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ON MODULI OF G-BUNDLES ON A CURVE FOR EXCEPTIONAL G

BY CHRISTOPH SORGER

ABSTRACT. – Let X be a complex, smooth, projective and connected curve, G be a simple and simply connected complex algebraic group and $\mathcal{M}_{G,X}$ be the stack of G-bundles on X. We show, using the decomposition formulas of Tsuchiya-Ueno-Yamada [T-U-Y] and Faltings [F], the existence of certain line bundles on $\mathcal{M}_{G,X}$ conjectured in [L-S]. The result is then applied to the question of local factoriality of the coarse moduli space of semi-stable G-bundles. © Elsevier, Paris

RÉSUMÉ. – Soient X une courbe algébrique projective complexe, lisse et connexe, G un groupe algébrique complexe simple et simplement connexe et $\mathcal{M}_{G,X}$ le champ des G-fibrés sur X. Nous montrons l'existence de certains fibrés inversibles sur $\mathcal{M}_{G,X}$, conjecturés dans [L-S], en utilisant les théorèmes de décomposition de Tsuchiya-Ueno-Yamada [T-U-Y] et Faltings [F]. Ce résultat est ensuite appliqué à la question de factorialité locale de l'espace de modules grossier des G-fibrés semi-stables sur X. © Elsevier, Paris

Contents

1. Introduction

Let G be a simple and simply connected complex algebraic group. Let $\mathcal{M}_{G,X}$ be the stack of G-bundles on the smooth connected and projective complex curve X of genus g. If $\rho : G \to SL_r$ is a representation of G, denote by $\mathcal{D}_\rho$ the pullback of the determinant bundle [D-N] under the morphism $\mathcal{M}_{G,X} \to \mathcal{M}_{SL_r,X}$ defined by extension of the structure group. Associate to G the number $d(G)$ and the representation $\rho(G)$ as follows:

Type of G	A_r	B_r $(r \geq 3)$	C_r	D_r $(r \geq 4)$	E_6	E_7	E_8	F_4	G_2
$d(G)$	1	2	1	2	6	12	60	6	2
$\rho(G)$	ϖ_1	ϖ_1	ϖ_1	ϖ_1	ϖ_6	ϖ_7	ϖ_8	ϖ_4	ϖ_1

THEOREM 1.1. – *There is a line bundle $\mathcal{L}$ on $\mathcal{M}_{G,X}$ such that* $\mathrm{Pic}(\mathcal{M}_{G,X}) \xrightarrow{\sim} \mathbb{Z}\mathcal{L}$. *Moreover we may choose $\mathcal{L}$ in such a way that $\mathcal{L}^{\otimes d(G)} = \mathcal{D}_{\rho(G)}$.*

The above theorem is proved, for classical G and G_2, in [L-S] where it is also proved that the space of sections $H^0(\mathcal{M}_{G,X}, \mathcal{L}^\ell)$ may be identified to the space of conformal blocks $B_{G,X}(\ell; p; 0)$ (*see* (2.2.1) for its definition). Now, once the generator of the Picard group is known in the exceptional cases, this identification is also valid in general, as well as what happens when we additionaly consider parabolic structures as we did in [L-S] (Theorems 1.1 and 1.2). The general case had been conjectured by Laszlo and the author [L-S] and figures as a question in Faltings [F] (5.(c)).

There is a topological approach to Theorem 1.1: as suggested in [L-S], in order to prove Theorem 1.1 it is sufficient to show that the group of algebraic morphisms from $X - \{p\}$ to G is simply connected, which would follow from the fact that this group is homotopy equivalent to the group of smooth morphisms from $X - \{p\}$ to G. A proof of the last statement, hence of Theorem 1.1, is discussed in [T]. Our proof however, avoids this question and is purely algebraic in nature: the basic idea is not only to identify the space of conformal blocks $B_{G,X}(\ell; p; 0)$ with sections of $\mathcal{L}^\ell$ *provided* that $\mathcal{L}$ exists, but also to *use* the space of conformal blocks and its properties as the decomposition formulas of [T-U-Y] and [F] *to prove the existence* of $\mathcal{L}$.

