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NICOLAS LERNER

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PERTURBATION AND ENERGY ESTIMATES

BY NICOLAS LERNER

ABSTRACT. – We prove in this paper that given a first order pseudo-differential operator P satisfying Nirenberg-Treves condition (ψ), there exists an L^2-bounded operator R so that $P + R$ is solvable. Solvability occurs with the loss of two derivatives. We prove along the way a natural factorization result for operators satisfying condition (ψ). © Elsevier, Paris

RÉSUMÉ. – On démontre que pour un opérateur pseudo-différentiel P de symbole dans la classe standard $S^1_{1,0}$, satisfaisant la condition (ψ) de Nirenberg-Treves, il existe un opérateur pseudo-différentiel R de symbole dans la classe $S^0_{1/2,1/2}$ (donc borné sur L^2), de sorte que l'opérateur $P + R$ soit résoluble. On obtient la résolubilité au prix de la perte de deux dérivées, c'est-à-dire des solutions u de régularité H^{s-1} pour l'équation $(P + R)u = f$, avec f dans H^s. Ce résultat est stable pour des perturbations classiques, *i.e.* $P + R + N$ est résoluble pour tout opérateur pseudo-différentiel N de symbole dans la classe $S^0_{1,0}$. L'élément-clé de la démonstration est un résultat de factorisation pour des opérateurs satisfaisant la condition (ψ). © Elsevier, Paris

Contents

Key words : Solvability, Energy estimates, condition (ψ), pseudo-differential operators, Wick symbols

1. Introduction and main results

Let p be a complex-valued homogeneous symbol defined on the cotangent bundle of a smooth manifold. We assume that p is of principal type, which means that the Hamiltonian vector field H_p is such that $H_p \wedge L \neq 0$, where L is the Liouville vector field. We shall assume that p satisfies condition (ψ), *i.e.* that $\operatorname{Im} p$ does not change sign from $-$ to $+$ along oriented bicharacteristics of $\operatorname{Re} p$. This was conjectured in the early seventies by

Nirenberg and Treves and it is still an open problem that condition (ψ) for p is equivalent to solvability for operators with principal symbol p . We refer the reader to [H2] for a historical perspective on the study of this problem.

This condition was proved necessary for solvability by Moyer in dimension two and by Hörmander in the general case (Corollary 26.4.8 in [H1]). On the other hand, it was shown that condition (ψ) implied solvability for differential operators (Nirenberg and Treves ([NT]) used an analyticity assumption, removed by Beals and Fefferman in [BF]), and also when the total dimension is two ([L1]) or in various special cases ([H2], [L2], [L5], [L6]). One can note that in all these cases, condition (ψ) implies "optimal" solvability (solvability with a loss of one derivative), that the existence of $\mathrm{H}^{s+\mathrm{order}P-1}$ solutions for equations $Pu = f$ with $f \in \mathrm{H}^s$. It was proved in [L3] that (ψ) does not imply optimal solvability: one should not expect solvability in its optimal version expressed above as a consequence of the geometric condition (ψ). The paper [L3] disproved various earlier claims on this matter.

Dencker ([D1]) was able to prove that the counterexamples of [L3], although not optimally solvable, were in fact solvable with a loss of *two* derivatives: he proved the existence of $\mathrm{H}^{s+\mathrm{order}P-2}$ solutions for equations $Pu = f$ with $f \in \mathrm{H}^s$, where P is the operator constructed in [L3]. He had to overcome several specific difficulties with handling these very weak estimates. First of all an a priori estimate yielding solvability of P with loss of two derivatives should be of the type

$$(1.1) \qquad \|P^*u\|_{\mathrm{H}^s} \geq C\|u\|_{\mathrm{H}^{s+m-2}}, \qquad m = \text{order } P, \qquad C \text{ "large constant"},$$

and is not obviously invariant by pseudo-differential perturbation of order $m-1$. Since the operator P is assumed of principal type and acting on functions, lower-order terms do not carry geometric significance since the subprincipal symbol is defined only on the doubly characteristic set which is the empty set here. Thus inequality (1.1) has to be invariant under classical pseudo-differential perturbations of order $m-1$: to prove this, one should somehow produce a different proof for each perturbation $P+R$ with R pseudo-differential of order $m-1$. Next, a very disturbing feature, clearly linked with the previous one, is that microlocalization of (1.1) does not automatically provide a local estimate. In fact the partitions of unity are made with operators of order 0 and thus generate commutators of order $m-1$. It is the same difficulty as above and one has to compensate the weakness of the estimate by some regularity of the multiplier in the energy method. A generalization of the results of [D1] is given in [L7]. Another difficulty is that the counterexample [L3] could probably be modified to yield the existence of a first-order pseudo-differential operator P with principal symbol satisfying condition (ψ) and a L^2-bounded operator R such that no estimate of type (1.1) could be true for $P^* + R$ even if $s+m-2$ is replaced by $s+m-N$, $N \geq 2$. We shall neither prove nor use this here, but one should keep in mind that to preserve a weak estimate of type (1.1) by a perturbation of order $m-1$ (say by an L^2-bounded operator if $s=0$ and $m=1$) one has to be very careful in choosing the type of perturbation. On the one hand, if the perturbation is a classical pseudo-differential operator of order $m-1$ the estimate must be preserved, and on the other hand, there are perturbations of "order"$m-1$, (*e.g.* L^2-bounded operator if $s=0$ and $m=1$) which could ruin any estimate of type (1.1).

In the present paper, we prove the following theorem.

THEOREM 1.1. – *Let P be a principal-type pseudo-differential operator of order 1 whose principal symbol satisfies condition* (ψ). *There exists an* L^2*-bounded operator R such that the equation* $(P+R)u = f$ *has a local solution in* H^{-1} *for f in* L^2.

This is in fact a consequence of the more precise theorem which follows.

THEOREM 1.2. – *Let $q(t,x,\xi)$ be a real-valued continuous function defined on* $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ *such that*

$$q(t,x,\xi)>0 \quad \text{and} \quad s>t \qquad \Longrightarrow \qquad q(s,x,\xi)\geq 0. \tag{1.2}$$

We assume that q is smooth with respect to x,ξ and that for all α,β,

$$\sup_{t,x,\xi}\left|\partial_x^\alpha\partial_\xi^\beta q(t,x,\xi)\right|(1+|\xi|)^{-1+|\beta|}<\infty.$$

Then there exists a positive constant C and a symbol $r(t,x,\xi)$ satisfying

$$\sup_{t,x,\xi}\left|\partial_x^\alpha\partial_\xi^\beta r(t,x,\xi)\right|(1+|\xi|)^{\frac{|\beta|-|\alpha|}{2}}<\infty, \tag{1.3}$$

such that for any $u\in \mathrm{C}_0^\infty\big((0,C^{-1}),\mathrm{L}^2(\mathbb{R}^n)\big)$ *(*$\mathrm{D}_t=-i\partial_t$ *and* $a(x,\mathrm{D}_x)$ *is the operator with symbol* $a(x,\xi)$*),*

$$C\int_{\mathbb{R}}|\mathrm{D}_t u+i(q+r)(t,x,\mathrm{D}_x)u|_{\mathrm{L}^2(\mathbb{R}^n)}dt\geq \sup_{t\in\mathbb{R}}|u(t)|_{\mathrm{H}^{-1}(\mathbb{R}^n)}. \tag{1.4}$$

We note that the symbol $r(t,\cdot)$ is in the Calderón-Vaillancourt class $S^0_{1/2,1/2}$ and in particular is L^2-bounded. Moreover (1.2) is the expression of condition (ψ) for the operator $\mathrm{D}_t-iq(t,x,\mathrm{D}_x)$. We shall see below (section 8) that the proof of theorem 1.2 follows the lines of the proof of the following theorem, expressed for pseudo-differential operators with large parameter.

THEOREM 1.3. – *Let $q(t,x,\xi,\Lambda)$ be a real-valued continuous function defined on* $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times[1,+\infty)$. *We assume that* (1.2) *is satisfied (for each Λ) and that*

$$\sup_{t,x,\xi,\Lambda}\left|\partial_x^\alpha\partial_\xi^\beta q(t,x,\xi,\Lambda)\right|\Lambda^{-1+|\beta|}<\infty. \tag{1.5}$$

Then there exists a positive constant C and a symbol $r(t,x,\xi,\Lambda)$ satisfying

$$\sup_{t,x,\xi}\left|\partial_x^\alpha\partial_\xi^\beta r(t,x,\xi,\Lambda)\right|\Lambda^{\frac{|\beta|-|\alpha|}{2}}<\infty, \tag{1.6}$$

such that for any $u\in \mathrm{C}_0^\infty\big((0,C^{-1}),\mathrm{L}^2(\mathbb{R}^n)\big)$,

$$C\int_{\mathbb{R}}|\mathrm{D}_t u+i(q+r)(t,x,\mathrm{D}_x)u|_{\mathrm{L}^2(\mathbb{R}^n)}dt\geq \Lambda^{-1/2}\sup_{t\in\mathbb{R}}|u(t)|_{\mathrm{L}^2(\mathbb{R}^n)}. \tag{1.7}$$

Let us observe that the power $\Lambda^{-1/2}$ amounts to a loss of only 3/2 of derivatives and is thus slightly better in these terms than theorem 1.2. Theorem 7.1 below gives also some stability properties of the estimate (1.7) by pseudo-differential perturbation of order 0.

The main step in the proofs of these theorems is to establish the following factorization theorem (see theorems 7.1 and 8.3):

THEOREM 1.4. – *Let q be a symbol satisfying the assumptions of theorem 1.3. There exist selfadjoint operators $\Omega(t), J(t)$ on $\mathrm{L}^2(\mathbb{R}^n)$ with*

$$\text{(1.8)} \qquad \mathrm{Id} \le \Omega(t) \le \Lambda\,\mathrm{Id}, \qquad \|J(t)\|_{\mathcal{L}(\mathrm{L}^2)} \le 1, \qquad \big(J(t_2) - J(t_1)\big)(t_2 - t_1) \ge 0$$

and there exists an operator $R(t)$ in $\mathcal{L}(\mathrm{L}^2)$ such that

$$\text{(1.9)} \qquad q(t,\cdot,\cdot,\Lambda)^w = \Omega(t)J(t) + R(t) = J(t)\Omega(t) + R^*(t),$$

where q^w stands here for the operator with Weyl symbol q.

The factorization (1.9) is rather natural for functions q satisfying (1.2): as a matter of fact, property (1.2) is indeed true for nondecreasing functions of the variable t, and also stable by multiplication by a nonnegative function. So all functions of type $q(t,x,\xi) = a(t,x,\xi)\ b(t,x,\xi)$ with $a \ge 0$ and $\partial b/\partial t \ge 0$ satisfy (1.2). In fact such a factorization is true for operators satisfying condition (P) (ruling out any change of sign of $\operatorname{Im} p$ along the characteristics of $\operatorname{Re} p$): in this case, the operator $J(t)$ above could be chosen independent of t. A very important feature of this factorization is that it is a consequence of a symbolic calculus which can be expressed in terms of a metric as introduced in [H1] (section 18.6). We have already used some properties of this new metric to handle some L^2 inequalities in [L6], improving results of [L5]. In particular the perturbation $R(t)$ is in a substantially better class than the $S^0_{1/2,1/2}$ set of symbols and belongs in fact to $S(1,g)$, where g is an "admissible" metric on $\mathbb{R}^{2n}$ with some asymptotic properties, *i.e.* such that $g \ll g^\sigma$ at some places (see definition 3.1 below).

Let us describe briefly the contents of the paper. We want to use energy inequalities for $\mathrm{D}_t + iq(t,x,\mathrm{D}_x)$ where q satisfies the assumptions of theorem 1.3. Using condition (ψ) for the adjoint operator (*i.e.* (1.2)) we find a sign function $s(t,x,\xi)$ for $q(t,x,\xi)$ increasing with t: the product sq is $|q|$ and $\partial s/\partial t \ge 0$. To quantify this function $s(t,\cdot)$, we use a positive quantization which amounts to regularizing s by convolution with a Gaussian function prior to using Weyl's formula. Thus we begin with Section 2, devoted to the study of Gaussian mollifiers for characteristic functions. We study Gaussian regularizations of $\operatorname{sign}(\beta)$ where β is a coordinate ($\mathrm{d}\beta \ne 0$). In section 3, we introduce a new metric $g^{\{t\}}$ defining a pseudo-differential calculus on $\mathbb{R}^{2n}_{x,\xi}$ for each real t. This metric is a refinement of the Calderón-Zygmund metric defined by Beals and Fefferman in [BF]. In section 4, we show that our function $q(t,\cdot)$ behaves like a symbol with respect to this metric. In section 5, we introduce the Wick quantization. Let Γ be a positive definite quadratic form on $\mathbb{R}^{2n}$ and a be a bounded measurable function ; one defines the Wick(Γ) quantization of a as the operator with Weyl symbol $a * 2^n \exp -2\pi\Gamma$. For a suitable choice of Γ we show that the Wick quantization of $s(t,\cdot)$ is the nondecreasing operator $J(t)$ appearing in theorem 1.4. The more technical and lengthy section is the sixth one, in which a precise factorization property for the symbol q is proved. We give a factorization (see theorem 1.4) for the operator $Q(t)$ with Weyl symbol $q(t,x,\xi)$ so that $Q(t) = \Omega(t)J(t) + R(t)$, where $R(t)$ is an L^2-bounded operator (in fact with symbol in $S(1,g^{\{t\}})$) and $\Omega(t)$ is positive. One should think of the operator $\Omega(t)$ as "almost" scalar ; for instance, the bracket $[\Omega(t), J(t)]$ is an L^2-bounded operator. The seventh section is devoted to proving energy estimates for a perturbation of $Q(t)$, *i.e.* for $\Omega(t)J(t)$. We use the energy method, *i.e.* we compute

$2\mathrm{Re}\,\langle \mathrm{D}_t u + i\Omega(t)J(t), iM(t)u\rangle$ with a carefully chosen multiplier $M(t)$. Some specific technicalities at this stage are linked to the loss of derivatives and to proving stability of our estimates by a class of zeroth-order perturbations. Section 8 deals with perturbations of operators with homogeneous symbols. It turns out that some substantial work remains to be done, even though the large parameter case was thoroughly investigated.

2. Gaussian mollifiers for characteristic functions

For $\xi \in \mathbb{R}$, we set

$$\sigma_0(\xi) = \int_{\mathbb{R}} \mathrm{sign}\,(\eta) 2^{1/2} e^{-2\pi|\xi-\eta|^2} \mathrm{d}\eta = \int_0^{\xi} 2^{3/2} e^{-2\pi t^2} \mathrm{d}t. \tag{2.1}$$

Note that σ_0 is odd, $\sigma_0(+\infty) = 1$ and its derivative σ_0' is in $\mathcal{S}(\mathbb{R})$ and is positive. We consider now a smooth real-valued function $b(\mathbf{x}, \lambda)$ defined on $\mathbb{R}^d \times [1, \infty)$, in the symbol class $S(\lambda^{1/2}, |\mathrm{d}\mathbf{x}|^2\lambda^{-1})$. It means that b satisfies the estimates

$$\sup_{\mathbf{x}\in\mathbb{R}^d,\ \lambda\geq 1} |\partial_{\mathbf{x}}^k b(\mathbf{x}, \lambda)|\,\lambda^{-\frac{1}{2}+\frac{k}{2}} = \gamma_k(b) < \infty, \tag{2.2}$$

for any integer k. We omit below the dependence of b on the parameter λ and refer to $\gamma_k(b)$ as the semi-norms of b. We set up then, for $(\mathbf{x}, \xi) \in \mathbb{R}^d \times \mathbb{R}$, $\beta \in \mathbb{R}$,

$$j(\mathbf{x}, \xi) = \iint_{\mathbb{R}^d\times\mathbb{R}} \mathrm{sign}\,\big(\eta + b(\mathbf{y})\big)\, 2^{\frac{d+1}{2}} e^{-2\pi(|\mathbf{x}-\mathbf{y}|^2 + |\xi-\eta|^2)} \mathrm{d}\mathbf{y}\mathrm{d}\eta, \tag{2.3}$$

$$\sigma(\beta, \mathbf{x}) = \int \sigma_0\big(\beta + b(\mathbf{x}+\mathbf{y}) - b(\mathbf{x})\big)\Xi(\mathbf{y})\mathrm{d}\mathbf{y}, \qquad \Xi(\mathbf{y}) = 2^{d/2} e^{-2\pi|\mathbf{y}|^2}. \tag{2.4}$$

Thus we have

$$j(\mathbf{x}, \xi) = \sigma\big(\xi + b(\mathbf{x}), \mathbf{x}\big). \tag{2.5}$$

Moreover,

$$B_0(\mathbf{x}, \mathbf{y}) = b(\mathbf{x}+\mathbf{y}) - b(\mathbf{x}) = b'(\mathbf{x})\cdot\mathbf{y} + \omega_0(\mathbf{x}, \mathbf{y})\mathbf{y}^2\lambda^{-1/2}, \tag{2.6}$$

where the bilinear form $\omega_0(\mathbf{x}, \mathbf{y}) = \int_0^1 (1-\theta) b''(\mathbf{x}+\theta\mathbf{y})\lambda^{1/2}\mathrm{d}\theta$ satisfies the estimates

$$|\partial_{\mathbf{x}}^k \partial_{\mathbf{y}}^l \omega_0(\mathbf{x}, \mathbf{y})| \leq \lambda^{-\frac{k+l}{2}} \gamma_{k+l+2}(b), \tag{2.7}$$

following from (2.2). We have from Taylor's formula and (2.4), (2.6),

$$\begin{aligned}\sigma(\beta, \mathbf{x}) = &\int \sigma_0\big(b'(\mathbf{x})\cdot\mathbf{y} + \omega_0(\mathbf{x}, \mathbf{y})\mathbf{y}^2\lambda^{-1/2}\big)\Xi(\mathbf{y})\mathrm{d}\mathbf{y} \\ &+ \beta \iint_0^1 \sigma_0'\big(\theta\beta + b(\mathbf{x}+\mathbf{y}) - b(\mathbf{x})\big)\Xi(\mathbf{y})\mathrm{d}\mathbf{y}\mathrm{d}\theta,\end{aligned} \tag{2.8}$$

which implies, since σ_0 is odd,

$$(2.9)\quad \sigma(\beta,\mathbf{x}) = \lambda^{-1/2}\iint_0^1 \sigma_0'\big(b'(\mathbf{x})\cdot\mathbf{y}+\theta\omega_0(\mathbf{x},\mathbf{y})\mathbf{y}^2\lambda^{-1/2}\big)\omega_0(\mathbf{x},\mathbf{y})\mathbf{y}^2\Xi(\mathbf{y})d\mathbf{y}d\theta + \beta\iint_0^1 \sigma_0'\big(\theta\beta+B_0(\mathbf{x},\mathbf{y})\big)\Xi(\mathbf{y})d\mathbf{y}d\theta.$$

On the other hand, from (2.4), (2.6), we get

$$(2.10)\qquad \sigma(\beta,\mathbf{x}) = \sigma_0(\beta) + \iint_0^1 \sigma_0'\big(\beta+\theta B_0(\mathbf{x},\mathbf{y})\big)B_0(\mathbf{x},\mathbf{y})\Xi(\mathbf{y})d\mathbf{y}d\theta.$$

LEMMA 2.1. – *Let b be a symbol satisfying* (2.2). *Then, if σ is defined by* (2.4), *we have*

$$(2.11)\qquad \sigma(\beta,\mathbf{x}) = \beta\sigma_1(\beta,\mathbf{x}) + \lambda^{-1/2}r_0(\mathbf{x}) = \sigma_0(\beta) + \sigma_2(\beta,\mathbf{x}),$$

where $r_0 \in S(1,|d\mathbf{x}|^2\lambda^{-1})$ with semi-norms depending only on the γ_k in (2.2). *Moreover $\sigma_1(\beta,\mathbf{x}) > 0$ and for all k,κ,l nonnegative integers*

$$(2.12)\qquad \begin{cases} \sup\limits_{\substack{\beta\in\mathbb{R},\mathbf{x}\in\mathbb{R}^d\\ \lambda\ge 1}} \big|(\partial_\beta^\kappa\partial_{\mathbf{x}}^k\sigma_1)(\beta,\mathbf{x})\big|\lambda^{k/2} = \gamma'_{k\kappa} < \infty,\\ \sup\limits_{\substack{\beta\in\mathbb{R},\mathbf{x}\in\mathbb{R}^d\\ \lambda\ge 1}} |\beta|^l\,\big|(\partial_\beta^\kappa\partial_{\mathbf{x}}^k\sigma_2)(\beta,\mathbf{x})\big|\lambda^{k/2} = \gamma''_{k\kappa l} < \infty, \end{cases}$$

depending only on the γ_k in (2.2). *Moreover, there exists a positive constant c_0 depending only on $d,\gamma_1(b),\gamma_2(b)$, such that, for all positive C,*

$$(2.13)\qquad \inf_{|\beta|\le C,\mathbf{x}\in\mathbb{R}^d} \sigma_1(\beta,\mathbf{x}) \ge c_0 e^{-4\pi C^2}.$$

Proof. – From (2.9–10), (2.6), to get (2.11) one can take

$$(2.14)\qquad r_0(\mathbf{x}) = \iint_0^1 \sigma_0'\big(b'(\mathbf{x})\cdot\mathbf{y}+\theta\omega_0(\mathbf{x},\mathbf{y})\mathbf{y}^2\lambda^{-1/2}\big)\omega_0(\mathbf{x},\mathbf{y})\mathbf{y}^2\Xi(\mathbf{y})d\mathbf{y}d\theta,$$

$$(2.15)\qquad \sigma_1(\beta,\mathbf{x}) = \iint_0^1 \sigma_0'\big(\theta\beta+B_0(\mathbf{x},\mathbf{y})\big)\Xi(\mathbf{y})d\mathbf{y}d\theta,$$

$$(2.16)\qquad \sigma_2(\beta,\mathbf{x}) = \iint_0^1 \sigma_0'\big(\beta+\theta B_0(\mathbf{x},\mathbf{y})\big)B_0(\mathbf{x},\mathbf{y})\Xi(\mathbf{y})d\mathbf{y}d\theta.$$

Leibniz and Faà de Bruno formulæ yield the first property in (2.12) from (2.1-2) and (2.7). From (2.15) and the positivity of σ_0', one gets the nonnegativity of σ_1. The second property in (2.12) follows from the identity (B_0 is given in (2.6))

$$(2.17)\qquad \beta\sigma_2(\beta,\mathbf{x}) = \iint_0^1 \big(\beta+\theta B_0(\mathbf{x},\mathbf{y})\big)\sigma_0'\big(\beta+\theta B_0(\mathbf{x},\mathbf{y})\big)B_0(\mathbf{x},\mathbf{y})\Xi(\mathbf{y})d\mathbf{y}d\theta - \iint_0^1 \sigma_0'\big(\beta+\theta B_0(\mathbf{x},\mathbf{y})\big)\theta\big(B_0(\mathbf{x},\mathbf{y})\big)^2\Xi(\mathbf{y})d\mathbf{y}d\theta.$$

Since $B_0(\cdot,\mathbf{y}) \in S((1+|\mathbf{y}|^2), |d\mathbf{x}|^2\lambda^{-1})$ and the functions $s\sigma_0'(s), \sigma_0'(s)$ are bounded with all their derivatives, we obtain (2.12) using induction on l to write $\beta^l\sigma_2(\beta,\mathbf{x})$ in a way similar to (2.17). On the other hand, from (2.15) and (2.1), using the nonnegativity of σ_0', assuming $|\beta| \leq C$, we get

$$\sigma_1(\beta,\mathbf{x}) \geq \iint_0^1 \mathbf{1}(|y| \leq 1)\ \sigma_0'\big(\theta\beta + B_0(\mathbf{x},\mathbf{y})\big)\ \Xi(\mathbf{y})d\mathbf{y}d\theta,$$

so that, using $|\theta\beta + B_0(\mathbf{x},\mathbf{y})| \leq C + \gamma_1 + \gamma_2$ and the monotonicity on $\mathbb{R}_+$ of $\sigma_0'(s) = 2^{3/2}e^{-2\pi s^2}$ we obtain (2.13), with

$$c_0 = e^{-4\pi(\gamma_1+\gamma_2)^2}2^{(3+d)/2}\int_{|y|\leq 1} e^{-2\pi|y|^2}d\mathbf{y}. \tag{2.18}$$

The proof of lemma 2.1 is complete.

