcs of the curves $\overline{\Gamma_2}$ and $\overline{\gamma_2}$, *i.e.*, there exists a point $P_1 \in \gamma_2 \cap \Gamma_2$, $P_1 \neq P_0$, such that $\partial \operatorname{Exp}(N'_{52}) = P_0\gamma_2 P_1 \cup P_0 \Gamma_2 P_1$. Then $\overline{\operatorname{Exp}(N'_{52})}$ does not contain the curves γ_2, Γ_2.

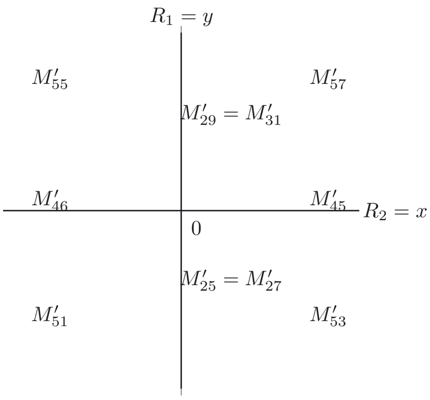

FIGURE 11. Decomposition of surface $\{\theta = 0\}$.

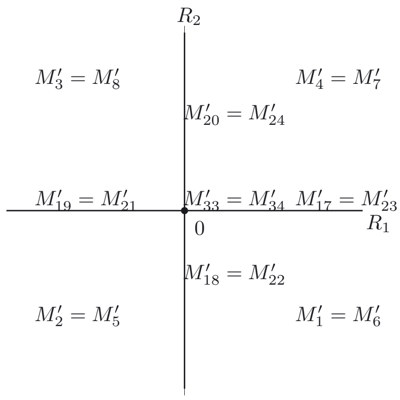

FIGURE 12. Decomposition of surface $\{\theta = \pi\}$.

This is a contradiction to the diffeomorphic property of the mapping $\mathrm{Exp} : n_3 = N'_{26} \to \Gamma_2 = M'_{26}$ and $\mathrm{Exp} : n_4 = N'_{47} \to \gamma_2 = M'_{47}$, see Lemmas 2.6 and 2.3 respectively.

Consequently, $\overline{\gamma_2} \cap \overline{\Gamma_2} = P_0$, and the domain $\mathrm{Exp}(N'_{52})$ is bounded by the curves $\overline{\gamma_2}$, $\overline{\Gamma_2}$.

The equalities $\dfrac{\mathrm{d}\, R_1^1}{\mathrm{d}\, \theta}(0) = +\infty$, $\dfrac{\mathrm{d}\, R_1^2}{\mathrm{d}\, \theta}(0) = 0$ (see Lem. 2.5 and Eq. (2.14)) imply that $R_1^2(\theta) < R_1^1(\theta)$ for sufficiently small $\theta > 0$. Further, the representations

$$\Gamma_2 = M'_{26} = \{q \in M \mid \theta \in (\pi, 2\pi),\ R_1 = R_1^1(2\pi - \theta),\ R_2 = 0\},$$
$$\gamma_2 = M'_{47} = \{q \in M \mid \theta \in (\pi, 2\pi),\ R_1 = R_1^2(2\pi - \theta),\ R_2 = 0\}$$

imply the required inequality

$$R_1^2(\theta) < R_1^1(\theta), \qquad \theta \in (0, \pi),$$

and the equality $\mathrm{Exp}(N'_{52}) = M'_{52}$.

Since the mapping $\mathrm{Exp} : N'_{52} \to M'_{52}$ is nondegenerate and proper, and the domains N'_{52}, M'_{52} are open, connected, and simply connected, it follows that this mapping is a diffeomorphism. □

The mutual disposition of the curves $\gamma_2 = M'_{47}$ and $\Gamma_2 = M'_{26}$ is shown in Figures 10 and 14.

2.3. Decomposition of the set M'

Now we have the functions $R_1^2(\theta) < R_1^1(\theta)$ required for definition of the following decomposition:

$$M' = \cup_{i=1}^{58} M'_i, \tag{2.21}$$

where the subsets M'_i are defined by Table 2. Notice that some of the sets M'_i coincide between themselves, unlike the sets N'_i, see (2.12). A precise definition of coinciding M'_i is given below in Theorem 3.1, item (1).

The structure of decomposition (2.21) in the surfaces $\{\theta = 0\}$, $\{\theta = \pi\}$, $\{R_1 = 0\}$, $\{R_2 = 0\}$ is shown respectively in Figures 11, 12, 13 and 14.

2.4. Exponential mapping of the sets N'_{36}, N'_{53}, N'_{18}, N'_{33}, N'_{34}, N'_{17}, N'_{27}, N'_1, N'_{10}

2.4.1. *Exponential mapping of the set* N'_{36}

Lemma 2.8. *The mapping* $\mathrm{Exp} : N'_{36} \to M'_{36}$ *is a diffeomorphism of 2-dimensional manifolds.*

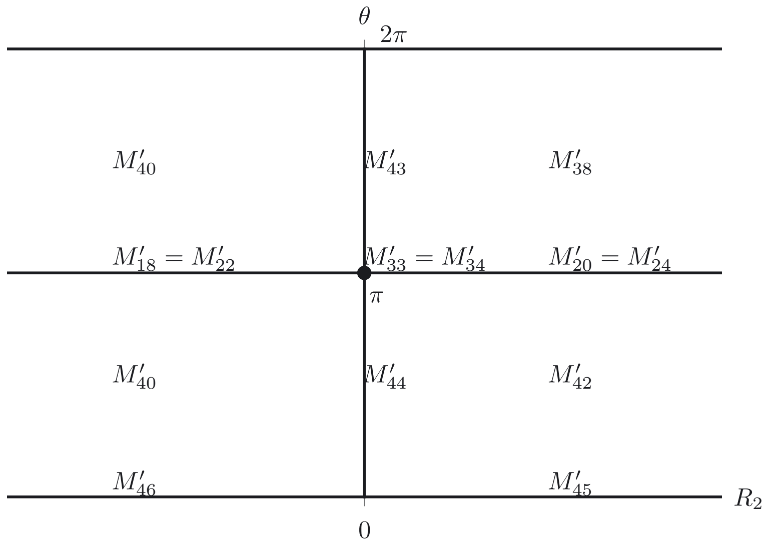

FIGURE 13. Decomposition of surface $\{R_1 = 0\}$.

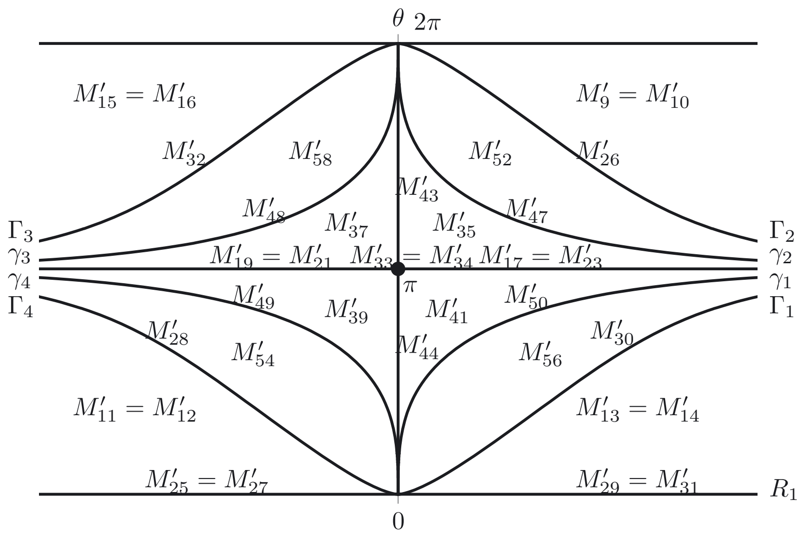

FIGURE 14. Decomposition of surface $\{R_2 = 0\}$.