Suppose $g(X) \geq 2$. For the coarse moduli spaces $M_{G,X}$ of semi-stable G-bundles, we will see that the roots of the determinant bundle of Theorem 1.1 do only exist on the open subset of regularly stable G-bundles. This will allow us to complete the following result of [B-L-S], which was proved there for classical G and G_2.

THEOREM 1.2. – *Let* G *be semi-simple. Then* $M_{G,X}$ *is locally factorial if and only if* G *is special in the sense of Serre.*

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2. Conformal Blocks

2.1. Affine Lie algebras

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of rank r over $\mathbb{C}$. Let P be the weight lattice, P_+ be the subset of dominant weights and $(\varpi_i)_{i=1,\ldots,r}$ be the fundamental weights. Given a dominant weight λ, we denote $L(\lambda)$ the associated simple $\mathfrak{g}$-module with highest weight λ. Finally $(\cdot,\cdot)$ will be the Cartan-Killing form normalized so that for the highest root θ we have $(\theta,\theta) = 2$. Let $\mathrm{L}\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$ be the *loop algebra* of $\mathfrak{g}$ and $\widehat{\mathrm{L}\mathfrak{g}}$ be the central extension of $\mathrm{L}\mathfrak{g}$

$$0 \longrightarrow \mathbb{C} \longrightarrow \widehat{\mathrm{L}\mathfrak{g}} \longrightarrow \mathrm{L}\mathfrak{g} \longrightarrow 0 \tag{2.1.1}$$

defined by the 2-cocycle $(\mathrm{X} \otimes f, Y \otimes g) \mapsto (X,Y)\mathrm{Res}_0(gdf)$.

Fix an integer ℓ. Call a representation of $\widehat{\mathrm{L}\mathfrak{g}}$ of level ℓ if the center acts by multiplication by ℓ. The theory of affine Lie algebras affirms that the irreducible and integrable representations of $\widehat{\mathrm{L}\mathfrak{g}}$ are classified by the dominant weights belonging to $\mathrm{P}_\ell = \{\lambda \in \mathrm{P}_+/(\lambda,\theta) \leq \ell\}$. For $\lambda \in \mathrm{P}_\ell$, denote $\mathcal{H}_\ell(\lambda)$ the associated representation.

2.2. Definition of conformal blocks

Fix an integer (the level) $\ell \geq 0$. Let $(\mathrm{X},\underline{p})$ be an n-pointed smooth and projective curve (we set $\underline{p} = (p_1,\ldots,p_n)$) and suppose that the points are labeled by $\underline{\lambda} = (\lambda_1,\ldots,\lambda_n) \in \mathrm{P}_\ell^n$ respectively. Choose an additional point $p \in \mathrm{X}$ and a local coordinate z at p. Let $\mathrm{X}^* = \mathrm{X} - \{p\}$ and $\mathrm{L}_\mathrm{X}\mathfrak{g}$ be the Lie algebra $\mathfrak{g} \otimes \mathcal{O}(\mathrm{X}^*)$. We have a morphism of Lie algebras $\mathrm{L}_\mathrm{X}\mathfrak{g} \to \mathrm{L}\mathfrak{g}$ by associating to $\mathrm{X} \otimes f$ the element $\mathrm{X} \otimes \hat{f}$, where $\hat{f}$ is the Laurent developpement of f at p. By the residue theorem, the restriction to $\mathrm{L}_\mathrm{X}\mathfrak{g}$ of the central extension (2.1.1) splits and we may see $\mathrm{L}_\mathrm{X}\mathfrak{g}$ as a Lie subalgebra of $\widehat{\mathrm{L}\mathfrak{g}}$. In particular, the $\widehat{\mathrm{L}\mathfrak{g}}$-module $\mathcal{H}_\ell(0)$ may be seen as a $\mathrm{L}_\mathrm{X}\mathfrak{g}$-module. Moreover, we may consider the $\mathfrak{g}$-modules $\mathrm{L}(\lambda_i)$ as a $\mathrm{L}_\mathrm{X}\mathfrak{g}$-modules by evaluation at p_i. The vector space of conformal blocks is defined as follows:

