STABILIZATION OF SECOND ORDER EVOLUTION EQUATIONS WITH UNBOUNDED FEEDBACK WITH DELAY

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Abstract. We consider abstract second order evolution equations with unbounded feedback with delay. Existence results are obtained under some realistic assumptions. Sufficient and explicit conditions are derived that guarantee the exponential or polynomial stability. Some new examples that enter into our abstract framework are presented.

1. Introduction

Time-delay often appears in many biological, electrical engineering systems and mechanical applications [1,11,21], and in many cases, in particular for distributed parameter systems, even arbitrarily small delays in the feedback may destabilize the system, see *e.g.* [8–10,12,15–17,20,23]. The stability issue of systems with delay is, therefore, of theoretical and practical importance.

We further remark that some techniques developed recently [16,17] in order to obtain some existence results and decay rates have some similarities. We therefore propose to consider an abstract setting as large as possible in order to contain a quite large class of problems with time delay feedbacks. In a second step we prove existence and stability results in this setting under realistic assumptions. Finally in order to show the usefulness of our approach, we give some examples where our abstract framework can be applied. For a similar approach, we refer to the paper in preparation [4]. Without delay such an approach was developed in [3].

Before going on, let us present our abstract framework. Let H be a real Hilbert space with norm and inner product denoted respectively by $\|.\|_H$ and $(.,.)_H$. Let $A : D(A) \to H$ be a self-adjoint positive operator with a compact inverse in H. Let $V := D(A^{\frac{1}{2}})$ be the domain of $A^{\frac{1}{2}}$. Denote by $D(A^{\frac{1}{2}})'$ the dual space of $D(A^{\frac{1}{2}})$ obtained by means of the inner product in H.

Further, for $i = 1, 2$, let U_i be a real Hilbert space (which will be identified to its dual space) with norm and inner product denoted respectively by $\|.\|_{U_i}$ and $(.,.)_{U_i}$ and let $B_i \in \mathcal{L}(U_i, D(A^{\frac{1}{2}})')$.

We consider the system described by

$$
\begin{cases}
\ddot{\omega}(t) + A\omega(t) + B_1 u_1(t) + B_2 u_2(t - \tau) = 0,\ t > 0 \\
\qquad\qquad \omega(0) = \omega_0,\ \dot{\omega}(0) = \omega_1, \\
\qquad u_2(t - \tau) = f^0(t - \tau),\ 0 < t < \tau,
\end{cases}
\tag{1.1}
$$

where $t \in [0, \infty)$ represents the time, τ is a positive constant which represents the delay, $\omega : [0, \infty) \to H$ is the state of the system and $u_1 \in L^2([0, \infty), U_1)$, $u_2 \in L^2([-\tau, \infty), U_2)$ are the input functions. Most of the linear equations modelling the vibrations of elastic structures with distributed control with delay can be written in the form (1.1), where ω stands for the displacement field.

In many problems, coming in particular from elasticity, the input u_i are given in the feedback form $u_i(t) = B_i^*\dot{\omega}(t)$, which corresponds to colocated actuators and sensors. We obtain in this way the closed loop system

$$
\begin{cases}
\ddot{\omega}(t) + A\omega(t) + B_1 B_1^*\dot{\omega}(t) + B_2 B_2^*\dot{\omega}(t - \tau) = 0,\ t > 0 \\
\qquad\qquad \omega(0) = \omega_0,\ \dot{\omega}(0) = \omega_1, \\
\qquad B_2^*\dot{\omega}(t - \tau) = f^0(t - \tau),\ 0 < t < \tau.
\end{cases}
\tag{1.2}
$$

The first natural question is the well-posedness of this system. In Section 2 we will give a sufficient condition that guarantees that this system (1.2) is well-posed, where we closely follow the approach developed in [16] for the wave equation. Secondly, we may ask if this system is dissipative. We show in Section 3 that the condition

$$
\exists\, 0 < \alpha < 1,\ \forall u \in V,\ \|B_2^* u\|_{U_2}^2 \le \alpha \|B_1^* u\|_{U_1}^2
\tag{1.3}
$$

guarantees the energy is decreasing; under this condition, using a result from [5] (see also [22]) we pertain a necessary and sufficient condition for the decay to zero of the energy. Note that this last condition is independent of the delay and therefore under the condition (1.3), our system is strongly stable if and only if the same system without delay is strongly stable. Note further that if (1.3) is not satisfied, there exist cases where some instabilities may appear (see [16,17,23] for the wave equation). Hence this assumption seems realistic.

In a third step, again under the condition (1.3) and a certain boundedness assumption from [3] between the resolvent operator of A and of the operators B_1 and B_2, see condition (4.1), we prove that the exponential decay of the system (1.2) follows from a certain observability estimate. Again this observability estimate is independent of the delay term $B_2 B_2^*\dot{\omega}(t - \tau)$ and therefore, under the conditions (1.3) and (4.1), the exponential decay of the system (1.2) follows from the exponential decay of the same system without delay. Nevertheless we give the dependence of the decay rate with respect to the delay, in particular we show that if the delay increases the decay rate decreases. This is the content of Section 4. A similar analysis for the polynomial decay is performed in Section 5 by weakening the observability estimate. Again we show that if the delay increases the decay rate decreases. In view of some applications, Section 6 is devoted to the proof of these two observability estimates by using a frequency domain method and a reduction to some conditions between the eigenvectors of A and the feedback operator B_1^*.

Finally we finish this paper by considering in Section 7 different examples where our abstract framework can be applied. To our knowledge, all the examples, with the exception of the first one, are new.

2. Well-posedness of the system

We aim to show that system (1.2) is well-posed. For that purpose, we use semi-group theory and an idea from [16] (see also [17]). Let us introduce the auxiliary variable $z(\rho, t) = B_2^*\dot{\omega}(t - \tau\rho)$ for $\rho \in (0, 1)$ and $t > 0$.

Note that z verifies the transport equation for $0 < \rho < 1$ and $t > 0$

$$\begin{cases} \tau \frac{\partial z}{\partial t} + \frac{\partial z}{\partial \rho} = 0 \\ z(0, t) = B_2^* \dot{\omega}(t) \\ z(\rho, 0) = B_2^* \dot{\omega}(-\tau \rho) = f^0(-\tau \rho). \end{cases} \tag{2.1}$$

Therefore, the system (1.2) is equivalent to

$$\begin{cases} \ddot{\omega}(t) + A\omega(t) + B_1 B_1^* \dot{\omega}(t) + B_2 z(1, t) = 0, \, t > 0 \\ \tau \frac{\partial z}{\partial t} + \frac{\partial z}{\partial \rho} = 0, \, t > 0, \, 0 < \rho < 1 \\ \omega(0) = \omega_0, \, \dot{\omega}(0) = \omega_1, \, z(\rho, 0) = f^0(-\tau \rho), \, 0 < \rho < 1 \\ z(0, t) = B_2^* \dot{\omega}(t), \, t > 0. \end{cases} \tag{2.2}$$

If we introduce

$$U := (\omega, \, \dot{\omega}, \, z)^T,$$

then U satisfies

$$U' = (\dot{\omega}, \, \ddot{\omega}, \, \dot{z})^T = \left(\dot{\omega}, \, -A\omega(t) - B_1 B_1^* \dot{\omega}(t) - B_2 z(1, t), \, -\frac{1}{\tau} \frac{\partial z}{\partial \rho} \right)^T.$$

Consequently the system (1.2) may be rewritten as the first order evolution equation

$$\begin{cases} U' = \mathcal{A}U \\ U(0) = (\omega_0, \, \omega_1, \, f^0(-\tau.)), \end{cases} \tag{2.3}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \begin{pmatrix} \omega \\ u \\ z \end{pmatrix} = \begin{pmatrix} u \\ -A\omega - B_1 B_1^* u - B_2 z(1) \\ -\frac{1}{\tau} \frac{\partial z}{\partial \rho} \end{pmatrix},$$

with domain

$$D(\mathcal{A}) := \{ (\omega, \, u, \, z) \in V \times V \times H^1((0, \, 1), \, U_2); \, z(0) = B_2^* u, \, A\omega + B_1 B_1^* u + B_2 z(1) \in H \}.$$

Now, we introduce the Hilbert space

$$\mathcal{H} = V \times H \times L^2((0, \, 1), \, U_2)$$

equipped with the usual inner product

$$\left\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \, \begin{pmatrix} \tilde{\omega} \\ \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle = \left(A^{\frac{1}{2}} \omega, \, A^{\frac{1}{2}} \tilde{\omega} \right)_H + (u, \, \tilde{u})_H + \int_0^1 (z(\rho), \, \tilde{z}(\rho))_{U_2} \, d\rho. \tag{2.4}$$

Let us suppose now that

$$\exists 0 < \alpha \le 1, \, \forall u \in V, \, \|B_2^* u\|_{U_2}^2 \le \alpha \|B_1^* u\|_{U_1}^2. \tag{2.5}$$

Under this condition, we will show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$.

For that purpose, we choose a positive real number ξ such that

$$1 \leq \xi \leq \frac{2}{\alpha} - 1. \tag{2.6}$$

This constant exists because $0 < \alpha \leq 1$.

We now introduce the following inner product on $\mathcal{H}$

$$\left\langle \begin{pmatrix} \omega \\ u \\ z \end{pmatrix}, \begin{pmatrix} \tilde{\omega} \\ \tilde{u} \\ \tilde{z} \end{pmatrix} \right\rangle_{\mathcal{H}} = \left(A^{\frac{1}{2}} \omega, A^{\frac{1}{2}} \tilde{\omega} \right)_H + (u, \tilde{u})_H + \tau \xi \int_0^1 (z(\rho), \tilde{z}(\rho))_{U_2} \, \mathrm{d}\rho.$$

This new inner product is clearly equivalent to the usual inner product (2.4) on $\mathcal{H}$.

Theorem 2.1. *Under the assumption* (2.5), *for an initial datum* $U_0 \in \mathcal{H}$, *there exists a unique solution* $U \in C([0, +\infty), \mathcal{H})$ *to system* (2.3). *Moreover, if* $U_0 \in D(\mathcal{A})$, *then*

$$U \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}).$$

Proof. By Lumer-Phillips' theorem, it suffices to show that $\mathcal{A}$ is m-dissipative (see Def. 3.3.1 and Thms. 1.4.3 and 1.4.6 of [18]).

We first prove that $\mathcal{A}$ is dissipative. Take $U = (\omega, u, z)^\top \in D(\mathcal{A})$. Then

$$
\begin{aligned}
\langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} u \\ -A\omega - B_1 B_1^* u - B_2 z(1) \\ -\frac{1}{\tau} \frac{\partial z}{\partial \rho} \end{pmatrix}, \begin{pmatrix} \omega \\ u \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \\
&= \left(A^{\frac{1}{2}} u, A^{\frac{1}{2}} \omega \right)_H - (A\omega + B_1 B_1^* u + B_2 z(1), u)_H - \xi \int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} \mathrm{d}\rho.
\end{aligned}
$$

Since $A\omega + B_1 B_1^* u + B_2 z(1) \in H$, we obtain

$$
\begin{aligned}
\langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= \left(A^{\frac{1}{2}} u, A^{\frac{1}{2}} \omega \right)_H - \langle A\omega, u \rangle_{V', V} - \langle B_1 B_1^* u, u \rangle_{V', V} - \langle B_2 z(1), u \rangle_{V', V} \\
&\quad - \xi \int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} \mathrm{d}\rho \\
&= \langle A\omega, u \rangle_{V', V} - \langle A\omega, u \rangle_{V', V} - \|B_1^* u\|_{U_1}^2 - (z(1), B_2^* u)_{U_2} \\
&\quad - \xi \int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} \mathrm{d}\rho,
\end{aligned}
$$

by duality. By integrating by parts, we obtain

$$\int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} \mathrm{d}\rho = - \int_0^1 \left(z(\rho), \frac{\partial z}{\partial \rho}(\rho) \right)_{U_2} \mathrm{d}\rho + (\|z(1)\|_{U_2}^2 - \|z(0)\|_{U_2}^2),$$

and thus

$$\int_0^1 \left(\frac{\partial z}{\partial \rho}(\rho), z(\rho) \right)_{U_2} \mathrm{d}\rho = \frac{1}{2}(\|z(1)\|_{U_2}^2 - \|B_2^* u\|_{U_2}^2).$$

Therefore, by Cauchy-Schwarz's inequality, we find

$$
\begin{aligned}
\langle \mathcal{A}U, U \rangle_{\mathcal{H}} &= -\|B_1^* u\|_{U_1}^2 - (z(1), B_2^* u)_{U_2} - \frac{\xi}{2} \|z(1)\|_{U_2}^2 + \frac{\xi}{2} \|B_2^* u\|_{U_2}^2 \\
&\leq -\|B_1^* u\|_{U_1}^2 + \left(\frac{1}{2} - \frac{\xi}{2} \right) \|z(1)\|_{U_2}^2 + \left(\frac{1}{2} + \frac{\xi}{2} \right) \|B_2^* u\|_{U_2}^2.
\end{aligned}
$$

By (2.5), we obtain

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq \left(\frac{\alpha}{2} + \frac{\xi\alpha}{2} - 1\right)\|B_1^*u\|_{U_1}^2 + \left(\frac{1}{2} - \frac{\xi}{2}\right)\|z(1)\|_{U_2}^2$$

with $\frac{\alpha}{2} + \frac{\xi\alpha}{2} - 1 \leq 0$ and $\frac{1}{2} - \frac{\xi}{2} \leq 0$ because ξ satisfies condition (2.6). This shows that $\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq 0$ and then the dissipativeness of $\mathcal{A}$.

Let us now prove that $\lambda I - \mathcal{A}$ is surjective for some $\lambda > 0$.

Let $(f, g, h)^T \in \mathcal{H}$. We look for $U = (\omega, u, z)^T \in D(\mathcal{A})$ solution of

$$(\lambda I - \mathcal{A})\begin{pmatrix} \omega \\ u \\ z \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

or equivalently

$$\begin{cases} \lambda\omega - u = f \\ \lambda u + A\omega + B_1 B_1^*u + B_2 z(1) = g \\ \lambda z + \frac{1}{\tau}\frac{\partial z}{\partial \rho} = h. \end{cases} \tag{2.7}$$

Suppose that we have found ω with the appropriate regularity. Then, we have

$$u = -f + \lambda\omega \in V.$$

We can then determine z, indeed z satisfies the differential equation

$$\lambda z + \frac{1}{\tau}\frac{\partial z}{\partial \rho} = h$$

and the boundary condition $z(0) = B_2^*u = -B_2^*f + \lambda B_2^*\omega$. Therefore z is explicitly given by

$$z(\rho) = \lambda B_2^*\omega e^{-\lambda\tau\rho} - B_2^*f e^{-\lambda\tau\rho} + \tau e^{-\lambda\tau\rho}\int_0^\rho e^{\lambda\tau\sigma}h(\sigma)\mathrm{d}\sigma.$$

This means that once ω is found with the appropriate properties, we can find z and u. In particular, we have

$$z(1) = \lambda B_2^*\omega e^{-\lambda\tau} - B_2^*f e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 e^{\lambda\tau\sigma}h(\sigma)\mathrm{d}\sigma = \lambda B_2^*\omega e^{-\lambda\tau} + z^0, \tag{2.8}$$

where $z^0 = -B_2^*f e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 e^{\lambda\tau\sigma}h(\sigma)\mathrm{d}\sigma$ is a fixed element of U_2 depending only on f and h.