□

LEMMA 2.2. – *Using the same assumptions and notations as in lemma 2.1, we have*

$$\begin{cases} \lambda^{1/2}\beta = \sigma(\beta,\mathbf{x})\omega(\beta,\mathbf{x}) + r(\beta,\mathbf{x}) \\ \qquad \textit{with} \inf_{\substack{\beta\in\mathbf{R},\mathbf{x}\in\mathbf{R}^d\\ \lambda\geq 1}} \omega(\beta,\mathbf{x})\lambda^{-1/2}(|\beta|+1)^{-1} > 0, \end{cases} \tag{2.19}$$

$$\begin{cases} \sup_{\substack{\beta\in\mathbf{R},\mathbf{x}\in\mathbf{R}^d\\ \lambda\geq 1}} \big|(\partial_\beta^\kappa\partial_\mathbf{x}^k\omega)(\beta,\mathbf{x})\big|\lambda^{\frac{k-1}{2}}(1+|\beta|)^{-1+\kappa} < \infty, \\ \sup_{\substack{\beta\in\mathbf{R},\mathbf{x}\in\mathbf{R}^d\\ \lambda\geq 1}} \big|(\partial_\beta^\kappa\partial_\mathbf{x}^k r)(\beta,\mathbf{x})\big|\lambda^{k/2} < \infty, \end{cases} \tag{2.20}$$

where these quantities depend only on the γ_k *in* (2.2). *Moreover there exists* $c_1 > 0$, *depending only on the* γ_k *in* (2.2) *such that for* $|\beta| \geq c_1$

$$|\sigma(\beta,\mathbf{x})| \geq \frac{\sigma_0(c_1)}{2} > 0. \tag{2.21}$$

Proof. – Let ρ_0 be in $C_0^\infty(\mathbb{R})$, nonnegative, identically 1 on $[-1,1]$, supported in $[-2,2]$. Using the first equality in (2.11), we have for an arbitrary positive c,

$$\lambda^{1/2}\rho_0(\beta/c)\sigma(\beta,\mathbf{x}) = \lambda^{1/2}\beta\rho_0(\beta/c)\sigma_1(\beta,\mathbf{x}) + r_0(\mathbf{x})\rho_0(\beta/c).$$

Thus using (2.12–13), which gives

$$\gamma_{00}'^{-1}\lambda^{1/2} \leq \lambda^{1/2}\sigma_1(\beta,\mathbf{x})^{-1} \leq \lambda^{1/2}c_0^{-1}e^{16\pi c^2} \qquad \text{if } |\beta| \leq 2c, \tag{2.22}$$

we obtain

$$\lambda^{1/2}\beta\rho_0(\beta/c) = \sigma(\beta,\mathbf{x})\lambda^{1/2}\sigma_1(\beta,\mathbf{x})^{-1}\rho_0(\beta/c) - r_0(\mathbf{x})\sigma_1(\beta,\mathbf{x})^{-1}\rho_0(\beta/c). \tag{2.23}$$

Moreover, using the second equality in (2.11), one gets, with $\rho_1 = 1-\rho_0$, using σ_0 odd

$$\begin{cases} \rho_1(\beta/c)\sigma(\beta,\mathbf{x})\lambda^{1/2}\beta = \rho_1(\beta/c)\lambda^{1/2}\beta\big(\sigma_0(\beta) + \sigma_2(\beta,\mathbf{x})\big) \\ \qquad = \rho_1(\beta/c)\lambda^{1/2}\beta\ \operatorname{sign}\beta\big(\sigma_0(|\beta|) + \operatorname{sign}\beta\sigma_2(\beta,\mathbf{x})\big) \\ \qquad = \lambda^{1/2}\beta\rho_1(\beta/c)\ \operatorname{sign}\beta\left[\sigma_0(|\beta|) + \dfrac{1}{c}\dfrac{|\beta|\,\sigma_2(\beta,\mathbf{x})}{\beta/c}\right]. \end{cases} \tag{2.24}$$

Using the second inequality in (2.12), we note that, on the support of ρ_1, where $|\beta| \geq c$, the term between brackets satisfies

$$(2.25) \qquad \sigma_0(|\beta|) + \frac{1}{c}\frac{|\beta|\sigma_2(\beta,\mathbf{x})}{\beta/c} \geq \sigma_0(|\beta|) - \frac{1}{c}|\beta||\sigma_2(\beta,\mathbf{x})| \geq \sigma_0(c) - \frac{\gamma''_{001}}{c} \geq \frac{\sigma_0(c)}{2}$$

if c is large enough (in fact if $c\sigma_0(c) \geq 2\gamma''_{001}$: from (2.1) we can assume this from now on for a fixed $c = c_1$). We obtain from (2.24)

$$(2.26) \qquad \rho_1(\beta/c_1)\sigma(\beta,\mathbf{x})\lambda^{1/2}\beta\left[\sigma_0(|\beta|) + \frac{1}{c_1}\frac{|\beta|\sigma_2(\beta,\mathbf{x})}{\beta/c_1}\right]^{-1} = \lambda^{1/2}\beta\rho_1(\beta/c_1)\ \mathrm{sign}\ \beta,$$

that is

$$(2.27) \qquad \lambda^{1/2}\beta\rho_1(\beta/c_1) = \sigma(\beta,\mathbf{x})\rho_1(\beta/c_1)\lambda^{1/2}|\beta|\left[\sigma_0(|\beta|) + \frac{1}{c_1}\frac{|\beta|\sigma_2(\beta,\mathbf{x})}{\beta/c_1}\right]^{-1}.$$

Adding up (2.23) and (2.27) we get

$$(2.28) \qquad \begin{cases} \lambda^{1/2}\beta = \sigma(\beta,\mathbf{x})\Big(\lambda^{1/2}\sigma_1(\beta,\mathbf{x})^{-1}\rho_0(\beta/c_1) + \\ \qquad\qquad \rho_1(\beta/c_1)\lambda^{1/2}|\beta|\left[\sigma_0(|\beta|) + \frac{1}{c_1}\frac{|\beta|\sigma_2(\beta,\mathbf{x})}{\beta/c_1}\right]^{-1}\Big) \\ \qquad -r_0(\mathbf{x})\sigma_1(\beta,\mathbf{x})^{-1}\rho_0(\beta/c_1), \end{cases}$$

so that setting-up

$$(2.29) \qquad \begin{cases} \omega(\beta,\mathbf{x}) = \lambda^{1/2}\sigma_1(\beta,\mathbf{x})^{-1}\rho_0(\beta/c_1) \\ \qquad +\rho_1(\beta/c_1)\lambda^{1/2}|\beta|\left[\sigma_0(|\beta|) + \frac{1}{c_1}\frac{|\beta|\sigma_2(\beta,\mathbf{x})}{\beta/c_1}\right]^{-1}, \end{cases}$$

$$(2.30) \qquad r(\beta,\mathbf{x}) = -r_0(\mathbf{x})\sigma_1(\beta,\mathbf{x})^{-1}\rho_0(\beta/c_1),$$

we obtain (2.19), which follows from (2.29) and (2.12). The estimates (2.20) are direct consequences of the estimates for σ_1 in (2.12-13), of (2.25) and the estimates for r_0 in lemma 2.1. Moreover from (2.11), (2.25) we have for $|\beta| \geq c_1$

$$(2.31) \qquad |\sigma(\beta,\mathbf{x})| \geq \sigma_0(|\beta|) - \frac{1}{c_1}\frac{|\beta||\sigma_2(\beta,\mathbf{x})|}{|\beta|/c_1} \geq \sigma_0(c_1) - \frac{\gamma''_{001}}{c_1} \geq \frac{\sigma_0(c_1)}{2}.$$

The proof of lemma 2.2 is complete.

□

We examine now to which extent the previous results depend on the choice of the positive definite quadratic form appearing in the integral of formula (2.3). This will not be used before section 8 of the paper and could be omitted before the reader gets there. Anyhow it clearly belongs to this section and we want to discuss this problem here. We set up then, for $(\mathbf{x},\xi) \in \mathbb{R}^d \times \mathbb{R}$, and a positive definite $(d+1)\times(d+1)$ matrix A with determinant 1

$$(2.32) \qquad A = \begin{pmatrix} M & \mathbf{n} \\ {}^t\mathbf{n} & \alpha \end{pmatrix},$$

where M is a $d \times d$ (positive definite) matrix, $\mathbf{n}$ is a column-vector in $\mathbb{R}^d$, $\alpha > 0$. We define also

(2.33)
$$j_A(\mathbf{x}, \xi) = \iint_{\mathbf{R}^d \times \mathbf{R}} \operatorname{sign}(\eta + b(\mathbf{y}))\, 2^{\frac{d+1}{2}} e^{-2\pi\left[M(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) + 2\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})(\xi-\eta) + \alpha(\xi-\eta)^2\right]} d\mathbf{y}\, d\eta.$$

Using

$$M\mathbf{y}\cdot\mathbf{y} + 2\mathbf{n}\cdot\mathbf{y}\eta + \alpha\eta^2 = (\alpha^{1/2}\eta + \alpha^{-1/2}\mathbf{n}\cdot\mathbf{y})^2 + \underbrace{(M - \alpha^{-1}\mathbf{n}^t\mathbf{n})}_{A_1^{-1}}\mathbf{y}\cdot\mathbf{y},$$

taking as new variables $\zeta = \alpha^{1/2}\eta + \alpha^{-1/2}\mathbf{n}\cdot\mathbf{y}$ and $\mathbf{z} = A_1{}^{-1/2}\mathbf{y}$ we get from the homogeneity of the sign and from $\det A = 1$

$$j_A(\mathbf{x}, \xi) = \iint_{\mathbf{R}^d \times \mathbf{R}} \operatorname{sign}\left(\zeta - \alpha^{-1/2} A_1^{1/2}\mathbf{n}\cdot\mathbf{z} + \alpha^{1/2}\xi + \alpha^{1/2} b(A_1^{1/2}\mathbf{z} + \mathbf{x})\right) 2^{\frac{d+1}{2}} e^{-2\pi(\zeta^2 + |\mathbf{z}|^2)} d\mathbf{z}\, d\zeta.$$

Using (2.1) this gives

$$(2.34) \quad j_A(\mathbf{x}, \xi) = \int_{\mathbf{R}^d} \sigma_0\left[\alpha^{1/2}\left(\xi + b(A_1^{1/2}\mathbf{z} + \mathbf{x}) - \alpha^{-1} A_1^{1/2}\mathbf{n}\cdot\mathbf{z}\right)\right] 2^{\frac{d}{2}} e^{-2\pi|\mathbf{z}|^2} d\mathbf{z}.$$

We note that, according to (2.6-7), we have

$$b(A_1^{1/2}\mathbf{z} + \mathbf{x}) - b(\mathbf{x}) - \alpha^{-1} A_1^{1/2}\mathbf{n}\cdot\mathbf{z} = [b'(\mathbf{x}) - \alpha^{-1}\mathbf{n}]\cdot A_1^{1/2}\mathbf{z} + \omega_0(\mathbf{x}, A_1^{1/2}\mathbf{z})(A_1^{1/2}\mathbf{z})^2\lambda^{-1/2},$$

so that

$$(2.35) \qquad j_A(\mathbf{x}, \xi) = \int_{\mathbf{R}^d} \sigma_0[M]\, 2^{\frac{d}{2}} e^{-2\pi|\mathbf{z}|^2} d\mathbf{z},$$

where $[M] = \left[\alpha^{1/2}\left(\overbrace{\xi + b(\mathbf{x})}^{\beta} + [b'(\mathbf{x}) - \alpha^{-1}\mathbf{n}]\cdot A_1^{1/2}\mathbf{z} + \omega_0(\mathbf{x}, A_1^{1/2}\mathbf{z})(A_1^{1/2}\mathbf{z})^2\lambda^{-1/2}\right)\right]$. This is in fact a slight modification of (2.4) where σ_0 is replaced by the (still odd with positive derivative) $\sigma_0 \circ \alpha^{1/2}$. A simple inspection of the proofs of (2.8-9) yields the result of lemma 2.1 for

$$(2.36) \qquad j_A(\mathbf{x}, \xi) = \sigma_A(\beta, \mathbf{x})$$

with

(2.37)
$$\sigma_A(\beta, \mathbf{x}) = \int_{\mathbf{R}^d} (\sigma_0 \circ \alpha^{1/2})\left(\beta + [b'(\mathbf{x}) - \alpha^{-1}\mathbf{n}]\cdot A_1^{1/2}\mathbf{z} + \omega_0(\mathbf{x}, A_1^{1/2}\mathbf{z})(A_1^{1/2}\mathbf{z})^2\lambda^{-1/2}\right) 2^{\frac{d}{2}} e^{-2\pi|\mathbf{z}|^2} d\mathbf{z}.$$

We could repeat lemmas 2.1-2 for σ_A but we shall need only the following

LEMMA 2.3. – *Let A be a positive definite $(d+1)\times(d+1)$ matrix given by* (2.32) *and b be a symbol satisfying* (2.2). *Then, if σ_A is defined by* (2.37), *we have*

$$\sigma_A(\beta,\mathbf{x}) = \beta\sigma_{1A}(\beta,\mathbf{x}) + \lambda^{-1/2}r_{0A}(\mathbf{x}) = (\sigma_0\circ\alpha^{1/2})(\beta) + \sigma_{2A}(\beta,\mathbf{x}), \tag{2.38}$$

where $r_{0A}\in S(1,|\mathrm{d}\mathbf{x}|^2\lambda^{-1})$ with semi-norms depending only on the γ_k in (2.2) *and on the norms of A, A^{-1}. Moreover $\sigma_{1A}(\beta,\mathbf{x})>0$ and for all k*

$$\sup_{\substack{\beta\in\mathbb{R},\mathbf{x}\in\mathbb{R}^d\\ \lambda\geq 1}} \left|(\partial_\beta^\kappa\partial_\mathbf{x}^k\sigma_{1A})(\beta,\mathbf{x})\right|\lambda^{k/2}<\infty,$$
$$\sup_{\substack{\beta\in\mathbb{R},\mathbf{x}\in\mathbb{R}^d\\ \lambda\geq 1}} |\beta|^l\left|(\partial_\beta^\kappa\partial_\mathbf{x}^k\sigma_{2A})(\beta,\mathbf{x})\right|\lambda^{k/2}<\infty,$$

depending only on the γ_k in (2.2) *and on the norms of A, A^{-1}.*

3. An admissible nonconformal metric

Let n be an integer and $\mathbb{R}^{2n}=\mathbb{R}^n_x\times\mathbb{R}^n_\xi$ be the standard phase space with its symplectic form

$$\varsigma = \sum_{1\leq j\leq n} \mathrm{d}\xi_j\wedge\mathrm{d}x_j.$$

We equip the phase space with a positive definite quadratic form Γ such that $\Gamma^\varsigma=\Gamma$: it means that there is a symplectic basis of $\mathbb{R}^{2n}$ in which the matrix of Γ is the identity (see (3.13) below or (18.5.7) in [H1] for a general definition of Γ^ς). We consider now a smooth real-valued function $q(X,\Lambda)$ defined on $\mathbb{R}^{2n}\times[1,\infty)$, in the symbol class $S(\Lambda,\Lambda^{-1}\Gamma)$. It means that q satisfies the estimates

$$\sup_{X\in\mathbb{R}^{2n},\Lambda\geq 1} |\partial_X^k q(X,\Lambda)|_\Gamma\Lambda^{-1+\frac{k}{2}} = \gamma_k(q)<\infty, \tag{3.1}$$

for any integer k (the norm of the multi-linear form $\partial_X^k q$ is evaluated with respect to Γ). As in the previous section, we omit below the dependence of q upon Λ as well as the index Γ for the norms of multilinear forms.

DEFINITION 3.1. – *Let $X\in\mathbb{R}^{2n}\mapsto g_X$ be a mapping from $\mathbb{R}^{2n}$ to the set of positive definite quadratic forms on $\mathbb{R}^{2n}$. The metric g is said to be admissible if g is slowly varying, temperate* (see [H1], chapters 1 and 18) *and such that, for each $X\in\mathbb{R}^{2n}$, $g_X\leq g_X^\varsigma$.*

The proper class of the symbol q is defined by the following metric, conformal to Γ,

$$G_X^{[q]} = \lambda(X)^{-1}\Gamma, \qquad \lambda(X) = 1+|q'(X)|_\Gamma^2+|q(X)|. \tag{3.2}$$

The metric $G_X^{[q]}$ is known to be admissible with constants depending only on γ_k, $k=0,1,2$ in (3.1) ([H1], section 26.10). Moreover q belongs to $S(\lambda,G^{[q]})$ with the same semi-norms as q in $S(\Lambda,\Lambda^{-1}\Gamma)$. We define a new metric by

$$g_X^{[q]}(T) = \frac{|\mathrm{d}q(X)\cdot T|^2}{\lambda(X)+|q(X)|^2} + \frac{\Gamma(T)}{\lambda(X)^{1/2}+|q(X)|}, \qquad T\in\mathbb{R}^{2n}. \tag{3.3}$$

We drop the superscript $[q]$ for g and G in the sequel of this section.

LEMMA 3.2. – *Let q be a symbol satisfying* (3.1). *If G is defined by* (3.2) *and g by* (3.3), *we have for $\Lambda \geq 1$,*

$$\gamma_{01}(q)^{-1}\Lambda^{-1}\Gamma \leq G_X \leq 2g_X \leq 4\Gamma = 4\Gamma^\varsigma \leq 8g_X^\varsigma \leq 16G_X^\varsigma \leq 16\gamma_{01}(q)\Lambda\Gamma, \tag{3.4}$$

with $\gamma_{01}(q) = 1 + \gamma_1(q)^2 + \gamma_0(q)$. Moreover, g is slowly varying and temperate.

In fact (3.1) implies $\lambda(X) \leq \gamma_{01}(q)\Lambda$, which gives the first inequality in (3.4). Moreover, from the expression of λ in (3.2), one gets $1 \leq \lambda(X)^{1/2} + |q(X)| \leq 2\lambda(X)$ and $|q'(X)|^2 \leq \lambda(X)$; this implies the second and third inequality in (3.4). The other inequalities follow then from the definition of g^ς. Let us prove now that g is slowly varying: let r be a positive number and assume $g_X(X - Y) \leq r$, that is

$$\frac{|q'(X)(X-Y)|^2}{\lambda(X) + |q(X)|^2} + \frac{\Gamma(X-Y)}{\lambda(X)^{1/2} + |q(X)|} \leq r.$$

Since $G \leq 2g$, and since G is slowly varying, we can assume that r is small enough (say $2r \leq r_0$) to ensure that $G_X \sim G_Y$, *i.e.* the ratios $G_X(T)/G_Y(T) = \lambda(Y)/\lambda(X)$ are bounded above and below by constants depending only on the γ_k in (3.1). From Taylor's formula and (3.1), we have

$$|q(X)| \leq |q(Y)| + |q'(X).(Y - X)| + \gamma_2(q)\Gamma(Y - X)/2.$$

Then we obtain, from $g_X(Y - X) \leq r$,

$$\lambda(X)^{1/2} + |q(X)| \leq C_1\lambda(Y)^{1/2} + |q(Y)| + r^{1/2}\big(\lambda(X)^{1/2} + |q(X)|\big) + \gamma_2(q)\ r\big(\lambda(X)^{1/2} + |q(X)|\big)/2,$$

where $C_1 \geq 1$ depends only on the γ_k in (3.1). This gives

$$2r \leq \min\big(1/2, 1/\gamma_2(q), r_0\big) \Longrightarrow \lambda^{1/2}(X) + |q(X)| \leq 4C_1\big(\lambda^{1/2}(Y) + |q(Y)|\big). \tag{3.5}$$

We have then, under this condition on r,

$$\begin{aligned} g_Y(T) &\leq 64C_1^2\,\frac{|q'(X).T|^2 + \gamma_2(q)^2\Gamma(Y-X)\Gamma(T)}{\lambda(X) + |q(X)|^2} + 4C_1\frac{\Gamma(T)}{\lambda^{1/2}(X) + |q(X)|} \\ &\leq 64C_1^2\, g_X(T) + 64\gamma_2(q)^2 r\frac{(\lambda^{1/2}(X) + |q(X)|)\Gamma(T)}{\lambda(X) + |q(X)|^2} \leq \big(64C_1^2 + 128\gamma_2(q)^2 r\big)g_X(T), \end{aligned}$$

thus proving that g is slowly varying.