$$\mathrm{B}_{\mathrm{G},\mathrm{X}}(\ell;\underline{p};\underline{\lambda}) = [\mathcal{H}_\ell(0) \otimes_{\mathbb{C}} \mathrm{L}(\lambda_1) \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathrm{L}(\lambda_n)]_{\mathrm{L}_\mathrm{X}\mathfrak{g}} \tag{2.2.1}$$

where $[]_{\mathrm{L}_\mathrm{X}\mathfrak{g}}$ means that we take co-invariants. It is known ([T-U-Y] or [S], 2.3.5) that the definition of $\mathrm{B}_{\mathrm{G},\mathrm{X}}(\ell;\underline{p};\underline{\lambda})$ may be extended to n-pointed *stable* $(\mathrm{X},\underline{p})$ and that these vector spaces are finite-dimensional ([T-U-Y] or [S], 2.5.1 for a simple proof). Important properties are as follows:

a) $\dim \mathrm{B}_{\mathrm{G},\mathbb{P}_1}(\ell;p_1;0) = 1$.
b) Uppon adding a non-singular point $q \in \mathrm{X}$, spaces $\mathrm{B}_{\mathrm{G},\mathrm{X}}(\ell;\underline{p};\underline{\lambda})$ and $\mathrm{B}_{\mathrm{G},\mathrm{X}}(\ell;\underline{p},q;\underline{\lambda},0)$ are canonically isomorphic.
c) Suppose X is singular in c and let $\widetilde{\mathrm{X}} \to \mathrm{X}$ be a partial desingularization of c. Let a and b be the points of $\widetilde{\mathrm{X}}$ over c. Then there is a canonical isomorphism

$$\bigoplus_{\mu\in\mathrm{P}_\ell} \mathrm{B}_{\mathrm{G},\widetilde{\mathrm{X}}}(\ell;\underline{p},a,b;\underline{\lambda},\mu,\mu^*) \xrightarrow{\sim} \mathrm{B}_{\mathrm{G},\mathrm{X}}(\ell;\underline{p};\underline{\lambda})$$

Remark. – If $\widetilde{\mathrm{X}}$ becomes disconnected and $a \in \widetilde{\mathrm{X}}'$ and $b \in \widetilde{\mathrm{X}}''$ are its connected components, then $\mathrm{B}_{\mathrm{G},\widetilde{\mathrm{X}}}(\ell;\underline{p},a,b;\underline{\lambda},\mu,\mu^*)$ should be understood as the tensor product $\mathrm{B}_{\mathrm{G},\widetilde{\mathrm{X}}'}(\ell;\underline{p}',a;\underline{\lambda}',\mu) \otimes_{\mathbb{C}} \mathrm{B}_{\mathrm{G},\widetilde{\mathrm{X}}''}(\ell;\underline{p}'',b;\underline{\lambda}'',\mu^*)$ where $\underline{p}'$ and $\underline{p}''$ are the points lying on $\widetilde{\mathrm{X}}'$ and $\widetilde{\mathrm{X}}''$ respectively.

d) The dimension of $\mathrm{B}_{\mathrm{G},\mathrm{X}}(\ell;\underline{p};\underline{\lambda})$ does not change when $(\mathrm{X},\underline{p})$ varies in the stack of n-pointed stable curves $\mathrm{M}_{g,n}$ ([T-U-Y]).

2.3. Application

Let X be a smooth connected curve with one marked point $p \in \mathrm{X}$. Using $d)$ and $c)$ and then $b)$ and $a)$ it follows that $\mathrm{B}_{\mathrm{G,X}}(1;p;0)$ is non trivial, which will be crucial in the proof of Theorem 1.1.

