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SUPERSYMMETRIC MEASURES AND MAXIMUM PRINCIPLES IN THE COMPLEX DOMAIN. EXPONENTIAL DECAY OF GREEN'S FUNCTIONS

BY J. SJÖSTRAND AND W.-M. WANG

ABSTRACT. – We study a class of holomorphic complex measures which are, in an appropriate sense, close to a complex Gaussian. We show that these measures can be reduced to a product measure of real Gaussians with the aid of a maximum principle in the complex domain. The formulation of this problem has its origin in the study of a certain class of random Schrödinger operators, for which we show that the expectation value of the Green's function decays exponentially. © Elsevier, Paris

RÉSUMÉ. – Nous étudions une classe de mesures holomorphes complexes, proches d'une gaussienne complexe. Nous montrons que ces mesures peuvent être réduites à un produit de gaussiennes réelles à l'aide d'un principe de maximum dans le domaine complexe. Notre principale motivation est l'étude d'une classe d'opérateurs de Schrödinger aléatoires, pour lesquels nous montrons que l'espérance de la fonction de Green décroît exponentiellement. © Elsevier, Paris

1. Introduction

We study in this paper a class of normalized complex holomorphic measures of the form $e^{-\psi_n(x)}\mathrm{d}^{2n}x$ in $\mathbf{R}^{2n}$, where $\psi_n(x)$ is holomorphic in x and $\mathrm{Re}\,\psi_n \geq 0$ and grows sufficiently fast at infinity, so that the integral is well defined. It is not presumed that $e^{-\psi_n(x)}\mathrm{d}^{2n}x$ is a product measure. Moreover we assume that $e^{-\psi_n(x)}$ is "close", in some sense, to a complex Gaussian in certain regions of the complex space. Assuming that f does not grow too fast at infinity, we are interested in estimates of integrals of the form

$$\int f(x)e^{-\psi_n(x)}\mathrm{d}^{2n}x,$$

which are *uniform* in n. So that eventually we can take the limit $n \to \infty$. Assume (for argument's sake) $|f(x)|_\infty = \mathcal{O}(1)$, then if $\psi_n(x)$ were real, we would immediately have

$$\int f(x)e^{-\psi_n(x)}\mathrm{d}^{2n}x = \mathcal{O}(1)$$

uniformly in n. However it is clear that, in the case where $\psi_n(x)$ is complex, the same argument will not give us a bound which is uniform in n. Since typically,

$$\int |e^{-\psi_n(x)}|\mathrm{d}^{2n}x \to \infty$$

as $n \to \infty$, even though

$$\int e^{-\psi_n(x)} \mathrm{d}^{2n} x = 1,$$

for all n.

In the following, we show that under appropriate conditions (convexity, domain of holomorphicity etc.), this class of measures can be reduced, **uniformly** with respect to the dimension of the space, to a product of real Gaussians. Hence the usual estimates of integrals with respect to positive measures become applicable.

The initial inspiration for this paper comes from random Schrödinger operators, where the expectation values of certain spectral quantities can be naturally expressed as the correlation functions of some normalized complex measures in even dimensions. Other examples of complex measures arise, for example, from considerations of analyticity of certain quantities in statistical mechanics. However for concreteness, we only state our results in the random Schrödinger case, although it is our belief that the method presented here should prove to be of a general nature, with possible applications to other fields.

We now describe the discrete random Schrödinger operator on $\ell^2(\mathbf{Z}^d)$:

$$H = t\Delta + V, \qquad (0 < t \leq 1) \tag{1.1}$$

where t is a parameter, Δ is the discrete Laplacian with matrix elements

$$\begin{aligned} \Delta_{i,j} &= 1 \qquad |i-j|_1 = 1, \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{1.2}$$

where $i,\ j \in \mathbf{Z}^d$, $|\cdot|_1$ is the ℓ^1 norm; V is a multiplication operator, $(Vu)(j) = v_j u_j$, with $v_j \in \mathbf{R}$. We assume that the v_j are independent random variables with a common distribution density g. We use $\langle \quad \rangle$ to denote the expectation with respect to (w.r.t.) the product probability measure. Such operators occur naturally in the quantum mechanical study of disordered systems. (See e.g. [FS,Sp].).

For small t, the spectrum of H is known to be almost surely pure point with exponentially localized eigenfunctions. (See e.g. [AM,DK,FMSS].) This is commonly known as *Anderson localization* after the physicist P. Anderson, who first realized the importance of the phenomenon [A]. Another related quantity of interest, which provides a necessary condition for the existing mechanisms for proving localization, is the density of states (d.o.s.). Roughly speaking, d.o.s. measures the number of states per unit energy per unit volume. More precisely, d.o.s. is the positive (non-random) Borel measure ρ such that

$$\langle \mathrm{tr} f(H) \rangle = \int f(E) \mathrm{d}\rho(E)$$

for all $f \in C_0(\mathbf{R})$. It is known generally that if g is smooth, then for t small enough or E large enough, ρ is also smooth. (See e.g. [CFS,BCKP].) In the continuum, one can prove similar results [W2], and moreover obtain an asymptotic expansion for ρ [W1].

Let Λ be a finite subset in $\mathbf{Z}^d$. Let Δ_Λ be the corresponding discrete Laplacian defined as in (1.2) for i, j in Λ. Define

$$H_\Lambda = t\Delta_\Lambda + V, \tag{1.3}$$

on $\ell^2(\Lambda)$. For E real, (assume $E \in \sigma(H_\Lambda)$ a.s.), let

$$G_\Lambda(E+i\eta) = (H_\Lambda - E - i\eta)^{-1}, \tag{1.4}$$

be the so called Green's function. We denote by $G_\Lambda(i,j;E+i\eta)$ the matrix elements of $G_\Lambda(E+i\eta)$. Then we have the following representation

$$\rho(E) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \lim_{\eta \searrow 0} \operatorname{Im}\langle G_\Lambda(0,0;E+i\eta)\rangle \qquad a.s.$$

In this paper we study $\langle G_\Lambda(\mu,\nu;E+i\eta)\rangle$ for t sufficiently small or E sufficiently large. Our aim is to obtain estimates which are uniform in η, Λ, so that we can pass to the limit:

$$\langle G(\mu,\nu;E+i0)\rangle := \lim_{\Lambda \nearrow \mathbf{Z}^d} \lim_{\eta \searrow 0} \operatorname{Im}\langle G_\Lambda(\mu,\nu;E+i\eta)\rangle.$$

The existence of the limiting function can be obtained directly [SW] and we will not enter into the details here. Although the present method can give that too.

Assuming g is sufficiently smooth, using the supersymmetric representation of the inverse of a matrix, which was first used in this context in [BCKP], we can express $\langle G_\Lambda(\mu,\nu;E+i0)\rangle$ as a correlation function of a normalized complex measure. (See sect. 2 and also appendix A.) Let

$$\widehat{g}(\tau) = \int e^{-iv\tau} g(v)\mathrm{d}v$$

denote the Fourier transform of g. Assume for example that $\widehat{g}(\tau) = e^{-k(\tau)} \neq 0$ for $\tau \in \mathbf{R}^+$; then (after taking the limit $\eta \searrow 0$)

$$\langle G_\Lambda(\mu,\nu;E+i0)\rangle = \\ i\int x_\mu \cdot x_\nu \left[\det(iM_\Lambda)e^{-i(\sum_{|j-k|_1=1} tx_j\cdot x_k - \sum_j Ex_j\cdot x_j - i\sum_j k(x_j\cdot x_j))}\right] \prod_{j\in\Lambda} \frac{\mathrm{d}^2 x_j}{\pi}, \tag{1.5}$$

where $x_j \in \mathbf{R}^2$, $x_j \cdot x_k$ is the usual scalar product in $\mathbf{R}^2$ and

$$M_\Lambda = t\Delta_\Lambda - E - i\,\mathrm{diag}\,(k'(x_j\cdot x_j)), \tag{1.6}$$

where $\mathrm{diag}\,(k'(x_j\cdot x_j))$ denotes the diagonal matrix whose jj-th entry is $k'(x_j\cdot x_j)$. We notice the appearance of the Fourier transform of the original probability measure in the above induced measure. We believe that this is the main accomplishment of the supersymmetric representation here. After an integration by parts, (see appendix A or B,) we have further:

$$\langle G_\Lambda(\mu,\nu;E+i0)\rangle = \\ \int M_\Lambda^{-1}(\mu,\nu;E)\left[\det(iM_\Lambda)e^{-i(\sum tx_j\cdot x_k - \sum Ex_j\cdot x_j - i\sum k(x_j\cdot x_j))}\right] \prod_{j\in\Lambda} \mathrm{d}^2 x_j. \tag{1.7}$$

Note that if the measure in the square brackets in (1.5), (1.7) were positive, then we would have immediately obtained that

$$|\langle G_\Lambda(\mu,\nu;E)\rangle| \le |M_\Lambda^{-1}(\mu,\nu;E)|_\infty$$

where the sup-norm is w.r.t. x. Hence the main idea is to make a change of contours in $(\mathbf{C}^2)^\Lambda$, so that on the new contour the measure becomes real positive. In order to do that we assume that g is such that $\widehat{g}$ is holomorphic in a region of $\mathbf{C}$ which includes the convex cone bounded by $\mathbf{R}^+$ and $e^{i\theta(E)}\mathbf{R}^+$, where $\theta(E) = \arg(1+iE) \subset]-\frac{\pi}{2}, \frac{\pi}{2}[$. Moreover we need to assume that g is ϵ $(0 \leq \epsilon << 1)$ "close" to

$$g_0 = \frac{1}{\pi(1+v^2)},$$

so that there exists an open neighborhood $\Omega(E) \subset \mathbf{C}$ of $e^{i\theta(E)}[0,\infty[$ which is conic at infinity and in which $\widehat{g}$ is ϵ-close to $\widehat{g}_0$. (See (3.7).) For the precise conditions on g, see (2.26)-(2.28). Note that assuming t, ϵ small, then the final contour where the phase becomes real should be "close" to $((e^{\frac{i\theta(E)}{2}}\mathbf{R})^2)^\Lambda$. (Recall that $x_j \in \mathbf{R^2}$.) Therefore in sect. 3, before we embark on the real work, we first rotate the contour from $(\mathbf{R}^2)^\Lambda$ to $((e^{\frac{i\theta(E)}{2}}\mathbf{R})^2)^\Lambda$. Using the assumptions on g, the measure then takes the simple form in (1.5), (1.7). Define

$$\phi := i\left(\sum_{|j-k|_1=1} tx_j\cdot x_k - \sum_j Ex_j\cdot x_j - i\sum_j k(x_j\cdot x_j)\right).$$

The change of contours is accomplished in two steps. In sect. 4, we look for a first vector field v_t (holomorphic both in x and t) in $(\mathbf{C}^2)^\Lambda$ such that

$$\partial_t(e^{-\phi}) + \nabla_x(e^{-\phi})\cdot v_t = 0, \tag{1.8}$$

or equivalently

$$\partial_t\phi + \nabla_x\phi\cdot v_t = 0. \tag{1.9}$$

where

$$v\cdot\nabla\phi := \sum_j (v_{j,1}\partial_{x_{j,1}}\phi + v_{j,2}\partial_{x_{j,2}}\phi).$$

Using the flow of the vector field to change variables, we get rid of the "interaction" term $\sum tx_j\cdot x_k$. The main difficulty here (as opposed to the case ϕ real) is to find v_t such that the corresponding flow stays in the appropriate region in $(\mathbf{C}^2)^\Lambda$ for t small enough so that the resulting integral is well defined and that the measure has no zeros there. This is achieved by using a cutoff function and solving (1.9) in some appropriate weighted space. Sect. 5 studies the corresponding flow, while sect. 6 expresses the resulting measure on the new contour.

Unfortunately, after this operation, the coupling between x_j and x_k $(j \neq k)$ still persists in the Jacobian of the above "change of variables". Writing the measure as $e^{-L}\prod \mathrm{d}^2x_j$ (with L holomorphic as the measure has no zeros there), in sect. 7, we look for a second vector field ν_t (holomorphic in x and t) such that

$$\partial_t(e^{-L}) + \nabla_x(e^{-L})\cdot\nu_t + e^{-L}\mathrm{div}\,\nu = 0.$$

or equivalently

$$\partial_t L + \nabla_x L\cdot\nu_t - \mathrm{div}\,\nu_t = 0. \tag{1.10}$$

We use a maximum principle in tube domains in the complex space to solve (1.10) under the condition that $\mathrm{Re}\,\mathrm{Hess}\,L > c > 0$ and some additional conditions on ∇L, which ensures that the resulting flow stays in tube domains around the real axis. This is in fact why we need to find the first vector field v_t to ensure that the new phase L is such that ∇L has the required properties. (See sect. 7, appendix C.)

Under these two changes of contours, the final measure takes the simple form

$$e^{-\sum_j z_j \cdot z_j} \prod_j \frac{\mathrm{d}^2 z_j}{\pi}.$$

We then obtain in sect. 8 that for $t/(|E|+1)$ sufficiently small and E in the appropriate range (depending on g), $\langle G_\Lambda(\mu,\nu;E+i\eta)\rangle$ decays exponentially in $|\mu-\nu|$ for all Λ sufficiently large, by using weighted estimates on $M_\Lambda^{-1}(\mu,\nu;E)$. The precise estimate is formulated in Theorem 2.1 in sect. 2.

We should mention here that the region of analyticity in t is uniform in Λ. The construction above does not depend on the fact that we have a nearest neighbour Laplacian (1.2). It works the same way if Δ is replaced by any other symmetric matrix with off-diagonal matrix elements decay sufficiently fast.

As we have seen earlier, $\langle G\rangle$ can be expressed as a correlation function of a normalized complex measure. In fact (1.5) shows clearly the link between the present problem and problems in statistical mechanics. (1.7) is special to the present problem. Our main constructions however do not depend on these special equalities arising from the symmetries of the present problem.

Before the first in a series of the works of B. Helffer and J. Sjöstrand [HS], where the equation (1.10), to our knowledge, first appeared in the context of statistical mechanics, one of the main tools to study correlation functions was cluster expansion–an algebraic way of rearranging the perturbation (e.g. in t) series. (1.10) provides an alternative way of treating such problems. The advantage, in our opinion, is that there is no combinatorics involved. The mathematics involved is purely analytical and self-contained. Moreover the convexity condition on L that one meets is the natural one.

Another general, but more probabilistic, approach to statistical mechanics is by using semi-groups or heat equations. It seems interesting to us to understand what would be the analogue of the construction presented here.

Although, as mentioned earlier, the inspiration for the present paper comes from quite a different source–random Schrödinger operators, in the end, the work presented here should be seen as a logical extension of the works of B. Helffer and J. Sjöstrand [HS,S1,S2] in statistical mechanics. (The work presented below might also be useful for the study of Feynmann formula.) Indeed one can take the standard example of studying the correlation function for the measure

$$\frac{e^{-\sum_{j,k\in\Lambda,|j-k|_1=1} t x_j\cdot x_k} \prod_{j\in\Lambda} e^{-k(x_j^2)} \mathrm{d}x_j}{\int e^{-\sum_{j,k\in\Lambda,|j-k|_1=1} t x_j\cdot x_k} \prod_{j\in\Lambda} e^{-k(x_j^2)} \mathrm{d}x_j}, \qquad x_j \in \mathbf{R},$$

assuming that k is such that the measure is well defined. It seems clear to us that under appropriate conditions on k, which essentially amounts to assuming k analytic and $k \neq 0$ on $\mathbf{R}^+$, k does not grow faster than linearly at infinity and some convexity conditions on k (See Lemma 3.1.), the analyticity of the correlation function in t for small t should be a direct consequence of the constructions here.

2. The supersymmetric representation and statement of the main result

Let $t \in]0,1]$ and let H be the discrete Schrödinger operator on $\ell^2(\mathbf{Z}^d)$ defined earlier in sect. 1. For convenience, we recall it here:

$$H = t\Delta + V, \tag{2.1}$$

where Δ is the discrete Laplacian with matrix elements

$$\Delta_{i,j} = 1, \text{ when } |i-j|_1 = 1, \text{ and } = 0 \text{ otherwise.} \tag{2.2}$$

V is a multiplication operator, $(Vu)(j) = v_j u_j$, with $v_j \in \mathbf{R}$ and $|\cdot|_1$ is the ℓ^1 norm. We assume that the v_j are independent random variables with a common distribution density g. For real E, we consider the inverse operator

$$G(E+i\eta) = (H - E - i\eta)^{-1}, \tag{2.3}$$

and more specifically, we are interested in the expectation value of the kernel (i.e. matrix) of $G(E+i\eta)$ (the so called Green's function): $\langle G(\mu,\nu;E+i\eta)\rangle$ in the limit $\eta \searrow 0$. We will write,

$$\langle G(\mu,\nu;E+i0)\rangle := \lim_{\eta\searrow 0}\langle G(\mu,\nu;E+i\eta)\rangle, \tag{2.4}$$

if the right hand side (RHS) exists.

We proceed by taking $\Lambda \subset \mathbf{Z}^d$ to be a finite set or to be a large discrete torus of the form $(\mathbf{Z}/N\mathbf{Z})^d$. The corresponding discrete Laplacian Δ_Λ on Λ is then defined as in (2.2), with i,j in Λ. Define

$$H_\Lambda = t\Delta_\Lambda + V, \tag{2.5}$$

on $\ell^2(\Lambda)$. Let

$$G_\Lambda = (H_\Lambda - E)^{-1}, \tag{2.6}$$

for complex E, whenever the inverse is well-defined. We also consider the expectation values $\langle G_\Lambda(\mu,\nu;E+i\eta)\rangle$ for $E \in \mathbf{R}$, $\eta > 0$, and the corresponding limits when $\eta \searrow 0$. The aim of the game is of course to have estimates which are uniform in Λ, and in this way we get information about $\langle G\rangle$ whenever we can take the infinite volume limit $\Lambda \to \mathbf{Z}^d$. (The possibility of taking this limit can be obtained by [SW] and we will not enter into the details in this paper, even though the present methods can give that limit too.)

We use the supersymmetric formalism to express $\langle G_\Lambda\rangle$. In order not to make too much of a digression, we will only write the few lines that are necessary to reach the representations (2.18,22,25), and we refer to appendix A and references therein for a more complete discussion. Using Gaussian integrals ((A.9) in appendix A), we have the following expression for the Green's function:

$$G_\Lambda(\mu,\nu;E+i\eta) = i\int x_\mu\cdot x_\nu \det[i(H-(E+i\eta))]\times \tag{2.7}$$
$$\exp\left[-i\sum_{j,k}(H-(E+i\eta))_{j,k}x_j\cdot x_k\right]\prod_{j\in\Lambda}\frac{\mathrm{d}^2x_j}{\pi},$$

where $x_j \in \mathbf{R}^2$, $\mu, \nu \in \Lambda$ and we sometimes drop the subscript Λ and write H instead of H_Λ.

Let $|\Lambda|$ be the number of points in Λ. We use the Grassmann algebra of $2|\Lambda|$ generators to express $\det[i(H-E)]$. This algebra is generated by $2|\Lambda|$ anticommuting variables ξ_i, η_i, $i \in \Lambda$ satisfying the relations:

$$\begin{aligned} [\xi_i, \eta_j] &= \xi_i\eta_j + \eta_j\xi_i = 0, \\ [\xi_i, \xi_j] &= \xi_i\xi_j + \xi_j\xi_i = 0, \\ [\eta_i, \eta_j] &= \eta_i\eta_j + \eta_j\eta_i = 0, \end{aligned} \tag{2.8}$$

with $[a,b] = ab + ba$ the anti-commutator. It is denoted by $\Lambda[\xi_1, \eta_1, .., \xi_{|\Lambda|}, \eta_{|\Lambda|}]$ (if we identify Λ with $\{1, .., |\Lambda|\}$). From (2.8), we see in particular that $\xi_i^2 = \eta_i^2 = 0$."C^∞ functions" $F(\xi_i, \eta_j)$ of these anticommuting variables are defined by Taylor's formula at $(0,0)$ which contains a finite number of terms because of nilpotency. In this way $F(\xi,\eta)$ becomes an element of the Grassmann algebra. For example if

$$F(\xi,\eta) := e^{A_{i,j}\xi_i\eta_j}, \tag{2.9}$$

then

$$F(\xi,\eta) = 1 + A_{i,j}\xi_i\eta_j. \tag{2.10}$$

This is the function that we need in writing the determinant. We also need to define the notions of differentiation and integration. Define:

$$\frac{\partial}{\partial \xi_i}(\xi_i) = 1, \tag{2.11}$$

$$\frac{\partial}{\partial \eta_i}(\eta_i) = 1. \tag{2.12}$$

We also require that these differentiations be linear operators and that Leibnitz' rule hold. We can then define integrals (with respect to ∂) as follows:

$$\int 1 \mathrm{d}\xi_i = 0, \quad \int \xi_i \mathrm{d}\xi_i = 1, \quad \int 1 \mathrm{d}\eta_i = 0, \quad \int \eta_i \mathrm{d}\eta_i = 1. \tag{2.13}$$

A multiple integral is defined to be a repeated integral. For example,

$$\int \xi_i\eta_j \mathrm{d}\xi_i \mathrm{d}\eta_j = -\int \eta_j\xi_i \mathrm{d}\xi_i \mathrm{d}\eta_j = -\int \eta_j \mathrm{d}\eta_j = -1. \tag{2.14}$$

Using (2.10), (2.14), we get

$$\det[i(H - E - i\eta)] = \int e^{-i\sum_{j,k\in\Lambda}(H-E-i\eta)_{j,k}\eta_j\xi_k} \prod_{j\in\Lambda} (\mathrm{d}\eta_j \mathrm{d}\xi_j). \tag{2.15}$$

We illustrate (2.15) in the case where $i(H - E - i\eta) := M$ is a 2×2 matrix, using (2.8)–(2.14). The integrand in the RHS of (2.15) is

$$e^{-\sum_{j,k=1,2} M_{j,k}\eta_j\xi_k} = \prod_{j,k=1,2} (1 - M_{j,k}\eta_j\xi_k),$$

where we used the commutativity of $e^{-M_{j,k}\eta_j\xi_k}$ and (2.10). Doing the integral in the RHS of (2.15) upon using (2.13) and (2.14), we obtain

$$\mathrm{RHS} = M_{11}M_{22} - M_{12}M_{21} = \det M,$$

as expected.

Combining (2.7) with (2.15), we obtain the following expression:

$$G(\mu,\nu;E+i\eta) = i\int x_\mu \cdot x_\nu e^{-i\sum_{j,k\in\Lambda}(H-E-i\eta)_{j,k}X_j\cdot X_k}\prod_{j\in\Lambda}\mathrm{d}^2X_j, \tag{2.16}$$

where

$$\begin{aligned} X_j &:= (x_j,\xi_j,\eta_j),\\ X_j\cdot X_k &:= x_j\cdot x_k + \frac{1}{2}(\eta_j\xi_k+\eta_k\xi_j),\\ \mathrm{d}^2X_j &:= \frac{\mathrm{d}^2x_j}{\pi}\mathrm{d}\eta_j\mathrm{d}\xi_j. \end{aligned} \tag{2.17}$$

Hence,

$$\begin{aligned} \langle G(\mu,\nu;E+i\eta)\rangle &= i\int x_\mu\cdot x_\nu e^{-i(\sum_{j,k\in\Lambda,|j-k|_1=1}tX_j\cdot X_k - \sum_{j\in\Lambda}(E+i\eta)X_j\cdot X_j)}\times \\ &\qquad \prod_{j\in\Lambda} e^{-iv_jX_j\cdot X_j}\prod g(v_j)\mathrm{d}v_j\prod \mathrm{d}^2X_j \\ &= i\int x_\mu\cdot x_\nu e^{-i(\sum_{j,k\in\Lambda,|j-k|_1=1}tX_j\cdot X_k - \sum_j(E+i\eta)X_j\cdot X_j)}\prod_j \widehat{g}(X_j\cdot X_j)\prod_j \mathrm{d}^2X_j, \end{aligned} \tag{2.18}$$

where

$$\widehat{g}(X_j\cdot X_j) = \widehat{g}(x_j\cdot x_j+\eta_j\xi_j) := \widehat{g}(x_j\cdot x_j) + \widehat{g}'(x_j\cdot x_j)\eta_j\xi_j, \tag{2.19}$$

is the (super-)Fourier transform. Assume that $\widehat{g}$ is in $\mathcal{S}$ away from 0. Then the above integral is well defined. We can take the limit $\eta \searrow 0$ and obtain

$$\langle G(\mu,\nu;E+i0)\rangle = i\int x_\mu\cdot x_\nu e^{-i(\sum tX_j\cdot X_k - \sum EX_j\cdot X_j)}\prod_j \widehat{g}(X_j\cdot X_j)\prod \mathrm{d}^2X_j. \tag{2.20}$$

Note that by using (2.17), the integrand in (2.20) is a sum of terms of the form

$$f_{j_1\cdots j_n,k_1\cdots k_n}(x)\xi_{j_1}\cdots\xi_{j_n}\eta_{k_1}\cdots\eta_{k_n} \qquad (n\le|\Lambda|),$$

where the f's are called coefficients. Note that apart from the factor $x_\mu\cdot x_\nu$, the integrand in (2.20) is only a "function" of the $X_j\cdot X_k$. Such "functions" are called supersymmetric functions. Using Theorem A.2 in appendix A, we have:

$$\int e^{-i(\sum_{|j-k|_1=1}tX_j\cdot X_k - \sum EX_j\cdot X_j)}\prod \widehat{g}(X_j\cdot X_j)\prod \mathrm{d}^2X_j = 1, \tag{2.21}$$

for all Λ, all t. Hence $\langle G(\mu,\nu;E+i0)\rangle$ can be seen as a correlation function associated to the normalized supersymmetric "measure" in (2.21). By integrating out the anti-commutative variables ξ, η, this measure can be further reduced to a (normalized) complex measure. Assume for example that $\widehat{g}(\tau) = e^{-k(\tau)} \neq 0$. Then using (2.15), (2.17), we obtain:

$$\langle G(\mu,\nu;E+i0)\rangle = \\ i\int x_\mu \cdot x_\nu [\det(iM) e^{-i(\sum_{|j-k|_1=1} tx_j\cdot x_k - \sum_j Ex_j\cdot x_j - i\sum_j k(x_j\cdot x_j))}] \prod_{j\in\Lambda} \frac{\mathrm{d}^2 x_j}{\pi}, \tag{2.22}$$

where

$$M = t\Delta - E - i\,\mathrm{diag}\,(k'(x_j\cdot x_j)). \tag{2.23}$$

We note also that by using (2.21)

$$\det(iM) e^{-i(\sum_{|j-k|_1=1} tx_j\cdot x_k - \sum_j Ex_j\cdot x_j - i\sum_j k(x_j\cdot x_j))} \prod_{j\in\Lambda} \frac{\mathrm{d}^2 x_j}{\pi} = 1, \tag{2.24}$$

for all t.

Using an integration by parts, established in Proposition A.3 in appendix A or equivalently (B.19) in appendix B, (2.22) can be further put in a more transparent form:

$$\langle G(\mu,\nu;E+i0)\rangle = \\ \int M^{-1}(\mu,\nu;E)\Big[\det(iM) e^{-i(\sum tx_j\cdot x_k - \sum Ex_j\cdot x_j - i\sum k(x_j\cdot x_j)}\Big] \prod_{j\in\Lambda} \mathrm{d}^2 x_j. \tag{2.25}$$

In appendix B, we give direct proofs of (2.22, 24,25) without using supersymmetry. The rest of the paper will be essentially devoted to the study of the resulting complex measuer as defined in (2.24) in an appropriate region in $(\mathbf{C}^2)^\Lambda$.

Note that if g is the Cauchy distribution, $g_0(v) = \frac{1}{\pi}\frac{1}{v^2+1}$, then $k(\tau) = |\tau|$ fo real τ and we have corresponding holomorphic extensions from each half axis (and we shall only use the one from the positive half axis, which is given by $k(\tau) = \tau$). Using (2.25), we then obtain another derivation of the fact that

$$\langle G(\mu,\nu;E)\rangle = (t\Delta - E - i)^{-1}{}_{\mu,\nu}, \tag{2.26}$$

for the Cauchy distribution. (A more direct proof based on the Cauchy formula can easily be found either as an exercise or by looking in [Ec]).

We now specify the class of densities g that we shall allow. We assume that g is of the form:

$$g(v) = (1+\mathcal{O}(\epsilon))g_0(v) + r_\epsilon(v), \tag{2.27}$$

where

$$g_0(v) = \frac{1}{\pi}\frac{1}{v^2+1}$$

and r_ϵ has the following properties:

(a) r_ϵ is smooth and real on $\mathbf{R}$ and satisfies

$$|\frac{\partial^k r_\epsilon}{\partial v^k}| \leq C_k\epsilon \text{ for all } k\in\mathbf{N}, \tag{2.28}$$

for some fixed constants $C_0, C_1, \ldots$.

(b) There is a compact ϵ-independent set $K \subset \mathbf{C}$, symmetric around $\mathbf{R}$ with $i \notin K$, such that r_ϵ has a holomorphic extension to $\mathbf{C} \setminus K$ (also denoted by r_ϵ) with

$$r_\epsilon(v) = \mathcal{O}(\epsilon)\frac{1}{1+|v|^2} \text{ in } \mathbf{C} \setminus K. \tag{2.29}$$

The $\mathcal{O}(\epsilon)$ in (2.27) is determined by the requirement that $\int g(v)\mathrm{d}v = 1$. Assuming also that $\epsilon \geq 0$ is small enough, as we shall always do in the following, we notice that it follows that $g(v) \geq 0$ and hence is a probability measure.

Remark. As it was mentioned in the introduction and as it will become clear later in the proof, the conditions for our constructions to be valid are rather on the Fourier transform $\widehat{g}$ of g. But for concreteness, we shall state our main theorem only for the class of densities above.

For all $\lambda > 2d$, introduce the convex open bounded set

$$W(\lambda) := \left\{\eta \in \mathbf{R}^d;\ 2\sum_1^d \cosh \eta_j < \lambda\right\}. \tag{2.30}$$

Let

$$p_\lambda(x) := \sup_{\eta \in W(\lambda)} x \cdot \eta \tag{2.31}$$

be the support function of $W(\lambda)$ so that $p_\lambda(x)$ is convex, even, positively homogeneous of degree 1. Moreover $p_\lambda(x) \geq 0$ with equality precisely at 0. In other words $p_\lambda(x)$ is a norm.

In sect. 8, by using weighted estimates, we show that there exist $C_0 \geq 1$, $C_1 \geq 0$, such that if $|E| \geq C_0^2$, $F \leq \frac{|E|}{C_0}$ and $V = \mathrm{diag}\,(v_j)$, with $|v_j| \leq F$, then

$$|(\Delta + V - E)^{-1}(\mu,\nu)| \leq C_1 \exp\left(-p_{|E|}(\mu-\nu) + \frac{C_1(1+F)}{|E|}|\mu-\nu|\right). \tag{2.32}$$

A special case of this is that if $E \in \mathbf{R}$, $V = \mathrm{diag}\,(v_j)$ with $|v_j| \leq \epsilon > 0$, $t \in]0,1]$, $t/|E+i| << 1$, $\epsilon/|E+i| << 1$, then

$$\left(\Delta + \frac{1}{t}V - \frac{E+i}{t}\right)^{-1}(\mu,\nu) = \mathcal{O}(1)\exp\left(-p_{|E+i|/t}(\mu-\nu) + \mathcal{O}(1)\frac{t+\epsilon}{|E+i|}|\mu-\nu|\right), \tag{2.33}$$

for all $\mu, \nu \in \mathbf{Z}^d$.

