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QUANTUM GROUPS IN HIGHER GENUS AND DRINFELD'S NEW REALIZATIONS METHOD ($\mathfrak{sl}_2$ CASE)

BY B. ENRIQUEZ AND V. N. RUBTSOV

ABSTRACT. – We define double (central and cocentral) extensions of Manin pairs attached to curves and meromorphic differentials, introduced by Drinfeld. We define "infinite twists" of these pairs and quantize them in the $\mathfrak{sl}_2$ case, adapting Drinfeld's "new realizations" technique. We study finite-dimensional representations of these algebras at level 0, and some elliptic examples.

RÉSUMÉ. – Nous définissons des extensions doubles (centrales et cocentrales) des paires de Manin introduites par Drinfeld, associées à une courbe et une différentielle méromorphe. Nous définissons des « twists infinis » de ces paires et nous les quantifions dans le cas $\mathfrak{sl}_2$, en adaptant la technique de Drinfeld des « nouvelles réalisations ». Nous étudions les représentations de dimension finie de ces algèbres en niveau 0, ainsi que certains exemples elliptiques.

Introduction

In [5], V. Drinfeld introduced examples of Manin pairs attached to the data of a curve, a meromorphic differential, and a finite dimensional reductive Lie algebra. He remarked that only in the cases where the curve had genus ≤ 1 could these Manin pairs be given the structure of a Manin triple; in these cases, the quantization of these Manin triples gives rise to known Hopf algebras (Yangians, quantum affine algebras and algebras connected with Sklyanin algebras). He raised the question of quantizing these Manin pairs in the higher genus case, in the sense of quasi-Hopf algebras.

In this paper, we first present a double (central and cocentral) extension of these Manin pairs. The general definition of these extensions, in the case of Manin triples, is due to M. Semenov-Tian-Shansky ([13]). This leads us to the problem of the quantization of these extended Manin pairs.

We then remark that this quantization problem can be approached in the spirit of the "new realizations" of Drinfeld (introduced in [4] and developed in [10], [3], [1]). This technique enabled Drinfeld to give a quantum analogue of the passage from the Serre to the loop presentations of an affine algebra; it can be presented as follows. The bialgebra structure corresponding to quantum affine algebras is a double bialgebra structure. Conjugating the corresponding Manin triple by a double group element, the bialgebra structure of the double gets changed by a twist (in the sense of [5]). Let us conjugate by affine Weyl group elements; when their length tends to infinity, we get a limit Manin triple which it is simple enough to quantize. The resulting Hopf algebra is then a twist of the one obtained by quantization of the initial Manin triple (the Drinfeld-Jimbo Hopf algebra).

In the present situation, we introduce a Lagrangean supplementary in our Manin pair, and conjugate it as before by affine Weyl group elements. (We note that a family of supplementaries is provided by a covering of the space of principal G-bundles over the curve X; we suggest a possible connection between the closedness of a 1-form, defined in terms of twist, and a generalized classical Yang-Baxter identity which underlies the integrability of the Hitchin system. We hope to return to this question in [6].) In the limit, we obtain a Manin triple, whose quantization (in the $\mathfrak{sl}_2$ case) is the main goal of this paper. Let us describe its contents more precisely.

Let X be a smooth compact complex curve, ω a meromorphic nonzero one-form on X, $\{x_i\} \subset X$ the set of its zeroes and poles. Let for each i, k_{x_i} be the local field at x_i, $\mathcal{O}_{x_i}$ the local ring at this point, $R \subset \oplus_i k_{x_i}$ the ring of functions that are regular outside $\{x_i\}$. We choose a supplementary Λ to R in $\oplus_i k_{x_i}$, Lagrangian for the scalar product defined by ω. To define a quantization of our Manin triple, we need operators $A : R \to \oplus_i k_{x_i}$ and $B : \Lambda \to \oplus_i k_{x_i}$, which serve to define the $h - e$ and $h - f$ relations (by $h - e$ relations, we understand relations between Fourier modes of the quantum analogues of fields $h(z)$ and $e(z)$, etc.; with e, h, f the Chevalley generators of $\mathfrak{sl}_2$). These operators also provide us with $e - e$ and $f - f$ relations which appear in the form

$$e(z)e(w) = a(z,w)e(w)e(z), \quad w \ll z. \tag{0.1}$$

Our aim is to put these relations in the form

$$(z - w + \sum_{i \geq 1} \hbar^i \alpha_i(z,w))e(z)e(w) = (z - w + \sum_{i \geq 1} \hbar^i \beta_i(z,w))e(w)e(z), \tag{0.2}$$

where α_i, β_i are formal series in z and w, ($\hbar$ is the quantization parameter) similar to the quantum affine algebra relations

$$(qz - w)e(z)e(w) = (z - qw)e(w)e(z).$$

Relations of the form (0.2) are usually called vertex relations. Fourier modes of such a relation provide relations between the commutators $[e_n, e_m]$, from which we derive (by suitable linear combinations) the expression of any such commutator in terms of formal series in $\hbar$, with coefficients combinations of $e_p e_q$, p, q larger that some integer. Such a derivation is usually not possible by taking Fourier modes of (0.1).

It turns out that to achieve this task, essentially one possibility for the operators A and B remains. The proof that it indeed leads to $e - e$ and $f - f$ relations of the desired form, relies on a statement about derivatives of a Green function (lemma 1), which allows us to give a universal treatment for all pairs (X, ω). The formal series α_i and β_i are then obtained from formal solutions to certain differential equations (eqs. (3.7)), where the variable is $\hbar$.