3. The Picard group of $\mathcal{M}_{\mathrm{G,X}}$

3.1. The uniformization theorem

Let us recall the description of $\mathrm{Pic}(\mathcal{M}_{\mathrm{G,X}})$ of [L-S], which uses as main tool the *uniformization* theorem which we now recall: let LG be the loop group $\mathrm{G}\big(\mathbb{C}((z))\big)$, seen as an ind-scheme over $\mathbb{C}$, $\mathrm{L}^+\mathrm{G}$ the sub-group scheme $\mathrm{G}\big(\mathbb{C}[[z]]\big)$ and $\mathcal{Q}_\mathrm{G} = \mathrm{LG}/\mathrm{L}^+\mathrm{G}$ be the infinite Grassmannian, which is a direct limit of projective integral varieties (*loc. cit.*). Finally let $\mathrm{L_XG}$ be the sub-ind group $\mathrm{G}(\mathcal{O}(\mathrm{X}^*))$ of LG. The uniformization theorem states that there is a canonical isomorphism of stacks $\mathrm{L_XG}\backslash\mathcal{Q}_\mathrm{G} \xrightarrow{\sim} \mathcal{M}_{\mathrm{G,X}}$ and moreover that $\mathcal{Q}_\mathrm{G} \to \mathcal{M}_{\mathrm{G,X}}$ is a $\mathrm{L_XG}$-bundle ([L-S], 1.3).

Let $\mathrm{Pic}_{\mathrm{L_XG}}(\mathcal{Q}_\mathrm{G})$ be the group of $\mathrm{L_XG}$-linearized line bundles on $\mathcal{Q}_\mathrm{G}$. Recall that a $\mathrm{L_XG}$-linearization of the line bundle $\mathcal{L}$ on $\mathcal{Q}_\mathrm{G}$ is an isomorphism $m^*\mathcal{L} \xrightarrow{\sim} \mathrm{pr}_2^*\mathcal{L}$, where $m : \mathrm{L_XG} \times \mathcal{Q}_\mathrm{G} \to \mathcal{Q}_\mathrm{G}$ is the action of $\mathrm{L_XG}$ on $\mathcal{Q}_\mathrm{G}$, satisfying the usual cocycle condition. It follows from the uniformization theorem that

$$\mathrm{Pic}(\mathcal{M}_{\mathrm{G,X}}) \xrightarrow{\sim} \mathrm{Pic}_{\mathrm{L_XG}}(\mathcal{Q}_\mathrm{G});$$

hence, in order to understand $\mathrm{Pic}(\mathcal{M}_{\mathrm{G,X}})$, it suffices to understand $\mathrm{Pic}_{\mathrm{L_XG}}(\mathcal{Q}_\mathrm{G})$. The Picard group of $\mathcal{Q}_\mathrm{G}$ itself is infinite cyclic; let us recall how its positive generator may be defined in terms of central extensions of LG.

3.2. The Picard group of the infinite Grassmannian

If $\mathcal{H}$ is an (infinite) dimensional vector space over $\mathbb{C}$, we define the $\mathbb{C}$-space $\mathrm{End}(\mathcal{H})$ by $\mathrm{R} \mapsto \mathrm{End}(\mathcal{H} \otimes_\mathbb{C} \mathrm{R})$, the $\mathbb{C}$-group $\mathrm{GL}(\mathcal{H})$ as the group of its units and $\mathrm{PGL}(\mathcal{H})$ by $\mathrm{GL}(\mathcal{H})/\mathrm{G}_m$. The $\mathbb{C}$-group LG acts on $\mathrm{L}\mathfrak{g}$ by the adjoint action which is extended to $\widehat{\mathrm{L}\mathfrak{g}}$ by the following formula:

$$\mathrm{Ad}(\gamma).(\alpha', s) = \Big(\mathrm{Ad}(\gamma).\alpha', s + \mathrm{Res}_{z=0}(\gamma^{-1}\frac{\mathrm{d}}{\mathrm{d}z}\gamma, \alpha')\Big)$$

where $\gamma \in \mathrm{LG(R)}$, $\alpha = (\alpha', s) \in \widehat{\mathrm{L}\mathfrak{g}}(\mathrm{R})$ and $(\cdot,\cdot)$ is the $\mathrm{R}((z))$-bilinear extension of the Cartan-Killing form. The main tool we use, due to Faltings, is that if $\overline{\pi} : \widehat{\mathrm{L}\mathfrak{g}} \to \mathrm{End}(\mathcal{H})$ is an integral highest weight representation, then for R a $\mathbb{C}$-algebra and $\gamma \in \mathrm{LG(R)}$ there is, locally over Spec(R), an automorphism u_γ of $\mathcal{H}_\mathrm{R} = \mathcal{H} \otimes_\mathbb{C} \mathrm{R}$, unique up to R^*, such that