Moreover, we show in sect. 8 that (2.33) is likely to be optimal by studying the inverse of $\Delta - E$ on $\ell^2(\mathbf{Z}^d)$, when $E \in \mathbf{C}$, $|E| >> 1$. After a suitable Fourier transform we see that this operator is unitarily equivalent to the operators of multiplication by $\delta(\xi) - E$ on $L^2(\mathbf{T}^d)$, where $\delta(\xi) = 2\sum \cos \xi_j$ and $\mathbf{T}^d = (\mathbf{R}/2\pi\mathbf{Z})^d$ is the standard torus. By Bochner's tube theorem we know that the largest open connected set of the form $\mathbf{R}^d + iW$ containing $\mathbf{R}^d$ where $\delta(\xi) - E \neq 0$, is of the form $\mathbf{R}^d + iW(E)$, where $W(E) \subset \mathbf{R}^d$ is an open convex neighborhood of 0. In sect. 8 we shall see that $W(E)$ is bounded, and we also note that $W(E)$ is symmetric around 0 since δ is an even function. As in the case E real, we define

$$p_E(x) := \sup_{\eta \in W(E)} x \cdot \eta \tag{2.34}$$

to be the support function of $W(E)$ so that $p_E(x)$ is convex, even, positively homogeneous of degree 1. Moreover $p_E(x) \geq 0$ with equality precisely at 0. In other words $p_E(x)$ is a norm.

In sect. 8, we shall see that

$$p_E(x) = p_{|E|}(x) + \mathcal{O}\left(\frac{1}{|E|}|x|\right), \tag{2.35}$$

$$W(|E|) = \left\{\eta \in \mathbf{R}^d;\ 2\sum_1^d \cosh \eta_j < |E|\right\}, \tag{2.36}$$

$$|(\Delta - E)^{-1}(\mu, \nu)| \leq \mathcal{O}(1) e^{-p_{|E|}(\mu-\nu) + \frac{\mathcal{O}(1)}{|E|}|\mu-\nu|}, \tag{2.37}$$

uniformly in E, μ, ν, when $|E|$ is large enough.

Equip the extended line $\overline{\mathbf{R}} := \{-\infty\} \cup \mathbf{R} \cup \{+\infty\}$ with the natural topology (i.e. the one induced from the topology on $[-1, +1]$ under the map $f : \overline{\mathbf{R}} \to [-1, 1]$, where $f(\pm\infty) = \pm 1$, $f(x) = x/\sqrt{1 + x^2}$, $x \in \mathbf{R}$). We define a subset $\mathcal{E} \subset \overline{\mathbf{R}}$ in the following way (see the figure at the end of this section):

When $E \in \mathbf{R}$, we say that $E \in \mathcal{E}$ if and only if (iff) the following holds: The line L_E through $-i$ which is orthogonal to the vector $E + i$ (the direction of the segment joining $-i$ to E) does not intersect $K_- := \{z \in K; \operatorname{Im} z \leq 0\}$ and separates K_- from E, in the sense that if P_+ is the open half-plane containing E with boundary L_E, and P_- the opposite open half-plane, then $K_- \subset P_-$.

When $E \in \{\pm\infty\}$, we say that $E \in \mathcal{E}$ iff the above holds with $L_E = i\mathbf{R}$.

Note that a necessary condition for $\mathcal{E}$ to be non-empty is that $-i$ does not belong to the convex hull of K_-. It is also clear that $\mathcal{E}$ is open and connected.

Let $d_{|E|}(\mu, \nu)$ be the distance on Λ associated to the norm $p_{|E|}(\mu - \nu)$, so that

$$d_{|E|}(\mu, \nu) = p_{|E|}(\mu - \nu)$$

when Λ is a finite set and

$$d_{|E|}(\mu, \nu) = \inf_{\widetilde{\mu} \in \pi_\Lambda^{-1}(\mu),\, \widetilde{\nu} \in \pi_\Lambda^{-1}(\nu)} p_{|E|}(\widetilde{\mu} - \widetilde{\nu}),$$

in the case when Λ is a torus, with $\pi_\Lambda : \mathbf{Z}^d \to \Lambda$ denoting the natural projection.

We can now state the main theorem of this paper.

THEOREM 2.1. – *For every $\mathcal{E}' \subset\subset \mathcal{E}$, there are constants $t_0 > 0$, $\epsilon_0 > 0$, such that if $0 \leq \epsilon \leq \epsilon_0$, $t \in]0, 1]$, $E \in \mathcal{E}'$, $\frac{t}{|E+i|} \leq t_0$, then for Λ sufficiently large we have uniformly in t, ϵ, E:*

$$|\langle G(\mu, \nu; E + i0)\rangle| \leq \frac{1}{t} \exp\left(-d_{|E+i|/t}(\mu, \nu) + \mathcal{O}\left(\frac{t + \epsilon}{|E + i|}\right)\rho(\mu, \nu)\right), \mu, \nu \in \Lambda. \tag{2.38}$$

Here ρ denotes the standard Euclidean distance in Λ.

3. Rotation of coordinates

We make the assumptions of Theorem 2.1. Then for $E \in \mathcal{E} \cap \mathbf{R}$, we have

$$\langle G(\mu,\nu;E+i0)\rangle = i\int x_\mu \cdot x_\nu e^{-i(\sum tX_j\cdot X_k - \sum EX_j\cdot X_j)} \prod_j \widehat{g}(X_j\cdot X_j)\prod \mathrm{d}^2X_j. \quad (3.1)$$

The corresponding normalized "measure" is

$$e^{-i(\sum tX_j\cdot X_k - \sum EX_j\cdot X_j)} \prod_j \widehat{g}(X_j\cdot X_j)\prod \mathrm{d}^2X_j. \quad (3.2)$$

where $\widehat{g}(X_j\cdot X_j)$ is as in (2.19). Our aim in this section is to make an appropriate change of contour, so that on the new contour, after integrating out the anti-commutative variables, the phase of the normalized complex measure is almost real.

Recall from the preceding section that,

$$\widehat{g}_0(\sigma) = e^{-\sigma}, \text{ for } \operatorname{Re}\sigma > 0, \quad (3.3)$$

and if we replace g by g_0 in (3.1), (3.2), we are naturally led to consider the factors,

$$e^{(iE-1)x_j\cdot x_j} = e^{-(1-iE)x_j\cdot x_j}, \quad (3.4)$$

(Recall that $X_i\cdot X_j = x_i\cdot x_j + \frac{1}{2}(\eta_i\xi_j+\eta_j\xi_i)$; see also (2.22), (2.25) with $k(x_j\cdot x_j) = x_j\cdot x_j$.) which in some sense can be expected to be dominant when $t>0$ is small or E is large. With $\sigma = x_j\cdot x_j$, this factor becomes real after the change of variables,

$$\sigma = e^{i\theta(E)}s = \frac{1+iE}{|1+iE|}s, \text{ where } \theta(E) = \arg(1+iE) \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[.$$

Put $\theta(\pm\infty) = \pm\frac{\pi}{2}$.

LEMMA 3.1. – *Let $E\in\mathcal{E}$ and let $S(E)$ be the closed convex sector in the complex plane bounded by the two half-lines $[0,+\infty[$ and $e^{i\theta(E)}[0,+\infty[$. The function $\widehat{g}(\sigma)$ has an entire extension from the positive half-axis, that we also denote by $\widehat{g}$, which has the following properties:*

(a) *If $E\neq 0$ and $|E|<\infty$, then for every $\gamma\in[0,1[$ and for all $k,N\in\mathbf{N}$:*

$$\partial_\sigma^k\widehat{g}(\sigma) = \mathcal{O}_{N,k,\gamma}(\langle\sigma\rangle^{-N}e^{-\frac{\gamma}{E}\operatorname{Im}\sigma}), \quad (3.5)$$

where $\sigma\in S(E)$ and $\langle\sigma\rangle = (1+|\sigma|^2)^{1/2}$.

(b) *If $E\in\{+\infty,-\infty\}$, then there exists $\epsilon_0>0$, such that for every $\delta>0$*

$$\partial_\sigma^k\widehat{g}(\sigma) = \mathcal{O}_{N,k,\delta}\langle\sigma\rangle^{-N}(e^{-\epsilon_0|\operatorname{Im}\sigma|} + e^{-(1-\delta)\operatorname{Re}\sigma}),\ \sigma\in S(E). \quad (3.6)$$

(c) *Recall that $g = g_\epsilon$. For every $\mathcal{E}'\subset\subset\mathcal{E}$, there exists an $\epsilon_0>0$ such that if E is confined to $\mathcal{E}'$ and $0\le\epsilon\le\epsilon_0$: There exists an open neighborhood $\Omega(E)\subset\mathbf{C}$ of*

$e^{i\theta(E)}[0,+\infty[$, *which is conic near infinity, and a holomorphic function* k *on* $\Omega(E)$ *such that*

$$\widehat{g}(\sigma)=e^{-k(\sigma)},\ k(\sigma)=\sigma+\mathcal{O}(\epsilon),\ \sigma\in\Omega(E). \tag{3.7}$$

Proof. – If $\sigma>0$, then in the defining integral,

$$\widehat{g}(\sigma)=\int_{\mathbf{R}}e^{-ix\sigma}g(x)\mathrm{d}x$$

we may replace the real axis by a closed curve γ in $\{\operatorname{Im} z\le 0\}$, which in the case when E is finite stays on the opposite side of the line L_E (introduced in the definition of the set $\mathcal{E}$ in the preceding section) from E, except in an arbitrarily small neighborhood of $-i$. We then get the entire extension from $]0,+\infty[$ by:

$$\widehat{g}(\sigma)=\int_{\gamma}e^{-iz\sigma}g(z)\mathrm{d}z, \tag{3.8}$$

so for every $N\in\mathbf{N}$,

$$|\widehat{g}(\sigma)|\le C_N\langle\sigma\rangle^{-N}e^{H_\gamma(\sigma)}, \tag{3.9}$$

where

$$H_\gamma(\sigma)=\sup_{z\in\gamma}\operatorname{Im}(z\sigma). \tag{3.10}$$

Now consider the situation in (a) and assume in order to fix the ideas that $E>0$. It is straight forward to study H_γ and we see that for every sufficiently small $\delta>0$, we can choose γ as above such that:

$$H_\gamma(\sigma)\le -\operatorname{Re}\sigma+\delta|\sigma|,\ \theta(E)-\delta\le\arg\sigma\le\theta(E)+\delta, \tag{3.11}$$

$$H_\gamma(\sigma)\le -\frac{1}{E}\operatorname{Im}\sigma,\ \text{when } 0\le\arg\sigma\le\theta(E)-\delta. \tag{3.12}$$

Note that $\operatorname{Re}\sigma=\frac{1}{E}\operatorname{Im}\sigma$ when $\arg\sigma=\theta(E)$. From this, we get part (a).

For part (b), we may assume for instance that $E=+\infty$. Then L_E is the imaginary axis, and we can choose γ confined to the intersection of the lower and the left half-planes except in a small neighborhood of $-i$. Then there exists $\epsilon_0>0$, such that for any small $\delta>0$, we can choose γ such that (3.11) holds and

$$H_\gamma(\sigma)\le -\epsilon_0\operatorname{Im}\sigma \text{ when } 0\le\arg\sigma\le\theta(E)-\delta. \tag{3.13}$$

Part (b) follows.

In order to get part (c), we use the decomposition (2.25) and (3.3) as well as the fact that, if we represent $\widehat{r}_\epsilon$ as in (3.8), then the contour can be pushed across $-i$ and consequently,

$$|\widehat{r}_\epsilon(\sigma)|\le\mathcal{O}(\epsilon)e^{-\operatorname{Re}\sigma-\delta|\sigma|}\text{ in }\Omega(E), \tag{3.14}$$

if $\delta>0$ and $\Omega(E)$ are small enough. $\square$

In the various integrals involving the density (3.2), we want to replace the integration variables $x \in (\mathbf{R}^2)^\Lambda$, by $x = e^{i\alpha/2}y$, with $y \in (\mathbf{R}^2)^\Lambda$ and $\alpha = \theta(E)$. As mentioned earlier in sect. 2, the integrand in (3.1) is a sum of terms of the form

$$f_{j_1\cdots j_n,k_1\cdots k_n}(x)\xi_{j_1}\cdots\xi_{j_n}\eta_{k_1}\cdots\eta_{k_n} \qquad (n \le |\Lambda|), \tag{3.15}$$

which is a polynomial in ξ, η and where the f's (coefficients) are holomorphic functions in x. For the purpose of change of contours in x, we can view ξ, η as mere "parameters". (See appendix A. for a more formal presentation of this simple fact.) The change of contours can be justified by means of the Stokes' formula, if we can show that the coefficients f decay fast enough on all the intermediate contours $x = e^{i\alpha/2}y$, $y \in (\mathbf{R}^2)^\Lambda$, for $0 \le \alpha \le \theta(E)$, (where we assume for simplicity that $E > 0$).

Using (2.19), f is proportional to

$$e^{-i(\sum tx_j\cdot x_k - \sum Ex_j\cdot x_j)}\prod_j h_j(x_j\cdot x_j), \tag{3.16}$$

where $h_j = \widehat{g}$ or $h_j = \widehat{g}'$. Assume $E \in \mathcal{E}$. Using (3.5) and without uniformity w.r.t. Λ, we have that

$$|f| \le \mathcal{O}_N(1)\left(e^{t\sum_{|j-k|_1=1}\operatorname{Im}(x_j\cdot x_k)}\prod_j(\langle x_j\cdot x_j\rangle^{-N}e^{-\gamma(E+1/E)\operatorname{Im}(x_j\cdot x_j)})\right),$$

where $\mathcal{O}_N(1)$ also depends on t, E. Here $\operatorname{Im}(x_j\cdot x_k) = (\sin\alpha)y_j\cdot y_k$, and we get

$$|f| \le \mathcal{O}_N(1)\exp\left[(\sin\alpha)\left(t\|\Delta\| - \gamma\left(E + \frac{1}{E}\right)\right)\|y\|^2\right]\prod\langle y_j\rangle^{-2N}.$$

Since $E + 1/E \ge 2$, and since we can choose γ arbitrarily close to 1, this quantity is $\mathcal{O}_N(1)\prod_j\langle y_j\rangle^{-2N}$ uniformly in α, when

$$t\|\Delta\| < 2, \tag{3.17}$$

or when

$$0 \le t \le 1 \text{ and } E \text{ is large enough.} \tag{3.18}$$

Stokes' formula can now be applied and we obtain

$$\langle G(\mu,\nu;E+i0)\rangle = i\int x_\mu\cdot x_\nu\, e^{-i(\sum tX_j\cdot X_k - \sum EX_j\cdot X_j)}\prod_j\widehat{g}(X_j\cdot X_j)\prod \mathrm{d}^2X_j, \tag{3.19}$$

where $x \in (e^{i\theta(E)/2}\mathbf{R}^2)^\Lambda$ and ξ_j, η_j are the corresponding Grassmann algebra generators. (See (A.4) in appendix A.)

From (c) of Lemma 3.1, we notice that for every compact subset $\mathcal{E}' \subset \mathcal{E}$, there exists $\epsilon_0 > 0$, such that $k(\sigma) = k_\epsilon(\sigma) = \sigma + r_\epsilon(\sigma)$ is holomorphic with $r_\epsilon(\sigma) = \mathcal{O}(\epsilon)$ in some neighborhood of $e^{i\theta(\mathcal{E}')}[0,+\infty[$, which is conic near infinity. Let $m = |\Lambda|$. Integrating out the anticommutative variables, we get for $E \in \mathcal{E}'$:

$$\begin{aligned}
&\langle G(\mu,\nu;E+i0)\rangle &(3.20)\\
&= i\int_{e^{i\theta(E)/2}\mathbf{R}^{2m}} x_\mu\cdot x_\nu \det(iM) e^{-i\left((t\Delta-E)x\cdot x - i\sum k(x_j\cdot x_j)\right)} \frac{\mathrm{d}^{2m}x}{\pi^m}\\
&= \int_{e^{i\theta(E)/2}\mathbf{R}^{2m}} (M^{-1})_{\mu,\nu} \det(iM) e^{-i\left((t\Delta-E)x\cdot x - i\sum k(x_j\cdot x_j)\right)} \frac{\mathrm{d}^{2m}x}{\pi^m},
\end{aligned}$$

where $M(x) = t\Delta - i\mathrm{diag}\,(k'(x_j\cdot x_j)) - E$. (See appendix A.6 for the integration by parts leading to the second equality.) The corresponding normalized complex measure becomes:

$$i^m(\det M)e^{-i((t\Delta-E)x\cdot x - i\sum k(x_j\cdot x_j))}\frac{\mathrm{d}^{2m}x}{\pi^m}. \tag{3.21}$$

We now specify the domains where we shall work. For $\alpha,\beta>0$, we introduce the neighborhood $\Omega(E,\alpha,\beta)$ of the half line $e^{i\theta(E)}[0,+\infty[$ asymptotically conic near infinity, by

$$|\mathrm{Im}\,(e^{-i\theta(E)}\tau)| < \alpha \mathrm{Re}\,(e^{-i\theta(E)}\tau)+\beta. \tag{3.22}$$

Then with $\mathcal{E}'$, $\mathcal{E}$ as above, $\mathcal{E}'$ connected and with $\alpha,\beta>0$ small enough, we have

$$k(\tau) = \tau + r(\tau),\ r(\tau) = \mathcal{O}(\epsilon) \text{ in } \Omega(\mathcal{E}',\alpha,\beta), \tag{3.23}$$

where we have put $\Omega(\mathcal{E}',\alpha,\beta) = \cup_{E\in\mathcal{E}'}\Omega(E,\alpha,\beta)$.

After the rotation of variables,

$$x_j = e^{i\theta(E)/2}y_j, \tag{3.24}$$

the density (3.21) becomes,

$$|1+iE|^m \det\left(1+\widetilde{t}\Delta + \mathrm{diag}\,(\widetilde{r}'(y_j\cdot y_j))\right)e^{-|1+iE|(y\cdot y+\widetilde{t}\Delta y\cdot y+\sum \widetilde{r}(y_j\cdot y_j))}\frac{\mathrm{d}^{2m}y}{\pi^m}, \tag{3.25}$$

where

$$\widetilde{t} = \frac{it}{(1-iE)}, \tag{3.26}$$

$$\widetilde{r}(\tau) = \frac{1}{|1+iE|}r(e^{i\theta(E)}\tau), \tag{3.27}$$

so that

$$\widetilde{r}(\tau) = \mathcal{O}(\widetilde{\epsilon}) \text{ for } \tau\in\Omega(0,\alpha,\beta), \tag{3.28}$$

where

$$\widetilde{\epsilon} = \frac{\epsilon}{|1+iE|}. \tag{3.29}$$

4. Elimination of $t\Delta$: The deforming vector field

We consider the density (3.25) together with (3.26-29). From now on we drop the superscripts "$\sim$" and write "x" instead of "y", so that the exponential in (3.25) can be written as $e^{-|1+iE|Q_t(x)}$, where

$$Q_t(x) = x \cdot x + t\Delta x \cdot x + \sum r(x_j \cdot x_j). \tag{4.1}$$

We look for a complex change of variables $x = x_t(y)$, generated by a t-dependent vector field $v = v_t(x) \cdot \frac{\partial}{\partial x}$, holomorphic in t, x:

$$\frac{\partial}{\partial t} x_t(y) = v_t(x_t(y)),\ x_0(y) = y, \tag{4.2}$$

such that

$$Q_t(x_t(y)) = Q_0(y). \tag{4.3}$$

Differentiating this equation with respect to t, we get

$$\partial_t Q_t + \nabla_x Q_t \cdot v_t = 0. \tag{4.4}$$

Letting $m \times m$ matrices act on $\mathbf{C}^{2m}$ in the natural way, we have

$$\nabla_x Q_t = 2(I + t\Delta + \operatorname{diag}(r'(x_j \cdot x_j)))x. \tag{4.5}$$

Note the appearence of the same matrix as in the determinant in (3.25).

Looking for $v = v_t$ of the form $v(x) = B(x)x$ where B is a (t-dependent) $m \times m$ matrix, and using that $\partial_t Q_t(x) = \Delta x \cdot x$, (4.4) becomes

$$-\langle \Delta x, x\rangle = 2\langle (I + t\Delta + \operatorname{diag}(r'(x_j \cdot x_j)))x, B(x)x\rangle, \tag{4.6}$$

and it suffices to find $B(x)$ such that

$$-\Delta = {}^tB(x) \circ (I + t\Delta + \operatorname{diag}(r'(x_j \cdot x_j))) + (I + t\Delta + \operatorname{diag}(r'(x_j \cdot x_j))) \circ B(x). \tag{4.7}$$

We shall take $B = B_t(x_1 \cdot x_1, \ldots, x_m \cdot x_m)$. (Note that this gives more choice for B then requiring ${}^tB(x) \circ (I + t\Delta + \operatorname{diag}(r'(x_j \cdot x_j))) = \Delta/2$.)

A possible choice would be $B(x) = -\frac{1}{2}(I + t\Delta + \operatorname{diag}(r'(x_j \cdot x_j)))^{-1}\Delta$, but it turns out that the corresponding vector field is not sufficiently small in some components, and that we cannot exclude that the corresponding flow will take us out of the region where r is well-defined. A better vector field can be constructed by means of a certain cut-off function, and before doing so, we specify in which region in $(\mathbf{C}^2)^\Lambda$, we want to work.

For $a \in]0,1[$, $b > 0$; let $V(a,b) \subset \mathbf{C}^2_{x_j}$ be the neighborhood of $\mathbf{R}^2$, given by

$$(\operatorname{Im} x_j)^2 \le a(\operatorname{Re} x_j)^2 + b, \tag{4.8}$$

where $(\operatorname{Im} x_j)^2 = (\operatorname{Im} x_{j,1})^2 + (\operatorname{Im} x_{j,2})^2$ and similarly we define $(\operatorname{Re} x_j)^2$. From simple estimates, we see that the map $\mathbf{C}^2 \ni x_j \mapsto x_j \cdot x_j \in \mathbf{C}$, takes $V(a,b)$ into $\Omega(0,\alpha,\beta)$, if

$$\alpha = \frac{2\sqrt{a}}{1-a},\ \beta = \frac{2\sqrt{a}b}{1-a} + \frac{b}{\sqrt{a}}, \tag{4.9}$$

and we can have α, β as small as we like by taking a, b, $b/\sqrt{a}$ sufficiently small.

We assume that a, b, $b/\sqrt{a}$, α, β in (4.9) are small enough, so that

$$\tau \in \Omega(0,\alpha,\beta) \Rightarrow |1+\tau| \geq \frac{1}{2}(1+|\tau|),\ |\arg(1+\tau)| \leq \frac{\pi}{2}, \tag{4.10}$$

$$x_j \in V(a,b) \Rightarrow |1+x_j \cdot x_j| \geq \frac{1}{2}(1+|x_j|^2). \tag{4.11}$$

For $x_j \in V(a,b)$ we can define in a natural way $\langle x_j \rangle = \sqrt{1+x_j \cdot x_j}$, and combining (4.10), (4.11), we see that for $x_j \in V(a,b)$:

$$\frac{1}{\sqrt{2}}\langle |x_j| \rangle \leq |\langle x_j \rangle| \leq \langle |x_j| \rangle,\ |\arg\langle x_j \rangle| \leq \frac{\pi}{4}. \tag{4.12}$$

Here $\langle |x_j| \rangle = \sqrt{1+|x_j|^2}$ is of the same order of magnitude as $1+|x_j|$. It follows from (4.12) that for $x_j, x_k \in V(a,b)$:

$$|\langle x_j \rangle + \langle x_k \rangle| \geq \frac{1}{2}(|\langle x_j \rangle| + |\langle x_k \rangle|). \tag{4.13}$$

Put

$$\chi(t,s) = \frac{t}{t+s}, \tag{4.14}$$

$$\chi_{i,j}(x) = \chi(\langle x_j \rangle, \langle x_k \rangle),\ x_j, x_k \in V(a,b). \tag{4.15}$$

Notice that $\chi_{j,k} + \chi_{k,j} = 1$ and that

$$|\chi_{j,k}(x)| \leq \frac{2|\langle x_j \rangle|}{|\langle x_j \rangle| + |\langle x_k \rangle|} \leq 2, \tag{4.16}$$

by (4.13).

Let $\chi(x)$ denote the $m \times m$ matrix $(\chi_{i,j}(x))_{1 \leq i,j \leq m}$ and let $*$ denote the operation of elementwise multiplication of $m \times m$-matrices: $(a * b)_{j,k} = a_{j,k} b_{j,k}$. We look for a solution $B(x)$ of (4.7) of the form

$$B(x) = \chi(x) * A(x),\ x \in V(a,b)^m, \tag{4.17}$$

with $A(x)$ symmetric. Then ${}^t(\chi * A) = {}^t\chi * A = A * {}^t\chi$, and (4.7) becomes

$$\begin{aligned} -\Delta = & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.18) \\ & (A * {}^t\chi) \circ (I + t\Delta + \operatorname{diag}(r'(x_j \cdot x_j))) + (I + t\Delta + \operatorname{diag} r'(x_j \cdot x_j))) \circ (\chi * A) \\ & = \mathcal{D}(x)(A) + t\mathcal{R}(x)(A),\ (A = A(x)), \end{aligned}$$

where

$$\mathcal{D}(x)(A) := (A * {}^t\chi) \circ D(x) + D(x) \circ (\chi * A),\ D(x) := I + \operatorname{diag}(r'(x_j \cdot x_j)), \tag{4.19}$$

$$\mathcal{R}(x)(A) := (A * {}^t\chi) \circ \Delta + \Delta \circ (\chi * A). \tag{4.20}$$

Write $D = D(x) = \operatorname{diag}(d_j(x))$. On the level of matrix elements, $\mathcal{D}(x)$ is the map

$$a_{j,k} \mapsto (d_j \chi_{j,k} + d_k \chi_{k,j}) a_{j,k}, \tag{4.21}$$

and we contemplate the multiplier:

$$d_j \chi_{j,k} + d_k \chi_{k,j} = 1 + \chi_{j,k} r'(x_j \cdot x_j) + \chi_{k,j} r'(x_k \cdot x_k). \tag{4.22}$$

Combining (3.23) (where the superscripts have been dropped), with the Cauchy inequality, we see after a slight decrease of a, b, α, β, that

$$|r'(x_j \cdot x_j)| \leq C\epsilon, \; x_j \in V(a,b), \tag{4.23}$$

where $C = C_{(4.23)}$ depends on a, b, α, β and on how much we decreased $V(a,b)$. Using this in (4.22) with (4.16), we get

$$|d_j \chi_{j,k} + d_k \chi_{k,j} - 1| \leq 4C_{(4.23)}\epsilon. \tag{4.24}$$

Clearly (4.24) imples the invertibility of the map $\mathcal{D}(x)$ and we shall introduce weighted ℓ^∞-norms on the $m \times m$-matrices, for which (4.18) can be solved by a perturbation argument. Let $\rho : \Lambda \times \Lambda \to \mathbf{R}$ be a symmetric function: $\rho(j,k) = \rho(k,j)$, satisfying

$$|\rho(j_1,k_1) - \rho(j_2,k_2)| \leq \delta(|j_1 - j_2|_1 + |k_1 - k_2|_1), \tag{4.25}$$

for some $\delta > 0$, where $|\cdot|_1 = |\cdot|_{\ell^1}$ is the ℓ^1 norm in $\mathbf{Z}^d$. The smallest possible δ will be denoted by $\|\rho\|_{\mathrm{Lip}}$. If $B = (b_{j,k})$ is an $m \times m$-matrix, we put

$$\|B\|_{\ell_\rho^\infty} = \max_{j,k} e^{\rho(j,k)} |b_{j,k}|. \tag{4.26}$$

Then according to (4.16):

$$\|\chi * A\|_{\ell_\rho^\infty}, \; \|A * {}^t\chi\|_{\ell_\rho^\infty} \leq 2\|A\|_{\ell_\rho^\infty}, \tag{4.27}$$

and (4.24) implies that

$$\|\mathcal{D}(x)\|_{\mathcal{L}(\ell_\rho^\infty, \ell_\rho^\infty)} \leq 1 + 4C_{(4.23)}\epsilon, \; \|\mathcal{D}(x)^{-1}\|_{\mathcal{L}(\ell_\rho^\infty, \ell_\rho^\infty)} \leq \frac{1}{1 - 4C_{(4.23)}\epsilon}. \tag{4.28}$$

In order to estimate the norm of $\mathcal{R}(x)$, write

$$(\Delta \circ B)_{j,k} = \sum_{|j-\ell|_1 = 1} b_{\ell,k},$$

$$e^{\rho(j,k)} (\Delta \circ B)_{j,k} = \sum_{|j-\ell|_1 = 1} e^{\rho(j,k) - \rho(\ell,k)} e^{\rho(\ell,k)} b_{\ell,k},$$

and conclude that

$$\|\Delta \circ B\|_{\ell_\rho^\infty} \leq 2d e^{\|\rho\|_{\mathrm{Lip}}} \|B\|_{\ell_\rho^\infty}, \tag{4.29}$$

where we recall that d is the dimension of the lattice. Similarly,

$$||B \circ \Delta||_{\ell^\infty_\rho} \leq 2de^{||\rho||_{\mathrm{Lip}}}||B||_{\ell^\infty_\rho}. \tag{4.30}$$

Combining this with (4.27), we get

$$||\mathcal{R}(x)||_{\mathcal{L}(\ell^\infty_\rho, \ell^\infty_\rho)} \leq 8de^{||\rho||_{\mathrm{Lip}}}. \tag{4.31}$$

Write (4.18):

$$-\Delta = \mathcal{D} \circ (I + t\mathcal{D}^{-1} \circ \mathcal{R})(A). \tag{4.32}$$

Assume from now on,

$$4C_{(4.23)}\epsilon \leq \frac{1}{2},\ 32|t|de \leq 1, \tag{4.33}$$

and choose ρ with

$$32|t|de^{||\rho||_{\mathrm{Lip}}} \leq 1. \tag{4.34}$$

Then $||\mathcal{D}^{-1}||_{\mathcal{L}(\ell^\infty_\rho, \ell^\infty_\rho)} \leq 2$, $||t\mathcal{D}^{-1}\mathcal{R}||_{\mathcal{L}(\ell^\infty_\rho, \ell^\infty_\rho)} \leq \frac{1}{2}$, and (4.32) has a unique solution $A = A(x)$, satisfying

$$||A||_{\ell^\infty_\rho} \leq 4||\Delta||_{\ell^\infty_\rho}. \tag{4.35}$$

Naturally, we may replace Δ in (4.32), (4.35) by any symmetric matrix.

Below we sum up the discussion of the existence of a deforming vector field:

PROPOSITION 4.1.