We must now verify that g is temperate. Going back to (3.3), we check

$$\left\{\begin{aligned} g_X(T) \leq 2&\frac{|q'(Y).T|^2 + \gamma_2(q)^2\Gamma(X-Y)\Gamma(T)}{\lambda(Y) + |q(Y)|^2}\Big(\frac{\lambda(Y) + |q(Y)|^2}{\lambda(X) + |q(X)|^2}\Big) \\ &+ \frac{\Gamma(T)}{\lambda(Y)^{1/2} + |q(Y)|}\Big(\frac{\lambda(Y)^{1/2} + |q(Y)|}{\lambda(X)^{1/2} + |q(X)|}\Big) \\ &\leq g_Y(T)\big(1 + \Gamma(X-Y)\big)\Big(\frac{\lambda(Y) + |q(Y)|^2}{\lambda(X) + |q(X)|^2} + 1\Big)C_2, \end{aligned}\right. \tag{3.6}$$

where C_2 depends only on $\gamma_2(q)$. Moreover, we have, using (3.1), (3.2) and Taylor's formula (first order for q', second order for q),

$$\lambda(Y) + |q(Y)|^2 = 1 + |q'(Y)|^2 + |q(Y)| + |q(Y)|^2 \leq \big(\lambda(X) + |q(X)|^2\big)\big(1 + \Gamma(X-Y)\big)^2 C_3,$$

where C_3 depends only on the γ_k. Thus we obtain immediately from the above inequality and (3.6) that

$$\frac{g_X(T)}{g_Y(T)} \le C_4\big(1+\Gamma(X-Y)\big)^3,$$

which implies, from the third inequality in (3.4), that

$$\frac{g_X(T)}{g_Y(T)} \le C_5\big(1+2g_X^\sigma(X-Y)\big)^3,$$

which is the temperance of the metric g (C_5 depends only on the γ_k). The proof of lemma 3.2 is complete.

□

Remark 3.3. – The metric G separates the phase space into specific regions, depending on the fact that the dominant term in the expression (3.2) of $\lambda(X)$ is $|q(X)|$, $|q'(X)|^2$ or 1. In fact, following lemma 26.10.2 in [H1], one gets G-elliptic regions in which $C|q(X)| \ge \lambda(X)$. In such places, the metric g is equivalent to G, *i.e.* the ratios $g_X(T)/G_X(T)$ are bounded above and below by fixed constants. This is also the case for the G-negligible regions, in which $\lambda(X)$ is bounded above. In fact, in both cases $\lambda(X)^{1/2}+|q(X)|$ is equivalent to $\lambda(X)$ and since $|q'(X)\cdot T|^2 \le \gamma_1^2\lambda(X)\Gamma(T)$, we get the equivalence of g and G there. The metric g is *not* equivalent to G on G-non-degenerate regions, that is on places where $|q'(X)|^2$ is the dominant term in (3.2). For instance, if q were the linear form $\lambda^{1/2}\xi_1$, the metric g would be, with symplectic coordinates $(x_1,\xi_1,X') \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^{2n-2}$,

$$g = \frac{|d\xi_1|^2}{1+\xi_1^2} + \frac{|dx_1|^2+|dX'|^2}{\lambda^{1/2}(1+|\xi_1|)} \gg \frac{|dX|^2}{\lambda} = G,$$

when $|\xi_1| \ll \lambda^{1/2}$.

LEMMA 3.4. – *Let $g/2$ be the admissible metric defined in* (3.3). *We define the positive numbers μ by*

$$\mu(X)^2 = 4\inf_T\big[g_X^\sigma(T)/g_X(T)\big]. \tag{3.7}$$

We have, with a constant C depending only on the γ_k in (3.1),

$$1 \le \mu(X) \le 4\lambda(X), \tag{3.8}$$

$$|q(X)|+1 \ge \lambda(X)/2 \Longrightarrow G_X \le 2g_X \le 4(1+4\gamma_1^2)G_X, \tag{3.9}$$

$$|q'(X)|^2 \ge \lambda(X)/2 \ \ \textit{and}\ \ |q(X)| \le \lambda(X)^{1/2} \Longrightarrow C^{-1} \le \frac{\mu(X)^2}{\lambda(X)^{1/2}} \le C, \tag{3.10}$$

$$|q'(X)|^2 \ge \lambda(X)/2 \ \ \textit{and}\ \ |q(X)| \ge \lambda(X)^{1/2} \Longrightarrow C^{-1} \le \frac{\mu(X)^2}{|q(X)|^3\lambda(X)^{-1}} \le C, \tag{3.11}$$

$$(3.12)\qquad |q(X)| \le C\lambda(X)^{1/3}\mu(X)^{2/3}, \qquad |q'(X)\cdot T| \le C\lambda(X)^{1/3}\mu(X)^{2/3}g_X(T)^{1/2}.$$

Proof. – The inequality (3.8) follows from (3.4), and (3.9) from (3.1–4). To get (3.10), we note that the linear form $L = q'(X)\lambda(X)^{-1/2}$ has a Γ norm in $[2^{-1/2}, 1]$. Let e_1 be the unit vector (for Γ) such that $L(e_1) = \|L\|$. Since $\Gamma = \Gamma^\varsigma$, we can identify Γ with the matrix I_{2n} and ς with the matrix

$$\begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix}.$$

Take now $\epsilon_1 = \varsigma e_1$; this is a unit vector orthogonal to e_1. Moreover, the space $V = (\mathrm{span}\{e_1, \varsigma e_1\})^{\perp\Gamma}$ is a symplectic space for the restriction of ς since it is invariant by ς: for $x \in V$ one has $\langle \varsigma x, e_1\rangle_\Gamma = -\langle x, \varsigma e_1\rangle_\Gamma = 0$ and $\langle \varsigma x, \varsigma e_1\rangle_\Gamma = \langle x, e_1\rangle_\Gamma = 0$. One can then choose in V a symplectic basis $\epsilon_2, \dots, \epsilon_n, e_2, \dots, e_n$, orthonormal for Γ. The matrix of ς in the basis $\epsilon_1, e_1, \epsilon_2, \dots, \epsilon_n, e_2, \dots, e_n$ is then

$$\begin{pmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & \mathrm{I}_{n-1} \\ -\mathrm{I}_{n-1} & 0 \end{matrix} \end{pmatrix}.$$

In fact, we proved the more invariantly stated lemma:

LEMMA 3.5. – *Let (E, ς) be a $2n$ dimensional symplectic space, i.e. $\varsigma : E \longrightarrow E^*$ with $\varsigma = -\varsigma^*$ and ς non degenerate. Let Γ be a positive definite quadratic form on E, i.e. $\Gamma : E \longrightarrow E^*$ with $\Gamma = \Gamma^*$ and $\langle \Gamma x, x\rangle_{E^*,E} > 0$ for $x \neq 0$. We define*

$$(3.13)\qquad \Gamma^\varsigma = \varsigma^*\Gamma^{-1}\varsigma, \qquad or \qquad \langle \Gamma^\varsigma T, T\rangle_{E^*,E} = \sup_{\langle \Gamma U, U\rangle_{E^*,E}=1} \langle \varsigma T, U\rangle^2_{E^*,E},$$

and we assume that Γ is symplectic, i.e. $\Gamma^\varsigma = \Gamma$. Let $e_1 \in E$ such that $\langle \Gamma e_1, e_1\rangle_{E^*,E} = 1$. Then there exists a basis of E $(\epsilon_1, \epsilon_2, \dots, \epsilon_n, e_1, e_2, \dots, e_n)$ orthonormal for Γ and symplectic, i.e.

$$\langle \varsigma\epsilon_j, e_k\rangle_{E^*,E} = \delta_{j,k}, \qquad \langle \varsigma\epsilon_j, \epsilon_k\rangle_{E^*,E} = 0, \qquad \langle \varsigma e_j, e_k\rangle_{E^*,E} = 0.$$

The argument is the same as above with $\epsilon_1 = \Gamma^{-1}\varsigma e_1$.

Going back to the proof of (3.10), we get that (X is fixed), using (η_1, y_1, η', y') as coordinates in the base $(e_1, \epsilon_1, e_2, \dots e_n, \epsilon_2, \dots \epsilon_n)$,

$$g_X = \frac{\lambda L \otimes L}{\lambda + |q|^2} + \frac{\Gamma}{\lambda^{1/2} + |q|} = \frac{|d\eta_1|^2\lambda\|L\|^2}{\lambda + |q|^2} + \frac{|d\eta_1|^2 + |dy_1|^2 + |d\eta'|^2 + |dy'|^2}{\lambda^{1/2} + |q|},$$

so that, since $|q| \le \lambda$, the metric g is equivalent to $\tilde g$:

$$(3.14)\qquad \tilde g = |d\eta_1|^2\frac{\lambda}{\lambda + |q|^2} + \frac{|dy_1|^2 + |dY'|^2}{\lambda^{1/2} + |q|}, \qquad \frac{1}{2}\tilde g \le g \le 2\tilde g,$$

with symplectic coordinates $(\eta_1, y_1, Y' = (y', \eta'))$; this gives (3.10) and (3.11). The inequalities (3.12) are obvious if (3.9) is satisfied. Moreover, from the definition of λ in

(3.2), $|q|+1<\lambda/2$ implies $|q'|^2 \geq \lambda/2$. If (3.10) is satisfied, we get the first part of (3.12), and to check the second part, we note from (3.3), that

$$|q'(X)\cdot T|^2 \leq g_X(T)\big(\lambda(X)+|q(X)|^2\big). \tag{3.15}$$

If (3.10) is satisfied, this gives (3.12). If (3.11) is satisfied, then $|q|^3\lambda^{-1}$ is equivalent to μ^2 which gives $|q|$ equivalent to $\lambda^{1/3}\mu^{2/3}$. If (3.11) is satisfied, since (3.15) is a consequence of (3.3), it is enough to check that $\lambda^{1/2}+|q|$ is controlled by $\lambda^{1/3}\mu^{2/3}$, which is obvious since $|q|\geq \lambda^{1/2}$ and $|q|$ is equivalent to $\lambda^{1/3}\mu^{2/3}$ there. This gives (3.12) under (3.11). The proof of lemma 3.4 is complete.

□

4. Symbol classes

LEMMA 4.1. – *Let q be a symbol satisfying* (3.1). *If g is defined by* (3.3), λ *by* (3.2), μ *by* (3.7), *then* $q\in S(\lambda^{1/3}\mu^{2/3},g)$ *with the same semi-norms as q in* (3.1).

Proof. – We have to find constants C_k such that, for all X,T in $\mathbb{R}^{2n}$,

$$|q^{(k)}(X)T^k| \leq \lambda(X)^{1/3}\mu(X)^{2/3}g_X(T)^{k/2}C_k. \tag{4.1}$$

This is obvious if X is a G-elliptic or a G-negligible point, *i.e.* if (3.9) is satisfied since $q\in S(\lambda,G)$. If X is a G-non degenerate point, *i.e.* if (3.10) or (3.11) are satisfied, (4.1) follows from (3.12) for $0\leq k\leq 1$. Let us check (4.1) for $k\geq 2$ when (3.10) or (3.11) are satisfied. We know from (3.1) that

$$|q^{(k)}(X)T^k| \leq \Lambda^{1-\frac{k}{2}}\Gamma(T)^{k/2}\,\gamma_k.$$

Moreover, from (3.3), we have

$$g_X(T)^{k/2} \geq \frac{\Gamma(T)^{k/2}}{(\lambda^{1/2}+|q|)^{k/2}}.$$

It suffices then to prove, for $k\geq 2$, that

$$\gamma_k\,\Lambda^{1-\frac{k}{2}} \leq C_k\lambda^{1/3}\mu^{2/3}\Big(\lambda^{1/2}+|q|\Big)^{-k/2},$$

that is, to verify

$$\Big(\lambda^{1/2}+|q|\Big)^{\frac{k}{2}-1}\;\Big(\lambda^{1/2}+|q|\Big) \leq \Lambda^{\frac{k}{2}-1}\lambda^{1/3}\mu^{2/3}C_k/\gamma_k.$$

Since $k-2\geq 0$ and $|q|\leq\lambda\leq\Lambda\gamma_{01}$ (see lemma 3.2), it is enough to check that $\lambda^{1/2}+|q|$ is controlled by $\lambda^{1/3}\mu^{2/3}$. When (3.10) is satisfied it amounts to controling $\lambda^{1/2}$ which is there equivalent to μ^2 being equivalent to $\lambda^{1/3}\mu^{2/3}$; when (3.11) is satisfied it amounts to controling $|q|$ which is there equivalent to $\lambda^{1/3}\mu^{2/3}$. The proof of lemma 4.1 is complete.

□

LEMMA 4.2. – *Let f be a bounded smooth function of one real variable so that f' belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. Let q be a symbol satisfying* (3.1) *and g be the metric defined in* (3.3). *Take* $\tilde{\lambda}(X) \in S(\lambda(X), G_X)$ *so that* $\tilde{\lambda}(X) \geq \mathrm{d}_0\lambda(X)$ *for some positive constant* d_0 *(e.g* $\tilde{\lambda}(X) = \sqrt{1+|q'(X)|_\Gamma^{\frac{4}{3}}+|q(X)|^2}$*). We have*

$$a(X) = f\big(\tilde{\lambda}(X)^{-1/2}\, q(X)\big) \in S(1, g), \tag{4.2}$$

with semi-norms depending only on those of q in (3.1), *on the* L^∞ *norm of f, on semi-norms of* f' *in* $\mathcal{S}(\mathbb{R})$ *and on* d_0.

Proof. – The fact that $\sqrt{1+|q'(X)|_\Gamma^{\frac{4}{3}}+|q(X)|^2}$ belongs to $S(\lambda, G)$ is a standard consequence of the slowly varying property for G. Since $q \in S(\lambda, G)$ and $f' \in \mathcal{S}(\mathbb{R})$, the estimates implying (4.2) are trivially satisfied when (3.9) is satisfied. Assuming thus that X is G-non-degenerate (*i.e.* that (3.10) or (3.11) are satisfied), we set $B(X) = \tilde{\lambda}(X)^{-1/2}\, q(X) \in S(\lambda^{1/2}, g)$ and we start from Faà de Bruno's formula,

$$a^{(k)}(X)T^k = \sum_{\substack{l_1+\ldots+l_r=k\\ l_j\geq 1}} \frac{k!}{r!l_1!\cdots l_r!} f^{(r)}(B)\; B^{(l_1)}T^{l_1}\ldots B^{(l_r)}T^{l_r}$$

that is

$$a^{(k)}(X)T^k = \sum_{\substack{l_1=\ldots=l_s=1\\ s+l_{s+1}+\ldots+l_r=k\\ l_j\geq 2 \text{ if } j\geq s+1}} \frac{k!}{r!l_{s+1}!\cdots l_r!} f^{(r)}(B)\; (B'(X).T)^s B^{(l_{s+1})}T^{l_{s+1}}\ldots B^{(l_r)}T^{l_r}. \tag{4.3}$$

Consequently, from $B \in S(\lambda^{1/2}, G)$, one gets

$$|a^{(k)}(X)T^k| \leq \sum_{\substack{l_1=\ldots=l_s=1\\ s+l_{s+1}+\ldots+l_r=k\\ l_j\geq 2 \text{ if } j\geq s+1}} \frac{k!}{r!l_{s+1}!\cdots l_r!} \left|f^{(r)}(B(X))\right|\, |B'(X).T|^s\; \lambda^{\frac{1}{2}-\frac{l_{s+1}}{2}}\ldots\lambda^{\frac{1}{2}-\frac{l_r}{2}}\; \Gamma(T)^{\frac{k-s}{2}}. \tag{4.4}$$

We obtain thus

$$\begin{aligned}|a^{(k)}(X)T^k| &\leq \sum_{\substack{l_1=\ldots=l_s=1\\ s+l_{s+1}+\ldots+l_r=k\\ l_j\geq 2 \text{ if } j\geq s+1}} \frac{k!}{r!l_{s+1}!\cdots l_r!} \left|f^{(r)}(B(X))\right|\, |B'(X).T|^s\; \overbrace{\lambda^{\frac{r-s}{2}}\lambda^{-\frac{1}{2}(k-s)}}^{\lambda^{\frac{r-k}{2}}}\; \Gamma(T)^{\frac{k-s}{2}}\\ &\leq C \sum_{\substack{1\leq r\leq k\\ 0\leq s\leq r\\ k\geq s+2(r-s)}} \left|f^{(r)}(B(X))\right|\, |B'(X).T|^s\; \lambda^{\frac{r-k}{2}}\; \Gamma(T)^{\frac{k-s}{2}}.\end{aligned} \tag{4.5}$$

Since $k \geq s + 2(r - s) = 2r - s$, *i.e.* since $\frac{r-k}{2} \leq \frac{s-k}{4}$, the inequality (4.5) implies, with d_k depending only on k,

$$|a^{(k)}(X)T^k| \leq \mathrm{d}_k \sum_{0 \leq s \leq r \leq k,\ r \geq 1} \left|f^{(r)}\big(B(X)\big)\right| \, |B'(X).T|^s \lambda^{\frac{s-k}{4}} \Gamma(T)^{\frac{k-s}{2}}. \tag{4.6}$$

Let us assume first that $|B(X)| \leq 1$: from (3.10) and (3.14) we get that $g_X(T)^{1/2}$ is equivalent to

$$|B'(X).T| + \frac{\Gamma(T)^{1/2}}{\lambda^{1/4}},$$

so that (4.6) gives the estimates for (4.2) in this case. When $|B(X)| \geq 1$, we saw in (3.14) and (3.11) that $g_X(T)^{1/2}$ was equivalent to

$$\frac{|B'(X).T|}{|B(X)|} + \frac{\Gamma(T)^{1/2}}{\lambda^{1/4}|B(X)|^{1/2}}.$$

Since f' is in the Schwartz space $\mathcal{S}(\mathbb{R})$ and $r \geq 1$ in (4.6), we can control any power of B and get (4.2). The proof of lemma 4.2 is complete.

□

5. Wick quantization

Before defining the Wick quantization, we recall the usual quantization formula,

$$a(x, \mathrm{D}_x)u(x) = \iint e^{2i\pi x\xi} a(x,\xi)\hat{u}(\xi)\mathrm{d}\xi, \qquad \hat{u}(\xi) = \int e^{-2i\pi x\xi} u(y)\mathrm{d}y,$$

and the Weyl formula

$$a^w u(x) = \iint e^{2i\pi(x-y)\xi} a\Big(\frac{x+y}{2}, \xi\Big) u(y)\mathrm{d}y\, \mathrm{d}\xi.$$

It may be useful at this point to notice that the definition of the Weyl quantization of the phase space $E \oplus E^*$, where E is a n-dimensional real vector space (the configuration space), can be given without further structure. Although it is clear with the formula above, we can use as well the following definition of the Weyl quantization, due to Unterberger [Un]: for $a \in \mathcal{S}'(E \oplus E^*)$

$$a^w = \int_{E \oplus E^*} a(X) 2^n \sigma_X \mathrm{d}X,$$

where the phase symmetry σ_X is defined as the unitary and selfadjoint operator on $\mathrm{L}^2(E)$

$$\sigma_X u(y) = u(2x - y) e^{-4i\pi\langle x-y,\xi\rangle_{E,E^*}}, \quad \text{where } X = (x, \xi).$$

Naturally the symplectic structure on $E \oplus E^*$ is given by the standard relation

$$[X, Y] = \langle \xi, y\rangle_{E,E^*} - \langle \eta, x\rangle_{E,E^*}, \quad \text{for } X = (x,\xi),\ Y = (y, \eta),$$

and the measure $\mathrm{d}X$ on the phase space is chosen as the n^{th} exterior product of the symplectic form divided by $n!$, which is also the tensor product of any Haar measure on E with its dual measure on E^*. A tempered distribution a can be quantized since it is easily checked that $X \in E \oplus E^* \mapsto \langle \sigma_X u, v\rangle_{\mathrm{L}^2(E)} \in \mathcal{S}(E \oplus E^*)$ if $u, v \in \mathcal{S}(E)$.

On the other hand the definition of the Wick quantization below requires the introduction of a symplectic norm (as defined in lemma 3.5) on the phase space. So as in section 3 and 4, we assume that the phase space $\mathbb{R}^{2n}$ is equipped with a symplectic norm Γ. The following definition is a regularization of the formula above for the Weyl quantization and contains also some classical properties.

DEFINITION 5.1. – *Let $Y = (y, \eta)$ be a point in $\mathbb{R}^{2n}$. The operator Σ_Y^Γ is defined as $\left[2^n e^{-2\pi\Gamma(\cdot - Y)}\right]^w$. This is a rank-one orthogonal projection: $\Sigma_Y^\Gamma u = (W_\Gamma u)(Y)\tau_Y^\Gamma\varphi$ with $(W_\Gamma u)(Y) = \langle u, \tau_Y^\Gamma \varphi\rangle_{\mathrm{L}^2(\mathbb{R}^n)}$, where $\varphi(x) = 2^{n/4}e^{-\pi|x|^2}$ and τ_Y^Γ is a metaplectic affine transformation quantifying the affine symplectic map $X \mapsto \Gamma^{1/2}(X - Y)$ (when $\Gamma = \mathrm{Id}$, $\tau^\Gamma_{(y,\eta)}\varphi(x) = \varphi(x-y)e^{2i\pi\langle x - \frac{y}{2}, \eta\rangle}$). Let a be in $\mathrm{L}^\infty(\mathbb{R}^{2n})$. The Wick quantization of a with respect to Γ is defined as*

$$a^{\mathrm{Wick}(\Gamma)} = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_Y^\Gamma \mathrm{d}Y. \tag{5.1}$$

PROPOSITION 5.2. – *Let a be in $\mathrm{L}^\infty(\mathbb{R}^{2n})$. Then $a^{\mathrm{Wick}(\Gamma)} = W_\Gamma^* a^\mu W_\Gamma$ and $1^{\mathrm{Wick}(\Gamma)} = \mathrm{Id}_{\mathrm{L}^2(\mathbb{R}^n)}$ where W_Γ is the isometric mapping from $\mathrm{L}^2(\mathbb{R}^n)$ to $\mathrm{L}^2(\mathbb{R}^{2n})$ given above, and a^μ the operator of multiplication by a in $\mathrm{L}^2(\mathbb{R}^{2n})$. The operator $\pi_{\mathrm{H}}^\Gamma = W_\Gamma W_\Gamma^*$ is the orthogonal projection on a closed proper subspace H of $\mathrm{L}^2(\mathbb{R}^{2n})$. Moreover, we have*

$$\|a^{\mathrm{Wick}(\Gamma)}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^n))} \le \|a\|_{\mathrm{L}^\infty(\mathbb{R}^{2n})}, a(X) \ge 0 \Longrightarrow a^{\mathrm{Wick}(\Gamma)} \ge 0, \tag{5.2}$$

$$\|\Sigma_Y^\Gamma \Sigma_Z^\Gamma\|_{\mathcal{L}((\mathrm{L}^2(\mathbb{R}^n)))} \le 2^n e^{-\frac{\pi}{2}\Gamma(Y-Z)}. \tag{5.3}$$

Proof. – This proposition is classical and easy, except possibly for the last property. For $Y, Z \in \mathbb{R}^{2n}$ a straightforward computation shows that the Weyl symbol of $\Sigma_Y^\Gamma\Sigma_Z^\Gamma$ is, as a function of the variable $X \in \mathbb{R}^{2n}$,

$$e^{-\frac{\pi}{2}\Gamma(Y-Z)}e^{-2i\pi[X-Y, X-Z]}2^n e^{-2\pi\Gamma(X - \frac{Y+Z}{2})}, \tag{5.4}$$

where $[\cdot,\cdot]$ stands for the symplectic form. Since for the Weyl quantization, one has $\|a^w\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^n))} \le 2^n\|a\|_{\mathrm{L}^1(\mathbb{R}^{2n})}$, we get the result.