It remains to find ω. By (2.7), ω must satisfy

$$\lambda^2\omega + A\omega + \lambda B_1 B_1^*\omega + B_2 z(1) = g + B_1 B_1^*f + \lambda f,$$

and thus by (2.8),

$$\lambda^2\omega + A\omega + \lambda B_1 B_1^*\omega + \lambda e^{-\lambda\tau}B_2 B_2^*\omega = g + B_1 B_1^*f + \lambda f - B_2 z^0 =: q,$$

where $q \in V'$. We take then the duality brackets $\langle ., .\rangle_{V', V}$ with $\phi \in V$:

$$\langle \lambda^2\omega + A\omega + \lambda B_1 B_1^*\omega + \lambda e^{-\lambda\tau}B_2 B_2^*\omega, \phi\rangle_{V', V} = \langle q, \phi\rangle_{V', V}.$$

Moreover:

$$
\begin{aligned}
\left\langle \lambda^2\omega + A\omega + \lambda B_1 B_1^* \omega + \lambda \mathrm{e}^{-\lambda\tau} B_2 B_2^*\omega,\, \phi \right\rangle_{V',V} &\\
= \lambda^2 \left\langle \omega,\, \phi \right\rangle_{V',V} + \left\langle A\omega,\, \phi \right\rangle_{V',V} &+ \lambda(\left\langle B_1 B_1^*\omega,\, \phi \right\rangle_{V',V} + \mathrm{e}^{-\lambda\tau}\left\langle B_2 B_2^*\omega,\, \phi \right\rangle_{V',V}) \\
= \lambda^2 \left(\omega,\, \phi\right)_H + \left(A^{\frac{1}{2}}\omega,\, A^{\frac{1}{2}}\phi\right)_H &+ \lambda((B_1^*\omega,\, B_1^*\phi)_{U_1} + \mathrm{e}^{-\lambda\tau}(B_2^*\omega,\, B_2^*\phi)_{U_2})
\end{aligned}
$$

because $\omega \in V \subset H$. Consequently, we arrive at the problem

$$
\lambda^2 \left(\omega,\, \phi\right)_H + \left(A^{\frac{1}{2}}\omega,\, A^{\frac{1}{2}}\phi\right)_H + \lambda((B_1^*\omega,\, B_1^*\phi)_{U_1} + \mathrm{e}^{-\lambda\tau}(B_2^*\omega,\, B_2^*\phi)_{U_2}) = \left\langle q,\, \phi \right\rangle_{V',V}. \tag{2.9}
$$

The left hand side of (2.9) is continuous and coercive on V. Indeed, we have

$$
\begin{aligned}
&\left| \lambda^2 \left(\omega,\phi\right)_H + \left(A^{\frac{1}{2}}\omega, A^{\frac{1}{2}}\phi\right)_H + \lambda((B_1^*\omega, B_1^*\phi)_{U_1} + \mathrm{e}^{-\lambda\tau}(B_2^*\omega, B_2^*\phi)_{U_2}) \right| \\
&\quad \leq \lambda^2 \|\omega\|_H \|\phi\|_H + \left\|A^{\frac{1}{2}}\omega\right\|_H \left\|A^{\frac{1}{2}}\phi\right\|_H + \lambda(\|B_1^*\omega\|_{U_1} \|B_1^*\phi\|_{U_1} + \mathrm{e}^{-\lambda\tau}\|B_2^*\omega\|_{U_2}\|B_2^*\phi\|_{U_2}) \\
&\quad \leq C\lambda^2 \|\omega\|_V \|\phi\|_H + \left\|A^{\frac{1}{2}}\right\|^2 \|\omega\|_V \|\phi\|_V + \lambda(\|B_1^*\|^2_{\mathcal{L}(V,U_1)} \|\omega\|_V \|\phi\|_V + \mathrm{e}^{-\lambda\tau}\|B_2^*\|^2_{\mathcal{L}(V,U_2)} \|\omega\|_V \|\phi\|_V) \\
&\quad \leq C \|\omega\|_V \|\phi\|_V,
\end{aligned}
$$

and for $\phi = \omega \in V$

$$
\lambda^2 \|\omega\|_H^2 + \left(A^{\frac{1}{2}}\omega,\, A^{\frac{1}{2}}\omega\right)_H + \lambda(\|B_1^*\omega\|_{U_1}^2 + \mathrm{e}^{-\lambda\tau}\|B_2^*\omega\|_{U_2}^2) \geq \left\|A^{\frac{1}{2}}\omega\right\|_H^2 \geq C\|\omega\|_V^2.
$$

Therefore, this problem (2.9) has a unique solution $\omega \in V$ by Lax-Milgram's lemma. Moreover $A\omega + B_1 B_1^* u + B_2 z(1) = g + \lambda f - \lambda^2\omega \in H$. In summary, we have found $(\omega,\, u,\, z)^T \in D(\mathcal{A})$ satisfying (2.7). $\qquad\square$

Remark 2.2. We deduce from Theorem 2.1 that $D(\mathcal{A})$ is dense in $\mathcal{H}$ (see [18]).

Remark 2.3. For initial data $(\omega_0,\, \omega_1,\, f^0(-\tau.))^T$ in $D(\mathcal{A})$, we easily show that the solution $(\omega(t),\, u(t),\, z(t))^T = T(t)(\omega_0,\, \omega_1,\, f^0(-\tau.))^T$, where $T(t)$ is the semigroup generated by $\mathcal{A}$, is indeed solution of (1.2) in the sense that

$$
u(t) = \dot\omega(t),
$$

and

$$
z(\rho,\, t) = B_2^*(t - \tau\rho),
$$

and ω satisfies (1.2).

3. The energy

We now restrict hypothesis (2.5) to obtain the decay of the energy. Namely, we suppose that (1.3) holds, namely

$$
\exists\, 0 < \alpha < 1,\ \forall u \in V,\ \|B_2^* u\|_{U_2}^2 \leq \alpha \|B_1^* u\|_{U_1}^2.
$$

For an initial datum $(\omega_0,\, \omega_1,\, f^0(-\tau.))^T \in \mathcal{H}$, Theorem 2.1 guarantees the existence of a weak solution $(\omega(t),\, u(t),\, z(t))^T = T(t)(\omega_0,\, \omega_1,\, f^0(-\tau.))^T$. Hence the associated energy (which corresponds to the inner product on $\mathcal{H}$) is defined by

$$
E(t) := \frac{1}{2}\left(\left\|A^{\frac{1}{2}}\omega(t)\right\|_H^2 + \|u(t)\|_H^2 + \tau\xi \int_0^1 \|z(\rho,\, t)\|_{U_2}^2\, \mathrm{d}\rho\right),
$$

where ξ is a positive constant satisfying

$$1 < \xi < \frac{2}{\alpha} - 1, \tag{3.1}$$

that exists because $0 < \alpha < 1$.

Note that by Remark 2.3 for initial data in $D(\mathcal{A})$, this energy takes the form

$$E(t) := \frac{1}{2} \left(\left\| A^{\frac{1}{2}} \omega \right\|_H^2 + \|\dot{\omega}\|_H^2 + \tau \xi \int_0^1 \|B_2^* \dot{\omega}(t - \tau\rho)\|_{U_2}^2 \, \mathrm{d}\rho \right). \tag{3.2}$$

3.1. Decay of the energy

Proposition 3.1. *If* (1.3) *holds, then for all* $(\omega_0, \omega_1, f^0(-\tau.))^T \in D(\mathcal{A})$, *the energy of the corresponding regular solution of* (1.2) *(i.e.* $(\omega, \dot{\omega}, B_2\dot{\omega}(t - \tau\rho))^T \in C([0, +\infty), D(\mathcal{A})) \cap C^1([0, +\infty), \mathcal{H}))$ *is non-increasing and there exist two positive constants* C_1 *and* C_2 *depending only on* α *and* ξ *such that*

$$-C_2 \left(\|B_1^* \dot{\omega}(t)\|_{U_1}^2 + \|B_2^* \dot{\omega}(t - \tau)\|_{U_2}^2 \right) \leq E'(t) \leq -C_1 \left(\|B_1^* \dot{\omega}(t)\|_{U_1}^2 + \|B_2^* \dot{\omega}(t - \tau)\|_{U_2}^2 \right). \tag{3.3}$$

Proof. Deriving (3.2), we obtain

$$
\begin{aligned}
E'(t) &= \left(A^{\frac{1}{2}} \omega, A^{\frac{1}{2}} \dot{\omega} \right)_H + (\dot{\omega}, \ddot{\omega})_H + \tau\xi \int_0^1 (B_2^* \dot{\omega}(t - \tau\rho), B_2^* \ddot{\omega}(t - \tau\rho))_{U_2} \, \mathrm{d}\rho \\
&= \langle A\omega, \dot{\omega} \rangle_{V', V} - (\dot{\omega}, A\omega + B_1 B_1^* \dot{\omega} + B_2 B_2^* \dot{\omega}(t - \tau))_H \\
&\quad + \xi\tau \int_0^1 (B_2^* \dot{\omega}(t - \tau\rho), B_2^* \ddot{\omega}(t - \tau\rho))_{U_2} \, \mathrm{d}\rho \\
&= \langle A\omega, \dot{\omega} \rangle_{V', V} - \langle \dot{\omega}, A\omega + B_1 B_1^* \dot{\omega} + B_2 B_2^* \dot{\omega}(t - \tau) \rangle_{V, V'} \\
&\quad + \xi\tau \int_0^1 (B_2^* \dot{\omega}(t - \tau\rho), B_2^* \ddot{\omega}(t - \tau\rho))_{U_2} \, \mathrm{d}\rho,
\end{aligned}
$$

because $A\omega + B_1 B_1^* \dot{\omega} + B_2 B_2^* \dot{\omega}(t - \tau) \in H$. Then

$$
\begin{aligned}
E'(t) &= \langle A\omega, \dot{\omega} \rangle_{V', V} - \langle \dot{\omega}, A\omega \rangle_{V, V'} - \langle \dot{\omega}, B_1 B_1^* \dot{\omega} \rangle_{V, V'} - \langle \dot{\omega}, B_2 B_2^* \dot{\omega}(t - \tau) \rangle_{V, V'} \\
&\quad + \xi\tau \int_0^1 (B_2^* \dot{\omega}(t - \tau\rho), B_2^* \ddot{\omega}(t - \tau\rho))_{U_2} \, \mathrm{d}\rho \\
&= -\|B_1^* \dot{\omega}\|_{U_1}^2 - (B_2^* \dot{\omega}, B_2^* \dot{\omega}(t - \tau))_{U_2} \\
&\quad + \xi\tau \int_0^1 (B_2^* \dot{\omega}(t - \tau\rho), B_2^* \ddot{\omega}(t - \tau\rho))_{U_2} \, \mathrm{d}\rho.
\end{aligned}
$$

Moreover, recalling that $z(\rho, t) = B_2^* \dot{\omega}(t - \tau\rho)$, we see that

$$
\begin{aligned}
\int_0^1 (B_2^* \dot{\omega}(t - \tau\rho), B_2^* \ddot{\omega}(t - \tau\rho))_{U_2} \, \mathrm{d}\rho &= \int_0^1 \left(z(\rho, t), \frac{\partial z}{\partial t}(\rho, t) \right)_{U_2} \, \mathrm{d}\rho \\
&= -\frac{1}{\tau} \int_0^1 \left(z(\rho, t), \frac{\partial z}{\partial \rho}(\rho, t) \right)_{U_2} \, \mathrm{d}\rho,
\end{aligned}
$$

because $\frac{\partial z}{\partial \rho}(\rho, t) = -\tau \frac{\partial z}{\partial t}(\rho, t)$. Then, we have

$$\int_0^1 (B_2^* \dot\omega(t - \tau\rho), B_2^* \ddot\omega(t - \tau\rho))_{U_2} \, \mathrm{d}\rho = -\frac{1}{2\tau} \int_0^1 \frac{\partial}{\partial \rho} \|z(\rho, t)\|_{U_2}^2 \, \mathrm{d}\rho$$

$$= -\frac{1}{2\tau}(\|z(1, t)\|_{U_2}^2 - \|z(0, t)\|_{U_2}^2)$$

$$= -\frac{1}{2\tau}(\|B_2^* \dot\omega(t - \tau)\|_{U_2}^2 - \|B_2^* \dot\omega(t)\|_{U_2}^2).$$

Consequently,

$$E'(t) = -\|B_1^* \dot\omega\|_{U_1}^2 - (B_2^* \dot\omega, B_2^* \dot\omega(t - \tau))_{U_2} - \frac{\xi}{2}\|B_2^* \dot\omega(t - \tau)\|_{U_2}^2 + \frac{\xi}{2}\|B_2^* \dot\omega(t)\|_{U_2}^2.$$

Cauchy-Schwarz's inequality yields

$$E'(t) \leq -\|B_1^* \dot\omega\|_{U_1}^2 + \left(\frac{1}{2} + \frac{\xi}{2}\right)\|B_2^* \dot\omega(t)\|_{U_2}^2 + \left(\frac{1}{2} - \frac{\xi}{2}\right)\|B_2^* \dot\omega(t - \tau)\|_{U_2}^2$$

and

$$E'(t) \geq -\|B_1^* \dot\omega\|_{U_1}^2 + \left(-\frac{1}{2} + \frac{\xi}{2}\right)\|B_2^* \dot\omega(t)\|_{U_2}^2 - \left(\frac{1}{2} + \frac{\xi}{2}\right)\|B_2^* \dot\omega(t - \tau)\|_{U_2}^2.$$

Therefore, by (1.3), these estimates leads to

$$E'(t) \leq -C_1 \left(\|B_1^* \dot\omega(t)\|_{U_1}^2 + \|B_2^* \dot\omega(t - \tau)\|_{U_2}^2\right)$$

with

$$C_1 = \min \left\{ \left(1 - \frac{\xi\alpha}{2} - \frac{\alpha}{2}\right), \left(\frac{\xi}{2} - \frac{1}{2}\right) \right\}$$

which is positive according to the assumption (3.1), and to

$$E'(t) \geq -C_2 \left(\|B_1^* \dot\omega\|_{U_1}^2 + \|B_2^* \dot\omega(t - \tau)\|_{U_2}^2\right)$$

with

$$C_2 = \max \left\{ 1, \frac{\xi + 1}{2} \right\}$$

which is also positive. □

Remark 3.2. Integrating the expression (3.3) between 0 and T, we obtain

$$\int_0^T \left(\|B_1^* \dot\omega(t)\|_{U_1}^2 + \|B_2^* \dot\omega(t - \tau)\|_{U_2}^2\right) \mathrm{d}t \leq \frac{1}{C_1}(E(0) - E(T)) \leq \frac{1}{C_1}E(0).$$

This estimate implies that $B_1^* \dot\omega(.) \in L^2((0, T), U_1)$ and $B_2^* \dot\omega(. - \tau) \in L^2((0, T), U_2)$.

Remark 3.3. If (1.3) is not satisfied, there exist cases where instabilities may appear, see [16,17,23] for the wave equation. Hence this condition appears to be quite realistic.

3.2. **Decay of the energy to 0**

We give a necessary and sufficient condition that guarantees the decay to 0 of the energy.

Proposition 3.4. *Assume that* (1.3) *holds. Then, for all initial data in* $\mathcal{H}$, *we have*

$$\lim_{t \to \infty} E(t) = 0 \tag{3.4}$$

if and only if for any (non zero) eigenvector $\varphi \in D(A)$ *of* A, *we have*

$$B_1^* \varphi \neq 0. \tag{3.5}$$

Remark 3.5. Notice that this necessary and sufficient condition is the same than in the case without delay (see [22]) and therefore, the system (1.2) with delay is strongly stable (*i.e.* the energy tends to zero) if and only if the system without delay (*i.e.* for $B_2 = 0$) is strongly stable.

Proof. $\boxed{\Leftarrow}$ Let us show that (3.5) implies (3.4). For that purpose we closely follow [22].

First, we show that $\mathcal{A}$ has no eigenvalue on the imaginary axis. If it is not the case, let $i\omega$ be an eigenvalue of $\mathcal{A}$ where $\omega \in \mathbb{R}$. Let φ be an eigenvector associated with $i\omega$. Then φ is of the form

$$\varphi = \begin{pmatrix} z \\ i\omega z \\ i\omega e^{-i\omega\tau\rho} B_2^* z \end{pmatrix},$$

with

$$-\omega^2 z + Az + i\omega B_1 B_1^* z + i\omega e^{-i\omega\tau} B_2 B_2^* z = 0. \tag{3.6}$$

It is an immediate consequence of the identity $(i\omega I - \mathcal{A})\varphi = 0$.

First we notice that $\omega \neq 0$ since for $\omega = 0$, the above identity reduces to $Az = 0$ with $z \in D(A)$. Since by hypothesis A is invertible, we get $z = 0$ and therefore 0 is not an eigenvalue of $\mathcal{A}$.

We now take the duality bracket $\langle ., . \rangle_{V', V}$ between (3.6) and $z \in V$:

$$
\begin{aligned}
0 &= \left\langle -\omega^2 z + Az + i\omega B_1 B_1^* z + i\omega e^{-i\omega\tau} B_2 B_2^* z, z \right\rangle_{V', V} \\
&= \left\langle (-\omega^2 I + A)z, z \right\rangle_{V', V} + i\omega \left\| B_1^* z \right\|_{U_1}^2 + i\omega e^{-i\omega\tau} \left\| B_2^* z \right\|_{U_2}^2.
\end{aligned}
$$

We look at the imaginary part of this expression to obtain

$$\omega \left(\left\| B_1^* z \right\|_{U_1}^2 + \cos(\omega\tau) \left\| B_2^* z \right\|_{U_2}^2 \right) = 0,$$

which implies, because $\omega \neq 0$,

$$\left\| B_1^* z \right\|_{U_1}^2 + \cos(\omega\tau) \left\| B_2^* z \right\|_{U_2}^2 = 0.$$

We deduce that

$$0 = \left\| B_1^* z \right\|_{U_1}^2 + \cos(\omega\tau) \left\| B_2^* z \right\|_{U_2}^2 \geq \left\| B_1^* z \right\|_{U_1}^2 - \left\| B_2^* z \right\|_{U_2}^2 \geq (1 - \alpha) \left\| B_1^* z \right\|_{U_1}^2 \geq 0,$$

by (1.3) with $\alpha < 1$. Consequently

$$\left\| B_1^* z \right\|_{U_1} = 0$$

which implies

$$B_1^* z = 0. \tag{3.7}$$

Moreover, by (3.6), (3.7) and (1.3), we have

$$Az = \omega^2 z.$$

Therefore, z is an eigenvector of A of associated eigenvalue ω^2 such that

$$B_1^* z = 0,$$

which contradicts (3.5). Thus, $\mathcal{A}$ has no eigenvalue on the imaginary axis.

Now, we can apply the main theorem of Arendt and Batty [5]: As $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is empty (because the surjectivity of $(i\omega I - \mathcal{A})$ is equivalent to the injectivity of $-\omega^2 + A - i\omega B_1 B_1^* - i\omega e^{i\omega\tau} B_2 B_2^*$), we obtain (3.4).