(i) *For $\alpha, \beta > 0$, define $\Omega(0, \alpha, \beta) \subset \mathbf{C}$ as in* (3.22) *to be: If $\tau \in \Omega(0, \alpha, \beta) \subset \mathbf{C}$, then*

$$|\mathrm{Im}\,\tau| < \alpha \mathrm{Re}\,\tau + \beta.$$

(ii) *For $a \in]0, 1[$, $b > 0$, define $V(a, b) \subset \mathbf{C}^2$ as in* (4.8) *to be: If $x_j \in V(a, b) \subset \mathbf{C}^2$, then*

$$(\mathrm{Im}\,x_j)^2 \leq a(\mathrm{Re}\,x_j)^2 + b.$$

(iii) *For α, β, a, b satisfying* (4.9)*, $\mathbf{C}^2 \ni x_j \mapsto x_j \cdot x_j \in \mathbf{C}$, takes $V(a, b)$ into $\Omega(0, \alpha, \beta)$. Let $V(a, b)$ be sufficiently small, so that* (4.10), (4.11) *hold.*

(iv) *Assume*

$$|r'(x_j \cdot x_j)| \leq C\epsilon,\ x_j \in V(a, b),$$
$$|r(x_j \cdot x_j)| \leq C\epsilon,\ x_j \in V(a, b).$$

(as in (3.28), (4.23) *with the tilde dropped), where ϵ satisfies* (4.33).

Then there is a holomorphic vector field $v = v_t(x) \cdot \frac{\partial}{\partial x}$, defined for $t \in \mathbf{C}$, $|t| < 1/(32\,de)$, $x \in V(a, b)^m$, which satisfies (4.4). $v(x)$ is of the form $v(x) = B(x)x = \chi(x) * A(x)x$, where $\chi(x)$ is as defined in (4.14), (4.15) and $A(x)$ is the unique symmetric $m \times m$-matrix satisfying (4.18). If ρ is such that

$$32|t|de^{||\rho||_{\mathrm{Lip}}} \leq 1$$

as in (4.34), then

$$\|A\|_{\ell^\infty_\rho} \le 4\|\Delta\|_{\ell^\infty_\rho}$$

as in (4.35).

In the remainder of this section, we shall derive various estimates on A and v under the assumptions of Proposition 4.1. According to (4.33), (4.34), a possible choice of ρ in (4.35) is $\rho(j,k) = |j-k|_1$. (4.35) then gives

$$|a_{j,k}(x)| \le 4e^{1-|j-k|_1}, \tag{4.36}$$

which implies that

$$\|A(x)\|_{\mathcal{L}(\ell^p,\ell^p)} \le 4e\Big(\frac{e+1}{e-1}\Big)^d, \tag{4.37}$$

for every $p \in [1,\infty]$.

We apply this with $p=\infty$ to the expression

$$v_j(x) = \sum_k a_{j,k}(x)\chi_{j,k}(x)x_k, \tag{4.38}$$

together with the estimate which follows from (4.16), (4.12):

$$\begin{aligned} |\chi_{j,k}(x)x_k| &\le \frac{2|\langle x_j\rangle|\,|x_k|}{|\langle x_j\rangle| + |\langle x_k\rangle|} \le \frac{2|\langle x_j\rangle||\langle|x_k|\rangle}{|\langle x_j\rangle| + |\langle x_k\rangle|} \\ &\le 2\sqrt{2}\frac{|\langle x_j\rangle|\,|\langle x_k\rangle|}{|\langle x_j\rangle| + |\langle x_k\rangle|} \le 2\sqrt{2}|\langle x_j\rangle|, \end{aligned} \tag{4.39}$$

and conclude that

$$|v_j(x)| \le 2\sqrt{2}|\langle x_j\rangle| \sum_k |a_{j,k}(x)| \le 8\sqrt{2}e\Big(\frac{e+1}{e-1}\Big)^d |\langle x_j\rangle|. \tag{4.40}$$

This estimate implies that if $0 < a' < a$, $0 < b' < b$, $y \in V(a',b')^m$, then for $|t|$ small enough depending on a, b, a', b', d, we have $x_t(y) \in V(a,b)^m$. It was in order to have this property that we introduced the "cutoffs" $\chi_{i,j}$ in (4.14), (4.15).

We now go on to estimate higher derivatives of A and v in a systematic way. The estimates are to be summarized in Lemmas 4.2, 4.3. We use (4.18). We first estimate derivatives of χ.

As can be seen from the Cauchy inequalities, we have

$$|\partial_{x_j}^\alpha \partial_{x_k}^\beta \chi(\langle x_j\rangle, \langle x_k\rangle)| \le C_{a,b,\alpha,\beta}|\langle x_j\rangle|^{-|\alpha|}|\langle x_k\rangle|^{-|\beta|}, \tag{4.41}$$

where we use standard multi-index notation, so that

$$\partial_{x_j}^\alpha = \partial_{x_{j,1}}^{\alpha_1}\partial_{x_{j,2}}^{\alpha_2},\ |\alpha| = \alpha_1+\alpha_2,\ \alpha = (\alpha_1,\alpha_2) \in \mathbf{N}^2,\ x_j = (x_{j,1}, x_{j,2}) \in V(a,b).$$

For $j \in \Lambda$, let Π_j be the orthogonal projection onto the corresponding copy of $\mathbf{C}^2$ in $(\mathbf{C}^2)^\Lambda$. When differentiating the matrix $\chi = (\chi(\langle x_\nu\rangle, \langle x_\mu\rangle))$ with respect to x_j, we see that we get zeros except on the j:th line or on the j:th column. Hence (with constants depending on α, a, b):

$$\partial_{x_j}^{\alpha}\chi = \Pi_j \circ \mathcal{O}(\langle x_j\rangle^{-|\alpha|}) + \mathcal{O}(\langle x_j\rangle^{-|\alpha|}) \circ \Pi_j, \tag{4.42}$$

where the $\mathcal{O}$'s refer to the ℓ^∞-norms for matrices. For $k \neq j$, $\alpha \neq 0 \neq \beta$, we get

$$\partial_{x_j}^{\alpha}\partial_{x_k}^{\beta}\chi = \Pi_j \circ \mathcal{O}(\langle x_j\rangle^{-|\alpha|}\langle x_k\rangle^{-|\beta|}) \circ \Pi_k + \Pi_k \circ \mathcal{O}(\langle x_j\rangle^{-|\alpha|}\langle x_k\rangle^{-|\beta|}) \circ \Pi_j. \tag{4.43}$$

For $k \neq j \neq \ell \neq k$, $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$, we have $\partial_{x_k}^{\alpha}\partial_{x_j}^{\beta}\partial_{x_\ell}^{\gamma}\chi = 0$.

For $D(x)$ in (4.19), we can also use the Cauchy inequalities, and obtain after a slight decrease of a, b in $V(a,b)$:

$$\partial_{x_j}^{\alpha}D(x) = \Pi_j \circ \mathcal{O}(\epsilon\langle x_j\rangle^{-|\alpha|}) \circ \Pi_j,\ \alpha \neq 0, \tag{4.44}$$

$$\partial_{x_j}^{\alpha}\partial_{x_k}^{\beta}D(x) = 0,\ \text{when } j \neq k,\ \alpha \neq 0 \neq \beta. \tag{4.45}$$

From (4.42)–(4.45), we get

$$\partial_{x_j}^{\alpha}({}^t\chi(x) \circ D(x)) = \Pi_j \circ \mathcal{O}(\langle x_j\rangle^{-|\alpha|}) + \mathcal{O}(\langle x_j\rangle^{-|\alpha|}) \circ \Pi_j,\ \alpha \neq 0, \tag{4.46}$$

$$\begin{aligned}&\partial_{x_j}^{\alpha}\partial_{x_k}^{\beta}({}^t\chi(x) \circ D(x)) = \\ &\Pi_j \circ \mathcal{O}(\langle x_j\rangle^{-|\alpha|}\langle x_k\rangle^{-|\beta|}) \circ \Pi_k + \Pi_k \circ \mathcal{O}(\langle x_j\rangle^{-|\alpha|}\langle x_k\rangle^{-|\beta|}) \circ \Pi_j,\ j \neq k,\ \alpha \neq 0 \neq \beta,\end{aligned} \tag{4.47}$$

$$\partial_{x_j}^{\alpha}\partial_{x_k}^{\beta}\partial_{x_\ell}^{\gamma}({}^t\chi(x) \circ D(x)) = 0,\ j \neq k \neq \ell \neq j,\ \alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0. \tag{4.48}$$

We can now study the derivatives of $\mathcal{D}(x) + t\mathcal{R}(x)$ in (4.18). If C is an $m \times m$-matrix, let $\mathcal{S}(C) = C + {}^tC$. Then,

$$\mathcal{D}(x)(A) = \mathcal{S}((A * {}^t\chi(x)) \circ D(x)) = \mathcal{S}(A * ({}^t\chi(x) \circ D(x))), \tag{4.49}$$

$$\mathcal{R}(x)(A) = \mathcal{S}((A * {}^t\chi(x)) \circ \Delta). \tag{4.50}$$

If ρ_1, ρ_2 are symmetric weights, then for all symmetric A's in the three cases in (4.46)–(4.48):

$$\begin{aligned}\|(\partial_{x_j}^{\alpha}\mathcal{D})(x)(A)\|_{\ell_{\rho_2}^\infty} &= \mathcal{O}(1)\langle x_j\rangle^{-|\alpha|}\|A\|_{\ell_{\rho_1}^\infty},\ \text{if} \\ &\rho_2 \leq \rho_1 \text{ on } L(j) := (\{j\} \times \Lambda) \cup (\Lambda \times \{j\}),\end{aligned} \tag{4.51}$$

$$\begin{aligned}\|(\partial_{x_j}^{\alpha}\partial_{x_k}^{\beta}\mathcal{D})(x)(A)\|_{\ell_{\rho_2}^\infty} &= \mathcal{O}(1)\langle x_j\rangle^{-|\alpha|}\langle x_k\rangle^{-|\beta|}\|A\|_{\ell_{\rho_1}^\infty},\ \text{if} \\ &\rho_2 \leq \rho_1 \text{ on } \{(j,k),\ (k,j)\} = L(j) \cap L(k),\end{aligned} \tag{4.52}$$

$$(\partial_{x_j}^{\alpha}\partial_{x_k}^{\beta}\partial_{x_\ell}^{\gamma}\mathcal{D})(x)(A) = 0. \tag{4.53}$$

We have the same estimates for the map $A \mapsto A * {}^t\chi(x)$, and if we assume in addition that $\|\rho_1\|_{\mathrm{Lip}}$, $\|\rho_2\|_{\mathrm{Lip}} \leq r$, then

$$\|(\partial_{x_j}^{\alpha}\mathcal{R})(x)(A)\|_{\ell^\infty_{\rho_2}} = \mathcal{O}(e^r)\langle x_j\rangle^{-|\alpha|}\|A\|_{\ell^\infty_{\rho_1}},\ \text{if } \alpha \neq 0,\ \text{and } \rho_2 \leq \rho_1 \text{ on } L(j), \tag{4.54}$$

$$\begin{gathered}\|(\partial_{x_j}^{\alpha}\partial_{x_k}^{\beta}\mathcal{R})(x)(A)\|_{\ell^\infty_{\rho_2}} = \mathcal{O}(e^r)\langle x_j\rangle^{-|\alpha|}\langle x_k\rangle^{-|\beta|},\ j \neq k,\ \alpha \neq 0 \neq \beta,\\ \text{if } \rho_2 \leq \rho_1 \text{ on } L(j) \cap L(k),\end{gathered} \tag{4.55}$$

$$\left(\partial_{x_j}^{\alpha}\partial_{x_k}^{\beta}\partial_{x_\ell}^{\gamma}\mathcal{R}(x)\right)(A) = 0,\ j \neq k \neq \ell,\ \alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0. \tag{4.56}$$

If we assume,

$$|t|e^r = \mathcal{O}(1), \tag{4.57}$$

then (4.51)–(4.53) are valid with $\mathcal{D}$ replaced by $\mathcal{E} := \mathcal{D} + t\mathcal{R}$, but now with the restriction $\|\rho_1\|_{\mathrm{Lip}}$, $\|\rho_2\|_{\mathrm{Lip}} \leq r$. For a given such ρ_1, the optimal choice of ρ_2 in (4.51) (with $\mathcal{D}$ replaced by $\mathcal{E}$) is

$$\rho_2(a) = \min_{b \in L(j)} \rho_1(b) + r|a-b|_1.$$

Similarly, the optimal choice of ρ_2 in (4.52) (with $\mathcal{D}$ replaced by $\mathcal{E}$) is

$$\min_{b \in L(j)\cap L(k)}(\rho_1(b) + r|a-b|_1) \geq \min_{b_1 \in L(j),\, b_2 \in L(k)}(\rho_1(b_1) + r|b_1 - b_2|_1 + r|b_2 - a|_1) =: \rho_2(a).$$

We shall now differentiate the equation (4.18), that we write as

$$\mathcal{E}_t(x)(A) = \Delta. \tag{4.58}$$

Let $r \geq 1$ satisfy

$$32\,|t|de^r \leq 1, \tag{4.59}$$

so that according to (4.35):

$$\|A\|_{\ell^\infty_{r|\cdot - \cdot\cdot|_1}} \leq 4 \cdot 2de^r \tag{4.60}$$

We use the remark after (4.35), on the differentiated equation, with $j_1, \dots, j_N$ pairwise distinct and with $\alpha_j \neq 0$:

$$\begin{gathered}\mathcal{E}_t(x)(\partial_{x_{j_1}}^{\alpha_1}\dots\partial_{x_{j_N}}^{\alpha_N}A) = \text{ a linear combination of terms}\\ (\partial_{x_{j_k}}^{\alpha_k'}\mathcal{E}_t(x))(\partial_{x_{j_1}}^{\alpha_1}\dots\partial_{x_{j_k}}^{\alpha_k-\alpha_k'}\dots\partial_{x_{j_N}}^{\alpha_N}A) \text{ and of terms}\\ (\partial_{x_{j_k}}^{\alpha_k'}\partial_{x_{j_\ell}}^{\alpha_\ell'}\mathcal{E}_t(x))(\partial_{x_{j_1}}^{\alpha_1}\dots\partial_{x_{j_k}}^{\alpha_k-\alpha_k'}\dots\partial_{x_{j_\ell}}^{\alpha_\ell-\alpha_\ell'}\dots\partial_{x_{j_N}}^{\alpha_N}A),\end{gathered} \tag{4.61}$$

with $0 < \alpha_k' \leq \alpha_k$ for the first kind of terms and with $k \neq \ell$, $0 < \alpha_k' \leq \alpha_k$, $0 < \alpha_\ell' \leq \alpha_\ell$ for the second kind.

Using the observation after (4.57) and an induction argument based on (4.61), we get

$$\|\partial_{x_{j_1}}^{\alpha_1}\dots\partial_{x_{j_N}}^{\alpha_N}A\|_{\ell_\rho^\infty}=\mathcal{O}(e^r)\langle x_{j_1}\rangle^{-|\alpha_1|}\dots\langle x_{j_N}\rangle^{-|\alpha_N|},\tag{4.62}$$

(where $\mathcal{O}(e^r)$ comes from $\|\Delta\|_{\ell_\rho^\infty}=\mathcal{O}(e^r)$,) when $j_1,\dots,j_N$ are distinct, $\alpha_1,\dots,\alpha_N\neq 0$ and

$$\rho(\mu,\nu)=r\min_{\pi\in\mathrm{Perm}\,(j_1,\dots,j_N)}\ \min_{\substack{b_N\in L(\pi(j_N))\\ \vdots\\ b_1\in L(\pi(j_1))\\ b_0\in\mathrm{diag}\,(\Lambda\times\Lambda)}}\big(|(\mu,\nu)-b_N|_1+|b_N-b_{N-1}|_1+\dots+|b_1-b_0|_1\big).\tag{4.63}$$

Here $\mathrm{Perm}\,(j_1,\dots,j_N)$ denotes the group of permutations of $(j_1,\dots,j_N)$. For given π and $b_0,b_1,\dots,b_N$ as in (4.63), we write $b_k=(b_{k,1},b_{k,2})$, so that

$$\begin{aligned}&|(\mu,\nu)-b_N|_1+|b_N-b_{N-1}|_1+\dots+|b_1-b_0|_1=\\&\quad|\mu-b_{N,1}|_1+|b_{N,1}-b_{N-1,1}|_1+\dots\\&\quad+|b_{1,1}-b_{0,1}|_1+|b_{0,1}-b_{1,2}|_1+|b_{1,2}-b_{2,2}|_1+\dots+|b_{N-1,2}-b_{N,2}|_1+|b_{N,2}-\nu|_1.\end{aligned}$$

Here for each $k\geq 1$, one of $b_{k,1},b_{k,2}$ is equal to $\pi(j_k)$ while the other component "is free". $b_{1,1}=b_{1,2}$ is also free. Taking the infimum over the free components, we get

$$|\mu-\widetilde{\pi}(j_N)|_1+|\widetilde{\pi}(j_N)-\widetilde{\pi}(j_{N-1})|_1+\dots+|\widetilde{\pi}(j_1)-\nu|_1,$$

for some new permutation (which can be arbitrary, when varying π and the choice of free and unfree components). We then arrive at the simpler expression for ρ in (4.62):

$$\rho(\mu,\nu)=r\min_{\pi\in\mathrm{Perm}\,(1,\dots,N)}|\mu-j_{\pi(N)}|_1+|j_{\pi(N)}-j_{\pi(N-1)})|_1+\dots+|j_{\pi(1)}-\nu|_1.\tag{4.64}$$

We may say that ρ is r times the ℓ^1 distance from μ to ν, when passing through the points $j_1,\dots,j_N$ in the shortest possible fashion. With this description of ρ it is quite obvious that we can drop the assumption that $j_1,\dots,j_N$ are distinct in (4.62).

We have proved:

LEMMA 4.2. – *We make the assumptions of Proposition* 4.1 *and choose ρ as in* (4.64) *with* $32|t|de^r<1$. *Then*

$$\|\partial_{x_{j_1}}^{\alpha_1}\dots\partial_{x_{j_N}}^{\alpha_N}A\|_{\ell_\rho^\infty}=\mathcal{O}(e^r)\langle x_{j_1}\rangle^{-|\alpha_1|}\dots\langle x_{j_N}\rangle^{-|\alpha_N|}.$$

We now go on to estimate v. It is easy to get the corresponding estimates for the matrix B in (4.17). We start by sharpening (4.41) by using the middle bound in (4.16) and the Cauchy inequalities, to get

$$|\chi_{j,k}(x)|\leq 2\min\left(1,\frac{|\langle x_j\rangle|}{|\langle x_k\rangle|}\right),\tag{4.65}$$

$$|\partial_{x_j}^\alpha\partial_{x_k}^\beta\chi_{j,k}(x)|=\mathcal{O}(1)\min\left(1,\frac{|\langle x_j\rangle|}{|\langle x_k\rangle|}\right)\langle x_j\rangle^{-|\alpha|}\langle x_k\rangle^{-|\beta|}.\tag{4.66}$$

Combining this with (4.62), (4.60), we get

$$\partial^{\alpha_1}_{x_{j_1}} \dots \partial^{\alpha_N}_{x_{j_N}} b_{\mu,\nu}(x) = \mathcal{O}(e^r) \min\left(1, \frac{|\langle x_\mu\rangle|}{|\langle x_\nu\rangle|}\right) e^{-\rho(\mu,\nu)} \langle x_{j_1}\rangle^{-|\alpha_1|} \dots \langle x_{j_N}\rangle^{-|\alpha_N|}, \quad (4.67)$$

with ρ given by (4.64). (We always have the option of replacing r by a smaller value in (4.64), (4.67).)

Recall that we have already estimated the vector field v in (4.40). We now estimate the derivatives. For $|\alpha| = 1$, consider

$$\partial^\alpha_{x_k} v_j(x) = \partial^\alpha_{x_k}(\sum_\mu b_{j,\mu}(x) x_\mu) = b_{j,k}(x) + \sum_\mu \partial^\alpha_{x_k}(b_{j,\mu}(x)) x_\mu. \quad (4.68)$$

Here the first term can be estimated by means of (4.67) and we use (4.67) also for the last sum in (4.68):

$$\begin{aligned}\sum_\mu \partial^\alpha_{x_k}(b_{j,\mu}(x)) x_\mu &= \sum_\mu \mathcal{O}(e^r) \frac{\langle x_j\rangle}{\langle x_\mu\rangle} \cdot \frac{1}{\langle x_k\rangle} e^{-r(|j-k|_1 + |k-\mu|_1)} x_\mu \\ &= \mathcal{O}(1) \frac{\langle x_j\rangle}{\langle x_k\rangle} \sum_\mu e^r e^{-r(|j-k|_1+|k-\mu|_1)} = \mathcal{O}(1) \frac{\langle x_j\rangle}{\langle x_k\rangle} e^r e^{-r|j-k|_1},\end{aligned}$$

where in the last estimate, we first assume a strictly positive lower bound on r. Writing the Jacobian matrix $\frac{\partial v}{\partial x} = (\frac{\partial v_j}{\partial x_k})$, where $\frac{\partial v_j}{\partial x_k}$ is a 2×2-matrix, we obtain

$$\frac{\partial v_j}{\partial x_k} = \mathcal{O}(1) \frac{\langle x_j\rangle}{\langle x_k\rangle} e^r e^{-r|j-k|_1}. \quad (4.69)$$

It follows that if $\|\rho\|_{\mathrm{Lip}} \le \theta r$, where $\theta \in [0, 1[$ is some fixed constant, then

$$\| \frac{1}{\langle x\rangle} \circ \frac{\partial v}{\partial x} \circ \langle x\rangle \|_{\mathcal{L}(\ell^p_\rho, \ell^p_\rho)} = \mathcal{O}(e^r), \quad (4.70)$$

for $1 \le p \le \infty$. Here we write $\langle x\rangle = \mathrm{diag}\,(\langle x_j\rangle)$.

We next generalize (4.69) to higher derivatives. Let $N \ge 1$ be fixed and let $k_1, \dots, k_N \in \Lambda$. With a slight abuse of notation, we have

$$\partial_{x_{k_1}} \dots \partial_{x_{k_N}} v_j = \sum_\mu (\partial_{x_{k_1}} \dots \partial_{x_{k_N}} b_{j,\mu}(x)) x_\mu + \sum_{\ell=1}^N \partial_{x_{k_1}} \dots \widehat{\partial_{x_{k_\ell}}} \dots \partial_{x_{k_N}} b_{j,k_\ell}(x), \quad (4.71)$$

where the hat indicates the absence of the corresponding factor. From (4.67), we get,

$$(\partial_{x_{k_1}} \dots \partial_{x_{k_N}} b_{j,\mu}(x)) x_\mu = \mathcal{O}(1) \frac{\langle x_j\rangle}{\langle x_{k_1}\rangle \dots \langle x_{k_N}\rangle} e^r e^{-\rho(j,\mu)}, \quad (4.72)$$

where

$$\rho(j,\mu) = \theta r \min_{\pi \in \mathrm{Perm}\,(1,\dots,N)} |j - k_{\pi(N)}|_1 + |k_{\pi(N)} - k_{\pi(N-1)}|_1 + \dots + |k_{\pi(1)} - \mu|_1. \quad (4.73)$$

It follows that the first term of the RHS in (4.71) is

$$\mathcal{O}(1)\frac{\langle x_j\rangle}{\langle x_{k_1}\rangle\dots\langle x_{k_N}\rangle}e^r e^{-\rho(j;k_1,\dots,k_N)}, \tag{4.74}$$

where

$$\rho(j;k_1,\dots,k_N) = \theta r \min_{\pi\in\mathrm{Perm}\,(1,\dots,N)} |j-k_{\pi(N)}|_1 + |k_{\pi(N)}-k_{\pi(N-1)}| + \dots + |k_{\pi(2)}-k_{\pi(1)}|_1. \tag{4.75}$$

Every term in the last sum in (4.71) is also of the form (4.74), so the same holds for $\partial_{x_{k_1}}\dots\partial_{x_{k_N}}v_j$. We did not assume $k_1,\dots,k_N$ to be distinct, and the resulting estimate can therefore be given the apparently more general form, which we state as:

LEMMA 4.3. – *We make the assumptions of Proposition 4.1 and choose ρ as in* (4.75), *with $0\le\theta<1$ fixed. Assume also that $32|t|de^r\le 1$. Then*

$$\partial^{\alpha_1}_{x_{k_1}}\dots\partial^{\alpha_N}_{x_{k_N}}\partial^{\beta_j}_{x_j}v_j = \mathcal{O}(1)\frac{\langle x_j\rangle^{1-|\beta_j|}}{\langle x_{k_1}\rangle^{|\alpha_1|}\dots\langle x_{k_N}\rangle^{|\alpha_N|}}e^r e^{-\rho(j;k_1,\dots,k_N)}, \tag{4.76}$$

when $|\alpha_1|,\dots,|\alpha_N|\ge 1$.

It follows that when $|\alpha_1|,\dots,|\alpha_N|\ge 1$:

$$\partial^{\alpha_1}_{x_{k_1}}\dots\partial^{\alpha_N}_{x_{k_N}}\operatorname{div}v = \mathcal{O}(1)\langle x_{k_1}\rangle^{-|\alpha_1|}\dots\langle x_{k_N}\rangle^{-|\alpha_N|}e^r e^{-\rho(k_1,\dots,k_N)}, \tag{4.77}$$

where

$$\rho(k_1,\dots,k_N) = \theta r \min_{\pi\in\mathrm{Perm}\,(1,\dots,N)}(|k_{\pi(N)}-k_{\pi(N-1)}| + \dots + |k_{\pi(2)}-k_{\pi(1)}|). \tag{4.78}$$

Note that there is no reason to expect some nice (i.e uniform in Λ) estimates for $\operatorname{div}v$ itself. We notice the special cases:

$$\partial^{\alpha}_{x_k}\operatorname{div}v = \mathcal{O}(1)\langle x_k\rangle^{-1},\ \text{when } |\alpha|=1, \tag{4.79}$$

$$\partial^{\alpha}_{x_j}\partial^{\beta}_{x_k}\operatorname{div}v = \mathcal{O}(1)\langle x_j\rangle^{-1}\langle x_k\rangle^{-1}e^r e^{-r|j-k|_1},\ \text{when } |\alpha|=|\beta|=1. \tag{4.80}$$

5. Elimination of $t\Delta$: The flow of the deforming vector field

In this section we shall study the flow of the vector field $v=v_t$ constructed in the preceding section. The constructions of that section extend to sufficiently small complex t, and we shall work here with complex t satisfying $|t|<T_0\le 2$, with

$$32T_0de^r\le 1. \tag{5.1}$$

Here r is some number ≥ 1. (5.1) ensures that Proposition 4.1 is at our disposal. Further we will work in a domain of the form $V(a,b)^m$ with $V(a,b)$ as in Proposition 4.1, and as we observed after (4.40), if $0<a'<a$, $0<b'<b$, then there exists $T_0>0$ depending only on a, b, a', b', d (but not on r in (5.1)), such that if $y \in V(a', b')^m$, $|t| < T_0$, then $x_t(y) \in V(a, b)^m$. Moreover, there is a constant $C > 0$ depending only on a, b, d, such that

$$\frac{1}{C} \leq \frac{|\langle (x_t(y))_j \rangle|}{|\langle y_j \rangle|} \leq C. \tag{5.2}$$

In order to estimate the differential and higher order derivatives (w.r.t. y) of $x(t, y) = x_t(y)$, we shall give a slightly weakened variant of (4.76). Introduce

$$d(j; k_1, \dots, k_N) = \min_{\pi \in \mathrm{Perm}(1,\dots,N)} (|j - k_{\pi(N)}| + |k_{\pi(N)} - k_{\pi(N-1)}| + \dots + |k_{\pi(2)} - k_{\pi(1)}|), \tag{5.3}$$

so that $\rho(j; k_1, \dots, k_N)$ in (4.76) is of the form $rd(j; k_1, \dots, k_N)$. Fix $\theta \in]0, 1[$. We claim that

$$\left\| \frac{1}{\langle x \rangle} \langle \nabla_x^N v, \tau_1 \otimes \dots \otimes \tau_N \rangle \right\|_{\ell_\rho^p} \leq C_N e^r \left\| \frac{1}{\langle x \rangle} \tau_1 \right\|_{\ell_{\rho_1}^{p_1}} \cdots \left\| \frac{1}{\langle x \rangle} \tau_N \right\|_{\ell_{\rho_N}^{p_N}}, \tag{5.4}$$

$$\tau_j \in (\mathbf{C}^2)^\Lambda, \text{ if } p, p_1, \dots, p_N \in [1, +\infty], \ \frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_N},$$

provided that the weights $\rho, \rho_1, \dots, \rho_N : \Lambda \to \mathbf{R}$ satisfy

$$\rho(j) \leq \theta rd(j; k_1, \dots, k_N) + \rho_1(k_1) + \dots + \rho_N(k_N), \ j, k_1, \dots, k_N \in \Lambda. \tag{5.5}$$

Here C_N is independent of the weights and the exponents and we recall the notations: $\langle x \rangle = \mathrm{diag}\,(\langle x_j \rangle)$, $1/\langle x \rangle = \langle x \rangle^{-1}$. $\nabla_x^N v$ is the symmetric tensor of the N:th order derivatives of v. To see (5.4), write with $\tau_\nu = (\tau_{\nu,1}, \dots, \tau_{\nu,m})$, $s = (s_1, \dots, s_m)$, and with a slight abuse of notation (since $\tau_{\nu,j}$, s_j are 2-vectors and not scalars):

$$\langle s, \langle \nabla_x^N v, \tau_1 \otimes \dots \otimes \tau_N \rangle \rangle = \sum_j \sum_{k_1} \dots \sum_{k_N} s_j (\partial_{x_{k_1}} \dots \partial_{x_{k_N}} v_j) \tau_{1,k_1} \dots \tau_{N,k_N} =$$

$$\sum_{j \in \Lambda} \sum_{k \in \Lambda^N} \mathcal{O}_N(1) e^r e^{\rho(j) - rd(j;k_1,\dots,k_N) - \rho_1(k_1) - \dots - \rho_N(k_N)} (\langle x_j \rangle s_j e^{-\rho(j)}) \times$$

$$\frac{e^{\rho_1(k_1)} \tau_{1,k_1}}{\langle x_{k_1} \rangle} \cdots \frac{e^{\rho_N(k_N)} \tau_{N,k_N}}{\langle x_{k_N} \rangle}.$$

Here the exponent is

$$\begin{aligned} -(1-\theta) rd(j; k_1, \dots, k_N) - (\theta rd(j; k_1, \dots, k_N) + \rho_1(k_1) + \dots + \rho_N(k_N) - \rho(j)) \\ \leq -(1-\theta) rd(j; k_1, \dots, k_N). \end{aligned}$$

It is easy to see that $\sum^{(\nu)} e^{-(1-\theta) rd(j;k_1,\dots,k_N)} = \mathcal{O}_N(1)$, $\nu = 0, \dots, N$, where $\sum^{(\nu)}$ denotes the sum over all the variables $j, k_1, \dots, \widehat{k_\nu}, \dots, k_N$ (with the exception of k_ν) and with the convention that $k_0 = j$. It follows that

$$\langle s, \langle \nabla_x^N v, \tau_1 \otimes \dots \otimes \tau_N \rangle \rangle = \mathcal{O}_N(1) e^r \|s\|_{\ell_{-\rho}^q} \|\tau_1\|_{\ell_{\rho_1}^{p_1}} \cdots \|\tau_N\|_{\ell_{\rho_N}^{p_N}},$$

for $q, p_1, \dots, p_N \in [1, +\infty]$ with $1 = 1/q + 1/p_1 + \dots + 1/p_N$, first in the case when precisely one of the $q, p_1, \dots, p_N$ is $= 1$ and the others $= +\infty$, then by interpolation in the general case. The last estimate is equivalent to (5.4), since $\ell_{-\rho}^q$ is the dual space to ℓ_ρ^p.

