ESAIM: COCV 15 (2009) 741–744
DOI: 10.1051/cocv/2009011

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

CORRIGENDUM TO: "ON THE CIRCLE CRITERION FOR BOUNDARY CONTROL SYSTEMS IN FACTOR FORM: LYAPUNOV STABILITY AND LUR'E EQUATIONS"

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Abstract. A corrected version of [P. Grabowski and F.M. Callier, *ESAIM: COCV* **12** (2006) 169–197], Theorem 4.1, p. 186, and Example, is given.

Mathematics Subject Classification. 93B, 47D, 35A, 34G.

Received September 8, 2008.
Published online June 18, 2009.

1. Introduction

The authors are deeply indebted to Hartmut Logemann, Department of Mathematics, University of Bath, UK for pointing out a counterexample, repeated below, showing that the statement of [2], Theorem 4.1, p. 186, is wrong.

With the notation of [2] all assumptions of that theorem are met for

$$\mathrm{H} = \mathbb{R}, \quad A = -1 = A^{-1},\ h = -1(\iff c^{\#}x = x),\ d = 1, \quad \delta = 1,\ e = \frac{8}{3},\ q = \frac{16}{3},$$

however the system (3.1) has exactly two solutions $(\mathcal{H}, \mathcal{G}) = (-\frac{8}{3}, 0)$, $(\mathcal{H}, \mathcal{G}) = (-\frac{2}{3}, 2)$ and none of them is such that $\mathcal{H} \geq 0$. This counterexample demonstrates that the assumptions of [2], Theorem 4.1, p. 186, are not enough to ensure non-negativity of $\mathcal{H}$.

The aim of this note is to correct the result by adding reasonable and non-restrictive assumptions which can be verified without solving (3.1) explicitly.

2. Corrigendum of [2], Theorem 4.1 (i), p. 186

Theorem 2.1. *Let assumptions* **(H1)**–**(H5)** *hold. Moreover assume that:*

(H6) *The operator* $A : (D(A) \subset H) \longrightarrow H$ *is such that the semigroup generated by* A^{-1} *is* **AS**.

Keywords and phrases. Infinite-dimensional control systems, semigroups, Lyapunov functionals, circle criterion.

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Then:

(i) *The system* (3.1) *has a solution* $(\mathcal{H},\mathcal{G})$, $\mathcal{H}\in\mathbf{L}(H)$, $\mathcal{H}=\mathcal{H}^*\geq 0$, *provided that if* $q>0$ *then, in addition, the assumption* **(A3)** *holds and*

$$\frac{1}{1+\mu_0\hat{g}}\in H^\infty(\mathbb{C}^+)\quad for\quad \mu_0:=\frac{k_1+k_2}{2},\tag{2.1}$$

$\mathcal{G}\in H$, *where in particular:* $\mathcal{G}$ *is the solution of the realization equation* (4.4), *where* ϕ *is the spectral factor of the Popov function* π *(given by* (4.2)*) such that* $\phi(0)=\sqrt{\delta}$, *and both* ϕ *and* $1/\phi$ *are in* $H^\infty(\mathbb{C}^+)$.

Remark 2.1. It should be emphasized that if $q\leq 0$ the statement of [2], Theorem 4.1**(i)**, p. 186, is fully correct, *i.e.*, the assertion holds without **(A3)** and (2.1). The claim [2], Theorem 4.1**(ii)**, p. 186, does not require any correction.

Proof. The whole reasoning of the existing proof remains correct after removing: the sentence starting from the words: "The symbol of the Toeplitz operator ...", the footnote on p. 186 and after dropping the inequality $\mathcal{H}\geq 0$ in the sentence just following (4.17). Having this done, we may correct the proof as follows. Since X is a solution of (4.15) given by (4.10) it is clear that

$$\mathcal{H}=-X=\psi^*\left[(q\mathbb{F}-eI)\mathcal{R}^{-1}(q\mathbb{F}-eI)^*-qI\right]\psi\geq 0\tag{2.2}$$

if $q\leq 0$, whence the claim of the remark above is met.

Now, consider the case $q>0$ ($\Longrightarrow \mu_0\neq 0$) where, in addition **(A3)** (*i.e.*, d is an admissible factor control vector) and (2.1) hold. Observe that

$$1-\mu_0\underbrace{c^\# d}_{=-\hat{g}(0)}\neq 0,$$

for if not, by (4.2), we would have $\pi(0)=\delta=\left(1-\frac{k_1}{\mu_0}\right)\left(1-\frac{k_2}{\mu_0}\right)=-\left(\frac{k_2-k_1}{k_1+k_2}\right)^2<0$, which contradicts (4.3). Since the LHS of (2.2) satisfies the Riccati equation

$$(A^{-1})^*\mathcal{H}+\mathcal{H}A^{-1}+\underbrace{\left[\frac{1}{\sqrt{\delta}}(-\mathcal{H}d+eh)\right]}_{=-\mathcal{G}}\left[\frac{1}{\sqrt{\delta}}(-\mathcal{H}d+eh)\right]^*-qhh^*=0\tag{2.3}$$

