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Annales scientifiques de l'É.N.S. 4e série, tome 31, nº 1 (1998), p. 17-45
<http://www.numdam.org/item?id=ASENS_1998_4_31_1_17_0>

Ann. scient. Éc. Norm. Sup.,
4e série, t. 31, 1998, p. 17 à 45.

ORBITAL VARIETIES OF THE MINIMAL ORBIT

By Anthony JOSEPH

ABSTRACT. – Let $\mathfrak{g}$ be a complex simple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. For any nilpotent orbit O an orbital variety $\mathcal{V}$ of O is defined to be an irreducible component of $\mathfrak{n}^+ \cap \mathrm{O}$. We say that $\mathcal{V}$ is strongly (resp. weakly) quantizable if there exists a $U(\mathfrak{g})$ module L isomorphic to $R[\mathcal{V}]$ as a $U(\mathfrak{h})$ module, up to a weight shift (resp. whose associated variety is $\mathcal{V}$). Here we obtain an explicit necessary and sufficient condition for strong (resp. weak) quantization of an orbital variety of the minimal non-zero nilpotent orbit. This shows that there is always at least one orbital variety admitting strong quantization, a result which hopefully should hold for any nilpotent orbit as the corresponding annihilator would be completely prime. On the other hand it also shows that even weak quantization can fail and even when this holds strong quantization can fail. In this latter case using the Demazure operators we show exactly how close the formal character of $R[\mathcal{V}]$ can approach that of a $U(\mathfrak{g})$ module and suggest that a similar behaviour holds in general. © Elsevier, Paris

RÉSUMÉ. – Soit $\mathfrak{g}$ une algèbre de Lie semi-simple complexe et $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ une décomposition triangulaire. Pour toute orbite nilpotente O une variété orbitale $\mathcal{V}$ de O est par définition un composant irréductible de $\mathfrak{n}^+ \cap \mathrm{O}$. On dira que $\mathcal{V}$ est fortement (resp. faiblement) quantifiable s'il existe un $U(\mathfrak{g})$ module L isomorphe à $R[\mathcal{V}]$ comme $U(\mathfrak{h})$ module, à une translation de poids près (resp. dont la variété associée est $\mathcal{V}$). Nous précisons une condition nécessaire et suffisante pour qu'une variété orbitale de l'orbite nilpotente minimale non-nulle admette une quantification forte (resp. faible). De cela il résulte qu'il existe toujours une variété orbitale fortement quantifiable, un résultat qu'on espère rester vrai pour toute orbite nilpotente, car l'annulateur correspondant serait complètement premier. En revanche on obtient également qu'il existe des variétés orbitales qui ne sont même pas faiblement quantifiables et d'autres qui sont faiblement mais pas fortement quantifiables. Dans ce dernier cas on se sert des opérateurs de Demazure pour préciser dans quelle mesure le caractère formel de $R[\mathcal{V}]$ peut se rapprocher avec celui d'un $U(\mathfrak{g})$ module et nous suggérons que ce comportement puisse se généraliser. © Elsevier, Paris

1. Introduction

1.1. Let G be a connected, simply connected complex semisimple algebraic group with Lie algebra $\mathfrak{g}$. One of the hard remaining problems of the orbit method is whether one can quantize a reasonable Lagrangian subvariety of a so-called nilpotent orbit in $\mathfrak{g}^*$. As noted for example in [J12, Section 8] this question can be formulated more precisely as follows. Fix a Borel subgroup B of G with torus T and let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the corresponding triangular decomposition of $\mathfrak{g}$, that is Lie $B = \mathfrak{b} := \mathfrak{n} \oplus \mathfrak{h}$, Lie $T = \mathfrak{h}$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form, $\mathfrak{n}$ with $(\mathfrak{n}^-)^*$ and $T^*(G/B)$ with $G \times_B \mathfrak{n}$. The multiplication map $p : G \times_B \mathfrak{n} \to G\mathfrak{n}$ is called the moment map. Its image $\mathcal{N}$ consists of all ad-nilpotent elements of $\mathfrak{g}$. It is a finite union of G orbits called nilpotent orbits. Now let K be an algebraic subgroup of G for which $K \setminus G/B$ is finite. For any K orbit $X := KxB/B$ in G/B let $C^*(X)$ be the corresponding conormal in $T^*(G/B)$.

Then $Y := p(C^*(X)) = \overline{KT_x^\perp}$ is the closure of a K stable Lagrangian subvariety of the unique dense nilpotent orbit in $GT_x^\perp$. One may ask if there exists an admissible $(\mathfrak{g}, K)$ module whose associated variety is Y. In view of say [BB, Thm. 1.90] one could expect to answer this question by showing that there exists an admissible $(\mathcal{D}, K)$ module whose characteristic variety is $C^*(X)$. However this does not have an immediate solution.

1.2. Now take $B = K$ in the above. Then $B \setminus G/B$ identifies with the Weyl group $W := N_G(T)/T$. Take a representative n_w of $w \in W$. Then $T_{n_w}^\perp = \mathfrak{n} \cap^w \mathfrak{n}$ and we let $O(w)$ denote the unique dense nilpotent in $GT_{n_w}^\perp$. After Steinberg [St] the map $w \mapsto O(w)$ of $W \to \mathcal{N}/G$ is surjective. One calls the Lagrangian subvariety $V(w) := \overline{BT_{n_w}^\perp} \cap O(w)$ an orbital variety of $O(w)$. Combined with a result of Spaltenstein [S], it follows that the set of orbital varieties of a given nilpotent orbit O is exactly the set of irreducible components of $O \cap \mathfrak{n}$ and of course takes the form $\{V(w) | w \in W \text{ with } O = O(w)\}$. For an exposition of these results and some further details the reader may consult [J4, Sect. 9]. An admissible $(\mathfrak{g}, B)$ module may be taken to be a module in the Bernstein-Gelfand-Gelfand $\mathcal{O}$ category. The associated variety $\mathcal{V}(M) : M \in Ob\mathcal{O}$ is automatically a B stable subvariety of $\mathfrak{n}$. By Gabber's theorem [G] any associated variety is involutive and so by [J4, 7.3] it follows that $\mathcal{V}(M)$ is a union of orbital variety closures. By a further theorem of Gabber [LS, Sect. A] if M is homogeneous for say Gelfand-Kirillov dimension then $\mathcal{V}(M)$ is equidimensional. If M is simple then for $\mathfrak{g} = \mathfrak{sl}(n)$ its associated variety $\mathcal{V}(M)$ is irreducible [M]; but the characteristic variety of the corresponding $\mathcal{D}$ module need not be [KT]. Outside type $\mathfrak{sl}(n)$ an example of Tanisaki [T] shows that even $\mathcal{V}(M)$ may fail to be irreducible. However in this example the components of $\mathcal{V}(M)$ also occur as associated varieties of other highest weight modules [J7, 8.6-8.8].

1.3. In the present article we study the orbital varieties of the minimal non-zero nilpotent orbit, that is the orbit of the highest root vector. We call these the minimal orbital varieties. They are parametrized by the set of long simple roots. Our first surprise was that not every minimal orbital variety V can be quantized in the above (weak) sense, namely that its closure is the associated variety of a highest weight module. These bad orbital varieties all occur for $\mathfrak{g}$ simple of type $B_\ell : \ell \geq 4$. Here one should perhaps recall [Sect. 4.10] that the decomposition of $\mathcal{V}(M)$ into irreducible components is completely determined by the decomposition of the Goldie rank polynomial associated to M into the characteristic polynomials of the orbital varieties. The former set are known implicitly through the Kazhdan-Lusztig polynomials. The latter are similarly determined [J10, 5.7] through analogous numerical (but as yet unknown) data. In simple cases one can calculate both sets of polynomials and from this Tanisaki's example can be obtained. It is also not difficult to check our present assertion. Apparently nobody bothered to look. It means that we now have an example of an involutive prime ideal of the symmetric algebra $S(\mathfrak{g})$ which is not the radical of a left ideal of the enveloping algebra $U(\mathfrak{g})$. Although such examples had been known previously even for the first Weyl algebra via [Str. Sect. 3, Prop. 3], they were somehow thought to be pathological and not candidates for reasonable Lagrangian subvarieties.

1.4. A further starting point of this paper is whether one may quantize an orbital variety V in the following strong sense. Does there exist a highest weight module L (not necessarily simple) with highest weight vector e_λ such that $gr\ Ann_{U(\mathfrak{n}^-)}e_\lambda$ coincides with the ideal I of definition of $\bar{V}$. One may consider that this gives $S(\mathfrak{n}^-)/I$ a $U(\mathfrak{g})$ module structure. It obviously implies in terms of formal characters that

$$(*) \qquad ch\ L = (ch\ S(\mathfrak{n}^-)/I)e^\lambda.$$

Conversely such an equality would imply the coincidence of the Goldie rank polynomial associated to L with the characteristic polynomial of V and hence that $gr\ Ann_{U(\mathfrak{n}^-)}e_\lambda \subset \sqrt{gr\ Ann_{U(\mathfrak{n}^-)}e_\lambda} = I$. Further use of $(*)$ then forces equality. A further surprise is that even when weak quantization is possible, strong quantization may fail. Indeed for minimal orbital varieties this is the case when $\mathfrak{g}$ is simple of types $B_\ell : \ell \geq 3$, $D_\ell : \ell \geq 5$, E_6, E_7, E_8, F_4. One may remark here that Benlolo [B] found two examples in A_5 where strong quantization fails if one requires L to be simple; but paradoxically can be recovered if L is just highest weight. This phenomenon cannot occur here basically because $Ann\ L$ is maximal. On the other hand we show that for each semisimple Lie $\mathfrak{g}$ there is always a minimal orbital variety which admits strong quantization. One can hope that this holds for a given nilpotent orbit as (*see* 1.6) this would construct a completely prime primitive ideal in $U(\mathfrak{g})$ whose associated variety is the nilpotent orbit in question.

1.5. The trouble with these questions and the orbit method in general is that they are rather like trying to fit a round peg into a square hole. Indeed there is no real reason to believe that involutive prime ideals of $S(\mathfrak{g})$ should follow left ideals in $U(\mathfrak{g})$ so closely. Nevertheless the tight structure of semisimple Lie algebras does suggest that whatever one calculates in either context the result must follow a very similar pattern; but discrepancies can occur because certain basic parameters may differ. This philosophy is particularly well illustrated by the comparison of Goldie rank and characteristic polynomials. Both are given by the same basic procedure; but the data needed to compute them differ slightly. Consequently the mysterious discrepancies between them take an entirely understandable nature.

1.6. A further aim of the present work is to apply the philosophy of 1.5 to strong quantization. Suppose we have an orbital variety V which is quantizable in the weak sense. This gives a so-called coherent family of simple highest weight modules whose associated varieties all coincide with V. Most of these modules will fail in a bad way to satisfy $(*)$. The strong quantization hypothesis asks if one can choose λ necessarily small in some sense so that $(*)$ is recovered. This is a challenging and difficult problem which also gives a particular meaning to λ. A nice consequence is that annihilator of such a module must be completely prime [J7, 8.1, 8.2]. However since it fails to have a positive answer we must adjust our sights.

Let us recall that to each orbital variety V there is a subset $\pi' := \tau(V)$ of the set of simple roots π such that V is contained in the nilradical $\mathfrak{m}_{\pi'}$ of the parabolic subalgebra $\mathfrak{p}_{\pi'} \supset \mathfrak{b}$ defined by π' and invariant under its reductive part $\mathfrak{r}_{\pi'}$. Consequently the formal character $ch(S(\mathfrak{n}^-)/I)$ of the algebra of regular functions on $\bar{V}$ is invariant under the

Demazure operator $\Delta_{\pi'}$, as defined in 3.2, and which we recall is idempotent. We make the suggestion that there exists a subset σ_V of the roots of $\mathfrak{m}_{\pi'}$ such that

$$(**) \qquad ch(S(\mathfrak{n}^-)/I(\bar{V})) = \Delta_{\pi'}\left(\frac{1}{\prod_{\alpha\in\sigma_V}(1-e^{-\alpha})}\right).$$

The hypothesis is not unreasonable. Let $R_{\pi'}$ be the connected algebraic subgroup of G with Lie $\mathfrak{r}_{\pi'}$ and set $\mathfrak{m}_V = \oplus\{\mathfrak{g}_\alpha : \alpha \in \sigma_V\}$ where $\mathfrak{g}_\alpha$ is the root subspace of α. Then $(**)$ suggests that $\bar{V} = \overline{R_{\pi'}\mathfrak{m}_V}$ whose truth is at least testable. Of course one cannot get $(**)$ from such a description of V without additional hypotheses one of which should be (Section 5.2) that $\mathfrak{m}_V$ is invariant under the $\mathfrak{g}_{-\alpha} : \alpha \in \pi'$.

Our calculations for the minimal weakly quantizable orbital varieties indicate that one may choose $\lambda \in \mathfrak{h}^*$ such that the simple highest weight module $L(\lambda)$ with highest weight λ satisfies

$$(***) \qquad ch\ L(\lambda) = \Delta_{\pi'}\frac{e^\lambda}{\prod_{\alpha\in\sigma_V}(1-e^{-\alpha})}.$$

Here $L(\lambda)$ should be chosen to be a locally finite $\mathfrak{r}_{\pi'}$ module and so $(\lambda, \alpha^\vee) \in \mathbb{N}$ for each coroot $\alpha^\vee$ with $\alpha \in \pi'$. Then the above two formulae imply $(*)$ exactly when $(\lambda, \alpha^\vee) = 0,\ \forall\ \alpha \in \pi'$; but otherwise $L(\lambda)$ is a little "fatter" than $S(\mathfrak{n}^-)/I(\bar{V})$.