□

PROPOSITION 5.3. – *Let p be a symbol in $S(\Lambda, \Lambda^{-1}\Gamma)$ (see (3.1) for the definition of a class of symbols with a large parameter Λ). Then $p^{\mathrm{Wick}(\Gamma)} = p^w + r(p)^w$, with $r(p) \in S(1, \Lambda^{-1}\Gamma)$ so that the mapping $p \mapsto r(p)$ is continuous. Moreover, $r(p) = 0$ if p is a linear form or a constant.*

Proof. – From the definition above, one has $p^{\mathrm{Wick}(\Gamma)} = \tilde{p}^w$, with

$$\begin{cases} \tilde{p}(X) = \displaystyle\int_{\mathbb{R}^{2n}} p(X+Y)\, e^{-2\pi\Gamma(Y)}2^n \mathrm{d}Y = \\ \qquad p(X) + \underbrace{\displaystyle\int_0^1\!\!\int_{\mathbb{R}^{2n}} (1-\theta)p''(X+\theta Y)Y^2 e^{-2\pi\Gamma(Y)}2^n\mathrm{d}Y\mathrm{d}\theta}_{r(p)(X)}. \end{cases} \tag{5.5}$$

Thus, from the estimates on p, we get that,

$$\left|r(p)^{(k)}(X)\right| \le \gamma_{k+2}(p)\Lambda^{1-\frac{k+2}{2}} \int_{\mathbb{R}^{2n}} |Y|^2 e^{-2\pi\Gamma(Y)} 2^{n-1} \mathrm{d}Y,$$

which implies $r \in S(1, \Lambda^{-1}\Gamma)$. The last point in the proposition follows from (5.5), which shows that $r(p)$ depends linearly on p''.

□

6. A factorization result

We consider in this section a smooth real-valued function $q(t, X, \Lambda)$ defined on $\mathbb{R}_t \times \mathbb{R}^{2n}_X \times [1, \infty)$ which satisfies (3.1) uniformly in t, *i.e.*

$$(6.1) \qquad \sup_{t\in\mathbb{R}, X\in\mathbb{R}^{2n}, \Lambda\ge 1} |\partial_X^k q(t, X, \Lambda)|_\Gamma \Lambda^{-1+\frac{k}{2}} = \gamma_k(q) < \infty,$$

where Γ is a symplectic norm on $\mathbb{R}^{2n}$(see §3). Moreover, we assume that $\tau - iq$ satisfies condition (ψ) (from now on, we omit the dependence of q on Λ),

$$(6.2) \qquad q(t, X) > 0 \text{ and } s > t \Longrightarrow q(s, X) \ge 0.$$

Let us consider, *for t fixed*, the function

$$(6.3) \qquad \lambda(t, X) = 1 + |q(t, X)| + |q'_X(t, X)|^2_\Gamma.$$

We have, according to (3.2),

$$(6.4) \qquad q(t, X) \in S\left(\lambda(t, X), \frac{\Gamma}{\lambda(t, X)} = G_X^{[q(t,\cdot)]}\right).$$

Following section 3, the metric $G^{[q(t,\cdot)]}$ is slowly varying on $\mathbb{R}^{2n}_X$, satisfies the uncertainty principle ($G \le G^\sigma$), and is temperate. All the metrics $G^{[q(t,\cdot)]}$ are conformal and have the same "median symplectic" norm Γ, according to lemma 3.2. The metric $G^{[q(t,\cdot)]}$ defines the proper class of the symbol $q(t, \cdot)$: this is a metric on the phase space $\mathbb{R}^{2n}$, depending on $t \in \mathbb{R}$. We shall refer below to $G^{[q(t,\cdot)]}$ as the proper metric of the symbol q at the level t. We define now the bounded measurable functions

$$(6.5) \qquad \theta(X) = \begin{cases} \inf\{t \in (-1, +1),\ q(t, X) > 0\} & \text{if this set is not empty,} \\ 1 & \text{if this set is empty,} \end{cases}$$

and

$$(6.6) \qquad s_\theta(t, X) = \begin{cases} 1 & \text{if } t > \theta(X), \\ 0 & \text{if } t = \theta(X), \\ -1 & \text{if } t < \theta(X). \end{cases}$$

We get from (6.2) and (6.6) that for $s = s_\theta$, for all $t, t_1, t_2 \in (-1, 1), X \in \mathbb{R}^{2n}$,

$$(6.7) \qquad q(t, X)s(t, X) = |q(t, X)|, \big(s(t_2, X) - s(t_1, X)\big)(t_2 - t_1) \ge 0, |s(t, X)| \le 1.$$

LEMMA 6.1. – *Let s be any measurable function satisfying* (6.7). *We define $J(t)$ as the operator (depending on the function s and on the symplectic norm Γ)*

$$(6.8)\quad J(t) = \big(s(t,X)\big)^{\mathrm{Wick}(\Gamma)} = \big(J(t,X)\big)^{w}, J(t,X) = \int_{\mathbb{R}^{2n}} s(t,Y)\, 2^n\, e^{-2\pi\Gamma(X-Y)}\mathrm{d}Y,$$

where the Wick quantization is given in (5.1). *The symbol $J(t,X)$ belongs to $S(1,\Gamma)$ with semi-norms bounded by constants depending only on the dimension n. The operator $J(t)$ is bounded selfadjoint on $\mathrm{L}^2(\mathbb{R}^n)$ and such that*

$$(6.9)\qquad \|J(t)\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}))} \le 1, \quad t_1 \le t_2 \Longrightarrow J(t_1) \le J(t_2).$$

The expression of the Weyl symbol in (6.8) follows from definition 5.1. The convolution formula in (6.8) implies readily the statement on the symbol class. Property (5.2) implies the first statement in (6.9) and J is selfadjoint since its Weyl symbol is real-valued. Since the function $s(t,X)$ used in the lemma satisfies (6.7), it is nondecreasing as a function of t and we have that $t_1 \le t_2$, implies that for all $X \in \mathbb{R}^{2n}$, $s(t_1,X) \le s(t_2,X)$. Since the Wick quantization is nonnegative (second property in (5.2)), we obtain that

$$\big(s(t_2,\cdot) - s(t_1,\cdot)\big)^{\mathrm{Wick}(\Gamma)} \ge 0.$$

The proof is complete.

□

Note. – We could have taken $s = s_\theta$ for the proof of the lemma and the remaining part of this section. However, we want to emphasize the fact that only property (6.7) is used. This fact will be important in section 8 when the symplectic metric will be allowed to vary. From now on in this section, we suppose that the variable t is fixed. We consider a partition of unity subordinated to the metric $G_X^{[q(t,\cdot)]}$ defined in (6.4). The following lemma is classical for an admissible metric (see section 18.4 in [H1]).

LEMMA 6.2. – *Let t be a number in $(-1,1)$. There exists a sequence $(X_\nu)_{\nu\in\mathbb{N}}$ of points in the phase space $\mathbb{R}^{2n}$ and positive numbers r_0, N_0, such that the following properties are satisfied* ($G_\nu = \lambda_\nu^{-1}\Gamma$, $\lambda_\nu = \lambda(X_\nu)$, will stand for $G_{X_\nu}^{[q(t,\cdot)]}$ defined in (6.4)). *We define $U_\nu, U_\nu^*, U_\nu^{**}$ as the G_ν balls with center X_ν and radius $r_0, 2r_0, 4r_0$. There exist two families of nonnegative smooth functions on $\mathbb{R}^{2n}$, $(\chi_\nu)_{\nu\in\mathbb{N}}$, $(\psi_\nu)_{\nu\in\mathbb{N}}$ such that*

$$(6.10)\qquad \sum_\nu \chi_\nu(X) = 1,\ \operatorname{supp}\chi_\nu \subset U_\nu, \psi_\nu \equiv 1 \ \text{ on } U_\nu^*,\ \operatorname{supp}\psi_\nu \subset U_\nu^{**}.$$

Moreover, $\chi_\nu, \psi_\nu \in S(1,G_\nu)$ with semi-norms bounded independently of ν (in fact depending only on the γ_k in (6.1)*). The overlap of the balls U_ν^{**} is bounded, i.e.*

$$\bigcap_{\nu\in\mathcal{N}} U_\nu^{**} \ne \emptyset \Longrightarrow \sharp\mathcal{N} \le N_0.$$

Moreover, $G_X \sim G_\nu$ all over U_ν^{**} (i.e. the ratios $G_X(T)/G_\nu(T)$ are bounded above and below by a fixed constant, provided that $X \in U_\nu^{**}$), so that $\psi_\nu q \in S(\lambda_\nu, G_\nu)$ uniformly (in fact with semi-norms depending only on the γ_k in (6.1)*).*

It should be kept in mind that the partition (χ_ν) as well as ψ_ν and the choice of X_ν depend on t ; we omit this dependence for simplicity of notation. We shall prove also the following convenient lemma, a version of which is given in [BC] (lemme 7.9) and which appears essentially as a consequence of Cotlar's lemma (see *e.g.* lemme 4.2.3 in [BL]).

LEMMA 6.3. – *Let G be an admissible metric on $\mathbb{R}^{2n}$ and $\sum_\nu \chi_\nu(x,\xi) = 1$ be a partition of unity related to G as in the previous lemma. There exists a positive constant C such that for all $u \in \mathrm{L}^2(\mathbb{R}^n)$*

$$C^{-1}\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n)} \le \sum_\nu \|\chi_\nu^w u\|^2_{\mathrm{L}^2(\mathbb{R}^n)} \le C\|u\|^2_{\mathrm{L}^2(\mathbb{R}^n)}. \tag{6.11}$$

The right inequality is indeed a direct consequence of Cotlar's lemma so we leave it to the reader and provide a proof of the more involved left inequality. Let us recall first (see Théorème 3.2.2 in [BL] or (18.6.10) in [H1]), using notations of lemma 6.2, that for

$$\Delta_{\mu\nu} = 1 + (G_\mu^\varsigma \wedge G_\nu^\varsigma)(U_\mu^{**} - U_\nu^{**}) = 1 + 2 \inf_{\substack{T\in\mathbb{R}^{2n}\\ Y_\mu\in U_\mu^{**}, Y_\nu\in U_\nu^{**}}} \left[G_\mu^\varsigma(Y_\mu - T) + G_\nu^\varsigma(T - Y_\nu)\right], \tag{6.12}$$

there exists $N_0 \ge 1$ such that

$$\sup_\mu \sum_\nu \Delta_{\mu\nu}^{-N_0} = C_0 < +\infty. \tag{6.13}$$

The fact that G is temperate and the so-called confinement estimates of Théorème 2.2.1 in [BL] are useful to get that for all $N \ge 1$, there exists C_N such that

$$\|\chi_\nu^w \chi_\mu^w\|_{\mathcal{L}(\mathrm{L}^2)} \le C_N \Delta_{\mu\nu}^{-N}. \tag{6.14}$$

Now since the χ_ν are real-valued and a partition of unity, we obtain with $\mathrm{L}^2(\mathbb{R}^n)$ norms and dot products

$$\|u\|^2 = \sum_{\nu,\mu} \langle \chi_\nu^w u, \chi_\mu^w u\rangle \le \sum_{\substack{\nu,\mu\\ \text{with } \Delta_{\mu\nu}\le\alpha}} \left(\frac{\alpha}{\Delta_{\mu\nu}}\right)^{N_0} \|\chi_\mu^w u\|\|\chi_\nu^w u\| + \overbrace{\sum_{\substack{\nu,\mu\\ \text{with } \Delta_{\mu\nu}>\alpha}} \langle \chi_\mu^w \chi_\nu^w u, u\rangle}^{\langle Ru, u\rangle}, \tag{6.15}$$

where α is a positive parameter to be chosen later. We check now the self-adjoint operator R defined above: from Cotlar's lemma

$$\|R\|_{\mathcal{L}(\mathrm{L}^2)} \le \sup_{\substack{\nu_o,\mu_o\\ \text{with } \Delta_{\mu_o\nu_o}>\alpha}} \left[\sum_{\substack{\nu,\mu\\ \text{with } \Delta_{\mu\nu}>\alpha}} \|\chi_{\mu_o}^w \chi_{\nu_o}^w \chi_\mu^w \chi_\nu^w\|^{1/2}_{\mathcal{L}(\mathrm{L}^2)}\right], \tag{6.16}$$

and since

$$\begin{aligned} \|\chi_{\mu_o}^w \chi_{\nu_o}^w \chi_\mu^w \chi_\nu^w\|^2_{\mathcal{L}(\mathrm{L}^2)} &\le \|\chi_{\mu_o}^w \chi_{\nu_o}^w\|\ \|\chi_\mu^w \chi_\nu^w\|\ \|\chi_{\mu_o}^w\|\ \|\chi_{\nu_o}^w \chi_\mu^w\|\ \|\chi_\nu^w\| \\ &\le C_N \Delta_{\nu_o\mu_o}^{-N} C_N \Delta_{\nu\mu}^{-N} C_N \Delta_{\nu_o\mu}^{-N} \sup \|\chi_\nu^w\|^2, \end{aligned} \tag{6.17}$$

we get that for $N/4 = N_0$

$$\|R\|_{\mathcal{L}(L^2)} \le C_N^{3/4} \sup \|\chi_\nu^w\|^{1/2} \sup_{\substack{\nu_o,\mu_o \\ \text{with } \Delta_{\mu_o\nu_o}>\alpha}} \Delta_{\nu_o\mu_o}^{-N/4}\Big[\sum_{\nu,\mu} \Delta_{\nu\mu}^{-N/4}\Delta_{\nu_o\mu}^{-N/4}\Big]$$

$$(6.18)\quad \le C_N^{3/4} \sup \|\chi_\nu^w\|^{1/2} \alpha^{-N/4} \sum_\mu \Delta_{\nu_o\mu}^{-N/4}\big[\sum_\nu \Delta_{\nu\mu}^{-N/4}\big] \le C_{4N_0}^{3/4} \sup \|\chi_\nu^w\|^{1/2}\ \alpha^{-N_0} C_0^2.$$

We choose now the parameter α so that $C_{4N_0}^{3/4} \sup \|\chi_\nu^w\|^{1/2}\ \alpha^{-N_0} C_0^2 = 1/2$ and we get from (6.15) and (6.18) that

$$(6.19)\qquad \|u\|^2 \le \sum_{\substack{\nu,\mu \\ \text{with } \Delta_{\mu\nu}\le\alpha}} \Big(\frac{\alpha}{\Delta_{\mu\nu}}\Big)^{N_0} \|\chi_\mu^w u\|\|\chi_\nu^w u\| + \frac{1}{2}\|u\|^2.$$

The kernel $\Delta_{\mu\nu}^{-N_0}$ is of Schur type on $l^2(\mathbb{N})$ from (6.13) and thus we obtain

$$(6.20)\qquad \|u\|^2 \le 2\alpha^{N_0} C_0 \sum_\nu \|\chi_\nu^w u\|^2,$$

which is the statement of the lemma. The proof is complete.

□

Let us now fix notations for the symbol $q(t,\cdot)$ in its proper class $G_X^{[q(t,\cdot)]}$ defined in (6.4). We use the notations of lemma 6.2 and sum-up below a construction due to Beals and Fefferman [BF], using the terminology of Hörmander (lemma 26.10.2 in [H1]). Let $r_1 \le 1/2$ be a positive number. We shall say that

- ν is an elliptic-plus index ($\nu \in E_+$ for short) whenever $q(t, X_\nu) \ge r_1\lambda_\nu$,
- ν is an elliptic-minus index ($\nu \in E_-$) whenever $q(t, X_\nu) \le -r_1\lambda_\nu$,
- ν is a nondegenerate index ($\nu \in D$) whenever
  $|q(t, X_\nu)| < r_1\lambda_\nu$ and $|q'_X(t, X_\nu)|_\Gamma^2 \ge \lambda_\nu/4$,
- ν is an L^2-bounded index ($\nu \in L$) in the remaining cases whenever
  $|q(t, X_\nu)| < r_1\lambda_\nu$ and $|q'_X(t, X_\nu)|_\Gamma^2 < \lambda_\nu/4$
  which imply from (6.3) that $\lambda_\nu < 1 + 3\lambda_\nu/4$, *i.e.* $\lambda_\nu < 4$.

One can choose r_1 depending only on r_0 so that, with $E_\pm = E_+ \cup E_-$,

$$(6.21)\qquad \text{if } \nu \in E_\pm, \text{ for } X \in U_\nu^{**},\ q(t,X) = \pm\lambda_\nu e_{0\nu}(X) \text{ with } e_{0\nu} \in S(1, G_\nu) \text{ and } r_1/2 \le e_{0\nu}(X) \le 2r_1,$$

$$(6.22)\qquad \text{if } \nu \in D, \text{ for } X \in U_\nu^{**},\ q(t,X) = \lambda_\nu^{1/2}\beta_\nu e_{0\nu}(X) \text{ with } e_{0\nu} \in S(1, G_\nu) \text{ and } 1/4 \le e_{0\nu}(X) \le 4,$$

$$(6.23)\qquad \text{with } \beta_\nu = \xi_1 + b_\nu(\underbrace{x_1, x', \xi'}_{\mathbf{x}}),\ b_\nu \in S(\lambda_\nu^{1/2}, G_\nu),$$

where (x_1, x', ξ_1, ξ') are linear symplectic coordinates in $(\mathbb{R} \times \mathbb{R}^{n-1}) \times (\mathbb{R} \times \mathbb{R}^{n-1})$ orthonormal for Γ (see lemma 3.5). Of course the semi-norms of $e_{0\nu}$ and b_ν above are bounded independently of ν by constants depending on the semi-norms of q in (6.1).

However the choice of coordinates in (6.23) depends on the box U_ν^{**}, but these coordinates are deduced from the "standard" ones by a symplectic matrix also orthogonal for Γ. Note also that we assume that b_ν is defined globally on $\mathbb{R}^{2n}$ which is easy to achieve by multiplication by a suitable cut-off function.

Note. — In the sequel of this paper, the notation β_ν will always refer to $\beta_\nu(X)$, that is to the function defined on the phase space $\mathbb{R}^{2n}$ given by (6.23). In particular, starting from the functions ω and σ defined on $\mathbb{R}\times\mathbb{R}^{2n-1}$ in section 2 (with $d=2n-1$) we define below $\omega(\beta_\nu,\mathbf{x})$ and $\sigma(\beta_\nu,\mathbf{x})$ as functions on the phase space $\mathbb{R}^{2n}$ given respectively by $\omega\big(\beta_\nu(X),\mathbf{x}\big)$ and $\sigma\big(\beta_\nu(X),\mathbf{x}\big)$.

We are now ready for the definition of the operator

(6.24)
$$\Omega(t)=\sum_\nu \chi_\nu^w W_\nu \chi_\nu^w,\quad\text{with}\quad \begin{cases} W_\nu &= \lambda_\nu e_\nu^{\mathrm{Wick}(\Gamma)}, \quad\text{if}\quad \nu\in E_\pm,\\ W_\nu &= [\omega(\beta_\nu,\mathbf{x})e_\nu\psi_\nu]^w+a_{0\nu}^w, \quad\text{if}\quad \nu\in D,\\ W_\nu &= \mathrm{Id}\,, \quad\text{if}\quad \nu\in L,\end{cases}$$

where χ_ν and ψ_ν are defined in lemma 6.2 ; the symbols e_ν will be chosen later and we assume only here that $e_\nu\in S(1,G_\nu)$ uniformly and is uniformly elliptic, *i.e.* satisfies for all ν,x,ξ, $e_\nu(x,\xi)\ge c_0>0$. The function $\omega(\beta_\nu,\mathbf{x})$ is defined in lemma 2.2 and (2.29) and β_ν is given in (6.23). The symbol $a_{0\nu}$ belongs to $S(1,g)$, ($g=g^{[q(t,\cdot]}$ is defined in (3.3)) and is given below; the metric g is given by (3.3) from $q=q(t,\cdot)$: g is an admissible metric on $\mathbb{R}^{2n}_{x,\xi}$ depending on the real variable t. From lemmas 4.2, 2.2 and 3.6 there exists a positive constant c_1 such that, for all $\nu\in D$, the symbol

$$\omega(\beta_\nu,\mathbf{x})e_\nu\psi_\nu-c_1\lambda_\nu^{1/2}(1+\beta_\nu^2)^{1/2}e_\nu\psi_\nu \quad \text{belongs to } S(\lambda^{1/3}\mu^{2/3},g) \text{ and is nonnegative.} \tag{6.25}$$

The Fefferman-Phong inequality applied to this second-order (in fact $\lambda^{1/3}\mu^{2/3}\le\mu^2$ from lemma 3.5) nonnegative symbol implies that there exists $a_\nu\in S(1,g)$ whose semi-norms depend only on those of ψ_ν and b_ν in (6.23), thus eventually only on those of q in (6.1), such that

$$a_\nu^w+\big(\omega(\beta_\nu,\mathbf{x})e_\nu\psi_\nu\big)^w-c_1\lambda_\nu^{1/2}\big[(1+\beta_\nu^2)^{1/2}e_\nu\psi_\nu\big]^w\ge 0, \tag{6.26}$$

and consequently

$$[a_\nu+1]^w+(\omega(\beta_\nu,\mathbf{x})e_\nu\psi_\nu)^w\ge \mathrm{Id}\ +c_1\lambda_\nu^{1/2}\big[(1+\beta_\nu^2)^{1/2}e_\nu\psi_\nu\big]^w. \tag{6.27}$$

Assuming as we may, using the ellipticity of e_ν, that $e_\nu\psi_\nu=\theta_\nu^2$, with $\theta_\nu\in S(1,G_\nu)$ uniformly, $\operatorname{supp}\theta_\nu\subset U_\nu^{**}$, we get from (6.27)

$$[a_\nu+1]^w+(\omega(\beta_\nu,\mathbf{x})e_\nu\psi_\nu)^w\ge \mathrm{Id}\ +c_1\lambda_\nu^{1/2}\big[(1+\beta_\nu^2)^{1/4}\theta_\nu\big]^w\big[(1+\beta_\nu^2)^{1/4}\theta_\nu\big]^w+r_\nu^w, \tag{6.28}$$

where $r_\nu\in S(1,g)$. We can then choose $a_{0\nu}$ in (6.24)

$$a_{0\nu}=a_\nu+1+\|r_\nu\|_{\mathcal{L}(\mathrm{L}^2)} \tag{6.29}$$

in such a way that for

$$\nu\in D, W_\nu\ge \mathrm{Id}\ +c_1\lambda_\nu^{1/2}\big[(1+\beta_\nu^2)^{1/4}\theta_\nu\big]^w\big[(1+\beta_\nu^2)^{1/4}\theta_\nu\big]^w\ge\mathrm{Id}\,. \tag{6.30}$$

Our main statement for this section is contained in the following proposition.