then, adding $\dfrac{\mu_0}{1-\mu_0c^\# d}hd^*\mathcal{H}+\dfrac{\mu_0}{1-\mu_0c^\# d}\mathcal{H}dh^*$ to both sides of (2.3), we conclude that $\mathcal{H}$ satisfies the Lyapunov operator equation

$$\left[A^{-1}+\frac{\mu_0}{1-\mu_0c^\# d}dh^*\right]^*\mathcal{H}+\mathcal{H}\left[A^{-1}+\frac{\mu_0}{1-\mu_0c^\# d}dh^*\right]=-(\mathcal{G}-q_1h)(\mathcal{G}-q_1h)^*-q_0hh^*$$

with

$$q_1:=\frac{\mu_0\sqrt{\delta}}{1-\mu_0c^\# d},\qquad q_0=\frac{(k_2-k_1)^2}{4(1-\mu_0c^\# d)^2}>0,$$

or equivalently,

$$\langle A_0x,\mathcal{H}x\rangle_{\mathrm{H}}+\langle\mathcal{H}x,A_0x\rangle_{\mathrm{H}}=-\left[(\mathcal{G}-q_1h)^*A_0x\right]^2-q_0\left[h^*A_0x\right]^2\qquad\forall x\in D(A_0),\tag{2.4}$$

where

$$A_0x:=A(x-\mu_0dc^\# x),\qquad D(A_0)=\left\{x\in D(d^*)\colon x-\mu_0dc^\# x\in D(A)\right\}\cdot$$

This is because $A_0^{-1} = A^{-1} + \dfrac{\mu_0}{1-\mu_0 c^{\#} d} \mathrm{d}h^* \in \mathbf{L}(\mathrm{H})$. The operator A_0 arises by applying negative linear feedback $u = -\mu_0 y$ to

$$\left\{ \begin{array}{rcl} \dot{x} & = & A(x+ud) \\ y & = & c^{\#} x \end{array} \right\} \tag{2.5}$$

and it corresponds to the Lur'e control system of [2], Figure 1.1, p. 170, with $f(y) = \mu_0 y$. Since $c^{\#}$ is admissible and $\hat{g} \in \mathrm{H}^{\infty}(\mathbb{C}^+)$, for $\mathrm{L}^2(0,\infty)$-controls the output is given by

$$y = \overline{P} x_0 + \overline{\mathbb{F}} u$$

where $\overline{P}$ and $\overline{\mathbb{F}}$ stand for the extended observability map and the extended input-output operator, both associated with (2.5). Thus, for the closed-loop system, by the Paley-Wiener theory, one has

$$(I + \mu_0 \overline{\mathbb{F}}) y = \overline{P} x_0 \iff (1 + \mu_0 \hat{g}) y = \widehat{\overline{P} x_0},$$

and, due to (2.1), the last equation has a unique solution $\hat{y} \in \mathrm{H}^2(\mathbb{C}^+)$. *Via* the feedback law equation $u = -\mu_0 y$ this implies that for any x_0: $u \in \mathrm{L}^2(0,\infty)$. Now [2], Lemma 2.11, p. 177, implies that for every initial condition x_0 the first equation of (2.5) has a unique weak solution, whence, by Ball's theorem [1], p. 371 (see also [4], p. 259), the operator A_0 generates a C_0-semigroup $\{S_0(t)\}_{t \geq 0}$ on H which is **AS**.

Now, for every $x_0 \in D(A_0)$ and each $t \geq 0$, (2.4) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle S_0(t) x_0, \mathcal{H} S_0(t) x_0 \rangle_{\mathrm{H}} = -\left[(\mathcal{G} - q_1 h)^* A_0 S_0(t) x_0 \right]^2 - q_0 \left[h^* A_0 S_0(t) x_0 \right]^2.$$

Integrating both sides from 0 to t and employing **AS** we obtain

$$\langle x_0, \mathcal{H} x_0 \rangle_{\mathrm{H}} = \int_0^{\infty} \left\{ \left[(\mathcal{G} - q_1 h)^* A_0 S_0(t) x_0 \right]^2 + q_0 \left[h^* A_0 S_0(t) x_0 \right]^2 \right\} \mathrm{d}t \geq 0 \qquad \forall x_0 \in D(A_0).$$

Since $D(A_0)$ is dense in H as a C_0-semigroup generator and $\mathcal{H} = \mathcal{H}^* \in \mathbf{L}(\mathrm{H})$ we get $\mathcal{H} \geq 0$. □