1.7. In this paper we present some technical computations to verify $(**)$ and $(***)$ for the weakly quantizable minimal orbital varieties in $\mathfrak{g}$ classical. These are made possible by the relative simple transformation properties of the relevant simple modules under coherent continuation and even then the calculations are quite difficult. It is also an essential point that $(***)$ should only hold for "small" λ and so its choice is rather delicate. Of course one can eventually weaken the hypothesis that $(**)$ holds to requiring just that the right hand side can be expressed as $\Delta_{\pi'}F$ and then that the right hand side of $(***)$ can be expressed as $\Delta_{\pi'}e^\lambda F$. A possible way to extend these considerations to the non-weakly quantizable orbital varieties for arbitrary nilpotent orbits is indicated by the result described in 4.8.

2. A geometric description of the minimal orbital varieties

2.1. Let us start with a general result due to B. Kostant. In the notation of 1.6 one may write $\mathfrak{p}_{\pi'} = \mathfrak{r}_{\pi'} \oplus \mathfrak{m}_{\pi'}$. Furthermore $\mathfrak{r}_{\pi'} = \mathfrak{s}_{\pi'} \oplus \mathfrak{z}_{\pi'}$ where $\mathfrak{s}_{\pi'}$ denotes the semisimple part of $\mathfrak{r}_{\pi'}$ generated by the root subspaces $\mathfrak{g}_\alpha : \alpha \in \pm\pi'$ and $\mathfrak{z}_{\pi'}$ the centre of $\mathfrak{r}_{\pi'}$. Then $\mathfrak{l}_{\pi'} = \mathfrak{s}_{\pi'} \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}_{\pi'}$ and $\mathfrak{h} = \mathfrak{l}_{\pi'} \oplus \mathfrak{z}_{\pi'}$ is an orthogonal direct sum decomposition for the Cartan form. Moreover using this form to identify $\mathfrak{h}$ and $\mathfrak{h}^*$, $\mathfrak{l}_{\pi'}$ becomes $\mathbb{C}\pi'$. Given $\lambda \in \mathfrak{h}^* \cong \mathfrak{h}$ let $\lambda = \lambda' + \lambda''$ be its decomposition according to the above direct sum. If λ is an integral weight which is dominant with respect to π', that is $(\lambda, \alpha^\vee) \in \mathbb{N}$ for all $\alpha \in \pi'$, then λ' is a positive integer linear combination of the fundamental weights for $\mathfrak{s}_{\pi'}$ in $\mathfrak{l}_{\pi'}$. In particular if λ, μ are π' dominant weights, then $(\lambda', \mu') \geq 0$. Again if $\lambda, \mu \in \mathbb{Z}\pi$, with $\lambda - \mu \in \mathbb{Z}\pi'$, then

$0 = (\lambda - \mu, (\mathbb{Z}\pi')^{\perp}) = (\lambda'' - \mu'', (\mathbb{Z}\pi')^{\perp})$ and so $\lambda'' = \mu''$. Consider the adjoint action of $\mathfrak{r}_{\pi'}$ on $\mathfrak{m}_{\pi'}$.

THEOREM (B. Kostant). – *The $\mathfrak{z}_{\pi'}$ isotypical components of $\mathfrak{m}_{\pi'}$ are simple $\mathfrak{s}_{\pi'}$ modules.*

Proof. – Let λ, μ be π' highest weights of a fixed isotypical component. Clearly $\lambda - \mu \in \mathbb{Z}\pi'$ and so $\nu := \lambda'' = \mu''$ by the above and moreover is non-zero. Then $(\lambda, \mu) = (\lambda', \mu') + (\nu, \nu) > 0$. Yet λ, μ are roots so $\lambda - \mu$ is a root of $\mathfrak{s}_{\pi'}$ forcing $\lambda = \mu$ and so giving the asserted simplicity. □

Remarks. – This result had been correctly attributed to Kostant in [W, Thm. 8.13.3] but the author had botched and bungled up the proof. I am grateful to B. Kostant for some correspondence on the matter and to P. Polo for some illuminating discussions.

2.2. It is often true that one can squeeze some extra mileage out of a good argument. In the present case we have the

LEMMA. – *Suppose $\mathfrak{p}_{\pi'}$ is maximal. Let γ, δ be π' highest roots of $\mathfrak{m}_{\pi'}$. Then $(\gamma, \delta) > 0$.*

Proof. – In our previous notation it is enough to show that $(\gamma,'', \delta'') > 0$. Let α be the unique simple root in $\pi \setminus \pi'$. Let ℓ_γ denote the coefficient of α in γ expressed as a sum of roots from π. Then $\ell_\gamma \in \mathbb{N}^+$ and $\gamma'' = \alpha''\ell_\gamma$. Thus $(\gamma'', \delta'') = \ell_\gamma \ell_\delta (\alpha'', \alpha'') > 0$ as required. □

2.3. In the remainder of this section we assume that $\mathfrak{p}_{\pi'}$ is maximal and write $\pi \backslash \pi' = \{\alpha\}$. We further assume that α is a long root. Let Γ_i denote the set of roots of $\mathfrak{m} = \mathfrak{m}_{\pi'}$ in which α occurs with coefficient i and $\mathfrak{m}_i$ the corresponding sum of root subspaces. It follows from 2.1 that each $\mathfrak{m}_i$ is a simple $\mathfrak{r} = \mathfrak{r}_{\pi'}$ module and we let γ_i denote its unique lowest weight, in particular $\gamma_1 = \alpha$. The subalgebra generated by $\mathfrak{m}_1$ coincides with $\mathfrak{m}$, since added to the maximal parabolic $\mathfrak{n}^- + \mathfrak{r}$ it must equal $\mathfrak{g}$. Thus setting $s = max\{i \mid \mathfrak{m}_i \neq 0\}$ we have $\mathfrak{m}_i \neq 0$ for $1 \leq i \leq s$ and moreover $\mathfrak{m}$ has length s as an $\mathfrak{r}$ module. Through the Weyl group for $\mathfrak{r}$, or directly, it follows from 2.2 that $(\gamma_i, \gamma_j) > 0$. In particular the $\delta_{i+1} = \gamma_{i+1} - \gamma_i : i = 1, 2, \cdots, s-1$ are positive roots. Set $\delta_1 = \gamma_1$.

LEMMA. – *The $\delta_i : i = 1, 2, \cdots, s$ form a simple system of type A_s.*

Proof. – Observe that the γ_i are all long roots. Indeed γ_1 is a long root by hypothesis and $\gamma_s : s > 1$ cannot be short for otherwise $2\gamma_s - \gamma_1$ would be a positive root with a coefficient of α strictly greater than s. Since $s \leq 2$ for $\mathfrak{g}$ classical, this leaves only one case (in F_4) which is easily checked.

Taking $(\gamma_i, \gamma_i) = 2$ we must have $(\gamma_i, \gamma_j) = 1$ for $i \neq j$. Then (δ_i, δ_j) has the required form (and in particular this forces the linear independence of the $\delta_i : i = 1, 2, \cdots, s$). □

Remark. – Curiously the conclusion fails (in G_2) when α is short.

2.4. We define a minimal orbital variety to be an irreducible component of $O \cap \mathfrak{n}$ where O is the unique non-zero nilpotent orbit of minimal dimension. Let V denote the closure of a minimal orbital variety. As noted in [BrJ, 6.2] these are exactly the $\overline{Be_\alpha} : \alpha \in \pi$

long. Taking $\pi' = \pi \setminus \{\alpha\}$ then $V := \overline{Be_\alpha}$ is $R := R_{\pi'}$ stable. Let $\{\gamma_i\}_{i=1}^s$ be defined as in 2.3 and set $V_0 = \overline{R(\sum_{i=1}^s e_{\gamma_i})} \subset m_{\pi'}$.

Proposition. – $V_0 \subset V$.

Proof. – It is enough to observe that $exp\left(\sum_{i=2}^s e_{\gamma_i-\gamma_1}\right)e_{\gamma_1} = \sum_{i=1}^s e_{\gamma_i}$ which hence belongs to Be_α. This follows from 2.3 which implies that $\gamma_j - \gamma_1 + \gamma_i$ is not a root for $i, j \geq 2$. □

2.5. Equality holds in 2.4 when dimensions coincide. The dimension of any given V above is known and that of V_0 may be calculated. The result is given in Table 1. Inspection gives the following curious result.

Proposition. – $V_0 = V$ *if and only if* $s \leq 2$.

2.6. When the conclusion of 2.5 holds one can regard V as being rather well understood. From a result of B. Kostant V_0 is a quadratic variety. (Actually as noted in [BrJ, 6.2] this result also holds for V without restriction on s). Again the formal character of the algebra $S[V_0]$ of regular functions on V_0, viewed as a quotient of $S(\mathfrak{m}^-)$ can be easily calculated. Indeed let $V(\mu)$ be the simple finite dimensional $\mathfrak{r}$ module with π' dominant weight μ. In the notation of 1.6 we then have the

Lemma.

$$ch\ S[V_0] = \Delta_{\pi'}\left(\prod_{i=1}^s (1 - e^{-\gamma_i})\right)^{-1}.$$

Proof. – Let $N_{\pi'}^-$ be the nilradical of the Borel subgroup $B_{\pi'}^-$ of R corresponding to the negative root vectors. Since the $\{\gamma_i\}_{i=1}^s$ are linearly independent we obtain $Stab_{B_{\pi'}^-}\left(\sum_{i=1}^s e_{\gamma_i}\right) = \bigcap_{i=1}^s Stab_{B_{\pi'}^-} e_{\gamma_i} \supset N_{\pi'}^-$. Consequently $S[V_0]$ has a $U(\mathfrak{r}) - U(\mathfrak{h})$ module structure and is generated by the $V(-\gamma_i) : i = 1, 2, \cdots, s$, whose direct sum we view as $m_{\pi'}^*$. Moreover the simple R submodules in $S[V_0]$ satisfy the Cartan multiplication rule $V(\mu)V(\nu) = V(\mu+\nu)$ forced by the right $U(\mathfrak{h})$ action. Thus $V(\mu)$ occurs in $S[V_0]$ if and only if $\mu = -\sum_{i=1}^s k_i\gamma_i : k_i \in \mathbb{N}$ and moreover with multiplicity one. Finally by 3.2 one has $ch\ V(\mu) = \Delta_{\pi'}e^\mu = \Delta_{\pi'} exp - \sum_{i=1}^s k_i\gamma_i$ and summing over all s-tuples gives the assertion of the lemma. □

2.7. In the tables below the notation follows Bourbaki [Bo, Planches I-X] except that ω_i replaces ϖ_i. We use also the convention that $A_0 = \emptyset$ and $\omega_0 = 0$. The orbital variety has closure $\overline{Be_\alpha}$. The set $\{\gamma_j\}$ is as defined in 2.3. In the column "Weight" each γ_j is viewed as a weight of $\mathfrak{r}_{\pi'}$. From it one may calculate the number of positive roots of $\mathfrak{r}_{\pi'}$ not commuting with some γ_j and this added to card $\{\gamma_j\}$ is just $dim\ V_0$. The values of $dim\ V$ may be obtained from [C, Chap. 13].

Type	$\{\alpha\}$	$\mathfrak{r}_{\pi'}$	$\{\gamma_j\}$	Weight	dim V_0	dim V
A_n	α_i	$A_{i-1} \times A'_{n-i}$	α_i	$\omega_{i-1} + \omega'_{n-i}$	n	n
B_n	α_1	B_{n-1}	α_1	ω_1	$2n-2$	$2n-2$
B_n	$\alpha_i : 1 < i < n$	$A_{i-1} \times B'_{n-i}$	$\left\{\begin{smallmatrix}\alpha_i \\ \varepsilon_{i-1}+\varepsilon_i\end{smallmatrix}\right.$	$\left.\begin{smallmatrix}\omega_{i-1}+\omega'_1 \\ \omega_{i-2}\end{smallmatrix}\right\}$	$2n-2$	$2n-2$
C_n	α_n	A_{n-1}	α_n	$2\omega_n$	n	n
D_n	α_1	D_{n-1}	α_1	ω_1	$2n-3$	$2n-3$
D_n	$\alpha_i : 1 < i < n-1$	$A_{i-1} \times D'_{n-i}$	$\left\{\begin{smallmatrix}\alpha_i \\ \varepsilon_{i-1}+\varepsilon_i\end{smallmatrix}\right.$	$\left.\begin{smallmatrix}\omega_{i-1}+\omega'_1 \\ \omega_{i-2}\end{smallmatrix}\right\}$	$2n-3$	$2n-3$
D_n	$\alpha_{n-1}(or\ \alpha_n)$	A_{n-1}	$\alpha_{n-1}(or\ \alpha_n)$	ω_{n-2}	$2n-3$	$2n-3$

Table 1.