Proof. By formulas (5.2)–(5.6) [7], exponential mapping in the domain N'_{36} reads as follows:

$$\sin(\theta/2) = \sqrt{1-k^2}\,\operatorname{sn} p \,/\, \operatorname{dn} p\,, \qquad \cos(\theta/2) = -\operatorname{cn} p \,/\, \operatorname{dn} p\,,$$

$$R_1 = 0, \qquad R_2 = -2 f_2(p,k)/(k \operatorname{dn} p\,),$$

where $f_2(p,k) = k^2 \operatorname{cn} p \operatorname{sn} p + \operatorname{dn} p\,(p - \operatorname{E}(p)) > 0$ by Lemma 5.2 [7], thus $\operatorname{Exp}(N'_{36}) \subset M'_{36}$.

TABLE 2. Definition of sets M'_i.

M'_i	$M'_1 = M'_6$	$M'_2 = M'_5$	$M'_3 = M'_8$	$M'_4 = M'_7$	$M'_9 = M'_{10}$	$M'_{11} = M'_{12}$
θ	π	π	π	π	$(\pi, 2\pi)$	$(0, \pi)$
R_1	$(0, +\infty)$	$(-\infty, 0)$	$(-\infty, 0)$	$(0, +\infty)$	$(R_1^1(2\pi - \theta), +\infty)$	$(-\infty, -R_1^1(\theta))$
R_2	$(-\infty, 0)$	$(-\infty, 0)$	$(0, +\infty)$	$(0, +\infty)$	0	0

M'_i	$M'_{13} = M'_{14}$	$M'_{15} = M'_{16}$	$M'_{17} = M'_{23}$	$M'_{18} = M'_{22}$	$M'_{19} = M'_{21}$
θ	$(0, \pi)$	$(\pi, 2\pi)$	π	π	π
R_1	$(R_1^1(\theta), +\infty)$	$(-\infty, -R_1^1(2\pi - \theta))$	$(0, +\infty)$	0	$(-\infty, 0)$
R_2	0	0	0	$(-\infty, 0)$	0

M'_i	$M'_{20} = M'_{24}$	$M'_{25} = M'_{27}$	M'_{26}	M'_{28}	$M'_{29} = M'_{31}$	M'_{30}	M'_{32}
θ	π	0	$(\pi, 2\pi)$	$(0, \pi)$	0	$(0, \pi)$	$(\pi, 2\pi)$
R_1	0	$(-\infty, 0)$	$R_1^1(2\pi - \theta)$	$-R_1^1(\theta)$	$(0, +\infty)$	$R_1^1(\theta)$	$-R_1^1(2\pi - \theta)$
R_2	$(0, +\infty)$	0	0	0	0	0	0

M'_i	$M'_{33} = M'_{34}$	M'_{35}	M'_{36}	M'_{37}	M'_{38}	M'_{39}
θ	π	$(\pi, 2\pi)$	$(\pi, 2\pi)$	$(\pi, 2\pi)$	$(\pi, 2\pi)$	$(0, \pi)$
R_1	0	$(0, R_1^2(2\pi - \theta))$	0	$(-R_1^2(2\pi - \theta), 0)$	0	$(-R_1^2(\theta), 0)$
R_2	0	0	$(-\infty, 0)$	0	$(0, +\infty)$	0

M'_i	M'_{40}	M'_{41}	M'_{42}	M'_{43}	M'_{44}	M'_{45}	M'_{46}	M'_{47}
θ	$(0, \pi)$	$(0, \pi)$	$(0, \pi)$	$(\pi, 2\pi)$	$(0, \pi)$	0	0	$(\pi, 2\pi)$
R_1	0	$(0, R_1^2(\theta))$	0	0	0	0	0	$R_1^2(2\pi - \theta)$
R_2	$(-\infty, 0)$	0	$(0, +\infty)$	0	0	$(0, +\infty)$	$(-\infty, 0)$	0

M'_i	M'_{48}	M'_{49}	M'_{50}	M'_{51}	M'_{52}	M'_{53}
θ	$(\pi, 2\pi)$	$(0, \pi)$	$(0, \pi)$	0	$(\pi, 2\pi)$	0
R_1	$-R_1^2(2\pi - \theta)$	$-R_1^1(\theta)$	$R_1^2(\theta)$	$(-\infty, 0)$	$(R_1^2(2\pi - \theta), R_1^1(2\pi - \theta))$	$(-\infty, 0)$
R_2	0	0	0	$(-\infty, 0)$	0	$(0, +\infty)$

M'_i	M'_{54}	M'_{55}	M'_{56}	M'_{57}	M'_{58}
θ	$(0, \pi)$	0	$(0, \pi)$	0	$(\pi, 2\pi)$
R_1	$(-R_1^1(\theta), -R_1^2(\theta))$	$(0, +\infty)$	$(R_1^2(\theta), R_1^1(\theta))$	$(0, +\infty)$	$(-R_1^1(2\pi - \theta), -R_1^2(2\pi - \theta))$
R_2	0	$(-\infty, 0)$	0	$(0, +\infty)$	0

In the topology of the manifold $\{R_1 = 0\}$, we have:

$$\partial N'_{36} = \cup_{i=1}^4 n_i,$$
$$n_1 = \{\nu \in N_1^0 \mid \tau = K,\ p = 0,\ k \in [0, 1]\},$$
$$n_2 = \{\nu \in N_1^0 \mid \tau = K,\ p \in [0, \pi/2],\ k = 0\},$$
$$n_3 = \{\nu \in N_1^0 \mid \tau = K,\ p = K,\ k \in [0, 1)\},$$
$$n_4 = \{\nu \in N_1^0 \mid \tau = K,\ p \in [0, +\infty),\ k = 1\},$$

see Figure 15.

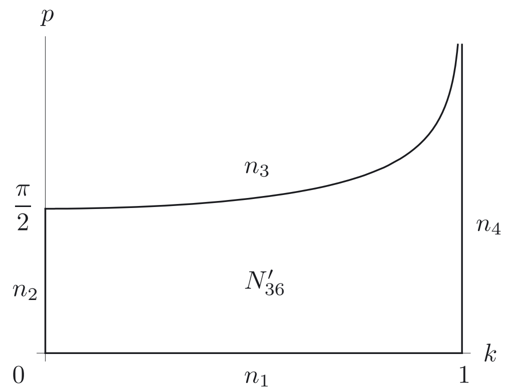

FIGURE 15. Domain N'_{36}.

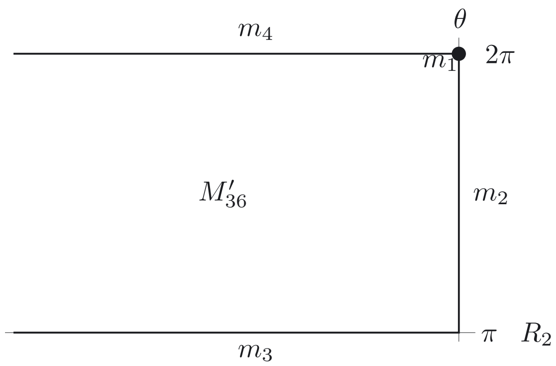

FIGURE 16. Domain M'_{36}.