PROPOSITION 6.4. – *The Weyl symbol $\Omega(t,X)$ of the operator $\Omega(t)$ defined by (6.24) belongs to $S(\lambda^{1/3}\mu^{2/3}, g)$, where $g = g^{[q(t,\cdot)]}$ in (3.3) (cf. (3.7) for the definition of μ). Moreover there exist positive constants c_0, c_1 and a nonnegative constant C_0 such that*

$$\Omega(t) \geq c_0 \operatorname{Id} \quad \text{and} \quad \Omega(t,X) + C_0 \geq c_1 \lambda(t,X)^{1/3}\mu(t,X)^{2/3}. \tag{6.31}$$

The Weyl symbol $J(t,X)$ of the operator $J(t)$ defined in lemma 6.1 belongs to $S(1,g)$ and the Poisson bracket

$$\{\Omega(t,X), J(t,X)\} \in S(1,g). \tag{6.32}$$

One can choose the elliptic symbols e_ν in (6.24) so that

$$q(t,X) - \Omega(t,X)J(t,X) \in S(1,g). \tag{6.33}$$

Eventually, with $Q(t) = q(t,X)^w$ one gets

$$Q(t) = \Omega(t)J(t) + R(t) = J(t)\Omega(t) + R^*(t), \tag{6.34}$$

where the Weyl symbol $R(t,X)$ of $R(t)$ belongs to $S(1,g)$ and thus $R(t)$ is L^2-bounded.

Proof. – Let ω_ν be the Weyl symbol of W_ν. We note from (6.24), lemmas 4.2, 2.2 and (6.23) that

$$\omega_\nu \in S(\lambda^{1/3}\mu^{2/3}, g) \text{ uniformly.} \tag{6.35}$$

We compute, using the fact that self-adjoint operators have real Weyl symbols, setting $\iota = 2i\pi$

$$\chi_\nu \sharp \omega_\nu \sharp \chi_\nu = \Big[\chi_\nu\omega_\nu + \frac{1}{2\iota}\{\chi_\nu, \omega_\nu\} + r_\nu\Big] \sharp \chi_\nu = \chi_\nu^2\omega_\nu + \chi_\nu \mathrm{Re}\, r_\nu + (\frac{1}{2\iota})^2\{\{\chi_\nu, \omega_\nu\}, \chi_\nu\} + \tilde{r}_\nu,$$

and we notice that the remainder

$$\sum_\nu \chi_\nu \mathrm{Re}\, r_\nu + (\frac{1}{2\iota})^2\{\{\chi_\nu, \omega_\nu\}, \chi_\nu\} + \tilde{r}_\nu \in S(1,g),$$

since each term of the series is confined in U_ν with respect to g, and belongs to $S(1,g)$. Thus, we get that

$$\Omega(t,X) = \sum_\nu \chi_\nu^2(X)\omega_\nu(X) + r_1(X), r_1 \in S(1,g). \tag{6.36}$$

From (6.35) and (6.36) we get that $\Omega(t,X) \in S(\lambda^{1/3}\mu^{2/3}, g)$. To prove the first part of (6.31), we note that, using lemma 6.3, it is enough to prove the same for each W_ν. For $\nu \in D \cup L$, it follows from (6.30) and (6.24), whereas for $\nu \in E_\pm$, we use only $e_\nu^{\mathrm{Wick}(\Gamma)} \geq c_0^{\mathrm{Wick}(\Gamma)} = c_0 \operatorname{Id}$. Moreover, from (6.24-25), there exists a positive constant c_1 and a nonnegative constant C_1 such that for $X \in U_\nu^*$,

$$\omega_\nu(X) + C_1 \geq c_1 \lambda(X)^{1/3}\mu(X)^{2/3}$$

so that (6.36) implies

$$\Omega(t,X) \geq \lambda(X)^{1/3}\mu(X)^{2/3}\sum_\nu c_1\chi_\nu^2(X) + r_2(X), r_2 \in S(1,g)$$

which gives with (a positive c_1) and a nonnegative C_0

$$\Omega(t,X) + C_0 \geq c_1\lambda(X)^{1/3}\mu(X)^{2/3}$$

which is the result of the second part in (6.31).

We check now the Weyl symbol of J(t), using the notations of (6.21–22). We note in particular that, from (6.21–22) and (6.7) we get that the the L^∞ function $s(t,X)$ satisfies

$$\begin{aligned} &\nu \in D, X \in U_\nu^{**}, \implies s(t,X) = \text{sign } \beta_\nu \\ &\nu \in E_\pm, X \in U_\nu^{**}, \implies s(t,X) = \pm 1. \end{aligned} \tag{6.37}$$

We have thus from (6.8)

$$\begin{aligned} J(t,X) &= s * 2^n e^{-2\pi\Gamma} = \sum_\nu \chi_\nu\left[s * 2^n e^{-2\pi\Gamma}\right] \\ &= \sum_\nu \chi_\nu\left[(\psi_\nu s) * 2^n e^{-2\pi\Gamma}\right] + \sum_\nu \chi_\nu\left[\left((1-\psi_\nu)s\right) * 2^n e^{-2\pi\Gamma}\right] \\ &= \sum_{\nu\in E_\pm} \chi_\nu\left[\pm\psi_\nu * 2^n e^{-2\pi\Gamma}\right] + \sum_{\nu\in D} \chi_\nu\left[(\psi_\nu \text{ sign}(\beta_\nu)) * 2^n e^{-2\pi\Gamma}\right] \\ &\quad + \sum_{\nu\in L} \chi_\nu\left[(\psi_\nu s) * 2^n e^{-2\pi\Gamma}\right] + \sum_\nu \chi_\nu\left[\left((1-\psi_\nu)s\right) * 2^n e^{-2\pi\Gamma}\right]. \\ &= \sum_{\nu\in E_\pm} \chi_\nu\left[\pm\psi_\nu * 2^n e^{-2\pi\Gamma}\right] + \sum_{\nu\in D} \chi_\nu\left[\text{sign}(\beta_\nu) * 2^n e^{-2\pi\Gamma}\right] \\ &\quad + \sum_{\nu\in D} \chi_\nu\left[\left((\psi_\nu - 1) \text{ sign}(\beta_\nu)\right) * 2^n e^{-2\pi\Gamma}\right] + \sum_{\nu\in L} \chi_\nu\left[(\psi_\nu s) * 2^n e^{-2\pi\Gamma}\right] \\ &\quad + \sum_\nu \chi_\nu\left[\left((1-\psi_\nu)s\right) * 2^n e^{-2\pi\Gamma}\right] \\ &= \sum_{\nu\in E_\pm} \chi_\nu \overbrace{\left[\pm 1 * 2^n e^{-2\pi\Gamma}\right]}^{\pm 1} + \sum_{\nu\in D} \chi_\nu\left[\text{sign}(\beta_\nu) * 2^n e^{-2\pi\Gamma}\right] \\ &\quad + \sum_{\nu\in E_\pm} \chi_\nu\left[\pm(\psi_\nu - 1) * 2^n e^{-2\pi\Gamma}\right] + \sum_{\nu\in D} \chi_\nu\left[\left((\psi_\nu - 1) \text{ sign}(\beta_\nu)\right) * 2^n e^{-2\pi\Gamma}\right] \\ &\quad + \sum_{\nu\in L} \chi_\nu\left[(\psi_\nu s) * 2^n e^{-2\pi\Gamma}\right] + \sum_\nu \chi_\nu\left[\left((1-\psi_\nu)s\right) * 2^n e^{-2\pi\Gamma}\right]. \end{aligned} \tag{6.38}$$

We shall give the same treatment to the third, fourth and sixth term in (6.38). We claim that for any sequence c_ν of uniformly L^∞ functions

$$\sum_\nu \chi_\nu\left[\left((1-\psi_\nu)\ c_\nu\right) * 2^n e^{-2\pi\Gamma}\right] \in S(\lambda^{-\infty}, G) = \bigcap_{N>0} S(\lambda^{-N}, G). \tag{6.39}$$

Since $G_\nu = \lambda_\nu^{-1}\Gamma$, χ_ν is supported in U_ν and

$$\left[\left((1-\psi_\nu)\ c_\nu\right) * 2^n e^{-2\pi\Gamma}\right] \in S(1,\Gamma) \tag{6.40}$$

uniformly, it suffices to estimate the following for all positive N:

$$\lambda_\nu^N \left| \chi_\nu(X) \int c_\nu(Y)\big(1-\psi_\nu(Y)\big) 2^n e^{-2\pi\Gamma(X-Y)} \mathrm{d}Y \right|. \tag{6.41}$$

Since in this expression

$$Y \notin U_\nu^*, \text{ and } X \in U_\nu, \tag{6.42}$$

we have

$$\Gamma(X-Y) \geq r_0^2 \lambda_\nu \quad \text{which implies} \quad \lambda_\nu^N \leq C(r_0,N)\, e^{\Gamma(X-Y)}, \tag{6.43}$$

yielding the desired estimate for (6.41). The fifth term in (6.38) is easy to handle, since for $\nu \in L$ we have that $G_\nu \sim G_\nu^\natural \sim \Gamma$ and $\lambda_\nu \sim 1$ in the sense of lemma 6.2 so that, for all k, N we have

$$\sup_{X \in \mathbf{R}^{2n}} \left| \left[(\psi_\nu s) * 2^n e^{-2\pi\Gamma} \right]^{(k)}(X) T^k \right| G_\nu(T)^{-k/2} \lambda_\nu^N < +\infty, \tag{6.44}$$

that is the estimates yielding

$$\sum_{\nu \in L} \chi_\nu \left[(\psi_\nu s) * 2^n e^{-2\pi\Gamma} \right] \in S(\lambda^{-\infty}, G). \tag{6.45}$$

We have proved so far from (6.38–45) that
(6.46)

$$J(t,X) = \sum_{\nu \in E_\pm} \pm \chi_\nu + \sum_{\nu \in D} \chi_\nu \left[\operatorname{sign}(\beta_\nu) * 2^n e^{-2\pi\Gamma} \right] + r_{-\infty}, \text{with } r_{-\infty} \in S(\lambda^{-\infty}, G).$$

According to (6.23) and lemma 6.5 below (it follows in fact directly from (2.3), (2.4), lemma 2.1 and lemma 4.2) we have, using lemma 3.4,

$$\operatorname{sign}(\beta_\nu) * 2^n e^{-2\pi\Gamma} = \sigma(\beta_\nu, \mathbf{x}) \in S\left(1, \frac{\mathrm{d}\beta_\nu \otimes \mathrm{d}\beta_\nu}{1+\beta_\nu^2} + \frac{\Gamma}{\lambda_\nu^{1/2}(1+|\beta_\nu|)}\right), \tag{6.47}$$

so that

$$\sum_{\nu \in D} \chi_\nu [\operatorname{sign}(\beta_\nu) * 2^n e^{-2\pi\Gamma}] = \sum_{\nu \in D} \chi_\nu(X) \sigma(\beta_\nu, \mathbf{x}) \in S(1,g). \tag{6.48}$$

The identities (6.46-47-48) prove that

$$J(t,X) \in S(1,g). \tag{6.49}$$

Let us verify now claim (6.32) in Proposition 6.4. Note that since J appears as an operator of order 0 (*i.e.* with symbol in $S(1=\mu^0,g)$) and Ω as an operator of order 2 (*i.e.* with symbol in $S(\mu^2,g)$), the symbolic calculus for the metric g would give *a priori* this commutator in $S(\mu^1,g)$. The sequel will show that the very particular structure of the operators Ω and J yields the key improvement (6.32). From (6.24) and (6.36), we know that the Weyl symbol of $\Omega(t)$ is

$$\begin{aligned}\Omega(t,X) = \sum_{\nu\in E_\pm} \chi_\nu^2(X)\lambda_\nu e_\nu(X) + \sum_{\nu\in D} \chi_\nu^2(X)\ \omega(\beta_\nu,\mathbf{x})\ e_\nu(X) + r_0(X),\\ \text{with } r_0 \in S(1,g),\end{aligned} \tag{6.50}$$

where $\omega(\beta_\nu,\mathbf{x})$ is given in lemma 2.2 and (2.29) and $e_\nu \in S(1,G_\nu)$ uniformly and is uniformly elliptic, *i.e.* satisfies for all ν, x, ξ, $e_\nu(x,\xi) \geq c_0 > 0$. The following identity for Poisson bracket

$$\{\chi^2 b, a\} = 2b\{\chi,\chi a\} + \chi\{b,\chi a\} + \chi a\{\chi, b\} \tag{6.51}$$

and (6.50) imply

$$\begin{cases}\{\Omega(t,X), J(t,X)\}\\ \quad = \sum_{\nu\in E_\pm} \{\chi_\nu^2(X)\lambda_\nu e_\nu(X), J(t,X)\}\\ \qquad + \sum_{\nu\in D} \{\chi_\nu^2(X)\ \omega(\beta_\nu,\mathbf{x})\ e_\nu(X), J(t,X)\} + \{r_0, J\}\\ \quad = \sum_{\nu\in E_\pm} 2\lambda_\nu e_\nu\{\chi_\nu,\chi_\nu J\} + \chi_\nu\{\lambda_\nu e_\nu,\chi_\nu J\} + \chi_\nu J\{\chi_\nu,\lambda_\nu e_\nu\}\\ \qquad + \sum_{\nu\in D} 2\omega(\beta_\nu,\cdot)\ e_\nu\{\chi_\nu,\chi_\nu J\} + \chi_\nu\{\omega(\beta_\nu,\cdot)\ e_\nu,\chi_\nu J\}\\ \qquad\quad + \chi_\nu J\{\chi_\nu,\omega(\beta_\nu,\cdot)\ e_\nu\} + \{r_0, J\}.\end{cases} \tag{6.52}$$

We can use the calculations in (6.38) giving $\chi_\nu(X)J(t,X) = \chi_\nu[s * 2^n e^{-2\pi\Gamma}]$ to obtain

$$\nu \in E_\pm \text{ implies } \chi_\nu(X)J(t,X) = \pm\chi_\nu(X) + S(\lambda^{-\infty}, G), \tag{6.53}$$

$$\nu \in D \text{ implies } \chi_\nu(X)J(t,X) = \chi_\nu(X)\sigma(\beta_\nu,\mathbf{x}) + S(\lambda^{-\infty}, G). \tag{6.54}$$

Looking now at (6.52) we have from (6.53-54) and (6.49)

$$\begin{cases}\text{for } \nu \in E_\pm,\\ \lambda_\nu e_\nu\{\chi_\nu,\chi_\nu J\} = \pm\lambda_\nu e_\nu\{\chi_\nu,\chi_\nu\} + S(\lambda^{-\infty},G) \in S(\lambda^{-\infty},G),\end{cases} \tag{6.55}$$

$$\begin{cases}\text{for } \nu \in E_\pm,\\ \chi_\nu\{\lambda_\nu e_\nu,\chi_\nu J\} = \pm\chi_\nu\{\lambda_\nu e_\nu,\chi_\nu\} + S(\lambda^{-\infty},G) \in S(1,G),\end{cases} \tag{6.56}$$

$$\begin{cases}\text{for } \nu \in E_\pm,\\ \chi_\nu J\{\chi_\nu,\lambda_\nu e_\nu\} = \pm\chi_\nu\{\chi_\nu,\lambda_\nu e_\nu\} + S(\lambda^{-\infty},G) \in S(1,G),\end{cases} \tag{6.57}$$

$$\begin{cases}\text{for } \nu \in D,\\ \omega(\beta_\nu,\cdot)\ e_\nu\{\chi_\nu,\chi_\nu J\} = \omega(\beta_\nu,\cdot)\ e_\nu\chi_\nu\{\chi_\nu,\sigma(\beta_\nu,\cdot)\} + S(\lambda^{-\infty},G),\end{cases} \tag{6.58}$$

$$(6.59)\qquad \begin{cases}\text{for } \nu \in D, \\ \quad \chi_\nu\{\omega(\beta_\nu,\cdot)\ e_\nu, \chi_\nu J\} = \chi_\nu\{\omega(\beta_\nu,\cdot)\ e_\nu, \chi_\nu\sigma(\beta_\nu,\cdot)\} + S(\lambda^{-\infty}, G),\end{cases}$$

$$(6.60)\qquad \begin{cases}\text{for } \nu \in D, \\ \quad \chi_\nu J\{\chi_\nu, \omega(\beta_\nu,\cdot)\ e_\nu\} = \chi_\nu\sigma(\beta_\nu,\cdot)\{\chi_\nu, \omega(\beta_\nu,\cdot)\ e_\nu\} + S(\lambda^{-\infty}, G),\end{cases}$$

$$(6.61)\qquad \{r_0, J\} \in S(1, g).$$

To get (6.32), we only need to examine the Poisson brackets in the right-hand side of (6.58–59–60). We shall first prove the following lemma

LEMMA 6.5. – *The symbols* $\sigma(\beta_\nu,\cdot)$ *and* $\omega(\beta_\nu,\cdot)$ *defined in lemmas* (2.1-2) *and* (2.11), (2.29) *with* β_ν *defined in (6.23) satisfy uniformly on* U_ν^{**}

$$(6.62)\qquad \begin{aligned}&\sigma(\beta_\nu,\cdot) \in S(1, g_\nu),\\ &\omega(\beta_\nu,\cdot) \in S(\lambda_\nu^{1/2}(1+\beta_\nu^2)^{1/2}, g_\nu),\\ &\{\sigma(\beta_\nu,\cdot), \omega(\beta_\nu,\cdot)\} \in S(1, g_\nu),\end{aligned}$$

with

$$(6.63)\qquad g_\nu = \frac{d\beta_\nu \otimes d\beta_\nu}{1+\beta_\nu^2} + \frac{\Gamma}{\lambda^{1/2}(1+{\beta_\nu}^2)^{1/2}}.$$

We prove first the second statement in (6.62) (the proof has the same flavour as lemma 4.2). Using the estimates (2.20) and (6.23) for b_ν, the derivative $D^k\omega(\beta_\nu, \mathbf{x})T^k$ is a sum of terms

$$(6.64)\qquad \partial_{\mathbf{x}}^l \partial_\beta^p \omega T^l \beta^{(k_1)} \dots \beta^{(k_p)} T^{k-l} \quad \text{with} \quad k_1 + \dots + k_p + l = k,$$

which is smaller than the absolute value of

$$(6.65)\qquad \begin{cases}(d\beta\cdot T)^j \beta^{(k_{j+1})}\dots\beta^{(k_p)}T^{k-l-j}(1+|\beta|)^{1-p}\lambda^{1/2-l/2}\Gamma(T)^{l/2}\\ \qquad\qquad \text{with}\quad j + \overbrace{k_{j+1}+\dots+k_p}^{k_{j+1},\dots,k_p \text{ are all } \geq 2} + l = k.\end{cases}$$

The absolute value of this term is bounded above by

$$\lambda^{1/2}(1+|\beta|)\frac{|d\beta\cdot T|^j}{(1+|\beta|)^j}(1+|\beta|)^{j-p}\overbrace{\lambda^{1/2-k_{j+1}/2}\dots\lambda^{1/2-k_p/2}\lambda^{-l/2}}^{(\lambda^{1/2})^{p-j-(k-l-j)-l}}\Gamma(T)^{(k-j)/2}$$

which is

$$\begin{aligned}&\lambda^{1/2}(1+|\beta|)\frac{|d\beta\cdot T|^j}{(1+|\beta|)^j}(1+|\beta|)^{j-p}\lambda^{(p-k)/2}\Gamma(T)^{(k-j)/2}\\ &\quad\leq \lambda^{1/2}(1+|\beta|)\frac{|d\beta\cdot T|^j}{(1+|\beta|)^j}\left(\frac{\Gamma(T)}{\lambda^{1/2}(1+|\beta|)}\right)^{(k-j)/2}\lambda^{\frac{p-k}{2}+\frac{k-j}{4}}(1+|\beta|)^{j-p+\frac{k-j}{2}}\\ &\qquad\qquad \leq \lambda^{1/2}(1+|\beta|)g_\nu(T)^{k/2}\left(\frac{1+|\beta|}{\lambda^{1/2}}\right)^{\frac{k+j-2p}{2}},\end{aligned}$$

which gives $\omega(\beta_\nu,\cdot)\in S(\lambda_\nu^{1/2}(1+\beta_\nu^2)^{1/2},g_\nu)$ since from (6.65) we have $j+2(p-j)\le k$, *i.e.* $k+j-2p\ge 0$ and

(6.66) on U_ν^{**} the ratio $\beta_\nu/\lambda_\nu^{1/2}$ is bounded.