3. Some identities of the Demazure operators

3.1. Let $P(\pi)$ (resp. $P^+(\pi)$) denote the lattice of weights (resp. dominant weights) and ρ the sum of the fundamental weights $\omega_\alpha : \alpha \in \pi$. For each $w \in W$, $\lambda \in P(\pi)$, set $w.\lambda = w(\lambda + \rho) - \rho$. As usual one may view $P(\pi)$ as a multiplicative group $e^{P(\pi)}$ with elements $e^\lambda : \lambda \in P(\pi)$. For each $\alpha \in \pi$ one has a Demazure operator Δ_α on $\mathbb{Z}e^{P(\pi)}$ defined through

$$\Delta_\alpha e^\lambda = (1 - e^{-\alpha})^{-1}(e^\lambda - e^{s_\alpha . \lambda})$$

which one easily checks lies in $\mathbb{Z}e^{P(\pi)}$. Each Δ_α is idempotent and together they satisfy the braid relations. The last fact was first checked purely combinatorially by M. Demazure [D, Thm. 1]. It also follows from the result noted below for which one now has a number of correct proofs. If $\alpha = \alpha_i \in \pi$ we set $s_{\alpha_i} = s_i$, $\omega_\alpha = \omega_i$ and $\Delta_{\alpha_i} = \Delta_i$.

3.2. For each $\pi' \subset \pi$, let $W_{\pi'}$ be the subgroup of W generated by the $s_\alpha : \alpha \in \pi'$ with $w_{\pi'}$ the unique longest element of $W_{\pi'}$. Fix a reduced decomposition $s_{i_1} s_{i_2} \cdots s_{i_m} : m = \ell(w)$ of $w \in W$ and set $\Delta_w = \Delta_{i_1} \Delta_{i_2} \cdots \Delta_{i_m}$. For each $\lambda \in P^+(\pi)$ let $V(\lambda)$ denote the simple (finite dimensional) $U(\mathfrak{g})$ module with highest weight λ. For each $w \in W$ one calls $V_w(\lambda) := U(\mathfrak{b})V(\lambda)_{w\lambda}$ the Demazure module for the weight $w\lambda \in P(\pi)$. In particular $V_{w_{\pi'}}(\lambda)$ is the simple $U(\mathfrak{r}_{\pi'})$ module with highest weight λ.

THEOREM. – *ch* $V_w(\lambda) = \Delta_w e^\lambda$.

Remarks. – Except [J6] for very dominant λ the only proofs are rather difficult involving positive characteristic or quantum groups. However we shall only need the cases $w = w_{\pi'}$ and for these using the Weyl character formula the result can be deduced purely combinatorically or from the very dominant case. For a review of these questions the reader may consult [J11, 4.5.7, 6.5.2]. We write $\Delta_{\pi'} = \Delta_{w_{\pi'}}$ for all $\pi' \subset \pi$. Demazure had himself considered the use of $\Delta_{\pi'}$ as being computationally advantageous to the Weyl character formula and in fact our present computations exhibit this rather well.

Type	$\{\alpha\}$	$\mathfrak{r}_{\pi'}$	$\{\gamma_i\}$	Weight	dim V_0	dim V
E_6	$\alpha_1 (or\ \alpha_6)$	D_5	$\alpha_1 (or\ \alpha_6)$	ω_4	11	11
E_6	$\alpha_3 (or\ \alpha_5)$	$A_1 \times A_4$	$\left\{\begin{matrix}\alpha_3\ (or\ \alpha_5)\\ \substack{12210\\1}\ (or\ \substack{01221\\1})\end{matrix}\right.$	$\left.\begin{matrix}\omega_1+\omega_2'\\ \omega_4'\end{matrix}\right\}$	11	11
E_6	α_2	A_5	$\left\{\begin{matrix}\alpha_2\\ \substack{12321\\2}\end{matrix}\right.$	$\left.\begin{matrix}\omega_3\\ 0\end{matrix}\right\}$	11	11
E_6	α_4	$A_2 \times A_1' \times A_2''$	$\left\{\begin{matrix}\alpha_4\\ \substack{01210\\1}\\ \substack{12321\\1}\end{matrix}\right.$	$\left.\begin{matrix}\omega_2+\omega_1'+\omega_1''\\ \omega_1+\omega_2''\\ \omega_1'\end{matrix}\right\}$	10	11
F_4	α_1	C_3	$\left\{\begin{matrix}\alpha_1\\ 2342\end{matrix}\right.$	$\left.\begin{matrix}\omega_1\\ 0\end{matrix}\right\}$	8	8
F_4	α_2	$A_1 \times A_2'$	$\left\{\begin{matrix}\alpha_2\\ 1220\\ 1342\end{matrix}\right.$	$\left.\begin{matrix}\omega_1+\omega_1'\\ \omega_2'\\ \omega_1\end{matrix}\right\}$	7	8
E_7	α_1	D_6	$\left\{\begin{matrix}\alpha_1\\ \substack{234321\\2}\end{matrix}\right.$	$\left.\begin{matrix}\omega_6\\ 0\end{matrix}\right\}$	17	17
E_7	α_2	A_6	$\left\{\begin{matrix}\alpha_2\\ \substack{123210\\2}\end{matrix}\right.$	$\left.\begin{matrix}\omega_3\\ \omega_6\end{matrix}\right\}$	17	17
E_7	α_3	$A_1 \times A_5'$	$\left\{\begin{matrix}\alpha_3\\ \substack{122100\\1}\\ \substack{134321\\2}\end{matrix}\right.$	$\left.\begin{matrix}\omega_1+\omega_2'\\ \omega_4'\\ \omega_1\end{matrix}\right\}$	16	17

Table 2.

Type	$\{\alpha\}$	$\mathfrak{r}_{\pi'}$	$\{\gamma_j\}$	Weight	dim V_0	dim V
E_7	α_4	$A_2 \times A_1' \times A_3''$	$\left\{\begin{matrix}\alpha_4\\ \substack{012100\\1}\\ \substack{123210\\1}\\ \substack{124321\\2}\end{matrix}\right.$	$\left.\begin{matrix}\omega_2+\omega_1'+\omega_1''\\ \omega_1+\omega_2''\\ \omega_1'+\omega_3''\\ \omega_2\end{matrix}\right\}$	14	17
E_7	α_5	$A_4 \times A_2'$	$\left\{\begin{matrix}\alpha_5\\ \substack{012210\\1}\\ \substack{123321\\1}\end{matrix}\right.$	$\left.\begin{matrix}\omega_3+\omega_1'\\ \omega_1+\omega_2'\\ \omega_4\end{matrix}\right\}$	15	17
E_7	α_6	$D_5 \times A_1$	$\left\{\begin{matrix}\alpha_6\\ \substack{012221\\1}\end{matrix}\right.$	$\left.\begin{matrix}\omega_5+\omega_1'\\ \omega_1\end{matrix}\right\}$	17	17
E_7	α_7	E_6	α_7	ω_6	17	17
E_8	α_1	D_7	$\left\{\begin{matrix}\alpha_1\\ \substack{2343210\\2}\end{matrix}\right.$	$\left.\begin{matrix}\omega_7\\ \omega_1\end{matrix}\right\}$	29	29
E_8	α_2	A_7	$\left\{\begin{matrix}\alpha_2\\ \substack{1232100\\2}\\ \substack{1354321\\3}\end{matrix}\right.$	$\left.\begin{matrix}\omega_3\\ \omega_6\\ \omega_1\end{matrix}\right\}$	26	29
E_8	α_3	$A_1 \times A_6'$	$\left\{\begin{matrix}\alpha_3\\ \substack{1221000\\1}\\ \substack{1343210\\2}\\ \substack{2454321\\2}\end{matrix}\right.$	$\left.\begin{matrix}\omega_1+\omega_2'\\ \omega_4'\\ \omega_1+\omega_6'\\ \omega_1'\end{matrix}\right\}$	24	29

Table 2 (*continued*).

Type	$\{\alpha\}$	$\mathfrak{r}_{\pi'}$	$\{\gamma_j\}$	Weight	dim V_0	dim V
E_8	α_4	$A_2 \times A_1' \times A_4''$	α_4 $01\underset{1}{2}1000$ $12\underset{1}{3}2100$ $12\underset{2}{4}3210$ $13\underset{2}{5}4321$ $24\underset{3}{6}4321$	$\omega_2+\omega_1'+\omega_1''$ $\omega_1+\omega_2''$ $\omega_1'+\omega_3''$ ω_4'' $\omega_1+\omega_1'$ ω_1''	20	29
E_8	α_5	$A_4 \times A_3'$	α_5 $01\underset{1}{2}2100$ $12\underset{1}{3}3210$ $12\underset{2}{4}4321$ $24\underset{3}{6}5321$	$\omega_3+\omega_1'$ $\omega_1+\omega_2'$ $\omega_4+\omega_3'$ ω_2 ω_1'	21	29
E_8	α_6	$D_5 \times A_2'$	α_6 $01\underset{1}{2}2210$ $12\underset{1}{3}3321$ $24\underset{3}{6}5421$	$\omega_5+\omega_1'$ $\omega_1+\omega_2'$ ω_4 ω_1'	24	29
E_8	α_7	$E_6 \times A_1'$	α_7 $01\underset{1}{2}2221$ $24\underset{3}{6}5431$	$\omega_6+\omega_1'$ ω_1 ω_1'	28	29
E_8	α_8	E_7	α_8 $24\underset{3}{6}5432$	ω_7 0	29	29
G_2	α_2	A_1	α_2 $3\alpha_1+2\alpha_2$	$3\omega_1$ 0	3	3

Table 2 (*continued*).

3.3. As is well-known

(1) $\Delta_i e^\lambda = -\Delta_i e^{s_i.\lambda}, \quad \forall\ \lambda \in P(\pi),$

and one easily checks that

(2) $(1+s_i)\left(\frac{e^\lambda}{1-e^{-\alpha_i}}\right) = \Delta_i e^\lambda, \quad \forall\ \lambda \in P(\pi).$

Departing from the usual tradition we shall view $F \mapsto \Delta_\alpha F = (1-e^{-\alpha})^{-1}(F - s_\alpha.F)$ as an operator on $Fract\ \mathbb{Z}e^{\mathfrak{h}^*}$, though all denominators will be products of the $(1-e^{-\alpha}) : \alpha \in \Delta^+$ and just viewed as infinite sums, that is belonging to a Krull completion of $\mathbb{Z}\ e^{\mathfrak{h}^*}$. Notice that $\mathbb{Z}e^{\mathfrak{h}^*}$ is itself not Δ_α stable. It seems that there are some remarkable new combinatorial identities the extent of which we do not yet understand. The simplest is the following. Suppose $\alpha \in \pi,\ \beta \in \Delta^+$ satisfy $(\alpha^\vee, \beta) = -1$. Then

$$\Delta_\alpha(1-e^{-\beta})^{-1} = (1-e^{-\beta})^{-1}(1-e^{-(\alpha+\beta)})^{-1}. \tag{3}$$

This is generalized below.

3.4. Let $\{1, 2, \cdots, n\}$ be a subset of the index set of π so that $\pi' = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is of type A_n. Set $\beta_i = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n$ and $s_{n+1} = 1$.

LEMMA. – *Assume $F \in Fract\ \mathbb{Z}e^{\mathfrak{h}^*}$ is $s_1, s_2, \cdots, s_{n-1}$ invariant. Then*

$$\Delta_{\pi'}F = \Delta_1\Delta_2\cdots\Delta_n F = \sum_{i=1}^{n+1} s_i s_{i+1}\cdots s_{n+1}\left(\frac{F}{\prod_{j=1}^n(1-e^{-\beta_j})}\right).$$

Proof. – Set $\pi'' = \pi' \setminus \{\alpha_n\}$. Then $w_{\pi'} = s_1 s_2 \cdots s_n w_{\pi''}$ as is well-known. This gives the first equality. The second equality is proved by induction on n. For $n = 1$ it is just 3.3(2). Then by the induction hypothesis

$$\begin{aligned}\Delta_1(\Delta_2\cdots\Delta_n F) &= \Delta_1\left(\sum_{i=2}^{n+1} s_i\cdots s_{n+1}\left(\frac{F}{\prod_{j=2}^n(1-e^{-\beta_j})}\right)\right)\\ &= (1-e^{-\alpha_1})^{-1}s_2\cdots s_{n+1}\left(\frac{F}{\prod_{j=2}^n(1-e^{-\beta_j})}\right)\\ &\quad + s_1(1-e^{-\alpha_1})^{-1}s_2\cdots s_{n+1}\ \frac{F}{(\prod_{j=2}^n(1-e^{-\beta_j}))}\\ &\quad + \sum_{i=3}^{n+1} s_i\cdots s_{n+1}\ \frac{F}{\prod_{j=3}^n(1-e^{-\beta_j})}\ \Delta_1(1-e^{-\beta_2})^{-1},\end{aligned}$$

which by 3.3(3) gives the required sum. □

3.5. Retain the notation and hypotheses of 3.4. Suppose $a_n \in Fract\ \mathbb{Z}e^{\mathfrak{h}^*}$ satisfies $s_i a_n = a_n$ for all $i < n$. Write $a_{n+1} = 0$ and define $a_{n-1}, a_{n-2}, \cdots$, recursively through $a_{i-1} + a_{i+1} = -(1+s_i)a_i$, for all $n \geq i > 1$.