Further, we have $\mathrm{Exp}(n_i) = m_i$, $i = 1, \dots, 4$, where

$$\begin{aligned} m_1 &= \{q \in M \mid \theta = 2\pi,\ R_1 = 0,\ R_2 = 0\}, \\ m_2 &= \{q \in M \mid \theta \in [\pi, 2\pi],\ R_1 = 0,\ R_2 = 0\}, \\ m_3 &= \{q \in M \mid \theta = \pi,\ R_1 = 0,\ R_2 \in (-\infty, 0]\}, \\ m_4 &= \{q \in M \mid \theta = 2\pi,\ R_1 = 0,\ R_2 \in (-\infty, 0]\}, \end{aligned}$$

see Figure 16.

The mapping $\mathrm{Exp} : N'_{36} \to M'_{36}$ is nondegenerate and proper, the domains N'_{36}, M'_{36} are open (in the 2-dimensional topology), connected and simply connected, thus it is a diffeomorphism. □

2.4.2. *Exponential mapping of the set* N'_{53}

Lemma 2.9. *The mapping* $\mathrm{Exp} : N'_{53} \to M'_{53}$ *is a diffeomorphism of 2-dimensional manifolds.*

Proof. The argument follows similarly to the proof of Lemma 2.8 *via* the following coordinate representation of the exponential mapping in the domain N'_{53}:

$$\theta = 0, \qquad R_1 = -2\sqrt{1-k^2}(p - \mathrm{E}(p))/\,\mathrm{dn}\, p < 0, \qquad R_2 = -2k f_1(p,k)/\,\mathrm{dn}\, p\,,$$

where $f_1(p, k) = \mathrm{cn}\, p\,(\mathrm{E}(p) - p) - \mathrm{dn}\, p\, \mathrm{sn}\, p < 0$ for $p \in (0, p_1^1)$, see Corollary 5.1 [7]. □

2.4.3. *Exponential mapping of the set* N'_{18}

Lemma 2.10. *The mapping* $\mathrm{Exp} : N'_{18} \to M'_{18}$ *is a diffeomorphism of 1-dimensional manifolds.*

Proof. By formulas (5.2)–(5.6) [7], we have in the set N'_{18}:

$$\theta = \pi, \qquad R_1 = 0, \qquad R_2 = -(2/k)(K(k) - E(k)),$$

and the diffeomorphic property of $\mathrm{Exp}|_{N'_{18}}$ follows as usual from its nondegeneracy and properness, and topological properties of the sets N'_{18}, M'_{18}. □

2.4.4. *Exponential mapping of the sets* N'_{33}, N'_{34}

Lemma 2.11. *The mappings* $\mathrm{Exp} : N'_{33} \to M'_{33}$ *and* $\mathrm{Exp} : N'_{34} \to M'_{34}$ *are diffeomorphisms (bijections) of 0-dimensional manifolds.*

Proof. Obvious. □

2.4.5. *Exponential mapping of the set* N'_{17}

Lemma 2.12. *The mapping* Exp : $N'_{17} \to M'_{17}$ *is a diffeomorphism of 1-dimensional manifolds.*

Proof. The statement follows as in the proof of Lemma 2.10 *via* the following coordinate representation of the exponential mapping in the domain N'_{17}:

$$\theta = \pi, \qquad R_1 = (2/k)(K(k) - E(k)), \qquad R_2 = 0. \qquad \square$$

2.4.6. *Exponential mapping of the set* N'_{27}

Lemma 2.13. *The mapping* Exp : $N'_{27} \to M'_{27}$ *is a diffeomorphism of 2-dimensional manifolds.*

Proof. Formulas (5.7)–(5.12) [7] yield:

$$\theta = \pi, \qquad R_1 = -2\sqrt{1-k^2}\,(p - \mathrm{E}(p))/\operatorname{dn} p\,|_{p=p_1^1(k)}\,, \qquad R_2 = 0.$$

Since $\tau = K$, then Lemma 2.4 [10] implies that the mapping $\mathrm{Exp}|_{N'_{27}}$ is nondegenerate. Then the diffeomorphic property of Exp : $N'_{27} \to M'_{27}$ follows as usual. $\square$

2.4.7. *Exponential mapping of the set* N'_1

Lemma 2.14. *The mapping* Exp : $N'_1 \to M'_1$ *is a diffeomorphism of 2-dimensional manifolds.*

Proof. Formulas (5.2)–(5.6) [7] yield:

$$\theta = \pi, \qquad R_1 = 2(K(k) - E(k))\operatorname{cn}\tau\,/(k\operatorname{dn}\tau\,) > 0,$$
$$R_2 = -2\sqrt{1-k^2}(K(k) - E(k))\operatorname{sn}\tau\,/(k\operatorname{dn}\tau\,) < 0,$$

and the statement follows as usual since the mapping $\mathrm{Exp}|_{N'_1}$ is nondegenerate and proper. $\square$

2.4.8. *Exponential mapping of the set* N'_{10}

Lemma 2.15. *The mapping* Exp : $N'_{10} \to M'_{10}$ *is a diffeomorphism of 2-dimensional manifolds.*

Proof. By formulas (5.7)–(5.12) [7] we get:

$$\sin(\theta/2) = k\operatorname{sn} p_1^1 \operatorname{cn}\tau\,/\sqrt{\Delta} > 0, \qquad \cos(\theta/2) = -\operatorname{dn} p_1^1/\sqrt{\Delta} < 0,$$
$$R_1 = 2(p - \mathrm{E}(p))\operatorname{dn}\tau\,/\sqrt{\Delta}\Big|_{p=p_1^1} > 0, \qquad R_2 = 0,$$

where $\Delta = 1 - k^2\operatorname{sn}^2 p \operatorname{sn}^2\tau$, and the statement follows by standard argument since $\mathrm{Exp}|_{N'_{10}}$ is nondegenerate and proper. $\square$

2.5. Action of the group of reflections in the preimage and image of the exponential mapping

In order to extend the results of the preceding subsections to all 58 pairs (N'_i, M'_i), $i \in I$, we describe the action of the group of reflections $G = \{\mathrm{Id}, \varepsilon^1, \ldots, \varepsilon^7\}$ on these sets.

Theorem 2.1. *Tables 3, 4, 5, 9 and 6, 7, 8, 10 define diffeomorphisms between the corresponding manifolds* N'_i *and* M'_i.

Proof. Follows from definitions of the manifolds N'_i and M'_i (Sects. 2.1 and 2.3) and Propositions 4.3 and 4.4 [7] describing action of the reflections $\varepsilon^i \in G$ in the image and preimage of the exponential mapping. Moreover, in the coordinates (θ, R_1, R_2) action of the reflections is described by Table 11. $\square$

TABLE 3. Action of ε^1, ε^4, ε^5 on N'_{35}, N'_{47}, N'_{52}, N'_{17}, N'_{26}.

D	N'_{35}	N'_{47}	N'_{52}	N'_{17}	N'_{26}
$\varepsilon^1(D)$	N'_{37}	N'_{48}	N'_{58}	N'_{19}	N'_{32}
$\varepsilon^4(D)$	N'_{39}	N'_{49}	N'_{54}	N'_{21}	N'_{28}
$\varepsilon^5(D)$	N'_{41}	N'_{50}	N'_{56}	N'_{23}	N'_{30}

TABLE 4. Action of ε^2, ε^4, ε^6 on N'_{36}, N'_{18}.