If $\rho_1, \ldots, \rho_N$ are given, then the optimal choice of ρ in (5.5) is given by

$$\begin{aligned}\rho(j) &= R_{\theta r,N}(\rho_1, \ldots, \rho_N)(j) := \\ &\inf_{k_1,\ldots,k_N \in \Lambda} \theta r d(j; k_1, \ldots, k_N) + \rho_1(k_1) + \ldots + \rho_N(k_N).\end{aligned} \tag{5.6}$$

PROPOSITION 5.1. – *Assume that*

$$\left\| \sum_{j \in K} \rho_j \right\|_{\mathrm{Lip}} \leq \theta r \textit{ for every } K \subset \{1, \ldots, N\}. \tag{5.7}$$

Then $R_{\theta r,N}(\rho_1, \ldots, \rho_N) = \rho_1 + \ldots + \rho_N$.

Proof. – It suffices to prove that $R_{\theta r,N}(\rho_1, \ldots, \rho_N) \geq \rho_1 + \ldots + \rho_N$, since the opposite inequality is obvious. We have the proposition in the case $N = 1$. Assume we have proved the proposition with N replaced by $N - 1$, for some $N \geq 2$. Let $\pi \in \mathrm{Perm}\,(1, 2, \ldots, N)$. Then if $\pi(N-1) = \nu$, $\pi(N) = \mu$:

$$\begin{aligned}&\theta r(|j - k_{\pi(1)}| + |k_{\pi(1)} - k_{\pi(2)}| + \ldots + |k_{\pi(N-1)} - k_{\pi(N)}|) + \rho_1(k_1) + \ldots + \rho_N(k_N) \\ &\geq \theta r(|j - k_{\pi(1)}| + \ldots + |k_{\pi(N-2)} - k_{\pi(N-1)}|) + \sum_{j \notin \{\nu,\mu\}} \rho_j(k_j) + (\rho_\nu + \rho_\mu)(k_\nu).\end{aligned}$$

Here $\{\pi(1), \ldots, \pi(N-1)\} = \{1, \ldots, \widehat{\mu}, \ldots, N\}$, so the last expression is

$$\geq R_{\theta r,N-1}(\rho_1, \ldots, \rho_\nu + \rho_\mu, \ldots, \widehat{\rho_\mu}, \ldots, \rho_N)(j) \geq (\rho_1 + \ldots + \rho_N)(j)$$

□

Now consider (4.2) and differentiate once w.r.t. $y \in V(a', b')^m$:

$$\begin{aligned}&\frac{\partial}{\partial t}\langle \nabla_y x(t, y), \tau_1 \rangle = \langle \nabla_x v_t(x(t, y)), \langle \nabla_y x(t, y), \tau_1 \rangle \rangle \\ &\langle \nabla_y x(0, y), \tau_1 \rangle = \tau_1.\end{aligned} \tag{5.8}$$

Here $\tau_1 \in (\mathbf{C}^2)^\Lambda$ is independent of t. If $\|\rho_1\|_{\mathrm{Lip}} \leq \theta r$, then we get from this (5.2), (5.4) and Proposition 5.1, that for $p \in [1, \infty]$

$$\frac{\partial}{\partial t}\left\| \frac{1}{\langle y \rangle}\langle \nabla_y x(t, y), \tau_1 \rangle \right\|_{\ell^p_{\rho_1}} \leq \mathcal{O}(1) e^r \left\| \frac{1}{\langle y \rangle}\langle \nabla_y x(t, y), \tau_1 \rangle \right\|_{\ell^p_{\rho_1}}. \tag{5.9}$$

Here, we also used that if $t \mapsto z(t) \in B$ is a C^1-curve in a Banach space B, then $t \mapsto \|z(t)\|_B$ is Lipschitz and the a.e. defined derivative satisfies

$$\left| \frac{d}{dt}\|z(t)\|_B \right| \leq \left\| \frac{\mathrm{d}z(t)}{\mathrm{d}t} \right\|_B.$$

Also recall that for Lipschitz functions, we have

$$f(t) - f(s) = \int_s^t \frac{\partial f}{\partial \sigma}(\sigma) \mathrm{d}\sigma.$$

Combining the differential inequality (5.9) and the initial condition in (5.8), we get

$$\left\|\frac{1}{\langle y\rangle}\langle\nabla_y x(t,y),\tau_1\rangle\right\|_{\ell^p_{\rho_1}}\leq e^{\mathcal{O}(e^r|t|)}\left\|\frac{1}{\langle y\rangle}\tau_1\right\|_{\ell^p_{\rho_1}}. \tag{5.10}$$

This can be reformulated as

$$\left\|\frac{1}{\langle y\rangle}\circ\frac{\partial x(t,y)}{\partial y}\circ\langle y\rangle\right\|_{\mathcal{L}(\ell^p_{\rho_1},\ell^p_{\rho_1})}\leq e^{\mathcal{O}(e^r|t|)}. \tag{5.11}$$

(Compare with (4.70).)

Considering also (5.8) with initial condition at some fixed t instead of at $t=0$, we get an estimate for the inverse of the differential in the same way:

$$\left\|\frac{1}{\langle y\rangle}\circ\left(\frac{\partial x(t,y)}{\partial y}\right)^{-1}\circ\langle y\rangle\right\|_{\mathcal{L}(\ell^p_{\rho_1},\ell^p_{\rho_1})}\leq e^{\mathcal{O}(e^r|t|)}. \tag{5.12}$$

Differentiating (5.8) $N-1$ times, we get for $N\geq 2$:

$$\left.\begin{array}{l}\frac{\partial}{\partial t}\langle\nabla_y^N x(t,y),\tau_1\otimes\ldots\otimes\tau_N\rangle\\ \qquad\qquad -\langle(\nabla_x v_t)(x(t,y)),\langle\nabla_y^N x(t,y),\tau_1\otimes\ldots\otimes\tau_N\rangle\rangle=\\ \text{a linear combination of terms of the type}\\ \qquad\langle(\nabla_x^L v_t)(x(t,y)),\langle\nabla_y^{\sharp K_1}x,\bigotimes_{k\in K_1}\tau_k\rangle\otimes\ldots\otimes\langle\nabla_y^{\sharp K_L}x,\bigotimes_{k\in K_L}\tau_k\rangle\rangle,\\ \text{with } L\geq 2,\ K_1\cup\ldots\cup K_L=\{1,\ldots,N\},\ K_\nu\cap K_\mu=\emptyset\text{ for }\nu\neq\mu,\ K_\nu\neq\emptyset.\end{array}\right\} \tag{5.13}$$

The initial condition is now:

$$\langle\nabla_y x(0,y),\tau_j\rangle=\tau_j,\ \nabla_y^M x(0,y)=0\text{ for }M\geq 2. \tag{5.14}$$

Let $\rho_1,\ldots,\rho_N:\Lambda\to\mathbf{R}$ be weights satisfying (5.7). Using (5.4), Proposition 5.1, (5.13), we get by induction over N:

$$\left\|\frac{1}{\langle y\rangle}\langle\nabla_y^N x(t,y),\tau_1\otimes\ldots\otimes\tau_N\rangle\right\|_{\ell^p_{\rho_1+\cdots+\rho_N}}\leq C_N e^r|t|\prod_1^N\left\|\frac{1}{\langle y\rangle}\tau_j\right\|_{\ell^{p_j}_{\rho_j}}, \tag{5.15}$$

for $N\geq 2$ and for weights $\rho_1,\ldots,\rho_N:\Lambda\to\mathbf{R}$ satisfying (5.7) and for exponents $p_1,\ldots,p_N,p\in[1,\infty]$ satisfying

$$\frac{1}{p}=\frac{1}{p_1}+\ldots+\frac{1}{p_N}. \tag{5.16}$$

The constant C_N in (5.15) only depends on θ in (5.7) and not on the choice of the $\rho_1,\ldots,\rho_N$ and $p_1,\ldots,p_N$.

6. Elimination of $t\Delta$: The end

We start with some formal considerations about how to transform integrals, to be justified in each case by convenient choices of contours along which the integrands decay fast enough near infinity. All functions are assumed to be holomorphic in x and sufficiently smooth in t where t varies in some interval. (The case of complex t with holomorphic dependence on t works the same way.) Let λ be some parameter $\neq 0$ and let v_t be a (holomorphic) vector field such that

$$\frac{\partial \phi_t}{\partial t} + v_t(\phi_t) - \frac{1}{\lambda}\mathrm{div}\,(v_t) + \frac{1}{\lambda} r_t = 0, \tag{6.1}$$

where r_t is a remainder and ϕ_t a phase.

Then

$$\frac{\partial}{\partial t}\int f_t(x) e^{-\lambda\phi_t(x)} \mathrm{d}x = \int \left(\frac{\partial f_t}{\partial t} + r_t f_t + v_t(f_t(x))\right) e^{-\lambda\phi_t(x)} \mathrm{d}x, \tag{6.2}$$

where we used an integration by parts in the next to the last integral. We conclude that the integral $\int f_t(x) e^{-\lambda\phi_t(x)}\mathrm{d}x$ is independent of t, if

$$\frac{\partial f_t}{\partial t} + v_t(f_t) + r_t f_t = 0. \tag{6.3}$$

Let $t \mapsto x_t(y)$ be an integral curve of v_t: $\partial_t x_t(y) = v_t(x_t(y))$. Writing $u(t) = f_t(x_t(y))$, (6.3) amounts to

$$\frac{\mathrm{d}}{\mathrm{d}t} u(t) + r_t(x_t(y)) u(t) = 0,$$

so

$$u(t) = u(0) e^{-\int_0^t r_s(x_s(y))\mathrm{d}s}.$$

In other words, the solutions of (6.3) (at least locally) are of the form $f_t(x)$, with

$$f_t(x_t(y)) = f_0(y) e^{-\int_0^t r_s(x_s(y))\mathrm{d}s}. \tag{6.4}$$

The way we have set up things, f_t is given for some t and we look for f_0, so we rewrite (6.4):

$$f_0(y) = e^{\int_0^t r_s(x_s(y))\mathrm{d}s} f_t(x_t(y)), \tag{6.5}$$

leading to the identity,

$$\int f_t(x_t) e^{-\lambda\phi_t(x_t)} \mathrm{d}x_t = \int f_t(x_t(y)) e^{-\lambda\phi_0(y) + \int_0^t r_s(x_s(y))\mathrm{d}s} \mathrm{d}y. \tag{6.6}$$

Let

$$M_t(x) := 1 + t\Delta + \mathrm{diag}\, r'(x_j \cdot x_j),$$

which first appeared in (3.25). Note that in $V(a,b)^m$, $\det M_t \neq 0$. $\log\det M_t$ is therefore holomorphic. We apply (6.6) with $\phi_t(x) = Q_t(x) - (1/\lambda)\log\det M_t$, where Q_t is as in

(4.1) and $\lambda = |1 + iE|$. Let v_t be the vector field constructed in sect. 4. Let $f(x)$ be a holomorphic function on $V(a,b)^m$ of at most polynomial growth at infinity. We also recall that $|t| \leq T_0$, with T_0 so small that $x_t(y) \in V(a,b)^m$ when $y \in V(a',b')^m$, for some fixed a', b' with $0 < a' < a$, $0 < b' < b$.

If a, b are not too large, $e^{-\lambda\phi_t(x)}$ will decay exponentially when $|x| \to \infty$, $x \in V(a,b)^m$ and by contour deformation (based on Stokes' formula) we first see that

$$\begin{aligned}\int_{(\mathbf{R}^2)^\Lambda} f(x)e^{-\lambda\phi_t(x)}\mathrm{d}x &= \int_{x_t((\mathbf{R}^2)^\Lambda)} f(x)e^{-\lambda\phi_t(x)}\mathrm{d}x \qquad (6.7)\\ &= \int_{(\mathbf{R}^2)^\Lambda} f(x_t(y))e^{-\lambda(\phi_0(y)+\int_0^t(\frac{\mathrm{d}}{\mathrm{d}s}\phi_s(x_s(y))\mathrm{d}s)}\left(\frac{\mathrm{d}x_t(y)}{\mathrm{d}y}\right)\mathrm{d}y.\end{aligned}$$

where $x_t((\mathbf{R}^2)^\Lambda)$ denotes the image of $(\mathbf{R}^2)^\Lambda$ under the flow x_t. Using now $\phi_s(x) = Q_s(x) - (1/\lambda)\log\det M_s$ and the fact that

$$\frac{\mathrm{d}}{\mathrm{d}s}Q_s = \frac{\partial}{\partial s}Q_s + v_s \cdot \nabla Q_s = 0,$$

(which is just (4.4),) we obtain that

$$\int_{(\mathbf{R}^2)^\Lambda} f(x)e^{-\lambda\phi_t(x)}\mathrm{d}x = \int_{(\mathbf{R}^2)^\Lambda} f(x_t(y))e^{-\lambda\phi_0(y)+\int_0^t(\mathrm{sdiv}\, v_s)(x_s(y))\mathrm{d}s}\mathrm{d}y, \qquad (6.8)$$

where

$$\begin{aligned}\mathrm{sdiv}\, v &:= \mathrm{div}\, v - \mathrm{tr}\,\mathcal{M},\\ \mathcal{M} &:= \Delta \circ M^{-1} + 2\mathrm{diag}\,(r''(x_j \cdot x_j)x_j \cdot v_j) \circ M^{-1},\end{aligned}$$

and where $f(x_t(y))$ is holomorphic of at most polynomial growth in $V(a',b')^m$, while $\int_0^t(\mathrm{sdiv}\, v_s)(x_s(y))\mathrm{d}s$ is holomorphic and bounded in the same set. Formula (6.8) represents the final elimination of $t\Delta$ from the exponent. Unfortunately this does not mean that we have decoupled the various x_j-variables in (3.25). Such couplings persist in the determinant and have appeared in the integral in the exponent in (6.8).

As a preparation for the final decoupling in the next section, we estimate derivatives in the transformed measure (3.25):

$$\begin{aligned}\rho(y) &:= \lambda^m e^{-\lambda\phi_0(y)+\int_0^t(\mathrm{sdiv}\, v_s)(x_s(y))\mathrm{d}s}\mathrm{d}y\\ &= \lambda^m\prod_j(1+r'(y_j\cdot y_j))e^{-\lambda(\sum y_j\cdot y_j+\sum r(y_j\cdot y_j))+\int_0^t \mathrm{sdiv}\, v_s(x_s(y))\mathrm{d}s}\prod_j\left(\frac{\mathrm{d}^2 y_j}{\pi}\right). \qquad (6.9)\end{aligned}$$

We first make a (separate) change of variables in each y_j. For every j, let

$$\widetilde{y}_j = \sqrt{1+\frac{r(y_j\cdot y_j)}{y_j\cdot y_j}}\,y_j \qquad (6.10)$$

in $\mathbf{C}^2$. Since $r(0) = 0$, $r(\tau) = \mathcal{O}(\epsilon)|\tau|$, the above change of variables is well defined for ϵ small enough and we have $\widetilde{y}_j \in V_j(a,b)$, if $y_j \in V_j(a',b')$. It is easy to check that the Jacobian of the above change of variables is precisely

$$\prod_j(1+r'(y_j\cdot y_j))^{-1}. \qquad (6.11)$$

We therefore obtain in the new coordinates:

$$\rho(\widetilde{y}) = \lambda^m e^{-\lambda \sum \widetilde{y}_j \cdot \widetilde{y}_j + \int_0^t \operatorname{sdiv} v_s(x_s(y(\widetilde{y})))\mathrm{d}s} \prod_j \left(\frac{\mathrm{d}^2 \widetilde{y}_j}{\pi}\right). \tag{6.12}$$

As in the proof of (5.4) and Proposition 5.1, we see from (4.77), that if the weights $\rho_1,..,\rho_N$ on Λ satify the condition (5.7) and $0 = \rho_1 + \cdots + \rho_N$, $1 = 1/p_1 + \cdots + 1/p_N$, then

$$\langle \nabla^N \operatorname{div} v_t, \tau_1 \otimes .. \otimes \tau_N \rangle = \mathcal{O}_N(1) e^r \prod_1^N \|\langle x \rangle^{-1} \tau_j\|_{\ell^{p_j}_{\rho_j}}, \tag{6.13}$$

(Here we first treat the case when one of the p_j's is 1 and the others $+\infty$, and then use interpolation.)

Similarly, we obtain for the same system of weights:

$$\langle \nabla^N \operatorname{tr} \mathcal{M}, \tau_1 \otimes .. \otimes \tau_N \rangle = \mathcal{O}_N(\epsilon) \prod_1^N \|\langle x \rangle^{-1} \tau_j\|_{\ell^{p_j}_{\rho_j}},\ 1 = \frac{1}{p_1} + .. + \frac{1}{p_N}, \tag{6.14}$$

where we also used the fact that $\nabla^N M$ is bounded for the same weights. Define $\widetilde{x}_t(\widetilde{y}) := x_t(y(\widetilde{y}))$. Using (6.10) $\widetilde{x}_t$ satisfies similar estimates as x_t. Using the analogue of (5.15) for $\widetilde{x}_t$, we can estimate $\nabla^N \operatorname{sdiv}(v_t(\widetilde{x}_t(y))$. It suffices to write $\langle \nabla^N \operatorname{sdiv} v_t(\widetilde{x}_t(\widetilde{y})), t_1 \otimes .. \otimes t_N \rangle$ as a linear combination of expressions

$$\langle (\nabla^M \operatorname{sdiv} v_t) \circ \widetilde{x}_t, \langle \nabla^{\sharp K_1} \widetilde{x}_t, \bigotimes_{k \in K_1} \tau_k \rangle \otimes .. \otimes \langle \nabla^{\sharp K_M} \widetilde{x}_t, \bigotimes_{k \in K_M} \tau_k \rangle,$$

with $K_j \neq \emptyset$, $K_\nu \cap K_\mu = \emptyset$ for $\nu \neq \mu$, $K_1 \cup .. \cup K_M = \{1,..,N\}$. It follows that if $\rho_1,..,\rho_N$ satisfy (5.7), then

$$\langle \nabla^N (\operatorname{sdiv} v_t(\widetilde{x}_t(\widetilde{y})), \tau_1 \otimes .. \otimes \tau_N \rangle = \mathcal{O}_N(1)(e^r + \epsilon) \prod_1^N \|\langle \widetilde{y} \rangle^{-1} \tau_j\|_{\ell^{p_j}_{\rho_j}}, \tag{6.15}$$

$$1 \leq p_j \leq \infty,\ 1 = \frac{1}{p_1} + .. + \frac{1}{p_N},\ \rho_1,..,\rho_N \text{ as in (5.7).}$$

This implies,

$$\left\langle \nabla^N \left(\int_0^t \operatorname{sdiv} v_s(x_s(y(\widetilde{y})))\mathrm{d}s \right), \tau_1 \otimes .. \otimes \tau_N \right\rangle = \mathcal{O}_N(1)(e^r + \epsilon)|t| \prod_1^N \|\langle \widetilde{y} \rangle^{-1} \tau_j\|_{\ell^{p_j}_{\rho_j}}, \tag{6.16}$$

with p_j, ρ_j as in (6.15).

Let

$$\mathcal{R}_t(\widetilde{y}) = \operatorname{sdiv} v_t(x_t(y(\widetilde{y}))),$$
$$R_t(\widetilde{y}) = \int_0^t \mathcal{R}_s(\widetilde{y})\mathrm{d}s. \tag{6.17}$$

Summing up our estimates, the "measure" ρ in (6.9), can be written as

$$\rho(x) = C_m \lambda^m e^{-\lambda(x\cdot x + \int_0^t \mathcal{R}(x))} \mathrm{d}^{2m}x, \ x \in V(a', b')^m, \tag{6.18}$$

where $\mathcal{R} = \mathcal{R}_{t,\epsilon}(x)$ is holomorphic in x and satisfies for every $N \geq 1$:

$$\langle \nabla^N \mathcal{R}_t(x), \tau_1 \otimes .. \otimes \tau_N \rangle = \mathcal{O}_N(1)\left(\frac{e^r + \epsilon}{\lambda}\right) \prod_1^N \left\| \langle x \rangle^{-1} \tau_j \right\|_{\ell_{\rho_j}^{p_j}}, \tag{6.19}$$

for all $\tau_j \in (\mathbf{C}^2)^\Lambda$, $p_j \in [1, \infty]$, with $1 = 1/p_1 + .. + 1/p_N$, and $\rho_1, .., \rho_N : \Lambda \to \mathbf{R}$, satisfying (5.7), for some fixed $\theta \in]0, 1[$. Here $\mathcal{O}_N(1)$ is uniformly bounded not only w.r.t. τ_j, but also w.r.t. p_j, ρ_j (and Λ).

7. The final decoupling

Let $R_t(x)$ be the function defined in (6.17), and put $\delta = \frac{1}{\lambda}(e^r + \epsilon)$, where we assume that $r \geq 1$, $\lambda \geq 1$. We assume that t is such that $|t|\delta$ is sufficiently small. Put

$$\phi_t(x) = x \cdot x + R_t(x). \tag{7.1}$$

We shall work in tubes around $(\mathbf{R}^2)^\Lambda$ of the form

$$\Omega(T) = (\mathbf{R}^2)^\Lambda + iB_{\ell^\infty}(0, T), \tag{7.2}$$

where $B_{\ell^\infty}(0, T)$ denotes the open ball of radius T in $(\mathbf{R}^2)^\Lambda$ for the ℓ^∞ norm: $\|s\|_{\ell^\infty} = \sup_{j\in\Lambda} |s_j|$.

In view of (6.1), (6.6), we look for a vector field $v = v_t$ in $\Omega(T)$, such that

$$\frac{\partial \phi_t}{\partial t} + v_t(\phi_t) - \frac{1}{\lambda}\mathrm{div}\,(v_t) - \frac{1}{\lambda}E_t = 0, \tag{7.3}$$

where E_t is a constant. We look for v_t of the form $v_t = \nabla u_t$ for some holomorphic function u_t on $\Omega(T)$, so that (7.3) becomes:

$$-\Delta u_t + \lambda \nabla \phi_t \cdot \frac{\partial}{\partial x} u_t - E_t = -\lambda \frac{\partial \phi_t}{\partial t} = -\lambda \mathcal{R}_t(x), \tag{7.4}$$

where $\mathcal{R}_t(x)$ is as defined in (6.17). Taking the gradient, we get

$$-\Delta(\nabla u_t) + \lambda \nabla \phi_t \cdot \frac{\partial}{\partial x}(\nabla u_t) + \lambda \phi_t''(x)(\nabla u_t) = -\lambda \nabla \mathcal{R}_t(x). \tag{7.5}$$

The LHS is $P(\nabla u_t)$, where $P = -\Delta + \nu(x, \frac{\partial}{\partial x}) + V(x)$ is of the form (C.29) in appendix C, with

$$\nu\left(x, \frac{\partial}{\partial x}\right) = \lambda \nabla \phi_t \cdot \frac{\partial}{\partial x} = \lambda\left(2x \cdot \frac{\partial}{\partial x} + \nabla R(x) \cdot \frac{\partial}{\partial x}\right), \tag{7.6}$$

$$V(x) = \lambda \phi_s''(x) = \lambda(2I + R''(x)). \tag{7.7}$$

In the following we shall work with some fixed $T > 0$, and write $\Omega = \Omega(T)$. Using appendix C, we have the following estimates for the solution to (7.4).

PROPOSITION 7.1. – *Fix $\theta\in]0,1[$, and let t be such that $|t|\delta=\frac{|t|}{\lambda}(\epsilon+e^r)$ be sufficiently small. Then* (7.4) *has a solution $u=u_t$, with $\nabla u\in C_b^\infty(\overline{\Omega})\cap$* Hol. *Moreover, for $N\in\{1,2,\dots\}$ there exists $C_N>0$, such that*

$$\left\|\nabla^N u\right\|_{(\ell^{p_1}_{\rho_1}\otimes\ldots\otimes\ell^{p_N}_{\rho_N})^*}\le C_N\delta, \tag{7.8}$$

for all weights $\rho_1,\dots,\rho_N:\Lambda\to\mathbf{R}$ satisfying (5.7)*, $\rho_1+\rho_2+\dots+\rho_N=0$ and all exponents $p_1,\dots,p_N\in[1,+\infty]$ satisfying $1=\frac{1}{p_1}+\ldots+\frac{1}{p_N}$.*

Remark. – Those readers who wish to delay reading the proof of the proposition, could go directly to the paragraph after (7.33).

PROOF OF PROPOSITION 4.1. – To verify the above proposition, we first check that the assumptions of Theorem C.8 are satisfied, when δ is small enough:

Let $x\in\partial\Omega$, so that $|x_j|\le T$ with equality for some $j=j_0$. The j_0:th component of ν is

$$\lambda(2x_{j_0}+\nabla_{x_{j_0}}R(x))=\lambda(2x_{j_0}+\mathcal{O}(\delta)).$$

For the corresponding real vector field $\nu_{\mathbf{R}}$, we therefore have $\nu_{\mathbf{R}}(|x_{j_0}|)>0$ at the point under consideration. The outgoing condition (C.32) follows. The conditions (C.25) and (C.31) are clearly fulfilled, and the vector field ν therefore satisfies all the required conditions.

Let $B=\ell^\infty_\rho$ with $\|\rho\|_{\mathrm{Lip}}\le\theta r$. Then as a special case of (6.15):

$$\|R''(x)\|_{\mathcal{L}(\ell^\infty_\rho,\ell^\infty_\rho)}=\mathcal{O}(\delta). \tag{7.9}$$

We then get (C.33) with "δ" there equal to λ:

$$\text{If } x\in\overline{\Omega},\ u\in B,\ v\in B^* \text{ and } \operatorname{Re}(u|v)=\|u\|_B\|v\|_{B^*},\quad \text{then } \operatorname{Re}(V(x)u|v)\ge\lambda\|u\|_B\|v\|_{B^*}. \tag{7.10}$$

It follows that if $v\in C_b(\overline{\Omega})\cap\mathrm{Hol}(\Omega)$, then there exists $u\in E$ (the space defined in Theorem C.8) such that

$$-\Delta u+\lambda\nabla\phi_t\cdot\frac{\partial}{\partial x}u+\lambda\phi_t''(x)u=v, \tag{7.11}$$

and

$$\sup_{\overline{\Omega}}\|u\|_{\ell^\infty_\rho}\le\frac{1}{\lambda}\sup_{\overline{\Omega}}\|v\|_{\ell^\infty_\rho}. \tag{7.12}$$

We also recall from the proof of Theorem C.8, that if $v\in\mathcal{S}(\overline{\Omega})\cap\mathrm{Hol}$, then u is in the same space.

We shall use next the maximum principle as in appendix C, to estimate the derivatives of u, when $u,v\in\mathcal{S}(\overline{\Omega})\cap\mathrm{Hol}$. To simplify the notations, we divide (7.11) by λ and then take the scalar product with the constant vector τ:

$$-\frac{1}{\lambda}\Delta\langle u,\tau\rangle+\langle\langle\nabla u,\nabla\phi\rangle,\tau\rangle+\langle\langle\nabla^2\phi,u\rangle,\tau\rangle=\left\langle\frac{v}{\lambda},\tau\right\rangle. \tag{7.13}$$

Now differentiate (7.13) in the constant direction s, using the identity

$$s_1(\partial_x)\circ\ldots\circ s_k(\partial_x)u=\langle\nabla^k u, s_1\otimes\ldots\otimes s_k\rangle, \tag{7.14}$$

when $s_1,\ldots,s_k$ are constant directions:

$$-\frac{1}{\lambda}\Delta\langle\langle\nabla u,s\rangle,\tau\rangle+\langle\langle\langle\nabla^2 u,s\rangle,\nabla\phi\rangle,\tau\rangle+\langle\langle\nabla u,\langle\nabla^2\phi,s\rangle\rangle,\tau\rangle+\langle\langle\nabla^2\phi,\langle\nabla u,s\rangle\rangle,\tau\rangle$$
$$=\frac{1}{\lambda}\langle\langle\nabla v,s\rangle,\tau\rangle-\langle\langle\langle\nabla^3\phi,s\rangle,u\rangle,\tau\rangle.$$

The second and third terms of the LHS can be rewritten, and we get,

$$\begin{aligned}-\frac{1}{\lambda}\Delta\langle\langle\nabla u,s\rangle,\tau\rangle+\nabla\phi\cdot\frac{\partial}{\partial x}\langle\langle\nabla u,s\rangle,\tau\rangle+\langle\langle\nabla^2\phi,s\rangle,\langle{}^t(\nabla u),\tau\rangle\rangle+&\\ \langle\langle\nabla^2\phi,\langle\nabla u,s\rangle\rangle,\tau\rangle=\frac{1}{\lambda}\langle\langle\nabla v,s\rangle,\tau\rangle-\langle\langle\langle\nabla^3\phi,s\rangle,u\rangle,\tau\rangle.&\end{aligned} \tag{7.15}$$

Let $B=\ell^\infty_\rho$ with $\|\rho\|_{\mathrm{Lip}}\le\theta r$, so that (7.10) holds with $V=\lambda\nabla^2\phi$. Let $x_0\in\overline{\Omega}$ be a point where $\|\nabla u\|_{\mathcal{L}(B,B)}$ $(=\|{}^t(\nabla u)\|_{\mathcal{L}(B^*,B^*)})$ is maximal $=: m$, and choose $s\in B$, $\tau\in B^*$ normalized, such that

$$\begin{aligned}&\langle\langle\nabla u(x_0),s\rangle,\tau\rangle=\langle s,\langle{}^t\nabla u(x_0),\tau\rangle\rangle=m\\ &=\|\langle\nabla u(x_0),s\rangle\|_B\|\tau\|_{B^*}=\|s\|_B\left\|\langle{}^t\nabla u(x_0),\tau\rangle\right\|_{B^*},\end{aligned}$$

so that $\overline{\Omega}\ni x\mapsto\mathrm{Re}\,\langle\langle\nabla u(x),\tau\rangle,s\rangle$ attains its maximum (m) at x_0. Hence the real part of the first term in (7.15) is ≥ 0 at $x=x_0$ and the same holds for the second term by the outgoing condition. In view of (7.10), the real parts of the third and the fourth terms in (7.15) at $x=x_0$, are both $\ge m$, so we end up with the estimate

$$2\sup_{\overline{\Omega}}\|\nabla u\|_{\mathcal{L}(B,B)}\le\frac{1}{\lambda}\sup_{\overline{\Omega}}\|\nabla v\|_{\mathcal{L}(B,B)}+\sup_{\overline{\Omega}}\left\|\nabla^3\phi\right\|_{(B\otimes E\otimes B^*)^*}\|u\|_E, \tag{7.16}$$

where $E\simeq(\mathbf{C}^2)^\Lambda$ is any Banach space with $(\mathbf{C}^2)^\Lambda$ as the underlying vector space and $\|\nabla^3\phi\|_{(B\otimes E\otimes B^*)^*}$ denotes the norm of $\nabla^3\phi$ as a trilinear form on $B\times E\times B^*$.