The quantization we propose depends both on a choice of Λ and on that of a certain element $\tau \in R \otimes R$. We show that the various quantizations obtained are related to each other by twist operations (in the sense of [5]).

We then turn to the problem of finite-dimensional representations of our algebras at level 0; these representations are indexed by points of formal discs. We construct a family of 2-dimensional representations. We expect that higher spin representations can be constructed as well and that their tensor products have properties similar to those explained in [2].

We close the paper by giving some explicit examples. In a certain elliptic case, we recover as $e - e$ relations certain elliptic $\mathcal{W}$-algebra relations discovered in [8]. We also propose a twist of the "automorphic" Manin triples of [12], and apply the techniques of this paper to derive a quantization in the case of $\mathfrak{sl}_2$. One may hope to identify it with the Hopf algebra arising in [16].

Further problems related to the present construction could be the following: applying the Reshetikhin-Semenov method for constructing central elements at the critical level ([11]); construction of level 1 representations as in [9], vertex operators and the corresponding quantum Knizhnik-Zamolodchikov (KZ) equations; generalization from $\mathfrak{sl}_2$ to an arbitrary semisimple Lie algebra. The resulting quantum KZ equations at the critical level might then be considered as q-deformations of the holonomic systems of equations occurring in the geometric Langlands program; viewing such q-deformations in this framework has been proposed by E. Frenkel and N. Reshetikhin. Finally, in [7], P. Etingof and D. Kazhdan showed how to attach a quantization procedure for bialgebras to any associator. It would be interesting to understand whether the construction presented here can be obtained from the KZ associator, as it is the case for finite dimensional Lie algebras.

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1. Manin triples

1. Generalities on Manin pairs and triples

Let us first recall general notions associated with Manin pairs and triples. A *Manin pair* is the data of a complex Lie algebra $\mathfrak{p}$, endowed with a scalar product $\langle , \rangle_{\mathfrak{p}}$, and of a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{p}$, wich is a maximal isotropic (or Lagrangean) subspace of $\mathfrak{p}$. The choice of a Lagrangean supplementary L to $\mathfrak{k}$ in $\mathfrak{p}$ determines a *Lie quasi-bialgebra* structure on $\mathfrak{k}$, that is the datum of a linear map $\delta_{\mathfrak{k}} : \mathfrak{k} \to \wedge^2\mathfrak{k}$, and of an element $\varphi_{\mathfrak{k}} \in \wedge^3\mathfrak{k}$, satisfying certain axioms (see [5]). The map $\delta_{\mathfrak{k}}$ and the element $\varphi_{\mathfrak{k}}$ are obtained by dualizing the bracket map from $\wedge^2 L$ to $L \oplus \mathfrak{k}$.

The notion of Lie quasi-bialgebra is the classical limit of that of quasi-bialgebra, that is an algebra A endowed with an algebra morphism $\Delta_A : A \to A^{\otimes 2}$, and an invertible element $\Phi \in A^{\otimes 3}$, such that $(\Delta_A \otimes 1) \circ \Delta_A = \Phi((1 \otimes \Delta_A) \circ \Delta_A)\Phi^{-1}$.

If $(\mathfrak{p}, \mathfrak{k})$ is a Manin pair, and L is a Lagrangean supplementary of $\mathfrak{k}$ in $\mathfrak{p}$; that $(p^i), (p_i)$ are dual bases of $\mathfrak{k}$ and L, and $R = \sum_i p^i \wedge p_i$, then $\delta_{\mathfrak{p}} = \mathrm{ad}(R)$ and $\varphi_{\mathfrak{k}}$ define a Lie quasi-bialgebra structure on $\mathfrak{p}$.

If $\mathfrak{p}$ has a Lie quasi-bialgebra structure $(\delta_{\mathfrak{p}}, \varphi_{\mathfrak{p}})$, any element $f \in \wedge^2 \mathfrak{p}$ defines a new Lie quasi-bialgebra structure on $\mathfrak{p}$ by the formuleas $\delta_{\mathfrak{p},f} = \delta_{\mathfrak{p}} - \mathrm{ad}(f)$, $\varphi_{\mathfrak{p},f} = \varphi - [f^{12}, f^{13}] - [f^{12}, f^{23}] - [f^{13}, f^{23}] + 1/2\mathrm{Alt}(\delta \otimes 1)f$. In particular, if $\mathfrak{p}$ contains two Lagrangean Lie subalgebras $\mathfrak{k}$ and $\mathfrak{l}$, with Lagrangean supplementaries $L_{\mathfrak{k}}$ and $L_{\mathfrak{l}}$, the corresponding double quasi-bialgebra structures on $\mathfrak{p}$ are related by the twist $f_{\mathfrak{k},\mathfrak{l}} = \sum_i p'^i \wedge p'_i$, with $(p'^i), (p'_i)$ dual bases of $\mathfrak{k} \cap L_{\mathfrak{l}}$ and $\mathfrak{l} \cap L_{\mathfrak{k}}$.

If $(\mathfrak{p}, \mathfrak{k})$ is a Manin pair and we can choose L to be a Lie subalgebra of $\mathfrak{p}$, then the triple $(\mathfrak{p}, \mathfrak{k}, L)$ is called a Manin triple. The corresponding $\varphi_{\mathfrak{k}}$ is then zero; the notion of a Manin triple is the classical counterpart of that of a bialgebra (that is a quasi-bialgebra with $\Phi = 1$).

2. *Drinfeld's Manin pairs*

Let X be a smooth compact complex curve, ω a meromorphic nonzero one-form on X, $\{x_i\} \subset X$ the set of its zeroes and poles. Let for each i, k_{x_i} be the local field at x_i, $\mathcal{O}_{x_i}$ the local ring at this point; let $k = \oplus_i k_{x_i}$, and $R \subset k$ be the ring formed by the Laurent expansions of the functions on X, regular outside $\{x_i\}$.