LEMMA. – *For all* $i \geq 1$ *one has*

$$a_i = (-1)^{n-i} \sum_{j=i+1}^{n+1} (s_j s_{j+1} \cdots s_{n+1}) a_n.$$

Proof. – By induction. It is clear for $i = n, n-1$. Then for $i < n$ one obtains

$$\begin{aligned} a_{i-1} &= -(1+s_i)a_i - a_{i+1} \\ &= (-1)^{n-(i-1)}(1+s_i)s_{i+1}\cdots s_{n+1}a_n \\ &\quad + (-1)^{n-(i-1)}\left[2\sum_{j=i+2}^{n+1} s_j \cdots s_{n+1}a_n - \sum_{j=i+2}^{n+1} s_j \cdots s_{n+1}a_n\right] \\ &= (-1)^{n-(i-1)}\sum_{j=i}^{n+1}(s_j \cdots s_{n+1})a_n, \end{aligned}$$

as required. □

3.6. Retain the above notation; but now just taking $\pi = \pi'$ so that $\mathfrak{g} \cong \mathfrak{sl}(n+1)$. Set $\pi_i = \pi \setminus \{\alpha_i\}$. Let $\lambda_n \in P(\pi)$ be a multiple of ω_n and set

$$a_n = \Delta_{\pi_n}\left(\frac{e^{\lambda_n}}{1 - e^{-\alpha_n}}\right).$$

Expansion of Δ_{π_n} and repeated application of 3.3(3) gives

$$a_n = e^{\lambda_n}\Delta_1\Delta_2\cdots\Delta_{n-1}(1 - e^{-\alpha_n})^{-1} = e^{\lambda_n}\prod_{i=1}^{n}(1 - e^{-\beta_i})^{-1}.$$

This may be recognized as the character of the generalized Verma module induced from the one dimensional $\mathfrak{p}_{\pi_n}$ module defined by λ_n, noting that the $\{\beta_i\}_{i=1}^n$ are the roots of the nilradical of $\mathfrak{p}_{\pi_n}$.

PROPOSITION. – *Take* $i \in \{1, 2, \cdots, n-1\}$ *and define* λ_i *inductively through* $\lambda_i = s_{i+1} \cdot \lambda_{i+1}$. *Then*

$$a_i = \Delta_{\pi_i}\left(\frac{e^{\lambda_i}}{1 - e^{-\alpha_i}}\right).$$

Proof. – For $1 < i < n$ the Dynkin diagram of π_i has two connected components π_i', π_i'' where we take π_i'' to contain α_n. Set also $\pi_1' = \pi_n'' = \emptyset$ and $\pi_n' = \pi_n$, $\pi_1'' = \pi_1$. Then as in the first part of 3.4 and making successive use of 3.3(3) we obtain

$$\begin{aligned} \Delta_{\pi_n}\left(\frac{e^{\lambda_n}}{1 - e^{-\alpha_n}}\right) &= \Delta_1\Delta_2\cdots\Delta_{n-1}\left(\frac{e^{\lambda_n}}{1 - e^{-\alpha_n}}\right) \\ &= \Delta_{\pi_i'}\frac{e^{\lambda_n}}{\prod_{k=i}^{n}(1 - e^{-\beta_k})}, \quad \textit{for all} \quad i = 1, 2, \cdots, n-1. \end{aligned}$$

Substitution from 3.5 gives

$$\begin{aligned} a_i &= (-1)^{n-i} \sum_{j=i+1}^{n+1} (s_j \cdots s_{n+1}) \Delta_{\pi_n} \left(\frac{e^{\lambda_n}}{1-e^{-\alpha_n}} \right) \\ &= (-1)^{n-i} \Delta_{\pi_i'} \left(\sum_{j=i+1}^{n+1} s_j \cdots s_{n+1} \frac{F}{\prod_{k=i+1}^{n}(1-e^{-\beta_k})} \right), \end{aligned}$$

where $F = \frac{e^{\lambda_n}}{1-e^{-\beta_i}}$. This is $s_{i+1}, \cdots, s_{n-1}$ invariant so by 3.4 we obtain

$$a_i = (-1)^{n-i} \Delta_{\pi_i'} \Delta_{\pi_i''} F.$$

Yet for any $\alpha \in \pi_i''$ we can write $\Delta_{\pi_i''}$ as a product of the $\Delta_{\alpha_j} : \alpha_j \in \pi_i''$ ending in Δ_α. Then successive use of 3.3(1) gives

$$\begin{aligned} a_i &= \Delta_{\pi_i'} \Delta_{\pi_i''} (s_{i+1} \cdots s_n).F \\ &= \Delta_{\pi_i} \left(\frac{e^{\lambda_i}}{1-e^{-\alpha_i}} \right), \end{aligned}$$

as required. □

3.7. A rather more complicated example arises when π is of type $D_{n+2} : n \geq 2$. In this case we take $\pi' = \pi \setminus \{\alpha_{n+1}, \alpha_{n+2}\}$ and set $\pi_i = \pi \setminus \{\alpha_i\}$ for $i \in \{1, 2, \cdots, n\}$. We construct $a_i \in Fract\ \mathbb{Z}e^{P(\pi)}$ as in 3.5 but taking a reverse ordering of $\{1, 2, \cdots, n\}$. Thus a_1 will be given and a_i is obtained inductively through the same recurrence relation as before but with $1 \leq i < n$ and $a_0 = 0$. Let $\lambda_1 \in P(\pi)$ be a multiple of ω_1 and set

$$a_1 = \Delta_{\pi_1} \left(\frac{e^{\lambda_1}}{1-e^{-\alpha_1}} \right).$$

Despite the similarity to our previous expression this is not the character of a generalized Verma module. However there is a choice of λ_1 for which it does have an interpretation as a formal character of a (simple) highest weight module (*see* Thm. 4.6).

It is convenient to describe the positive roots through the Bourbaki notation. Then the set of positive roots is given by $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j \big| 1 \leq i < j \leq n+2\}$. Here $\varepsilon_i - \varepsilon_{i+1} = \alpha_i$ for $1 \leq i \leq n+1$ and $\varepsilon_{n+1} + \varepsilon_{n+2} = \alpha_{n+2}$. Then $\varepsilon_{i-1} + \varepsilon_i = \alpha_{i-1} + 2\alpha_i + \cdots + 2\alpha_n + \alpha_{n+1} + \alpha_{n+2}$ for $2 \leq i \leq n$. Set $s_0 = 1$.

PROPOSITION. – *Take $i \in \{2, 3, \cdots, n\}$ and define λ_i inductively through $\lambda_i = s_{i-1}.\lambda_{i-1}$. Then*

$$a_i = \Delta_{\pi_i} \frac{e^{\lambda_i}}{(1-e^{-\alpha_i})(1-e^{-(\varepsilon_{i-1}+\varepsilon_i)})}.$$

Proof. – For $2 \leq i \leq j \leq n+1$, set

$$\Delta_{j,i} = \Delta_j \Delta_{j+1} \cdots \Delta_n \Delta_{n+1} \Delta_{n+2} \Delta_n \Delta_{n-1} \cdots \Delta_i$$

and

$$E_{j,i} = \Delta_{j,i}(1-e^{-(\varepsilon_{i-1}-\varepsilon_i)})^{-1}, \quad E_i = E_{i,i}.$$

We claim for $2 \leq i < n$ that

(1) $$E_i = [(1 - e^{-(\varepsilon_{i-1} - \varepsilon_i)})(1 - e^{-(\varepsilon_{i-1} + \varepsilon_i)})]^{-1} \; s_{i-1}(E_{i+1}).$$

First by successive application of 3.3(3) we obtain

$$E_{n+1,i} = (D^-_{i,n})^{-1} \Delta_{n+2} [(1 - e^{-(\varepsilon_{i-1} - \varepsilon_{n+1})})(1 - e^{-(\varepsilon_{i-1} - \varepsilon_{n+2})})]^{-1}$$

where

$$D^{\pm}_{i,n} = \prod_{j=i}^{n} (1 - e^{-(\varepsilon_{i-1} \pm \varepsilon_j)}).$$

Evaluating the second factor appearing in $E_{n+1,i}$ gives

$$\begin{aligned} E_{n+1,i} &= (D^-_{i,n+2})^{-1} (1 - e^{-(\varepsilon_{n+1} + \varepsilon_{n+2})})^{-1} [(1 - e^{-(\varepsilon_{i-1} + \varepsilon_{n+1})})(1 - e^{-(\varepsilon_{i-1} + \varepsilon_{n+2})}) \\ &\quad - e^{-(\varepsilon_{n+1} + \varepsilon_{n+2})} (1 - e^{-(\varepsilon_{i-1} - \varepsilon_{n+1})})(1 - e^{-(\varepsilon_{i-1} - \varepsilon_{n+2})})] \\ &= (D^-_{i,n+2})^{-1} [(1 - e^{-(\varepsilon_{i-1} + \varepsilon_{n+1})})(1 - e^{-(\varepsilon_{i-1} + \varepsilon_{n+2})})]^{-1} (1 - e^{-2\varepsilon_{i-1}}). \end{aligned}$$

In this $D^-_{i,n+2}$ is $s_i, s_{i+1}, \cdots, s_n$ invariant and of the remaining factors only the first fails to be s_n invariant. Successive application of 3.3(3) then gives $E_{j,i} = (1 - e^{-(\varepsilon_{i-1} + \varepsilon_j)}))^{-1} \; E_{j+1,i}$ for $i \leq j \leq n$ and so

$$E_i = (D^-_{i,n+2} D^+_{i,n+2})^{-1} (1 - e^{-2\varepsilon_{i-1}}).$$

Since $s_{i-1}\varepsilon_{i-1} = \varepsilon_i$ the assertion of (1) easily follows.

As in 3.6 the Dynkin diagram of π_i for $1 < i < n$ has two connected components similarly designated π'_i, π''_i. Set also $\pi'_1 = \emptyset$, $\pi''_1 = \pi_1$ and $\pi'_n = \{\alpha_1, \alpha_2, \cdots, \alpha_{n-1}\}$, $\pi''_n = \{\alpha_{n+1}, \alpha_{n+2}\}$. From an appropriate reduced decomposition of $w_{\pi''_i}$ it follows from 3.2 that

$$\Delta_{\pi''_i} = \Delta_{i+1,i+1} \Delta_{\pi''_{i+1}}, \quad \textit{for all} \quad i \in \{1, 2, \cdots, n-1\}$$

and so

$$E_{i+1} = \Delta_{\pi''_i} (1 - e^{-\alpha_i})^{-1}.$$

Consequently (1) gives for $1 \leq j < n$ that

(2) $$\Delta_{\pi_1} (1 - e^{-\alpha_1})^{-1} = \left[\prod_{i=1}^{j} (1 - e^{-(\varepsilon_1 - \varepsilon_{i+1})})(1 - e^{-(\varepsilon_1 + \varepsilon_{i+1})}) \right]^{-1} s_1 s_2 \cdots s_j \Delta_{\pi''_{j+1}} (1 - e^{-\alpha_{j+1}})^{-1}.$$

Taking $s_0 = 1$ we obtain from 3.5 that

$$\begin{aligned} a_{j+1} &= (-1)^j \sum_{k=0}^{j} s_k s_{k-1} \cdots s_0 a_1 \\ &= (-1)^j \sum_{k=0}^{j} s_k s_{k-1} \cdots s_0 \left(\frac{F}{\prod_{\ell=1}^{j} (1 - e^{-\beta_j})} \right) \end{aligned}$$

where $\beta_i = \varepsilon_1 - \varepsilon_{i+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_i$ and

$$F = e^{\lambda_1}(D^+_{2,j+1})^{-1}\Delta_{\pi''_{j+1}}(1 - e^{-\beta_{j+1}})^{-1}$$

which is $s_2, s_3, \cdots, s_j$ invariant (for $1 \leq j < n$). We conclude that 3.4 applies to give

$$a_{j+1} = (-1)^j \Delta_{\pi'_{j+1}} F = \Delta_{\pi'_{j+1}}(s_j s_{j-1} \cdots s_1).F \tag{3}$$

where the last equality follows (as in 3.6) by successive use of 3.3(1). Now

$$\begin{aligned}
&s_j s_{j-1} \cdots s_1 \left[\prod_{i=1}^{j}(1 - e^{-(\varepsilon_1+\varepsilon_{i+1})})\right]^{-1} \\
&= s_j s_{j-1} \cdots s_2 \left[(1 - e^{-(\varepsilon_1+\varepsilon_2)}) \prod_{i=2}^{j}(1 - e^{-(\varepsilon_2+\varepsilon_{i+1})}\right]^{-1} \\
&= \left[\prod_{i=1}^{j}(1 - e^{-(\varepsilon_i+\varepsilon_{j+1})}\right]^{-1} \\
&= \Delta_1\Delta_2 \cdots \Delta_{j-1}(1 - e^{-(\varepsilon_j+\varepsilon_{j+1})})^{-1} := H.
\end{aligned}$$

Thus

$$\begin{aligned}
(s_j s_{j-1} \cdots s_1).F &= e^{\lambda_{j+1}} H \Delta_{\pi''_{j+1}}(1 - e^{-\alpha_{j+1}})^{-1} \\
&= \Delta_1\Delta_2 \cdots \Delta_{j-1}[e^{\lambda_{j+1}}(1 - e^{-(\varepsilon_j+\varepsilon_{j+1})})^{-1}\Delta_{\pi''_{j+1}}(1 - e^{-\alpha_{j+1}})^{-1}] \\
&= \Delta_1\Delta_2 \cdots \Delta_{j-1}\Delta_{\pi''_{j+1}}\left(\frac{e^{\lambda_{j+1}}}{(1 - e^{-\alpha_{j+1}})(1 - e^{-(\varepsilon_j+\varepsilon_{j+1})})}\right).
\end{aligned}$$

Combined with (3) and noting that $\Delta_{\pi'_{j+1}}\Delta_i = \Delta_{\pi'_{j+1}}$ for all $1 \leq i \leq j$, the required formula for a_{j+1} is obtained. □

3.8. We now consider π of type $B_{n+1} : n \geq 2$. The resulting calculation will be very similar even though the conclusion concerning the possibility of quantizing an orbital variety is very different, a phenomenon which suggests a possible advantage of our present approach. We take $\pi' = \pi \setminus \{\alpha_{n+1}\}$ and set $\pi_i = \pi \setminus \{\alpha_i\}$ for $i \in \{1, 2, \cdots, n\}$. We construct the a_i in 3.7. Let $\lambda_1 \in P(\pi)$ be a multiple of ω_1 and set

$$a_1 = \Delta_{\pi_1}\left(\frac{e^{\lambda_1}}{1 - e^{-\alpha_1}}\right).$$

Ultimately the difference in types B_{n+1}, D_{n+2} arises because in the former case it is λ_1 being a certain *half*-integer multiple of ω_1 which gives the above expression the meaning of a formal character of a (simple) highest weight module. It is notable that in the non-integral case the denominator of Δ_α need not cancel.