D	N'_{36}	N'_{18}
$\varepsilon^2(D)$	N'_{38}	N'_{20}
$\varepsilon^4(D)$	N'_{40}	N'_{22}
$\varepsilon^6(D)$	N'_{42}	N'_{24}

TABLE 5. Action of ε^1, ε^2, ε^3 on N'_{53}, N'_{27}.

D	N'_{53}	N'_{27}
$\varepsilon^1(D)$	N'_{57}	N'_{31}
$\varepsilon^2(D)$	N'_{51}	N'_{25}
$\varepsilon^3(D)$	N'_{55}	N'_{29}

TABLE 6. Action of ε^1, ε^4, ε^5 on M'_{35}, M'_{47}, M'_{52}, M'_{17}, M'_{26}.

D	M'_{35}	M'_{47}	M'_{52}	M'_{17}	M'_{26}
$\varepsilon^1(D)$	M'_{37}	M'_{48}	M'_{58}	M'_{19}	M'_{32}
$\varepsilon^4(D)$	M'_{39}	M'_{49}	M'_{54}	M'_{21}	M'_{28}
$\varepsilon^5(D)$	M'_{41}	M'_{50}	M'_{56}	M'_{23}	M'_{30}

TABLE 7. Action of ε^2, ε^4, ε^6 on M'_{36}, M'_{18}.

D	M'_{36}	M'_{18}
$\varepsilon^2(D)$	M'_{38}	M'_{20}
$\varepsilon^4(D)$	M'_{40}	M'_{22}
$\varepsilon^6(D)$	M'_{42}	M'_{24}

TABLE 8. Action of ε^1, ε^2, ε^3 on M'_{53}, M'_{27}.

D	M'_{53}	M'_{27}
$\varepsilon^1(D)$	M'_{57}	M'_{31}
$\varepsilon^2(D)$	M'_{51}	M'_{25}
$\varepsilon^3(D)$	M'_{55}	M'_{29}

2.6. The final result for exponential mapping of the sets N'_i

Theorem 2.2. *For any $i \in I$, the mapping* Exp : $N'_i \to M'_i$ *is a diffeomorphism of manifolds of appropriate dimension 2, 1, or 0.*

Proof. For $i \in \{35, 47, 26, 52, 36, 53, 18, 17, 27, 1, 10\}$ the statement follows from Lemmas 2.2, 2.3, 2.6, 2.7, 2.8, 2.9, 2.10, 2.12, 2.13, 2.14 and 2.15 respectively.

For $i \in \{33, 34\}$ the statement was proved in Lemma 2.11.

For all the rest i the statement follows from the above lemmas and Theorem 2.1 since the reflections $\varepsilon^i \in G$ are symmetries of the exponential mapping, see Proposition 4.5 [7]. □

2.7. Reflections ε^k as permutations

In addition to the index sets I, C, J, R, X introduced in (2.9) and (2.10), we will need also the set

$$T = \{(i, j) \in I \times I \mid i < j,\ M'_i = M'_j\}.$$

Table 9. Action of $\varepsilon^1, \dots, \varepsilon^7$ on N'_1, N'_{10}.

D	N'_1	N'_{10}
$\varepsilon^1(D)$	N'_2	N'_{15}
$\varepsilon^2(D)$	N'_4	N'_9
$\varepsilon^3(D)$	N'_3	N'_{16}
$\varepsilon^4(D)$	N'_5	N'_{12}
$\varepsilon^5(D)$	N'_6	N'_{13}
$\varepsilon^6(D)$	N'_8	N'_{11}
$\varepsilon^7(D)$	N'_7	N'_{14}

Table 10. Action of $\varepsilon^1, \dots, \varepsilon^7$ on M'_1, M'_{10}.

D	M'_1	M'_{10}
$\varepsilon^1(D)$	M'_2	M'_{15}
$\varepsilon^2(D)$	M'_4	M'_9
$\varepsilon^3(D)$	M'_3	M'_{16}
$\varepsilon^4(D)$	M'_5	M'_{12}
$\varepsilon^5(D)$	M'_6	M'_{13}
$\varepsilon^6(D)$	M'_8	M'_{11}
$\varepsilon^7(D)$	M'_7	M'_{14}

Table 11. Action of $\varepsilon^1, \dots, \varepsilon^7$ on $M = \{(R_1, R_2, \theta)\}$.

	ε^1	ε^2	ε^3	ε^4	ε^5	ε^6	ε^7
R_1	$-R_1$	R_1	$-R_1$	$-R_1$	R_1	$-R_1$	R_1
R_2	R_2	$-R_2$	$-R_2$	R_2	R_2	$-R_2$	$-R_2$
θ	θ	θ	θ	$2\pi - \theta$	$2\pi - \theta$	$2\pi - \theta$	$2\pi - \theta$

Table 12. Multiplication table in the group G.

	ε^1	ε^2	ε^3	ε^4	ε^5	ε^6	ε^7
ε^1	Id	ε^3	ε^2	ε^5	ε^4	ε^7	ε^6
ε^2		Id	ε^1	ε^6	ε^7	ε^4	ε^5
ε^3			Id	ε^7	ε^6	ε^5	ε^4
ε^4				Id	ε^1	ε^2	ε^3
ε^5					Id	ε^3	ε^2
ε^6						Id	ε^1
ε^7							Id

From the definition of the sets M'_i in Section 2.3 we obtain the explicit representation:

$$T = \{(1,6),\ (2,5),\ (3,8),\ (4,7),\ (9,10),\ (11,12),\ (13,14),\ (15,16),\\ (17,23),\ (18,22),\ (19,21),\ (20,24),\ (25,27),\ (29,31),\ (33,34)\}.$$

Notice that $X = \{i \in I \mid \exists j \in I \ : \ (i,j) \in T \text{ or } (j,i) \in T\}$.

Now we show that reflections $\varepsilon^k \in G$ permute elements in any pair $(i,j) \in T$.

We will need multiplication Table 12 in the group G, which follows from definitions of the reflections ε^k (Sect. 4 [7]). The lower diagonal entries of the table are not filled since G is Abelian.

Lemma 2.16. *For any $(i,j) \in T$ there exists a reflection $\varepsilon^k \in G$ such that the following diagram is commutative:*

$$\begin{array}{ccc} N'_i & \xrightarrow{\text{Exp}} & M'_i \\ {\scriptstyle \varepsilon^k}\downarrow & & \downarrow{\scriptstyle \text{Id}} \\ N'_j & \xrightarrow{\text{Exp}} & M'_j \end{array} \qquad\qquad \begin{array}{ccc} (\lambda, t) & \xmapsto{\text{Exp}} & q_t \\ {\scriptstyle \varepsilon^k}\downarrow & & \downarrow{\scriptstyle \text{Id}} \\ (\lambda^k, t) & \xmapsto{\text{Exp}} & q_t \end{array}$$

Proof. From definitions of the sets N_i' (Sect. 2.1) and the reflections ε^k (Sect. 4 [7]), Tables 3, 4, 5, 9, 12 and Proposition 4.5 [7], we obtain the following indices k of required symmetries ε^k for pairs $(i,j) \in T$:

$$(i,j) \in \{(1,6),\ (2,5),\ (3,8),\ (4,7),\ (17,23),\ (19,21)\} \quad \Rightarrow \quad k=5,$$
$$(i,j) \in \{(9,10),\ (15,16),\ (11,12),\ (13,14),\ (25,27),\ (29,31)\} \quad \Rightarrow \quad k=2,$$
$$(i,j) \in \{(33,34),\ (18,22),\ (20,24)\} \quad \Rightarrow \quad k=4.$$
□

3. Solution to optimal control problem

In this section we present the final results of this study of the sub-Riemannian problem on SE(2).