Let us show here that R is a Lagrangean supspace of k. Let $\mathbf{C}(X)$ be the function field of X and $\mathbf{A}$ be its adeles ring. Let $\langle , \rangle_{\mathbf{A}}$ be the pairing defined on $\mathbf{A}$ by $\langle f, g \rangle_{\mathbf{A}} = \sum_{x \in X} \mathrm{res}_x(fg\omega)$. Then we have

LEMMA 1.2.1. – *$\mathbf{C}(X)$ is a Lagrangean subspace of $\mathbf{A}$.*

Proof. – Recall first the duality theorem ([14], II-8, thm. 2). Let D be any divisor on X, and $\Omega(D)$ be the space of all meromorphic forms ω equal to zero or such that their divisor is $\geq D$. Let on the other hand, $\mathbf{A}_{\geq -D}$ be the space of adeles with divisor $\geq -D$. Then $\langle , \rangle_{\mathbf{A}}$ induces a non-degenerate pairing

$$\Omega(D) \times (\mathbf{A}/(\mathbf{A}_{\geq -D} + \mathbf{C}(X))) \to \mathbf{C}.$$

Let us now prove the lemma. The isotropy of $\mathbf{C}(X)$ follows from the residue formula. Let Ω be the space of all meromorphic forms on X, and let us now show that the pairing

$$\Omega \times (\mathbf{A}/\mathbf{C}(X)) \to \mathbf{C} \tag{1.2.1}$$

is also non-degenerate. Let $f \in \mathbf{A}/\mathbf{C}(X)$ have vanishing pairing with Ω. Then for any divisor D, the pairing of its image in $\mathbf{A}/(\mathbf{A}_{\geq -D} + \mathbf{C}(X))$ with any element of $\Omega(D)$ is zero, which means that f belongs to $\mathbf{A}_{\geq -D}/(\mathbf{A}_{\geq -D} \cap \mathbf{C}(X))$ for any D, and so is zero.

The lemma now follows from the non-degeneracy of (1.2.1). ■

We then have

LEMMA 1.2.2. – *R is a Lagrangean subspace of k.*

Proof. – We follow [5], sect. 2, example. First observe the following general fact. Let E be a vector space with a scalar product $\langle , \rangle_E$ et let F be a Lagrangean subspace of E. Let C be a subspace of E, such that $C \supset C^{\perp}$. Then $\langle , \rangle_E$ naturally defines a scalar product $\langle , \rangle_{E'}$ on $E' = C/C^{\perp}$, and $F' = ((F \cap C) + C^{\perp})/C^{\perp}$ is a Lagrangean subspace of E'.

Apply this statement to the case $E = \mathbf{A}$, $\langle , \rangle_E = \langle , \rangle_{\mathbf{A}}$; by Lemma 1.2.1, we may take $F = \mathbf{C}(X)$. Let $C = \oplus_i k_{x_i} \oplus \bigoplus_{x \in X - \{x_i\}} \mathcal{O}_x$. Then $C^{\perp} = \oplus_{x \in X - \{x_i\}} \mathcal{O}_x$, because the orthogonal of k_{x_i} is $\mathbf{A}^{x_i}$ (the adeles with x_i-component equal to zero), and the orthogonal of $\mathcal{O}_x$ is $\mathbf{A}^x \oplus \mathcal{O}_x$ for $x \in X - \{x_i\}$, because ω has no zero or pole at x. Therefore $C \supset C^\perp$, and $C/C^\perp = k$. Since F' is then identified with R, the lemma follows. ∎

Let us introduce now Drinfeld's Manin pairs ([5]). Let $\mathfrak{a}$ be a simple complex Lie algebra, $\langle,\rangle_\mathfrak{a}$ be its Killing form. For A any ring over $\mathbf{C}$, we use the notation

$$\mathfrak{a}(A) = \mathfrak{a} \otimes A.$$

Endow

$$\mathfrak{g}_0 = \mathfrak{a}(k)$$

with the bilinear form $\langle x_1, x_2\rangle_0 = \sum_i \mathrm{res}_{x_i}(\langle x_1, x_2\rangle_\mathfrak{a}\omega)$. Then $\mathfrak{a}(R)$ is a Lagrangean subalgebra of $\mathfrak{g}_0$; this defines Drinfeld's Manin pair

$$(\mathfrak{a}(k), \mathfrak{a}(R)).$$

3. *Double extension*

We extend this Manin pair in the following way. Let ∂ be the derivation of $\mathfrak{a}(k)$ defined by $\partial f = df/\omega$. We denote in the same way the derivation of k, defined by the same formula. Let $\mathfrak{g}$ be the skew product of $\mathfrak{g}_0$ by ∂; we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathbf{C}\widetilde{D},$$