As in 3.7 we use the Bourbaki notation so that the set of positive roots is $\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, \varepsilon_k \mid 1 \leq i < j \leq n+1,\ 1 \leq k \leq n+1\}$. Here $\varepsilon_i - \varepsilon_{i+1} = \alpha_i$ for $1 \leq i \leq n$

and $\varepsilon_{n+1} = \alpha_{n+1}$ which is the unique short root. Then $\varepsilon_{i-1}+\varepsilon_i = \alpha_{i-1}+2\alpha_i+\cdots+2\alpha_{n+1}$ for $2 \leq i \leq n+1$.

PROPOSITION. – *Take* $\{i = 2, 3, \cdots, n\}$ *and define* λ_i *inductively through* $\lambda_i = s_{i-1}.\lambda_{i-1}$. *Then*

$$a_i = \Delta_{\pi_i} \frac{e^{\lambda_i}}{(1-e^{-\alpha_i})(1-e^{-(\varepsilon_{i-1}+\varepsilon_i)})}.$$

Proof. – The calculation starts slightly differently. For $2 \leq i \leq j \leq n+1$ set

$$\Delta_{j,i} = \Delta_j \Delta_{j+1} \cdots \Delta_n \Delta_{n+1} \Delta_n \cdots \Delta_i$$

with respect to which $E_{j,i}$ and E_i are defined as in 3.7. We claim for $2 \leq i < n$ that the conclusion of 3.7(1) holds though the calculation which leads to it is slightly different. First by successive application of 3.3(3) we obtain

$$E_{n+1,i} = (D^-_{i,n})^{-1} \Delta_{n+1} (1 - e^{-(\varepsilon_{i-1}-\varepsilon_{n+1})})^{-1}$$

where $D^{\pm}_{i,n}$ are defined as in 3.7. The conclusion of 3.3(3) does not apply to evaluating the second factor in $E_{n+1,i}$ because α_{n+1} is a short root. Instead we obtain

$$E_{n+1,i} = (D^-_{i,n+1})^{-1} (1 - e^{-(\varepsilon_{i-1}+\varepsilon_{n+1})})^{-1} (1 + e^{-\varepsilon_{i-1}}).$$

As in 3.7 these three factors are $s_i, s_{i+1}, \cdots, s_n$ invariant, except the central one which fails to be s_n invariant. As in 3.7 this gives for $i \leq j \leq n$ that

$$E_i = (D^-_{i,n+1} D^+_{i,n+1})^{-1} (1 + e^{-\varepsilon_{i-1}})$$

and as before 3.7(1) results.

As in 3.7 the Dynkin diagram of π_i for $1 < i < n+1$ has two connected components similarly designated π'_i, π''_i. Set also $\pi'_1 = \emptyset$, $\pi''_1 = \pi_1$. From an appropriate reduced decomposition of $w_{\pi''_i}$ it follows that

$$\Delta_{\pi''_i} = \Delta_{i+1,i+1} \Delta_{\pi''_{i+1}}, \quad for\ all \quad i \in \{1, 2, \cdots, n-1\},$$

and so

$$E_{i+1} = \Delta_{\pi''_i} (1 - e^{-\alpha_i})^{-1}.$$

From this and (1) it follows that 3.7(2) is also valid here. Then the remaining calculation is exactly as before because it only involves reflections and Demazure operators from π' which are the same in both cases. (Of course $\Delta_{\pi''_{j+1}}$ is different but only the fact that it commutes with $\Delta_i : 1 \leq i < j$ is used). $\square$

3.9. The corresponding calculation in types C_n, G_2 are unnecessary because there is just one minimal orbital variety. In types E_6, F_4 (which also exhibit some similarities) we succeeded in calculating all the required expressions except one, though it is obvious what the answer should be in this case. The difficulty resides in establishing an invariance property of the numerators which appear, similar to that exhibited in 3.7, 3.8 and which led to equation (1) of 3.7.

This numerator invariance becomes increasingly difficult to establish in types E_7 and E_8. Moreover for the latter even the denominators become increasingly difficult to control. Indeed we expect formulae similar to those of 3.6-3.8 but with as many as 15 terms in the denominator.

4. Character formulae for some simple highest weight modules

4.1. For each $\lambda \in \mathfrak{h}^*$ let $M(\lambda)$ denote the Verma module with highest weight λ and $L(\lambda)$ its unique simple quotient. Our method for computing the $ch\ L(\lambda)$ associated to the minimal orbit is based on the use of the Enright functors $C_\alpha : \alpha \in \pi_\lambda$ defined on the $\mathcal{O}_\Lambda$-category as in [J8, 3.2]. Here $\Lambda = \lambda + P(\pi)$ and π_λ is a choice of simple roots for the integral root system defined by λ. Let W_λ be the subgroup of W generated by the $s_\alpha : \alpha \in \pi_\lambda$. We shall mainly be concerned with the case $\lambda = 0$, for which $\pi_\lambda = \pi$ and $W_\lambda = W$. Let δ denote $\mathcal{O}_\Lambda$-duality functor. If the canonical map $M \to C_\alpha M$ (resp. $\delta M \to C_\alpha \delta M$) is injective one calls M α-free (resp. α-cofree). If $\alpha \in \pi$ then on α-free modules C_α coincides with Deodhar's more elementary definition described for example in [J11, 4.4.12]. If $L \in Ob\mathcal{O}_\Lambda$ is simple then either $C_\alpha L = 0$ (if $\alpha \in \pi$ this equality corresponds to L being $\mathfrak{r}_\alpha$ locally finite) or L is both α-free and α-cofree.

LEMMA. – *Take $L \in Ob\mathcal{O}_\Lambda$ and $\alpha \in \pi_\lambda$. If $C_\alpha L \neq 0$, then*

$$ch(C_\alpha L)/L = -(1 + s_\alpha) ch\ L.$$

Proof. – This is immediate from [J8, 3.2.3] though some explanation is in order. There we defined the Verma module $M(\lambda)$ to have highest weight $\lambda - \rho$ and an action of W_λ on the Grothendieck group of $\mathcal{O}_\Lambda$ which in the present convention gives $w[M(\lambda)] = [M(w.\lambda)]$, for all $w \in W_\lambda$. Yet one checks that $ch\ M(s_\alpha.\lambda) = -s_\alpha\ ch\ M(\lambda)$ and so this gives the action $(w, ch\ M) \mapsto (-1)^{\ell(w)}\ w(ch\ M)$ of W_λ on the set of formal characters $ch\ M : M \in Ob\mathcal{O}_\Lambda$. Our assertion now follows from [J8, 3.2.3(ii)]. □

4.2. In general $C_\alpha L/L$ above is a semisimple module whose precise form is given by the Kazhdan-Lusztig polynomials (*see* [J8, 3.2.17] for example, the first statement follows from a result of Vogan and the truth of Kazhdan-Lusztig conjectures). From the definition of $C_\alpha L$ it follows that $Ann_{U(\mathfrak{g})} C_\alpha L \subset Ann_{U(\mathfrak{g})} L$. Consequently if $Ann_{U(\mathfrak{g})} L$ is a maximal ideal then it is also the annihilator of each simple factor of $C_\alpha L$. This introduces a significant simplification which is further enhanced (*see* 4.3) if any (and hence every) element of the unique dense nilpotent orbit [J5, 3.11] in the zero variety of $gr\ Ann_{U(\mathfrak{g})} L$ has a connected centralizer. Finally $C_\alpha^2 = C_\alpha$ and so $C_\alpha(C_\alpha L/L) = 0$ (that is $C_\alpha L/L$ is $\mathfrak{r}_\alpha$ locally finite if $\alpha \in \pi$). As we shall see these facts make 4.1 easy to use in the case when the above nilpotent orbit is the minimal one.

4.3. Assume that $\mathfrak{g}$ is simple. If $\mathfrak{g}$ is not of type A_n then [J1] there is a unique completely prime ideal J_0 of $U(\mathfrak{g})$ whose associated variety is the closure of the minimal non-zero nilpotent orbit O_0. In type A_n there is a one parameter family of such ideals obtained by inducing an ideal of codimension 1 of $U(\mathfrak{p}_{\pi'})$ with $\pi' = \pi \setminus \{\alpha_1\}(or\ \ \pi \setminus \{\alpha_n\})$ in the Bourbaki notation. We apply 4.1 to the case when L satisfies $J_0 = Ann\ L$. First we must describe all such occurrences, that is determine the set $\Lambda_0 = \{\lambda \in \mathfrak{h}^* \big| Ann\ L(\lambda) = J_0\}$. Assume $\mathfrak{g}$ not of type A_n. One member of Λ_0 which we shall denote by λ_0 is given in [J1, Table], where we note that $Ann\ L(\lambda_0) = J_0$ holds there because J_0 is maximal. Through the action of the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ and the Harish-Chandra isomorphism all other members of Λ_0 take the form $w.\lambda_0 : w \in W$. Set $\mathcal{W}_0 := \{w \in W \big| w.\lambda_0 \in \Lambda_0\}$. Define an order relation $\geq$ on $\mathfrak{h}^*$ by $\lambda \geq \mu$ if $\lambda - \mu \in \mathbb{N}\pi$. Call $\lambda + \rho$ dominant (resp. regular) if $w.\lambda \geq \lambda$ implies equality (resp. $w.\lambda = \lambda$ implies $w = e$). We remark that $\lambda_0 + \rho$ was chosen to be always dominant. When $\lambda_0 + \rho$ is also regular (and then non-integral) which occurs exactly when $\mathfrak{g}$ is not simply-laced, the set $\mathcal{W}_0$ is particularly easy to determine. Indeed suppose that $\lambda + \rho$ is dominant and regular. One may recall [Ja2, Satz. 1.3] that $W_\lambda = \{w \in W \big| w\lambda - \lambda \in \mathbb{Z}\pi\}$. Since $\lambda + \rho$ is dominant and regular, the left cell of W_λ corresponding to the maximal ideal $J(\lambda) := Ann_{U(\mathfrak{g})}L(\lambda)$ is reduced to the identity element. Consequently $J(\lambda) = Ann_{U(\mathfrak{g})}L(w.\lambda)$ if and only if $w \in W$ belongs to the coset D_λ of the identity. One may further characterize D_λ as the set of $w \in W$ for which $w\alpha > 0$ given $\alpha > 0$ and $(\alpha^\vee, \lambda) \in \mathbb{Z}$.

We remark that if $\lambda_0+\rho$ is not regular, then $\lambda_0+\rho$ is integral, and non-zero except on the unique simple root α_0 with 3 neighbours. Then the coherent family of primitive ideals of which J_0 is a member are "almost maximal". More precisely the $J(s_{\alpha_0}.\lambda) : \lambda+\rho \in P^+(\pi)$ are almost maximal (contained only in the maximal ideal $J(\lambda)$) and J_0 is obtained by translation (in the sense of [BJ, Sect.2]) to the α_0 wall. It follows that the Goldie rank polynomial determined by this family (via [J3, Thm. 5.1]) generates the tensor product of the standard and sign representation of W. Thus the two-sided cell of W corresponding to O_0 breaks into just $|\pi|$ left cells. Since each $x \in \Omega_0$ has a connected centralizer [C, Chap.13] each such left cell is irreducible [L1, Chap. 12] carrying therefore just the Goldie rank representation and in particular has cardinality $|\pi|$. In particular $|\mathcal{W}_0| = |\pi|$. Moreover there is a natural bijective correspondence between $\mathcal{W}_0$ and π described as follows. First the $J(w^{-1}.\lambda) : w \in \mathcal{W}_0,\ \lambda + \rho \in P^+(\pi)$ run over the almost maximal ideals in the fibre defined by λ. These are characterized by their τ-invariant [D, Sect.II.3, Cor.3; BJ,2.14] and so for each $\alpha \in \pi$ there is a unique $w \in \mathcal{W}_0$ such that $w^{-1}\alpha < 0$. Using the Duflo order relation [D, Sect.II.3, Cor.1] it follows that $w = s_{\alpha_i}s_{\alpha_{i-1}} \cdots s_{\alpha_0}$ where the sequence $\alpha_i, \alpha_{i-1}, \cdots, \alpha_0$ joins $\alpha = \alpha_i$ to α_0 in the Dynkin diagram.

In type A_n the result is similar to either of the above descriptions. One may assume that α_0 is at the end-point of the Dynkin diagram, say $\alpha_0 = \alpha_n$, and that λ is a multiple of the corresponding fundamental weight *excluding* non-negative integer multiples. Then $\mathcal{W}_0 = \{1, s_{\alpha_i} \cdots s_{\alpha_n} \mid i = 2, \cdots, n\}$ except if $\lambda = \omega_n$ and in the latter case $\mathcal{W}_0 = \{s_{\alpha_i} \cdots s_{\alpha_n} \mid i = 1, 2, \cdots, n\}$.