$\boxed{\Rightarrow}$ Let us show that (3.4) implies (3.5). For that purpose we use a contradiction argument. Suppose that there exists an eigenvector φ of A of associated eigenvalue λ^2 such that

$$B_1^* \varphi = 0.$$

Let us set

$$\omega(.,\, t) = \varphi \cos(\lambda t).$$

Then ω is solution of (1.2) and satisfies
$$E(t) = E(0)$$

because

$$\|B_1^* \dot{\omega}(t)\|_{U_1}^2 = \lambda^2 \sin^2(\lambda t) \|B_1^* \varphi\|_{U_1}^2 = 0$$

and

$$\begin{aligned}
\|B_2^* \dot{\omega}(t-\tau)\|_{U_2}^2 &= \lambda^2 \sin^2(\lambda(t-\tau)) \|B_2^* \varphi\|_{U_2}^2 \\
&\leq \alpha\lambda^2 \sin^2(\lambda(t-\tau)) \|B_1^* \varphi\|_{U_1}^2 = 0,
\end{aligned}$$

by (1.3). This means that we have obtained a solution of system (1.2) with a constant energy, which contradicts (3.4). $\qquad\square$

4. THE EXPONENTIAL STABILITY

4.1. *A priori* estimate

In order to obtain the characterization of decay properties of the damped system *via* observability inequalities for the conservative system we will use the following assumption from [3]:

If $\beta > 0$ is fixed and $C_\beta = \{\lambda \in \mathbb{C} \,|\, \Re\lambda = \beta\}$, the function

$$\lambda \in C_\beta \to H(\lambda) = \lambda B^*(\lambda^2 I + A)^{-1} B \in \mathcal{L}(U) \text{ is bounded,} \tag{4.1}$$

where $B \in \mathcal{L}(U, V')$ with U a Hilbert space.

We consider the evolution system

$$\begin{cases} \ddot{y}(t) + Ay(t) = Bv(t) \\ y(0) = \dot{y}(0) = 0, \end{cases} \tag{4.2}$$

and the following conservative system

$$\begin{cases} \ddot{\phi}(t) + A\phi(t) = 0 \\ \phi(0) = \omega_0,\ \dot{\phi}(0) = \omega_1. \end{cases} \tag{4.3}$$

Let us recall the two following results proved in [3]:

Lemma 4.1. *Suppose that $v \in L^2(0, T; U)$ and that the solutions ϕ of (4.3) are such that $B^*\phi(.) \in H^1(0, T; U)$ and there exists a constant $C > 0$ such that*

$$\|(B^*\phi)'(.)\|_{L^2(0,T;U)} \leq C \|(\omega_0,\, \omega_1)\|_{V \times H} \quad \forall(\omega_0,\, \omega_1) \in V \times H.$$

Then the system (4.2) *admits a unique solution having the regularity*

$$y \in C(0, T; V) \cap C^1(0, T; H).$$

Proposition 4.2. *Suppose that* $v \in L^2(0, T; U)$ *and that the system* (4.2) *admits a unique solution having the regularity*

$$y \in C(0, T; V) \cap C^1(0, T; H).$$

Then hypothesis (4.1) *holds if and only if* $B^*y(.) \in H^1(0, T; U)$ *and there exists a constant* $C > 0$ *independent of* T *such that*

$$\|(B^*y)'(.)\|_{L^2(0,T;U)} \leq Ce^{\beta T} \|v\|_{L^2(0,T;U)}.$$

Let $\omega \in C(0, T; V) \cap C^1(0, T; H)$ be the solution of (1.2) with $(\omega_0, \omega_1, f^0(-\tau.))^T \in D(\mathcal{A})$. Then it can be split up in the form

$$\omega = \phi + \psi,$$

where ϕ is solution of the system without damping (4.3), and ψ satisfies

$$\begin{cases} \ddot{\psi}(t) + A\psi(t) = -B_1 B_1^* \dot{\omega}(t) - B_2 B_2^* \dot{\omega}(t - \tau) \\ \psi(0) = 0, \ \dot{\psi}(0) = 0. \end{cases} \tag{4.4}$$

We now set $B = (B_1 \ B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$. It is easy to verify that $B^* = \begin{pmatrix} B_1^* \\ B_2^* \end{pmatrix} \in \mathcal{L}(V, U)$. Therefore ψ is solution of

$$\begin{cases} \ddot{\psi}(t) + A\psi(t) = Bv(t) \\ \psi(0) = 0, \ \dot{\psi}(0) = 0, \end{cases} \tag{4.5}$$

where $v(t) = \begin{pmatrix} -B_1^* \dot{\omega}(t) \\ -B_2^* \dot{\omega}(t - \tau) \end{pmatrix}$. In other words, ψ is solution of system (4.2) with

$$B = (B_1 \ B_2)$$

and by Remark 3.2

$$v = \begin{pmatrix} -B_1^* \dot{\omega}(\cdot) \\ -B_2^* \dot{\omega}(\cdot - \tau) \end{pmatrix} \in L^2((0, T), U).$$

Then $\psi = \omega - \phi \in C(0, T; V) \cap C^1(0, T; H)$. Suppose that hypothesis (4.1) is satisfied for $B = (B_1 \ B_2)$ and $U = U_1 \times U_2$. By applying Proposition 4.2, we obtain

$$\int_0^T \|(B^*\psi)'\|_U^2 \, dt \leq Ce^{2\beta T} \int_0^T \|v(t)\|_U^2 \, dt,$$

which is equivalent to

$$\int_0^T (\|(B_1^*\psi)'\|_{U_1}^2 + \|(B_2^*\psi)'\|_{U_2}^2) dt \leq Ce^{2\beta T} \int_0^T (\|B_1^* \dot{\omega}(t)\|_{U_1}^2 + \|B_2^* \dot{\omega}(t - \tau)\|_{U_2}^2) dt.$$

In particular, we have

$$\int_0^T \|(B_1^*\psi)'\|_{U_1}^2 \, dt \leq Ce^{2\beta T} \int_0^T (\|B_1^* \dot{\omega}(t)\|_{U_1}^2 + \|B_2^* \dot{\omega}(t - \tau)\|_{U_2}^2) dt.$$

Therefore, since $\omega = \phi + \psi$, we have

$$
\begin{aligned}
\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, \mathrm{d}t \quad &\leq \quad 2\left(\int_0^T \|(B_1^*\omega)'(t)\|_{U_1}^2 \, \mathrm{d}t + \int_0^T \|(B_1^*\psi)'(t)\|_{U_1}^2 \, \mathrm{d}t\right) \\
&\leq \quad C\mathrm{e}^{2\beta T}\int_0^T \left(\|(B_1^*\dot\omega)(t)\|_{U_1}^2 + \|(B_2^*\dot\omega)(t-\tau)\|_{U_2}^2\right)\mathrm{d}t.
\end{aligned}
$$

Thus, we have proved the following result:

Lemma 4.3. *Suppose that assumption* (4.1) *is satisfied for* $B = (B_1\ B_2)$, $U = U_1 \times U_2$. *Then the solutions* ω *of* (1.2) *and* ϕ *of* (4.3) *satisfy*

$$
\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, \mathrm{d}t \leq C\mathrm{e}^{2\beta T}\int_0^T (\|B_1^*\dot\omega(t)\|_{U_1}^2 + \|B_2^*\dot\omega(t-\tau)\|_{U_2}^2)\mathrm{d}t,
$$

with $C > 0$ *independent of* T.

4.2. The stability result

Theorem 4.4. *Assume that hypotheses* (1.3) *and* (4.1) *are verified for* $B = (B_1\ B_2)$, $U = U_1 \times U_2$. *If there exist a time* $T > 0$ *and a constant* $C > 0$ *such that the observability estimate*

$$
\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2 + \|\omega_1\|_H^2 \leq C\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, \mathrm{d}t \tag{4.6}
$$

holds, where ϕ *is solution of* (4.3), *then system* (1.2) *is exponentially stable in the energy space: there exist* $C > 0$ *independent of* τ *and* $\nu > 0$ *such that, for all initial data in* $\mathcal{H}$,

$$
E(t) \leq CE(0)\mathrm{e}^{-\nu t} \quad \forall t > 0. \tag{4.7}
$$

Proof. Let ω be a solution of (1.2) with initial data $(\omega_0, \omega_1, f^0(-\tau\cdot)) \in D(\mathcal{A})$.

Without loss of generality, we can always assume that (4.6) holds with $T > \tau$ and C independent of τ.

Integrating inequality (3.3) of Proposition 3.1 between 0 and T, we obtain

$$
\begin{aligned}
E(0) - E(T) \quad &\geq C\int_0^T \left(\|B_1^*\dot\omega(t)\|_{U_1}^2 + \|B_2^*\dot\omega(t-\tau)\|_{U_2}^2\right)\mathrm{d}t \\
&\geq \frac{C}{2}\int_0^T \left(\|B_1^*\dot\omega(t)\|_{U_1}^2 + \|B_2^*\dot\omega(t-\tau)\|_{U_2}^2\right)\mathrm{d}t + \frac{C}{2}\int_0^T \|B_2^*\dot\omega(t-\tau)\|_{U_2}^2 \, \mathrm{d}t \\
&\geq C\mathrm{e}^{-2\beta T}\left(\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, \mathrm{d}t + \int_0^T \|B_2^*\dot\omega(t-\tau)\|_{U_2}^2 \, \mathrm{d}t\right) \text{ by Lemma 4.3.}
\end{aligned}
$$

By assumption (4.6), we obtain

$$
E(0) - E(T) \geq C\mathrm{e}^{-2\beta T}\left(\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2 + \|\omega_1\|_H^2 + \int_0^T \|B_2^*\dot\omega(t-\tau)\|_{U_2}^2 \, \mathrm{d}t\right),
$$

with C independent of τ. As $T > \tau$, by change of variables, we have

$$
\begin{aligned}
\int_0^T \|B_2^*\dot\omega(t-\tau)\|_{U_2}^2 \, \mathrm{d}t \quad &= \quad \int_{-\tau}^{T-\tau} \|B_2^*\dot\omega(t)\|_{U_2}^2 \, \mathrm{d}t \\
&\geq \quad \int_{-\tau}^0 \|B_2^*\dot\omega(t)\|_{U_2}^2 \, \mathrm{d}t = \tau\int_0^1 \|B_2^*\dot\omega(-\tau\rho)\|_{U_2}^2 \, \mathrm{d}\rho.
\end{aligned}
$$

The two previous inequalities and (3.1) directly imply that

$$E(0) - E(T) \geq C' \mathrm{e}^{-2\beta T} \left(\left\| A^{\frac{1}{2}} \omega_0 \right\|_H^2 + \|\omega_1\|_H^2 + \xi\tau \int_0^1 \|B_2^* \dot\omega(-\tau\rho)\|_{U_2}^2 \, \mathrm{d}\rho \right),$$

with $C' = C/(2/\alpha - 1)$. This means that for $T > \tau$, we have

$$E(0) - E(T) \geq C' \mathrm{e}^{-2\beta T} E(0).$$

This estimate is equivalent to

$$E(0) - E(T) \geq C' \mathrm{e}^{-2\beta T} E(T),$$

because the energy is decreasing, which leads to

$$E(T) \leq \gamma E(0),$$

where $\gamma = \frac{1}{1+C'\mathrm{e}^{-2\beta T}} < 1$. Applying this argument on $[(m-1)T, mT]$, for $m = 1, 2, ...$ (which is valid because the system is invariant by translation in time), we will get

$$E(mT) \leq \gamma E((m-1)T) \leq ... \leq \gamma^m E(0).$$

Therefore, we have

$$E(mT) \leq \mathrm{e}^{-\nu mT} E(0), \ m = 1, 2, ...$$

with $\nu = \frac{1}{T} \ln\frac{1}{\gamma} = \frac{1}{T} \ln(1 + C'\mathrm{e}^{-2\beta T}) > 0$ which depends on T and thus on τ (because $T > \tau$). For an arbitrary positive t, there exists $m \in \mathbb{N}^*$ such that $(m-1)T < t \leq mT$ and by the non-increasing property of the energy, we conclude

$$E(t) \leq E((m-1)T) \leq \mathrm{e}^{-\nu(m-1)T} E(0) \leq \frac{1}{\gamma} \mathrm{e}^{-\nu t} E(0).$$

Hence the energy decays exponentially with the decay rate $\nu = \frac{1}{T} \ln\frac{1}{\gamma} = \frac{1}{T} \ln(1 + C'\mathrm{e}^{-2\beta T}) < \frac{1}{\tau} \ln(1 + C'\mathrm{e}^{-2\beta\tau})$, because $T > \tau$. Notice that the constant C of (4.7) can be chosen such that $C \geq 1 + C'$ (which is independent of τ) because $\frac{1}{\gamma} = 1 + C'\mathrm{e}^{-2\beta T} \leq 1 + C'$.

By density of $D(\mathcal{A})$ into $\mathcal{H}$, we deduce that (4.7) holds for any initial data in $\mathcal{H}$. $\qquad\square$

Remark 4.5. In the previous theorem, we have seen that the decay rate is $\nu = \frac{1}{T} \ln\frac{1}{\gamma} = \frac{1}{T} \ln(1 + C'\mathrm{e}^{-2\beta T}) < \frac{1}{\tau} \ln(1 + C'\mathrm{e}^{-2\beta\tau})$ because $T > \tau$ and where C' is independent of τ. Therefore, when τ becomes larger, the decay rate is slower.

Remark 4.6. Notice that the sufficient condition (4.6) for the exponential decay of the energy is the same than the case without delay (see [3]). Therefore, if the hypothesis (4.1) holds and if the dissipative system without delay (*i.e.* with $B_2 = 0$) is exponentially stable, then the system (1.2) is exponentially stable.

5. THE POLYNOMIAL STABILITY

In some cases, the decay of the energy is not exponential, but can be polynomial. Our aim here is to give a sufficient condition that yields the explicit decay rate.

The proof of this stability result requires the next technical lemma proved in [2], Lemma 5.2.

Lemma 5.1. *Let $(\varepsilon_k)_k$ be a sequence of positive real numbers satisfying*

$$\varepsilon_{k+1} \leq \varepsilon_k - C\varepsilon_{k+1}^{2+\mu} \quad \forall k \geq 0, \tag{5.1}$$

where $C > 0$ and $\mu > -1$ are constants. Then there exists a positive constant M (depending on μ and C) such that

$$\varepsilon_k \leq \frac{M}{(1+k)^{\frac{1}{1+\mu}}} \quad \forall k \geq 0,$$

with $M > \left(\frac{4}{(1+\mu)C} \right)^{\frac{1}{1+\mu}}$.

Moreover we recall the following interpolation result.

Lemma 5.2. *For $(\omega_0, \omega_1) \in D(A) \times V$, we have*

$$\begin{aligned}
\|\omega_0\|_{D(A^{\frac{1}{2}})}^{m+1} &\leq C \|\omega_0\|_{D(A)}^m \|\omega_0\|_{D(A^{\frac{1-m}{2}})}, \\
\|\omega_1\|_H^{m+1} &\leq C \|\omega_1\|_{D(A^{\frac{1}{2}})}^m \|\omega_1\|_{D(A^{-\frac{m}{2}})},
\end{aligned}$$

where $C > 0$.

Proof. If we denote by $\{\lambda_{k_n}\}_n$ the eigenvalues of $A^{\frac{1}{2}}$ counted without multiplicity, l_n the multiplicity of the eigenvalue λ_{k_n} and $\{\varphi_{k_n+j}\}_{0 \leq j \leq l_n-1}$ the orthonormal eigenvectors associated with the eigenvalue λ_{k_n}, this lemma is a direct consequence of the equivalence $\|u\|_{D(A^s)}^2 \sim \sum_{n \geq 1} \sum_{j=0}^{l_n-1} |u_{k_n+j}|^2 \lambda_{k_n}^{4s}$ for all $s \in \mathbb{R}$, when $u = \sum_{n \geq 1} \sum_{j=0}^{l_n-1} u_{k_n+j}\varphi_{k_n+j}$ and of Hölder's inequality with $p = 1 + \frac{1}{m}$ and $q = m + 1$. $\square$

As for $(\omega_0, \omega_1, f^0(-\tau\cdot))^\top \in D(\mathcal{A})$, ω_0 is not necessarily in $D(A)$, we can not apply Lemma 5.2 to ω_0, and therefore we need to make the following hypothesis: there exists $C > 0$ such that for all $(\omega_0, \omega_1, z)^\top \in D(\mathcal{A})$, we have

$$\|\omega_0\|_V^{m+1} \leq C \|(\omega_0, \omega_1, z)\|_{D(\mathcal{A})}^m \|\omega_0\|_{D(A^{\frac{1-m}{2}})}. \tag{5.2}$$

Theorem 5.3. *Let ω be a solution of (1.2) with $(\omega_0, \omega_1, f^0(-\tau\cdot))^\top \in D(\mathcal{A})$. Assume that the hypotheses (1.3), (4.1) and (5.2) are verified for $B = (B_1\, B_2)$, $U = U_1 \times U_2$. If there exist a positive real number m, a time $T > 0$ and a constant $C > 0$ such that*

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, dt \geq C \left(\|\omega_0\|_{D(A^{\frac{1-m}{2}})}^2 + \|\omega_1\|_{D(A^{-\frac{m}{2}})}^2 \right) \tag{5.3}$$

holds where ϕ is solution of (4.3), then the energy decays polynomially, i.e., there exists $C > 0$ depending on m and τ such that, for all initial data in $D(\mathcal{A})$,

$$E(t) \leq \frac{C}{(1+t)^{\frac{1}{m}}} \left\| (\omega_0, \omega_1, f^0(-\tau\cdot)) \right\|_{D(\mathcal{A})}^2 \quad \forall t > 0. \tag{5.4}$$

Proof. As the hypothesis (4.1) is satisfied for $B = (B_1\, B_2)$, $U = U_1 \times U_2$, by using Lemma 4.3, we obtain

$$\int_0^T \left(\|B_1^*\dot{\omega}(t)\|_{U_1}^2 + \|B_2^*\dot{\omega}(t-\tau)\|_{U_2}^2 \right) dt \geq C e^{-2\beta T} \left(\|\omega_0\|_{D(A^{\frac{1-m}{2}})}^2 + \|\omega_1\|_{D(A^{-\frac{m}{2}})}^2 \right).$$

On the other hand, integrating the inequality (3.3) of Proposition 3.1 between 0 and T^* for T^* large enough: $T^* \geq \max(T, \tau)$, we have

$$
\begin{aligned}
E(0) - E(T^*) \quad &\geq \quad C \int_0^{T^*} (\|B_1^* \dot{\omega}(t)\|_{U_1}^2 + \|B_2^* \dot{\omega}(t - \tau)\|_{U_2}^2) \mathrm{d}t \\
&\geq \quad \frac{C}{2} \int_0^{T^*} (\|B_1^* \dot{\omega}(t)\|_{U_1}^2 + \|B_2^* \dot{\omega}(t - \tau)\|_{U_2}^2) \mathrm{d}t + \frac{C}{2} \int_0^{T^*} \|B_2^* \dot{\omega}(t - \tau)\|_{U_2}^2 \, \mathrm{d}t \\
&\geq \quad C \mathrm{e}^{-2\beta T^*} \left(\|\omega_0\|_{D(A^{\frac{1-m}{2}})}^2 + \|\omega_1\|_{D(A^{-\frac{m}{2}})}^2 + \tau \int_0^1 \|B_2^* \dot{\omega}(-\tau \rho)\|_{U_2}^2 \, \mathrm{d}\rho \right),
\end{aligned}
$$

by change of variable (because $T^* > \tau$). Therefore

$$
E(T^*) \leq E(0) - K_1 \mathrm{e}^{-2\beta T^*} \left(\|\omega_0\|_{D(A^{\frac{1-m}{2}})}^2 + \|\omega_1\|_{D(A^{-\frac{m}{2}})}^2 + \tau \xi \int_0^1 \|B_2^* \dot{\omega}(-\tau \rho)\|_{U_2}^2 \, \mathrm{d}\rho \right), \tag{5.5}
$$

for some $K_1 > 0$ independent of T^* and τ.