3.1. Global structure of the exponential mapping

We say that a mapping $F : X \to Y$ is double if any point $y \in Y$ has exactly two preimages:

$$\forall\, y \in Y \qquad F^{-1}(y) = \{x_1, x_2\}, \qquad x_1 \neq x_2.$$

Theorem 3.1.

(1) *There is the following decomposition of preimage of the exponential mapping* $\operatorname{Exp} : \widehat{N} \to \widehat{M}$:

$$\begin{aligned}
&\widehat{N} = \widetilde{N} \sqcup N',\\
&\widetilde{N} = \sqcup_{i=1}^{8} D_i,\\
&N' = N_{\text{Max}} \sqcup N_{\text{conj}} \sqcup N_{\text{rest}},\\
&N_{\text{Max}} = \sqcup_{i\in X} N_i',\\
&N_{\text{conj}} = \sqcup_{i\in J} N_i',\\
&N_{\text{rest}} = \sqcup_{i\in R} N_i',
\end{aligned}$$

and in the image of the exponential mapping:

$$\begin{aligned}
&\widehat{M} = \widetilde{M} \sqcup M',\\
&\widetilde{M} = \sqcup_{i=1}^{8} M_i,\\
&M' = M_{\text{Max}} \sqcup M_{\text{conj}} \sqcup M_{\text{rest}},\\
&M_{\text{Max}} = \cup_{i\in X} M_i',\\
&M_i' \cap M_j' \neq \emptyset,\ i<j \quad \Rightarrow \quad (i,j) \in T, \quad \{i,j\} \subset X,\\
&(i,j) \in T \quad \Rightarrow \quad M_i' = M_j',\\
&M_{\text{conj}} = \sqcup_{i\in J} M_i',\\
&M_{\text{rest}} = \sqcup_{i\in R} M_i'.
\end{aligned}$$

(2) *In terms of these decompositions the exponential mapping* $\operatorname{Exp} : \widehat{N} \to \widehat{M}$ *has the following structure:*

$$\operatorname{Exp} : D_i \to M_i \text{ is a diffeomorphism } \forall\ i = 1,\dots,8, \tag{3.1}$$
$$\operatorname{Exp} : N_i' \to M_i' \text{ is a diffeomorphism } \forall\ i \in I. \tag{3.2}$$

Thus

$$\mathrm{Exp} : \widetilde{N} \to \widetilde{M} \text{ is a bijection,} \tag{3.3}$$
$$\mathrm{Exp} : N_{\mathrm{Max}} \to M_{\mathrm{Max}} \text{ is a double mapping,} \tag{3.4}$$
$$\mathrm{Exp} : N_{\mathrm{conj}} \to M_{\mathrm{conj}} \text{ is a bijection,} \tag{3.5}$$
$$\mathrm{Exp} : N_{\mathrm{rest}} \to M_{\mathrm{rest}} \text{ is a bijection.} \tag{3.6}$$

(3) *Any point* $q \in \widetilde{M}$ *(*$q \in M_{\mathrm{conj}}$*,* $q \in M_{\mathrm{rest}}$*) has a unique preimage* $\nu = \mathrm{Exp}^{-1}(q)$ *for the mapping* $\mathrm{Exp}|_{\widehat{N}} : \widehat{N} \to \widehat{M}$*. Moreover,* $\nu \in \widetilde{N}$ *(resp.,* $\nu \in N_{\mathrm{conj}}$*,* $\nu \in N_{\mathrm{rest}}$*).*

(4) *Any point* $q \in M_{\mathrm{Max}}$ *has exactly two preimages* $\{\nu', \nu''\} = \mathrm{Exp}^{-1}(q)$ *for the mapping* $\mathrm{Exp}|_{\widehat{N}} : \widehat{N} \to \widehat{M}$*. Moreover,* $\nu', \nu'' \in N_{\mathrm{Max}}$ *and* $\nu'' = \varepsilon^k(\nu')$ *for some* $\varepsilon^k \in G$*.*

The domains D_i and M_i were defined in Tables 1 and 2 [10].

Proof. Equalities in item (1) follow immediately from definitions of the corresponding decompositions.

(2) Property (3.1) was proved in Theorem 3.1 [10], and property (3.3) is its corollary, with account of item (1). Property (3.2) was proved in Theorem 3.2, and properties (3.4)–(3.6) are its corollaries, with account of item (1).

(3) The statement follows from (3.3), (3.4), (3.5), (3.6), with account of item (1).

(4) The statement follows from (3.4), (3.5), (3.6), and Lemma 2.16. □

3.2. **Optimal synthesis**

Theorem 3.2. *Let* $q \in \widehat{M} = M \setminus \{q_0\}$*.*

(1) *Let* $q \in \widetilde{M} \cup M_{\mathrm{conj}} \cup M_{\mathrm{rest}} = \widehat{M} \setminus M_{\mathrm{Max}}$*. Denote* $\nu = (\lambda, t) = \mathrm{Exp}^{-1}(q) \in \widetilde{N} \cup N_{\mathrm{conj}} \cup N_{\mathrm{rest}} = \widehat{N} \setminus N_{\mathrm{Max}}$*. Then* $q_s = \mathrm{Exp}(\lambda, s)$*,* $s \in [0, t]$*, is the unique optimal trajectory connecting* q_0 *with* q*. If* $q \in \widetilde{M} \cup M_{\mathrm{rest}}$*, then* $t < \mathbf{t}(\lambda)$*; if* $q \in M_{\mathrm{conj}}$*, then* $t = \mathbf{t}(\lambda) = t_1^{\mathrm{conj}}(\lambda)$*.*

(2) *Let* $q \in M_{\mathrm{Max}}$*. Denote* $\{\nu', \nu''\} = \mathrm{Exp}^{-1}(q) \subset N_{\mathrm{Max}}$*,* $\nu' = (\lambda', t) \neq \nu'' = (\lambda'', t)$*. Then there exist exactly two distinct optimal trajectories connecting* q_0 *and* q*; namely,* $q'_s = \mathrm{Exp}(\lambda', s)$ *and* $q''_s = \mathrm{Exp}(\lambda'', s)$*,* $s \in [0, t]$*. Moreover,* $t = \mathbf{t}(\lambda) < t_1^{\mathrm{conj}}(\lambda)$*.*

(3) *An optimal trajectory* $q_s = \mathrm{Exp}(\lambda, s)$ *is generated by the optimal controls*

$$u_1(s) = \sin(\gamma_s/2), \qquad u_2(s) = -\cos(\gamma_s/2),$$

where γ_s *is the solution to the equation of pendulum* $\ddot{\gamma}_s = -\sin\gamma_s$ *with the initial condition* $(\gamma_0, \dot{\gamma}_0) = \lambda$*.*

Proof. For any point $q \in \widehat{M}$ there exists an optimal trajectory $q_s = \mathrm{Exp}(\lambda, s)$, $s \in [0, t]$, $\nu = (\lambda, t) \in N$, such that $q_t = q$ and $t \leq t_{\mathrm{cut}}(\lambda)$. By Theorem 5.4 [7], we have $t \leq \mathbf{t}(\lambda)$, thus $\nu \in \widehat{N}$.