As in the simply-laced case the elements of $\mathcal{W}_0$ are also separated by their τ-invariant (as above) though generally $|\mathcal{W}_0| \leq |\pi|$. If d_0 is the degree of the corresponding Goldie rank representation then

$$d_0 = \begin{cases} |\mathcal{W}_0|, & \text{if } \mathfrak{g} \text{ is simply-laced,} \\ |\mathcal{W}_0| - 1, & \text{otherwise.} \end{cases}$$

We give Λ_0 in Table 3 based on the above considerations. We omit type E_n since we do not calculate the corresponding characters; besides being simply-laced these obtain from the general formula above. It is convenient not to start from λ_0 but to write $\Lambda_0 = \mathcal{W}_0'.\mu_0$, where $\mu_0 \in \mathfrak{h}^*$ and $\mathcal{W}_0' \subset W$ are given in Table 3. We adopt the Bourbaki notation; but writing ω_i for the i^{th} fundamental weight. We set $s_i = s_{\alpha_i} : i = 1, 2, \cdots, n$ and $s_0 = s_{n+1} = 1$.

	μ_0	$\mathcal{W}'_0$
A_n	$k\omega_n : k+1 \in \mathbb{C}\backslash\mathbb{N}$ $-\omega_n$	$\{s_{i+1}s_{i+2}\cdots s_{n+1} \mid i=1,2,\cdots,n\}$ $\{s_i s_{i+1}\cdots s_n \mid i=1,2,\cdots,n\}$
B_n	$-(n-1\frac{1}{2})\omega_1$	$\{s_{i-1}s_{i-2}\cdots s_0 \mid i=1,2,\cdots,n\}$
C_n	$\frac{1}{2}\,\omega_n$	$\{1,\ s_n\}$
D_n	$-(n-2)\omega_1$	$\{w_i := s_{i-1}\cdots s_0 \mid i=1,2,\cdots,n-2\} \cup \{s_{n-1}w_{n-2}, s_n w_{n-2}\}$
F_4	$-1\frac{1}{2}\,\omega_1$	$\{1, s_1, s_2 s_1\}$
G_2	$-\frac{2}{3}\omega_2$	$\{1,\ s_2\}$.

Table 3.

4.4. Assume $\mathfrak{g}$ simple and simply-laced. Outside type A_n we have $\Lambda_0 \subset P(\pi)$. In type A_n we shall just replace Λ_0 by $\Lambda_0 \cap P(\pi)$ which is non-empty. Given $L \in Ob\mathcal{O}_{P(\pi)}$ simple, set $\tau(L) = \{\alpha \in \pi \mid C_\alpha L = 0\}$. If $\lambda \in \Lambda_0 \cap P(\pi)$ then from the above table (or more directly from the remarks concerning the τ-invariant) it follows that $|\tau(L(\lambda))| = |\pi| - 1$, that is there is a unique $\alpha \in \pi \setminus \tau(L(\lambda))$. It is convenient to write $L(\lambda)$ as L_α. If $\mathfrak{g}$ is not of type A_n then for each $\alpha \in \pi$ there exists exactly one such L_α. In type A_n this is also true in the intersection of each W. orbit with $\Lambda_0 \cap P(\pi)$. With these conventions we have the

LEMMA. – *For each $\alpha \in \pi$, $C_\alpha L_\alpha / L_\alpha$ is the direct sum of the L_β as β runs over the neighbours of α in the Dynkin diagram.*

Proof. – This is immediate from the above remarks and Vogan's $T_{\alpha\beta}$ maps [Vo, Sect.3] and the result in [J8, 3.2.3] relating the Enright functor to coherent continuation. □

4.5. Retain the hypotheses and notation of 4.3 and 4.4. Let us first apply 4.4 for $\mathfrak{g}$ simple of type A_n. The condition that $\mu_0 \in P(\pi)$ forces $\mu_0 = -k\omega_n : k \in \mathbb{N}^+$. Assume $k \neq 1$. Then in the conventions of 4.4 we obtain $L_{\alpha_i} = L(s_{i+1}\cdots s_{n+1}.\mu_0) : i = 1,2,\cdots,n$. Thus setting $a_i = ch\ L_{\alpha_i}$ it follows from 4.1 and 4.4 that these elements satisfy the recurrence relations of 3.5. Furthermore 3.6(*) holds since L_{α_n} is the generalized Verma module induced from the one-dimensional $\mathfrak{p}_{\pi\setminus\{\alpha_n\}}$ module of highest weight μ_0. Thus 3.6 gives the following result.

THEOREM. – *($\mathfrak{g}$ simple of type A_n). Take $k \in \{-2,-3,\cdots\}$ and set $\lambda_i = s_{i+1}\cdots s_{n+1}.(-k\omega_n)$. Then*

$$ch\ L(\lambda_i) = \Delta_{\pi_i}\left(\frac{e^{\lambda_i}}{1-e^{-\alpha_i}}\right).$$

Remarks. – From 2.5, 2.6 it follows that for each i the above expression for $\lambda_i = 0$ is exactly $ch\ S[\bar{V}_i]$, where $V_i = \mathcal{O}_0 \cap \mathfrak{m}_{\pi_i}$ (which is an orbital variety). This illustrates the claim in 1.7. Observe that these two characters coincide up to a translation by λ_i if and only if the latter is multiple of ω_i. For $i < n$ this occurs exactly when $k = n-1-i$ and then $\lambda_i = -\omega_i$. Consequently V_i is quantizable in the strong sense. Of course this result is well-known having several other quite different proofs (for details and a historical discussion, *see* [J10, 1.2; BrJ, 6.3]).

A similar result obtains for $k = 1$ except that one must take $\lambda_i = s_i s_{i+1}\cdots s_n.(-\omega_n)$: $i = 1,2,\cdots,n$. At first sight this seems to lead to a slight discrepancy. However one checks that it is not so. For example if $n = 2$, $\lambda_2 = -\omega_2$ then 3.6, 4.1 and 4.4 give as above that

$$ch\ L_{\alpha_1} = \Delta_{\alpha_2}\left(\frac{e^{-\omega_2}}{1-e^{-\alpha_1}}\right).$$

Yet $L_{\alpha_1} = L(\lambda_1) = L(s_1s_2.(-\omega_2)) = L(-2\omega_1)$ which is an induced module and so

$$ch\ L_{\alpha_1} = e^{-2\omega_1}\Delta_{\alpha_2}\left(\frac{1}{1-e^{-\alpha_1}}\right).$$

Nevertheless these two formulae do coincide.

4.6. Retain the notation of 3.7. A similar reasoning based on 3.7 using the fact that the orbital variety $V_1 := \mathcal{O}_0 \cap \mathfrak{m}_{\pi_1}$ is strongly quantizable with μ_0 of Table 3 coinciding with λ in the conclusion of [BrJ, 6.6] gives the

THEOREM. – *($\mathfrak{g}$ simple of type D_n : $n \geq 4$). Set $\lambda_i = (s_{i-1}\cdots s_0).(-(n-2)\omega_1)$: $i = 1,2,\cdots,n-2$. Then*

$$ch\ L(\lambda_i) = \Delta_{\pi_i}\ \frac{e^{\lambda_i}}{(1-e^{-\alpha_i})(1-e^{-(\varepsilon_{i-1}+\varepsilon_i)})}.$$

Remarks. – Again from 2.5, 2.6 the above expression for $\lambda_i = 0$ is exactly $ch\ S[\bar{V}_i]$ where $V_i := O_0 \cap \mathfrak{m}_{\pi_i}$: $i = 1,2,\cdots,n-2$. These two expressions coincide up to a shift exactly when λ_i is a multiple of ω_i. This holds exactly when $i = 1$ or $i = n-2$. Thus V_{n-2} is also strongly quantizable (which is a new result) whereas V_i : $1 < i < n-2$ is not strongly quantizable though it is weakly quantizable. These facts can be also obtained from the general results (Thms. 4.13, 4.14) noted below. However the latter do not give the detailed information obtained here and illustrating the claim in 1.7. We remark that V_{n-1}, V_n are also strongly quantizable (by say [BrJ, Sect.6]). The really curious aspect of the above calculation is that we cannot just start from a different value of λ_1 as in type A_n so that λ_i becomes proportional to ω_i even though Proposition 3.7 is valid for such a choice. What happens is that a_1 (of Proposition 3.7) and $ch\ L(\lambda_1)$ do not coincide for $\lambda_1 = k\omega_1$ unless $k = -(n-2)$. The easiest way to see this is that otherwise by [J7, 8.1, 8.2] the annihilator of $L(\lambda_1)$ would be completely prime and hence coincide with J_0 contradicting the assertion in Table 3.

4.7. Now assume $\mathfrak{g}$ simple and not simply-laced. In this case if $\lambda \in \Lambda_0$, then $\lambda + \rho$ is dominant and regular. A result which goes back to Jantzen [Ja1, Satz 2] asserts that for such λ one has

$$ch\ L(\lambda) = \frac{\sum_{w \in W_\lambda} (-1)^{\ell(w)} e^{w.\lambda}}{D}$$

where D is the Weyl denominator. Suppose $\alpha \in \pi$ is such that $s_\alpha(\lambda + \rho)$ is again dominant and regular. Then $W_{s_\alpha.\lambda} = s_\alpha W_\lambda s_\alpha$. It follows easily that we have the

LEMMA. – ($\mathfrak{g}$ *simple and not simply-laced). Suppose* $\lambda,\ s_\alpha.\lambda \in \Lambda_0$ *for some* $\alpha \in \pi$. *Then*

$$s_\alpha ch\ L(\lambda) = -ch\ L(s_\alpha.\lambda).$$

4.8. Suppose $\mathfrak{g}$ simple of type B_n. Set $\lambda_i = s_{i-1}s_{i-2}\cdots s_0.(-(n-1\frac{1}{2})\omega_1) : i = 1, 2, \cdots, n$ which by Table 3 are just the elements of Λ_0. Define the $a_i : i = 1, 2, \cdots, n-1$ as in 3.8 starting from a_1 and with λ_1 as above. Set $a_0 = 0$ and define a_n by replacing λ_{n-1} by λ_n in the formula for a_{n-1}.

LEMMA. – ($\mathfrak{g}$ *simple of type* B_n).

(i) *For all* $i = 1, 2, \cdots, n-1$ *one has* $ch\ L(\lambda_i) = a_i + a_{i-1}$.

(ii) $ch\ L(\lambda_n) = a_n$.

Proof. – By say [BrJ, Sect.6] and Table 3 one has $a_1 = ch\ L(\lambda_1)$. Then 3.4 gives

$$\begin{aligned} a_i &= (-1)^{i-1} \sum_{j=0}^{i-1} s_j s_{j-1} \cdots s_0.a_1, \quad \mathit{for\ all}\quad i = 1, 2, \cdots, n-1 \\ &= \sum_{j=0}^{i-1} (-1)^{i-j} ch\ L(\lambda_j), \quad \mathit{by}\quad 4.7. \end{aligned}$$

Inversion affords (i). For (ii) recall the notation of 3.8. We have

$$a_1 = \Delta_{\pi_1}\left(\frac{e^{\lambda_1}}{1 - e^{-\alpha_1}}\right) = e^{\lambda_1} E_2 = \frac{e^{\lambda_1}(1 + e^{-\varepsilon_1})}{\prod_{j=2}^{n}(1 - e^{-(\varepsilon_1 - \varepsilon_j)})(1 - e^{-(\varepsilon_1 + \varepsilon_j)})}.$$

Then by 4.7 we obtain

$$\begin{aligned} ch\ L(\lambda_n) &= (-1)^{n-1} s_{n-1} s_{n-2} \cdots s_1 a_1 \\ &= \frac{e^{\lambda_n}(1 + e^{-\varepsilon_n})}{\prod_{i=1}^{n-1}(1 - e^{-(\varepsilon_i - \varepsilon_n)}(1 - e^{-(\varepsilon_i + \varepsilon_n)})} \end{aligned}$$

as in the calculation following (3) of 3.7.