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\overline{\pi}(\alpha)} & \mathcal{H} \\ {\scriptstyle u_\gamma}\downarrow & & \downarrow{\scriptstyle u_\gamma} \\ \mathcal{H} & \xrightarrow[\overline{\pi}(\mathrm{Ad}(\gamma).\alpha)]{} & \mathcal{H} \end{array}$$

is commutative for any $\alpha \in \widehat{\mathrm{L}\mathfrak{g}}(\mathrm{R})$ ([L-S], Prop. 4.3).

By the above, the representation $\overline{\pi}$ may be "integrated" to a (unique) *algebraic* projective representation of LG, *i.e.* there is a morphism of $\mathbb{C}$-groups $\pi : \mathrm{LG} \to \mathrm{PGL}(\mathcal{H})$ whose derivative coincides with $\overline{\pi}$ up to homothety. Indeed, thanks to the unicity property, the automorphisms u associated locally to γ glue together to define an element $\pi(\gamma) \in \mathrm{PGL}(\mathcal{H})(\mathrm{R})$ and, again because of the unicity property, π defines a morphism of $\mathbb{C}$-groups. The assertion on the derivative is consequence of (3.2.1). We apply this to the basic representation $\mathcal{H}_1(0)$ of $\widehat{\mathrm{L}\mathfrak{g}}$. Consider the central extension

$$1 \longrightarrow \mathrm{G}_m \longrightarrow \mathrm{GL}(\mathcal{H}_1(0)) \longrightarrow \mathrm{PGL}(\mathcal{H}_1(0)) \longrightarrow 1. \tag{3.2.2}$$

The pull back of (3.2.2) to LG defines a central extension to which we refer as the *canonical* central extension of LG:

$$1 \longrightarrow \mathrm{G}_m \longrightarrow \widehat{\mathrm{LG}} \longrightarrow \mathrm{LG} \longrightarrow 1. \tag{3.2.3}$$

A basic fact is that the extension (3.2.3) splits canonically over $\mathrm{L}^+\mathrm{G}$ ([L-S], 4.9), hence we may define a line bundle on the homogeneous space $\mathcal{Q}_\mathrm{G} = \widehat{\mathrm{LG}}/\widehat{\mathrm{L}^+\mathrm{G}}$ via the character $\mathrm{G}_m \times \mathrm{L}^+\mathrm{G} \to \mathrm{G}_m$ defined by the first projection. Then this line bundle generates $\mathrm{Pic}(\mathcal{Q}_\mathrm{G})$ ([L-S], 4.11); we denote its dual by $\mathcal{O}_{\mathcal{Q}_\mathrm{G}}(1)$.

3.3. Restriction of the canonical central extension to $\mathrm{L}_\mathrm{X}\mathrm{G}$

By ([L-S], 6.2) the forgetful morphism $\mathrm{Pic}_{\mathrm{L}_\mathrm{X}\mathrm{G}}(\mathcal{Q}_\mathrm{G}) \to \mathrm{Pic}(\mathcal{Q}_\mathrm{G})$ is injective. Recall the Kumar-Narasimhan-Ramanathan lemma ([L-S], 6.8): if $\rho : \mathrm{G} \to \mathrm{SL}_r$ is a representation, then the pullback of the determinant bundle $\mathcal{D}$ to $\mathcal{Q}_\mathrm{G}$ under $\mathcal{Q}_\mathrm{G} \to \mathcal{Q}_{\mathrm{SL}_r,\mathrm{X}} \to \mathcal{M}_{\mathrm{SL}_r,\mathrm{X}}$ is $\mathcal{O}_{\mathcal{Q}_\mathrm{G}}(d_\rho)$, where d_ρ is the Dynkin index of ρ [D]. As $\gcd(d_\rho)$ is $d(\mathfrak{g})$ when ρ runs over all (finite dimensional) representations of $\mathfrak{g}$ ([L-S], 2.6), proving theorem 1.1 is equivalent to showing that there is a $\mathrm{L}_\mathrm{X}\mathrm{G}$-linearization of $\mathcal{O}_{\mathcal{Q}_\mathrm{G}}(1)$. This in turn is equivalent ([L-S], 6.4) to the splitting of the central extension (3.2.3) when restricted to $\mathrm{L}_\mathrm{X}\mathrm{G}$ and so, the proof is complete once we know the following.