Therefore, by (5.2), the previous interpolation result of Lemma 5.2 and a convexity inequality, we have:

$$
\begin{aligned}
\|(\omega_0, \omega_1)\|_{V \times H}^{m+1} \quad &\leq \quad C \left(\|\omega_0\|_V^{m+1} + \|\omega_1\|_H^{m+1} \right) \\
&\leq \quad C \left(\|(\omega_0, \omega_1, f^0(-\tau.))\|_{D(\mathcal{A})}^m \|\omega_0\|_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|_{D(A^{\frac{1}{2}})}^m \|\omega_1\|_{D(A^{-\frac{m}{2}})} \right) \\
&\leq \quad C \|(\omega_0, \omega_1, f^0(-\tau.))\|_{D(\mathcal{A})}^m \left(\|\omega_0\|_{D(A^{\frac{1-m}{2}})} + \|\omega_1\|_{D(A^{-\frac{m}{2}})} \right).
\end{aligned}
$$

Denoting by $X_{-m} = D(A^{\frac{1-m}{2}}) \times D(A^{-\frac{m}{2}})$, we have shown that

$$
\|(\omega_0, \omega_1)\|_{X_{-m}}^2 \geq C \frac{\|(\omega_0, \omega_1)\|_{V \times H}^{2m+2}}{\|(\omega_0, \omega_1, f^0(-\tau.))\|_{D(\mathcal{A})}^{2m}}. \tag{5.6}
$$

Now introduce the modified energy

$$
\tilde{E}(t) = \frac{1}{2} \|U(t)\|_{D(\mathcal{A})}^2 = \frac{1}{2} (\|U(t)\|_{\mathcal{H}}^2 + \|\mathcal{A} U(t)\|_{\mathcal{H}}^2).
$$

As in Proposition 3.1, this energy $\tilde{E}$ is decaying.

Combining estimates (5.5) and (5.6), we obtain

$$
E(T^*) \leq E(0) - K_2 \mathrm{e}^{-2\beta T^*} \left(\frac{\|(\omega_0, \omega_1)\|_{V \times H}^{2m+2}}{\tilde{E}(0)^m} + \xi \tau \int_0^1 \|B_2^* \dot{\omega}(-\tau \rho)\|_{U_2}^2 \, \mathrm{d}\rho \right),
$$

for some $K_2 > 0$ independent of T^* and τ, or equivalently

$$
E(T^*) \leq E(0) - K_2 \mathrm{e}^{-2\beta T^*} \left(\frac{\|(\omega_0, \omega_1)\|_{V \times H}^{2m+2}}{\tilde{E}(0)^m} + \xi \tau \|f^0(-\tau.)\|_{L^2((0,1), U_2)}^2 \right). \tag{5.7}
$$

Using the trivial estimate

$$
\begin{aligned}
(\xi \tau)^{m+1} \|f^0(-\tau.)\|_{L^2((0,1), U_2)}^{2m+2} \quad &= \quad \xi \tau \|f^0(-\tau.)\|_{L^2((0,1), U_2)}^2 (\xi \tau)^m \|f^0(-\tau.)\|_{L^2((0,1), U_2)}^{2m} \\
&\leq \quad \tau \xi \|f^0(-\tau.)\|_{L^2((0,1), U_2)}^2 \tilde{E}(0)^m
\end{aligned}
$$

the above inequality (5.7) becomes

$$
\begin{aligned}
E(T^*) &\leq E(0) - K_2 e^{-2\beta T^*} \left(\frac{\|(\omega_0, \omega_1)\|_{V \times H}^{2m+2} + (\xi\tau)^{m+1} \|f^0(-\tau.)\|_{L^2((0,1),U_2)}^{2m+2}}{\tilde{E}(0)^m} \right) \\
&\leq E(0) - K' e^{-2\beta T^*} \frac{E(0)^{m+1}}{\tilde{E}(0)^m},
\end{aligned}
$$

with $K' > 0$ independent of T^* and τ. Since the energy of our system is decaying, we obtain

$$
E(T^*) \leq E(0) - K' e^{-2\beta T^*} \frac{E(T^*)^{m+1}}{\tilde{E}(0)^m}. \tag{5.8}
$$

We now follow the method used in [2]. The estimate (5.8) being valid on the intervals $[kT^*, (k+1)T^*]$, for any $k \geq 0$, we have

$$
E((k+1)T^*) \leq E(kT^*) - K' e^{-2\beta T^*} \frac{E((k+1)T^*)^{m+1}}{\tilde{E}(kT^*)^m}. \tag{5.9}
$$

Setting

$$
\varepsilon_k = \frac{E(kT^*)}{\tilde{E}(0)},
$$

and dividing (5.9) by $\tilde{E}(0)$, we obtain

$$
\varepsilon_{k+1} \leq \varepsilon_k - K' e^{-2\beta T^*} \varepsilon_{k+1}^{m+1}, \tag{5.10}
$$

because $\tilde{E}(kT^*) \leq \tilde{E}(0)$. By Lemma 5.1 with $\mu = m - 1 > -1$ (as $m > 0$), there exists a constant $M' > 0$ (depending on m and $K' e^{-2\beta T^*}$, and verifying $M' > \left(\frac{4e^{2\beta T^*}}{mK'} \right)^{\frac{1}{m}}$) such that

$$
\varepsilon_k \leq \frac{M'}{(1+k)^{\frac{1}{m}}} \quad \forall k \geq 0,
$$

or equivalently

$$
E(kT^*) \leq \frac{M'}{(1+k)^{\frac{1}{m}}} \tilde{E}(0).
$$

This estimate and again the decay of the energy lead to estimate (5.4), where $C = M'(1+T^*)^{\frac{1}{m}}$. $\square$

Remark 5.4. Since the proof of the above theorem reveals that $C = M'(1+T^*)^{\frac{1}{m}}$ with $M' > \left(\frac{4e^{2\beta T^*}}{mK'} \right)^{\frac{1}{m}}$ and $T^* > \tau$, the constant C depends on τ and when τ becomes larger, the decay rate becomes slower.

6. Checking the observability inequalities

In this section, we show how to obtain the observability inequalities used in Theorems 4.4 and 5.3. Our method is based on the generalized gap condition. Before giving spectral conditions to obtain exponential or polynomial decay, we recall some results about Ingham's inequality.

6.1. **Preliminaries about Ingham's inequality**

Let $\{\lambda_k\}_{k\geq 1}$ be the set of eigenvalues of $A^{\frac{1}{2}}$ counted with their multiplicities (*i.e.* we repeat the eigenvalues according to their multiplicities). We further rewrite the sequence of eigenvalues $\{\lambda_k\}_{k\geq 1}$ as follows:

$$\lambda_{k_1} < \lambda_{k_2} < ... < \lambda_{k_i} < ...$$

where $k_1 = 1$, k_2 is the lowest index of the second distinct eigenvalue, k_3 is the lowest index of the third distinct eigenvalue, etc. For all $i \in \mathbb{N}^*$, let l_i be the multiplicity of the eigenvalue λ_{k_i}, *i.e.*

$$\lambda_{k_i-1} < \lambda_{k_i} = \lambda_{k_i+1} = ... = \lambda_{k_i+l_i-1} < \lambda_{k_i+l_i} = \lambda_{k_{i+1}}.$$

We have $k_1 = 1$, $k_2 = l_1$, $k_3 = l_1 + l_2$, etc. Let $\{\varphi_{k_i+j}\}_{0\leq j\leq l_i-1}$ be the orthonormal eigenvectors associated with the eigenvalue λ_{k_i}. We assume that the following generalized gap condition holds:

$$\exists M \in \mathbb{N}^*, \ \exists \gamma_0 > 0, \ \forall k \geq 1, \ \lambda_{k+M} - \lambda_k \geq M\gamma_0. \tag{6.1}$$

Fix a positive real number $\gamma_0' \leq \gamma_0$ and denote by A_k, $k = 1, ..., M$ the set of natural numbers k_m satisfying (see for instance [6])

$$\begin{cases} \lambda_{k_m} - \lambda_{k_{m-1}} \geq \gamma_0' \\ \lambda_{k_n} - \lambda_{k_{n-1}} < \gamma_0' \qquad \text{for } m+1 \leq n \leq m+k-1, \\ \lambda_{k_{m+k}} - \lambda_{k_{m+k-1}} \geq \gamma_0'. \end{cases}$$

Then one easily checks that the sets $A_k + j$, $j = 0, ..., k-1$, $k = 1, ..., M$ form a partition of $\mathbb{N}^*$. Notice that some sets A_k may be empty because, for the generalized gap condition, the choice of M takes into account multiple eigenvalues.

Now for $k_m \in A_k$, we recall that the finite differences $e_{m+j}(t)$, $j = 0, ..., k-1$, corresponding to the exponential functions $e^{-i\lambda_{k_{m+j}}t}$, $j = 0, ..., k-1$ are given by

$$e_{m+j}(t) = \sum_{p=m}^{m+j} \prod_{\substack{q=m \\ q\neq p}}^{m+j} (\lambda_{k_p} - \lambda_{k_q})^{-1} e^{i\lambda_{k_p}t}.$$

Write for shortness, $e_{-n}(t)$ the same finite differences functions corresponding to $-\lambda_{k_n}$.

Now we are ready to recall the next inequality of Ingham's type, see for instance Theorem 1.5 of [6]:

Theorem 6.1. *If the sequence $(\lambda_n)_{n\geq 1}$ satisfies (6.1), then for all sequence $(a_n)_{n\in\mathbb{Z}^*}$ (where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$), the function*

$$f(t) = \sum_{n\in\mathbb{Z}^*} a_n e_n(t),$$

satisfies the estimates

$$\int_0^T |f(t)|^2 dt \sim \sum_{n\in\mathbb{Z}^*} |a_n|^2, \tag{6.2}$$

for $T > \frac{2\pi}{\gamma_0}$.

Going back to the original functions $e^{i\lambda_{k_n}t}$, the above equivalence (6.2) means that, for $T > \frac{2\pi}{\gamma_0}$, the function (from now on $\lambda_{-k_n} = -\lambda_{k_n}$)

$$f(t) = \sum_{n\in\mathbb{Z}^*} \alpha_n e^{i\lambda_{k_n}t},$$

satisfies the estimates

$$\int_0^T |f(t)|^2 \mathrm{d}t \sim \sum_{k=1}^{M} \sum_{|k_n|\in A_k} \|B_{k_n}^{-1} C_{k_n}\|_2^2, \tag{6.3}$$

where $\|\cdot\|_2$ means the Euclidean norm of the vector, for $k_n \in A_k$ the vector C_{k_n} is given by

$$C_{k_n} = (\alpha_n, \ldots, \alpha_{n+k-1})^T,$$

and the $k \times k$ matrix B_{k_n} allows to pass from the coefficients a_{k_n} to α_{k_n}, namely

$$C_{k_n} = B_{k_n} \cdot (a_n, \ldots, a_{n+k-1})^T,$$

and is given by $B_{k_n} = (B_{k_n,\,ij})_{1\le i,\,j\le k}$ the matrix of size $k \times k$ such that

$$B_{k_n,\,ij} = \begin{cases} \displaystyle\prod_{\substack{q=n \\ q\neq n+i-1}}^{n+j-1} (\lambda_{k_{n+i-1}} - \lambda_{k_q})^{-1} & \text{if } i \le j, \ (i,j) \neq (1,1), \\[4pt] 1 & \text{if } (i,j) = (1,1), \\ 0 & \text{if } i > j. \end{cases}$$

More explicitly, we have

$$B_{k_n} = \begin{pmatrix} 1 & \frac{1}{\lambda_{k_n}-\lambda_{k_{n+1}}} & \frac{1}{(\lambda_{k_n}-\lambda_{k_{n+1}})(\lambda_{k_n}-\lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_n}-\lambda_{k_{n+1}})\cdots(\lambda_{k_n}-\lambda_{k_{n+k-1}})} \\ 0 & \frac{1}{\lambda_{k_{n+1}}-\lambda_{k_n}} & \frac{1}{(\lambda_{k_{n+1}}-\lambda_{k_n})(\lambda_{k_{n+1}}-\lambda_{k_{n+2}})} & \cdots & \frac{1}{(\lambda_{k_{n+1}}-\lambda_{k_n})\cdots(\lambda_{k_{n+1}}-\lambda_{k_{n+k-1}})} \\ 0 & 0 & \frac{1}{(\lambda_{k_{n+2}}-\lambda_{k_n})(\lambda_{k_{n+2}}-\lambda_{k_{n+1}})} & \cdots & \frac{1}{(\lambda_{k_{n+2}}-\lambda_{k_n})\cdots(\lambda_{k_{n+2}}-\lambda_{k_{n+k-1}})} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{(\lambda_{k_{n+k-1}}-\lambda_{k_n})\cdots(\lambda_{k_{n+k-1}}-\lambda_{k_{n+k-2}})} \end{pmatrix}.$$

We proceed similarly for $n \le -1$, the indices being decreasing from n to $n-k+1$.

Remark 6.2. If the standard gap condition

$$\exists \gamma_0 > 0, \ \forall n \ge 1, \ \lambda_{k_{n+1}} - \lambda_{k_n} \ge \gamma_0 \tag{6.4}$$

holds, then $A_1 = \mathbb{Z}^*$ and $B_1 = 1$ and in that case $f(t) = \sum_{n\in\mathbb{Z}^*} \alpha_n \mathrm{e}^{\mathrm{i}\lambda_{k_n} t}$ satisfies Ingham's inequality (see [13]):

$$\int_0^T |f(t)|^2 \mathrm{d}t \sim \sum_{n\in\mathbb{Z}^*} |\alpha_n|^2, \tag{6.5}$$

for $T > \frac{2\pi}{\gamma_0}$.

Now, let U be a separable Hilbert space (in the sequel, U will be U_1). For a vector $c = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$ in U^m, we set $\|.\|_{U,\,2}$ the norm in U^m defined by

$$\|c\|_{U,\,2}^2 = \sum_{l=1}^{m} \|c_l\|_U^2.$$

Then we obtain the inequality of Ingham's type in U:

Proposition 6.3. *If we have the standard gap condition* (6.4)*, then for all sequence* $(a_n)_n$ *in* U*, the function*

$$u(t) = \sum_{n \in \mathbb{Z}^*} a_n e^{i\lambda_{k_n} t}$$

satisfies the estimates

$$\int_0^T \|u(t)\|_U^2 \, dt \sim \sum_{n \in \mathbb{Z}^*} \|a_n\|_U^2,$$

for $T > \frac{2\pi}{\gamma_0}$.