On the other hand by 3.8 and the definition of a_n we have

$$a_n = \Delta_{\pi_{n-1}} \frac{e^{\lambda_n}}{(1-e^{-\alpha_{n-1}})(1-e^{-(\varepsilon_{n-2}+\varepsilon_{n-1})})}.$$

One checks that $(\alpha_n^\vee, \lambda_n) = 1$. Recalling (3.8) we have $\Delta_{\pi_{n-1}} = \Delta_{\pi'_{n-1}}\Delta_n$. Now $(\alpha_n, \varepsilon_{n-2}+\varepsilon_{n-1}) = 0$ and

$$\Delta_n\left(\frac{e^{\lambda_n}}{1-e^{-\alpha_{n-1}}}\right) = \frac{e^{\lambda_n}}{(1-e^{-\alpha_n})}\left[\frac{1}{(1-e^{-\alpha_{n-1}})} - \frac{e^{-2\alpha_n}}{(1-e^{-\alpha_{n-1}-2\alpha_n})}\right]$$

and so

$$a_n = \Delta_{\pi'_{n-1}}\left[\frac{e^{\lambda_n}(1+e^{-\varepsilon_n})}{(1-e^{-(\varepsilon_{n-2}+\varepsilon_{n-1})})(1-e^{-(\varepsilon_{n-1}-\varepsilon_n)})(1-e^{-(\varepsilon_{n-1}+\varepsilon_n)})}\right].$$

Since $(\alpha_i, \varepsilon_n) = (\alpha_i, \lambda_n) = 0$ for $i < n-1$ and the denominator is s_i invariant for $i < n-3$ we may replace $\Delta_{\pi'_{n-1}}$ by $\Delta_1\Delta_2\cdots\Delta_{n-2}\Delta_1\Delta_2\cdots\Delta_{n-3}$ which further equals $(\Delta_2\Delta_1)(\Delta_3\Delta_2)\cdots(\Delta_{n-2}\Delta_{n-3})\Delta_{n-2}$. Now $(\alpha_2, \varepsilon_{n-2}+\varepsilon_{n-1}) = 0$ and for $1 \le i \le n-2$ we have

$$(*) \qquad \Delta_i \frac{1}{(1-e^{-(\varepsilon_{i+1}-\varepsilon_n)})(1-e^{-(\varepsilon_{i+1}+\varepsilon_n)})}$$
$$= \frac{1-e^{-(2\varepsilon_{i+1}+\alpha_i)}}{(1-e^{-(\varepsilon_i-\varepsilon_n)})(1-e^{-(\varepsilon_i+\varepsilon_n)})(1-e^{-(\varepsilon_{i+1}-\varepsilon_n)}(1-e^{-(\varepsilon_{i+1}+\varepsilon_n)}))}.$$

In particular Δ_{n-2} applied to the denominator in the expression for a_n gives the factor

$$D := \frac{1}{(1-e^{-(\varepsilon_{n-2}-\varepsilon_n)})(1-e^{-(\varepsilon_{n-1}-\varepsilon_n)})(1-e^{-(\varepsilon_{n-2}+\varepsilon_n)})(1-e^{-(\varepsilon_{n-1}+\varepsilon_n)})}.$$

Combining the above formulae we obtain

$$a_n = e^{\lambda_n}(1+e^{-\varepsilon_n})(\Delta_2\Delta_1)\cdots(\Delta_{n-2}\Delta_{n-3})D.$$

Hence (ii) obtains on noting that for all $i = 1, 2, \cdots, n-3$ we have

$$\begin{aligned}(\Delta_2\Delta_1)\ &\cdots\ (\Delta_{n-2}\Delta_{n-3})D \\ &= (\Delta_2\Delta_1)\cdots(\Delta_{i+1}\Delta_i)\left(\prod_{j=i+1}^{n-1}(1-e^{-(\varepsilon_j-\varepsilon_n)})(1-e^{-(\varepsilon_j+\varepsilon_n)})\right)^{-1} \\ &= (\Delta_2\Delta_1)\cdots(\Delta_{i+1})\frac{(1-e^{-(\varepsilon_i+\varepsilon_{i+1})})}{\prod_{j=i}^{n-1}(1-e^{-(\varepsilon_j-\varepsilon_n)})(1-e^{-(\varepsilon_j+\varepsilon_n)})} \\ &= (\Delta_2\Delta_1)\cdots(\Delta_i\Delta_{i-1})\left(\prod_{j=i}^{n-1}(1-e^{-(\varepsilon_j-\varepsilon_n)})(1-e^{-(\varepsilon_j+\varepsilon_n)})\right)^{-1},\end{aligned}$$

since the denominator in the penultimate step is s_{i+1} invariant and $(\alpha_{i+1}^\vee, \varepsilon_i+\varepsilon_{i+1}) = 1$ (giving $\Delta_{i+1}(1-e^{-(\varepsilon_i+\varepsilon_{i+1})}) = 1$). $\square$

Remark. – Thus in type B_n the only minimal orbital variety which is strongly quantizable is $\overline{Be_{\alpha_1}} \cap O_0$. Since $\lambda_n = -1\frac{1}{2}\omega_{n-1} + \omega_n$, it follows that $\overline{Be_{\alpha_{n-1}}} \cap O_0$ is weakly but not strongly quantizable. Then (ii) shows that 1.7 is verified. We shall see for $n \geq 4$ that (i) implies that the minimal orbital variety $\overline{Be_{\alpha_i}} \cap O_0 : 1 < i < n-1$ is not even weakly quantizable. In particular the associated variety of $L(\lambda_i) : 1 < i < n-1$ is not irreducible being instead $\overline{Be_{\alpha_{i-1}}} \cup \overline{Be_{\alpha_i}}$. This can also be shown using just the calculation of the characteristic polynomials of [J4] which is a little easier.

4.9. Let us recall [J4, Sect.2] that to each $M \in Ob\mathcal{O}$ there is a polynomial $p_M \in S(\mathfrak{h})$ determined by the asymptotics of $ch\ M$ and defined as follows. To each $\nu \in P^+(\pi) + \rho$ consider the function

$$n \mapsto \sum_{(\mu,\nu)\leq n} dim\ M_{-\mu}.$$

The latter is a polynomial in n whose leading coefficient we denote by $r_M(\nu)$. As in [J4, 2.4(i)] one obtains $p_M := \big(\prod_{\alpha\in\Delta^+}\alpha\big) r_M \in S(\mathfrak{h})$, where Δ^+ is the set of positive roots (*i.e.* the weights of $\mathfrak{n}$).

Fix $\lambda \in \mathfrak{h}^*$ with $\lambda + \rho$ regular and set $\Lambda = \lambda + P(\pi)$. Let Λ^+ (resp. Λ^{++}) denote the set of elements $\mu \in \Lambda$ for which $\mu + \rho$ is dominant (resp. and regular). After Jantzen [Ja2, 2.6] one may write Λ as a disjoint union $\Lambda^+_w : w \in W_\lambda$ so that the $L(\mu) : \mu \in \Lambda^+_w$ form a coherent family. In this $w.\Lambda^{++} \subset \Lambda^+_w$ and for any $\nu \in w.\Lambda^{++}$ each $L(\mu) : \mu \in \Lambda^+_w$ can be expressed as a direct summand of $E \otimes L(\nu)$ for some finite dimensional module E.

The theory of Goldie rank polynomials forces a factorization property [J4, 5.1($*$)] on the asymptotics of $ch\ L(\mu)$. This implies that the $p_{L(\mu)} : \mu \in \Lambda^+_w$ are scalar multiples (polynomially dependent on μ) of a fixed polynomial $p_{\Lambda^+_w}$. Moreover two such polynomials coincide (up to scalars) if and only if the corresponding elements of W_λ lie in the same right cell [J4, 5.5]. Finally the non-proportional polynomials obtained as the $w \in W_\lambda$ run over a two-sided cell $\mathcal{DC}$ of W_λ, form a basis of a simple W_λ module which in turn generates a simple W module. This simple W module determines via the Springer correspondence a nilpotent orbit O. Moreover the G saturation set of the associated variety of $L(\mu) : \mu \in \Lambda^+_w : w \in \mathcal{DC}$ is just $\bar{O}$. In this fashion one may determine the $L(\mu)$ whose associated varieties is a union of orbital varieties corresponding to a given nilpotent orbit.

4.10. Let V be an orbital variety closure and $I(V)$ its ideal of definition. As in 4.9 one obtains a polynomial $p_V \in S(\mathfrak{h})$ from $ch(S(\mathfrak{n}^-)/I(V))$ introduced in [J4, 2.4] and called the characteristic polynomial of V. By [J4, 3.1] the p_V, as V runs over the closures of irreducible components of $O \cap \mathfrak{n}$, span a W submodule of $S(\mathfrak{h})$. As shown by Hotta [H] (and later for example in [J10; V]) the p_V are in fact a basis of the simple W submodule of $S(\mathfrak{h})$ corresponding to the Springer representation of W attached to O (and to the trivial representation of its component group). In particular the p_V are linearly independent as V runs over all orbital varieties. From the definitions of $p_{L(\mu)}$ and p_V it follows [J4, 5.2] that

$$p_{L(\mu)} = \sum n_i p_{V_i}$$

where $n_i \in \mathbb{N}^+$ is the multiplicity of V_i occurring in the associated scheme of $L(\mu)$. Thus V is weakly quantizable if and only if there exists a simple highest weight module $L(\mu)$ such that $p_{L(\mu)}$ is proportional to p_V.

4.11. Fix $\alpha \in \pi$ and let a be the character of a finitely generated $S(\mathfrak{m}_\alpha^-)$ module. As noted in [J4, 2.2] there exists a finite set $F \subset \mathfrak{h}^*$ and scalars $c(\nu) : \nu \in F$ such that a takes the form

$$a = \frac{(\sum_{\nu \in F} c(\nu)e^\nu)(1 - e^{-\alpha})}{\prod_{\beta \in \Delta^+}(1 - e^{-\beta})}.$$

More generally consider any such expression and let r_a (resp. p_a) be the rational (resp. polynomial) function on $\mathfrak{h}^*$ obtained from a by the asymptotic procedure described in 4.9.

LEMMA. – *Suppose* $(1 + s_\alpha)p_a \neq 0$. *Then*

(i) $p_{\Delta_\alpha a} = \frac{1}{\alpha}(1 + s_\alpha)p_a$,

(ii) $r_{\Delta_\alpha a} = \frac{1}{\alpha}(1 - s_\alpha)r_a$.

Proof. – By [J4, 2.3(ii)] one obtains p_a as the first non-vanishing term in the expansion of the numerator of a. Thus set $q = \frac{1}{m!} \sum_{\nu \in F} c(\nu)\nu^m$, where m is the least integer ≥ 0 for which such an expression is non-zero. Then $p_a = q\alpha$. The corresponding numerator for $\Delta_\alpha a$ is $\sum_{\nu \in F} c(\nu)(e^\nu - e^{s_\alpha \nu - \alpha})$. This has leading term $q - s_\alpha q = \frac{1}{\alpha}(1 + s_\alpha)p_a$ which is non-vanishing by the hypothesis. Hence (i). Clearly (ii) follows from (i).

Remark. – More generally let π' be a subset of π and $\Delta'^+ = \Delta^+ \cap \mathbb{N}\pi'$ the corresponding set of positive roots. If a is the character of a finitely generated $S(\mathfrak{m}_{\pi'}^-)$ module then its numerator has the factor $\prod_{\alpha \in \Delta'^+}(1 - e^{-\alpha})$ and this property is preserved on successive application of the $\Delta_\alpha : \alpha \in \pi'$. If $s_\alpha a = a$ then $(1 + s_\alpha)p_a = 0$. Yet $\Delta_\alpha a = a$ and so $p_{\Delta_\alpha a} = p_a$ trivially; but we cannot in general conclude that $(1 + s_\alpha)p_\alpha = 0$ implies that $p_{\Delta_\alpha a} = p_a$. However suppose that a is the character of the $\mathfrak{h}$ module of regular functions on a closed B stable irreducible subvariety V of $\mathfrak{m}_{\pi'}$ and $\Delta_\alpha a$ the corresponding character for $P_\alpha V$. Then either $V = P_\alpha V$ and $s_\alpha\ a = a$ or $dim\ P_\alpha V = dim\ V + 1$. The latter implies [J4, 2.3(iii)] that $deg\ p_{\Delta_\alpha a} = deg\ p_a - 1$ and so forces the conclusion of (i). This reasoning may be applied to the asymptotics of the expression in 4.5, 4.6, 4.8 since for $\lambda_i = 0$ they admit the above interpretation. Alternatively one may check that the hypothesis of the lemma holds whenever $s_\alpha a \neq a$.

The operators $A_\alpha := \frac{1}{\alpha}(1 - s_\alpha) : \alpha \in \pi$ occurring in (ii) above were introduced by Bernstein-Gelfand-Gelfand [BGG, Sect.3] who showed that they have square zero and satisfy the braid relations. They may be viewed as infinitesimal versions of the Demazure operators and taking a reduced decomposition A_w may be defined for each $w \in W$. If w_0 is the unique longest element of W one may then check the formula

$$A_{w_0} = \frac{1}{(\prod_{\alpha \in \Delta^+} \alpha)} \sum_{w \in W} (-1)^{\ell(w)} w.$$

4.12. Assume $\mathfrak{g}$ classical. Through 2.6 and Table 1 one may calculate the character of the algebra of regular functions on any minimal orbital variety closure $\overline{Be_\alpha} : \alpha$ long.

Comparison shows that these are exactly the formulae given in 3.6 - 3.8 taking $\lambda_i = 0$. Then from 4.5, 4.6 and 4.8 and taking account of 4.11 which shows that λ_i "disappears" in the asymptotics we obtain the

COROLLARY. – *Every minimal orbital variety is weakly quantizable except the* $V_i := \overline{Be_{\alpha_i}} : i = 2, 3, n-2$ *in type* $B_n : n \geq 4$. *In the latter case there exists* $\lambda_i \in \mathfrak{h}^*$ *such that*

$$p_{L(\lambda_i)} = p_{V_{i-1}} + p_{V_i} : i = 2, 3, \cdots, n-2.$$

Proof. – To show in type B_n that there is no further choice of λ which would allow $p_{L(\lambda)}$ to be proportional to some $p_{V_i} : i = 2, 3, \cdots, n-2$ it is enough to show that we have already exhausted all coherent families for which the associated variety of $Ann\ L(\lambda)$ is the minimal orbit closure $\bar{O}_0$. Since the minimal orbit is not special it cannot occur in the integral fibre [L1, Chaps. 4, 5; L2, Sect. 3] (that is λ cannot be integral; this is already contained in the results of Barbasch and Vogan [BV]). In general setting $\Delta_\lambda = \{\alpha \in \Delta | (\alpha^\vee, \lambda) \in \mathbb{Z}\}$ it follows from say [J2, 4.1] that we must have $dim\ O_0 \geq |\Delta| - |\Delta_\lambda|$ and if equality holds then $\lambda + \rho$ must be dominant and regular. Listing subroot systems one checks that this condition is only satisfied for coherent families obtained from Λ_0 of 4.3. (Unfortunately it is not known if every coherent family of primitive ideals has a completely prime member. Then the assertion would have already followed from 4.3 and obliviated the need for the above verification.) □

Remark. – As we shall see the conclusion is valid without the restriction on $\mathfrak{g}$ being classical.