Remark 2.2. The above proof may be slightly, but not essentially, modified by concluding **AS** of the semigroup $\{\mathrm{e}^{tA_0^{-1}}\}_{t \geq 0}$ from the reciprocal system

$$\left\{ \begin{array}{rcl} \dot{x} & = & A^{-1} x + ud \\ y & = & -h^* x \end{array} \right\}$$

with the feedback law $u = -\dfrac{\mu_0}{1-\mu_0 c^{\#} d} y$, with an aid of [3], Lemma 12, p. 959. This is possible if d is admissible with respect to $\{\mathrm{e}^{tA^{-1}}\}_{t \geq 0}$ and $u \in \mathrm{L}^2(0,\infty)$ for any initial condition $x_0 \in \mathrm{H}$. It is not difficult to see, using duality between observation and control (see [2], p. 173) and the arguments which led to [2], Lemma 2.6, p. 174, that the first condition holds iff d is admissible. Since in the frequency-domain the closed-loop output equation reads as

$$\begin{aligned} \hat{y}(s) &= -h^* \left(sI - A^{-1} \right)^{-1} x_0 - h^* \left(sI - A^{-1} \right)^{-1} d \left[-\frac{\mu_0}{1-\mu_0 c^{\#} d} \hat{y}(s) \right] \\ &= \left(U \widehat{\overline{P} x_0} \right)(s) + G(s) \left[-\frac{\mu_0}{1-\mu_0 c^{\#} d} \right] \hat{y}(s), \end{aligned}$$

where U is the unitary operator introduced in [2], p. 174, and G is given by [2], (4.12), p. 187, then the second condition holds if $\dfrac{1}{1 + \dfrac{\mu_0}{1-\mu_0 c^{\#} d} G} \in \mathrm{H}^{\infty}(\mathbb{C}^+)$. By [2], (4.13), p. 187, the last condition is equivalent to (2.1).

Next, our Lyapunov operator equation

$$\left(A_0^{-1}\right)^* \mathcal{H} + \mathcal{H}A_0^{-1} = -(\mathcal{G} - q_1 h)(\mathcal{G} - q_1 h)^* - q_0 hh^*$$

allows to get directly

$$\langle x_0, \mathcal{H}x_0\rangle_{\mathrm{H}} = \int_0^\infty \left\{ \left[(\mathcal{G} - q_1 h)^* \mathrm{e}^{tA_0^{-1}} x_0 \right]^2 + q_0 \left[h^* \mathrm{e}^{tA_0^{-1}} x_0 \right]^2 \right\} \mathrm{d}t \geq 0 \qquad \forall x_0 \in \mathrm{H}.$$

3. Correction of [2], Example

Just before the sentence starting from the words ([2], Sect. 5.2, p. 1927): "Thus all assumptions of Theorem 4.1 are met ..." the following text should be inserted[3].

Recall that d is an admissible factor control vector and for $b \in (0,1)$ the assumption (2.1) holds. Indeed, here

$$\frac{1}{1 + \mu_0 \hat{g}(s)} = \frac{1}{1 + \dfrac{4b}{a(1+b)} \dfrac{a\mathrm{e}^{-sr}}{1 + b\mathrm{e}^{-2sr}}} = \frac{1 + b\mathrm{e}^{-2sr}}{b\mathrm{e}^{-2sr} + \dfrac{4b}{(1+b)}\mathrm{e}^{-sr} + 1}\cdot$$

The numerator is bounded by $1 + b$ on $\overline{\mathbb{C}^+}$, while for the denominator one has

$$b\mathrm{e}^{-2sr} + \frac{4b}{(1+b)}\mathrm{e}^{-sr} + 1 = b\left(z_0 - \mathrm{e}^{-sr}\right)\left(\overline{z_0} - \mathrm{e}^{-sr}\right), \qquad \operatorname{Re} z_0 = \frac{-2b}{1+b}, \quad |z_0|^2 = \frac{1}{b},$$

whence

$$\left| b\mathrm{e}^{-2sr} + \frac{4b}{(1+b)}\mathrm{e}^{-sr} + 1 \right| = b\left|z_0 - \mathrm{e}^{-sr}\right| \left|\overline{z_0} - \mathrm{e}^{-sr}\right| \geq b\left(|z_0| - 1\right)^2 = (1 - \sqrt{b})^2,$$

and consequently: $\left\| \dfrac{1}{1 + \mu_0 \hat{g}} \right\|_{\mathrm{H}^\infty(\mathbb{C}^+)} \leq \dfrac{1+b}{(1-\sqrt{b})^2} < \infty.$

References

[1] J.M. Ball, Strongly continuous semigroups, weak solutions, and the variation of constants formula. *P. Am. Math. Soc.* **63** (1977) 370–373.

[2] P. Grabowski and F.M. Callier, On the circle criterion for boundary control systems in factor form: Lyapunov stability and Lur'e equations. *ESAIM: COCV* **12** (2006) 169–197.

[3] J.C. Oostveen and R.F. Curtain, Riccati equations for strongly stabilizable bounded linear systems. *Automatica* **34** (1998) 953–967.

[4] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer Verlag, New York (1983).

[3]Since $q = k_1 k_2 < 0$ for $b \in (0, 3 - 2\sqrt{2})$ and sufficiently small ν then, in fact, corrections are needed only for $b \in [3 - 2\sqrt{2}, 1)$.