To get the first statement in (6.62), one needs only to repeat a simpler version of the calculations above using the first equality in (2.11) and (2.12) for $|\beta_\nu|\le 1$ and the second equality in (2.11) and (2.12) for $|\beta_\nu|\ge 1$. We compute now the Poisson bracket, using $\{\beta,\beta\}=0$,

(6.67)
$$\begin{aligned}
&\{\sigma(\beta,\mathbf{x}),\omega(\beta,\mathbf{x})\}\\
&=\left[\frac{\partial\sigma}{\partial\beta}\frac{\partial\beta}{\partial\xi}+\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial\xi}\right]\cdot\left[\frac{\partial\omega}{\partial\beta}\frac{\partial\beta}{\partial x}+\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]\\
&\quad-\left[\frac{\partial\omega}{\partial\beta}\frac{\partial\beta}{\partial\xi}+\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial\xi}\right]\cdot\left[\frac{\partial\sigma}{\partial\beta}\frac{\partial\beta}{\partial x}+\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]\\
&=\left[\frac{\partial\sigma}{\partial\beta}\frac{\partial\beta}{\partial\xi}\right]\cdot\left[\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]+\left[\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial\xi}\right]\cdot\left[\frac{\partial\omega}{\partial\beta}\frac{\partial\beta}{\partial x}+\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]\\
&\quad-\left[\frac{\partial\omega}{\partial\beta}\frac{\partial\beta}{\partial\xi}\right]\cdot\left[\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]-\left[\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial\xi}\right]\cdot\left[\frac{\partial\sigma}{\partial\beta}\frac{\partial\beta}{\partial x}+\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]\\
&=\overbrace{\left[\frac{\partial\sigma}{\partial\beta}\frac{\partial\beta}{\partial\xi}\right]\cdot\left[\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]}^{\Pi_1}+\overbrace{\left[\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial\xi}\right]\cdot\left[\frac{\partial\omega}{\partial\beta}\frac{\partial\beta}{\partial x}\right]}^{\Pi_2}+\overbrace{\left[\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial\xi}\right]\cdot\left[\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]}^{\Pi_3}\\
&\quad\underbrace{-\left[\frac{\partial\omega}{\partial\beta}\frac{\partial\beta}{\partial\xi}\right]\cdot\left[\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]}_{\Pi_4}\underbrace{-\left[\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial\xi}\right]\cdot\left[\frac{\partial\sigma}{\partial\beta}\frac{\partial\beta}{\partial x}\right]}_{\Pi_5}\underbrace{-\left[\frac{\partial\omega}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial\xi}\right]\cdot\left[\frac{\partial\sigma}{\partial\mathbf{x}}\cdot\frac{\partial\mathbf{x}}{\partial x}\right]}_{\Pi_6}.
\end{aligned}$$

We have from lemmas 2.1-2 and (6.23)

(6.68)
$$\left\{\begin{array}{ll}\dfrac{\partial\sigma}{\partial\beta}\in S\big((1+|\beta|)^{-1},g\big), & \dfrac{\partial\beta}{\partial\xi}\in S(1,g),\\ \dfrac{\partial\omega}{\partial\mathbf{x}}\in S\big((1+|\beta|),g\big), & \dfrac{\partial\mathbf{x}}{\partial x}\in S(1,g)\end{array}\right\}\Longrightarrow\quad \Pi_1\in S(1,g),$$

(6.69)
$$\left\{\begin{array}{ll}\dfrac{\partial\sigma}{\partial\mathbf{x}}\in S(\lambda^{-1/2},g), & \dfrac{\partial\mathbf{x}}{\partial\xi}\in S(1,g),\\ \dfrac{\partial\omega}{\partial\beta}\in S(\lambda^{1/2},g), & \dfrac{\partial\beta}{\partial x}\in S(1,g)\end{array}\right\}\Longrightarrow\quad \Pi_2\in S(1,g),$$

(6.70)
$$\left\{\begin{array}{ll}\dfrac{\partial\sigma}{\partial\mathbf{x}}\in S(\lambda^{-1/2},g), & \dfrac{\partial\mathbf{x}}{\partial\xi}\in S(1,g),\\ \dfrac{\partial\omega}{\partial\mathbf{x}}\in S\big((1+|\beta|),g\big)\subset S(\lambda^{1/2},g), & \dfrac{\partial\mathbf{x}}{\partial x}\in S(1,g)\end{array}\right\}\Longrightarrow\quad \Pi_3\in S(1,g),$$

$$\left\{\begin{array}{ll}\dfrac{\partial\omega}{\partial\beta}\in S\big(\lambda^{1/2},g\big), & \dfrac{\partial\beta}{\partial\xi}\in S(1,g),\\ \dfrac{\partial\sigma}{\partial\mathbf{x}}\in S\big(\lambda^{-1/2},g\big), & \dfrac{\partial\mathbf{x}}{\partial x}\in S(1,g)\end{array}\right\}\Longrightarrow\quad \Pi_4\in S(1,g),$$

$$\left\{\begin{array}{ll}\frac{\partial\omega}{\partial\mathbf{x}}\in S\big((1+|\beta|),g\big), & \frac{\partial\mathbf{x}}{\partial\xi}\in S(1,g),\\ \frac{\partial\sigma}{\partial\beta}\in S\big((1+|\beta|)^{-1},g\big), & \frac{\partial\beta}{\partial x}\in S(1,g)\end{array}\right\}\implies \Pi_5\in S(1,g),$$

$$\left\{\begin{array}{l}\frac{\partial\omega}{\partial\mathbf{x}}\in S\big((1+|\beta|),g\big),\ \frac{\partial\mathbf{x}}{\partial\xi}\in S(1,g),\ \frac{\partial\mathbf{x}}{\partial x}\in S(1,g)\\ \frac{\partial\sigma}{\partial\mathbf{x}}\in S\big(\lambda^{-1/2},g\big)\subset S\big((1+|\beta|)^{-1},g\big),\end{array}\right\}\implies \Pi_6\in S(1,g),$$

which gives (6.62) from (6.67). The proof of lemma 6.5. is complete.

□

Note also that from (6.68–70) above, one gets by direct computation

$$\{\omega(\beta_\nu,\cdot),\chi_\nu\}\in S(1,g),\{\sigma(\beta_\nu,\cdot),e_\nu\}\in S(\lambda^{-1/2}(1+|\beta|)^{-1},g). \tag{6.71}$$

Looking back at (6.52) and (6.58–59–60), omitting the index ν of β_ν, and writing for each symbol its weight m in the class $S(m,g)$, we see that using lemma 6.5, (6.23), (6.66) and (6.68–71)

(6.72)
$$\left\{\begin{array}{l}\omega(\beta,\cdot)\ e_\nu\chi_\nu\{\chi_\nu,\sigma(\beta,\cdot)\}\\ \quad=\overbrace{\omega(\beta,\mathbf{x})}^{\lambda^{1/2}(1+|\beta|)}\ \overbrace{e_\nu\chi_\nu}^{1}\Big[\overbrace{\{\chi_\nu,\beta\}}^{\lambda^{-1/2}}\ \overbrace{\frac{\partial\sigma}{\partial\beta}}^{(1+|\beta|)^{-1}}+\overbrace{\frac{\partial\chi_\nu}{\partial\xi}}^{\lambda^{-1/2}}\overbrace{\frac{\partial\sigma}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial x}}^{\lambda^{-1/2}}-\overbrace{\frac{\partial\chi_\nu}{\partial x}}^{\lambda^{-1/2}}\overbrace{\frac{\partial\sigma}{\partial\mathbf{x}}\frac{\partial\mathbf{x}}{\partial\xi}}^{\lambda^{-1/2}}\Big]\in S(1,g),\end{array}\right.$$

$$\left\{\begin{array}{l}\chi_\nu\{\omega(\beta,\cdot)\ e_\nu,\chi_\nu\sigma(\beta,\cdot)\}\\ \quad=\underbrace{\{\omega,\chi_\nu\}}_{1}\underbrace{\sigma e_\nu\chi_\nu}_{1}+\underbrace{\chi_\nu\sigma\omega}_{\lambda^{1/2}(1+|\beta|)}\ \underbrace{\{e_\nu,\chi_\nu\}}_{\lambda^{-1}}\\ \qquad+\underbrace{\chi_\nu^2\omega}_{\lambda^{1/2}(1+|\beta|)}\ \overbrace{\{e_\nu,\sigma\}}^{\lambda^{-1/2}(1+|\beta|)^{-1}}+\overbrace{\chi_\nu^2e_\nu}^{1}\overbrace{\{\omega,\sigma\}}^{1}\in S(1,g),\end{array}\right. \tag{6.73}$$

$$\left\{\begin{array}{l}\chi_\nu\sigma(\beta,\cdot)\{\chi_\nu,\omega(\beta,\cdot)\ e_\nu\}\\ \quad=\underbrace{\chi_\nu\sigma e_\nu}_{1}\underbrace{\{\chi_\nu,\omega(\beta,\cdot)\}}_{1}+\underbrace{\chi_\nu\sigma\omega}_{\lambda^{1/2}(1+|\beta|)}\ \underbrace{\{\chi_\nu,e_\nu\}}_{\lambda^{-1}}\in S(1,g).\end{array}\right. \tag{6.74}$$

We reach the conclusion of (6.32).

From (6.36), (6.24) the Weyl symbol $\Omega(t,X)$ of $\Omega(t)$ is

(6.75)
$$\left\{\begin{array}{l}\Omega(t,X)=\\ \quad\sum_{\nu\in E_\pm}\chi_\nu^2(X)\lambda_\nu e_\nu(X)+\sum_{\nu\in D}\chi_\nu^2(X)\omega(\beta_\nu,\mathbf{x})e_\nu(X)+r,\quad(\text{with } r\in S(1,g)),\end{array}\right.$$

so that from (6.49)

(6.76)
$$\begin{array}{l}\Omega(t,X)J(t,X)\\ \quad=\sum_{\nu\in E_\pm}\chi_\nu(X)J(t,X)\lambda_\nu e_\nu(X)\chi_\nu(X)+\sum_{\nu\in D}\chi_\nu(X)J(t,X)\omega(\beta_\nu,\mathbf{x})e_\nu(X)\chi_\nu(X)+r_1,\\ \qquad(\text{with}\quad r_1\in S(1,g)).\end{array}$$

According to the calculations in (6.38–45), we have
(6.77)
for $\nu \in E_\pm$, $\chi_\nu(X)J(t,X) = \pm\chi_\nu + \tilde{r}_\nu$, and for $\nu \in D$, $\chi_\nu(X)J(t,X) = \chi_\nu\sigma(\beta_\nu,\cdot) + \tilde{r}_\nu$,

in such a way that $\sum_{\nu \in E_\pm \cup D} \tilde{r}_\nu \in S(1,g)$. Plugging (6.77) in (6.76) we obtain

$$(6.78)\quad \Omega(t,X)J(t,X) = \sum_{\nu \in E_\pm} \pm\lambda_\nu e_\nu(X)\chi_\nu(X)^2 + \sum_{\nu \in D} \sigma(\beta_\nu, \mathbf{x})\omega(\beta_\nu, \mathbf{x}) e_\nu(X)\chi_\nu(X)^2 + r_2, \text{ with } r_2 \in S(1,g).$$

Using (2.19) and (6.21-22), we get
(6.79)
$$\Omega(t,X)J(t,X) = \sum_{\nu \in E_\pm} q(t,X)e_{0\nu}(X)^{-1}e_\nu(X)\chi_\nu(X)^2 + \sum_{\nu \in D} q(t,X)e_{0\nu}(X)^{-1}e_\nu(X)\chi_\nu(X)^2 + r_3, \quad (\text{with } r_3 \in S(1,g)),$$

that is
(6.80)
$$\Omega(t,X)J(t,X) = q(t,X)\Big[\sum_{\nu \in E_\pm} e_{0\nu}(X)^{-1}e_\nu(X)\chi_\nu(X)^2 + \sum_{\nu \in D} e_{0\nu}(X)^{-1}e_\nu(X)\chi_\nu(X)^2\Big] + r_3.$$

Let us introduce the symbol

$$(6.81)\qquad \alpha(X) = \sum_\nu \chi_\nu^2(X).$$

We have $\alpha, \alpha^{-1} \in S(1,G)$ since for $X \in \mathbb{R}^{2n}$, setting $\mathcal{N}_X = \{\nu, X \in U_\nu\}$, we know from lemma 6.2 that $\sharp\mathcal{N}_X \leq N_0$ which implies

$$1 = \sum \chi_\nu(X) = \sum_{\nu \in \mathcal{N}_X} \chi_\nu(X) \leq [\sum_{\nu \in \mathcal{N}_X} \chi_\nu(X)^2]^{1/2} N_0^{1/2} = \alpha(X)^{1/2} N_0^{1/2} \implies \alpha(X) \geq N_0^{-1} > 0.$$

We can now *choose* the elliptic symbols e_ν in (6.24) to be

$$(6.82)\qquad e_\nu(X) = \alpha(X)^{-1}\, e_{0\nu}(X).$$

From (6.82) and (6.80), we obtain
(6.83)
$$\Omega(t,X)J(t,X) = q(t,X)\Big[\sum_{\nu \in E_\pm} \alpha(X)^{-1}\chi_\nu(X)^2 + \sum_{\nu \in D} \alpha(X)^{-1}\chi_\nu(X)^2 + \sum_{\nu \in L} \alpha(X)^{-1}\chi_\nu(X)^2\Big] + r_4 = q(t,X) + r_4, \quad \text{with } r_4 \in S(1,g),$$

which proves (6.33). The Weyl symbol of the selfadjoint operator $\Omega(t)J(t) + J(t)\Omega(t)$ is

$$(6.84)\qquad 2\Omega(t,X)J(t,X) + r_5, \text{ with } r_5 \in S(1,g),$$

since $\Omega(t,X) \in S(\mu^2,g)$ and $J(t,X) \in S(1,g)$. The latter implies also that the Weyl symbol of the antiadjoint operator $[\Omega(t),J(t)]$ is, up to a symbol in $S(1,g)$,

$$\frac{1}{\iota}\{\Omega(t,X),J(t,X)\} \tag{6.85}$$

But we have just proved above the assertion (6.32) that (6.85) was indeed itself in $S(1,g)$. Putting together this and (6.83-84), we obtain with $R_0 \in \mathrm{Op}(S(1,g))$ selfadjoint

$$\begin{cases} Q(t) = \frac{1}{2}\big(\Omega(t)J(t)+J(t)\Omega(t)\big)+R_0(t) \\ \qquad = \Omega(t)J(t)+R_0+\frac{1}{2}[J(t),\Omega(t)] = \Omega(t)J(t)+R_0(t)+iS_0(t), \end{cases} \tag{6.86}$$

with $R_0, S_0 \in \mathrm{op}(S(1,g))$ selfadjoint. Eventually taking adjoints we get also

$$Q(t) = J(t)\Omega(t)+R_0(t)-iS_0(t). \tag{6.87}$$

The proof of proposition 6.4 is complete.

□

Let ρ_0 be a $C_0^\infty(\mathbb{R})$ nonnegative function identically 1 on $[-1,1]$, supported in $[-2,2]$. Using the notations of (6.24), (6.35) and (6.50), we define with $\rho_1 = 1-\rho_0$, c chosen in (2.25) (such that $c\sigma_0(c) \geq 2\gamma''_{001}$, *cf.* (2.1)),

$$\Omega(t,X) = \Omega^{(0)}(t,X)+\Omega^{(1)}(t,X) \text{ with} \tag{6.88}$$

$$\Omega^{(0)}(t,X) = r_0(X)+\sum_{\nu\in D}\chi_\nu^2(X)\omega(\beta_\nu,\mathbf{x})e_\nu(X)\rho_0(\beta_\nu/c), \tag{6.89}$$

$$\Omega^{(1)}(t,X) = \sum_{\nu\in E_\pm}\chi_\nu^2(X)\lambda_\nu e_\nu(X)+\sum_{\nu\in D}\chi_\nu^2(X)\omega(\beta_\nu,\mathbf{x})e_\nu(X)\rho_1(\beta_\nu/c). \tag{6.90}$$

LEMMA 6.6. – *The operator $\Omega^{(0)}(t)$ whose Weyl symbol $\Omega^{(0)}(t,X)$ is given by* (6.89) *satisfies*

$$\|\Omega^{(0)}(t)\|_{\mathcal{L}(\mathrm{L}^2)} \leq C_0\Lambda^{1/2},$$

where Λ is given in (6.1) *and C_0 is a semi-norm of q in* (6.1).

From (6.50) we know that $r_0 \in S(1,g)$ and thus r_0^w is L^2-bounded. From lemma 2.2, we know that $|\omega(\beta_\nu,\mathbf{x})| \leq C\lambda_\nu^{1/2}(1+|\beta_\nu|)$. Since $\lambda_\nu \leq \gamma_{01}(q)\Lambda$ from lemma 3.2 and $e_\nu \in S(1,G_\nu)$ uniformly (*cf.* (6.24) and (6.82)), we obtain that

$$\Lambda^{-1/2}\omega(\beta_\nu,\mathbf{x})e_\nu(X)\rho_0(\beta_\nu/c) \in S(1,g)$$

uniformly, using lemma 4.2 to estimates the derivatives. Thus the operator whose Weyl symbol is

$$\Lambda^{-1/2}\sum_{\nu\in D}\chi_\nu^2(X)\omega(\beta_\nu,\mathbf{x})e_\nu(X)\rho_0(\beta_\nu/c)$$

is bounded on L^2. The proof of lemma 6.6 is complete.

□

LEMMA 6.7. – *With $J(t)$ defined in* (6.8) *and $\Omega^{(1)}(t)$ with symbol* (6.90), *the symbol of $J(t)\Omega^{(1)}(t)J(t)$ is $\Omega^{(1)}(t,X)J(t,X)^2$, up to a symbol in $S(1,g)$. Moreover, there exist positive constants C_1, C_2 depending only on the semi-norms of q in* (6.1) *such that the operator*

$$J(t)\Omega^{(1)}(t)J(t) - \frac{1}{C_1}\Omega^{(1)}(t) + C_2 \geq 0.$$

Proof. – Using the calculations of (6.52), the identities (6.55–60) and (6.68–70), we obtain that the Poisson bracket $\{J(t,X), \Omega^{(1)}(t,X)\}$ is in $S(1,g)$ which gives readily, using (6.49), that the symbol of $J(t)\Omega^{(1)}(t)J(t)$ is indeed $\Omega^{(1)}(t,X)J(t,X)^2$, up to a symbol in $S(1,g)$. Moreover, from the calculations in (6.38),

$$\nu \in E_\pm \Longrightarrow J(t,X)^2\chi_\nu(X) = \chi_\nu(X) + r_\nu(X), \tag{6.91}$$

where $r_\nu \in S(\lambda^{-\infty}, g)$. From (6.38), (6.47) and lemma 2.2

$$\nu \in D \Longrightarrow J(t,X)^2\chi_\nu(X) = \chi_\nu(X)\sigma(\beta_\nu, \mathbf{x})^2 + r_\nu(X), \tag{6.92}$$

where $r_\nu \in S(\lambda^{-\infty}, g)$. Using (2.21) in lemma 2.2, we get that

$$J(t,X)^2\Omega^{(1)}(t,X) - \frac{\sigma_0(c)}{2}\Omega^{(1)}(t,X)$$

is bounded from below. The Fefferman-Phong inequality gives the result of the lemma. □

7. Energy estimates for perturbations of operators with large parameters

In this section and the following one, we improve some estimates obtained by Dencker in [D2] (Theorem A2). We keep using the notations of section 6. From (6.34) in proposition 6.4 we consider $g^{[q(t,\cdot)]}$ the admissible metric on $\mathbb{R}^{2n}$ defined for each $t \in \mathbb{R}$ by (3.3) for $q(t,\cdot)$ and $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n)$

$$\tilde{Q}(t) = \Omega(t)J(t) = Q(t) - R(t), \text{ with } R(t) \in \mathrm{Op}\big(S(1, g^{[q(t,\cdot)]})\big) \subset \mathcal{L}(\mathcal{H}). \tag{7.1}$$

We compute for $u \in \mathrm{C}_0^\infty(\mathbb{R}, \mathcal{H})$ with $\mathrm{L}^2(\mathbb{R}, \mathcal{H})$ dot products, T a real parameter, $\mathrm{D}_t = -i\partial/\partial t$,

$$\begin{aligned} &2\mathrm{Re}\Big\langle \overbrace{\mathrm{D}_t u + i\tilde{Q}(t)u(t)}^{\tilde{L}u}, \overbrace{iJ(t)u(t) + i(1/2)\Lambda^{-1/2}\operatorname{sign}(t-T)u(t)}^{iMu} \Big\rangle \\ &\quad = \big\langle \dot{J}u, u\big\rangle + \Lambda^{-1/2}|u(T)|^2_{\mathcal{H}} + 2\langle J\Omega J u, u\rangle + \big\langle s_0\Lambda^{-1/2}\mathrm{Re}\,(\Omega J)u, u\big\rangle, \end{aligned} \tag{7.2}$$

where $2\mathrm{Re}\,(\Omega J) = (\Omega J) + (\Omega J)^* = \Omega J + J\Omega$ and $s_0 = \operatorname{sign}(t-T)$. We have, using $s_0^2 = 1$ and (6.31),

$$\begin{aligned} &J\Omega J + 2s_0\Lambda^{-1/2}\mathrm{Re}\,(\Omega J) \\ &\quad = (J\Omega^{1/2} + s_0\Lambda^{-1/2}\Omega^{1/2})(\Omega^{1/2}J + s_0\Lambda^{-1/2}\Omega^{1/2}) - \Lambda^{-1}\Omega \\ &\quad \geq -\Lambda^{-1}\Omega. \end{aligned} \tag{7.3}$$

From lemma 6.1 the operator $J(t)$ is bounded nondecreasing with t, although not a priori differentiable. The term $\left\langle \dot{J}u, u \right\rangle$ in (7.2) should be understood as a distribution derivative. It is an elementary matter (see *e.g.* lemma 2.3.1 in [L1]) to prove that for $u \in \mathrm{C}_0^\infty(\mathbb{R}, \mathcal{H})$,

$$2\mathrm{Re}\,\langle \mathrm{D}_t u, iJu\rangle = \left\langle \dot{J}u, u \right\rangle = -2\int_{\mathbb{R}} \langle \partial_t u(t), J(t)u(t)\rangle_{\mathcal{H}}\, \mathrm{d}t \geq 0. \tag{7.4}$$

From (7.2–4) we get

$$2\mathrm{Re}\left\langle \tilde{L}u, iMu \right\rangle \geq \Lambda^{-1/2}|u(T)|_{\mathcal{H}}^2 - \frac{1}{2}\int \Lambda^{-1}\langle \Omega(t)u(t), u(t)\rangle_{\mathcal{H}}\, \mathrm{d}t + \frac{3}{2}\langle J\Omega Ju, u\rangle. \tag{7.5}$$

From (6.88–90) and the estimates of lemmas 6.6 and 6.7, we get

$$\begin{cases} \displaystyle\int \Lambda^{-1}\langle \Omega(t)u(t), u(t)\rangle_{\mathcal{H}}\, \mathrm{d}t \\ \quad \displaystyle = \int \Lambda^{-1}\left\langle \Omega^{(0)}(t)u(t), u(t)\right\rangle_{\mathcal{H}} \mathrm{d}t + \int \Lambda^{-1}\left\langle \Omega^{(1)}(t)u(t), u(t)\right\rangle_{\mathcal{H}} \mathrm{d}t \\ \quad \displaystyle \leq C_0 \int \Lambda^{-1/2}\|u(t)\|_{\mathcal{H}}^2\, \mathrm{d}t + \int \Lambda^{-1}C_1\left\langle J(t)\Omega^{(1)}(t)J(t)u(t), u(t)\right\rangle_{\mathcal{H}} \mathrm{d}t \\ \quad\quad \displaystyle + \int \Lambda^{-1}C_1C_2\|u(t)\|_{\mathcal{H}}^2\, \mathrm{d}t \\ \quad \displaystyle = \Lambda^{-1/2}\left(C_0 + C_1C_2\Lambda^{-1/2}\right)\int \|u(t)\|_{\mathcal{H}}^2\, \mathrm{d}t \\ \quad\quad\quad \displaystyle + \int \Lambda^{-1}C_1\left\langle J(t)\Omega^{(1)}(t)J(t)u(t), u(t)\right\rangle_{\mathcal{H}} \mathrm{d}t. \end{cases} \tag{7.6}$$

Moreover from (6.88–90), lemma 6.5 and the estimate (6.9) we obtain

$$\begin{aligned} \left\langle J(t)\Omega^{(1)}(t)J(t)u(t), u(t)\right\rangle_{\mathcal{H}} &= \langle J(t)\Omega(t)J(t)u(t), u(t)\rangle_{\mathcal{H}} - \left\langle J(t)\Omega^{(0)}(t)J(t)u(t), u(t)\right\rangle_{\mathcal{H}} \\ &\leq \langle J(t)\Omega(t)J(t)u(t), u(t)\rangle_{\mathcal{H}} + C_0\Lambda^{1/2}\|J(t)u(t)\|_{\mathcal{H}}^2 \\ &\leq \langle J(t)\Omega(t)J(t)u(t), u(t)\rangle_{\mathcal{H}} + C_0\Lambda^{1/2}\|u(t)\|_{\mathcal{H}}^2. \end{aligned} \tag{7.7}$$

From (7.5–7), we get

$$\begin{aligned} &\Lambda^{-1/2}|u(T)|_{\mathcal{H}}^2 + \frac{3}{2}\langle J\Omega Ju, u\rangle \leq \frac{1}{2}\int \Lambda^{-1}\langle \Omega(t)u(t), u(t)\rangle_{\mathcal{H}}\, \mathrm{d}t + 2\mathrm{Re}\left\langle \tilde{L}u, iMu \right\rangle \\ &\leq \frac{1}{2}\Lambda^{-1/2}\left(C_0 + C_1C_2\Lambda^{-1/2} + C_0C_1\right)\int \|u(t)\|_{\mathcal{H}}^2\, \mathrm{d}t + \frac{\Lambda^{-1}C_1}{2}\langle J\Omega Ju, u\rangle + 2\mathrm{Re}\left\langle \tilde{L}u, iMu \right\rangle, \end{aligned}$$

that is, with θ standing for the diameter of the support of u, choosing T so that $|u(T)|_{\mathcal{H}} = \sup|u(t)|$,

$$\begin{aligned} &\Lambda^{-1/2}\sup_{t\in\mathbb{R}}|u(t)|_{\mathcal{H}}^2\left[1 - \theta(C_0 + C_1C_2 + C_0C_1)/2\right] + \langle \Omega Ju, Ju\rangle\left[3/2 - \Lambda^{-1}C_1/2\right] \\ &\qquad \leq 2\mathrm{Re}\left\langle \tilde{L}u, iMu \right\rangle. \end{aligned} \tag{7.8}$$

We can now prove the following theorem, implying theorems 1.3 and 1.4 in the introduction, and summarize the situation.