Notice that u is the gradient of a holomorphic function in Ω iff ∇u is symmetric. The same holds for v of course, and we now rewrite (7.15) in the form:

$$-\frac{1}{\lambda}\Delta\nabla u+\left(\nabla\phi\cdot\frac{\partial}{\partial x}\right)(\nabla u)+\nabla u\circ\nabla^2\phi+\nabla^2\phi\circ\nabla u+\langle\nabla^3\phi,u\rangle=\frac{1}{\lambda}\nabla v. \tag{7.17}$$

Here $\langle\nabla^3\phi,u\rangle$ is symmetric, so if we transpose the last equation and then take the difference between (7.17) and its transpose, we get

$$\begin{aligned}-\frac{1}{\lambda}\Delta(\nabla u-{}^t\nabla u)+\nabla\phi\cdot\frac{\partial}{\partial x}(\nabla u-{}^t\nabla u)+(\nabla u-{}^t\nabla u)\circ\nabla^2\phi+&\\ \nabla^2\phi\circ(\nabla u-{}^t\nabla u)=\frac{1}{\lambda}(\nabla v-{}^t\nabla v).&\end{aligned} \tag{7.18}$$

The maximum principle (used after going back to an equation of the type (7.15)) shows that

$$2\sup_{\overline{\Omega}} \left\|\nabla u - {}^t\nabla u\right\|_{\mathcal{L}(B,B)} \leq \frac{1}{\lambda}\sup_{\overline{\Omega}} \left\|\nabla v - {}^t\nabla v\right\|_{\mathcal{L}(B,B)}. \tag{7.19}$$

In particular, if v is a gradient, so that $\nabla v - {}^t\nabla v = 0$, then the same holds for u. In this case, if $u = \nabla f$, $v = \nabla g$, we see that the LHS in (7.11) is the gradient of $-\Delta f + \lambda \nabla\phi_t \cdot \frac{\partial}{\partial x} f$ and we get

$$-\Delta f + \lambda \nabla\phi_t \cdot \frac{\partial}{\partial x} f - E_t = g, \tag{7.20}$$

where E_t is a constant.

We now want to estimate higher derivatives and we start from (7.20) with $\nabla f, \nabla g \in \mathcal{S}(\overline{\Omega}) \cap \mathrm{Hol}(\Omega)$. Let $s_1, \ldots, s_N$ be constant directions, and apply $s_1(\partial_x) \circ \ldots \circ s_N(\partial_x)$ to (7.20):

$$\begin{aligned}
-\frac{1}{\lambda}\Delta(s_1(\partial_x) \circ \ldots \circ s_N(\partial_x) f) + \nabla\phi \cdot \frac{\partial}{\partial x}(s_1(\partial_x) \circ \ldots \circ s_N(\partial_x) f) + \\
\sum_{j=1}^{N} \nabla(s_j(\partial_x)\phi) \cdot \nabla(s_1(\partial_x) \circ \ldots \widehat{s_j(\partial_x)} \circ \ldots \circ s_N(\partial_x) f) + \\
\sum_{J\cup K = \{1,\ldots,N\}, J\cap K = \emptyset, \sharp J \geq 2} \nabla\left(\left(\prod_J s_j(\partial_x)\right)\phi\right) \cdot \nabla\left(\prod_K s_k(\partial_x) f\right) \\
= \frac{1}{\lambda} s_1(\partial_x) \circ \ldots \circ s_N(\partial_x) g,
\end{aligned}$$

with the convention that $\prod_K s_k(\partial_x) = 1$, when $K = \emptyset$. This can also be written

$$\begin{aligned}
-\lambda\Delta\langle \nabla^N f, s_1 \otimes \ldots \otimes s_N\rangle + \nabla\phi \cdot \frac{\partial}{\partial x}\langle \nabla^N f, s_1 \otimes \ldots \otimes s_N\rangle \\
+ \sum_{j=1}^{N} \langle\langle \nabla^2\phi, s_j\rangle, \langle \nabla^N, s_1 \otimes \ldots \widehat{s_j} \ldots \otimes s_N\rangle\rangle = \\
\frac{1}{\lambda}\langle \nabla^N g, s_1 \otimes \ldots \otimes s_N\rangle - \sum_{J\cup K = \{1,\ldots,N\}, J\cap K : \emptyset, \sharp J \geq 2} \langle\langle \nabla^{1+\sharp K} f, \bigotimes_{k\in K} s_k\rangle, \langle \nabla^{1+\sharp J}\phi, \bigotimes_{j\in J} s_j\rangle\rangle.
\end{aligned} \tag{7.21}$$

Let $\rho_1, \ldots, \rho_N : \Lambda \to \mathbf{R}$ satisfy (5.7), and put $\rho_K = \sum_{k\in K} \rho_k$, when $K \subset \{1, \ldots, N\}$ is non-empty, and $\rho_\emptyset = 0$. Let $p_1, \ldots, p_N \in [1, +\infty]$ satisfy

$$1 = \frac{1}{p_1} + \ldots + \frac{1}{p_N}. \tag{7.22}$$

If $K \subset \{1, \ldots, N\}$, define $p_K \in [1, +\infty]$, by

$$\frac{1}{p_K} = \sum_{k\in K} \frac{1}{p_k}, \text{ for } K \neq \emptyset,\ p_\emptyset = +\infty. \tag{7.23}$$

Let $x_0 \in \overline{\Omega}$ be a point where

$$\sup_{x\in\overline{\Omega}} \left\|\nabla^N f(x)\right\|_{(\ell^{p_1}_{\rho_1} \otimes \ldots \otimes \ell^{p_N}_{\rho_N})^*} =: m, \tag{7.24}$$

is attained, and observe that $\|\nabla^N f(x)\|_{(\ell^{p_1}_{\rho_1}\otimes\ldots\otimes\ell^{p_N}_{\rho_N})^*}$ (defined as after (7.16)) is also the norm of $\nabla^N f(x)$ as a multilinear map: $\ell^{p_1}_{\rho_1}\times\ldots\widehat{\ell^{p_j}_{\rho_j}}\ldots\times\ell^{p_N}_{\rho_N}\to\ell^{q_j}_{-\rho_j}$, where q_j is the conjugate index to p_j: $\frac{1}{q_j}+\frac{1}{p_j}=1$, so that $q_j=p_{\{1,\ldots\widehat{j}\ldots,N\}}$. (When $N=1$, we interpret $\ell^{p_1}_{\rho_1}\times\ldots\widehat{\ell^{p_j}_{\rho_j}}\ldots\times\ell^{p_N}_{N}$ as $\mathbf{C}$ and our identification remains valid trivially.) The latter norm will be denoted

$$\|\nabla^N f(x)\|_{\mathcal{L}(\ell^{p_1}_{\rho_1}\otimes\ldots\widehat{\ell^{p_j}_{\rho_j}}\ldots\otimes\ell^{p_N}_{\rho_N};\ell^{q_j}_{-\rho_j})}.$$

Let $s_j\in\ell^{p_j}_{\rho_j}$ be normalized vectors with

$$\langle\nabla^N f(x_0), s_1\otimes\ldots\otimes s_N\rangle = m. \tag{7.25}$$

We notice here that (7.9), (7.10) remain valid, if we replace "∞" there by some arbitrary $p\in[1,\infty]$. Since

$$\begin{aligned} m &= \mathrm{Re}\,\langle s_j,\langle\nabla^N f(x_0), s_1\otimes\ldots\widehat{s_j}\ldots\otimes s_N\rangle\rangle\\ &= \|s_j\|_{\ell^{p_j}_{\rho_j}}\|\langle\nabla^N f(x_0), s_1\otimes\ldots\widehat{s_j}\ldots\otimes s_N\rangle\|_{\ell^{q_j}_{-\rho_j}}, \end{aligned} \tag{7.26}$$

and $\ell^{q_j}_{-\rho_j}$ is the dual of $\ell^{p_j}_{\rho_j}$, it follows from the above mentioned extension of (7.10), that

$$\mathrm{Re}\,\langle\langle\nabla^2\phi(x_0), s_j\rangle,\langle\nabla^N f(x_0), s_1\otimes\ldots\widehat{s_j}\ldots\otimes s_N\rangle\rangle\ge m. \tag{7.27}$$

(When $N=1$, we use the convention: $\langle\nabla^N f(x), s_1,\ldots\widehat{s_j}\ldots\otimes s_N\rangle=\nabla f(x)$.)

Taking the real part of (7.21) and putting $x=x_0$, we can apply the maximum principle as before, and get

$$\begin{aligned} N\sup_{x\in\overline{\Omega}}\|\nabla^N f\|_{(\ell^{p_1}_{\rho_1}\otimes\cdots\otimes\ell^{p_N}_{\rho_N})^*} &\le \frac{1}{\lambda}\sup_{x\in\overline{\Omega}}\|\nabla^N g\|_{(\ell^{p_1}_{\rho_1}\otimes\ldots\otimes\ell^{p_N}_{\rho_N})^*}+\\ \sum_{J\cup K=\{1,\ldots,N\},\,J\cap K=\emptyset,\,\sharp J\ge 2}&\inf_{\rho}\sup_{x\in\overline{\Omega}}\Big(\|\nabla^{1+\sharp K}f\|_{\mathcal{L}(\bigotimes_{k\in K}\ell^{p_k}_{\rho_k};\ell^{p_K}_{\rho})}\|\nabla^{1+\sharp J}\phi\|_{\mathcal{L}(\bigotimes_{j\in J}\ell^{p_j}_{\rho_j};\ell^{p_J}_{-\rho})}\Big), \end{aligned} \tag{7.28}$$

where we also used that $1/p_K+1/p_J=1$, so that $(\ell^{p_k}_{\rho})^*=\ell^{p_J}_{-\rho}$. If we also have $\rho_1+\ldots+\rho_N=0$, then a natural choice for ρ, to bound the infimum above, may be $\rho=\rho_K$, since then $-\rho=\rho_J$.

We return to the equation (7.4). Approximating $\mathcal{R}_t$ by the functions $e^{-\epsilon x^2}\mathcal{R}_t(x)\in\mathcal{S}(\overline{\Omega})\cap\mathrm{Hol}\,(\Omega)$, we see that (7.4) has a solution $u=u_t$ with $\nabla u\in C_b^\infty(\overline{\Omega})\cap\mathrm{Hol}\,(\Omega)$ and such that the estimates we made for the equation (7.20), can be applied with $g=-\lambda\mathcal{R}_t$, $f=u$.

Let $\rho_1,\ldots,\rho_N:\Lambda\to\mathbf{R}$ be a system of weights which satisfies (5.7) for some fixed θ and assume,

$$\rho_1+\ldots+\rho_N=0. \tag{7.29}$$

Let $p_1,\ldots,p_N\in[1,+\infty]$ satisfy (7.22). We shall derive estimates for $\nabla^N u$, which depend on θ in (5.7), but not on the choice of ρ_j and p_j satisfying (5.7), (7.29) and (7.22). If

$\emptyset \neq K \subset \{1,\dots,N\}$, then ρ_k, $k \in K$, $-\rho_K$ satisfy (5.7), (7.9) with N replaced by $1+\sharp K$, and if q_K is the exponent conjugate to p_K, then p_k, $k \in K$, q_K satisfy (7.21): $\sum_K \frac{1}{p_k} + \frac{1}{q_K} = 1$. Using this remark, we can make an "induction over N": Let $m(N)$ be the infimum of all constants $C = C_t$ such that

$$|\langle \nabla^N u, \tau_1 \otimes \dots \otimes \tau_N \rangle| \leq C \|\tau_1\|_{\ell^{p_1}_{\rho_1}} \dots \|\tau_N\|_{\ell^{p_N}_{\rho_N}}, \tag{7.30}$$

for all $\tau_j \in \mathbf{C}^2$, $p_j \in [1,+\infty]$ satisfying (7.22), ρ_j satisfying (5.7) (where θ is fixed) and (7.29).

In (7.28) we choose ρ as in the subsequent remark, and get

$$\begin{aligned} N \sup_{\overline{\Omega}} \|\nabla^N u\|_{(\ell^{p_1}_{\rho_1}\otimes\dots\otimes\ell^{\rho_N}_{p_N})^*} \leq \\ \sup_{\overline{\Omega}} \|\nabla^N \mathcal{R}\|_{(\ell^{p_1}_{\rho_1}\otimes\dots\otimes\ell^{\rho_N}_{p_N})^*} + \\ \sum_{J\cup K=\{1,\dots,N\}, J\cap K=\emptyset, \sharp J\geq 2} m(1+\sharp K) \sup_{\overline{\Omega}} \|\nabla^{1+\sharp J}\phi\|_{\mathcal{L}(\bigotimes_{j\in J}\ell^{p_j}_{\rho_j};\ell^{p_J}_{\rho_J})}. \end{aligned} \tag{7.31}$$

Now recall (6.19):

$$\|\nabla^N \mathcal{R}_t\|_{(\ell^{p_1}_{\rho_1}\otimes\dots\otimes\ell^{p_N}_{\rho_N})^*} \leq C_N \frac{1}{\lambda}(e^r + \epsilon) = C_N,$$

and that $\delta = \mathcal{O}(1)$, by assumption. It follows from (7.1), that

$$\|\nabla^N \phi\|_{(\ell^{p_1}_{\rho_1}\otimes\dots\otimes\ell^{p_N}_{\rho_N})^*} \leq C_N,$$

so

$$\|\nabla^{1+\sharp J}\phi\|_{\mathcal{L}(\bigotimes_{j\in J}\ell^{p_j}_{\rho_j};\ell^{p_J}_{\rho_J})} \leq C_{1+\sharp J}.$$

From (7.31), we get with a new constant C_N:

$$m(N) \leq C_N\left(\delta + \sum_1^{N-1} m(k)\right), \tag{7.32}$$

so with a new constant C_N:

$$m(N) \leq C_N \delta. \tag{7.33}$$

Hence the Proposition. □

We now establish that the second deforming vector field ∇u_t depends holomorphically on t for t such that $|t|\delta$ is sufficiently small.

Let v_t be a holomorphic function in (t,x), more precisely $v_t \in \mathcal{S}(\overline{\Omega}) \cap \mathrm{Hol}\,(\Omega)$ and is holomorphic in t for t sufficiently small. Let u_t be the solution with $\nabla u_t \in \mathcal{S}(\overline{\Omega}) \cap \mathrm{Hol}\,(\Omega)$ of

$$-\frac{1}{\lambda}\Delta u_t + \nabla\phi_t \cdot \frac{\partial}{\partial x} u_t - E_t = v_t.$$

Return to the equation for the gradient:

$$-\frac{1}{\lambda}\Delta\nabla u_t + \nabla\phi_t \cdot \frac{\partial}{\partial x}\nabla u_t + \nabla^2\phi_t\nabla u_t = \nabla v_t. \tag{7.34}$$

Let us first show that ∇u_t depends continuously on t in a slightly smaller tube $\Omega' = \Omega(T')$, $T' < T$:

$$\begin{aligned}&-\frac{1}{\lambda}\Delta(\nabla u_{t_2} - \nabla u_{t_1}) + \nabla\phi_{t_1} \cdot \frac{\partial}{\partial x}(\nabla u_{t_2} - \nabla u_{t_1}) + \nabla^2\phi_{t_1}(\nabla u_{t_2} - \nabla u_{t_1}) \\ &+(\nabla\phi_{t_2} - \nabla\phi_{t_1}) \cdot \frac{\partial}{\partial x}\nabla u_{t_2} + (\nabla^2\phi_{t_2} - \nabla^2\phi_{t_1})\nabla u_{t_2} = \nabla v_{t_2} - \nabla v_{t_1}.\end{aligned} \tag{7.35}$$

Here the $\sup_{\Omega'} \|\cdot\|_{\ell^\infty}$ of the last two terms on the LHS and the RHS are $\mathcal{O}(t_2 - t_1)$, (where we do not require any uniformity w.r.t. the dimension,) and it follows that

$$\sup_{\Omega'} \|\nabla u_{t_2} - \nabla u_{t_1}\|_{\ell^\infty} = \mathcal{O}(|t_2 - t_1|).$$

Dividing (7.35) by $t_2 - t_1$ and letting $t_2 \to t_1$, we see that $\frac{\partial}{\partial t}\nabla u_t$ exists in $\mathcal{S}(\overline{\Omega'}) \cap \mathrm{Hol}\,(\Omega')$ and that

$$\begin{aligned}&-\frac{1}{\lambda}\Delta(\frac{\partial}{\partial t}\nabla u_t) + \nabla\phi_t \cdot \frac{\partial}{\partial x}(\frac{\partial}{\partial t}\nabla u_t) + \nabla^2\phi_t(\frac{\partial}{\partial t}\nabla u_t) \\ &\quad + \frac{\partial}{\partial t}(\nabla\phi_t) \cdot \frac{\partial}{\partial x}\nabla u_t + \frac{\partial}{\partial t}(\nabla^2\phi_t)\nabla u_t = \frac{\partial}{\partial t}\nabla v_t \text{ in } \Omega'.\end{aligned} \tag{7.36}$$

These arguments also work, when we let t be complex while keeping δ sufficiently small, and (7.36) remains valid with $\frac{\partial}{\partial t}$ replaced by $\frac{\partial}{\partial \bar{t}}$. Hence $\frac{\partial}{\partial \bar{t}}\nabla u_t = 0$, $x \in \Omega'$. Letting $\Omega' \nearrow \Omega$, we conclude that ∇u_t is holomorphic in (t, x) $(\in \mathbf{C} \times \Omega)$ for t such that $|t|\delta$ is sufficiently small. The Cauchy inequalities w.r.t. t now allow us to take as many t-derivatives as we like and all the estimates that we have obtained for $\nabla^N u_t$ extend to $\left(\frac{\partial}{\partial t}\right)^k \nabla^N u_t$.

Finally, if $v \in C_b(\overline{\Omega}) \cap \mathrm{Hol}$, we approximate v narrowly by $v_\epsilon \in \mathcal{S}(\overline{\Omega}) \cap \mathrm{Hol}$ and we see that the limiting solution of (7.41) depends holomorphically on t, and that all estimates for $\nabla^N u_t$ are also valid for $\left(\frac{\partial}{\partial t}\right)^k \nabla^N u_t$.

In terms of the deforming vector field $\nabla u = \nabla u_t$, (7.8) can be written,

$$\left\|\nabla^N \nabla u\right\|_{\mathcal{L}(\ell^{p_1}_{\rho_1} \otimes \ldots \otimes \ell^{p_N}_{\rho_N}; \ell^p_\rho)} \le C_{N+1}\delta,\ N \ge 0, \tag{7.37}$$

when $p_1, \ldots, p_N \in [1, +\infty]$, $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_N}$, $\rho = \rho_1 + \ldots + \rho_N$ and $\rho_1, \ldots, \rho_N$ satisfy (5.7). This is analogous to (5.4) (except that we have lost the gain in powers of $\langle x \rangle$, since the Cauchy inequalities do not produce such a gain when working in a tube). The argument leading to (5.15) now gives an analogous result for the (new) flow $\widetilde{x}(t, y)$ of the (new) vector field $\nabla u_t(x)$, for t such that δ is sufficiently small.

$$\|\widetilde{x}(t, y) - y\|_{\ell^\infty} \le C_1|t|\delta, \tag{7.38}$$

$$\left\| \frac{\partial \widetilde{x}(t,y)}{\partial y} - I \right\|_{\mathcal{L}(\ell^p_\rho, \ell^p_\rho)} \leq C_2 |t| \delta, \tag{7.39}$$

when $1 \leq p \leq \infty$, $\|\rho\|_{\mathrm{Lip}} \leq \theta r$,

$$\left\| \nabla_y^N \widetilde{x}(t,y) \right\|_{\mathcal{L}(\ell^{p_1}_{\rho_1} \otimes \ldots \otimes \ell^{p_N}_{\rho_N}; \ell^p_\rho)} \leq C_{N+1} |t| \delta \tag{7.40}$$

for $N \geq 1$ and p_j, p, ρ_j, ρ as in (7.37). We observe that by (7.38), if $0 < T' < T$, then for t such that $|t|\delta > 0$ small enough,

$$\widetilde{x}_t = \widetilde{x}(t, \cdot) : \Omega(T') \to \Omega(T). \tag{7.41}$$

We fix such a T'.

The final decoupling can now be carried out: We recall that the RHS of (6.8) is of the form

$$\int_{(\mathbf{R})^\Lambda} g(x) e^{-\lambda \phi_t(x)} \mathrm{d}x, \tag{7.42}$$

where ϕ_t is givien by (7.1), and where $g(x) = f(x_t(x))$, with x_t here denoting the earlier v-flow, so that g is holomorphic and of at most polynomial growth in the tube $\Omega(T)$. Using Stokes' formula, we replace $(\mathbf{R}^2)^\Lambda$ in (7.42) by $\widetilde{x}_t((\mathbf{R}^2)^\Lambda)$, then a second application of Stokes' formula gives us as in sect. 6, that the integral (7.42) is equal to

$$\int_{(\mathbf{R}^2)^\Lambda} g(\widetilde{x}_t(x)) e^{-\lambda x \cdot x - \int_0^t E_s \mathrm{d}s} \mathrm{d}x.$$

Here we also use that the vector field in Proposition 7.1 is holomorphic in t.

Using finally that $\int e^{-\lambda x \cdot x} \mathrm{d}x = \int e^{-\lambda \phi_t(x_t)} \mathrm{d}x_t$, (see (2.24) and also appendix B,) we see that $\int_0^t E_s ds = 0$. The RHS of (6.8) is then of the form

$$\int_{(\mathbf{R}^2)^\Lambda} f(x_t \circ \widetilde{x}_t(x)) e^{-\lambda x \cdot x} \mathrm{d}x. \tag{7.43}$$

8. Exponentially weighted estimates and end of the proof of Theorem 2.1.

We consider weighted estimates for $\Delta + V - E$, where $E \in \mathbf{C}$. Let

$$q(\eta) := 2 \sum_1^d \cosh \eta_j. \tag{8.1}$$

We then have

$$\|e^{(\cdot)\eta} \Delta e^{-(\cdot)\eta}\|_{\mathcal{L}(\ell^2, \ell^2)} \leq q(\eta).$$

Writing

$$e^{(\cdot)\eta}(\Delta + V - E) e^{-(\cdot)\eta} = V - E + e^{(\cdot)\eta} \Delta e^{-(\cdot)\eta}, \tag{8.2}$$

we observe that $|v - E| \geq |E| - |v|_\infty$ everywhere on $\mathbf{Z}^d$. Let $2d < \lambda < |E|$. Assume $q(\eta) \leq \lambda$.

Passing to the matrices, and using that every entry of a matrix is bounded by the norm of the matrix in $\mathcal{L}(\ell^2, \ell^2)$, we get from (8.2):

$$|(\Delta + V - E)^{-1}(\mu, \nu)| \leq \frac{1}{|E| - \lambda - |v|_\infty} e^{-(\mu-\nu)\cdot\eta}. \tag{8.3}$$

Define the convex set

$$W(\lambda) = \{\eta \in \mathbf{R}^d; \quad q(\eta) < \lambda\}.$$

We introduce the support function of $W(\lambda)$:

$$p_\lambda(x) = \sup_{\eta \in W(\lambda)} x \cdot \eta, \; x \in \mathbf{R}^d. \tag{8.4}$$

Then p_λ is even, continuous, convex, positively homogeneous of degree 1, and $p_\lambda(x) > 0$ for $x \neq 0$. In other words, p_λ is a norm on $\mathbf{R}^d$. Varying $\eta \in \mathcal{W}(\lambda)$ in (8.3), we get

$$|(\Delta + V - E)^{-1}(\mu, \nu)| \leq \frac{1}{|E| - \lambda - |v|_\infty} e^{-p_\lambda(\mu-\nu)}. \tag{8.5}$$

We now assume $|E| >> d$. In order to get a precise control on p_λ, we first consider $(\Delta - E)^{-1}$ on $\ell^2(\mathbf{Z}^d)$, when $E \in \mathbf{C}$ and $|E| >> 1$. Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. The Fourier transform $\mathcal{F} : \ell^2(\mathbf{Z}^d) \to L^2(\mathbf{T}^d; \frac{1}{(2\pi)^d} \mathrm{d}\xi)$ given by:

$$\mathcal{F}u(\xi) = \sum_{j \in \mathbf{Z}^d} e^{ij\xi} u(j), \tag{8.6}$$

is unitary and has the inverse,

$$\mathcal{F}^{-1}v(j) = \frac{1}{(2\pi)^d} \int e^{-ij\xi} v(\xi) \mathrm{d}\xi. \tag{8.7}$$

Conjugation by $\mathcal{F}$ shows that Δ is unitarily equivalent to the operator of multiplication on $L^2(\mathbf{T}^d)$ by

$$p(\xi) := 2 \sum_1^d \cos \xi_j. \tag{8.8}$$

Whenever convenient, we view p as a $(2\pi\mathbf{Z})^d$-periodic function on $\mathbf{R}^d$ and it will be natural to consider p also as a function on $\mathbf{C}^d$:

$$p(\zeta) = 2 \sum_1^d \cos \zeta_j = 2 \sum_1^d (\cos \xi_j \cosh \eta_j - i \sin \xi_j \sinh \eta_j), \tag{8.9}$$

with $\zeta = \xi + i\eta \in \mathbf{C}^d$.

We are interested in points where $p(\zeta) - E \neq 0$. Our analysis will be based on a certain approximate translation invariance. Observe that

$$\cosh \eta_j = \frac{1}{2} e^{|\eta_j|} + \mathcal{O}(e^{-|\eta_j|}), \; \sinh \eta_j = \frac{1}{2} (\operatorname{sgn} \eta_j) e^{|\eta_j|} + \mathcal{O}(e^{-|\eta_j|}), \tag{8.10}$$

so that

$$p(\zeta) = \sum_1^d e^{-i(\operatorname{sgn}\eta_j)\xi_j} e^{|\eta_j|} + \mathcal{O}(1). \tag{8.11}$$

Here $\operatorname{sgn}\eta_j = +1$, when $\eta_j \geq 0$, and $= -1$, when $\eta_j < 0$. (The choice for $\eta_j = 0$ is unimportant.) Put

$$s(\eta) = (\operatorname{sgn}\eta_1, \ldots, \operatorname{sgn}\eta_d). \tag{8.12}$$

Then uniformly for $t \in \mathbf{R}$:

$$p(\xi + ts(\eta) + i\eta) = e^{-it} p(\xi + i\eta) + \mathcal{O}(1). \tag{8.13}$$

For $E \in \mathbf{C} \setminus [-2d, 2d]$, let $\Omega(E) = \mathbf{R}^d + iW(E)$ be the largest connected open tube (*i.e.* set of the form $\mathbf{R}^d + iW$) containing $\mathbf{R}^d$, where $p(\zeta) - E \neq 0$. Bochner's tube theorem implies that $W(E)$ is convex.

When $E > 2d$, this coincides with the earlier definition:

$$W(E) = \left\{ \eta \in \mathbf{R}^d;\ 2\sum_1^d \cosh\eta_j < E \right\}. \tag{8.14}$$

This is so because if η belongs to the RHS of (8.14), then for every $\xi \in \mathbf{R}^d$:

$$|p(\zeta)| \leq 2\sum_1^d |\cos\xi_j \cosh\eta_j - i\sin\xi_j \sinh\eta_j| \leq 2\sum_1^d \cosh\eta_j < E,$$

so $p(\zeta) \neq E$ and hence $\eta \in W(E)$. On the other hand, if $2\sum_1^d \cosh\eta_j = E$, then $p(i\eta) = E$, so $\eta \notin W(E)$, and (8.14) follows. As in (8.4), we introduce the support function of $W(E)$:

$$p_E(x) = \sup_{\eta \in W(E)} x \cdot \eta,\ x \in \mathbf{R}^d.$$

If $q(\eta) = 2\sum_1^d \cosh\eta_j$, as before, we notice, from (8.10), that

$$\|\nabla q(\eta)\|_{\ell^1} = q(\eta) + \mathcal{O}(1). \tag{8.15}$$

It follows that for $E_2 \geq E_1 >> 2d$:

$$W(E_1) \subset W(E_2) \subset W(E_1) + B\left(0, \mathcal{O}(1)\log\frac{E_2}{E_1}\right). \tag{8.16}$$

(The first inclusion holds more generally for $E_2 \geq E_1 > d$.)

(8.16) is all we need to have a precise control on p_λ in (8.5). However for completeness we now go on to consider the case of complex E. We first recall the estimate obtained in the proof of (8.14):

$$|p(\zeta)| < E, \text{ when } E > d,\ \zeta \in \mathbf{R}^d + iW(E). \tag{8.17}$$

Consider now the case of general $E \in \mathbf{C} \setminus [-2d, 2d]$. It follows from (8.17) that $|p(\zeta)| < |E|$ for $\zeta \in \mathbf{R}^d + iW(|E|)$, so

$$W(|E|) \subset W(E). \tag{8.18}$$

In the other direction, we have:

PROPOSITION 8.1. – *There exists a constant $C > 0$, such that*

$$|p(\zeta)| < |E| + C \text{ for all } \zeta \in \mathbf{R}^d + iW(E). \tag{8.19}$$

In particular,

$$W(E) \subset W(|E| + C). \tag{8.20}$$

Proof. – Let $\zeta = \xi + i\eta \in \mathbf{R}^d + iW(E)$ and assume that $|p(\zeta)| = R >> 1$. Consider the closed curve

$$\gamma : \mathbf{R}/2\pi\mathbf{Z} \ni t \mapsto p(\xi + its(\eta) + i\eta) = e^{-it}p(\zeta) + \mathcal{O}(1),$$

which winds once around 0 in the negative direction at a distance $\geq R - C$ from 0. Since the set of values of $p_{|\mathbf{R}^n + iW(E)}$ is simply connected and contains the image of γ, it also has to contain the closed disc $\overline{D(0, R - C)}$. By definition of $W(E)$, E cannot belong to $p(\mathbf{R}^d + iW(E))$, so $|E| > R - C$, and $|p(\zeta)| < |E| + C$, as claimed. □

We now go back to estimate (8.5). We assume that $|E| >> 2d$ and that $1 + |v|_\infty \leq \frac{1}{2}|E|$. Choose $\lambda = |E| - |v|_\infty - 1$. (8.16) gives

$$W(\lambda) \subset W(|E|) \subset W(\lambda) + B\left(0, \mathcal{O}(1) \log \frac{|E|}{|E| - |v|_\infty - 1}\right).$$

Here

$$\frac{|E|}{|E| - (|v|_\infty + 1)} = 1 + \mathcal{O}\left(\frac{1 + |v|_\infty}{|E|}\right),$$

so

$$W(\lambda) \subset W(|E|) \subset W(\lambda) + B\left(0, \mathcal{O}(1)\frac{1 + |v|_\infty}{|E|}\right).$$

Consequently,

$$p_\lambda(x) \leq p_{|E|}(x) \leq p_\lambda(x) + \mathcal{O}(1)\frac{1 + |v|_\infty}{|E|}|x|.$$

Substitution into (8.5) gives on $\mathbf{Z}^d \times \mathbf{Z}^d$:

$$|(\Delta + V - E)^{-1}(\mu, \nu)| \leq e^{-p_{|E|}(\mu - \nu) + \mathcal{O}(1)\frac{1 + |v|_\infty}{|E|}|\mu - \nu|}. \tag{8.21}$$

We shall derive similar estimates for $(\Delta_\Lambda + V - E)^{-1}$, when Λ is a discrete torus or a subset of $\mathbf{Z}^d$. As a preparation, we first establish that,

$$(\log \lambda - \log 2d)|x|_{\ell^1} \leq p_\lambda(x) \leq (\log \lambda)|x|_{\ell^1}, \tag{8.22}$$

when $\lambda >> 1$. Recall that p_λ is the support function of the set $W(\lambda)$ in $\mathbf{R}^d$, defined by $2\sum_1^d \cosh \eta_j \leq \lambda$, so if $\eta \in W(\lambda)$, we have $e^{|\eta_j|} \leq \lambda$ for every j, or equivalently,

$\eta \in B_{\ell^\infty}(0, \log \lambda)$. In the other direction we notice that if $|\eta_j| \le \log \lambda - \log(2d)$, then $2\cosh \eta_j \le 2e^{|\eta_j|} \le \frac{\lambda}{d}$, so $2\sum_1^d \cosh \eta_j \le \lambda$ and hence $\eta \in W(\lambda)$. We have shown that

$$B_{\ell^\infty}(0, \log \lambda - \log 2d) \subset W(\lambda) \subset B_{\ell^\infty}(0, \log \lambda). \tag{8.23}$$

(8.22) now follows, since the support function of $B_{\ell^\infty}(0,1)$ is $|x|_{\ell^1}$.