Proof. As U is a separable Hilbert space, there exists a Hilbert basis $(\psi_k)_{k \geq 1}$ of U. Therefore, $a_n \in U$ can be written as

$$a_n = \sum_{k=1}^{+\infty} a_n^k \psi_k.$$

We truncate a_n as follows: for $K \in \mathbb{N}^*$, let $a_n^{(K)} = \sum_{k=1}^{K} a_n^k \psi_k$ and set $u_K(t) = \sum_{k=1}^{K} \left(\sum_{n \in \mathbb{Z}^*} a_n^k e^{i\lambda_{k_n} t} \right) \psi_k$. Since $(\psi_k)_{k \geq 1}$ is a Hilbert basis, we have by Fubini's theorem

$$\|u_K(t)\|_U^2 = \sum_{k=1}^{K} \left| \sum_{n \in \mathbb{Z}^*} a_n^k e^{i\lambda_{k_n} t} \right|^2.$$

Thus, by applying Ingham's inequality (6.5), we have

$$
\begin{aligned}
\int_0^T \|u_K(t)\|_U^2 \, dt &= \sum_{k=1}^{K} \int_0^T \left| \sum_{n \in \mathbb{Z}^*} a_n^k e^{i\lambda_{k_n} t} \right|^2 dt \\
&\sim \sum_{k=1}^{K} \sum_{n \in \mathbb{Z}^*} (a_n^k)^2 \\
&\sim \sum_{n \in \mathbb{Z}^*} \sum_{k=1}^{K} (a_n^k)^2.
\end{aligned}
$$

Therefore

$$\int_0^T \|u_K(t)\|_U^2 \, dt \sim \sum_{n \in \mathbb{Z}^*} \left\| a_n^{(K)} \right\|_U^2.$$

As $u_K \to u$ and $a_n^{(K)} \to a_n$ when $K \to +\infty$, we obtain the result. $\qquad \square$

In the same way, we obtain an Ingham's type inequality in a Hilbert space U in the case of the generalized gap condition (6.1).

Corollary 6.4. *If the sequence* $(\lambda_n)_{n \geq 1}$ *satisfies* (6.1)*, then for all sequence* $(\alpha_n)_{n \in \mathbb{Z}^*}$ *in* U*, the function*

$$f(t) = \sum_{n \in \mathbb{Z}^*} \alpha_n e^{i\lambda_{k_n} t},$$

satisfies the estimates

$$\int_0^T |f(t)|^2 dt \sim \sum_{k=1}^{M} \sum_{|k_n| \in A_k} \|B_{k_n}^{-1} C_{k_n}\|_{U,2}^2, \tag{6.6}$$

for $T > \frac{2\pi}{\gamma_0}$, *where*

$$C_{k_n} = (\alpha_n, \ldots, \alpha_{n+k-1})^T \in U^k.$$

6.2. A first observability inequality

Proposition 6.5. *Assume that the generalized gap condition* (6.1) *holds and that* U_1 *is separable. Let* ϕ *be the solution of* (4.3) *with* $(\omega_0, \omega_1) \in V \times H$. *Then there exists a time* $T > 0$ *and a constant* $C > 0$ *(depending on* T*) such that* (4.6) *holds if and only if*

$$\exists \gamma > 0, \forall k = 1, \ldots, M, \forall k_n \in A_k, \forall \xi \in \mathbb{R}^{L_n}, \left\| B_{k_n}^{-1} \Phi_{k_n} \xi \right\|_{U_1, 2} \geq \gamma \left\| \xi \right\|_2, \tag{6.7}$$

where the matrix Φ_{k_n} *with coefficients in* U_1 *and size* $k \times L_n$, *where* $L_n = \sum\limits_{i=1}^{k} l_{n+i-1} - 1$, *is given as follow: for all* $i = 1, \ldots, k$, *we set*

$$(\Phi_{k_n})_{ij} = \begin{cases} B_1^* \varphi_{k_{n+i-1}+j-L_{n,i-1}} & \text{if } L_{n,i-1} < j \leq L_{n,i}, \\ 0 & \text{else,} \end{cases}$$

where

$$\begin{aligned} L_{n,0} &= 0, \\ L_{n,i} &= \sum_{i'=1}^{i} l_{n+i'-1} - 1 \quad \forall i \geq 1. \end{aligned} \tag{6.8}$$

Proof. We first show that (6.7)$\Rightarrow$(4.6). Writing

$$\omega_0 = \sum_{i \geq 1} \sum_{j=0}^{l_i - 1} a_{k_i+j} \varphi_{k_i+j}$$

and

$$\omega_1 = \sum_{i \geq 1} \sum_{j=0}^{l_i - 1} b_{k_i+j} \varphi_{k_i+j}$$

where $(\lambda_{k_i} a_{k_i+j})_{i,j}, (b_{k_i+j})_{i,j} \in l^2(\mathbb{N}^*)$, then the solution ϕ of system without damping (4.3) is given by

$$\phi(\cdot, t) = \sum_{i \geq 1} \sum_{j=0}^{l_i - 1} \left(a_{k_i+j} \cos(\lambda_{k_i} t) + \frac{b_{k_i+j}}{\lambda_{k_i}} \sin(\lambda_{k_i} t) \right) \varphi_{k_i+j}.$$

Consequently

$$(B_1^* \phi)'(t) = \sum_{i \geq 1} \sum_{j=0}^{l_i - 1} \left(-a_{k_i+j} \lambda_{k_i} \sin(\lambda_{k_i} t) + b_{k_i+j} \cos(\lambda_{k_i} t) \right) B_1^* \varphi_{k_i+j}.$$

By grouping the terms corresponding to the same eigenvalue, we get

$$\begin{aligned} (B_1^* \phi)'(t) &= \sum_{i \geq 1} \left(\sum_{j=0}^{l_i - 1} -a_{k_i+j} B_1^* \varphi_{k_i+j} \right) \lambda_{k_i} \sin(\lambda_{k_i} t) + \sum_{i \geq 1} \left(\sum_{j=0}^{l_j - 1} b_{k_i+j} B_1^* \varphi_{k_i+j} \right) \cos(\lambda_{k_i} t) \\ &= \sum_{n \in \mathbb{Z}^*} \alpha_n e^{i \lambda_{k_n} t}, \end{aligned}$$

where

$$\alpha_n = \frac{1}{2}\left(\left(\sum_{j=0}^{l_n-1} b_{k_n+j} B_1^* \varphi_{k_n+j}\right) + \mathrm{i}\left(\sum_{j=0}^{l_n-1} a_{k_n+j} B_1^* \varphi_{k_n+j}\right)\lambda_{k_n}\right) \quad \forall n \geq 1,$$

$$\alpha_{-n} = \frac{1}{2}\left(\left(\sum_{j=0}^{l_n-1} b_{k_n+j} B_1^* \varphi_{k_n+j}\right) - \mathrm{i}\left(\sum_{j=0}^{l_n-1} a_{k_n+j} B_1^* \varphi_{k_n+j}\right)\lambda_{k_n}\right) \quad \forall n \geq 1.$$

Integrating the square of the norm of this identity between 0 and $T > 0$ and using Ingham's inequality (6.6) in U_1, for T large enough, we get

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, \mathrm{d}t \geq C \sum_{k=1}^M \sum_{|k_n|\in A_k} \left\|B_{k_n}^{-1} C_{k_n}\right\|_{U_1,2}^2,$$

where $C_{k_n} = (\alpha_n, \ldots, \alpha_{n+k-1})^T$ is a vector of U_1^k.

But for all $k_n \in A_k$, setting

$$\tilde{A}_{k_n} = \left(\lambda_{k_n} a_{k_n}, \ldots, \lambda_{k_n} a_{k_n+l_n-1}, \lambda_{k_{n+1}} a_{k_{n+1}}, \ldots, \lambda_{k_{n+1}} a_{k_{n+1}+l_{n+1}-1}, \ldots,\right.$$

$$\left. \lambda_{k_{n+k-1}} a_{k_{n+k-1}}, \ldots, \lambda_{k_{n+k-1}} a_{k_{n+k-1}+l_{n+k-1}-1}\right)^T,$$

$$\tilde{B}_{k_n} = \left(b_{k_n}, \ldots, b_{k_n+l_n-1}, b_{k_{n+1}}, \ldots, b_{k_{n+1}+l_{n+1}-1}, \ldots, b_{k_{n+k-1}}, \ldots,\right.$$

$$\left. b_{k_{n+k-1}+l_{n+k-1}-1}\right)^T,$$

we readily check that

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, \mathrm{d}t \geq C \sum_{k=1}^M \sum_{|k_n|\in A_k} \left(\left\|B_{k_n}^{-1}\Phi_{k_n}\tilde{A}_{k_n}\right\|_{U_1,2}^2 + \left\|B_{k_n}^{-1}\Phi_{k_n}\tilde{B}_{k_n}\right\|_{U_1,2}^2\right).$$

Hence the assumption (6.7) yields

$$\int_0^T \|(B_1^*\phi)'(t)\|_{U_1}^2 \, \mathrm{d}t \geq C \sum_{k=1}^M \sum_{|k_n|\in A_k} \left(\left\|\tilde{A}_{k_n}\right\|_2^2 + \left\|\tilde{B}_{k_n}\right\|_2^2\right)$$

$$= C\left(\left\|A^{\frac{1}{2}}\omega_0\right\|_H^2 + \|\omega_1\|_H^2\right)$$

because $(\varphi_{k_n+i})_{n,i}$ is an orthonormal basis associated with the operator $A^{\frac{1}{2}}$.

It remains to show that (4.6)$\Rightarrow$(6.7).

Let $k = 1, \ldots, M$ and $k_n \in A_k$ be fixed. Take $\omega_0 = \sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1} a_{k_i+j}\varphi_{k_i+j}$ and $\omega_1 = \sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1} b_{k_i+j}\varphi_{k_i+j}$. Then the solution ϕ of system (4.3) is given by

$$\phi(\cdot, t) = \sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1}\left(a_{k_i+j}\cos(\lambda_{k_i}t) + \frac{b_{k_i+j}}{\lambda_{k_i}}\sin(\lambda_{k_i}t)\right)\varphi_{k_i+j},$$

and then

$$(B_1^*\phi)'(t) = \sum_{i=n}^{n+k-1}\sum_{j=0}^{l_i-1}\left(-a_{k_i+j}\lambda_{k_i}\sin(\lambda_{k_i}t) + b_{k_i+j}\cos(\lambda_{k_i}t)\right)B_1^*\varphi_{k_i+j}.$$

Applying again Ingham's inequality, we get for T large enough and $\tilde{A}_{k_n}$, $\tilde{B}_{k_n}$ defined above

$$\int_0^T \left\| (B_1^* \phi)'(t) \right\|_{U_1}^2 \, dt \sim \left\| B_{k_n}^{-1} \Phi_{k_n} \tilde{A}_{k_n} \right\|_{U_1,2}^2 + \left\| B_{k_n}^{-1} \Phi_{k_n} \tilde{B}_{k_n} \right\|_{U_1,2}^2.$$

By (4.6), we obtain

$$\left\| B_{k_n}^{-1} \Phi_{k_n} \tilde{A}_{k_n} \right\|_{U_1,2}^2 + \left\| B_{k_n}^{-1} \Phi_{k_n} \tilde{B}_{k_n} \right\|_{U_1,2}^2 \geq C \left\| A^{\frac{1}{2}} \omega_0 \right\|_H^2 + \| \omega_1 \|_H^2$$
$$= C \sum_{i=n}^{n+k-1} \sum_{j=0}^{l_i-1} (a_{k_i+j}^2 \lambda_{k_i}^2 + b_{k_i+j}^2), \qquad (6.9)$$

for some $C > 0$. Hence we conclude that

$$\left\| B_{k_n}^{-1} \Phi_{k_n} \xi \right\|_{U_1,\,2} \geq \gamma \| \xi \|_2.$$

This ends the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.6. If the standard gap condition (6.4) holds, then $A_1 = \mathbb{N}^*$ and $B_1 = 1$. In this case, the assumption (6.7) becomes

$$\exists \gamma > 0, \, \forall k_n \geq 1, \, \forall \xi \in \mathbb{R}^{l_n}, \, \| \Phi_{k_n} \xi \|_{U_1} \geq \gamma \| \xi \|_2.$$

Moreover, if the standard gap condition (6.4) holds and if the eigenvalues are simple, the assumption (6.7) becomes

$$\exists \gamma > 0, \, \forall k \geq 1, \, \| B_1^* \varphi_k \|_{U_1} \geq \gamma.$$

Remark 6.7. The above Proposition 6.5 yields a time $T > 0$ and a constant $C > 0$ depending on T such that (4.6) holds but the time T and the constant C do not depend on the delay τ. Hence if the minimal time T is not strictly greater than τ, by choosing $T' > \max\{T, \tau\}$, we still have

$$\left\| A^{\frac{1}{2}} \omega_0 \right\|_H^2 + \| \omega_1 \|_H^2 \leq C \int_0^{T'} \left\| (B_1^* \phi)'(t) \right\|_{U_1}^2 \, dt,$$

with the same constant C as before and that does not depend on τ. This means that under the generalized gap condition (6.1), the condition (6.7) is equivalent to the sufficient condition of Theorem 4.4.

6.3. A second observability inequality

Proposition 6.8. *Assume that the generalized gap condition (6.1) holds and that U_1 is separable. Let ϕ be the solution of (4.3) with $(\omega_0, \omega_1) \in V \times H$. Then for a fixed real number $m > 0$, there exist a time $T > 0$ and a constant $C > 0$ such that (5.3) holds if and only if*

$$\exists \gamma > 0, \, \forall k = 1, \ldots, M, \, \forall k_n \in A_k, \, \forall \xi \in \mathbb{R}^{L_n}, \, \left\| B_{k_n}^{-1} \Phi_{k_n} \xi \right\|_{U_1,\,2} \geq \frac{\gamma}{\lambda_{k_n}^m} \| \xi \|_2. \qquad (6.10)$$

Proof. The proof is similar to the one of Proposition 6.5 because

$$\sum_{n \geq 1} \frac{1}{\lambda_{k_n}^{2m}} \sum_{j=0}^{l_n-1} (a_{k_n+j}^2 \lambda_{k_n}^2 + b_{k_n+j}^2) = \sum_{n \geq 1} \sum_{j=0}^{l_n-1} (a_{k_n+j}^2 \lambda_{k_n}^{2(1-m)} + b_{k_n+j}^2 \lambda_{k_n}^{-2m})$$
$$\sim \| \omega_0 \|_{D(A^{\frac{1-m}{2}})}^2 + \| \omega_1 \|_{D(A^{-\frac{m}{2}})}^2.$$

The details are therefore omitted. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.9. If the standard gap condition (6.4) holds, the assumption (6.10) becomes

$$\exists \gamma > 0,\, \forall k_n \geq 1,\, \forall \xi \in \mathbb{R}^{l_n},\, \|\Phi_{k_n}\xi\|_{U_1} \geq \frac{\gamma}{\lambda_{k_n}^m}\, \|\xi\|_2\,.$$

Moreover, if the standard gap condition (6.4) holds and if the eigenvalues are simple, the assumption (6.10) becomes

$$\exists \gamma > 0,\, \forall k \geq 1,\, \|B_1^*\varphi_k\|_{U_1} \geq \frac{\gamma}{\lambda_{k_n}^m}\,.$$

A remark similar to Remark 6.7 can be made for polynomial stability.

7. Examples

We end up this paper by considering different examples for which our abstract framework can be applied. To our knowledge, all the examples, with the exception of the first one, are new.

7.1. **A wave equation on 1-d networks with nodal feedbacks**

In this section we show that the result obtained in [17] enter in the framework of this paper. Obviously we will use the same notations for a network that in [17] that we briefly recall (for more details see [17]). We denote by $\mathcal{E} = \{e_j;\, 1 \leq j \leq N\}$ the set of edges e_j of length $l_j > 0$ of a given network $\mathcal{R}$ and $\mathcal{V}$ the set of vertices of $\mathcal{R}$. For a function $u : \mathcal{R} \to \mathbb{R}$, we set $u_j = u_{|e_j}$ the restriction of u to e_j. For a fixed vertex v, we set

$$\mathcal{E}_v = \{j \in \{1, ..., N\}\,;\, v \in \bar{e}_j\}.$$

If $\operatorname{card}(\mathcal{E}_v) \geq 2$, v is an interior node. Let $\mathcal{V}_{\text{int}}$ be the set of interior nodes. If $\operatorname{card}(\mathcal{E}_v) = 1$, v is an exterior node. Let $\mathcal{V}_{\text{ext}}$ be the set of exterior nodes. For $v \in \mathcal{V}_{\text{ext}}$, we set $\mathcal{E}_v = \{j_v\}$.

We now fix a partition of $\mathcal{V}_{\text{ext}}$:

$$\mathcal{V}_{\text{ext}} = \mathcal{D} \cup \mathcal{N} \cup \mathcal{V}_{\text{ext}}^c, \text{ where } \mathcal{D} \neq \emptyset.$$

We actually will impose Dirichlet boundary condition at the nodes of $\mathcal{D}$, Neumann boundary condition at the nodes of $\mathcal{N}$ and finally a feedback boundary condition at the nodes of $\mathcal{V}_{\text{ext}}^c$. We further fix a subset $\mathcal{V}_{\text{int}}^c$ of $\mathcal{V}_{\text{int}}$ where a feedback transmission condition will be imposed. By shortness, we denote by $\mathcal{V}_c$ the set of controlled nodes, namely

$$\mathcal{V}_c = \mathcal{V}_{\text{int}}^c \cup \mathcal{V}_{\text{ext}}^c.$$

We here consider the following initial and boundary system:

$$\begin{cases}
\frac{\partial^2 u_j}{\partial t^2}(x,\,t) - \frac{\partial^2 u_j}{\partial x^2}(x,\,t) = 0 & 0 < x < l_j,\, t > 0,\, \forall j \in \{1, ..., N\} \\[2mm]
u_j(v,\,t) = u_l(v,\,t) = u(v,\,t) & t > 0,\, \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{\text{int}} \\[2mm]
\displaystyle\sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = -\alpha_1^{(v)}\frac{\partial u}{\partial t}(v,\,t) - \alpha_2^{(v)}\frac{\partial u}{\partial t}(v,\,t-\tau) & t > 0,\, \forall v \in \mathcal{V}_c \\[2mm]
\displaystyle\sum_{j \in \mathcal{E}_v} \frac{\partial u_j}{\partial n_j}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{V}_{\text{int}} \backslash \mathcal{V}_{\text{int}}^c \\[2mm]
u_{j_v}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{D} \\[2mm]
\frac{\partial u_{j_v}}{\partial n_{j_v}}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{N} \\[2mm]
u(t=0) = u^{(0)},\, \frac{\partial u}{\partial t}(t=0) = u^{(1)} & \\[2mm]
\frac{\partial u}{\partial t}(v,\,t-\tau) = f_v^0(t-\tau) & \forall v \in \mathcal{V}_c,\, 0 < t < \tau,
\end{cases} \tag{7.1}$$

where $\alpha_i^{(v)} \geq 0$ are fixed non-negative real numbers and the delay τ is positive.