PROPOSITION. – *The restriction of the central extension* (3.2.3) *to* $\mathrm{L}_\mathrm{X}\mathrm{G}$ *splits.*

Proof. – Let $\mathrm{B} = \mathrm{B}_{\mathrm{G},\mathrm{X}}(\ell; p; 0)$. We know from (2.3) that $\mathrm{B} \neq 0$. Remark that the commutativity of (3.2.1) implies that for $\gamma \in \mathrm{L}_\mathrm{X}\mathrm{G}(\mathrm{R})$ the associated automorphism u_γ of $\mathcal{H}$ maps coinvariants to coinvariants. We get a morphism of $\mathbb{C}$-groups $\pi : \mathrm{L}_\mathrm{X}\mathrm{G} \to \mathrm{PGL}(B)$ and so we may consider the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{G}_m & \longrightarrow & \widehat{\mathrm{L}_\mathrm{X}\mathrm{G}} & \longrightarrow & \mathrm{L}_\mathrm{X}\mathrm{G} & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle \hat{\pi}} & & \downarrow{\scriptstyle \pi} & & \\
1 & \longrightarrow & \mathrm{G}_m & \longrightarrow & \mathrm{GL}(\mathrm{B}) & \longrightarrow & \mathrm{PGL}(\mathrm{B}) & \longrightarrow & 1
\end{array}$$

By construction, the central extension of $\mathrm{L}_\mathrm{X}\mathrm{G}$ above coincides with the central extension obtained by restriction of (3.2.3) to $\mathrm{L}_\mathrm{X}\mathrm{G}$. By definition of B, the derivative of π is trivial. As $\mathrm{L}_\mathrm{X}\mathrm{G}$ is an *integral* ind-group ([L-S], 5.1) it follows that π has to be the constant map with value the identity. Indeed, write $\mathrm{L}_\mathrm{X}\mathrm{G}$ as the direct limit of integral schemes V_n and remark that π has to be constant on V_n; for large n, as V_n contains 1, this constant is $\pi(1) = 1$. So π being the identity, $\hat{\pi}$ factors through G_m which gives the desired splitting. □

4. Proof of theorem 1.2

According to ([B-L-S], §13) it remains to prove that the coarse moduli space $\mathrm{M}_{\mathrm{G,X}}$ of semi-stable G-bundles is not locally factorial for $\mathrm{G} = \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7$ or E_8. Let $\mathrm{M}^{\mathrm{reg}}_{\mathrm{G,X}}$ be the open subset of $\mathrm{M}_{\mathrm{G,X}}$ corresponding to regularly stable G-bundles E (*i.e.* E is stable and $\mathrm{Aut(E)} = \mathrm{Z(G)}$). Denote by Cl the group of Weil divisor classes. We have a commutative diagram, with r_* the restriction, c and c_{reg} the canonical, and f and f_{reg} the forgetful morphisms:

$$\begin{array}{ccccc}
\mathrm{Pic}(\mathcal{M}_{\mathrm{G,X}}) & & & & \\
\downarrow{\scriptstyle r_1} & & & & \\
\mathrm{Pic}(\mathcal{M}^{\mathrm{ss}}_{\mathrm{G,X}}) & \xleftarrow{f^*} & \mathrm{Pic}(\mathrm{M}_{\mathrm{G,X}}) & \xrightarrow{c} & \mathrm{Cl}(\mathrm{M}_{\mathrm{G,X}}) \\
\downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle r_3} & & \downarrow{\scriptstyle r_4} \\
\mathrm{Pic}(\mathcal{M}^{\mathrm{reg}}_{\mathrm{G,X}}) & \xleftarrow{f^*_{\mathrm{reg}}} & \mathrm{Pic}(\mathrm{M}^{\mathrm{reg}}_{\mathrm{G,X}}) & \xrightarrow{c_{\mathrm{reg}}} & \mathrm{Cl}(\mathrm{M}^{\mathrm{reg}}_{\mathrm{G,X}})
\end{array}$$