THEOREM 7.1. – *Let q be a symbol satisfying* (6.1), *$Q(t)$ be the operator with Weyl symbol $q(t,\cdot)$ and $L = \mathrm{D}_t + iQ(t)$ with $\mathrm{D}_t = -i\partial/\partial t$. There exists an operator $R(t) \in \mathrm{Op}\big(S(1, g^{[q(t,\cdot)]})\big)$ such that*

$$Q(t) = \Omega(t)J(t) + R(t) = J(t)\Omega(t) + R^*(t)$$

where the nondecreasing operator $J(t)$ is defined in (6.8), *the positive operator $\Omega(t)$ is given in* (6.24) *and their main properties are described in proposition* 6.4 *and lemmas* 6.6-7. *We set*

$$\tilde{L} = \mathrm{D}_t + i\Omega(t)J(t) = L - iR(t).$$

There exists a positive constant C depending only on the semi-norms of q in (6.1) *such that for all $u \in C_0^\infty(\mathbb{R}, \mathcal{H})$, with $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^n)$, and assuming $\Lambda \geq C$ and $\theta = \operatorname{diameter}(\operatorname{supp} u) \leq C^{-1}$,*

$$\theta^{-1}\Lambda^{-1/2}\|u\|_{\mathrm{L}^2(\mathbb{R},\mathcal{H})} \leq C\|\tilde{L}u\|_{\mathrm{L}^2(\mathbb{R},\mathcal{H})}. \tag{7.9}$$

Moreover the estimate (7.9) *is true under the same assumptions (with an a priori different C) with $\tilde{L}$ replaced by $\tilde{L} + N_0(t)$ where the operator $N_0(t)$ has a Weyl symbol $N_0(t, X, \Lambda)$ satisfying*

$$\sup_{t\in\mathbb{R}, X\in\mathbb{R}^{2n}, \Lambda\geq 1} |\partial_X^k N_0(t, X, \Lambda)|_\Gamma \Lambda^{\frac{k}{2}} < \infty, \tag{7.10}$$

where Γ is the symplectic norm occurring in (6.1).

Proof. – From (7.2) and lemma 6.1, we have, using $\Lambda \geq 1$, $\|M(t)\|_{\mathcal{L}(\mathcal{H})} \leq 3/2$. Assuming

$$\theta(C_0 + C_1C_2 + C_0C_1) \leq 1 \text{ and } \Lambda \geq C_1 \tag{7.11}$$

we get from (7.8) the inequality

$$\begin{cases} 3\displaystyle\int_{\mathbb{R}} |\tilde{L}u(t)|_{\mathcal{H}} dt \ \sup_{t\in\mathbb{R}} |u(t)|_{\mathcal{H}} \\ \quad \geq \dfrac{1}{2}\Lambda^{-1/2} \sup_{t\in\mathbb{R}} |u(t)|^2_{\mathcal{H}} + \langle \Omega J u, Ju\rangle \\ \qquad \geq \dfrac{1}{2}\Lambda^{-1/2} \sup_{t\in\mathbb{R}} |u(t)|^2_{\mathcal{H}}, \end{cases} \tag{7.12}$$

which is

$$6\int_{\mathbb{R}} |\tilde{L}u(t)|_{\mathcal{H}} dt \geq \Lambda^{-1/2} \sup_{t\in\mathbb{R}} |u(t)|_{\mathcal{H}}. \tag{7.13}$$

The Cauchy-Schwarz inequality and (7.13) yield eventually

$$6\|\tilde{L}u\|_{\mathrm{L}^2(\mathbb{R},\mathcal{H})} \geq \|u\|_{\mathrm{L}^2(\mathbb{R},\mathcal{H})}\Lambda^{-1/2}\theta^{-1},$$

which is the estimate (7.9).

Let $a_0(t, X, \Lambda)$ and $b_0(t, X, \Lambda)$ be real-valued in $S(1, \Lambda^{-1}\Gamma)$ (*i.e.* satisfying (7.10)). We shall use the notation

$$S(\Lambda^m, \Lambda^{-1}\Gamma) = S^m \tag{7.14}$$

for the class of symbols $p(t, X, \Lambda)$ so that (7.10) is satisfied for $\Lambda^{-m}p(t, X, \Lambda)$. We set

$$u_0(t, X) = \exp\Big(-\int_0^t b_0(s, X)\mathrm{d}s\Big). \tag{7.15}$$

The symbols u_0 and $1/u_0$ belong to S^0 locally in t (*i.e.* (7.10) is satisfied for t in a bounded set) with semi-norms depending only on those of b_0. We calculate

$$(u_0)^w \mathrm{D}_t\ (1/u_0)^w = \mathrm{D}_t - i\ (u_0)^w \Big(\frac{b_0}{u_0}\Big)^w = \mathrm{D}_t - ib_0^w + r_{-1}^w, \tag{7.16}$$

where $r_{-1} \in S^{-1}$, with semi-norms depending only on those of b_0. We have for q defined in (6.1) (in particular q is real-valued),

$$(u_0)^w \Big(\mathrm{D}_t + iq^w + a_0^w + ib_0^w\Big)\ (1/u_0)^w = \mathrm{D}_t + a_0^w + (u_0)^w \Big(iq^w\Big)(1/u_0)^w + \mathrm{Op}(S^{-1}) \tag{7.17}$$

Let us now recall the following simple formula from the Weyl symbolic calculus: for $p_1, p_2 \in S^{m_1}, S^{m_2}$

$$p_1^w p_2^w = (p_1 \sharp p_2)^w = \Big(p_1 p_2 + \frac{1}{2\iota}\{p_1, p_2\} + \rho\Big)^w, \ \rho \in S^{m_1+m_2-2},\ \iota = 2i\pi. \tag{7.18}$$

From this we find, with obvious notations,

(7.19)

$$u_0 \sharp iq \sharp \frac{1}{u_0} = \big[u_0 iq + \frac{1}{\iota}\{u_0, iq\} + S^{-1}\big]\frac{1}{u_0} + \frac{1}{\iota}\{u_0 iq, \frac{1}{u_0}\} + S^{-1} = iq + \underbrace{\frac{1}{\pi u_0}\{u_0, q\}}_{\text{real-valued}} + S^{-1}.$$

This implies from (7.17)

(7.20)

$$(u_0)^w \Big(\mathrm{D}_t + iq^w + a_0^w + ib_0^w\Big)\ (1/u_0)^w = \mathrm{D}_t + a_{01}^w + iq^w + \mathrm{Op}(S^{-1}),\ a_{01} \in S^0 \text{ and real-valued.}$$

LEMMA 7.2. – *Let α_0 be a symbol in $S(1, \Gamma)$. Then, using the above notations*

$$(u_0)^w \alpha_0^w (1/u_0)^w = \alpha_0^w + \Lambda^{-1/2}\mathrm{Op}\big(S(1, \Gamma)\big). \tag{7.21}$$

In fact we apply proposition 18.5.7 in [H1] to $g_1 = \Lambda^{-1}\Gamma$ and $g_2 = \Gamma$ to obtain first $u_0 \sharp \alpha_0 = u_0\alpha_0 + \Lambda^{-1/2}\tilde{\alpha}_0$, with $\tilde{\alpha}_0 \in S(1, \Gamma)$, so that applying the same proposition once more and the calculus in the $S(1, \Gamma)$ class we get

$$\underbrace{u_0 \sharp \alpha_0}_{S(1,\Gamma)} \sharp \underbrace{1/u_0}_{S(1,\Lambda^{-1}\Gamma)} = \alpha_0 + \Lambda^{-1/2}\tilde{\tilde{\alpha}}_0,\ \tilde{\tilde{\alpha}}_0 \in S(1, \Gamma),$$

which concludes the proof of the lemma.

□

Going back to $\tilde{L}$ and $N_0 = a_0^w + ib_0^w$ defined in theorem 7.1, we get

$$\tilde{L} = L - iR,$$

where $R \in S(1,g) \subset S(1,\Gamma)$ from lemma 3.2. Thus from (7.20) and lemma 7.2

$$\begin{aligned}(u_0)^w\left(\tilde{L} + a_0^w + ib_0^w\right)(1/u_0)^w \\ = L - iR + a_{01}^w + \Lambda^{-1/2}\mathrm{Op}\big(S(1,\Gamma)\big) + \mathrm{Op}(S^{-1}) \\ = \tilde{L} + a_{01}^w + \Lambda^{-1/2}R_{00}\end{aligned} \tag{7.22}$$

where a_{01} is real-valued and belongs to S^0, R_{00} has a symbol in $S(1,\Gamma)$ which implies in particular that it is L^2-bounded. It suffices then to prove (7.12) for $\tilde{L} + a_{01}^w$ since the term $\Lambda^{-1/2}R_{00}$ could be absorbed by shrinking the diameter of the support of the function u. To prove this estimate, we need to go back to our proof at the beginning of the section and check the structure of the multiplier M in (7.2). In fact one needs only to check from (7.8) that the extra term introduced by a_{01} can be absorbed in the left-hand side. From (7.8), under the assumptions (7.10) we have

$$\begin{aligned}\Lambda^{-1/2}\sup_{t\in\mathbb{R}}|u(t)|_{\mathcal{H}}^2 \le 4\mathrm{Re}\left\langle \tilde{L}u, iMu\right\rangle \\ = 4\mathrm{Re}\left\langle \tilde{L} + a_{01}^w u, iMu\right\rangle - 4\mathrm{Re}\left\langle a_{01}^w u, iJu + \frac{i}{2}\Lambda^{-1/2}\,\mathrm{sign}\,(t-T)u\right\rangle.\end{aligned} \tag{7.23}$$

Since a_{01} is real valued we have

$$4\mathrm{Re}\left\langle a_{01}^w u, iJu + \frac{i}{2}\Lambda^{-1/2}\,\mathrm{sign}\,(t-T)u\right\rangle = 2\langle [a_{01}^w, J]u, u\rangle.$$

Moreover, since $J(t)$ has a symbol in $S(1, g^{[q(t,\cdot)]}) \subset S(1,\Gamma)$ from lemma 3.2 and $a_{01} \in S^0 = S(1,\Lambda^{-1}\Gamma)$, we apply the proposition 18.5.7 in [H1] to get that the bracket $[a_{01}^w, J(t)]\Lambda^{1/2}$ is bounded on L^2: shrinking the diameter of the support of the function u, the inequality (7.23) implies

$$\Lambda^{-1/2}\sup_{t\in\mathbb{R}}|u(t)|_{\mathcal{H}}^2 \le 8\,\mathrm{Re}\left\langle \tilde{L} + a_{01}^w u, iMu\right\rangle \le 12\,\|(\tilde{L} + a_{01}^w)u\|_{L^1(\mathbb{R},\mathcal{H})}\sup_{t\in\mathbb{R}}|u(t)|_{\mathcal{H}},$$

that is

$$\Lambda^{-1/2}\sup_{t\in\mathbb{R}}|u(t)|_{\mathcal{H}} \le 12\,\|(\tilde{L} + a_{01}^w)u\|_{L^1(\mathbb{R},\mathcal{H})}. \tag{7.24}$$

Using (7.22) we would get the desired estimate for $\tilde{L} + N_0$ if we were able to prove the invertibility of u_0^w and $(1/u_0)^w$. Since the Poisson bracket $\{u_0, u_0^{-1}\} = 0$, we have

$$u_0^w(1/u_0)^w = \mathrm{Id} + \Lambda^{-2}r_0^w, \text{ with } r_0 \in S^0, \tag{7.25}$$

whose semi-norms depend only of those of b_0. This ensures that u_0^w and $(1/u_0)^w$ are both invertible operators with spectra in (1/2,2) provided Λ is large enough with respect to a finite number of semi-norms of b_0. The proof of theorem 7.1 is complete.

□

8. Energy estimates for perturbations of operators with homogeneous symbols

Let M be a smooth manifold of dimension $d = n + 1$ and L a principal type classical pseudo-differential operator whose principal symbol satisfies condition $(\overline{\psi})$ on a neighborhood of a point γ_0 in the cosphere bundle. Let m be the order of L. We know (see *e.g.* [H1], section 26.4) that there exists an elliptic pseudo-differential operator E of order $(1 - m)$, an invertible Fourier integral operator U_0 of order 0, a classical pseudo-differential operator R such that there exists a compact neighborhood K_0 of γ_0 satisfying $WF(R) \cap K_0 = \emptyset$ with

$$U_0^* E L U_0 = \mathrm{D}_t + iq(t,x,\xi)^w + r_0(t,x,\tau,\xi)^w + r(t,x,\tau,\xi)^w + R \tag{8.1}$$

where (t,x,τ,ξ) are homogeneous symplectic coordinates ($(t,\tau) \in \mathbb{R}^2$ are dual variables as well as $(x,\xi) \in \mathbb{R}^{2n}$), q is a real-valued symbol of degree 1, r_0 is a complex-valued symbol of degree 0, r is a symbol of order -1. Condition $(\overline{\psi})$ reads

$$q(t,x,\xi) > 0,\ \ s > t \implies q(s,x,\xi) \geq 0. \tag{8.2}$$

Using the Malgrange-Weierstrass preparation theorem as in the remark following theorem 26.4.7′ page 103 in [H1], we can as well assume that r_0 in (8.1) does not depend on τ. This means that solvability of L^* at γ_0 with loss of $1+\kappa$ derivatives ($\kappa \leq 1$) amounts to proving under (8.2) and for r_0 of order 0 the existence of a positive C such that for all $u \in \mathrm{C}_0^\infty(\mathbb{R}^d)$ with diameter(supp u) $\leq 1/C$,

$$C\|\big(\mathrm{D}_t + iq(t,x,\xi)^w + r_0(t,x,\xi)^w\big)u\|_{\mathrm{L}^2} \geq \|u\|_{\mathrm{H}^{-\kappa}}. \tag{8.3}$$

We consider now the classical admissible metrics on $\mathbb{R}^{2n}_{X=(x,\xi)}$

$$\begin{aligned} \mathcal{G}_X &= |dx|^2 + \big(1+|\xi|\big)^{-2}|d\xi|^2 = \big(1+|\xi|\big)^{-1}\Gamma_X,\\ \Gamma_X &= (1+|\xi|)|dx|^2 + \big(1+|\xi|\big)^{-1}|d\xi|^2 = \Gamma_X^\sigma. \end{aligned} \tag{8.4}$$

We have the following equality of class of symbols referring to the standard $S^m_{\rho,\delta}$:

$$S((1+|\xi|)^m, \mathcal{G}) = S^m_{1,0},\ \ S((1+|\xi|)^m, \Gamma) = S^m_{1/2,1/2}.$$

In fact it will be enough for our purpose to use as $\mathcal{G}$ boxes the classical Littlewood-Paley decomposition: one can find $\varphi_0, \varphi, \phi_0, \phi, \Phi_0, \Phi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ valued in $[0,1]$ such that

$$\begin{cases} \sum_{k\in\mathbb{N}} \varphi_k(\xi) = 1,\ \varphi_k(\xi) = \varphi(\xi 2^{-k}) \text{ for } k \geq 1,\ \operatorname{supp}\varphi \subset \{1/2 \leq |\xi| \leq 2\},\\ \phi_k(\xi) = \phi(\xi 2^{-k}) \text{ for } k \geq 1,\ \operatorname{supp}\phi \subset \{1/4 \leq |\xi| \leq 4\},\ \operatorname{supp}\varphi_k \subset \phi_k^{-1}(1),\\ \Phi_k(\xi) = \Phi(\xi 2^{-k}) \text{ for } k \geq 1,\ \operatorname{supp}\Phi \subset \{1/8 \leq |\xi| \leq 8\},\ \operatorname{supp}\phi_k \subset \Phi_k^{-1}(1). \end{cases} \tag{8.5}$$

Moreover we shall need some global versions of the metrics introduced in section 3.

PROPOSITION 8.1. – *Let $q(t,x,\xi)$ be a real-valued symbol in $S(1+|\xi|, \mathcal{G})$ (uniformly with respect to t) satisfying* (8.2) *and let $G^{\{t\}}$ and $g^{\{t\}}$ be the metrics on $\mathbb{R}^{2n}$ defined by* (Γ_X is defined in (8.4))

$$G_X^{\{t\}} = \lambda(t,X)^{-1}\Gamma_X,\ \textit{with}\ \lambda(t,X) = 1 + |q(t,X)| + |q'_X(t,X)|^2_{\Gamma_X}, \tag{8.6}$$

$$g_X^{\{t\}}(T) = \frac{|d_X q(t,X)\cdot T|^2}{\lambda(t,X) + |q(t,X)|^2} + \frac{\Gamma_X(T)}{\lambda(t,X)^{1/2} + |q(t,X)|}, \quad T \in \mathbb{R}^{2n}. \tag{8.7}$$

The metrics $G^{\{t\}}$ and $g^{\{t\}}$ are admissible (see definition 3.1).

It is a well-known fact for the metric (8.6) (see [H1], section 26.10) so we give the proof only for (8.7). After lemma 3.2, the only point to be checked is the temperance of g, that is the comparison of g_X with g_Y. When $\mathcal{G}_X(X-Y) \le 1$, the proof of lemma 3.2 works. When $\mathcal{G}_X(X-Y) \ge 1$, we have $(X=(x,\xi), Y=(y,\eta))$

$$\Gamma_X(X-Y) \ge \Lambda(X) = 1+|\xi|, \text{ and } |\eta| \le |\eta-\xi|+|\xi| \le (1+|\xi|)^{1/2}\Gamma_X(X-Y)^{1/2}+|\xi|$$

which gives

$$\Lambda(X)+\Lambda(Y) \le \Gamma_X(X-Y)+1+(1/2)(1+|\xi|)+(1/2)\Gamma_X(X-Y)+|\xi| \le 3\Gamma_X(X-Y).$$

Using now lemma 3.2 we have (C_2, C_3 are some semi-norms of q)

$$\frac{g_X(T)}{g_Y(T)} \le \frac{2\Gamma_X(T)}{C_2^{-1}\Lambda(Y)^{-1}\Gamma_Y(T)} \le 6C_2\Gamma_X(X-Y)\frac{\Gamma_X(T)}{\Gamma_Y(T)} \le C_3\big(1+\Gamma_X(X-Y)\big)^{3/2},$$

which implies the temperance of g since from lemma 3.1, Γ_X is controlled by g_X^σ. The proof of proposition 8.1 is complete.

Going back to section 5, we recall that we can define the Wick quantization with respect to any symplectic norm. We shall consider the symplectic norms, slightly abusing notations of (8.4) (here the index k is an integer)

$$\Gamma_k = 2^k|dx|^2 + 2^{-k}|d\xi|^2, \ k \in \mathbb{N} \tag{8.8}$$

and refer to formula (5.1) for the Wick(Γ_k) quantization. One should notice that the metric Γ is uniformly equivalent to Γ_k on the support of Φ_k. Since Φ_k is identically 1 on the support of ϕ_k, this implies that the metric $G^{[\Phi_k(\cdot)q(t,\cdot)]}$ (resp. $g^{[\Phi_k(\cdot)q(t,\cdot)]}$) defined in (3.2) (resp. (3.3)) is uniformly equivalent to $G^{\{t\}}$ (resp. $g^{\{t\}}$) defined in (8.6) (resp. (8.7)) on the support of ϕ_k.

We use the definition (6.5-6) for $s(t,X)$, and $\theta(X)$ defined with respect to q. We note that since (6.7) is satisfied for s and q, it is also satisfied for s and q_k with (here $\Phi_k(x,\xi)$ stands for $\Phi_k(\xi)$)

$$q_k(t,X) = \Phi_k(X)q(t,X). \tag{8.9}$$

The symbol q_k satisfies the estimates (6.1) with Γ replaced by Γ_k and $\Lambda = 2^k$. We set

$$J_k(t) = s(t,\cdot)^{\mathrm{Wick}(\Gamma_k)}. \tag{8.10}$$

Note here that the operators $J_k(t)$ are the Wick(Γ_k) quantizations of the *same* function $s(t,X)$. We apply proposition 6.4 to $q_k(t,\cdot)$ to get with a positive constant γ_1 and a nonnegative constant C_0

$$\Phi_k(X)q(t,X) = \Omega_k(t,X)J_k(t,X) + R_k(t,X) \tag{8.11}$$

$$\Omega_k(t,X) \in S(\lambda_k^{1/3}\mu_k^{2/3}, g^{[\Phi_k(\cdot)q(t,\cdot)]}), \ \Omega_k(t,X)+C_0 \ge \gamma_1\lambda_k(t,X)^{1/3}\mu_k(t,X)^{2/3}, \tag{8.12}$$

$$J_k(t,X) \in S(1, g^{[\Phi_k(\cdot)q(t,\cdot)]}), \ \{\Omega_k(t,X), J_k(t,X)\} \in S(1, g^{[\Phi_k(\cdot)q(t,\cdot)]}), \tag{8.13}$$

$$R_k(t,X) \in S(1, g^{[\Phi_k(\cdot)q(t,\cdot)]}), \tag{8.14}$$

where λ_k and μ_k stand here for the functions λ and μ defined respectively in (3.2) and (3.7) with respect to the metric $g^{[\Phi_k(\cdot)q(t,\cdot)]}$. In fact the constants c_0, c_1 and C_0 given by (6.31) in proposition 6.4 depend only on the semi-norms of the symbol which is here $\Phi_k q(t,\cdot)$. These constants can be chosen independently of k since the semi-norms of $\Phi_k q$ are bounded from above independently of k.