To rewrite this system in the form (1.2), we introduce

$$H = L^2(\mathcal{R}) = \{u \, : \, \mathcal{R} \to \mathbb{R}; \, u_j \in L^2(0, l_j), \, \forall j = 1, ..., N\}$$

and the operator

$$A : \begin{cases} D(A) & \to & H \\ (\varphi_j)_j & \mapsto & (-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi_j)_j \end{cases} \tag{7.2}$$

where

$$D(A) = \left\{ \varphi \in V \cap \prod_{j=1}^N H^2(0, l_j); \, \sum_{j \in \mathcal{E}_v} \frac{\partial \varphi_j}{\partial n_j}(v) = 0, \, \forall v \in \mathcal{V}_{\text{int}}; \frac{\partial \varphi_{j_v}}{\partial n_{j_v}}(v) = 0, \, \forall v \in \mathcal{N} \cup \mathcal{V}_{\text{ext}}^c \right\}, \tag{7.3}$$

and

$$V := \left\{ \varphi \in \prod_{j=1}^N H^1(0, l_j) \, : \, \varphi_j(v) = \varphi_k(v) \, \forall j, \, k \in \mathcal{E}_v, \, \forall v \in \mathcal{V}_{\text{int}} \, ; \, \varphi_{j_v}(v) = 0, \, \forall v \in \mathcal{D} \right\}.$$

The operator A is self-adjoint and positive with a compact inverse in H. Moreover

$$D(A^{\frac{1}{2}}) = V.$$

We now define $U = U_1 = U_2 = \mathbb{R}^{V_c}$, where V_c is the cardinal of $\mathcal{V}_c$, with norm $\|.\|_U = \|.\|_2$ and the operators B_i for $i = 1, 2$ as

$$B_i : \begin{cases} U & \to & D(A^{\frac{1}{2}})' \\ (k_v)_{v \in \mathcal{V}_c} & \mapsto & \sum_{v \in \mathcal{V}_c} \sqrt{\alpha_i^{(v)}} k_v \delta_v. \end{cases} \tag{7.4}$$

It is easy to verify that $B_i^*(\varphi) = (\sqrt{\alpha_i^{(v)}}\varphi(v))_{v \in \mathcal{V}_c}^T$ for $\varphi \in D(A^{\frac{1}{2}})$ and thus $B_i B_i^*(\varphi) = \sum_{v \in \mathcal{V}_c} \alpha_i^{(v)} \varphi(v) \delta_v$ for $\varphi \in D(A^{\frac{1}{2}})$. Hence the system (7.1) can be rewritten in the form (1.2).

We notice that (2.5) is here reduced to

$$\exists 0 < \alpha \le 1, \, \forall \varphi \in V, \, \sum_{v \in \mathcal{V}_c} (\alpha_2^{(v)}\varphi(v))^2 \le \alpha \sum_{v \in \mathcal{V}_c} (\alpha_1^{(v)}\varphi(v))^2,$$

and therefore, the system (7.1) is well posed for $\alpha_2^{(v)} \le \alpha_1^{(v)}$ for all $v \in \mathcal{V}_c$ by Theorem 2.1, and the energy is decreasing for $\alpha_2^{(v)} < \alpha_1^{(v)}$ for all $v \in \mathcal{V}_c$ by Proposition 3.1.

By Proposition 6.2 of [7], the generalized gap condition (6.1) holds with $M = N + 1$.

We know, by [17], that the hypothesis (4.1) is satisfied. Moreover, the hypothesis (5.2) is verified because

$$\begin{aligned} \|\omega_0\|_V^{m+1} &\le & C \, \|\omega_0\|_X^m \, \|\omega_0\|_{D(A^{\frac{1-m}{2}})} \\ &\le & C \, \|(\omega_0, \, \omega_1, \, z)\|_{D(A)}^m \, \|\omega_0\|_{D(A^{\frac{1-m}{2}})} , \end{aligned}$$

where $X = V \cap \left(\prod_{j=1}^N H^2(0, l_j) \right)$, by using Corollary 6.4 of [17].

Now we define $\Psi_{k_n}(v)$ the matrix of size $k \times L_n$ by: for all $i = 1, ..., k$, we set

$$(\Psi_{k_n}(v))_{ij} = \begin{cases} \varphi_{k_{n+i-1}+j-L_{n, i-1}}(v) & \text{if } L_{n, i-1} < j < L_{n, i}, \\ 0 & \text{else,} \end{cases}$$

where $L_{n,0} = 1$ and $L_{n,i} = \sum_{i'=1}^{i}(l_{n+i'-1} - 1)$ for $i \geq 1$. Then, for all $\xi \in \mathbb{R}^{L_n}$, we have

$$
\begin{aligned}
\left\|B_{k_n}^{-1}\Phi_{k_n}\xi\right\|_{U_1,2}^2 &= \sum_{v \in \mathcal{V}_c} \alpha_1^l \left\|B_{k_n}^{-1}\Psi_{k_n}(v)\xi\right\|_2^2 \\
&= \sum_{v \in \mathcal{V}_c} \alpha_1^l \xi^T \Psi_{k_n}(v)^T B_{k_n}^{-T} B_{k_n}^{-1} \Psi_{k_n}(v)\xi \\
&= \xi^T \left(\sum_{v \in \mathcal{V}_c} \alpha_1^l \Psi_{k_n}(v)^T B_{k_n}^{-T} B_{k_n}^{-1} \Psi_{k_n}(v)\right)\xi.
\end{aligned}
$$

Therefore by setting

$$
\tilde{\mathcal{M}}^{k_n} = \sum_{v \in \mathcal{V}_c} \alpha_1^l \Psi_{k_n}(v)^T B_{k_n}^{-T} B_{k_n}^{-1} \Psi_{k_n}(v),
$$

we see that assumption (6.7) becomes

$$
\exists \gamma > 0, \, \forall k \in \{1, ..., M\}, \, \forall k_n \in A_k, \, \lambda_{\min}(\tilde{\mathcal{M}}^{k_n}) \geq \gamma,
$$

and the assumption (6.10) becomes

$$
\exists m \in \mathbb{R}_+^*, \, \exists \gamma > 0, \, \forall k \in \{1, ..., M\}, \, \forall k_n \in A_k, \, \lambda_{\min}(\tilde{\mathcal{M}}^{k_n}) \geq \frac{\gamma}{\lambda_{k_n}^{2m}}
$$

which corresponds respectively to the conditions (6.8) and (7.4) from the paper [17], because $\lambda_k \sim \frac{k\pi}{L}$, where $L = \sum_{j=1}^{N} l_j$.

Note that if the standard gap condition (6.4) holds and if all eigenvalues are simple (*i.e.*, $l_k = 1$), then the condition (6.7) becomes

$$
\exists \gamma > 0, \, \forall k \geq 1, \, \sum_{v \in \mathcal{V}_c} \alpha_1^{(v)} |\varphi_k(v)|^2 \geq \gamma, \tag{7.5}
$$

while the conditions (6.10) becomes

$$
\exists m \in \mathbb{R}_+^*, \, \exists \gamma > 0, \, \forall k \geq 1, \, \sum_{v \in \mathcal{V}_c} \alpha_1^{(v)} |\varphi_k(v)|^2 \geq \frac{\gamma}{\lambda_k^{2m}}. \tag{7.6}
$$

Consequently, we find again all the results from [17] (see for instance the examples treated in Sect. 7 of [17]), here we can even precise the dependence of the decay rate with respect to the delay τ.

7.2. An Euler-Bernoulli beam with interior damping

We consider an Euler-Bernoulli beam of length 1 with interior damping and a delay term at ξ. Two types of boundary conditions will be considered. Without delay, these two problems were analyzed in [2,3], where some decay rates similar to the ones proved below were obtained.

7.2.1. *Mixed boundary conditions*

We consider the following initial and boundary system:

$$
\begin{cases}
\frac{\partial^2 \omega}{\partial t^2}(x,\,t) + \frac{\partial^4 \omega}{\partial x^4}(x,\,t) + \alpha_1 \frac{\partial \omega}{\partial t}(\xi,\,t)\delta_\xi + \alpha_2 \frac{\partial \omega}{\partial t}(\xi,\,t-\tau)\delta_\xi = 0 & 0 < x < 1,\, t > 0 \\[2mm]
\omega(0,\,t) = \frac{\partial \omega}{\partial x}(1,\,t) = \frac{\partial^2 \omega}{\partial x^2}(0,\,t) = \frac{\partial^3 \omega}{\partial x^3}(1,\,t) = 0 & t > 0 \\[2mm]
\omega(x,\,0) = \omega_0(x),\, \frac{\partial \omega}{\partial t}(x,\,0) = \omega_1(x) & 0 < x < 1 \\[2mm]
\frac{\partial \omega}{\partial t}(\xi,\,t-\tau) = f^0(t-\tau) & 0 < t < \tau,
\end{cases}
\tag{7.7}
$$

where $\xi \in (0,\,1)$, $\alpha_1,\, \alpha_2 > 0$ and $\tau > 0$. To enter into the framework of Section 1, we rewrite this system in the form (1.2). For that purpose, we introduce $H = L^2(0,\,1)$ and the operator

$$
A : D(A) \to H \,:\, \varphi \mapsto \frac{\mathrm{d}^4}{\mathrm{d}x^4}\varphi
\tag{7.8}
$$

where $D(A) = \{\varphi \in H^4(0,\,1)\,;\, \varphi(0) = \frac{\partial \varphi}{\partial x}(1) = \frac{\partial^2 \varphi}{\partial x^2}(0) = \frac{\partial^3 \varphi}{\partial x^3}(1) = 0\}$. The operator A is self-adjoint and positive with a compact inverse in H. We now define $U = U_1 = U_2 = \mathbb{R}$ and the operators B_1 and B_2 as

$$
B_i : U \to D(A^{\frac{1}{2}})' \,:\, k \mapsto \sqrt{\alpha_i}\, k\, \delta_\xi,\, i = 1, 2.
\tag{7.9}
$$

It is easy to verify that $B_i^*(\varphi) = \sqrt{\alpha_i}\varphi(\xi)$ for $\varphi \in D(A^{\frac{1}{2}})$ and thus $B_i B_i^*(\varphi) = \alpha_i \varphi(\xi)\delta_\xi$ for $\varphi \in D(A^{\frac{1}{2}})$ and $i = 1, 2$. Then the system (7.7) can be rewritten in the form (1.2). We notice that (2.5) is equivalent to

$$
\exists\, 0 < \alpha \leq 1,\, \alpha_2 \leq \alpha\alpha_1,
$$

and consequently, this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.1.

Let us now state the next well-known results about the spectral properties of A.

Proposition 7.1. *The eigenvalues of the operator A defined in (7.8) are simple and are given by $\lambda_k^2 = (\frac{2k+1}{2}\pi)^4$ of associated eigenvector $\varphi_k(x) = \sqrt{2}\sin(\frac{2k+1}{2}\pi x)$, for all $k \in \mathbb{N}$. Consequently the standard gap condition (6.4) holds, i.e., there exists a constant $\gamma_0 > 0$ such that*

$$
\lambda_{k+1} - \lambda_k \geq \gamma_0 > 0 \quad \forall k \geq 0,
$$

and moreover for all $k \geq 0$, $\|B_1^*\varphi_k\|_{U_1} = \sqrt{2\alpha_1}\left|\sin((k\pi + \frac{\pi}{2})\xi)\right|$.

The hypothesis (4.1) was verified in [2]. Moreover, we have by Lemma 2.9 of [19]:

Lemma 7.2. *ξ is a rational number with an irreducible fraction*

$$
\xi = \frac{p}{q}, \text{ where } p \text{ is odd}
$$

if and only if there exists a constant $\gamma > 0$ such that

$$
\forall k \geq 1,\, \left|\sin\left((k\pi + \frac{\pi}{2})\xi\right)\right| > \gamma.
$$

Therefore, by applying Proposition 3.4 and Theorem 4.4, we obtain the following results:

Proposition 7.3. *Assume that $\alpha_2 < \alpha_1$. Then*

(i) The energy of system (7.7) decays to 0 if and only if

$$\xi \neq \frac{2m}{2k+1}, \; m, k \in \mathbb{N}.$$

(ii) The energy of system (7.7) decays exponentially if ξ is a rational number with an irreductible fraction

$$\xi = \frac{p}{q}, \; where \; p \; is \; odd.$$

Remark 7.4. As mentioned before, in the case $\alpha_2 = 0$ we recover the results from [2].

7.2.2. *Other boundary conditions*

We here consider the following initial and boundary system:

$$\begin{cases} \frac{\partial^2 \omega}{\partial t^2}(x, t) + \frac{\partial^4 \omega}{\partial x^4}(x, t) + \alpha_1 \frac{\partial \omega}{\partial t}(\xi, t)\delta_\xi + \alpha_2 \frac{\partial \omega}{\partial t}(\xi, t-\tau)\delta_\xi = 0 & 0 < x < 1, t > 0 \\ \omega(0, t) = \omega(1, t) = \frac{\partial^2 \omega}{\partial x^2}(0, t) = \frac{\partial^2 \omega}{\partial x^2}(1, t) = 0 & t > 0 \\ \omega(x, 0) = \omega^0(x), \; \frac{\partial \omega}{\partial t}(x, 0) = \omega^1(x) & 0 < x < 1 \\ \frac{\partial \omega}{\partial t}(\xi, t-\tau) = f^0(t-\tau) & 0 < t < \tau, \end{cases} \tag{7.10}$$

where $\xi \in (0, 1)$, $\alpha_1, \alpha_2 > 0$ and $\tau > 0$. This system (7.10) is not exponentially stable if $\alpha_2 = 0$ as shown in [3]. Hence we only consider the polynomial decay of system (7.10). As before we rewrite this system in the form (1.2) by introducing $H = L^2(0, 1)$ and the operator

$$A : D(A) \to H \; : \; \varphi \mapsto \frac{\mathrm{d}^4}{\mathrm{d}x^4}\varphi \tag{7.11}$$

with $D(A) = \{\varphi \in H^4(0, 1) \cap V \; ; \; \frac{\partial^2 \varphi}{\partial x^2}(0) = \frac{\partial^2 \varphi}{\partial x^2}(1) = 0\}$, $V = H^2(0, 1) \cap H^1_0(0, 1)$. The operator A is self-adjoint and positive with a compact inverse in H. We then define $U = U_1 = U_2 = \mathbb{R}$ and the operators B_1, B_2 by (7.9).

Then the system (7.10) can be rewritten in the form (1.2) and consequently, this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.1.

The spectral properties of A are well-known and can be summarized as follows:

Proposition 7.5. *The eigenvalues of the operator A defined in (7.11) are simple and given by $\lambda_k^2 = k^4\pi^4$ of eigenvector $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, for all $k \in \mathbb{N}^*$. Therefore there exists a constant $\gamma_0 > 0$ such that the standard gap condition (6.4) is verified and moreover*

$$\|B_1^*\varphi_k\|_{U_1} = \sqrt{2\alpha_1} \, |\sin(k\pi\xi)| \, .$$

Hypothesis (4.1) was verified in [3]. Let us prove that the condition (5.2) is satisfied.

Lemma 7.6. *Let $m \in \mathbb{R}_+^*$. Then there exists $C > 0$ such that for all $\omega_0 \in X = \{u \in V : u \big|_{(0, \xi)} \in H^4(0, \xi),$ $u \big|_{(\xi, 1)} \in H^4(\xi, 1), \frac{\partial^2 u}{\partial x^2}(0) = \frac{\partial^2 u}{\partial x^2}(1) = 0\}$, we have*

$$\|\omega_0\|_V^{m+1} \leq C \, \|\omega_0\|_X^m \, \|\omega_0\|_{D(A^{\frac{1-m}{2}})} \, ,$$

where the natural norm in X is given by $\|u\|_X^2 = \|u\|_{H^4(0,\xi)}^2 + \|u\|_{H^4(\xi,1)}^2$.