Let $\Lambda = (\mathbf{Z}/N\mathbf{Z})^d$, $N >> 1$ be a discrete torus, and consider $\Delta_\Lambda + V - E$, where $V = \mathrm{diag}\,(v_j)$, $j \in \Lambda$. We also view v as an $N\mathbf{Z}^d$-periodic function on $\mathbf{Z}^d$ in the natural way. If $\pi : \mathbf{Z}^d \to \Lambda$ is the natural projection, and $\widetilde{\nu} \in \mathbf{Z}^d$ some point in the pre-image of ν,

$$(\Delta_\Lambda + V - E)^{-1}(\mu, \nu) = \sum_{\widetilde{\mu} \in \pi^{-1}(\mu)} (\Delta + V - E)^{-1}(\widetilde{\mu}, \widetilde{\nu}). \tag{8.24}$$

Let

$$d_\lambda(\mu, \nu) = \min_{\widetilde{\mu} \in \pi^{-1}(\mu),\, \widetilde{\nu} \in \pi^{-1}(\nu)} p_\lambda(\mu - \nu)$$

be the distance on Λ, induced by the norm p_λ. Observe that in (8.21) we can introduce an arbitrarily small (but fixed) prefactor in the RHS, by modifying the choice of λ by $\mathcal{O}(1)$, which increases the $\mathcal{O}(1)$ in the exponent. Using also (8.22), we see that (8.24) converges as a geometric series and that only a fixed finite number of terms may contribute to the leading behaviour. It follows that

$$|(\Delta_\Lambda + V - E)^{-1}(\mu, \nu)| \le e^{-d_{|E|}(\mu,\nu) + \mathcal{O}(1)\frac{1+|v|_\infty}{|E|}\rho(\mu,\nu)}, \tag{8.25}$$

where ρ denotes the Euclidean distance on Λ.

Consider next the case when Λ is a subset of $\mathbf{Z}^d$. Let $V = \mathrm{diag}\,(v_j)$, $v \in \ell^\infty$ and let Δ_Λ be the discrete Laplacian on Λ. The observation after (8.1) extends:

$$\|e^{(\cdot)\eta}\Delta_\Lambda e^{-(\cdot)\eta}\|_{\mathcal{L}(\ell^2,\ell^2)} \le q(\eta),$$

and the argument there shows that

$$\|e^{(\cdot)\eta}(\Delta_\Lambda + V - E)^{-1}e^{-(\cdot)\eta}\|_{\mathcal{L}(\ell^2,\ell^2)} \le \frac{1}{|E| - \lambda - |v|_\infty}, \tag{8.26}$$

when $\eta \in W(\lambda)$, $\lambda + |v|_\infty < E$, and we get the analogue of (8.20),

$$|(\Delta_\Lambda + V - E)^{-1}(\mu, \nu)| \le \frac{1}{|E| - \lambda - |v|_\infty} e^{-p_\lambda(\mu - \nu)}, \tag{8.27}$$

and for $|E|$, $|v|_\infty$ as in (8.21):

$$|(\Delta_\Lambda + V - E)^{-1}(\mu, \nu)| \le e^{-p_{|E|}(\mu - \nu) + \mathcal{O}(1)\frac{1+|v|_\infty}{|E|}|\mu - \nu|}. \tag{8.28}$$

To establish Theorem 2.1, it only remains to combine the above estimates with the changes of variables in the preceding sections. The starting point is the identity (3.20), where

$$M(x \cdot x) = t\Delta - i\mathrm{diag}\,(k'(x_j \cdot x_j)) - E,$$
$$k'(x_j \cdot x_j) = 1 + \mathcal{O}(\epsilon),$$

so that $M = t\Delta - (E+i) + \mathrm{diag}\,(\mathcal{O}(\epsilon))$. The subsequent changes of variables lead to (7.43) (with the λ there equal to $|1+iE|$), and we get

$$\langle (t\Delta + V - (E+i0))^{-1}(\mu,\nu)\rangle = \int f(e^{i\theta(E)/2}x_t \circ \widetilde{x}_t(y))e^{-|1+iE|y\cdot y}\prod_{j\in\Lambda}\left(\frac{|1+iE|}{\pi}\mathrm{d}^2 y_j\right), \tag{8.29}$$

where

$$\begin{aligned} f(x) &= M^{-1}(x)(\mu,\nu) \\ &= (t\Delta - (E+i) + \mathrm{diag}\,\mathcal{O}(\epsilon))^{-1}(\mu,\nu) \\ &= \frac{1}{t}\left(\Delta - \frac{E+i}{t} + \mathrm{diag}\,\mathcal{O}\Big(\frac{\epsilon}{t}\Big)\right)^{-1}(\mu,\nu). \end{aligned}$$

The modulus of this expression can be bounded by

$$\frac{1}{t}e^{-d_{\left|\frac{E+i}{t}\right|}(\mu,\nu)+\mathcal{O}(1)\frac{t+\epsilon}{|E+i|}|\mu-\nu|},$$

and since we integrate f against a positive normalized measure in (8.29), we get the conclusion in the theorem.

Appendix A. The supersymmetric formalism

We give here a brief account of an algebraic formalism, which amongst its many virtues is convenient for expressing the inverse of a matrix. For more details, see *e.g.* [Be,V]. For the usage of supersymmetry in the study of random Schrödinger operators, see *e.g.* [K,KS].

1. Terminologies and notations

All algebras considered here are $\mathbf{Z}_2$ -graded associative algebras, *i.e.* can be written

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$$

with

$$\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j} \quad \text{for} \quad i,j\in\mathbf{Z}_2.$$

The grading of a homogeneous element a is called *parity* and is denoted by $\tilde{a}$. The supercommutator of two homogeneous elements of an associative, graded algebra is defined as

$$[a,b] = ab - (-1)^{\tilde{a}\tilde{b}}ba.$$

If the commutator is equal to zero, then the elements *commute*. If all the commutators are equal to zero, then the algebra is *commutative*.

An algebra homomorphism from $\mathcal{A}$ to $\mathcal{A}$ is even if it preserves the grading, and odd if it exchanges even and odd elements.

2. Differential calculus on $\mathbf{R}^{n|m}$ and $\mathbf{C}^{n|m}$

Functions of odd variables

Polynomials are the simplest functions of ordinary analysis. A polynomial algebra with n generators is generated by n (*even*) commuting variables x^a $(a = 1, \ldots, n)$, and is written $\mathbf{R}[x_1, \ldots, x_n]$. (With the above convention, *even* means that $[x^a, x^b] = x^a x^b - x^b x^a = 0$.)

The *Grassmann algebra with m generators* is analogously generated by m *odd* commuting variables ξ^μ $(\mu = 1, \ldots, m)$ which satisfy the relations

$$[\xi^\mu, \xi^\nu] = \xi^\mu \xi^\nu + \xi^\nu \xi^\mu = 0.$$

It is denoted $\Lambda[\xi_1, \ldots, \xi_m]$. It is natural to consider its elements to be analogues of polynomials, and in fact of all C^∞-functions of even variables. Indeed, in the even case all functions can be obtained from polynomials by taking limits, but the Grassmann algebra is complete in itself. We will think of the Grassmann algebra $\Lambda[\xi_1, \ldots, \xi_m]$ as the algebra of "C^∞-functions of m anti-commuting variables $\xi_1, \ldots, \xi_m$". The general form of such a function of odd variables is

$$f(\xi) = f_0 + \xi^\mu f_\mu + \xi^{\mu_1} \xi^{\mu_2} f_{\mu_1 \mu_2} + \cdots + \xi^{\mu_1} \cdots \xi^{\mu_m} f_{\mu_1 \cdots \mu_m}, \tag{A.1}$$

where all repeated indices are summed over and the coefficients $f_{\mu_1 \cdots \mu_m}$ are real and antisymmetric in $\mu_1 \cdots \mu_m$. The parity of the function depends on the number of ξ^μ factors. The space of functions of variables ξ^μ, $\mu = 1, \cdots, m$ is a $\mathbf{Z}_2$ -algebra; it has dimension 2^m, with even and odd parts both having dimension 2^{m-1}. We define $f_0 = f(0)$ to be the value of the function at zero.

The algebra $\mathrm{C}^\infty(\mathbf{R}^{n|m})$

We will now assume that we consider expressions of the type (A.1) with the coefficients being C^∞-functions of n variables $x_1, \ldots, x_n$. The set of all such "functions" is called $\mathrm{C}^\infty(\mathbf{R}^{n|m})$. An element of $\mathrm{C}^\infty(\mathbf{R}^{n|m})$ is called a smooth (super)-function of the variables $x_1, \ldots, x_n, \xi_1, \ldots, \xi_m$. We will write

$$\begin{aligned} f(x, \xi) &= f(x_1, \ldots, x_n, \xi_1, \ldots, \xi_m) \\ &= f_0(x) + \xi^1 f_1(x) + \cdots \xi^m f_m(x) + \cdots \xi^{\mu_1} \ldots \xi^{\mu_m} f_{\mu_1 \ldots \mu_m}(x), \end{aligned} \tag{A.2}$$

where the variables $x^1, \ldots, x^n$ are the even variables, and the variables $\xi^1, \ldots, \xi^m$ are the odd variables. The map $f_0(x) = f(x, 0)$ defined on $\mathbf{R}^n$ is called the scalar function associated to f. We have :

$$\mathrm{C}^\infty(\mathbf{R}^{n|m}) = \mathrm{C}^\infty(\mathbf{R}^n) \otimes \Lambda[\xi_1, \ldots, \xi_m].$$

$\mathrm{C}^\infty(\mathbf{R}^{n|m})$ is itself a $\mathbf{Z}_2$-graded algebra: $\mathrm{C}^\infty(\mathbf{R}^{n|m}) = \mathrm{C}^\infty(\mathbf{R}^{n|m})_0 \oplus \mathrm{C}^\infty(\mathbf{R}^{n|m})_1$. Elements of $\mathrm{C}^\infty(\mathbf{R}^{n|m})_0$ are called even functions, and elements of $\mathrm{C}^\infty(\mathbf{R}^{n|m})_1$ are called odd functions.

Examples

1. $m = 0$. Then $\mathrm{C}^\infty(\mathbf{R}^{n|0})$ is the ordinary $\mathrm{C}^\infty(\mathbf{R}^n)$.

2. $n=0$. Then $C^\infty(\mathbf{R}^{0|m})$ is the usual Grassmann algebra $\Lambda[\xi_1,\ldots,\xi_m]$.

3. $n=m$. Then $C^\infty(\mathbf{R}^{n|n})$ is the algebra of differential forms on $\mathbf{R}^n$.

In this paper, we are mainly interested in the case $n=m$. Hence all the algebraic operations that we describe below can be seen as (perhaps) a more compact way of writing the standard operations on differential forms.

Non-linear transformations

Assume that we consider two algebras $C^\infty(\mathbf{R}^{n|m})$ and $C^\infty(\mathbf{R}^{n'|m'})$, where the variables for the first algebra are denoted by x^a,ξ^μ, and the variables for the second algebra are denoted by t^b,τ^ν. Suppose we are given $f(x,\xi)\in C^\infty(\mathbf{R}^{n|m})$ and n *even* functions in $C^\infty(n'|m')$: x^a and m *odd* functions ξ^μ:

$$\begin{aligned} x^a &= x^a(t,\tau),\\ \xi^\mu &= \xi^\mu(t,\tau), \end{aligned} \tag{A.3}$$

We can then define $f(x^1(t,\tau),\ldots,x^n(t,\tau),\xi^1(t,\tau),\ldots,\xi^m(t,\tau))$ by substitution as follows. If $f=f(x,\xi)$ is a polynomial (*i.e.* all the coefficients are polynomials in x), then the result of substitution (A.3) is obvious. For an arbitrary smooth function the result of the substitution is determined by Taylor's formula. The even function $x^a(t,\tau)$ is separated into a numerical part $x^a(t,0)$ and a nilpotent supplement $h^a(t,\tau)=x^a(t,\tau)-x^a(t,0)$. For each coefficient $f_{\mu_1\ldots\mu_k}$, we can expand

$$\begin{aligned} f_{\mu_1\ldots\mu_k}(x(t,\tau)) &= f_{\mu_1\ldots\mu_k}(x(t,0)+x(t,\tau)-x(t,0))\\ &= f_{\mu_1\ldots\mu_k}(x(t,0)) + \mathrm{d}f_{\mu_1\ldots\mu_k}(x(t,0))(x(t,\tau)-x(t,0))\\ &\quad+\cdots. \end{aligned}$$

Because of nilpotency the above Taylor series contains, in fact, only a finite number of terms.

Example. $\sin(t+\tau^1\tau^2) = \sin t + \tau^1\tau^2\cos t$.

In this fashion, we define a change of variables. The use of Taylor's formula to extend a function from numerical values to all even elements of a Grassmann algebra is called *Grassmann analytic continuation.*

From the rules for manipulating power series it follows that substitution (A.3) possesses the natural property "associativity": the result of two consecutive substitutions does not depend on the "arrangement of brackets". Thus one can deal with non-linear transformations of even and odd variables just as with changes of variables in classical analysis.

Differentiation

Derivatives with respect to odd variables are defined by algebraic rules:

$$\frac{\partial}{\partial\xi}(\xi)=1$$

together with linearity and the super-Leibniz formula (see below). For differentiating with respect to even variables one differentiates the coefficients in (A.1). We now use collective notations–we let x^A stand for both x^a and ξ^μ. For simplicity, we let $|A|$ denote the parity

of x^A, *i.e.* $|A| = \widetilde{x^A}$. If x^A is even, then $|A| = 0$. If x^A is odd, then $|A| = 1$. Let c be a (numerical) constant. The properties of partial derivatives (in collective notation) are:
(linearity)

$$\frac{\partial}{\partial x^A}(cf) = c\frac{\partial f}{\partial x^A} \quad \text{and} \quad \frac{\partial}{\partial x^A}(f+g) = \frac{\partial f}{\partial x^A} + \frac{\partial g}{\partial x^A},$$

(*the Leibniz formula*)

$$\frac{\partial}{\partial x^A}(fg) = \frac{\partial f}{\partial x^A}g + (-1)^{|A|\tilde{f}} f\frac{\partial g}{\partial x^A},$$

(*derivative of the composition of two functions*)

$$\frac{\partial}{\partial x^A}(f(y(x))) = \frac{\partial y^B}{\partial x^A}\frac{\partial f}{\partial y^B}.$$

(Note the order.) The parity of the derivative is equal to the parity of the corresponding variable (*i.e.* $\partial/\partial x^A$ maps even to even and odd to odd, or exchanges even and odd according to whether x^A is even or odd). The partial derivatives *commute*:

$$\frac{\partial^2 f}{\partial x^A \partial x^B} = (-1)^{|A||B|}\frac{\partial^2 f}{\partial x^B \partial x^A},$$

and *Taylor's formula* is valid:

$$f(x+h) = f(x) + h^A\frac{\partial f}{\partial x^A}(x) + \frac{1}{2}h^B h^A \frac{\partial^2 f}{\partial x^A \partial x^B}(x) + \cdots + \mathbf{O}(h^{k+1}).$$

(Note the order. The symbol $\mathbf{O}$ has its natural meaning.)

By analogy, one can also define the notion of (super)vector fields, which we do not elaborate here. See however (A.5) for an example of such a vector field.

In general all naturally formulated analogues of the assertions in an analysis course carry over to the supercase. The most important of them is the *implicit function theorem*: the system of equations

$$F^A(x,y) = 0$$

is uniquely solvable with respect to the variables $x = (x^A)$ if the matrix of partial derivatives $(\partial F^A/\partial x^B)$ is invertible (see below). Then the solution (x^A) can be expressed as a smooth function of the variables $y = (y^K)$ (a square matrix is invertible if and only if its even-even and odd-odd blocks are invertible, see below).

Example

The change of variables

$$x^a = x^a(x', \xi') = x_0^a(x') + \mathbf{O}(\xi'^2),$$
$$\xi^\mu = \xi^\mu(x', \xi') = \xi'^{\mu'} T^\mu_{\mu'}(x') + \mathbf{O}(\xi'^3).$$

The variables x', ξ' will be expressible in terms of the variables x, ξ if the numerical matrices $(\partial x_0^a/\partial x^{b'})$ and $(T^{\mu}_{\mu'})$ are invertible. (This should be compared with the fact that an element of the Grassmann algebra of the form $g = g_1 + g_2$, where g_1 is a scalar and g_2 the nilpotent part has an inverse, if and only if $g_1 \neq 0$.) Such a change is called *non-degenerate.*

The algebra $\mathcal{H}(U^{n|m})$

In this paper, we are in fact more concerned with expressions of the type (A.2) with the coefficients being holomorphic functions of n variables $z_1 \cdots z_n$ in an open set $U^n \subset \mathbf{C}^n$. Complex odd coordinates are $\zeta_j = \xi_j + i\eta_j$ and $\bar{\zeta}_j = \xi_j - i\eta_j$ where ξ_j, η_j $(j = 1 \cdots m)$ are the generators of a Grassmann algebra. A holomorphic function, *i.e.* an element of $\mathcal{H}(U^{n|m})$ is then of the form

$$\begin{aligned} f(z,\zeta) &= f(z_1,\cdots,z_n,\zeta_1,\cdots,\zeta_m) \\ &= f_0(z) + \zeta^1 f_1(z) + \cdots \zeta^m f_m(z) + \cdots \zeta^{\mu_1} \cdots \zeta^{\mu_m} f_{\mu_1 \ldots \mu_m}(z), \end{aligned}$$

where the coefficients are holomorphic functions of z in $U^n \subset \mathbf{C}^n$. We have therefore

$$\mathcal{H}(U^{n|m}) = \mathcal{H}(U^n) \otimes \Lambda[\zeta_1,\cdots,\zeta_m].$$

Naturally, all the statements that we have made so far carry over with holomorphic functions replacing C^∞ functions.

3. The Berezin Integral

The integral for a differential algebra

The definition of an integral with respect to odd variables emerges from the following general algebraic construction, obtained from a formal variational calculation. Suppose we have a commutative algebra A with an operator ∂ – a 'differential' (but not endowed with any kind of $\partial^2 = 0$ property). Then the equivalence f mod ∂A is called the *integral* of the element $f \in A$. If ∂ is a differentiation of the algebra A, then 'integration by parts' works. This construction is used to model the integral of functions of a single variable.

Example

On the algebra of functions of compact support $C_0^\infty(\mathbf{R})$, taking ∂ to be the ordinary derivative, the integral coincides with the ordinary integral over $\mathbf{R}$.

The Berezin integral over $\mathbf{R}^{n|m}$

We first consider the algebra $C^\infty(\mathbf{R}^{0|1})$. It is spanned by the functions 1 and ξ. The operator $\partial/\partial\xi$ annihilates 1 and turns ξ into 1. The corresponding integral of the function $f = f_0 + \xi f_1$ is therefore equal to the coefficient f_1 up to normalization. We write:

$$\int_{\mathbf{R}^{0|1}} \mathrm{d}\xi \quad 1 = 0, \qquad \int_{\mathbf{R}^{0|1}} \mathrm{d}\xi \quad \xi = 1.$$

The operation of integration is odd. We assign parity 1 to the symbol $\mathrm{d}\xi$. Therefore its permutation with functions follows the supercommutator rule.

We define a multiple integral over $\mathbf{R}^{n|m}$ to be a repeated integral. To do this we assign parity 1 to $\mathrm{d}x$ in $\mathbf{R} = \mathbf{R}^{1|0}$. We define

$$\begin{aligned} \mathrm{d}(x,\xi) &= \mathrm{d}(x^1,\cdots,x^n,\xi^1,\cdots,\xi^m) \\ &= \mathrm{d}x^1\,\mathrm{d}x^2\cdots\mathrm{d}x^n\,\mathrm{d}\xi^1\cdots\mathrm{d}\xi^m \end{aligned}$$

for $\mathbf{R}^{n|m}$.

Let $f \in \mathrm{C}^\infty(\mathbf{R}^{n|m})$ be such that all of its coefficients are in $\mathcal{S}(\mathbf{R}^n)$. If the term of highest degree in ξ is $\xi^m\cdots\xi^1 a(x)$, we obtain by using the parity conventions for $\mathrm{d}x$, $\mathrm{d}\xi$,

$$\int_{\mathbf{R}^{n|m}} \mathrm{d}(x,\xi) f(x,\xi) = (-1)^{n(n-1)/2}\int_{\mathbf{R}^n} a(x)\,\mathrm{d}x^1\cdots\mathrm{d}x^n.$$

Let $x = (x^A)$ be the collective symbol of (x,ξ). Let $\mathrm{d}x$ denote $\mathrm{d}(x,\xi)$. Let c be a (numerical) constant. Then the following properties can be verified directly:
(*linearity*)

$$\begin{aligned} \int_{\mathbf{R}^{n|m}} (f(x)+g(x))\,\mathrm{d}x &= \int_{\mathbf{R}^{n|m}} f(x)\,\mathrm{d}x + \int_{\mathbf{R}^{n|m}} g(x)\,\mathrm{d}x\,, \\ \int_{\mathbf{R}^{n|m}} cf(x)\,\mathrm{d}x &= c\int_{\mathbf{R}^{n|m}} f(x)\,\mathrm{d}x\,, \end{aligned}$$

(*differentiation under the integral sign*)

$$\frac{\partial}{\partial y}\int_{\mathbf{R}^{n|m}} f(x,y)\,\mathrm{d}x = (-1)^{n\tilde{y}}\int_{\mathbf{R}^{n|m}} \frac{\partial f}{\partial y}(x,y)\,\mathrm{d}x\,,$$

(*integral of a derivative and integration by parts*)

$$\begin{aligned} \int_{\mathbf{R}^{n|m}} \frac{\partial}{\partial x^A}(f(x))\,\mathrm{d}x &= 0, \\ \int_{\mathbf{R}^{n|m}} \frac{\partial f}{\partial x^A} g\,\mathrm{d}x &= (-1)^{|A|\tilde{f}}\int_{\mathbf{R}^{n|m}} f\frac{\partial g}{\partial x^A}\,\mathrm{d}x\,, \end{aligned}$$

and (*Fubini's theorem-reduction to a repeated integral*)

$$\int_{\mathbf{R}^{n|m}\times\mathbf{R}^{p|q}} \mathrm{d}x\;\mathrm{d}y\, f(x,y) = (-1)^{(n+m)p}\int_{\mathbf{R}^{n|m}} \mathrm{d}x\int_{\mathbf{R}^{p|q}} \mathrm{d}y\, f(x,y).$$

Clearly all the above properties hold in the case $\mathbf{C}^{n|m}$ under appropriate conditions on the coefficients.

In the special case $n = m$, the Berezin integral can be seen as follows. We consider an inhomogeneous differential form on $\mathbf{R}^n$ as a function of the variables x^a and $\mathrm{d}x^a$, where $\widetilde{\mathrm{d}x^a} = 1$:

$$\omega(x,\,\mathrm{d}x) = \omega^{(0)} + \omega^{(1)} + \cdots + \omega^{(n)}.$$

Then

$$\int_{\mathbf{R}^n}\omega = \int_{\mathbf{R}^n}\omega^{(n)} = \pm\int_{\mathbf{R}^{n|n}} \omega(x,\,\mathrm{d}x)\mathrm{d}(x,\,\mathrm{d}x).$$

Change of variables in the integral

Suppose we have a non-degenerate coordinate transformation

$$x^a = x^a(x', \xi')$$
$$\xi^\mu = \xi^\mu(x', \xi')$$

with Jacobian matrix

$$J := \frac{\partial(x,\xi)}{\partial(x',\xi')} := \begin{pmatrix} \partial x/\partial x' & \partial\xi/\partial x' \\ \partial x/\partial\xi' & \partial\xi/\partial\xi' \end{pmatrix}.$$

It can be shown (see *e.g.* [V]) from general algebraic considerations that there exists an essentially unique scalar function, (*i.e.* a function which is of degree zero in ξ), associated with J, called the *Berezinian* of J, denoted by Ber J. It is the generalization (counterpart) in $\mathbf{R}^{n|m}$ of the notion of the determinant in $\mathbf{R}^n$. Let $g_{i,j}$ $(i, j = 0, 1)$ be the blocks of J, *i.e.*

$$J(x', \xi') := \begin{pmatrix} g_{00}(x',\xi') & g_{01}(x',\xi') \\ g_{10}(x',\xi') & g_{11}(x',\xi') \end{pmatrix}.$$

Then it can be shown by using Gauss's method that

$$\begin{aligned}(\mathrm{Ber}\, J)(x') &= \frac{\det(g_{00}(x',0) - [g_{01}g_{11}^{-1}g_{10}](x',0))}{\det g_{11}(x',0)} \\ &= \frac{\det g_{00}(x',0)}{\det(g_{11}(x',0) - [g_{10}g_{00}^{-1}g_{01}](x',0))}.\end{aligned}$$

Define

$$\frac{\mathrm{d}(x,\xi)}{\mathrm{d}(x',\xi')} := \mathrm{Ber}\, J = \mathrm{Ber}\left\{\frac{\partial(x,\xi)}{\partial(x',\xi')}\right\}.$$

We then have

THEOREM A.1. – *Let the function $f(x,\xi)$ on $\mathbf{R}^{n|m}$ be such that all of its coefficients are in $\mathcal{S}(\mathbf{R}^n)$. Then we have the equality*

$$\begin{aligned}\int_{\mathbf{R}^{n|m}} f(x,\xi)\mathrm{d}(x,\xi) =&\mathrm{sign}\{\det[(\partial x/\partial x')(x',0)]\} \\ &\int_{\mathbf{R}^{n|m}} f(x(x',\xi'),\xi(x',\xi'))\frac{\mathrm{d}(x,\xi)}{\mathrm{d}(x',\xi')}\mathrm{d}(x',\xi').\end{aligned}$$

Change of contours in $U^{n|m}$

Let $f(z,\zeta) \in \mathcal{H}(U^{n|m})$. Let Γ^n be an open set in U^n. Assume all the coefficients of f are rapidly decreasing in Γ^n so that contour integration in Γ^n is well defined. The superanalogue of the usual Stokes' formula (see e.g. [V]) then allows us to make a change of contours in $U^{n|m}$. Specifically, assuming $\mathbf{R}^n \subset \Gamma^n \subset U^n$, $(e^{i\theta}\mathbf{R})^n \subset \Gamma^n \subset U^n$ for some $\theta \neq 0$, we make the following change of contours in sect. 3:

$$\int_{\mathbf{R}^{n|m}} f(z,\zeta)d(z,\zeta) = \int_{(e^{i\theta}\mathbf{R})^{n|m}} f(z,\zeta)d(z,\zeta) \tag{A.4}$$

by using the superstokes' formula.

4. Supersymmetry on $\mathbf{R}^{2n|2n}$

We will now consider the special case of the superspace $\mathbf{R}^{2n|2n}$. It will be more convenient to change our notations. We group the $4n$ (super)commuting variables into $2n$ pairs of coordinates: let $x_i \in \mathbf{R}^2$ $(i = 1, \cdots, n)$ be the *even* commuting coordinates; ξ_i, η_i $(i = 1, \cdots, n)$ be the *odd* commuting coordinates:

$$\begin{aligned}
[\xi_i, \xi_j] &= 0 \\
[\eta_i, \eta_j] &= 0 \\
[\eta_i, \xi_j] &= 0.
\end{aligned}$$

We use the composite notation $X_i = (x_i, \xi_i, \eta_i)$. We define the (super)dot product:

$$\begin{aligned}
X_i \cdot X_j &:= \mathrm{D}(X_i, X_j) \\
&:= f_0(x_i, x_j) + f_1(x_i, x_j)\eta_i\xi_j + f_2(x_i, x_j)\eta_j\xi_i \\
&:= x_i \cdot x_j + \frac{1}{2}(\eta_i\xi_j + \eta_j\xi_i)
\end{aligned}$$

where $x_i \cdot x_j$ denotes the usual inner product of x_i and x_j in $\mathbf{R}^2$. Note that when $i = j$,

$$X_i \cdot X_i = x_i \cdot x_i + \eta_i\xi_i.$$

Supersymmetries are defined to be the set of coordinate transformations that leave the above dot product invariant. Two obvious transformations that leave D invariant are the usual rotations O in $\mathbf{R}^2$,

$$x_i = x_i'O \qquad (i = 1, \cdots, n)$$

and the transformations $A \in Sp(2)$ acting on ξ_i, η_i $(i = 1, \cdots, n)$ such that

$$(\xi_i, \eta_i) = (\xi_i', \eta_i')A,$$

where $\{x_i', \xi_i', \eta_i'\}_{i=1}^n$ is another set of coordinates on $\mathbf{R}^{2n|2n}$, x_i' being the even ones and ξ_i', η_i' the odd ones. We put $X_i' = (x_i', \xi_i', \eta_i')$, $(i = 1, \cdots, n)$. Aside from these two linear transformations, supersymmetries also include transformations generated by (super)vector fields of the type:

$$\begin{aligned}
V &= \sum_i V_i \\
&= \sum_i (\xi_i a + \eta_i b)\frac{\partial}{\partial x_i} + 2(b \cdot x_i)\frac{\partial}{\partial \xi_i} - 2(a \cdot x_i)\frac{\partial}{\partial \eta_i},
\end{aligned} \tag{A.5}$$

where a, $b \in \mathbf{R}^2$, and

$$\begin{aligned}
a\frac{\partial}{\partial x_i} &:= a_1\frac{\partial}{\partial x_{i,1}} + a_2\frac{\partial}{\partial x_{i,2}}, \\
b\frac{\partial}{\partial x_i} &:= b_1\frac{\partial}{\partial x_{i,1}} + b_2\frac{\partial}{\partial x_{i,2}}.
\end{aligned} \tag{A.6}$$

(Note that it is the same transformation in all the X_i.) As before the above transformation is to be understood in the algebraic sense. We check that $VD(X_i, X_j) = 0$. We check also that the Berezinian corresponding to such a change of variables is 1.

Let τ be a supersymmetric transformation. Let $X_i = \tau X_i'$ $(i = 1, \cdots, n)$.

DEFINITION. – *A superfunction F is supersymmetric if it is invariant under all supersymmetries:*

$$F(X_1, \cdots, X_n) = F(X_1', \cdots, X_n'),$$

for all τ supersymmetric transformations.