$\mathfrak{g}_0 \subset \mathfrak{g}$ is a Lie algebra homomorphism, and $[\widetilde{D}, x] = \partial x$ for $x \in \mathfrak{g}_0$. Since ∂ preserves $\mathfrak{a}(R)$, $\mathfrak{a}(R) \oplus \mathbf{C}\widetilde{D}$ is a Lie subalgebra of $\mathfrak{g}$. Let $\hat{\mathfrak{g}}$ be the central extension of $\mathfrak{g}$ by $\mathbf{C}K$ using the cocycle defined by $c(x,y) = \sum_i \mathrm{res}_{x_i}\langle x, dy\rangle_\mathfrak{a} K$, $c(x, \widetilde{D}) = 0$ for $x \in \mathfrak{g}_0$. We have

$$\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{C}K,$$

with the usual commutation rules. Since c vanishes on $\mathfrak{a}(R) \oplus \mathbf{C}\widetilde{D}$, this algebra has a section to $\hat{\mathfrak{g}}$, that we denote by $\mathfrak{g}_R$. Identifying $\hat{\mathfrak{g}}$ with $\mathfrak{g} \oplus \mathbf{C}K$, $\mathfrak{g}_R$ is identified with

$$(\mathfrak{a}(R) \oplus \mathbf{C}\widetilde{D}) \times \{0\}.$$

Let $D = (\widetilde{D}, 0)$. Let us consider now on $\hat{\mathfrak{g}}$ the symmetric bilinear form, defined by $\langle K, D\rangle = 2$, $\langle D, \mathfrak{g} \times \{0\}\rangle = 0$, $\langle K, \mathfrak{g}_0 \times \{0\}\rangle = 0$, $\langle (x_1, 0), (x_2, 0)\rangle = \langle x_1, x_2\rangle_0$ for $x_1, x_2 \in \mathfrak{g}_0$. Then $\langle,\rangle$ is invariant, and $\mathfrak{g}_R$ is a subalgebra of $\hat{\mathfrak{g}}$, Lagrangean w.r.t. this scalar product.

$$(\hat{\mathfrak{g}}, \mathfrak{g}_R)$$

is a double extension (central and cocentral) of the above Manin pair.

4. *Lagrangean supplementaries*

Consider on k, the scalar product defined by $\langle f_1, f_2\rangle_k = \sum_i \mathrm{res}_{x_i}(f_1 f_2 \omega)$. R is a subspace of k, Lagrangean w.r.t. this scalar product. Fix a Lagrangean supplementary Λ to

R, commensurable with $\oplus_i \mathcal{O}_{x_i}$ (that is, such that their intersection has finite codimension in each of them). Then

$$(\mathfrak{a} \otimes \Lambda) \oplus \mathbf{C}K$$

is a Lagrangean supplementary to $\mathfrak{g}_R$ in $\hat{\mathfrak{g}}$. We will denote by $\delta_R : \hat{\mathfrak{g}} \to \wedge^2 \hat{\mathfrak{g}}$ the double Lie quasi-bialgebra structure on $\hat{\mathfrak{g}}$ corresponding to this choice of a supplementary.

We note here, that a family of Lagrangean supplementaries can be defined in the following way. Let Λ_0 be a Lagrangean subspace of k, containing $\oplus_i \mathcal{O}_{x_i}$. Let G be a Lie group, with Lie algebra $\mathfrak{a}$. For $g \in G(k)$, let

$${}^g\mathfrak{a}(\Lambda_0) = \mathrm{Ad}(g)\mathfrak{a}(\Lambda_0);$$

for generic g this defines a Lagrangean supplementary to $\mathfrak{g}_R$ in $\hat{\mathfrak{g}}$, which up to equivalence depends only on the class of g in $G(R) \setminus G(k)/\mathrm{Stab}\ \mathfrak{a}(\Lambda_0)$. All the resulting bialgebra structures on $\mathfrak{g}_R$ are then associated by twist. The twist between two bialgebra structures associated to nearby points defines an element of $\wedge^2 \mathfrak{a}(R)$; we thus get a 1-form $\Phi \in \Omega^1(G(k), \wedge^2 \mathfrak{a}(R))$, equivariant w.r.t. left $G(R)$-translations. This 1-form is closed; we hope that the expression of this fact can be interpreted as the generalized Yang-Baxter identity for the dynamical r-matrices of the Hitchin system ([6]).

5. Infinite twist

Let $z = (z_i)$ be a system of local coordinates at each point x_i. Let us conjugate the triple formed by $\hat{\mathfrak{g}}$, $\mathfrak{g}_R$ and the Lagrangean supplementary $(\mathfrak{a} \otimes \Lambda) \oplus \mathbf{C}K$ by the collection $(w_{i,n_i})_i$ of affine Weyl group elements, where w_{i,n_i} is the element of $G(k_{x_i})$ such that

$$\mathrm{Ad}(w_{i,n_i})(x_+ z_i^n) = x_+ z_i^{n+n_i}, \quad \mathrm{Ad}(w_{i,n_i})(x_- z_i^n) = x_- z_i^{n-n_i},$$
$$\mathrm{Ad}(w_{i,n_i})(x_0 z_i^n) = x_0 z_i^{n+n_i},$$

for $x_\pm \in \mathfrak{n}_\pm, x_0 \in \mathfrak{h}$. In the limit where all n_i tend to infinity, we obtain the Manin triple

$$(\hat{\mathfrak{g}}, \mathfrak{g}_+, \mathfrak{g}_-)$$

with

$$\mathfrak{g}_+ = \mathfrak{h}(R) \oplus \mathfrak{n}_+(k) \oplus \mathbf{C}D, \quad \mathfrak{g}_- = (\mathfrak{h} \otimes \Lambda) \oplus \mathfrak{n}_-(k) \oplus \mathbf{C}K.$$

Here $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are the Cartan and opposite nilpotent subalgebras of $\mathfrak{a}$. Note that the conjugation of a Manin triple by a double group element, is equivalent to its twist by a certain cocycle, as it is explained in [10]. Let $\delta : \hat{\mathfrak{g}} \to \wedge^2 \hat{\mathfrak{g}}$ be the cobracket corresponding to the above Manin triple structure, then

$$\delta(x) = \delta_R(x) + \mathrm{ad}(x)(f_1),$$

where

$$f_1 = \sum_i e[e_i] \wedge f[e^i],$$

with $(e^i), (e_i)$ dual bases of R and Λ.