(1) If $q \in \widetilde{M} \sqcup M_{\mathrm{conj}} \sqcup M_{\mathrm{rest}}$, then by Theorem 3.1, there exists a unique $\nu = (\lambda, t) \in \widehat{N}$ such that $q = \mathrm{Exp}(\nu)$, moreover, $\nu \in \widetilde{N} \sqcup N_{\mathrm{conj}} \sqcup N_{\mathrm{rest}}$. Consequently, $q_s = \mathrm{Exp}(\lambda, s)$, $s \in [0, t]$, is a unique optimal trajectory connecting q_0 with q.

The inequality $t < \mathbf{t}(\lambda)$ for $\nu = (\lambda, t) \in \widetilde{N} \sqcup N_{\mathrm{rest}}$, and the equality $t = \mathbf{t}(\lambda) = t_1^{\mathrm{conj}}(\lambda)$ for $\nu \in N_{\mathrm{conj}}$ follow from definitions of the sets $\widetilde{N}$, N_{rest}, N_{conj}.

(2) If $q \in M_{\mathrm{Max}}$, then the statement follows similarly to item (1) from Theorem 3.1 and definition of the set N_{Max}.

(3) The expressions for optimal controls were obtained in Section 2 [7]. □

It follows from the definition of cut time that for any $\lambda \in C$ and $t \in (0, t_{\mathrm{cut}}(\lambda))$, the trajectory $q(s) = \mathrm{Exp}(\lambda, s)$ is optimal at the segment $s \in [0, t]$. For the case of finite $t_{\mathrm{cut}}(\lambda)$, we obtain a similar statement for $t = t_{\mathrm{cut}}(\lambda)$.

Theorem 3.3. *If* $t_{\mathrm{cut}}(\lambda) < +\infty$*, then the extremal trajectory* $\mathrm{Exp}(\lambda, s)$ *is optimal for* $s \in [0, t_{\mathrm{cut}}(\lambda)]$*.*

Proof. Let $t_{\mathrm{cut}}(\lambda) = \mathbf{t}(\lambda) < +\infty$, *i.e.*, $\lambda \in C_1 \cup C_2 \cup C_4$, and let $t = t_{\mathrm{cut}}(\lambda)$. Then $(\lambda, t) \in N_{\mathrm{Max}}$, and the statement follows from item (2) of Theorem 3.2. □

3.3. Cut locus

Now we are able to describe globally the first Maxwell set

$$\mathrm{Max} = \{q \in M \mid \exists t > 0,\ \exists \text{ optimal trajectories } q_s \not\equiv q'_s,\ s \in [0, t], \text{ such that } q_t = q'_t = q\}, \tag{3.7}$$

the cut locus

$$\mathrm{Cut} = \{\mathrm{Exp}(\lambda, t) \mid \lambda \in C,\ t = t_{\mathrm{cut}}(\lambda)\}, \tag{3.8}$$

and its intersection with caustic (the first conjugate locus)

$$\mathrm{Conj} = \{\mathrm{Exp}(\lambda, t) \mid \lambda \in C,\ t = t_1^{\mathrm{conj}}(\lambda)\}. \tag{3.9}$$

Theorem 3.4.
(1) $\mathrm{Max} = M_{\mathrm{Max}}$,
(2) $\mathrm{Cut} = M_{\mathrm{cut}}$,
(3) $\mathrm{Cut} \cap \mathrm{Conj} = M_{\mathrm{conj}}$.

Proof. Items (1) and (2) follow from Theorem 3.2 and Corollary 3.3.

(3) Let $q \in M_{\mathrm{conj}}$, and let $q_s = \mathrm{Exp}(\lambda, s)$, $s \in [0, \mathbf{t}(\lambda)]$, be the optimal trajectory connecting q_0 with q. Thus there are no conjugate points at the interval $(0, \mathbf{t}(\lambda))$. By item (1) of Theorem 3.2, we have $t_1^{\mathrm{conj}}(\lambda) = \mathbf{t}(\lambda)$. Thus $M_{\mathrm{conj}} \subset \mathrm{Conj}$, and in view of item (2) of this theorem we get $M_{\mathrm{conj}} \subset \mathrm{Cut} \cap \mathrm{Conj}$. Now we prove that $\mathrm{Cut} \cap \mathrm{Conj} \subset M_{\mathrm{conj}}$, *i.e.*, $M_{\mathrm{cut}} \cap \mathrm{Conj} \subset M_{\mathrm{conj}}$.

Fix any point $q \in M_{\mathrm{cut}}$. Then $q = \mathrm{Exp}(\lambda, t)$ for some $(\lambda, t) \in N_{\mathrm{cut}} = N_{\mathrm{Max}} \sqcup N_{\mathrm{conj}}$. In order to complete the proof, we assume that $(\lambda, t) \in N_{\mathrm{Max}}$ and show that $q \notin \mathrm{Conj}$. Since $(\lambda, t) \in N_{\mathrm{Max}}$, then $t = \mathbf{t}(\lambda)$. We prove that $t < t_1^{\mathrm{conj}}(\lambda)$.

If $\lambda \in C_1 \cup C_4$, then $t_1^{\mathrm{conj}}(\lambda) = +\infty$ by Theorem 2.5 [10] (see (1.14)).

Let $\lambda \in C_2$. If $\operatorname{sn}\tau = 0$, then $(\lambda, t) \in N_{\mathrm{conj}}$, which is impossible since $(\lambda, t) \in N_{\mathrm{Max}}$. And if $\operatorname{sn}\tau \neq 0$, then $t < t_1^{\mathrm{conj}}(\lambda)$ by Proposition 2.2 [10].

The inclusion $\mathrm{Cut} \cap \mathrm{Conj} \subset M_{\mathrm{conj}}$ follows. □

Theorem 3.5. *The cut locus has three connected components:*

$$\mathrm{Cut} = \mathrm{Cut}_{\mathrm{glob}} \sqcup \mathrm{Cut}_{\mathrm{loc}}^{+} \sqcup \mathrm{Cut}_{\mathrm{loc}}^{-}, \tag{3.10}$$
$$\mathrm{Cut}_{\mathrm{glob}} = \{q \in M \mid \theta = \pi\}, \tag{3.11}$$
$$\mathrm{Cut}_{\mathrm{loc}}^{+} = \{q \in M \mid \theta \in (-\pi, \pi),\ R_2 = 0,\ R_1 > R_1^1(|\theta|)\}, \tag{3.12}$$
$$\mathrm{Cut}_{\mathrm{loc}}^{-} = \{q \in M \mid \theta \in (-\pi, \pi),\ R_2 = 0,\ R_1 < -R_1^1(|\theta|)\}, \tag{3.13}$$

where the function R_1^1 is defined by equation (2.20). The initial point q_0 is contained in the closure of the components $\mathrm{Cut}_{\mathrm{loc}}^{+}$, $\mathrm{Cut}_{\mathrm{loc}}^{-}$, and is separated from the component $\mathrm{Cut}_{\mathrm{glob}}$.