It is known (*see* [L-S], 9.2 and 9.3) that r_3 is injective (normality of $\mathrm{M}_{\mathrm{G,X}}$), that c_{reg} is an isomorphism (smoothness of $\mathrm{M}^{\mathrm{reg}}_{\mathrm{G,X}}$) and that r_4 is an isomorphism (the complement of $\mathrm{M}^{\mathrm{reg}}_{\mathrm{G,X}}$ in $\mathrm{M}_{\mathrm{G,X}}$ is of codimension ≥ 2). So in order to prove that $\mathrm{M}_{\mathrm{G,X}}$ is not locally factorial, it is sufficient to show that r_3 is not surjective. In order to see this, we will consider the generator $\mathcal{L}$ of theorem 1.1. Indeed, there is an element $\mathcal{L}'$ of $\mathrm{Pic}(\mathrm{M}^{\mathrm{reg}}_{\mathrm{G,X}})$ such that $f^*_{\mathrm{reg}}(\mathcal{L}') = r_2 \circ r_1(\mathcal{L})$: as the center Z(G) of G acts trivially on $\mathcal{H}_1(0)$, the restriction of $\mathcal{O}_{\mathcal{Q}_{\mathrm{G}}}(1)$ to $\mathcal{Q}^{\mathrm{reg}}_{\mathrm{G}}$ is $\mathrm{L_X G/Z(G)}$-linearized, hence descends to a line bundle $\mathcal{L}'$ to $\mathrm{M}^{\mathrm{reg}}_{\mathrm{G,X}}$ (use that $\mathcal{Q}^{\mathrm{reg}}_{\mathrm{G}} \to \mathrm{M}^{\mathrm{reg}}_{\mathrm{G,X}}$ is a $\mathrm{L_X G/Z(G)}$-bundle). On the other hand, $\mathcal{L}'$ cannot be in the image of r_3. Let us suppose the contrary. Then there is a line bundle $\mathcal{L}''$ such that $f^*(\mathcal{L}'') = r_1(\mathcal{L})$. Now consider the well known tower of inclusions

$$\mathrm{Spin}_8 \xrightarrow{\alpha} \mathrm{F}_4 \xrightarrow{\beta} \mathrm{E}_6 \xrightarrow{\gamma} \mathrm{E}_7 \xrightarrow{\delta} \mathrm{E}_8. \tag{4.1.1}$$

An easy calculation (using for example [Sl], tables 77-128) shows that the restriction of the representation $\mathrm{L}(\varpi_8)$ of E_8 to Spin_8 is $28\mathrm{L}(0) \oplus 8\mathrm{L}(\varpi_1) \oplus \mathrm{L}(\varpi_2) \oplus 8\mathrm{L}(\varpi_3) \oplus 8\mathrm{L}(\varpi_4)$, hence has Dynkin index 60, since $d(\varpi_i) = 2$ for $i = 1, 3, 4$ and $d(\varpi_2) = 12$ ([L-S], 2.6). It follows from Theorem 1.1 (and the discussion in 3.3) that $\mathcal{L}$ pulls back to the (positive) generator $\mathcal{P}$ of $\mathrm{Pic}(\mathcal{M}_{\mathrm{Spin}_8})$, which is the pfaffian line bundle of [L-S]. The pullback of $\mathcal{L}''$ then defines a line bundle $\mathcal{P}''$ on $\mathrm{M}_{\mathrm{Spin}_8,\mathrm{X}}$ such that $f^*(\mathcal{P}'') = \mathcal{P}$. But this is a contradiction, as the pfaffian line bundle does not descend to the coarse moduli space $\mathrm{M}_{\mathrm{Spin}_8,\mathrm{X}}$ ([B-L-S], 8.2). □

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