Our aim will be to give a quantization of the Lie bialgebras $(\mathfrak{g}_\pm, \delta_\pm)$, where $\delta_\pm = \delta_{|\mathfrak{g}_\pm}$, in the case where $\mathfrak{a} = \mathfrak{sl}_2(\mathbf{C})$.

The Lie algebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ can be presented as follows. $\mathfrak{g}_+$ has generators D, $h^+[r]$, $r \in R$, and $e[\varepsilon]$, $\varepsilon \in k$, with

$$h^+[\alpha_1 r_1 + \alpha_2 r_2] = \alpha_1 h^+[r_1] + \alpha_2 h^+[r_2], \quad \alpha_i \in \mathbf{C}, r_i \in R, \tag{1.4.1}$$

and

$$e[\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2] = \alpha_1 e[\varepsilon_1] + \alpha_2 e[\varepsilon_2], \quad \alpha_i \in \mathbf{C}, \varepsilon_i \in k; \tag{1.4.2}$$

$\mathfrak{g}_-$ has generators K, $h^-[\lambda]$, $\lambda \in \Lambda$, and $f[\varepsilon]$, $\varepsilon \in k$, with

$$h^-[\alpha_1\lambda_1 + \alpha_2\lambda_2] = \alpha_1 h^-[\lambda_1] + \alpha_2 h^-[\lambda_2], \quad \alpha_i \in \mathbf{C}, \lambda_i \in \Lambda, \tag{1.4.3}$$

$$f[\alpha_1\varepsilon_1 + \alpha_2\varepsilon_2] = \alpha_1 f[\varepsilon_1] + \alpha_2 f[\varepsilon_2], \quad \alpha_i \in \mathbf{C}, \varepsilon_i \in k. \tag{1.4.4}$$

The relations are the following. Let us set $\sum_j \varepsilon^j \otimes \varepsilon_j = \sum_i e^i \otimes e_i + e_i \otimes e^i$. We define the formal series

$$\begin{aligned} e(z) = \sum_j e[\varepsilon^j]\varepsilon_j(z), \qquad f(z) = \sum_j f[\varepsilon^j]\varepsilon_j(z), \\ h^+(z) = \sum h^+[e^i]e_i(z), \quad h^-(z) = \sum h^-[e_i]e^i(z); \end{aligned} \tag{1.4.5}$$

then the Lie algebra relations for $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are respectively

$$\begin{aligned} [h^+[r], h^+[r']] = 0, \quad [h^+[r], e(z)] = 2r(z)e(z), \quad r, r' \in R \\ [D, h^+[r]] = h^+[\partial r], \quad [D, e(z)] = -\partial_z e(z), \quad [e(z), e(w)] = 0 \end{aligned} \tag{1.4.6}$$

and

$$\begin{aligned} [h^-[\lambda], h^-[\lambda']] = 2\sum_i \mathrm{res}_{x_i}(\lambda d\lambda')K, \quad [h^-[\lambda], f(z)] = -2\lambda(z)f(z), \\ [f(z), f(w)] = 0, \quad [K, h^-[\lambda]] = [K, f[\lambda']] = 0, \quad \lambda, \lambda' \in \Lambda, \quad [K, \text{anything}] = 0. \end{aligned} \tag{1.4.7}$$

The relations between the generators of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are given by

$$\begin{aligned} [h_+[r], f(w)] = -2r(w)f(w), \quad [h_-[\lambda], e(w)] = 2\lambda(w)e(w), \\ [h^+[r], h^-[\lambda]] = 2\sum_i \mathrm{res}_{x_i}(rd\lambda)K, \quad [K, h^+[r]] = [K, f([\varepsilon]] = 0, \\ [D, h^-[\lambda]] = h^-[\partial\lambda], \quad [D, f(z)] = -\partial_z f(z), \\ [e(z), f(w)] = (h^+(z) + h^-(z))\delta(z,w) + K\partial_z\delta(z,w), \end{aligned} \tag{1.4.8}$$

$r \in R, \lambda \in \Lambda, \varepsilon \in k$. Certain of the above identities should be understood in the sense of generating series. We have set

$$\delta(z,w) = \sum_j \varepsilon^j(z)\varepsilon_j(w).$$

Let $G(z,w) = \sum_i e^i(z)e_i(w)$. The product $G(z,w)\omega_z$ may be viewed as an expansion, for w near x_i of the Green kernel of X. Let $z = (z_i)$ be a system of local coordinates at each point x_i; $z \in k$, and let us set $r_0(z) = \omega/(dz/z)$. Then

$$\delta(z,w) = G(z,w) + G(w,z) = \frac{1}{r_0(z)} \sum_{i\in\mathbf{Z}} (z/w)^i; \tag{1.4.9}$$

the last identity can be proved viewing the l.h.s. as a kernel.