Proof. By Theorems 3.4, 3.1 and Lemma 2.1, we have

$$\mathrm{Cut} = M_{\mathrm{cut}} = \mathrm{Exp}(N_{\mathrm{cut}}) = \cup_{i \in C} M'_i.$$

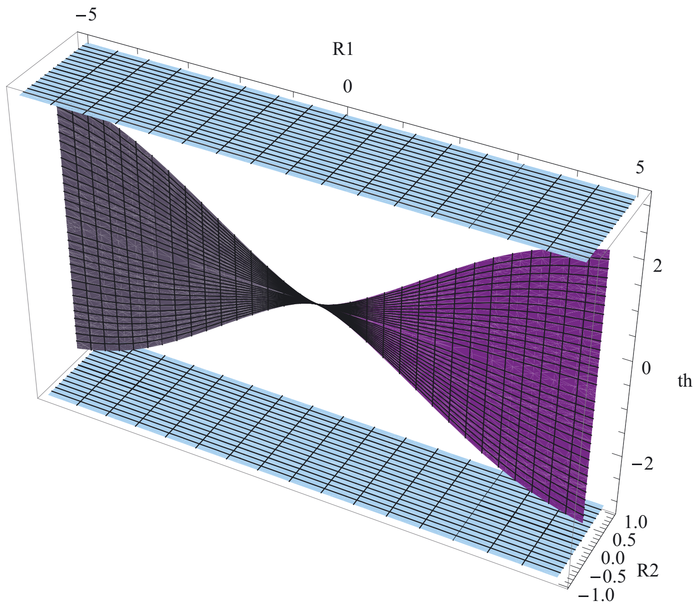

FIGURE 17. Cut locus in rectifying coordinates (R_1, R_2, θ).

Denote

$$C_{\text{glob}} = \{1, \dots, 8, 17, \dots, 24, 33, 34\},$$
$$C_{\text{loc}}^{+} = \{13, \dots, 16, 29, \dots, 32\},$$
$$C_{\text{loc}}^{-} = \{9, \dots, 12, 25, \dots, 28\}.$$

Then

$$\cup_{i \in C_{\text{glob}}} M_i' = \text{Cut}_{\text{glob}}, \qquad \cup_{i \in C_{\text{loc}}^{+}} M_i' = \text{Cut}_{\text{loc}}^{+}, \qquad \cup_{i \in C_{\text{loc}}^{-}} M_i' = \text{Cut}_{\text{loc}}^{-},$$

and decomposition (3.10)–(3.13) follows.

The topological properties of Cut_{glob}, $\text{Cut}_{\text{loc}}^{\pm}$ follow from equalities (3.10)–(3.13). □

The cut locus in rectifying coordinates (R_1, R_2, θ) is presented in Figure 17, notice that here the horizontal planes $\theta = 0$ and $\theta = 2\pi$ should be identified. Global embedding of the cut locus to the solid torus (diffeomorphic image of the state space $M = \text{SE}(2)$) is shown in Figure 18.

The curve $\Gamma = \text{Cut} \cap \text{Conj}$ has the following asymptotics near the initial point q_0:

$$R_1 = R_1^1(\theta) = \sqrt[3]{\pi}/2 \; \theta^{2/3} + o(\theta^{2/3}), \quad \theta \to 0, \qquad R_2 = 0,$$

see item (5) of Lemma 2.5. This agrees with the result on asymptotics of cut and conjugate loci for contact sub-Riemannian structures in $\mathbb{R}^3$ obtained by Agrachev [1] and by El-Alaoui *et al.* [5].

Illustrations of cut points and the corresponding optimal trajectories are given in Figures 19–27.

4. Explicit optimal solutions for special terminal points

In this section we describe optimal solutions for particular terminal points $q_1 = (x_1, y_1, \theta_1)$. Where applicable, we interpret the optimal trajectories in terms of the corresponding optimal motion of a car in the plane.

For generic terminal points, we developed a software in computer system Mathematica [13] for numerical evaluation of solutions to the problem.

FIGURE 18. Cut locus: global view.

4.1. $x_1 \neq 0,\ y_1 = 0,\ \theta_1 = 0$

In this case $\nu \in N_5$, and the optimal trajectory is

$$x_t = t \operatorname{sgn} x_1, \quad y_t = 0, \quad \theta_t = 0, \qquad t \in [0, t_1], \quad t_1 = |x_1|,$$

the car moves uniformly forward or backward along a segment.

4.2. $x_1 = 0,\ y_1 = 0,\ |\theta_1| \in (0, \pi)$

We have $\nu \in N_4$, and the optimal solution is

$$x_t = 0, \quad y_t = 0, \quad \theta_t = t \operatorname{sgn} \theta_1, \qquad t \in [0, t_1], \quad t_1 = |\theta_1|,$$

the car rotates uniformly around itself by the angle θ_1.

4.3. $x_1 = 0,\ y_1 = 0,\ \theta_1 = \pi$

We have $\nu \in N_4$, and there are two optimal solutions:

$$x_t = 0, \quad y_t = 0, \quad \theta_t = \pm t, \qquad t \in [0, t_1], \quad t_1 = \pi,$$

the car rotates uniformly around itself clockwise or counterclockwise by the angle π, see Figure 19.

4.4. $x_1 \neq 0,\ y_1 = 0,\ \theta_1 = \pi$

There are two optimal solutions:

$$x_t = (\operatorname{sgn} x_1)/k(t + E(k) - E(K + t, k)), \quad y_t = (s/k)(\sqrt{1 - k^2} - \operatorname{dn}(K + t, k)),$$
$$\theta_t = s \operatorname{sgn} x_1 (\pi/2 - \operatorname{am}(K + t, k)), \quad s = \pm 1, \qquad t \in [0, t_1], \quad t_1 = 2K,$$

and $k \in (0, 1)$ is the root of the equation

$$(2/k)(K(k) - E(k)) = |x_1|,$$

see Figure 21.

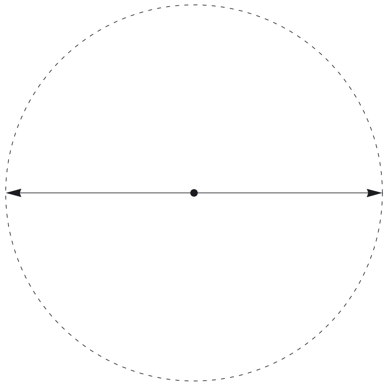

FIGURE 19. Cut point for $\lambda \in C_4$ (optimal solutions for $x_1 = y_1 = 0$, $\theta_1 = \pi$).

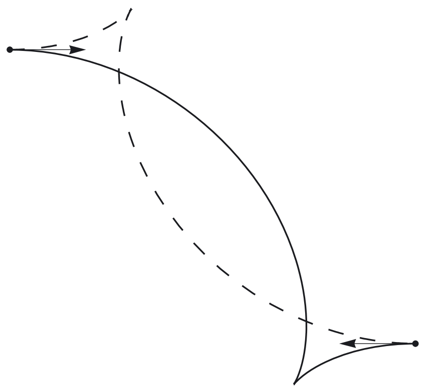

FIGURE 20. Cut point for $\lambda \in C_1$, generic case (optimal solutions for $\theta_1 = \pi$).

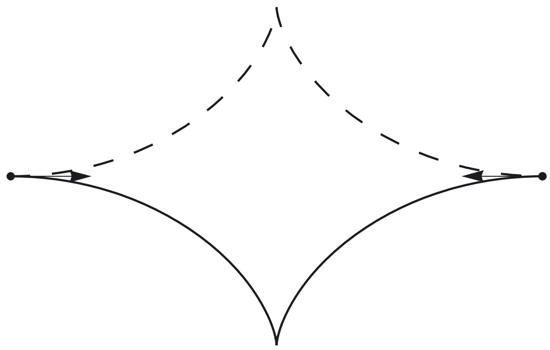

FIGURE 21. Cut point for $\lambda \in C_1$, symmetric case with cusp (optimal solutions for $x_1 \neq 0$, $y_1 = 0$, $\theta_1 = \pi$).