Proof. Let us fix a cut-off function $\eta \in \mathcal{D}(0, 1)$ such that $\eta = 1$ in a neighbourhood of ξ, $\eta = 0$ on $\left[\frac{2}{3} + \frac{\xi}{3}, 1\right]$ and $\eta = 0$ on $\left[0, \frac{\xi}{3}\right]$. Since $(1 - \eta)\omega_0 \in D(A)$, by Lemma 5.2, we have

$$\|(1-\eta)\omega_0\|_V^{m+1} \leq C \, \|(1-\eta)\omega_0\|_X^m \, \|(1-\eta)\omega_0\|_{D(A^{\frac{1-m}{2}})} .$$

Since

$$
\begin{aligned}
\|(1-\eta)\omega_0\|_{D(A^{\frac{1-m}{2}})} &= \sup_{\varphi \in D(A^{\frac{m-1}{2}})} \frac{((1-\eta)\omega_0, \varphi)}{\|\varphi\|_{D(A^{\frac{m-1}{2}})}} \\
&= \sup_{\varphi \in D(A^{\frac{m-1}{2}})} \frac{(\omega_0, (1-\eta)\varphi)}{\|\varphi\|_{D(A^{\frac{m-1}{2}})}} \\
&\leq C \, \|\omega_0\|_{D(A^{\frac{1-m}{2}})} ,
\end{aligned}
$$

for some $C > 0$ (depending on η) we get

$$\|(1-\eta)\omega_0\|_V^{m+1} \leq C \, \|\omega_0\|_X^m \, \|\omega_0\|_{D(A^{\frac{1-m}{2}})} . \tag{7.12}$$

In a second step, we set

$$
\begin{aligned}
\omega_1(x) &= \eta(\xi - x)\omega_0(\xi - x) && \text{if } 0 < x < l_1 := \xi \\
\omega_2(x) &= \eta(x + \xi)\omega_0(x + \xi) && \text{if } 0 < x < l_2 := 1 - \xi.
\end{aligned}
$$

For any $j = 1, 2$, we introduce the following extension of ω_j:

$$
\begin{aligned}
(E\omega)_j(x) &= \omega_j(x) && \text{if } x \in (0, l_j), \\
(E\omega)_{-j}(x) &= \sum_{i=0}^{n-1} \nu_i \omega_j(-2^i x) && \text{if } x \in (-2^{-(n-1)} l_j, 0),
\end{aligned}
$$

where ω_j is extended by zero outside its support and the real numbers ν_i are the unique solution of the system

$$
\begin{cases}
\sum_{i=0}^{n-1} \nu_i = 1 \\
-\sum_{i=0}^{n-1} 2^i \nu_i = 1 \\
\sum_{i=0}^{n-1} 2^{2i} \nu_i = 1 \\
-\sum_{i=0}^{n-1} 2^{3i} \nu_i = 1 \\
\sum_{i=0}^{n-1} 2^{-2ki} \nu_i = 1, \ \forall k = 1, \ldots, n - 4,
\end{cases}
$$

and finally $n \in \mathbb{N}^*$ is chosen large enough such that $n \geq m + 3$.

We obtain an extension of ω to a function $E\omega$, which belongs to $\mathcal{D}(\tilde{A})$ (due to the four first properties of the ν_i), where $\tilde{A}$ is the positive operator $\frac{d^4}{dx^4}$ on the star shaped network $\tilde{\mathcal{S}} = \cup_{j=1, 2}(0, l_j) \bigcup \cup_{j=1, 2}(-2^{-(n-1)} l_j, 0)$, with interior vertex ξ (identified to 0) and Dirichlet boundary conditions at all other vertices.

Therefore, we can apply the interpolation lemma 5.2 to $\tilde{A}$ and then write

$$\|E\omega\|_{D(\tilde{A}^{\frac{1}{2}})}^{m+1} \leq C \, \|E\omega\|_{D(\tilde{A})}^m \, \|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})} .$$

But we easily check that

$$\|\eta\omega_0\|_{D(A^{\frac{1}{2}})}^{m+1} \leq \|E\omega\|_{D(\tilde{A}^{\frac{1}{2}})}^{m+1} .$$

Consequently, we have

$$\|\eta\omega_0\|_{D(A^{\frac{1}{2}})}^{m+1} \leq C \, \|E\omega\|_{D(\tilde{A})}^m \, \|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})} .$$

Moreover, as E is an extension operator, we have

$$\|E\omega\|_{D(\tilde{A})} \leq K \|\omega_0\|_X,$$

and thus

$$\|\eta\omega_0\|^{m+1}_{D(A^{\frac{1}{2}})} \leq C \|\omega_0\|^m_X \|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})}.$$

To estimate the last factor, we use a duality argument. We write

$$\|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})} = \sup_{\varphi \in D(\tilde{A}^{\frac{m-1}{2}})} \frac{|(E\omega,\, \varphi)|}{\|\varphi\|_{D(\tilde{A}^{\frac{m-1}{2}})}}.$$

For $\varphi \in D(\tilde{A}^{\frac{m-1}{2}})$, we have

$$\int_{\tilde{\mathcal{S}}} E\omega\varphi = \sum_{j=1,\,2} \int_0^{l_j} \omega_j(x)\varphi_j(x)\,\mathrm{d}x + \sum_{j=1,\,2} \int_{-2^{-(n-1)}l_j}^0 (E\omega)_{-j}(x)\varphi_{-j}(x)\,\mathrm{d}x.$$

By changes of variables, we obtain

$$\int_{\tilde{\mathcal{S}}} E\omega\varphi = \sum_{j=1,\,2} \int_0^{l_j} \omega_j(x)(F\varphi)_j(x)\,\mathrm{d}x,$$

where

$$(F\varphi)_j(x) = \varphi_j(x) + \chi_j(x) \sum_{i=0}^{n-1} \nu_i 2^{-i}\varphi_j(-2^{-i}x) \quad \forall x \in (0, l_j),$$

the cut-off function χ_j being fixed such that $\chi_j \equiv 1$ on $[0, 2l_j/3]$ and $\chi_j \equiv 0$ on $[5l_j/6, l_j]$ (reminding that $\omega_j(x) \equiv 0$ for $x > 2l_j/3$). Now we notice that the conditions on ν_i guarantees that $F\varphi$ belongs to $D(A^{\frac{m-1}{2}})$ and by Leibniz's rule we have

$$\|F\varphi\|_{D(A^{\frac{m-1}{2}})} \leq C \|\varphi\|_{D(\tilde{A}^{\frac{m-1}{2}})}.$$

Therefore

$$\int_{\tilde{\mathcal{S}}} E\omega\varphi \leq C \|\omega\|_{D(A^{\frac{1-m}{2}})} \|\varphi\|_{D(\tilde{A}^{\frac{m-1}{2}})}.$$

By duality, we conclude that

$$\|E\omega\|_{D(\tilde{A}^{\frac{1-m}{2}})} \leq C \|\omega\|_{D(A^{\frac{1-m}{2}})}.$$

Consequently, with the previous inequalities, we obtain

$$\|\eta\omega_0\|^{m+1}_{D(A^{\frac{1}{2}})} \leq C \|\omega\|^m_X \|\omega\|_{D(A^{\frac{1-m}{2}})}. \tag{7.13}$$

The conclusion follows from (7.12) and (7.13). $\square$

This lemma leads to (5.2). Indeed for $(\omega_0, \omega_1, z) \in D(\mathcal{A})$, we have

$$\begin{aligned}
\|\omega_0\|^{m+1}_V &\leq C \|\omega_0\|^m_X \|\omega_0\|_{D(A^{\frac{1-m}{2}})} \\
&\leq C \|(\omega_0, \omega_1, z)\|^m_{D(\mathcal{A})} \|\omega_0\|_{D(A^{\frac{1-m}{2}})}.
\end{aligned}$$

Now, we denote by $\mathcal{S}$ the set of all real numbers ρ such that $\rho \notin \mathbb{Q}$ and if $[0, a_1, ..., a_n, ...]$ is the expansion of ρ as a continued fraction, then the sequence (a_n) is bounded. It is well-known that $\mathcal{S}$ is uncountable and that its Lebesgue measure is zero. Roughly speaking, the set $\mathcal{S}$ contains all irrational numbers which are badly

approximated by rational numbers. In particular, by the Euler-Lagrange theorem, $\mathcal{S}$ contains all irrational quadratic numbers (*i.e.* the roots of a second order equation with rational coefficients). By a classical result, we have:

Lemma 7.7. *If $s \in \mathcal{S}$, then there exists a positive constant γ such that*

$$|\sin(k\pi s)| \geq \frac{\gamma}{k} \quad \forall k \geq 1.$$

Therefore, by applying Proposition 3.4 and Theorem 5.3 with $m = \frac{1}{2}$, we obtain the next results:

Proposition 7.8. *Assume that $\alpha_2 < \alpha_1$. Then*
  (*i*) *The energy of system* (7.10) *decays to 0 if and only if ξ is irrational.*
  (*ii*) *The energy of system* (7.10) *decays polynomially like $\frac{1}{(1+t)^2}$ if ξ belongs to $\mathcal{S}$.*

Remark 7.9. Again in the case $\alpha_2 = 0$ we recover the results from [2,3].

7.3. **Examples with distributed damping terms**

7.3.1. *A non homogeneous string with distributed damping terms (1-d)*

We consider the following initial and boundary system:

$$\begin{cases} \frac{\partial^2 \omega}{\partial t^2}(x,\,t) - \frac{\partial^2 \omega}{\partial x^2}(x,\,t) + \alpha_1 \frac{\partial \omega}{\partial t}(x,\,t)\chi_{|I_1} + \alpha_2 \frac{\partial \omega}{\partial t}(x,\,t-\tau)\chi_{|I_2} = 0 & \text{in } (0,\,1) \times (0,\,\infty) \\ \omega(0,\,t) = \omega(1,\,t) = 0 & t > 0 \\ \omega(x,\,0) = \omega_0(x),\, \frac{\partial \omega}{\partial t}(x,\,0) = \omega_1(x) & \text{in } (0,\,1) \\ \frac{\partial \omega}{\partial t}(x,\,t-\tau) = f^0(x,\,t-\tau) & \text{in } I_2 \times (0,\,\tau), \end{cases} \quad (7.14)$$

where here and below $\chi_{|I}$ denotes the characteristic function of the set I. In the remainder of this subsection we assume that $\alpha_1,\, \alpha_2 > 0$, $\tau > 0$ and

$$I_2 \subset I_1 \subset [0,\,1].$$

Later we will need that

$$\exists \delta \in [0,\,1] \text{ and } \epsilon > 0 \text{: } [\delta,\,\delta + \epsilon] \subset I_1. \quad (7.15)$$

We rewrite this system in the form (1.2). For that purpose, we introduce $H = L^2(0,\,1)$ and the operator

$$A : D(A) \to H \;:\; \varphi \mapsto -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\varphi \quad (7.16)$$

where $D(A) = H_0^1(0,\,1) \cap H^2(0,\,1)$ and $V = D(A^{\frac{1}{2}}) = H_0^1(0,\,1)$. The operator A is self-adjoint and positive with a compact inverse in H. We then define $U_i = L^2(I_i)$ and the operators B_i as

$$B_i : U_i \to H \subset V' \;:\; k \mapsto \sqrt{\alpha_i}\,\tilde{k}\,\chi_{|I_i} \quad (7.17)$$

where $\tilde{k}$ is the extension of k by zero outside I_i (which defines an element of $L^2(0,\,1)$).

It is easy to verify that $B_i^*(\varphi) = \sqrt{\alpha_i}\varphi_{|I_i}$ for $\varphi \in V$ and thus $B_i B_i^*(\varphi) = \alpha_i \varphi_{|I_i}\chi_{|I_i} = \alpha_i \varphi \chi_{|I_i}$ for $\varphi \in V$ and $i = 1,\, 2$. Then the system (7.14) can be rewritten in the form (1.2). Moreover

$$\|B_i^* \varphi\|_{U_i}^2 = \alpha_i \int_{I_i} |\varphi|^2 \,\mathrm{d}x.$$

Therefore, we notice that (2.5) is equivalent to

$$\exists 0 < \alpha \leq 1, \ \alpha_2 \int_{I_2} |\varphi|^2 \, dx \leq \alpha \alpha_1 \int_{I_1} |\varphi|^2 \, dx,$$

and consequently, this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.1.

Proposition 7.10. *The eigenvalues of the operator A defined in (7.16) are simple and given by $\lambda_k^2 = (k\pi)^2$ of associated eigenvector $\varphi_k(x) = \sqrt{2} \sin(k\pi x)$, for all $k \in \mathbb{N}^*$. Hence there exists a constant $\gamma_0 > 0$ such that the standard gap condition (6.4) holds. Moreover if (7.15) holds, then there exists $\gamma > 0$ such that $\forall k \geq 1, \|B_1^* \varphi_k\|_{U_1} \geq \gamma$.*

Proof. It is well-known that the eigenvectors of the operator A are $\varphi_k(x) = \sqrt{2} \sin(k\pi x)$ of eigenvalue $(k\pi)^2$, $k \geq 1$ of multiplicity 1. Hence, the standard gap condition (6.4) is verified.

From the definition of B_1^* we have

$$
\begin{aligned}
\|B_1^* \varphi_k\|_{U_1}^2 &= \alpha_1 \int_{I_1} |\varphi_k(x)|^2 \, dx \\
&\geq 2\alpha_1 \int_\delta^{\delta+\epsilon} |\sin(k\pi x)|^2 \, dx = \alpha_1 \left[x - \frac{\sin(2k\pi x)}{2k\pi} \right]_\delta^{\delta+\epsilon} \\
&\geq \alpha_1 \left(\epsilon - \frac{1}{k\pi} \right) \geq \alpha_1 \frac{\epsilon}{2},
\end{aligned}
$$

for $k \geq \mathbf{E}(\frac{2}{\epsilon\pi}) + 1 =: k_\epsilon$, where $\mathbf{E}(x)$ is the entire part of x. This leads to the conclusion because for $k \in \{1, ..., k_\epsilon\}$, we have

$$\int_\delta^{\delta+\epsilon} |\sin(k\pi x)|^2 \, dx > 0. \qquad \square$$

Lemma 7.11. *The operators A and $B = (B_1 B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$ satisfy assumption (4.1).*

Proof. Let $i \in \{1, 2\}$ and $\varphi \in L^2(I_i)$. It can be easily checked that $v = (\lambda^2 + A)^{-1} B_i \varphi$ satisfies

$$
\begin{cases}
\lambda^2 v - \frac{d^2 v}{dx^2} = \sqrt{\alpha_i} \tilde{\varphi} \chi_{|I_i} \\
v(0) = v(1) = 0.
\end{cases} \tag{7.18}
$$

As $v \in L^2(0, 1)$, v can be written as

$$v = \sum_{k=1}^\infty c_k \varphi_k.$$

By replacing v in (7.18), c_k must satisfy

$$c_k = \frac{\sqrt{\alpha_i}}{\lambda^2 + \lambda_k^2} \left(\tilde{\varphi}, \varphi_k \right),$$

and therefore

$$v = \sum_{k=1}^\infty \frac{\sqrt{\alpha_i}}{\lambda^2 + \lambda_k^2} \left(\tilde{\varphi}, \varphi_k \right) \varphi_k.$$

Moreover

$$\|\lambda v\|_{L^2(0,1)}^2 = \sum_{k=1}^\infty \left| \frac{\lambda}{\lambda^2 + \lambda_k^2} \right|^2 \alpha_i |(\tilde{\varphi}, \varphi_k)|^2.$$

Now, we set $z = \frac{\lambda}{\lambda^2 + \lambda_k^2}$ and, if $\lambda = \beta + iy$, with $y \in \mathbb{R}$, then

$$|z|^2 = \frac{\beta^2 + y^2}{(\beta^2 - y^2 + \lambda_k^2)^2 + 4\beta^2 y^2} \leq C(\beta),$$

where $C(\beta)$ is a positive constant depending only on β. Indeed, if $y^2 \leq \frac{\beta^2 + \lambda_k^2}{2}$, then

$$|z|^2 \leq \frac{\beta^2 + y^2}{\left(\frac{\beta^2 + \lambda_k^2}{2}\right)^2} \leq \frac{\beta^2 + \frac{\beta^2 + \lambda_k^2}{2}}{\left(\frac{\beta^2 + \lambda_k^2}{2}\right)^2}$$

which is bounded uniformly in k, and if $y^2 \geq \frac{\beta^2 + \lambda_k^2}{2}$, then

$$|z|^2 \leq \frac{\beta^2 + y^2}{4\beta^2 y^2},$$

which is a decreasing function with respect to y and thus

$$|z|^2 \leq \frac{\beta^2 + \frac{\beta^2 + \lambda_k^2}{2}}{4\beta^2 \left(\frac{\beta^2 + \lambda_k^2}{2}\right)},$$

which is again uniformly bounded in k. Therefore, we have

$$\|\lambda v\|_{L^2(0,\,1)}^2 \leq C(\beta)\alpha_i \sum_{k=1}^{\infty} |(\tilde{\varphi},\, \varphi_k)|^2 \leq C(\beta)\alpha_i \|\varphi\|_{L^2(I_i)}^2,$$

which leads to

$$\left\|\lambda B_j^* v\right\|_{L^2(I_j)}^2 \leq \alpha_j \|\lambda v\|_{L^2(0,\,1)}^2 \leq C(\beta)\alpha_i\alpha_j \|\varphi\|_{L^2(I_i)}^2, \quad \forall j \in \{1,\, 2\}.$$

Consequently, the operator $\lambda \rightarrow \lambda B_j^*(\lambda I + A)^{-1}B_i$ is bounded on C_β and the lemma is proved. $\qquad \square$

Therefore, by applying Theorem 4.4, we obtain:

Proposition 7.12. *If $\alpha_2 < \alpha_1$ and (7.15) holds, then the energy of system (7.14) decays exponentially.*