The pairing between $\mathfrak{g}_+$ and $\mathfrak{g}_-$ is given by

$$\langle D, K\rangle = 2, \quad \langle e(z), f(w)\rangle = \delta(z,w), \quad \langle h^+(z), h^-(w)\rangle = 2G(w,z) \tag{1.4.10}$$

The formulae for the cobracket of $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are then respectively

$$\begin{gathered} \delta_+(e(z)) = e(z)\wedge h^+(z), \quad \delta_+(h^+[r]) = 0, \\ \delta_+(D) = \sum_{i,j} \mathrm{res}_{z=x_i}\mathrm{res}_{w=x_j}\bar\gamma(z,w)(h^+(z)\wedge h^+(w))\omega_z\omega_w, \end{gathered} \tag{1.4.11}$$

and

$$\begin{gathered} \delta_-(f(z)) = h^-(z)\wedge f(z) + K\wedge\partial_z f(z), \\ \delta_-(h^-(z)) = K\wedge\partial_z h^-(z), \quad \delta_-(K) = 0. \end{gathered} \tag{1.4.12}$$

In (1.4.11), we have set

$$\bar\gamma(z,w) = (\partial_z + \partial_w)G(z,w).$$

We have $\bar\gamma \in \wedge^2 R$. Indeed, set $\partial e^i = \sum_j a^i_j e^j$, $\partial e_i = \sum_j c^j_i e_j + d_{ij}e^j$; $a^i_j + c^i_j = 0$ and $d_{ij} + d_{ji} = 0$ because ∂ is anti-self-adjoint. $(\partial_z + \partial_w)G(z,w)$ is then equal to $-\sum_{i,j} d_{ij}e^i\otimes e^j$, and so belongs to $R\otimes R$ and is antisymmetric.

Note that the element $\bar\gamma$ may serve to rewrite the $h^- - h^-$ relations as

$$[h^-(z), h^-(w)] = \bar\gamma(z,w)K.$$

Let us describe how a change of Λ affects $G(z,w) \in R_z((w))$. Let Λ' be another Lagrangean supplementary to R, commensurable with $\mathcal{O}_x$; let π' be the projection of k onto R parallel to Λ'.

Let $(e^i), (e'_i)$ be dual bases of R and Λ', and let $G_{\Lambda'}(z,w) = \sum_i e^i(z)e'_i(w)$, and $\bar\gamma_{\Lambda'}(z,w) = -(\partial_z+\partial_w)G_{\Lambda'}(z,w)$. The projection of Λ on R parallel to Λ' may be viwed as the product with the first component of some antisymmetric element $r_1 \in R\otimes R$, and to any such element corresponds a Lagrangean Λ'. We have then

$$G_{\Lambda'}(z,w) = G(z,w) - r_1(z,w), \quad \bar\gamma_{\Lambda'}(z,w) = \bar\gamma(z,w) + (\partial_z+\partial_w)r_1(z,w). \tag{1.4.13}$$

2. Quantization of $\mathfrak{g}_+$ and $\mathfrak{g}_-$

Let $\hbar$ be a formal parameter and T be the operator $\frac{\mathrm{sh}\hbar\partial}{\hbar\partial} : k[[\hbar]] \to k[[\hbar]]$; since T is symmetric for $\langle , \rangle_{k[[\hbar]]}$, the expression

$$\sum_i Te^i \otimes e_i - e^i \otimes Te_i$$

belongs to $R^{\otimes 2}[[\hbar]]$ and is symmetric. Let us fix $\tau \in (R \otimes R)[[\hbar]]$, such that

$$\tau + \widetilde{\tau} = \sum_i Te^i \otimes e_i - e^i \otimes Te_i, \tag{2.0.1}$$

where we denote $\widetilde{f}(z, w) = f(w, z)$. Let U be the operator from Λ to $R[[\hbar]]$, such that

$$\tau = \sum_i Ue_i \otimes e^i;$$

U verifies

$$\sum (T + U)e_i \otimes e^i + e^i \otimes (T + U)e_i = \sum Te^i \otimes e_i + Te_i \otimes e^i. \tag{2.0.2}$$

We will employ the following notation: for E any vector space, and $\xi \in E \otimes k$, we define ξ_R and ξ_Λ to be the projections of ξ on $E \otimes R$ parallel to $E \otimes \Lambda$ (resp. on $E \otimes \Lambda$ parallel to $E \otimes R$).

1. The Hopf algebra $U_\hbar \mathfrak{g}_+$

Let $U_\hbar \mathfrak{g}_+$ be the algebra with generators $h^+[r]$, $r \in R$, $e[\varepsilon]$, $\varepsilon \in k$, and D, subject to relations (1.4.1-2), organized in generating series (1.4.5), and subject to the relations

$$[h^+[r], h^+[r']] = 0, \quad [h^+[r], e(w)] = 2r(w)e(w), \tag{2.1.1}$$

$$e(z)e(w) = e^{2\hbar \sum ((T+U)e_i)(z)e^i(w)} e(w)e(z), \tag{2.1.2}$$

$$[D, h^+[r]] = h^+[\partial r], \tag{2.1.3}$$

$$[D, e(z)] = -\partial_z e(z) + \hbar[\partial(T + U)h^+ - (T + U)(\partial h^+)_\Lambda](z)e(z); \tag{2.1.4}$$

this algebra has a Hopf structure, with coproduct Δ_+ defined by
(2.1.5)
$$\Delta_+(h^+[r]) = h^+[r] \otimes 1 + 1 \otimes h^+[r], \Delta_+(e(z)) = e(z) \otimes \exp(\hbar((T + U)h^+(z)) + 1 \otimes e(z),$$

and
(2.1.6)
$$\Delta_+(D) = D \otimes 1 + 1 \otimes D - \frac{\hbar}{4}\{h^+[((T + U)e_i)_R] \otimes h^+[\partial e^i] + h^+[(\partial(T + U)e_i)_R] \otimes h^+[e^i]\},$$

counit ε_+ defined to be zero on all generators, and skew antipode defined by

(2.1.7) $$S'_+(h^+[r]) = -h^+[r], \quad S'_+(e(z)) = -\exp(-\hbar((T+U)h^+(z))e(z),$$

(2.1.8) $$S'_+(D) = -D - \frac{\hbar}{2}\sum_i \{h^+[\partial e^i]h^+[((T+U)e_i)_R] + h^+[e^i]h^+[(\partial(T+U)e_i)_R]\}.$$