4.13. A rather deep result of Bezrukavnikov [Be] and Inamdar-Mehta [IM] allows us to determine when a minimal orbital variety is strongly quantizable.

THEOREM. – *The minimal orbital variety* $\overline{Be_\alpha} : \alpha \in \pi$ *long, is strongly quantizable if and only if there exists* $\lambda \in \Lambda_0$ *such that* $(\lambda, \beta) = 0,\ \forall \beta \in \pi' := \pi \setminus \{\alpha\}$.

Proof. – By [J7, 8.1, 8.2] if $gr\ Ann_{u(\mathfrak{n}^-)} e_\lambda$ is prime then $Ann\ L(\lambda)$ is completely prime. Furthermore $\mathfrak{p}_{\pi'} \subset gr\ Ann_{U(\mathfrak{n}^-)} e_\lambda$ forces $\mathfrak{p}_{\pi'} e_\lambda = 0$ and so $(\lambda, \beta) = 0$, for all $\beta \in \pi'$. This proves the only if part. Conversely the above conditions on λ imply that $\mathfrak{p}_{\pi'} + gr\ J(\lambda_0) \subset gr\ Ann_{U(\mathfrak{n}^-)} e_\lambda$. Yet $gr\ J(\lambda_0)$ is [Ga; BrJ, 1.2] just the ideal of definition of $\bar{O}_o$ and then by [BrJ,6.2] the above sum is just the ideal of definition of $\overline{Be_\alpha}$. Consequently equality holds in the above and this expresses the strong quantizibility of $\overline{Be_\alpha}$.

Remark. – From Table 3 one checks that strong quantizability exactly holds for all minimal orbital varieties in types A_n, C_n, G_2, for $\overline{Be_{\alpha_1}}$ is types B_n and F_4, and for $\overline{Be_\alpha}$ in types D and E whenever α has an odd number of neighbours in the Dynkin diagram.

4.14. One can give an equally simple criterion for when a minimal orbital variety is weakly quantizable.

THEOREM. – *The minimal orbital variety* $\overline{Be_\alpha} : \alpha$ *long, is weakly quantizable if and only if there exists* $\lambda \in \Lambda_0$ *such that* $(\lambda, \beta^\vee) \in \mathbb{N}$ *for all* $\beta \in \pi' := \pi \setminus \{\alpha\}$.

Proof. – The second hypothesis on λ implies that $\mathfrak{p}_{\pi'} \subset \sqrt{gr\ Ann_{U(\mathfrak{n}^-)} e_\lambda}$ and so the if part obtains as in the proof of 4.13. From Table 3 one checks that the hypotheses can always be satisfied except for the orbital varieties in type B_n described in 4.12. □

5. Some remarks on character formulae for orbital varieties

5.1. Let us examine the plausibility of the hypothesis following 1.6(**). Namely let V be an orbital variety closure contained in some $\mathfrak{m}_{\pi'}$. Then V is $R_{\pi'}$ stable and we assume that there exists some subspace $\mathfrak{m}_V \subset \mathfrak{m}_{\pi'}$ such that $V = \overline{R_{\pi'} m_V}$. Identify $\mathfrak{m}^*_{\pi'}$ with $\mathfrak{m}^-_{\pi'}$ through the Killing form and let I be the ideal of definition of V in $S(\mathfrak{m}^-_{\pi'})$.

LEMMA. – *There exists an isomorphism*

$$S(\mathfrak{m}^-_{\pi'})/I \xrightarrow{\sim} (ad\ U(\mathfrak{r}_{\pi'}))S(\mathfrak{m}^-_{\pi'}/\mathfrak{m}^\perp_V)$$

of $U(\mathfrak{r}_{\pi'})$ *modules.*

Proof. – Since I is the largest $R_{\pi'}$ stable ideal contained in $S(\mathfrak{m}^-_{\pi'})\mathfrak{m}^\perp_V$ we have

$$(*) \qquad I = \bigcap_{g \in R_{\pi'}} g(S(\mathfrak{m}^-_{\pi'}/\mathfrak{m}^\perp_V).$$

Extend the non-degenerate $R_{\pi'}$ invariant pairing $\mathfrak{m}_{\pi'} \times \mathfrak{m}^-_{\pi'} \to \mathbb{C}$ given by the Killing form, to a non-degenerate $R_{\pi'}$ invariant pairing $S(\mathfrak{m}_{\pi'}) \times S(\mathfrak{m}^-_{\pi'}) \to \mathbb{C}$. Then $S(\mathfrak{m}^-_{\pi'})\mathfrak{m}^\perp_V$ identifies with $S(\mathfrak{m}_V)^\perp$. Now the action of $R_{\pi'}$ preserves each graded (by degree) component of $S(\mathfrak{m}^-_{\pi'})$ which is finite dimensional and paired to the corresponding graded component of $S(\mathfrak{m}_{\pi'})$. Since $(\sum V_i)^\perp = \cap V_i^\perp$ for any finite sum of finite dimensional vector spaces we conclude that

$$I = \bigcap_{g \in R_{\pi'}} g(S(\mathfrak{m}_V)^\perp) = \bigcap_{g \in R_{\pi'}} g(S(\mathfrak{m}_V))^\perp = \left(\sum_{g \in R_{\pi'}} gS(\mathfrak{m}_V)\right)^\perp$$

(since the sum can be assumed finite for any fixed degree). Thus I is the orthogonal complement of $(ad\ U(\mathfrak{r}_{\pi'}))S(\mathfrak{m}_V)$ in $S(\mathfrak{m}^-_{\pi'})$ and which hence identifies with the graded dual of $S(\mathfrak{m}^-_{\pi'})/I$. On the other hand $S(\mathfrak{m}_V)$ identifies with the graded dual of $S(\mathfrak{m}^-_{\pi'}/\mathfrak{m}^\perp_V)$ and hence $(ad\ U(\mathfrak{r}_{\pi'}))S(\mathfrak{m}_V)$ identifies with the graded dual of $(ad\ U(\mathfrak{r}_{\pi'}))S(\mathfrak{m}^-_{\pi'}/\mathfrak{m}^\perp_V)$. Combined with our first observation, this gives the required assertion. □

5.2. Now let us suppose that $\mathfrak{m}_V$ is stable under the Borel subgroup $B^-_{\pi'}$ of $R_{\pi'}$ whose Lie algebra consists of $\mathfrak{h}$ and the root subspaces $\mathfrak{g}_{-\alpha} : \alpha \in \Delta'^+$. Then $\mathfrak{m}^-_{\pi'}/\mathfrak{m}^\perp_V$ identifies with an $\mathfrak{h}$ stable subspace which is stable under the Borel subalgebra $\mathfrak{b}_{\pi'}$ of $\mathfrak{r}_{\pi'}$ whose Lie algebra consists of $\mathfrak{h}$ and the root subspaces $\mathfrak{g}_\alpha : \alpha \in \Delta'^+$. One would like to conclude that 1.6(**) holds with σ_V being the subset of Δ^+ specifying $\mathfrak{m}_V$. A criterion under which this holds was given in [J6, Sect. 5] in terms of certain derived functors $\mathcal{D}^i_w : w \in W,\ i \in \mathbb{N}$. Writing $F = S(\mathfrak{m}^-_{\pi'}/\mathfrak{m}^\perp_V)$ which is a locally finite $\mathfrak{b}_{\pi'}$ module one must show that $\mathcal{D}_w F$, for each w belonging to the Weyl subgroup W' generated by the $s_\alpha : \alpha \in \pi'$, identifies with a $\mathfrak{b}_{\pi'}$ submodule of $S(\mathfrak{m}^-_{\pi'})$. Let w'_0 be the unique longest element of W'. Then by [J6, 5.5) the above would imply that $\mathcal{D}^i_{w'_0} F = 0$ for $i > 0$ and $\mathcal{D}_{w'_0} F = (ad\ U(\mathfrak{r}_{\pi'}))F$ giving the latter to have character $\Delta_{\pi'}(ch\ F)$. Even though we cannot yet prove this, we can show that the characteristic polynomial p_V of V defined in 4.10 takes the form prescribed by 4.11 and such a character formula. This follows from successive application of [J4, 2.6]. Indeed let $P^-_\alpha : \alpha \in \pi'$ be the parabolic subgroup of $R_{\pi'}$ whose Lie algebra is Lie $B^-_{\pi'} + \mathfrak{g}_\alpha$. Take a reduced decomposition $s_{\alpha_k} s_{\alpha_{k-1}} \cdots s_{\alpha_1}$ of w'_0 and set $V_0 = \mathfrak{m}_V$, $V_i = P^-_{\alpha_i} V_{i-1}$ for $i = 1, 2, \cdots, k$. Then $V_k = V$. Each V_i is irreducible and B^- stable. Moreover by [J4, 2.6] either $V_i = V_{i-1}$ and $p_{V_i} = p_{V_{i-1}}$ or $dim\ V_i = dim\ V_{i-1} + 1$ and p_{V_i} is positive integer multiple of $\frac{1}{\alpha_i}(1 + s_{\alpha_i}) p_{V_{i-1}}$. Finally since $p_{V_0} = \prod_{\alpha \in \Delta^+ \setminus \sigma_V} \alpha$ up to a non-zero scalar, the required formula is obtained.

5.3. The significance of 5.2 is that the characteristic polynomial of an orbital variety closure can be computed (at least in low rank) by other means. In type $\mathfrak{sl}(n)$ a result of Melnikov [M] asserts that every orbital variety closure V is weakly quantizable and so then p_V is a Goldie rank polynomial. The latter are given [via J3, Thm. 5.1] by the Kazhdan-Lusztig polynomials and so can be computed (in principle).

5.4. In view of the rather simple transformation properties 4.4, 4.7 of the $ch\ L$, when L is a simple highest weight module associated to a minimal orbital variety, one can ask if the characters associated to these varieties also transform in a simple manner. This is at best so only in the following weak sense. Fix $\alpha \in \pi$ and let V_0 be a closed irreducible subvariety of $\mathfrak{m}^+_\alpha$ stable under the Borel subgroup B of G whose Lie algebra is $\mathfrak{h} \oplus \mathfrak{n}^+$. Let P_α be the parabolic subgroup of G whose Lie algebra is Lie $B \oplus g_{-\alpha}$. Assume $P_\alpha V_0 \subsetneq V_0$. Let I_0 (resp. J_0) be the ideal of definition V_0 (resp. $P_\alpha V_0$) in $S(\mathfrak{m}^-_\alpha)$. Then $J_0 \subset I_0$ and as noted in [J4, 2.6] the canonical projection restricts to an embedding $(S(\mathfrak{m}^-_\alpha)/J_0)^{e_{-\alpha}} \hookrightarrow S(\mathfrak{m}^-_\alpha)/I_0$ of integral domains. Now suppose V to be a closed irreducible B stable subvariety of $\mathfrak{n}^+$ (for example an orbital variety closure) and let J denote its ideal of definition in $S(\mathfrak{n}^-)$. Set $I = J + S(\mathfrak{n}^-)e_{-\alpha}$ which one may view as an ideal of $S(\mathfrak{m}^-_\alpha)$. A careful reworking of [J4, 2.6, 2.9] gives the following result.

LEMMA.

(i) $ch(S(n^-)/J) = (1 - e^{-\alpha})^{-1} ch(S(\mathfrak{n}^-)/I)$

(ii) *Suppose* $(S(\mathfrak{m}^-_\alpha)/J_0)^{e_{-\alpha}} \xrightarrow{\sim} S(\mathfrak{m}^-_\alpha)/I_0$. *Then*

$$ch(S(\mathfrak{m}^-_\alpha)/J_0) = (1 - e^{-\alpha})^{-1}(s_\alpha - e^{-\alpha})(ch\ S(\mathfrak{m}^-_\alpha)/I_0).$$

Remark. – One may observe that the operator $(1 - e^{-\alpha})^{-1}(s_\alpha - e^{-\alpha})$ occurring in the right hand side of (ii) is not quite the (negative of the) Demazure operator.

5.5. In the above I determines the scheme theoretic intersection $\mathfrak{m}_\alpha \cap V$. The simplest case to analyze is when I is semiprime and the intersection has at most two components. The latter occurs when $V = \overline{Be_\alpha}$ and α has at most two neighbours in the Dynkin diagram. Let us suppose $I = I_1 \cap I_2$ with I_1, I_2 prime. Then as an $\mathfrak{h}$ module we may write

$$S(\mathfrak{n}^-)/(I_1 \cap I_2) = S(\mathfrak{n}^-)/I_1 \oplus (I_1 + I_2)/I_2.$$

Now $I_1 + I_2 \subsetneq S(\mathfrak{n}^-)$, for example the identity does not occur in the left hand side. Thus

$$ch\ S(\mathfrak{n}^-)/I = ch\ S(\mathfrak{n}^-)/I_1 + ch\ S(\mathfrak{n}^-)/I_2 - ch\ S(\mathfrak{n}^-)/(I_1 + I_2)$$

where the last term is a "correction factor" not present in 4.4. Assuming that this can be calculated and that the hypothesis of 5.4(ii) holds then one obtains an inductive procedure to calculate the characters associated to the minimal orbital varieties analogous to, though less simple than, 4.4. One may in principle check the validity of this formula for $\mathfrak{g}$ classical since all such characters are known; but we only did this in the simplest non-trivial case, namely in $\mathfrak{sl}(4)$.

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(Manuscript received June 21, 1996;
revised July 8, 1997.)

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