Clearly, supersymmetric functions belong to a rather restricted class of functions. For example, in $\mathbf{R}^{2|2}$, F is supersymmetric if and only if there exists f: $[0, \infty) \mapsto \mathbf{R}$ of class C^∞, such that

$$F(X) = f(X \cdot X) = f(x \cdot x) + f'(x \cdot x)\eta\xi.$$

For the general classification in $\mathbf{R}^{2n|2n}$, see *e.g.* [KS].

Define $\mathrm{d}X_i = (\mathrm{d}^2 x_i/\pi)\mathrm{d}\eta_i\, \mathrm{d}\xi_i$ $(i = 1, \cdots, n)$. One of the most useful properties of the supersymmetric functions is the following:

THEOREM A.2. – (see *e.g.*[K]) *If F is supersymmetric with all of its coefficients in $\mathcal{S}(\mathbf{R}^{2n})$, then*

$$\int F(X_1, \cdots, X_n)\, \mathrm{d}X_1 \cdots \mathrm{d}X_n = F(0, \cdots 0). \tag{A.7}$$

5. An Expression for the Inverse of a Matrix

Let A be an operator on $\ell^2(\Lambda)$, where Λ is some finite index set. Let $|\Lambda|$ be the number of elements in Λ. Assume $A = A_1 + iA_2$, where A_1, A_2 are real symmetric matrices with $A_1 > 0$. We then have the following well-known Gaussian integrals on $\mathbf{R}^{2|\Lambda|}$:

$$\int e^{-\sum_{i,j\in\Lambda} A_{ij} x_i \cdot x_j} \prod_{j\in\Lambda} \frac{\mathrm{d}^2 x_j}{\pi} = \frac{1}{\det A} \tag{A.8}$$

and

$$\int x_a \cdot x_b e^{-\sum_{i,j\in\Lambda} A_{ij} x_i \cdot x_j} \prod_{j\in\Lambda} \frac{\mathrm{d}^2 x_j}{\pi} = \frac{(A^{-1})_{ab}}{\det A} \tag{A.9}$$

where a, $b \in \Lambda$. Using the construction made so far in this section, we also have the following counterpart on $\mathbf{R}^{0|(2|\Lambda|)}$:

$$\int e^{-\sum_{i,j\in\Lambda} A_{ij} \eta_i \cdot \xi_j} \prod_{j\in\Lambda} \mathrm{d}\eta_j\, \mathrm{d}\xi_j = \det A \tag{A.10}$$

and

$$\int \xi_a \eta_b e^{-\sum_{i,j\in\Lambda} A_{ij} \eta_i \xi_j} \prod_{j\in\Lambda} \mathrm{d}\eta_j\, \mathrm{d}\xi_j = (A^{-1})_{ab}(\det A). \tag{A.11}$$

Combining (A.5) and (A.6), (A.4) and (A.7), we finally have the following expressions for the inverse of A, expressed as a Berezin integral:

$$\begin{aligned}(A^{-1})_{ab} &= \int x_a \cdot x_b e^{-\sum_{i,j\in\Lambda} A_{ij} X_i \cdot X_j} \prod_{j\in\Lambda} \mathrm{d}X_j \\ &= \int \xi_a \eta_b e^{-\sum_{i,j\in\Lambda} A_{ij} X_i \cdot X_j} \prod_{j\in\Lambda} \mathrm{d}X_j. \end{aligned} \tag{A.12}$$

This is precisely the representation that we used in sect. 2.

6. An Integration by Parts

We now give the details of the integration by parts which led (2.22) to (2.24) in sect. 2. It was first derived by using superanalysis (*i.e.* using supervector fields etc.). In order not to venture too far in that direction, we present below a "translated" version which uses standard analysis. Define

$$L = i\Big(\sum t x_j \cdot x_k - \sum E x_j \cdot x_j - i \sum k(x_j \cdot x_j)\Big) - \log\det M(x),$$

where

$$M(x) = t\Delta - E - i\,\mathrm{diag}\,(k'(x_j \cdot x_j)), \qquad (\det M \neq 0),$$

as in (2.23) in sect. 2. Let $\langle G(\mu,\nu;E+i0)\rangle$ be as in (2.22). Let $m = |\Lambda|$. Then we have

PROPOSITION A.3.

$$\langle G(\mu,\nu;E+i0)\rangle = i^m \int M^{-1}(\mu,\nu;E) e^{-L(x)} \prod_{j\in\Lambda} \frac{\mathrm{d}^2 x_j}{\pi}.$$

Proof. – Define

$$\phi(x) = i\Big(\sum t x_j \cdot x_k - \sum E x_j \cdot x_j - i \sum k(x_j \cdot x_j)\Big).$$

We first look for a vector field v, such that

$$x_\mu e^{-\phi} = v \cdot \nabla(e^{-\phi}), \tag{A.13}$$

so

$$x_\mu = -v \cdot \nabla\phi. \tag{A.14}$$

Since

$$\nabla\phi = 2iMx, \tag{A.15}$$

we look for v of the form: $v = Bu$, where B is a matrix and $u_j = 1$ for all j. Let π_μ be the matrix, such that $(\pi_\mu)_{ij} = \delta_{i\mu}\delta_{j\mu}$. Then (A.14) can be written as

$$(\pi_\mu)x = -2i(B^t \circ M)x.$$

Therefore

$$B = \frac{i}{2} M^{-1} \circ \pi_\mu$$

is a solution. Hence

$$v = \frac{i}{2} M^{-1} \circ \pi_\mu u \tag{A.16}$$

satisfies (A.14). We now show that v in fact verifies

$$x_\mu e^{-L} = v \cdot \nabla(e^{-L}) + (\operatorname{div} v) e^{-L}. \tag{A.17}$$

Comparing (A.14) with (A.17), we see that we only need to show that

$$v \cdot (\nabla \log \det M) + \operatorname{div} v = 0. \tag{A.18}$$

Using the fact that

$$\partial_j \log \det M = -2ik''(x_j \cdot x_j)(M^{-1})_{jj} x_j$$

and the expression for v in (A.16), we easily verify that (A.18) holds. Hence (A.17) holds. We then have

$$\begin{aligned}
\langle G(\mu, \nu; E) \rangle &= i^{m+1} \int x_\mu \cdot x_\nu e^{-L(x)} \prod_{j \in \Lambda} \mathrm{d}^2 x_j \\
&= i^{m+1} \int x_\nu \cdot (v \cdot \nabla + \operatorname{div} v) e^{-L(x)} \prod_{j \in \Lambda} \mathrm{d}^2 x_j \\
&= i^m \int M^{-1}(\mu, \nu; E) e^{-L(x)} \prod_{j \in \Lambda} \mathrm{d}^2 x_j
\end{aligned}$$

by integration by parts. □

Appendix B. Direct approach to some basic formulas

In this appendix we shall give direct proofs of the normalization property (2.24) and of the formulas (2.22), (2.25). The main step will be to establish the following proposition:

PROPOSITION B.1. – *Let $x_1, x_2, .., x_m$ denote variables in $\mathbf{R}^2$, let*

$$\kappa : \mathbf{R}^{2m} \ni (x_1, .., x_m) \mapsto (x_j \cdot x_k)_{1 \le j,k \le m} \in \mathbf{R}^{m^2}. \tag{B.1}$$

Let μ be a probability measure on the space of complex symmetric $m \times m$-matrices with compact support contained in the open subset of matrices with positive definite real part and denote by $\langle \cdot \rangle$ the corresponding expectation value. For $\tau \in \mathbf{R}^{m2}$, let

$$f_A(\tau) = e^{-\sum a_{j,k} \tau_{j,k}} = e^{-A \cdot \tau}. \tag{B.2}$$

Introduce the Laplace transform:

$$\widehat{\mu}(\tau) = \int f_A(\tau) \mu(\mathrm{d}A) = \int e^{-A \cdot \tau} \mu(\mathrm{d}A). \tag{B.3}$$

Then if $\mathrm{d}^{2m}x$ *denotes the Lebesgue measure on* $\mathbf{R}^{2m}$, *we have*

$$\int \left(\det\left(-\frac{\partial}{\partial\tau_{j,k}}\right)\widehat{\mu}\right) \circ \kappa \,\frac{\mathrm{d}^{2m}x}{\pi^m} = 1, \tag{B.4}$$

$$\begin{aligned}\langle (A^{-1})_{j,k}\rangle &= \int \left(\frac{1}{2}\left[M_{j,k}\left(-\frac{\partial}{\partial\tau}\right) + M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)\right]\widehat{\mu}(\tau)\right) \circ \kappa \,\frac{\mathrm{d}^{2m}x}{\pi^m} \\ &= \int x_j\cdot x_k \left(\det\left(-\frac{\partial}{\partial\tau}\right)\widehat{\mu}(\tau)\right) \circ \kappa \,\frac{\mathrm{d}^{2m}x}{\pi^m}.\end{aligned} \tag{B.5}$$

Here $M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)$ *is the* $m\times m$ *matrix obtained from* $\left(-\frac{\partial}{\partial\tau}\right) = \left(-\frac{\partial}{\partial\tau_{\mu,\nu}}\right)_{1\le\mu,\nu\le m}$, *by replacing* $-\frac{\partial}{\partial\tau_{j,k}}$ *by 1 and all other elements on the j:th line and on the k:th column by 0.*

Notice that the integrals converge exponentially. In fact, if τ is a point in the image of κ and A belongs to the support of μ, then $\mathrm{Re}\,(A\cdot\tau) = (\mathrm{Re}\,A)x\cdot x \ge c\|x\|^2$ for some $c>0$ independent of τ and A. Here we let A act on $(\mathbf{R}^2)^m$ in the natural way.

Proof and Proposition B.1. – If $A=(a_{j,k})$ is a complex symmetric $m\times m$-matrix with $\mathrm{Re}\,A>0$, then

$$\int \det A\, e^{-\sum a_{j,k}x_j\cdot x_k}\,\frac{\mathrm{d}^{2m}x}{\pi^m} = 1, \tag{B.6}$$

which can be written

$$\int \left(\det\left(-\frac{\partial}{\partial\tau_{j,k}}\right) f_A\right) \circ \kappa \,\frac{\mathrm{d}^{2m}x}{\pi^m} = f_A(0). \tag{B.7}$$

Let $\mu(A)$ be a distribution with compact support on the space of complex symmetric $m\times m$-matrices A with $\mathrm{Re}\,A>0$. Then we can define the Laplace transform $\widehat{\mu}(\tau)$ as in the proposition, and if we apply μ to (B.7), we get

$$\int \left(\det\left(-\frac{\partial}{\partial\tau_{j,k}}\right)\widehat{\mu}\right) \circ \kappa \,\frac{\mathrm{d}^{2m}x}{\pi^m} = \widehat{\mu}(0). \tag{B.8}$$

In the case when μ is a probability measure, this reduces to (B.4).

In the general case we notice that $\frac{1}{2}(\tau_{j,k}+\tau_{k,j})\widehat{\mu}(\tau)$ is also the Laplace transform of a distribution with compact support in the space of symmetric matrices with positive real part, so (B.8) gives:

$$\int \left(\det\left(-\frac{\partial}{\partial\tau}\right)\left(\frac{1}{2}(\tau_{j,k}+\tau_{k,j})\widehat{\mu}\right)\right) \circ \kappa \,\frac{\mathrm{d}^{2m}x}{\pi^m} = 0. \tag{B.9}$$

We write this as

$$\begin{aligned}&\int \left(\left[\det\left(-\frac{\partial}{\partial\tau}\right), \frac{1}{2}(\tau_{j,k}+\tau_{k,j})\right]\widehat{\mu}\right) \circ \kappa \,\frac{\mathrm{d}^{2m}x}{\pi^m} + \\ &\qquad \int \left(\frac{1}{2}(\tau_{j,k}+\tau_{k,j})\det\left(-\frac{\partial}{\partial\tau}\right)\widehat{\mu}\right) \circ \kappa \,\frac{\mathrm{d}^{2m}x}{\pi^m} = 0,\end{aligned} \tag{B.10}$$

or

$$\int \left(\frac{1}{2}(\tau_{j,k}+\tau_{k,j})\det\left(-\frac{\partial}{\partial\tau}\right)\widehat{\mu}\right)\circ\kappa\,\frac{\mathrm{d}^{2m}x}{\pi^m} = \int \left([\frac{1}{2}(\tau_{j,k}+\tau_{k,j}),\det\left(-\frac{\partial}{\partial\tau}\right)]\widehat{\mu}\right)\circ\kappa\,\frac{\mathrm{d}^{2m}x}{\pi^m}. \tag{B.11}$$

Put

$$M_{j,k}\left(-\frac{\partial}{\partial\tau}\right) = [\tau_{k,j},\det\left(-\frac{\partial}{\partial\tau}\right)] = [\det\left(-\frac{\partial}{\partial\tau}\right),-\tau_{k,j}]$$

and notice that $M_{j,k}$ is indeed the differential operator in the proposition. Consequently, by summing over a column:

$$\det\left(-\frac{\partial}{\partial\tau}\right) = \sum_j M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)\left(-\frac{\partial}{\partial\tau_{j,k}}\right),$$

and more generally,

$$\det\left(-\frac{\partial}{\partial\tau}\right)\delta_{k,\widetilde{k}} = \sum_j M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)\left(-\frac{\partial}{\partial\tau_{j,\widetilde{k}}}\right),$$

so if $M = (M_{j,k})$, we obtain:

$$M\left(-\frac{\partial}{\partial\tau}\right)\circ\left(-\frac{\partial}{\partial\tau}\right) = \left(-\frac{\partial}{\partial\tau}\right)\circ M\left(-\frac{\partial}{\partial\tau}\right) = \det\left(-\frac{\partial}{\partial\tau}\right)\otimes I.$$

Formally we can write:

$$M\left(-\frac{\partial}{\partial\tau}\right) = \left(-\frac{\partial}{\partial\tau}\right)^{-1}\det\left(-\frac{\partial}{\partial\tau}\right)\otimes I.$$

Rewrite (B.11):

$$\begin{aligned}\int \frac{1}{2}\left(M_{j,k}\left(-\frac{\partial}{\partial\tau}\right)+M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)\right)\widehat{\mu}\circ\kappa\,\frac{\mathrm{d}^{2m}x}{\pi^m}& \\ = \int \left(\frac{1}{2}(\tau_{j,k}+\tau_{k,j})\det\left(-\frac{\partial}{\partial\tau}\right)\widehat{\mu}\right)\circ\kappa\,\frac{\mathrm{d}^{2m}}{\pi^m}.&\end{aligned} \tag{B.12}$$

Since

$$M_{j,k}\left(-\frac{\partial}{\partial\tau}\right)e^{-\sum a_{j,k}\tau_{j,k}} = \left(A^{-1}\right)_{j,k}e^{-\sum a_{j,k}\tau_{j,k}}\det A,$$

we can apply this to $\widehat{\mu} = f_A$ (so that μ is the δ-measure at A,) and get with (B.4):

$$\begin{aligned}A^{-1}{}_{j,k} &= \int \frac{1}{2}\left(M_{j,k}\left(-\frac{\partial}{\partial\tau}\right)+M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)\right)e^{-\sum a_{j,k}\tau_{j,k}}\circ\kappa\,\frac{\mathrm{d}^{2m}x}{\pi^m} \\ &= \int x_j\cdot x_k\left(\det\left(-\frac{\partial}{\partial\tau}\right)e^{-\sum a_{j,k}\tau_{j,k}}\right)\circ\kappa\,\frac{\mathrm{d}^{2m}x}{\pi^m}.\end{aligned} \tag{B.13}$$

If μ is a probability measure as in the proposition, we get (B.5) by taking the expectation value of (B.13) w.r.t. μ. □

We now apply Proposition B.1 to get the formulas (2.24), (2.22), (2.25). We will take $m = |\Lambda|$, $\mathbf{R}^{2m} = (\mathbf{R}^2)^\Lambda$. Let $g(v)\mathrm{d}v$ be a probability measure on $\mathbf{R}$, with compact support to start with. For $E \in \mathbf{R}$, $t, \eta > 0$, let $\mu = \mu_{t,E+i\eta}(\mathrm{d}A)$ be the probability measure with support on the set of matrices of the form,

$$it\Delta_\Lambda + i\mathrm{diag}\,(v_j) - iE + \eta \ v_j \in \mathbf{R}, \tag{B.14}$$

given by $\prod g(v_j)\mathrm{d}v_j$. Then we get the Laplace transform:

$$\begin{aligned}\widehat{\mu}_{t,E+i\eta}(\tau) &= \int e^{-(it\Delta_\Lambda - iE+\eta+i\mathrm{diag}\,(v_j))\cdot\tau} \prod g(v_j)\mathrm{d}v_j \\ &= e^{-(it\Delta_\Lambda - iE+\eta)\cdot\tau} \prod \widehat{g}(\tau_{j,j}),\end{aligned}$$

where $\widehat{g}$ denotes the Fourier transform.

If $\langle\cdot\rangle_g$ denotes the expectation value w.r.t. $\prod g(v_j)\mathrm{d}v_j$, we get from Proposition B.1:

$$1 = \int \left(\det\left(-\frac{\partial}{\partial\tau}\right)\left(e^{(it\Delta_\Lambda - iE+\eta)\cdot\tau} \prod \widehat{g}(\tau_{j,j})\right)\right) \circ \kappa \prod \left(\frac{\mathrm{d}^2 x_j}{\pi}\right), \tag{B.15}$$

$$\begin{aligned}&\langle (t\Delta_\Lambda + \mathrm{diag}\,(v_j) - (E+i\eta))^{-1}{}_{j,k}\rangle_g = \qquad\qquad (\mathrm{B}.16)\\ &i\int \left(\left(\frac{1}{2}M_{j,k}\left(-\frac{\partial}{\partial\tau}\right) + \frac{1}{2}M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)\right)\left(e^{(it\Delta_\Lambda - iE+\eta)\cdot\tau} \prod \widehat{g}(\tau_{j,j})\right)\right) \circ \kappa \prod \left(\frac{\mathrm{d}^2 x_j}{\pi}\right) \\ &= i\int x_j \cdot x_k \left(\det\left(-\frac{\partial}{\partial\tau}\right)\left(e^{(it\Delta_\Lambda - iE+\eta)\cdot\tau} \prod \widehat{g}(\tau_{j,j})\right)\right) \circ \kappa \prod \left(\frac{\mathrm{d}^2 x_j}{\pi}\right).\end{aligned}$$

Now replace the assumption that g has compact support by the assumption that $\widehat{g}_{|]0,\infty[}$ belongs to $\mathcal{S}([0,\infty[)$; the space of smooth functions on $[0,\infty[$ which decay rapidly near infinity together with all their derivatives. Since g is real, we also have $\widehat{g}_{|]-\infty,0[} \in \mathcal{S}(]-\infty,0])$. Notice that $\widehat{g}$ is continuous, being the Fourier transform of a measure of finite mass. Let $0 \le \chi \in C_0^\infty(\mathbf{R})$ be even with $\chi(0) = 1$, and approximate g by

$$g_\epsilon = \frac{1}{\int \chi(\epsilon v)g(v)\mathrm{d}v}\chi(\epsilon v)g(v),$$

which is a probability measure with compact support when $\epsilon > 0$ is small enough. Then

$$\widehat{g}_\epsilon = \frac{1}{2\pi}\frac{1}{\epsilon}\widehat{\chi}\left(\frac{\cdot}{\epsilon}\right) * \widehat{g},$$

where $*$ indicates convolution, and $\widehat{g}_\epsilon \to \widehat{g}$, $\frac{\partial}{\partial\tau}\widehat{g}_\epsilon \to \frac{\partial}{\partial\tau}\widehat{g}$ pointwise away from 0, when $\epsilon \to 0$. Moreover

$$|\widehat{g}_\epsilon(\tau)| + |\frac{\partial}{\partial\tau}\widehat{g}_\epsilon(\tau)| \le C_N\langle\tau\rangle^{-N}, \tag{B.17}$$

uniformly in ϵ for every $N \ge 0$.

(B.15,16) hold with g replaced by g_ϵ, and since we never get any higher derivatives of $\widehat{g}$ in these formulas, we can apply the dominated convergence theorem and (B.17), to conclude that (B.15), (B.16) also hold for g. Using again that $\widehat{g}_{|]0,\infty[} \in \mathcal{S}([0,\infty[)$, we can let η tend to 0 in these equations and get:

$$1 = \int \left(\det\left(-\frac{\partial}{\partial\tau}\right)\left(e^{(it\Delta_\Lambda - iE)\cdot\tau}\prod \widehat{g}(\tau_{j,j})\right)\right)\circ\kappa \prod\left(\frac{\mathrm{d}^2 x_j}{\pi}\right), \tag{B.18}$$

$$\begin{aligned}
&\langle (t\Delta_\Lambda + \operatorname{diag}(v_j) - (E+i0))^{-1}{}_{j,k}\rangle_g = \qquad\qquad (\text{B.19})\\
&i\int\left(\left(\frac{1}{2}M_{j,k}\left(-\frac{\partial}{\partial\tau}\right) + \frac{1}{2}M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)\right)\left(e^{(it\Delta_\Lambda - iE)\cdot\tau}\prod\widehat{g}(\tau_{j,j})\right)\right)\circ\kappa\prod\left(\frac{\mathrm{d}^2x_j}{\pi}\right)\\
&= i\int x_j\cdot x_k\left(\det\left(-\frac{\partial}{\partial\tau}\right)\left(e^{(it\Delta_\Lambda - iE)\cdot\tau}\prod\widehat{g}(\tau_{j,j})\right)\right)\circ\kappa\prod\left(\frac{\mathrm{d}^2x_j}{\pi}\right),
\end{aligned}$$

where by definition the first member in (B.19) is the limit of the first member in (B.16), when $\eta \to 0$.

Wherever $\widehat{g}(\tau_{j,j}) = e^{-k(\tau_{j,j})}$, with k smooth, the integrands in (B.15,16,18,19) become more explicit:

$$\begin{aligned}
&\det\left(-\frac{\partial}{\partial\tau}\right)\left(e^{-i(t\Delta_\Lambda - E)\cdot\tau}\prod\widehat{g}(\tau_{j,j})\right)\\
&\qquad\qquad = e^{-i(t\Delta_\Lambda - E)\cdot\tau}e^{-\sum k(\tau_{j,j})}\det i(t\Delta_\Lambda - E - i\operatorname{diag}k'(\tau_{j,j})).
\end{aligned}$$

Here

$$(t\Delta_\Lambda - E - i\operatorname{diag}k'(\tau_{j,j}))_{\tau=\kappa(x)} = M,$$

where M is given by (2.23). From this we get (2.24), (2.22). To get (2.25) it suffices to observe that when $\widehat{g}(\tau_{j,j}) = e^{-k(\tau_{j,j})}$:

$$\begin{aligned}
&\left[\frac{1}{2}M_{j,k}\left(-\frac{\partial}{\partial\tau}\right) + \frac{1}{2}M_{k,j}\left(-\frac{\partial}{\partial\tau}\right)\right]\left[e^{-i(t\Delta_\Lambda - E)\cdot\tau}\prod\widehat{g}(\tau_{\ell,\ell})\right]\\
&\quad = e^{-i(t\Delta_\Lambda - E)\cdot\tau - \sum k(\tau_{\ell,\ell})}\det\left[i[t\Delta_\Lambda - E - i\operatorname{diag}(k'(\tau_{\ell,\ell}))]\right]\\
&\qquad\qquad\qquad \times\left[i\left[t\Delta_\Lambda - E - i\operatorname{diag}(k'(\tau_{\ell,\ell}))^{-1}_{j,k}\right]\right].
\end{aligned}$$

Appendix C. An equation in a tube

We start by developping some L^2-theory on the real space $\mathbf{R}^N$, and later we use these results to study more precise L^∞ estimates, using a version of the maximum principle. It is only in this last part that we make estimates which are uniform with respect to the dimension.

Let $\mathrm{C}_b^\infty = \mathrm{C}_b^\infty(\mathbf{R}^N)$ denote the space of all C^∞-functions a on $\mathbf{R}^N$ such that for every multiindex $\alpha \in \mathbf{N}^N$, there exists a constant $C = C_{a,\alpha}$, such that $|\partial^\alpha a(x)| \le C$ on $\mathbf{R}^N$. Here a is a scalar (real or complex) function, but we may similarly define the space of vector-valued functions $\mathrm{C}_b^\infty(\mathbf{R}^N; E)$, if E is a finite dimensional vector space.

We consider a differential operator of the form

$$P = -\Delta + \nu\left(x, \frac{\partial}{\partial x}\right) + V(x),\ \ x \in \mathbf{R}^N,$$

where $\nu = \sum_1^N \nu_j(x)\frac{\partial}{\partial x_j}$ is a complex but scalar vector field and $V \in \mathrm{C}_b^\infty(\mathbf{R}^N; \mathrm{Mat}_M(\mathbf{C}))$ a function of class C_b^∞ with values in the space of complex $M \times M$-matrices. This means of course that the operator P acts on functions with values in $\mathbf{C}^M$. We assume that the vector field ν satisfies: $\mathrm{Im}\nu_j \in \mathrm{C}_b^\infty$, $\nabla\mathrm{Re}\nu_j \in \mathrm{C}_b^\infty$. Here Δ denotes the usual Laplace operator, and ∇ the standard gradient.

We start by deriving two basic *a priori* estimates. Let $u,\ v \in \mathcal{S}(\mathbf{R}^N)$, $z = z_1 + iz_2 \in \mathbf{C}$, and consider the equation

$$(P + z)u = v. \tag{C.1}$$

We will assume that $z_1 \geq C_0$ for some sufficiently large constant $C_0 \geq 0$. It will also be convenient to use the notation $\mathrm{D}_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, so that $i\nu(x, \mathrm{D}_x) = \nu(x, \frac{\partial}{\partial x})$. Notice that the complex adjoint of $\nu(x, \mathrm{D})$ is given by

$$\nu^*(x, \mathrm{D}_x) = -i\overline{\mathrm{div}\,\nu}(x) - 2i(\mathrm{Im}\nu)(x, \mathrm{D}_x) + \nu(x, \mathrm{D}_x).$$

The assumptions on ν tell us that the first term to the right belongs to C_b^∞ and that the second term is a vector field with coefficients in C_b^∞. Write $\nu(x, \mathrm{D}_x) = \nu_1(x, \mathrm{D}_x) + i\nu_2(x, \mathrm{D}_x)$, with $\nu_1 = \frac{1}{2}(\nu + \nu^*)$, $\nu_2 = \frac{1}{2i}(\nu - \nu^*)$. Then

$$\nu_1 \equiv \nu \bmod \left(\mathrm{C}_b^\infty + \sum_1^N \mathrm{C}_b^\infty \frac{\partial}{\partial x_j}\right)$$
$$\nu_2 \equiv 0 \bmod \left(\mathrm{C}_b^\infty + \sum_1^N \mathrm{C}_b^\infty \frac{\partial}{\partial x_j}\right),$$

where until further notice, we write $\nu = \nu(x, \mathrm{D}_x)$ and similarly for ν_j, $j = 1, 2$. Similarly, write $V(x) = V_1(x) + iV_2(x)$, where V_1, V_2 are Hermitian, and $P + z = (P_1 + z_1) + i(P_2 + z_2)$, where $P_1 = -\Delta - \nu_2 + V_1(x)$, $P_2 = \nu_1 + V_2(x)$. From (C.1), we get:

$$\|v\|^2 = \|(P_1+z_1)u\|^2 + \|(P_2+z_2)u\|^2 + i((P_2+z_2)u|(P_1+z_1)u) - i((P_1 + z_1)u|(P_2 + z_2)u). \tag{C.2}$$

The sum of the last two terms can be written $(i[P_1, P_2]u|u)$; here,

$$\begin{aligned}
&[P_1, P_2] = \\
&[-\Delta, \nu_1(x, \mathrm{D}_x)] + [-\Delta, V_2(x)] - (\nu_2(x, \mathrm{D}_x)\nu_1(x, \mathrm{D}_x) - \nu_1(x, \mathrm{D}_x)\nu_2(x, \mathrm{D}_x)) \\
&- [\nu_2(x, \mathrm{D}_x), V_2(x)] + V_1(x) \circ \nu_1(x, \mathrm{D}_x) - \nu_1(x, \mathrm{D}_x) \circ V_1(x) + [V_1(x), V_2(x)] \\
&= (-\nu_2(x, \mathrm{D}_x) + V_1(x)) \circ \nu_1(x, \mathrm{D}_x) - \nu_1(x, \mathrm{D}_x) \circ (-\nu_2(x, \mathrm{D}_x) + V_1(x)) + \frac{1}{i}Q(x, \mathrm{D}_x),
\end{aligned}$$

where Q is a second order formally self-adjoint operator with coefficients in $\mathrm{C}_b^\infty(\mathbf{R}^N)$. We rewrite (C.2) as

$$\begin{aligned}
\|v\|^2 = \|(-\Delta - \nu_2(x, \mathrm{D}_x) + V_1(x) + z_1)u\|^2 + \|(\nu_1(x, \mathrm{D}_x) + V_2(x) + z_2)u\|^2 \quad \text{(C.3)}\\
+ (Q(x, \mathrm{D}_x)u|u) + i((\nu_1(x, \mathrm{D}_x) + z_2)u|(-\nu_2(x, \mathrm{D}_x) + V_1(x))u)\\
- i((-\nu_2(x, \mathrm{D}_x) + V_1(x))u|(\nu_1(x, \mathrm{D}_x) + z_2)u),
\end{aligned}$$

where we judged it convenient to reintroduce z_2.

Since ν_2 has coefficients in C_b^∞, we can apply a standard a priori estimate to the first term of the RHS, and using also the famous inequality $\|a+b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$:

$$\frac{1}{2}\|(\nu_1(x,\mathrm{D}_x)+z_2)u\|^2 \leq \|(\nu_1(x,\mathrm{D}_x)+V_2(x)+z_2)u\|^2 + \|V_2(x)u\|^2,$$

we get for $z_1 \geq C_0$ large enough:

$$\begin{aligned}\|v\|^2 \geq \frac{1}{\mathrm{C}_1}\|u\|_{\mathrm{H}^2}^2 + \frac{z_1}{\mathrm{C}_1}\|u\|_{\mathrm{H}^1}^2 + \frac{z_1^2}{\mathrm{C}_1}\|u\|^2 + \frac{1}{2}\|(\nu_1(x,\mathrm{D}_x)+z_2)u\|^2 \\ - \mathcal{O}(1)\|u\|^2 - \mathcal{O}(1)\|u\|_{\mathrm{H}^2}\|u\| - \mathcal{O}(1)\|(\nu_1(x,\mathrm{D}_x)+z_2)u\|\|u\|_{\mathrm{H}^1}.\end{aligned} \tag{C.4}$$

After increasing C_0, C_1, we can absorb the last three terms and get the basic a priori estimate

$$\mathrm{C}_1\|v\|^2 \geq \|u\|_{\mathrm{H}^2}^2 + z_1\|u\|_{\mathrm{H}^1}^2 + z_1^2\|u\|^2 + \|(\nu_1(x,\mathrm{D}_x)+z_2)u\|^2, \tag{C.5}$$

for solutions to (C.1) of class $\mathcal{S}$, when $z = z_1 + iz_2$ and $z_1 \geq C_0$ with C_0 sufficiently large. In this estimate, we can also replace ν_1 by ν, if we so wish. We notice that this estimate is equally valid when $u \in \mathrm{H}^2_{\mathrm{comp}}(\mathbf{R}^N)$.