2. The Hopf algebra $U_\hbar\mathfrak{g}_-$

Let $U_\hbar\mathfrak{g}_-$ be the algebra with generators K, $h^-[\lambda]$, $\lambda \in \Lambda$, $f[\varepsilon]$, $\varepsilon \in k$, subject to relations (1.4.3-4), organized in generating series (1.4.5), and subject to the relations

(2.2.1) $$K \text{ is central}, \; [h^-[\lambda], f(w)] = -2q^{K\partial}((T+U)(q^{-K\partial}\lambda)_\Lambda)(w)f(w),$$

(2.2.2) $$[h^-(z), h^-(w)] = \frac{2}{\hbar}(q^{K(\partial_z+\partial_w)} - q^{-K(\partial_z+\partial_w)})\sum e^i(z)(T+U)e_i(w),$$

(2.2.3) $$f(z)f(w) = q^{K(\partial_z+\partial_w)}\{e^{2\hbar\sum e^i(z)((T+U)e_i)(w)}\}f(w)f(z);$$

this algebra has a Hopf structure, defined by the coproduct Δ_- given by

(2.2.4) $$\Delta_-(K) = K\otimes 1 + 1\otimes K, \; \Delta_-(h^-[\lambda]) = h^-[(q^{K_2\partial}\lambda)_\Lambda]\otimes 1 + 1\otimes h^-[(q^{-K_1\partial}\lambda)_\Lambda],$$

(2.2.5) $$\Delta_-(f(z)) = (q^{-K_2\partial}f)(z)\otimes \exp(-\hbar(q^{K_1\partial}h^-)(z)) + 1\otimes (q^{K_1\partial}f)(z),$$

where, due to the formula for $\Delta_-(K)$, we view Δ_- as a system of maps from $(U_\hbar\mathfrak{g}_-)_{K_1+K_2}$ to $(U_\hbar\mathfrak{g}_-)_{K_1}\otimes(U_\hbar\mathfrak{g}_-)_{K_2}$, for variable scalars K_i, where $(U_\hbar\mathfrak{g}_-)_k = U_\hbar\mathfrak{g}_-/(K-k)$ for any scalar k. The counit ε_- of $U_\hbar\mathfrak{g}_-$ is defined to be zero on all generators, and the skew antipode S'_- is defined by

(2.2.6) $$S'_-(K) = -K, \quad S'_-(h^-[\lambda]) = -h^-[\lambda], \quad S'_-(f(z)) = -\exp(\hbar h^-(z))f(z).$$

3. Hopf algebra pairing between $U_\hbar\mathfrak{g}_+$ and $U_\hbar\mathfrak{g}_-$

The pairing

(2.3.1) $$\langle e[\epsilon], f[\epsilon']\rangle = \langle\epsilon,\epsilon'\rangle_k, \quad \langle h^+[r], h^-[\lambda]\rangle = \frac{2}{\hbar}\langle r,\lambda\rangle_k,$$

(2.3.2) $$\langle D, K\rangle = 1, \quad \langle D, h^-[\lambda]\rangle = \langle D, f(z)\rangle = 0,$$

(2.3.3) $$\langle e[\epsilon], h^-[\lambda]\rangle = \langle e[\epsilon], K\rangle = \langle h^+[r], f[\epsilon']\rangle = \langle h^+[r], K\rangle = 0,$$

extends to a Hopf algebra pairing between $U_\hbar\mathfrak{g}_+$ and $U_\hbar\mathfrak{g}_-$.

4. Double algebra $U_\hbar\mathfrak{g}$

The double $U_\hbar\mathfrak{g}$ of $U_\hbar\mathfrak{g}_+$ then contains both algebras $U_\hbar\mathfrak{g}_+$ and $U_\hbar\mathfrak{g}_-$ as subalgebras; we will denote the same way elements of these algebras and their images in $U_\hbar\mathfrak{g}$. The algebra $U_\hbar\mathfrak{g}$ may then be viewed as the algebra with generators $h^+[r], h^-[\lambda], e[\epsilon], f[\epsilon'], D$ and K subject to the algebra relations above, as well as to the additional relations

(2.4.1) $$K \text{ is central,}$$

(2.4.2) $$[h^+[r], h^-[\lambda]] = \frac{2}{\hbar}\langle (q^{K\partial} - q^{-K\partial})r, \lambda\rangle_k,$$

(2.4.3) $$[h^+[r], f(z)] = -2(q^{K\partial}r)(z)f(z),$$

(2.4.4) $$[h^-[\lambda], e(w)] = 2[(T+U)((q^{K\partial}\lambda)_\Lambda)](w)e(w),$$

(2.4.5) $$[e(z), f(w)] = (q^{K\partial_w}\delta(z,w))q^{(T+U)h^+(z)} - (q^{-K\partial_w}\delta(z,w))q^{-h^-(w)},$$

(2.4.6) $$[D, q^{-h^-(z)}] = -\partial(q^{-h^-})(z) + \hbar\sum_i [(q^{-K\partial} - q^{K\partial})Ae_i](z)h^+[e^i]q^{-h^-(z)}$$

where $A : \Lambda[[\hbar]] \to R[[\hbar]]$ is defined by $A(\lambda) = \partial(T+U)(\lambda) - (T+U)((\partial\lambda)_\Lambda)$, and

(2.4.7) $$[D, f(z)] = -\partial f(z) + \hbar q^{K\partial}[\partial(T+U)h^+ - (T+U)(\partial h^+)_\Lambda](z)f(z).$$

Remarks.

1) We will show later on how to put the $e-e$ and $f-f$ relations in a correct form.