4.5. $x_1 = 0$, $y_1 \neq 0$, $\theta_1 = \pi$

There are two optimal solutions:

$$x_t = s(1 - \mathrm{dn}(t,k))/k, \quad y_t = (\mathrm{sgn}\, y_1/k)(t - E(t,k)),$$
$$\theta_t = s\, \mathrm{sgn}\, y_1\, \mathrm{am}(t,k), \quad s = \pm 1, \qquad t \in [0, t_1], \quad t_1 = 2K,$$

and $k \in (0,1)$ is the root of the equation

$$(2/k)(K(k) - E(k)) = |y_1|,$$

see Figure 22.

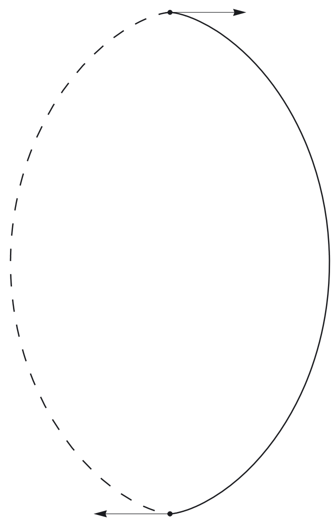

FIGURE 22. Cut point for $\lambda \in C_1$, symmetric case without cusp (optimal solutions for $x_1 = 0$, $y_1 \neq 0$, $\theta_1 = \pi$).

4.6. $x_1 = 0,\ y_1 \neq 0,\ \theta_1 = 0$

There are two optimal solutions given by formulas for (x_t, y_t, θ_t) for the case $\lambda \in C_2$ in Section 3.3 [7] for the following values of parameters:

$$t \in [0, t_1], \quad t_1 = 2kp_1^1(k),$$

with the function $p_1^1(k)$ defined in Lemma 5.3 [7],

$$s_2 = -\operatorname{sgn} y_1, \qquad \psi = \pm K(k) - p_1^1(k),$$

and $k \in (0,1)$ is the root of the equation

$$2(p_1^1(k) - E(p_1^1(k), k)\sqrt{1-k^2}/\operatorname{dn}(p_1^1(k), k)) = |y_1|,$$

see Figure 23.

4.7. $(x_1, y_1) \neq 0,\ \theta_1 = \pi$

Introduce the polar coordinates $x_1 = \rho_1 \cos\chi_1$, $y_1 = \rho_1 \sin\chi_1$. There are two optimal solutions given by formulas for (x_t, y_t, θ_t) for the case $\lambda \in C_1$ in Section 3.3 [7] for the following values of parameters:

$$t \in [0, t_1], \quad t_1 = 2K(k),$$

and $k \in (0,1)$ is the root of the equation

$$2(p_1^1(k) - E(p_1^1(k), k)\sqrt{1-k^2}/\operatorname{dn}(p_1^1(k), k)) = \rho_1,$$
$$s_1 = \pm 1, \qquad \varphi = s_1 F(\pi/2 - \chi_1, k),$$

see Figures 20–22. In the cases $y_1 = 0$ and $x_1 = 0$ we get respectively the cases considered in Sections 4.4 and 4.5.

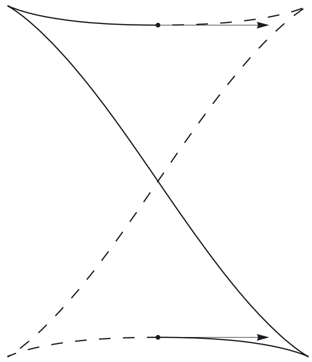

FIGURE 23. Optimal solutions for $x_1 = 0$, $y_1 \neq 0$, $\theta_1 = 0$.

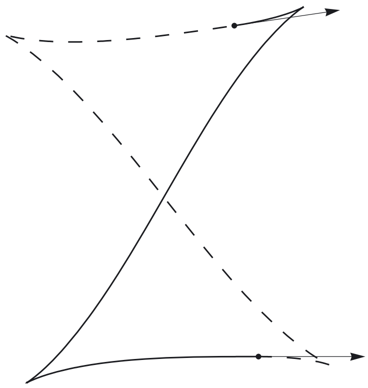

FIGURE 24. Cut point for $\lambda \in C_2$, generic case.

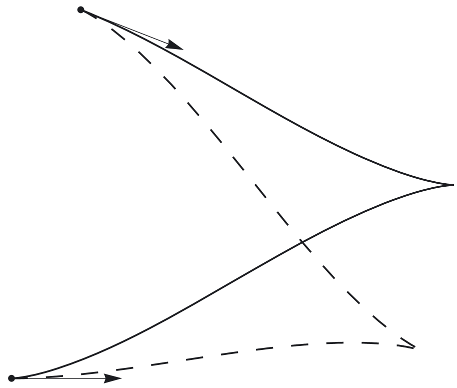

FIGURE 25. Cut point for $\lambda \in C_2$, special case with one cusp.

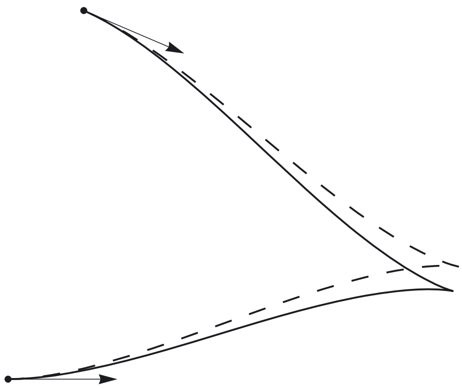

FIGURE 26. Cut point for $\lambda \in C_2$ approaching conjugate point.

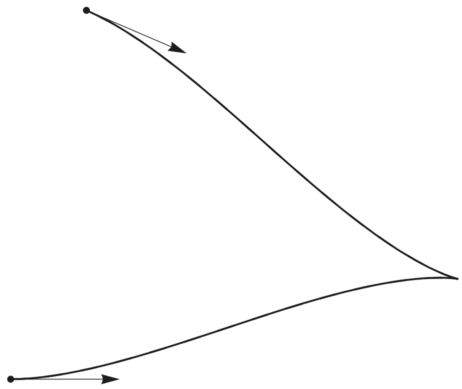

FIGURE 27. Cut point for $\lambda \in C_2$ coinciding with conjugate point.

4.8. $y_1 \neq 0,\ \theta_1 = 0$

There is a unique optimal solution given by formulas for (x_t, y_t, θ_t) for the case $\lambda \in C_2$ in Section 3.3 [7] for the following values of parameters:

$$s_2 = -\operatorname{sgn} y_1,$$

$k \in (0,1)$ and $p \in (0, p_1^1(k)]$ are solutions to the system of equations

$$s(\operatorname{sgn} y_1) 2k f_1(p,k)/\operatorname{dn}(p,k) = x_1, \qquad s = \pm 1,$$
$$2(p - \mathrm{E}(p))\sqrt{1-k^2}/\operatorname{dn}(p,k) = |y_1|,$$

and

$$t \in [0, t_1], \quad t_1 = 2kp, \qquad \psi = sK(k) - p.$$

Acknowledgements. The author is grateful to Prof. Andrei Agrachev for bringing the sub-Riemannian problem on SE(2) to authors' attention. The author wishes to thank Prof. Arrigo Cellina for hospitality and excellent conditions for work at the final version of this paper.

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