7.3.2. *The wave equation with distributed damping terms*

Let Ω be an open bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a boundary Γ of class C^2. We assume that Γ is divided into two parts Γ_D and Γ_N, *i.e.* $\Gamma = \Gamma_D \cup \Gamma_N$, with $\bar{\Gamma}_D \cap \bar{\Gamma}_N = \emptyset$ and $\Gamma_D \neq \emptyset$. Let

$$O_2 \subset O_1 \subset \Omega$$

such that O_1 is an open neighborhood of Γ_N (*i.e.* $\Gamma_N \subset \partial O_1$). Moreover, we assume that $x_0 \in \mathbb{R}^n$ is such that

$$(x - x_0) \cdot \nu(x) \leq 0 \quad \forall x \in \Gamma_D. \tag{7.19}$$

We consider the following initial and boundary system:

$$\begin{cases} \frac{\partial^2 \omega}{\partial t^2}(x,\, t) - \Delta\omega(x,\, t) + \alpha_1 \frac{\partial \omega}{\partial t}(x,\, t)\chi_{|O_1} + \alpha_2 \frac{\partial \omega}{\partial t}(x,\, t-\tau)\chi_{|O_2} = 0 & \text{in } \Omega \times (0,\, \infty), \\ \qquad\qquad\qquad\qquad \omega(x,\, t) = 0 & \text{on } \Gamma_D \times (0,\, \infty), \\ \qquad\qquad\qquad\qquad \frac{\partial \omega}{\partial \nu}(x,\, t) = 0 & \text{on } \Gamma_N \times (0,\, \infty), \\ \qquad \omega(x,\, 0) = \omega_0(x),\ \frac{\partial \omega}{\partial t}(x,\, 0) = \omega_1(x) & \text{in } \Omega, \\ \qquad\qquad \frac{\partial \omega}{\partial t}(x,\, t-\tau) = f^0(x,\, t-\tau) & \text{in } O_2 \times (0,\, \tau), \end{cases} \tag{7.20}$$

where $\frac{\partial \omega}{\partial \nu}$ is the normal derivative of ω and α_1, $\alpha_2 > 0$, $\tau > 0$. In order to reformulate this system in the form (1.2), we introduce $H = L^2(\Omega)$ and the operator

$$A : D(A) \to H \ : \ \varphi \mapsto -\Delta \varphi \tag{7.21}$$

where $D(A) = \{\varphi \in V \cap H^2(\Omega): \frac{\partial \varphi}{\partial \nu}(x) = 0 \text{ on } \Gamma_N\}$ and $V = D(A^{\frac{1}{2}}) = \{\varphi \in H^1(\Omega): \varphi = 0 \text{ on } \Gamma_D\}$. The operator A is self-adjoint and positive with a compact inverse in H. We then define $U_i = L^2(O_i)$ and the operators B_i as

$$B_i : U_i \to H \subset V' \ : \ k \mapsto \sqrt{\alpha_i}\, \tilde{k}\, \chi_{|O_i} \tag{7.22}$$

where $\tilde{k}$ is the extension of k by zero outside O_i (which defines an element of $L^2(\Omega)$).

It is easy to verify that $B_i^*(\varphi) = \sqrt{\alpha_i}\varphi_{|O_i}$ for $\varphi \in V$ and thus $B_i B_i^*(\varphi) = \alpha_i \varphi_{|O_i}\chi_{|O_i}$ for $\varphi \in V$ and $i = 1$, 2. Then the system (7.20) can be rewritten in the form (1.2). Moreover

$$\|B_i^*\varphi\|_{U_i}^2 = \alpha_i \int_{O_i} |\varphi|^2 \,\mathrm{d}x,$$

and therefore (2.5) is equivalent to

$$\exists 0 < \alpha \leq 1, \ \alpha_2 \int_{O_2} |\varphi|^2 \,\mathrm{d}x \leq \alpha\alpha_1 \int_{O_1} |\varphi|^2 \,\mathrm{d}x.$$

Consequently, this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.1.

To obtain the exponential decay of system (7.20), we simply check the observability inequality (4.6) and hypothesis (4.1), which are the aim of the two following propositions.

Proposition 7.13. *There exists a time T_0 such that for all times $T > T_0$ there exists a positive constant C (depending on T) for which the observability inequality (4.6) holds for any regular solution of system (4.3).*

Proof. This proposition is proved in [14,16]. □

Proposition 7.14. *The operators A and $B = (B_1 B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$ satisfy assumption (4.1).*

Proof. Let $i \in \{1, 2\}$ and $\varphi \in L^2(O_i)$. It can be easily checked that $v = (\lambda^2 + A)^{-1}B_i\varphi$ satisfies

$$\begin{cases} \lambda^2 v - \Delta v = \sqrt{\alpha_i}\tilde{\varphi}\chi_{|O_i} & \text{in } \Omega, \\ \qquad\quad v = 0 & \text{on } \Gamma_D, \\ \qquad\ \ \frac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_N. \end{cases} \tag{7.23}$$

As $v \in L^2(\Omega)$, it can be written as

$$v = \sum_{k=1}^{\infty} c_k \varphi_k.$$

By replacing v in (7.23), c_k must satisfy

$$c_k = \frac{\sqrt{\alpha_i}}{\lambda^2 + \lambda_k^2}\, (\tilde{\varphi}, \, \varphi_k)\,,$$

and therefore

$$v = \sum_{k=1}^{\infty} \frac{\sqrt{\alpha_i}}{\lambda^2 + \lambda_k^2}\, (\tilde{\varphi}, \, \varphi_k)\, \varphi_k.$$

Moreover, by Parseval's identity, we have

$$\|\lambda v\|^2_{L^2(\Omega)} = \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda^2 + \lambda_k^2} \right|^2 \alpha_i \left| (\tilde{\varphi}, \, \varphi_k) \right|^2.$$

Now, for $\lambda = \beta + \mathrm{i}y$, with $y \in \mathbb{R}$, we have checked in the previous subsection that

$$\left| \frac{\lambda}{\lambda^2 + \lambda_k^2} \right|^2 \leq C(\beta),$$

where $C(\beta)$ is a positive constant depending only on β. Therefore, we have

$$\|\lambda v\|^2_{L^2(\Omega)} \leq C(\beta)\alpha_i \sum_{k=1}^{\infty} |(\tilde{\varphi}, \, \varphi_k)|^2 \leq C(\beta)\alpha_i \|\varphi\|^2_{L^2(O_i)},$$

which leads to

$$\left\| \lambda B_j^* v \right\|^2_{L^2(O_j)} \leq \alpha_j \|\lambda v\|^2_{L^2(\Omega)} \leq C(\beta)\alpha_i \alpha_j \|\varphi\|^2_{L^2(O_i)}, \quad \forall j \in \{1, \, 2\}.$$

Consequently, the operator $\lambda \to \lambda B_j^*(\lambda I + A)^{-1}B_i$ is bounded on C_β and the lemma is proved. □

Therefore, by applying Theorem 4.4, we obtain:

Proposition 7.15. *If $\alpha_2 < \alpha_1$, then the energy of system* (7.20) *decays exponentially.*

Remark 7.16. This result is a generalization of [16] because in [16] the authors supposed that $O_2 = O_1$.

7.3.3. *A beam with distributed damping terms*

We consider the following initial and boundary system:

$$\begin{cases} \frac{\partial^2 \omega}{\partial t^2}(x, \, t) + \frac{\partial^4 \omega}{\partial x^4}(x, \, t) + \alpha_1 \frac{\partial \omega}{\partial t}(x, \, t)\chi_{|I_1} + \alpha_2 \frac{\partial \omega}{\partial t}(x, \, t - \tau)\chi_{|I_2} = 0 & \text{in } (0, \, 1) \times (0, \, \infty), \\ \omega(0, \, t) = \omega(1, \, t) = \frac{\partial^2 \omega}{\partial x^2}(0, \, t) = \frac{\partial^2 \omega}{\partial x^2}(1, \, t) = 0 & t > 0, \\ \omega(x, \, 0) = \omega_0(x), \, \frac{\partial \omega}{\partial t}(x, \, 0) = \omega_1(x) & \text{in } (0, \, 1), \\ \frac{\partial \omega}{\partial t}(x, \, t - \tau) = f^0(x, \, t - \tau) & \text{in } I_2 \times (0, \, \tau), \end{cases} \quad (7.24)$$

where $\alpha_1, \, \alpha_2 > 0$, $\tau > 0$ and where

$$I_2 \subset I_1 \subset [0, \, 1].$$

We rewrite this system in the form (1.2) by introducing $H = L^2(0, \, 1)$, the operator A by (7.11) and the operators B_1 and B_2 by (7.17). Hence this system is well posed for $\alpha_2 \leq \alpha_1$ by Theorem 2.1, and the energy is decreasing for $\alpha_2 < \alpha_1$ by Proposition 3.1.

By the results of the previous subsections, we know that the standard gap condition holds and if (7.15) holds that the eigenvector $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$ of A associated with $\lambda_k^2 = k^4\pi^4$ satisfies

$$\|B_1^* \varphi_k\|_{U_1} \geq \gamma,$$

for some $\gamma > 0$.

Lemma 7.17. *The operators A and $B = (B_1 B_2) \in \mathcal{L}(U, \, V')$ where $U = U_1 \times U_2$ satisfy assumption* (4.1).

Proof. The proof is the same than in the previous subsection. □

Therefore, by applying Proposition 3.4 and Theorem 4.4, we obtain the

Proposition 7.18. *If $\alpha_2 < \alpha_1$ and* (7.15) *holds, then the energy of system* (7.24) *decays exponentially.*

7.3.4. *A wave equation on 1-d networks with internal damping terms*

In this last subsection we consider again a wave equation on a given network $\mathcal{R}$ but we suppose that the feedbacks are located in the edges. Namely with the notations from Section 7.1 we consider the system:

$$
\begin{cases}
\frac{\partial^2 u_j}{\partial t^2}(x,\,t) - \frac{\partial^2 u_j}{\partial x^2}(x,\,t) + \alpha_1^{(j)}\frac{\partial u_j}{\partial t}(x,\,t)\chi_{|I_1^{(j)}} \\
\quad + \alpha_2^{(j)}\frac{\partial u_j}{\partial t}(x,\,t-\tau)\chi_{|I_2^{(j)}} = 0 & 0 < x < l_j,\, t > 0,\, \forall j \in \{1,\ldots,N\}, \\
u_j(v,\,t) = u_l(v,\,t) = u(v,\,t) & t > 0,\, \forall j,\, l \in \mathcal{E}_v,\, v \in \mathcal{V}_{\text{int}}, \\
\sum_{j\in\mathcal{E}_v}\frac{\partial u_j}{\partial n_j}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{V}\backslash\mathcal{D}, & (7.25) \\
u_{j_v}(v,\,t) = 0 & t > 0,\, \forall v \in \mathcal{D}, \\
u(t=0) = u^{(0)},\, \frac{\partial u}{\partial t}(t=0) = u^{(1)}, \\
\frac{\partial u_j}{\partial t}(x,\,t-\tau) = f_j^0(x,\,t-\tau) & \text{in } I_2^{(j)} \times (0,\,\tau),\, \forall j \in \{1,\ldots,N\},
\end{cases}
$$

where $\alpha_i^{(j)}$ are fixed non-negative real numbers, the delay τ is positive and the intervals $I_i^{(j)}$ satisfy

$$
I_2^{(j)} \subset I_1^{(j)} \subset e_j.
$$

As before we can rewrite this system in the form (1.2), by introducing $H = L^2(\mathcal{R})$, the operator A defined by (7.2), $U_i = L^2(\cup_{j=1}^N I_i^{(j)})$ and the operators B_i for $i = 1,\,2$ as

$$
B_i : \begin{cases}
U & \to & D(A^{\frac{1}{2}})' \\
k & \mapsto & \sum_{j=1}^N \sqrt{\alpha_i^{(j)}}\tilde{k}\chi_{|I_i^{(j)}}
\end{cases} \tag{7.26}
$$

where $\tilde{k}$ means the extension of k by zero outside $\cup_{j=1}^N I_i^{(j)}$. It is easy to verify that

$$
B_i^*(\varphi) = \sum_{j=1}^N \sqrt{\alpha_i^{(j)}}\varphi_{|I_i^{(j)}}\chi_{|I_i^{(j)}}
$$

for $\varphi \in D(A^{\frac{1}{2}})$. Hence the system (7.25) can be rewritten in the form (1.2).

As before it is easy to see that the system (7.1) is well posed for $\alpha_2^{(j)} \leq \alpha_1^{(j)}$ for all $j = 1,\ldots,N$ by Theorem 2.1, and the energy is decreasing for $\alpha_2^{(j)} < \alpha_1^{(j)}$ for all $j = 1,\ldots,N$ by Proposition 3.1.

As mentioned before by Proposition 6.2 from [7], the generalized gap condition (6.1) holds with $M = N+1$, but in this general setting, the conditions (6.7) or (6.10) seem difficult to check. Hence, for the sake of simplicity, if we suppose here that the standard gap condition (6.4) holds and that all eigenvalues are simple (*i.e.*, $l_k = 1$), then condition (6.7) becomes

$$
\exists \gamma > 0,\, \forall k \geq 1,\, \sum_{j=1}^N \alpha_1^{(j)}\int_{I_1^{(j)}}\varphi_k(x)^2\,\mathrm{d}x \geq \gamma, \tag{7.27}
$$

while condition (6.10) becomes

$$
\exists m \in \mathbb{R}_+^*,\, \exists \gamma > 0,\, \forall k \geq 1,\, \sum_{j=1}^N \alpha_1^{(j)}\int_{I_1^{(j)}}\varphi_k(x)^2\,\mathrm{d}x \geq \frac{\gamma}{\lambda_k^{2m}}. \tag{7.28}
$$

For instance using an argument like in Proposition 7.10, we easily see that (7.27) (resp. (7.28)) holds if (7.5) (resp. (7.6)) holds and if $\cup_{j=1}^{N} I_1^{(j)}$ contains a neighborhood of the set of control points $\mathcal{V}_c$.

In the same manner, we have:

Proposition 7.19. *If there exists $\varepsilon > 0$ and for all $j \in \{1, \ldots, N\}$, there exists $\delta_j \in (0, l_j)$ such that*

$$[\delta_j, \delta_j + \varepsilon] \subset I_1^{(j)} \quad \forall j \in \{1, \ldots, N\},$$

then (7.27) holds.

Proof. For any $k \in \mathbb{N}^*$, let φ_k be the eigenvector of A associated with the eigenvalue λ_k^2. Then its restriction $\varphi_{k,j}$ to the edge e_j can be written in the form

$$\varphi_{k,j}(x) = c_{k,j} \cos(\lambda_k x) + d_{k,j} \sin(\lambda_k x) \quad \forall x \in (0, l_j),$$

for some real numbers $c_{k,j}$ and $d_{k,j}$. Hence the normalization of φ_k yields

$$1 = \sum_{j=1}^{N} \int_0^{l_j} |\varphi_{k,j}(x)|^2 \, dx \leq 2(\max_j l_j) \sum_{j=1}^{N} (c_{k,j}^2 + d_{k,j}^2). \tag{7.29}$$

On the other hand by the above expression of $\varphi_{k,j}$ and direct calculations, we see that

$$\int_{I_1^{(j)}} \varphi_k(x)^2 \, dx \quad \geq \quad \int_{\delta_j}^{\delta_j + \varepsilon} \varphi_k(x)^2 \, dx$$

$$\geq \quad (c_{k,j}^2 + d_{k,j}^2) \left(\frac{\varepsilon}{2} - \frac{1}{2\lambda_k} \right) - \frac{|c_{k,j} d_{k,j}|}{\lambda_k}$$

$$\geq \quad (c_{k,j}^2 + d_{k,j}^2) \left(\frac{\varepsilon}{2} - \frac{1}{\lambda_k} \right).$$

Therefore for k large enough such that

$$\frac{\varepsilon}{2} - \frac{1}{\lambda_k} \geq \frac{\varepsilon}{4},$$

which is equivalent to $k \geq k_\varepsilon$, for some $k_\varepsilon \in \mathbb{N}^*$, we deduce that

$$\int_{I_1^{(j)}} \varphi_k(x)^2 \, dx \geq \frac{\varepsilon}{4} (c_{k,j}^2 + d_{k,j}^2).$$

By summing this estimate on j and using the normalization estimate (7.29), we obtain (7.27) for $k \geq k_\varepsilon$. The proof is complete since for $k \leq k_\varepsilon$,

$$\sum_{j=1}^{N} \alpha_1^{(j)} \int_{I_1^{(j)}} \varphi_k(x)^2 \, dx > 0. \qquad \square$$

The analysis of the condition (7.27) in some particular cases reveals that the condition of the above proposition is far from being optimal but in its full generality we cannot easily obtain a weaker condition.

As in the previous subsection one easily shows (see the proof of Lem. 7.11) that the operators A and $B = (B_1 B_2) \in \mathcal{L}(U, V')$ where $U = U_1 \times U_2$ satisfy the assumption (4.1).

In conclusion applying either Theorem 4.4 or Theorem 5.3, we obtain:

Proposition 7.20. *Assume that $\alpha_2^{(j)} < \alpha_1^{(j)}$ for all $j = 1, \ldots, N$ and that the standard gap condition* (6.4) *holds and that all eigenvalues are simple (i.e., $l_k = 1$). Then*

(i) *The energy of system* (7.25) *decays exponentially if* (7.27) *holds.*

(ii) *The energy of system* (7.25) *decays like $t^{-\frac{1}{m}}$ if* (7.28) *holds.*

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