Our second basic L^2-estimate will be of semi-boundedness type, and very simple to obtain: For $u \in \mathcal{S}$, we simply notice that

$$\begin{aligned}\mathrm{Re}((P+z)u|u) &= (-\Delta u|u) + (-\nu_2(x,\mathrm{D}_x)u|u) + (V_1(x)u|u) + z_1\|u\|^2 \\ &\geq \frac{1}{2}\|u\|_{\mathrm{H}^1}^2 + (z_1 - \mathcal{O}(1))\|u\|^2 - \|u\|_{\mathrm{H}^1}\|u\| \\ &\geq \frac{1}{3}\|u\|_{\mathrm{H}^1}^2 + (z_1 - \mathcal{O}(1))\|u\|^2.\end{aligned} \tag{C.6}$$

Let $\mathrm{H}^1_{z_1}$ be the space H^1 equipped with the norm $\|(|\mathrm{D}_x| + \sqrt{z_1})u\|$, and let $\mathrm{H}^{-1}_{z_1}$ be the corresponding dual space, equipped with the norm $\|(|\mathrm{D}_x| + \sqrt{z_1})^{-1}u\|$. Assuming as before that $z_1 \geq C_0$, with C_0 sufficiently large, we can write the preceding estimate,

$$\|u\|_{\mathrm{H}^1_{z_1}}^2 \leq \mathcal{O}(1)\|(P+z)u\|_{\mathrm{H}^{-1}_{z_1}}\|u\|_{\mathrm{H}^1_{z_1}},$$

so

$$\|u\|_{\mathrm{H}^1_{z_1}} \leq C\|(P+z)u\|_{\mathrm{H}^1_{z_1}},\ u \in \mathcal{S}. \tag{C.7}$$

We have the same estimate for the adjoint:

$$\|u\|_{\mathrm{H}^1_{z_1}} \leq C\|(P+z)^*u\|_{\mathrm{H}^1_{z_1}},\ u \in \mathcal{S}. \tag{C.8}$$

Using this estimate we now start to consider the existence of solutions to (C.1). Let $v \in \mathrm{H}^{-1}_{z_1}$, and consider the antilinear form: $\ell_v : \mathcal{S} \ni \phi \mapsto (v|\phi)$. Then

$$|\ell_v(\phi)| \leq \|v\|_{\mathrm{H}^{-1}_{z_1}}\|\phi\|_{\mathrm{H}^1_{z_1}} \leq C\|v\|_{\mathrm{H}^{-1}_{z_1}}\|(P+z)^*\phi\|_{\mathrm{H}^{-1}_{z_1}}.$$

By the Hahn-Banach theorem, there exist $u \in \mathrm{H}^1_{z_1}$ with $\|u\|_{\mathrm{H}^1_{z_1}} \leq C\|v\|_{\mathrm{H}^{-1}_{z_1}}$, such that $\ell_\phi(\phi) = (u|(P+z)^*\phi)$, $\forall \phi \in \mathcal{S}$. Consequently, we have shown:

PROPOSITION C.1. – *There exists a constant $C_0 > 0$, such that if $z_1 \geq C_0$, and $v \in \mathrm{H}_{z_1}^{-1}$, then there exists $u \in \mathrm{H}_{z_1}^1$, such that*

$$(P+z)u = v,$$

in the sense of distributions, and

$$\|u\|_{\mathrm{H}_{z_1}^1} \leq C_0 \|v\|_{\mathrm{H}_{z_1}^{-1}}. \tag{C.9}$$

Notice that this applies if $v \in \mathrm{L}^2$, since v then also belongs to $\mathrm{H}_{z_1}^{-1}$, and

$$\|v\|_{\mathrm{H}_{z_1}^{-1}} \leq \|(|\mathrm{D}_x| + \sqrt{z_1})^{-1} v\| \leq \frac{1}{\sqrt{z_1}} \|v\|.$$

Consequently, for $v \in \mathrm{L}^2$, we get a solution $u \in \mathrm{H}_{z_1}^1$ of (C.1), which satisfies

$$\sqrt{z_1} \|u\|_{\mathrm{H}_{z_1}^1} \leq C_0 \|v\|,$$

or more explicitly,

$$z_1 \|u\| + \sqrt{z_1} \| |\mathrm{D}_x| u\| \leq \mathcal{O}(1) \|v\|. \tag{C.10}$$

In order to complete most of the L^2-theory, we have to consider the regularity of H^1-solutions of (C.1). Let $u \in \mathrm{H}^1$, $v \in \mathrm{L}^2$ and assume that (C.1) holds. Let $\chi \in \mathrm{C}_0^\infty(\mathbf{R}^N)$ be equal to 1 near 0 and put $\chi_R(x) = \chi(\frac{1}{R}x)$, $R \geq 1$. Using the fact that ν_j grow at most linearly, we see that

$$[P, \chi_R] = \frac{1}{R} \mathcal{O}(1) \cdot \partial \leq \partial x + \mathcal{O}(1).$$

where $\mathcal{O}(1)$ indicate functions which belong to some bounded set in C_b^∞. It follows that

$$(P+z)(\chi_R u) = \chi_R v + \mathcal{O}\left(\frac{1}{R}\right) \cdot \partial \leq \partial x u + \mathcal{O}(1) u, \tag{C.11}$$

so the RHS is $\mathcal{O}(1)$ in L^2. Since $\chi_R u$ has compact support, the local ellipticity implies that $\chi_R u \in \mathrm{H}^2$ and we can apply the basic a priori estimate, with v replaced by the RHS of the preceding equation, and we get:

$$\begin{aligned} &\|\chi_R u\|_{\mathrm{H}^2}^2 + z_1 \|\chi_R u\|_{\mathrm{H}^1}^2 + z_1^2 \|\chi_R u\|^2 + \|(\nu_1(x, \mathrm{D}_x) + z_2) \chi_R u\|^2 \\ &\qquad \leq \mathcal{O}(1 (\|v\|^2 + \|u\|_{\mathrm{H}^1}^2). \end{aligned} \tag{C.12}$$

Here $(\nu_1(x, \mathrm{D}_x) + z_2) \chi_R u = \chi_R (\nu_1(x, \mathrm{D}_x) + z_2) u + \mathcal{O}(1) u$, so

$$\begin{aligned} &\|\chi_R u\|_{\mathrm{H}^2} + \sqrt{z_1} \|\chi_R u\|_{\mathrm{H}^1} + z_1 \|\chi_R u\| + \|\chi_R (\nu_1(x, \mathrm{D}_x) + z_2) u\| \\ &\leq \mathcal{O}(1)(\|v\| + \|u\|_{\mathrm{H}^1}). \end{aligned} \tag{C.13}$$

Letting R tend to infinity, we see that $u \in \mathrm{H}^2$, $(\nu_1(x, \mathrm{D}_x) + z_2) u \in \mathrm{L}^2$, and

$$\|u\|_{\mathrm{H}^2} + \sqrt{z_1} \|u\|_{\mathrm{H}^1} + z_1 \|u\| + \|(\nu_1(x, \mathrm{D}_x) + z_2) u\| \leq \mathcal{O}(1)(\|v\| + \|u\|_{\mathrm{H}^1}).$$

Possibly after increasing C_0, we get:

PROPOSITION C.2. – *There exists a constant $C_0 > 0$, such that if $z_1 \geq C_0$, and $u \in \mathrm{H}^1$ solves* (C.1) *in the sense of distributions with $v \in \mathrm{L}^2$, then we have $u \in \mathrm{H}^2$, $(\nu_1(x, \mathrm{D}_x) + z_2)u \in \mathrm{L}^2$, and*

$$\|u\|_{\mathrm{H}^2} + \sqrt{z_1}\|u\|_{\mathrm{H}^1} + z_1\|u\| + \|(\nu_1(x, \mathrm{D}_x) + z_2)u\| \leq C_0\|v\| \tag{C.14}$$

Notice that (C.14) implies uniqueness. Summing up, we have proved:

THEOREM C.3. – *There exists $C_0 > 0$, such that if $z_1 \geq C_0$, and $v \in \mathrm{L}^2$, then* (C.1) *has a unique solution u of class* H^1. *Moreover $u \in \mathrm{H}^2$, $(\nu_1(x, \mathrm{D}_x) + z_2)u \in \mathrm{L}^2$ and* (C.14) *holds.*

When v has more regularity, we can differentiate (C.1). If for instance $v \in \mathrm{H}^1$, we get for every $\alpha \in \mathbf{N}^N$ of length 1:

$$(P - z)(\mathrm{D}^\alpha u) = \mathrm{D}^\alpha v - [P, \mathrm{D}^\alpha]u, \tag{C.15}$$

and

$$[P, \mathrm{D}^\alpha] = i[\nu(x, \mathrm{D}), \mathrm{D}^\alpha] + [V, \mathrm{D}^\alpha] \in \sum \mathrm{C}_b^\infty \mathrm{D}_{x_j} + \mathrm{C}_b^\infty,$$

and knowing that $u \in \mathrm{H}^2$, we see that the RHS of (C.15) is in L^2. Since we also know that $\mathrm{D}^\alpha u \in \mathrm{H}^1$, the preceding proposition implies that $\mathrm{D}^\alpha u \in \mathrm{H}^2$, $(\nu_1(x, \mathrm{D}) + z_2)\mathrm{D}^\alpha u \in \mathrm{L}^2$. By iteration, we get:

THEOREM C.4. – *Let C_0 be as in the preceding theorem, let $m \in \mathbf{N}$, $v \in \mathrm{H}^m$, $z_1 \geq C_0$ and let u be the solution of* (C.1), *given by the preceding theorem. Then $u \in \mathrm{H}^{m+2}$, $(\nu_1(x, \mathrm{D}) + z_2)u \in \mathrm{H}^m$ and we have*

$$\|u\|_{\mathrm{H}^{m+2}} + \sqrt{z_1}\|u\|_{\mathrm{H}^{m+1}} + z_1\|u\|_{\mathrm{H}^m} + \|(\nu_1(x, \mathrm{D}_x) + z_2)u\|_{\mathrm{H}^m} \leq \mathrm{C}_m\|v\|_{\mathrm{H}^m}. \tag{C.16}$$

There remains to make two routine extensions. The first one concerns the decay of u if v decays. Let $f : [1, +\infty[\to]0, +\infty[$ with f, $1/f$ bounded by some constant that will not enter into the estimates and assume that f is smooth with $f^{(k)}(t) = \mathcal{O}_k(1)f(t)t^{-k}$. Then $f(\langle x\rangle)^{-1} \circ P \circ f(\langle x\rangle)$ has the same properties as P. We can approximate the function $F(t) = t$ by functions $f_\epsilon(t) = t/(1 + \epsilon t)$, $0 < \epsilon \leq 1$, for which $f^{(k)}(t) = \mathcal{O}_k(1)f_\epsilon(t)t^{-k}$ uniformly w.r.t. ϵ. From this it is easy to see that we can gain power decay for u, if v has such a power decay. More precisely, we can prove the following theorem, where we let $\mathrm{H}^{k,m}$ for k, $m \in \mathbf{N}$ denote the weighted Sobolev space of all $u \in \mathcal{S}'$ s.t. $\langle x\rangle^k \mathrm{D}^\alpha u \in \mathrm{L}^2$ for $|\alpha| \leq m$:

THEOREM C.5. – *Same as the preceding theorem after the substitutions: $m \mapsto (k, m) \in \mathbf{N}^2$, $\mathrm{H}^m \mapsto \mathrm{H}^{k,m}$, $\mathrm{H}^{m+1} \mapsto \mathrm{H}^{k,m+1}$, $\mathrm{H}^{m+2} \mapsto \mathrm{H}^{k,m+2}$ everywhere.*

The second extension concerns parameters. Let $W \subset \mathbf{R}^N$ be open, and let $\nu(x, y, \partial \leq \partial x)$ be a complex vector field, $V = V(x, y)$. We assume

$$V,\ \mathrm{Im}\nu,\ \nabla\mathrm{Re}\nu \in \mathrm{C}_b^\infty(\mathbf{R}^N \times W), \tag{C.17}$$

$$\mathrm{Re}\nu = \mathcal{O}(\langle x\rangle). \tag{C.18}$$

Of course, we have the estimate in (C.18) for every fixed y, by (C.17), but the point of (C.18) is that the estimate holds uniformly with respect to y. It is clear that the preceding estimates hold uniformly with respect to y. If the function $v = v(x, y)$ depends sufficiently smoothly on y, we can also differentiate the equation (C.1) with repect to y, and we get the following result:

THEOREM C.6. – *There exist $C_k > 0$ for all $k \in \mathbf{N}$ such that the following holds: Let $\ell, k, m \in \mathbf{N}$, and let $v = v(x,y)$ be a measurable function on $\mathbf{R}^N \times W$, such that $\mathrm{D}_y^\beta v \in \mathrm{H}^{k,m}(\mathbf{R}^N)$ with locally bounded norm, for $y \in W$, $|\beta| \le \ell$. Let $z_1 \ge C_k$ and let $u = u(x,y)$, be the unique solution of* (C.1) *which belongs to* H^1 *for every y. Then $\mathrm{D}_y^\beta u \in \mathrm{H}^{k,m+2}$, $\nu_1(x,y,\mathrm{D}_x) + z_2)\mathrm{D}_y^\beta u \in \mathrm{H}^{k,m}$ for $|\beta| \le \ell$ with locally bounded norms for $y \in W$, and we have*

$$\sum_{|\beta|\le\ell} (\|\mathrm{D}_y^\beta u\|_{\mathrm{H}^{k,m+2}} + \sqrt{z_1}\|\mathrm{D}_y^\beta u\|_{\mathrm{H}^{k,m+1}} + z_1\|\mathrm{D}_y^\beta u\|_{\mathrm{H}^{k,m}} + \tag{C.19}$$

$$\|(\nu_1(x,y,\mathrm{D}_x) + z_2)\mathrm{D}_y^\beta u\|_{\mathrm{H}^{k,m}}) \le \mathrm{C}_{\ell,k,m} \sum_{|\beta|\le\ell} \|\mathrm{D}_y^\beta v\|_{\mathrm{H}^{k,m}},\ y \in W,$$

where $\mathrm{C}_{\ell,k,m}$ is independent of y.

We return temporarily to the parameter independent situation. By combining Theorem C.3 and the second important a-priori estimate (C.6), we see that P is a closed unbounded operator on $\mathrm{L}^2(\mathbf{R}^N)$ with domain $\{u \in \mathrm{H}^2;\ \nu_1 u \in \mathrm{L}^2\}$, such that $\{z \in \mathbf{C};\ z_1 < -C_0\}$ is contained in the resolvent set and such that for $z_1 > C_0$:

$$\|(z+P)^{-1}\|_{\mathcal{L}(\mathrm{L}^2)} \le \frac{1}{z_1 - C_0}. \tag{C.20}$$

We can apply the Hille-Yoshida theorem to conclude that $-P$ is the generator of a strongly continuous semi-group,

$$[0,+\infty[\ni t \mapsto T_t = e^{-tP}, \tag{C.21}$$

with

$$\|e^{-tP}\|_{\mathcal{L}(\mathrm{L}^2)} \le e^{C_0 t}. \tag{C.22}$$

Applying Theorem 4 with $m = 0$, and the observation leading to that result, we see that e^{-tP} is also a strongly continuous semigroup on $\mathrm{H}^{k,0}$ for every $k \in \mathbf{N}$, and

$$\|e^{-tP}\|_{\mathcal{L}(\mathrm{H}^{k,0})} \le e^{C_k t}. \tag{C.23}$$

To obtain this, we consider P as an unbounded operator in $\mathrm{H}^{k,0}$ with the analogous domain, and we identify the two semigroups using a limiting sequence of weights as above. In both cases, we notice that e^{-tP} is a strongly continuous semigroup on $\mathcal{D}(P^m)$ for every fixed m. Playing with k, m, we conclude that if $u \in \mathcal{S}(\mathbf{R}^N)$, then $e^{-tP}u \in \mathrm{C}^\infty([0,+\infty[;\mathcal{S}(\mathbf{R}^N))$ and we have in the classical sense:

$$\left(\frac{\partial}{\partial t} + P(x,\mathrm{D}_x)\right)(e^{-tP}u(x)) = 0 \tag{C.24}$$

We now consider equations in tube domains and we start by applying the L^2 theory above. Let $W \subset\subset \mathbf{R}^N$ be open, connected and satisfy a cone condition, so that if $u \in \mathrm{H}^m(\Omega)$, $\Omega = \mathbf{R}^N + iW$ and $m > N$, then $u \in C(\overline{\Omega})$. Let $V(z) \in \mathrm{C}_b^\infty(\overline{\Omega}; \mathrm{Mat}_M(\mathbf{C}))$ be holomorphic in Ω and let $\nu(z,\partial \leq \partial z) = \sum_1^N \nu_j(z)\partial \leq \partial z_j$ have holomorphic coefficients ν_j which are also of class $\mathrm{C}^\infty(\overline{\Omega})$, and which satisfy:

$$\mathrm{Im}\nu_j,\ \nabla\mathrm{Re}\nu_j\ \in \mathrm{C}_b^\infty(\overline{\Omega}). \tag{C.25}$$

A typical example of such a vector field is $\nu(z,\frac{\partial}{\partial z}) = \sum_1^N z_j\partial \leq \partial z_j$. If u is holomorphic in Ω, we notice that

$$\nu\left(z,\frac{\partial}{\partial z}\right)u = \widetilde{\nu}\left(x,y,\frac{\partial}{\partial x}\right)u = \nu_{\mathbf{R}}\left(x,y,\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)u, \tag{C.26}$$

where we write $z = x + iy$, and where

$$\widetilde{\nu}\left(x,y,\frac{\partial}{\partial x}\right) = \sum_1^N \nu_j(z)\frac{\partial}{\partial x_j}, \tag{C.27}$$

$$\nu_{\mathbf{R}}\left(x,y,\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) = \sum_1^N (\mathrm{Re}(\nu_j(z))\frac{\partial}{\partial x_j} + \left(\mathrm{Im}\nu_j(z))\frac{\partial}{\partial y_j}\right). \tag{C.28}$$

$\nu_{\mathbf{R}}$ is the real vector field determined by the direction $(\nu_1,..,\nu_N) \in \mathbf{C}^N \simeq \mathbf{R}^{2N}$.

Let $\mathcal{H}^m(\Omega) = \{u \in \mathrm{H}^m(\Omega); u \text{ is holomorphic }\}$, $m \in \mathbf{N}$, $\mathcal{H}(\Omega) = \mathcal{H}^0(\Omega)$ and more generally for $k, m \in \mathbf{N}$:

$$\mathcal{H}^{k,m}(\Omega) = \{u \in \mathrm{H}^m(\Omega); u \text{ is holomorphic, } \langle z\rangle^k \mathrm{D}_z^\alpha u \in \mathrm{L}^2(\Omega),\ |\alpha| \leq m\}.$$

Similarly, define

$$\widetilde{\mathrm{H}}^{k,m}(\Omega) = \{u \in \mathrm{H}^m(\Omega); \langle z\rangle^k \mathrm{D}_x^\alpha u \in \mathrm{L}^2(\Omega),\ |\alpha| \leq m\}.$$

Let

$$P = -\Delta_{\mathbf{C}} + \nu\left(z,\frac{\partial}{\partial z}\right) + V(z), \tag{C.29}$$

$$\widetilde{P} = -\Delta_{\mathbf{R}} + \widetilde{\nu} + V, \tag{C.30}$$

where $\Delta_{\mathbf{C}} = \sum_1^N(\frac{\partial}{\partial z_j})^2$, $\Delta_{\mathbf{R}} = \sum_1^N(\frac{\partial}{\partial x_j})^2$. Notice that our two Laplace operators have the same action on holomorphic functions. For this reason we shall sometimes drop the subscripts $\mathbf{R}$, $\mathbf{C}$. Also, when u is holomorphic, $Pu = \widetilde{P}u$. We can apply the preceding results and see that $\widetilde{P} : \mathrm{H}^0(\Omega) \to \mathrm{H}^0(\Omega)$ is a closed operator with domain $\{u \in \widetilde{\mathrm{H}}^2(\Omega); \widetilde{\nu}(x,y,\frac{\partial}{\partial x})u \in \mathrm{H}^0(\Omega)\}$ and resolvent set containing the half plane $z_1 < -C_0$. Moreover $\|(\widetilde{P} - z)^{-1}\| \leq 1/(-C_0 - z_1)$ for z in that half plane. We have the completely analogous result for $\widetilde{P} : \widetilde{\mathrm{H}}^{k,0} \to \widetilde{\mathrm{H}}^{k,0}$. If $v \in \mathcal{H}^0(\Omega)$, let $u \in \mathcal{D}(\widetilde{P})$ be the solution of $(\widetilde{P} - z)u = v$ for $z_1 < -C_0$. Notice that $\frac{\partial}{\partial \overline{z}_j}$ formally commutes with $\widetilde{P}$. If $W' \subset\subset W$, $\Omega' = \mathbf{R}^N + iW'$, then $\frac{\partial}{\partial \overline{z}_j}u \in \widetilde{\mathrm{H}}^1(\Omega')$, and we get $(\widetilde{P} - z)(\frac{\partial}{\partial \overline{z}_j}u) = 0$, implying $\frac{\partial}{\partial \overline{z}_j}u = 0$ in Ω' and hence in Ω if we take a sequence of Ω' converging to Ω. We have shown that u is holomorphic and $(P - z)u = v$. We get

THEOREM C.7. – $P : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ *is a closed operator with domain*

$$\{u \in \mathcal{H}^2(\Omega);\ \nu(z,\partial \leq \partial z)u \in \mathcal{H}^0(\Omega)\}$$

and resolvent set containing the half-plane $z_1 < -C_0$. *Moreover* $\|(\widetilde{P} - z)^{-1}\| \leq 1/(-C_0 - z_1)$ *for* z *in that half-plane. The same result is valid with the substitutions:* $\mathcal{H}^0 \mapsto \mathcal{H}^{k,0}$, $\mathcal{H}^2 \mapsto \mathcal{H}^{k,2}$, $C_0 \mapsto C_k$.

The Hille-Yoshida theorem allows us to define the strongly continuous semigroup $T_t = e^{-tP} : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$, $t \geq 0$, with $\|e^{-tP}\|_{\mathcal{L}(\mathcal{H}(\Omega))} \leq e^{C_0 t}$, and more generally $\|e^{-tP}\|_{\mathcal{L}(\mathcal{H}^{k,0})} \leq e^{C_k t}$. Notice also that T_t acts as a strongly continuous semi-group in the domain of any positive integer power of $P : \mathcal{H}^{k,0} \to \mathcal{H}^{k,0}$. It follows that if $u \in \mathcal{S}(\overline{\Omega})$ in the sense that $u \in \mathrm{C}^\infty(\overline{\Omega})$ and all derivatives tend to zero at infinity faster than any negative power of $\langle z \rangle$, and if u is holomorphic in Ω, then $e^{-tP}u \in \mathrm{C}^\infty([0, +\infty[; \mathcal{S}(\overline{\Omega}) \cap \mathrm{Hol}(\Omega))$, and the heat equation $(\frac{\partial}{\partial t} + P)e^{-tP}u = 0$ holds in the classical sense. Moreover for such u's we also have $e^{-tP}Pu = Pe^{-tP}u$.

Finally, we are ready for the L^∞ estimates, but we will have to add an assumption about ν and an assumption about V.

$$\begin{aligned}&\text{There is a real vector field } \mu \text{ in } \mathbf{C}^N \text{ with smooth coefficients}\\ &\text{of at most linear growth, such that } \mu_{|\Omega} = \nu_{\mathbf{R}},\end{aligned} \tag{C.31}$$

$$\text{If } z \in \Omega, \text{ then } \exp(-t\mu)(z) \in \Omega, \ t \geq 0. \tag{C.32}$$

Now equip $\mathbf{C}^M$ with some norm and view correspondingly $\mathbf{C}^N$ as a Banach space B, with dual B^*. Let $(u|v)$ be the corresponding sesquilinear scalar product on $B \times B^*$. We view $V(z)$ as a map $B \to B$, and make the following assumption on V:

$$\begin{aligned}&\text{There exists } \delta > 0, \text{ such that if } z \in \overline{\Omega},\ u \in B,\ v \in B^*, \text{ and}\\ &\mathrm{Re}(u|v) = \|u\|_B \|v\|_B^*, \text{ then } \mathrm{Re}(V(z)u|v) \geq \delta \|u\|_B \|v\|_{B^*}.\end{aligned} \tag{C.33}$$

Let $u(t,z) \in \mathrm{C}^\infty([0, +\infty[; \mathcal{S}(\overline{\Omega}; B))$ be holomorphic in z, and assume that u solves the equation:

$$\frac{\partial}{\partial t}u + Pu = 0. \tag{C.34}$$

Let

$$m(t) = \sup_{z \in \overline{\Omega}} \|u(z)\|_B. \tag{C.35}$$

Notice that

$$m(t) = \max_{(z,e) \in \overline{\Omega} \times S(B^*)} \mathrm{Re}(u(t,z)|e), \tag{C.36}$$

where $S(B^*) = \{e \in B^*;\ \|e\|_{B^*} = 1\}$. Let $M(t)$ be the set of points in $\overline{\Omega} \times S(B^*)$, where the maximum is attained in (C.36). It follows that $m(t)$ is a locally Lipschitz function on $[0, +\infty[$ whose (a.e. defined) derivative satisfies:

$$m'(t) \leq \sup_{(x,e) \in M(t)} \mathrm{Re}\left(\frac{\partial}{\partial t}u(t,z)|e\right). \tag{C.37}$$

Consider,

$$\begin{aligned}&\mathrm{Re}(Pu(t,z)|e)\\ &\quad = -\Delta_{\mathbf{R}}\mathrm{Re}(u(t,z)|e) + \nu_{\mathbf{R}}\left(x, y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\mathrm{Re}(u(t,z)|e) + \mathrm{Re}(V(z)u(t,z)|e).\end{aligned}$$

If $(z,e) \in M(t)$, then $w \mapsto \mathrm{Re}(u(w)|e)$ has a maximum at z, so $-\Delta_{\mathbf{R}}\mathrm{Re}(u(t,z)|e) \geq 0$. On the other hand the assumptions (C.32) imply that $\nu_{\mathbf{R}}\mathrm{Re}(u(z)|e) \geq 0$, and since $\mathrm{Re}(u(t,z)|e) = \|u(t,z)\|_B\|e\|_{B^*}$, we have $\mathrm{Re}(V(z)u(t,z)|e) \geq \delta\|u(t,z)\|_B\|e\|_{B^*}$. From (C.34), we get $\mathrm{Re}(\frac{\partial u}{\partial t}|e) = -\mathrm{Re}(Pu|e)$, so for $(z,e) \in M(t)$: $\mathrm{Re}(\frac{\partial u}{\partial t}|e) \leq -\delta m(t)$, so (C.37) implies that

$$m'(t) \leq -\delta m(t), \tag{C.38}$$

and hence that $m(t) \leq e^{-\delta t}m(0)$.

Summing up, we have shown that if $u \in \mathcal{S}(\overline{\Omega})$ is holomorphic in Ω, then

$$\sup_{x\in\Omega}\|e^{-tP}u(x)\|_B \leq e^{-\delta t}\sup_{x\in\Omega}\|u(x)\|_B. \tag{C.39}$$

For the same u's we have $Pe^{-tP}u = e^{-tP}Pu = -\frac{\partial}{\partial t}e^{-tP}u$, so if we put

$$Qu = \int_0^\infty e^{-tP}u\,\mathrm{d}t,$$

we get

$$PQu = QPu = -\int_0^\infty \frac{\partial}{\partial t}(e^{-tP}u)\,\mathrm{d}t = u. \tag{C.40}$$

We also have,

$$\sup_{z\in\Omega}\|Qu(z)\|_B \leq \frac{1}{\delta}\sup_{z\in\Omega}\|u(z)\|_B. \tag{C.41}$$

Put $f_\epsilon(t) = \frac{t}{1+\epsilon t}$, and $F_\epsilon(x) = f_\epsilon(\langle x/C\rangle)$ (with $\langle x\rangle = \sqrt{1+x^2}$) where C is large enough, so that the latter function is well-defined in $\overline{\Omega}$, when $0 < \epsilon \leq 1$. Then (C.39) remains valid if we replace P by $F_\epsilon^{-1}\circ P\circ F_\epsilon$ and δ by $\delta/2$, provided that ϵ is small enough. Examining the earlier arguments, we see that (C.41) also holds with Q replaced by $F_\epsilon^{-1}\circ Q\circ F_\epsilon$ and with δ replaced by $\delta/2$.

Definition. – Let $u_j, u \in \mathrm{C}_b(\overline{\Omega})\cap\mathrm{Hol}(\Omega)$, for $j \in \mathbf{N}$. We say that $u_j \to u$ narrowly when $j \to \infty$ if $\sup_\Omega \|u_j\|_B$ is bounded by a constant independent of j and $u_j \to u$ uniformly on every compact subset of $\overline{\Omega}$.

Let $u_j, u \in \mathcal{S}(\overline{\Omega})\cap\mathrm{Hol}$ and assume that $u_j \to u$ narrowly, when $j \to \infty$. Then $\sup_\Omega\|F_\epsilon^{-1}(u_j-u)\|_B \to 0$, so $\sup_\Omega\|F_\epsilon^{-1}(Qu_j-Qu)\|_B \to$ and we see that $Qu_j \to Qu$ narrowly when $j \to \infty$. In other words, Q preserves narrow convergence of sequences in $\mathcal{S}(\overline{\Omega})$ with limits in the same space. From (C.40), we then get:

THEOREM C.8. – *Let $E \subset \mathrm{C}_b(\overline{\Omega})\cap\mathrm{Hol}(\Omega)$ be the closure of $\mathcal{S}(\overline{\Omega})\cap\mathrm{Hol}(\Omega)$ for narrow convergence. Then:*

a) $Q : E \to E$ is well-defined and (C.41) *holds for $u \in E$.*

b) If $v \in E$, then $PQv = v$,

c) Let $u \in E$ and assume that there is a sequence $u_j \in \mathcal{S}(\overline{\Omega})\cap\mathrm{Hol}\,(\Omega)$ with $u_j \to u$ and $Pu_j \to Pu$ narrowly (so that $Pu \in E$). Then, $QPu = u$.

Naturally we want to know if there is a simpler characterization of the spaces that appear here.

PROPOSITION C.9. – *If W is starshaped with respect to $y = 0$, then* $E = \mathrm{C}_b(\overline{\Omega}) \cap \mathrm{Hol}(\Omega)$.

Proof. – Let $u \in \mathrm{C}_b(\overline{\Omega}) \cap \mathrm{Hol}(\Omega)$. Then $\widetilde{u}_j \to u$ narrowly, where $\widetilde{u}_j(z) = u(\theta_j z)$ and $\theta_j = (1 - \frac{1}{j})$. Put $u_j(z) = e^{-\epsilon_j \langle z/C \rangle} \widetilde{u}_j(z) \in \mathcal{S}(\overline{\Omega}) \cap \mathrm{Hol}(\Omega)$, where $C > 0$ is sufficiently large and $\epsilon_j \searrow 0$. Then $u_j \to u$ narrowly. □

We leave the following question open until an answer is needed: Make the assumptions of the last proposition and assume that $\partial^\alpha u,\ Pu \in \mathrm{C}_b(\overline{\Omega}) \cap \mathrm{Hol}\,(\Omega)$, for $|\alpha| \leq 2$. Is it true that u satisfies the assumption of c) in the last theorem?

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