LEMMA 8.2. – *Let $J_k(t)$ be given by* (8.10) *and ϕ_k by* (8.5). *We define $\mathcal{J}(t)$ as the operator*
(8.15)

$$\mathcal{J}(t) = \sum_{k\geq 0} \phi_k^w J_k(t)\phi_k^w = \Big(\mathcal{J}(t,X)\Big)^w, \quad \mathcal{J}(t,X) = \sum_{k\geq 0} \phi_k(\xi)^2 J_k(t,X) + \mathcal{R}(t,X),$$

with $\mathcal{R}(t,X) \in S((1+|\xi|)^{-1},\Gamma)$. The symbol $\mathcal{J}(t,X)$ belongs to $S(1,\Gamma)$ with semi-norms bounded by constants depending only on the dimension n and on the choice of functions ϕ_0 and ϕ in (8.5). *The operator $\mathcal{J}(t)$ is bounded selfadjoint on $\mathrm{L}^2(\mathbb{R}^n)$ and such that*

$$\|\mathcal{J}(t)\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}))} \leq 6, \quad t_1 \leq t_2 \Longrightarrow \mathcal{J}(t_1) \leq \mathcal{J}(t_2). \tag{8.16}$$

From (8.10), lemma 6.1, $\phi_k^w\phi_l^w = [\phi_k(\xi)\phi_l(\xi)]^w = 0$ for $k \geq l+4$, ϕ_k real-valued and

$$\sum_{k\geq 0}\phi_k^2(\xi) \leq 1 + \sum_{k\geq 1}\mathbf{1}_{2^{k-2}\leq|\xi|\leq 2^{k+2}} \leq 6$$

we get that $\mathcal{J}(t)$ is a bounded self-adjoint operator on $\mathrm{L}^2(\mathbb{R}^n)$ nondecreasing with t from (6.9). Since the Weyl symbol $J_k(t,X)$ of $J_k(t)$ belongs to $S(1,\Gamma_k)$ the symbol $\phi_k(\xi)\sharp J_k(t,x,\xi)\sharp\phi_k(\xi) \in S(1,\Gamma_k)$ belongs to $S(1,\Gamma_k)$ and is rapidly decreasing outside the support of ϕ_k so that $\mathcal{J}(t,X) \in S(1,\Gamma)$. More precisely, we have with $\Lambda_k = 2^k$

$$\begin{aligned}\mathcal{J}(t,X) &= \sum_k \phi_k\sharp J_k\sharp\phi_k\left[\phi_k J_k + \frac{1}{2\iota}\{\phi_k, J_k, +\}\Lambda_k^{-1} r_k\right]\sharp\phi_k \\ &\equiv \sum_k \phi_k^2 J_k + \frac{1}{(2\iota)^2}\big\{\{\phi_k, J_k\},\phi_k\big\}\end{aligned}$$

modulo a symbol in $S((1+|\xi|)^{-1},\Gamma)$. Since $\sum_k \big\{\{\phi_k, J_k\},\phi_k\big\}$ itself is in $S((1+|\xi|)^{-1},\Gamma)$, we get (8.15). The proof of lemma 8.2 is complete.

□

We have thus using (8.11–14) and (8.5),

$$\begin{aligned}&\mathcal{J}(t,X)\\ &= \sum_k \phi_k^2(X)\big(\Phi_k(X)q(t,X) - R_k(t,X) + C_0 J_k(t,X)\big)\big(\Omega_k(t,X)+C_0\big)^{-1} + \mathcal{R}(t,X)\\ &= q(t,X)\sum_k \phi_k^2(X)(\Omega_k(t,X)+C_0)^{-1}\\ &\qquad + \sum_k \phi_k^2(X)(\Omega_k(t,X)+C_0)^{-1}(-R_k(t,X)+C_0 J_k(t,X)) + \mathcal{R}(t,X).\end{aligned}$$

Since $G^{\{t\}}$ and $g^{\{t\}}$ are respectively equivalent to $G^{[\Phi_k(\cdot)q(t,\cdot)]}$ and $g^{[\Phi_k(\cdot)q(t,\cdot)]}$ on the support of ϕ_k, we can set

$$\mathcal{A} = \left[\sum_k \phi_k^2(X)\big(\Omega_k(t,X)+C_0\big)^{-1}\right]^{-1} \in S(\lambda^{1/3}\mu^{2/3}, g) \tag{8.17}$$

so that

$$\mathcal{J}(t,X) = q(t,X)\mathcal{A}(t,X)^{-1} + S(\lambda^{-1/3}\mu^{-2/3}, g) + S((1+|\xi|)^{-1}, \Gamma)$$

which gives

$$\begin{aligned}\mathcal{A}(t,X)\mathcal{J}(t,X) \\ &= q(t,X) + S(1,g) + S((1+|\xi|)^{-1}\lambda^{1/3}\mu^{2/3}, \Gamma) \\ &= q(t,X) + S(1,\Gamma).\end{aligned} \tag{8.18}$$

With $\mathcal{J}_0 = \mathcal{J} - \mathcal{R}$ (see (8.15)), we take now a look at the Poisson bracket

$$\begin{aligned}&\{\mathcal{A}, \mathcal{J}_0\} = \mathcal{A}^2\{\mathcal{J}_0, \mathcal{A}^{-1}\} = \mathcal{A}^2 \sum_{k,l} \{\phi_k^2 J_k, \phi_l^2(\Omega_l + C_0)^{-1}\} \\ &= \mathcal{A}^2 \sum_{|k-l|\le 4} \{\phi_k^2 J_k, \phi_l^2\}(\Omega_l + C_0)^{-1} + \{\phi_k^2 J_k, (\Omega_l + C_0)^{-1}\}\phi_l^2 \\ &= \mathcal{A}^2 \sum_{|k-l|\le 4} \underbrace{\{\phi_k^2 J_k, \phi_l^2\}}_{S(\Lambda^{-1},g)} \underbrace{(\Omega_l + C_0)^{-1}}_{S(\lambda^{-1/3}\mu^{-2/3},g)} + \mathcal{A}^2 \sum_{|k-l|\le 4} \{\Omega_l + C_0, \phi_k^2 J_k\}(\Omega_l + C_0)^{-2}\phi_l^2,\end{aligned}$$

since $\phi_k = \phi(\xi\Lambda_k^{-1})$ and

$$\{\phi_k^2 J_k, \phi_l^2\} = \phi_k^2\{J_k, \phi_l^2\} + + J_k \underbrace{\{\phi_k^2, \phi_l^2\}}_{=0} = -\phi_k^2 \underbrace{\frac{\partial J_k}{\partial x}}_{\in S(1,g)} \cdot \underbrace{\frac{\partial \phi_l^2}{\partial \xi}}_{\in S(\Lambda^{-1},\mathcal{G})}.$$

The first term in factor of $\mathcal{A}^2$ above is an element of $S(\lambda^{1/3}\mu^{2/3}\Lambda^{-1}, g) \subset S(1,g)$ since λ and μ are bounded above by Λ. We thus get that modulo $S(1,g)$

$$\begin{aligned}&\{\mathcal{A}, \mathcal{J}_0\} \equiv \mathcal{A}^2 \sum_{|k-l|\le 4} \{\Omega_l, \phi_k^2 J_k\}(\Omega_l + C_0)^{-2}\phi_l^2 \\ &= \mathcal{A}^2 \sum_{|k-l|\le 4} \underbrace{\{\Omega_l, \phi_k^2\}}_{S(\lambda^{1/3}\mu^{2/3}\Lambda^{-1},g)} \underbrace{J_k(\Omega_l + C_0)^{-2}\phi_l^2}_{S(\lambda^{-2/3}\mu^{-4/3},g)} + \mathcal{A}^2 \sum_{|k-l|\le 4} \{\Omega_l, J_k\}(\Omega_l + C_0)^{-2}\phi_k^2\phi_l^2.\end{aligned}$$

The first term is thus in $S(1,g)$. Eventually, we obtain modulo $S(1,g)$

$$\{\mathcal{A}, \mathcal{J}_0\} \equiv \sum_{|k-l|\le 4} \{\Omega_l, J_k\} \underbrace{(\Omega_l + C_0)^{-2}\mathcal{A}^2\phi_k^2\phi_l^2}_{S(1,g)}.$$

If we go back now to the definition (8.10) of J_k we get that, for $|k-l| \le 4$ the operator $J_k(t)$ is defined as $s(t,\cdot)^{\mathrm{Wick}(\Gamma_k)}$ where the norm Γ_k is equivalent to Γ_l. Using the calculations (6.38–46) this does not affect J_l except for the term with $\operatorname{sign} \beta_\nu$ in (6.46) which is convoluted with a different Gaussian function with an equivalent profile Γ_k instead of Γ_l. Using lemma 2.3, we get the identity (6.54) with σ replaced by σ_A where $A = \Gamma_l^{-1/2}\Gamma_k\Gamma_l^{-1/2}$. Since $|k-l| \le 4$ the norms of A and A^{-1} are bounded above by 2^4. Eventually we obtain that

$$\{\mathcal{A}, \mathcal{J}_0\} \in S(1,g). \tag{8.19}$$

We check now using $\mathcal{J}_0 \in S(1,g)$ and (8.17–19),

$$\begin{aligned}\mathcal{A}(t,X)\sharp\mathcal{J}(t,X) &= \mathcal{A}(t,X)\sharp\mathcal{J}_0(t,X) + \overbrace{\mathcal{A}(t,X)\sharp\mathcal{R}(t,X)}^{S(1,\Gamma)}\\ &= \mathcal{A}(t,X)\mathcal{J}_0(t,X) + \frac{1}{2\iota}\{\mathcal{A}(t,X),\mathcal{J}_0(t,X)\} + S(1,g) + S(1,\Gamma)\\ &= \mathcal{A}(t,X)\mathcal{J}_0(t,X) + S(1,\Gamma)\\ &= \mathcal{A}(t,X)\mathcal{J}(t,X) + S(1,\Gamma) = q(t,X) + S(1,\Gamma).\end{aligned}$$

We obtain

$$q(t,\cdot)^w = [\mathcal{A}(t,\cdot) + C_1]^w \mathcal{J}(t,\cdot)^w + \mathrm{Op}\big(S(1,\Gamma)\big), \tag{8.20}$$

where C_1 is a constant such that $\mathcal{A}(t,\cdot)^w + C_1 \geq 0$ which follows from the Fefferman-Phong inequality (applicable here since $\lambda^{1/3}\mu^{2/3} \leq \mu^2$). We can now prove the following theorem

THEOREM 8.3. – *Let q be a symbol satisfying the assumptions of proposition 8.1, $Q(t)$ be the operator with Weyl symbol $q(t,\cdot)$ and $L = \mathrm{D}_t + iQ(t)$ with $\mathrm{D}_t = -i\partial/\partial t$. There exists an operator $\mathcal{R}_0(t) \in \mathrm{Op}\big(S(1,\Gamma)\big)$* (see (8.4)) *such that*

$$Q(t) = \mathcal{A}_1(t)\mathcal{J}(t) + \mathcal{R}_0(t)$$

where the nondecreasing operator $\mathcal{J}(t)$ is defined in (8.15), *the positive operator*

$$\mathcal{A}_1(t) = \mathcal{A}_1(t,\cdot)^w = \mathcal{A}(t,\cdot)^w + C_1 \geq \mathrm{Id} \tag{8.21}$$

is given in (8.17), (8.20). *In particular from proposition 8.1*

$$\mathcal{A}_1(t,X) \in S(\lambda^{1/3}\mu^{2/3}, g^{\{t\}}) \subset S(1+|\xi|,\Gamma). \tag{8.22}$$

We set

$$\tilde{L} = \mathrm{D}_t + i\mathcal{A}_1(t)\mathcal{J}(t) = L - i\mathcal{R}_0(t).$$

There exists a positive constant C depending only on the semi-norms of q such that for all $u \in \mathrm{C}_0^\infty(\mathbb{R}^{n+1})$ with θ =diameter(supp u) $\leq C^{-1}$,

$$C\|\tilde{L}u\|_{\mathrm{H}^{1/2}(\mathbb{R}^{n+1})} \geq \|u\|_{\mathrm{H}^{-1/2}(\mathbb{R}^{n+1})}, \tag{8.23}$$

Proof. – It remains only to prove (8.23). We shall essentially repeat the arguments of section 7. We set $\mathcal{H} = \mathrm{L}^2(\mathbb{R})$ and we compute for $u \in \mathrm{C}_0^\infty(\mathbb{R},\mathcal{H})$ with $\mathrm{L}^2(\mathbb{R},\mathcal{H})$ dot products, T a real parameter, $\Lambda = \big[(1+|\xi|^2)^{1/2}\big]^w$, $\tilde{Q}(t) = Q(t) - \mathcal{R}_0(t)$,

$$\begin{aligned}2\mathrm{Re}\left\langle \mathrm{D}_t u + i\tilde{Q}(t)u(t), \overbrace{i\mathcal{J}(t)u(t) + i(1/2)\Lambda^{-1}\ \mathrm{sign}\ (t-T)u(t)}^{i\mathcal{M}u}\right\rangle \\ = \left\langle \dot{\mathcal{J}}u, u\right\rangle + |\Lambda^{-1/2}u(T)|^2_{\mathcal{H}} + 2\langle \mathcal{J}\mathcal{A}_1\mathcal{J}u, u\rangle + \langle s_0 \mathrm{Re}\,(\Lambda^{-1}\mathcal{A}_1\mathcal{J})u, u\rangle,\end{aligned} \tag{8.24}$$

where $2\mathrm{Re}\,(\Lambda^{-1}\mathcal{A}_1\mathcal{J}) = (\Lambda^{-1}\mathcal{A}_1\mathcal{J}) + (\Lambda^{-1}\mathcal{A}_1\mathcal{J})^* = \Lambda^{-1}\mathcal{A}_1\mathcal{J} + \mathcal{J}\mathcal{A}_1\Lambda^{-1}$ and $s_0 = \mathrm{sign}\,(t-T)$. We have, using $s_0^2 = 1$ and (8.21),

$$\begin{aligned}&\mathcal{J}\mathcal{A}_1\mathcal{J} + 2s_0\mathrm{Re}\,(\Lambda^{-1}\mathcal{A}_1\mathcal{J})\\ &\quad= (\mathcal{J}{\mathcal{A}_1}^{1/2} + s_0\Lambda^{-1}{\mathcal{A}_1}^{1/2})({\mathcal{A}_1}^{1/2}\mathcal{J} + s_0{\mathcal{A}_1}^{1/2}\Lambda^{-1}) - \Lambda^{-1}\mathcal{A}_1\Lambda^{-1}\\ &\quad\geq -\Lambda^{-1}\mathcal{A}_1\Lambda^{-1}.\end{aligned} \tag{8.25}$$

From lemma 8.2 the operator $\mathcal{J}(t)$ is bounded nondecreasing with t so that arguing like in (7.4) we get that

$$\left\langle \dot{\mathcal{J}}u, u\right\rangle \geq 0.$$

From (8.24-25) we get

$$2\mathrm{Re}\left\langle \tilde{L}u, i\mathcal{M}u\right\rangle \geq |\Lambda^{-1/2}u(T)|^2_{\mathcal{H}} - \frac{1}{2}\int \langle \Lambda^{-1}\mathcal{A}_1(t)\Lambda^{-1}u(t), u(t)\rangle_{\mathcal{H}}\,\mathrm{d}t + \frac{3}{2}\langle \mathcal{J}\mathcal{A}_1\mathcal{J}u, u\rangle. \tag{8.26}$$

Since $\mathcal{A}_1$ is a pseudo-differential operator with symbol in $S(\lambda^{1/3}\mu^{2/3}, g) \subset S(1+|\xi|, \Gamma)$ (cf. (8.17) and proposition 8.1), we obtain

$$\begin{aligned}&\int \langle \Lambda^{-1}\mathcal{A}_1\Lambda^{-1}u(t), u(t)\rangle_{\mathcal{H}}\mathrm{d}t\\ &\qquad= \int \left\langle \Lambda^{-1/2}\mathcal{A}_1\Lambda^{-1/2}\Lambda^{-1/2}u(t), \Lambda^{-1/2}u(t)\right\rangle_{\mathcal{H}}\\ &\qquad\leq C_2\theta \sup_t |\Lambda^{-1/2}u(t)|^2_{\mathcal{H}}.\end{aligned}$$

We eventually get, with θ standing for the diameter of the support of u,

$$|\Lambda^{-1/2}u(T)|^2_{\mathcal{H}} + \frac{3}{2}\langle \mathcal{J}\mathcal{A}_1\mathcal{J}u, u\rangle \leq \frac{C_2\theta}{2}\sup_t |\Lambda^{-1/2}u(t)|^2_{\mathcal{H}} + 2\mathrm{Re}\left\langle \tilde{L}u, i\mathcal{M}u\right\rangle,$$

so that choosing T so that $|\Lambda^{-1/2}u(T)|_{\mathcal{H}} = \sup_t |\Lambda^{-1/2}u(t)|_{\mathcal{H}}$,

$$\sup_{t\in\mathbb{R}} |\Lambda^{-1/2}u(t)|^2_{\mathcal{H}}[1 - \theta C_2/2] \leq 2\mathrm{Re}\left\langle \tilde{L}u, i\mathcal{M}u\right\rangle. \tag{8.27}$$

Assuming $\theta C_2 \leq 1$, we get from (8.27),

$$4\int_{\mathbb{R}} |\Lambda^{1/2}\tilde{L}u(t)|_{\mathcal{H}}\mathrm{d}t \sup_{t\in\mathbb{R}}\Big|\overbrace{\Lambda^{-1/2}\mathcal{M}\Lambda^{1/2}}^{\in\,\mathrm{Op}(S(1,\Gamma))}\Lambda^{-1/2}u(t)\Big|_{\mathcal{H}} \geq \sup_{t\in\mathbb{R}} |\Lambda^{-1/2}u(t)|^2_{\mathcal{H}}.$$

The operator $\Lambda^{-1/2}\mathcal{M}(t)\Lambda^{1/2}$ has a symbol in $S(1,\Gamma)$ for each t since it is the case for $\mathcal{J}(t)$ (lemma 8.2); its semi-norms are also bounded from above independently of t since it is true for $\mathcal{J}(t)$. Consequently we obtain

$$C_3\int_{\mathbb{R}} |\Lambda^{1/2}\tilde{L}u(t)|_{\mathcal{H}}\mathrm{d}t \geq \sup_{t\in\mathbb{R}} |\Lambda^{-1/2}u(t)|_{\mathcal{H}}. \tag{8.28}$$

The Cauchy-Schwarz inequality gives

$$C_3^2\int_{\mathbb{R}} |\Lambda^{1/2}\tilde{L}u(t)|^2_{\mathcal{H}}\mathrm{d}t \geq \int_{\mathbb{R}} |\Lambda^{-1/2}u(t)|^2_{\mathcal{H}}\mathrm{d}t\theta^{-2},$$

which implies

$$C_3 \|\tilde{L}u\|_{\mathrm{H}^{1/2}(\mathbb{R}^{n+1})} \geq \theta^{-1} \|u\|_{\mathrm{H}^{-1/2}(\mathbb{R}^{n+1})}, \tag{8.29}$$

which is (8.23). The proof of theorem 8.3 is complete. To get theorem 1.2 in the introduction, one needs to add the following

Remark 8.4. – It is indeed possible to find for each real number $s \in [0,1]$ a perturbation $\mathcal{K}_s(t)$ with symbol uniformly in Op$(S(1,\Gamma))$ such that

$$C\|Lu + \mathcal{K}_s u\|_{\mathrm{H}^s(\mathbb{R}^{n+1})} \geq \theta^{-1} \|u\|_{\mathrm{H}^{s-1}(\mathbb{R}^{n+1})}. \tag{8.30}$$

For this purpose, we write (8.28) as

$$C_3 \int_{\mathbb{R}} |\Lambda^{1/2}(L - i\mathcal{R}_0)\Lambda^{s-1/2} u(t)|_{\mathcal{H}} \mathrm{d}t \ \geq \sup_{t\in\mathbb{R}} |\Lambda^{-1/2}\Lambda^{s-1/2} u(t)|_{\mathcal{H}}, \tag{8.31}$$

and we note that

$$\begin{aligned} &\Lambda^{1/2}(L - i\mathcal{R}_0)\Lambda^{s-1/2} \\ &\quad = \Lambda^s \Lambda^{1/2-s}(L - i\mathcal{R}_0)\Lambda^{s-1/2} \\ &\quad\quad = \Lambda^s \Big(\mathrm{D}_t + iQ(t) + i[\Lambda^{1/2-s}, Q]\Lambda^{s-1/2} - \Lambda^{1/2-s} i\mathcal{R}_0 \Lambda^{s-1/2}\Big). \end{aligned}$$

The operator $[\Lambda^{1/2-s}, Q]$ has a symbol in $S((1+|\xi|)^{1/2-s}, \mathcal{G})$ defined in (8.4) whereas the symbol of $\Lambda^{1/2-s}\mathcal{R}_0\Lambda^{s-1/2}$ is in $S(1,\Gamma)$ defined in proposition 8.1. We define then

$$\mathcal{K}_s = i[\Lambda^{1/2-s}, \mathcal{Q}]\Lambda^{s-1/2} - \Lambda^{1/2-s} i\mathcal{R}_0\Lambda^{s-1/2} \in \mathrm{Op}\mathcal{S}(1,\Gamma).$$

The inequality (8.31) and the previous identities give for any real s

$$C_3 \int_{\mathbb{R}} |\Lambda^s (L + \mathcal{K}_s) u(t)|_{\mathcal{H}} \mathrm{d}t \ \geq \sup_{t\in\mathbb{R}} |\Lambda^{s-1} u(t)|_{\mathcal{H}}, \tag{8.32}$$

yielding (8.30) for $1 \geq s \geq 0$.

Remark 8.5. – One could prove as we did in theorem 7.1 that the estimates (8.30) are stable by a perturbation with symbol in $S(1,\mathcal{G})$. The proof follows the lines of lemma 7.2. It seems also possible to reduce the loss of derivatives from 2 to $3/2$ as it is done in theorem 7.1. This would prove that the perturbation $\mathcal{K}_s$ of the previous remark could be chosen independently of s.

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N. Lerner
Université de Rennes,
Irmar, Campus de Beaulieu,
35042 Rennes cedex, France
lerner@univ-rennes1.fr