2) Note that the above relations have also variants defined using generating series. For example, $(2.2.4b)$ can also be written

(2.5.1) $$\Delta_- h^-(z) = (q^{-K_2\partial}h^-)(z) \otimes 1 + 1 \otimes (q^{K_1\partial}h^-)(z).$$

We can also write the $h^- - h^-$ commutator as

(2.5.2) $$\begin{aligned}[h^-(z), h^-(w)] =& \frac{1}{\hbar}(T_z + T_w)(q^{K(\partial_z+\partial_w)} - q^{-K(\partial_z+\partial_w)})G(z,w) \\ &+ \frac{2}{\hbar}(q^{K(\partial_z+\partial_w)} - q^{-K(\partial_z+\partial_w)})\sum_i e^i \otimes Ue_i;\end{aligned}$$

recall that $G(z,w) = \sum_i e^i(z)e_i(w)$ and that $(\partial_z + \partial_w)^k G(z,w) \in R \otimes R$ for $k \geq 1$; this shows that the r.h.s. of the formula for $[h^-(z), h^-(w)]$ belongs to $\wedge^2 R$, as it should be.

3. $e - e$ and $f - f$ relations

As we explained in the introduction, the forms (2.1.2) and (2.2.3) of the $e-e$ and $f-f$ relations do not enable us to derive quadratic relations for the $e[\epsilon], f[\epsilon]$. To do that, we need to put these relations in the vertex form (0.2). This will be done by proving some results on the kernels on the curve X.

To see what kind of kernels will be useful, observe that in the quantum affine situation, the quantity of interest will be $k(z,w) = \frac{qz-w}{z-qw}$. We have $\ln k(z,w) = \ln q + \sum_{i\geq 1} \frac{q^n - q^{-n}}{n}(w/z)^n$. Since in that situation, we have $X = \mathbf{C}P^1$, $\partial = z\frac{d}{dz}$, and $G(z,w) = \frac{1}{2} + \sum_{i\geq 1}(w/z)^i$, $k(z,w)$ coincides with

$$\ln q + \frac{q^{\partial_w} - q^{-\partial_w}}{\partial_w} G(z,w)$$

(in this expression, the indices w mean that the operators act on the variable w). We are therefore led to the study of the expression $\frac{q^{\partial_w} - q^{-\partial_w}}{\partial_w} G(z,w)$, in the general situation.

For that, we first study $\partial_z G$. In the case where $X = \mathbf{C}P^1$ and $\omega = dz$, the Green kernel G is equal to $1/(z-w)$, and $\partial_z = d/dz$, so that $\partial_z G$ is exactly equal to G^2. In general, these two quantities have the same most singular terms on the diagonal, but they usually no longer coincide. We now study their difference.

3.1. *Construction of* γ

Let us compute the endomorphism of R, defined by

$$\rho(f)(z) = \text{res}_w G^2(z,w) f(w)\omega_w. \tag{3.1}$$

ρ is the restriction to R of the endomorphism $\bar{\rho}$ of k_x, defined by

$$\bar{\rho}(f) = \text{res}_w (G^2 - \widetilde{G}^2)(z,w) f(w)\omega_w. \tag{3.2}$$

Let $\alpha(z,w) = (z-w)G(z,w)$; we have $\alpha(z,w) = -(z-w)\widetilde{G}(z,w)$ and

$$[z, \bar{\rho}](f) = \text{res}_w \alpha(z,w)(G + \widetilde{G})(z,w) f(w)\omega_w.$$

Now $(G + \widetilde{G})(z,w) = \delta(z/w)/r_0(z)$, so

$$[z, \bar{\rho}](f)(z) = \alpha(z,z) f(z).$$

But $G(z,w) = (\sum_{i\leq -N}(z/w)^i + \sum_{i>-N} z^i \lambda_i(w))/r_0(z)$, with $\lambda_i \in \Lambda$, so that

$$\begin{aligned}(z-w)G(z,w) &= -(w/r_0(z))(z/w)^{-N} + (z-w)\sum_{i\geq -N} z^i \lambda_i(w)/r_0(z) \\ &\in -z/r_0(z) + (z-w)\mathbf{C}((z,w)),\end{aligned}$$

so $\alpha(z,z) = -z/r_0(z)$; so that $[\bar{\rho}, z] = [\partial, z]$; so we have $\bar{\rho} = \partial+$function. Since Λ is isotropic, $\bar{\rho}(1) = 0$; this shows $\bar{\rho} = \partial$.

Let us consider now $\partial_z G(z,w) - G(z,w)^2$; this expression belongs to $R_z \hat{\otimes} k_w$ and the endomorphism of R it defines is zero, so it belongs to $R \otimes R$. We have shown:

PROPOSITION 1. – *There exists $\gamma \in R \otimes R$, such that $\partial_z G(z,w) = G(z,w)^2 + \gamma(z,w)$.*

3.2. The kernel of ∂^n, $n \geq 0$

In this section we will study the expressions $\sum \partial^k e^i \otimes e_i$. For $k=0$, this expression is equal to G; for $k=1$, it is equal to $\partial_z G = G^2 + \gamma$; for $k=2$, it is equal to $\partial_z(G^2+\gamma) = 2G(G^2+\gamma) + \partial_z\gamma = 2G^3 + 2G\gamma + \partial_z\gamma$. More generally, we have:

PROPOSITION 2. – *Let $(P_k^{(n)})_{k\in\mathbf{Z},n\geq 0}$ be the system of polynomials in $\mathbf{C}[\gamma_0,\gamma_1,\dots]$ defined by $P_{<0}^{(0)} = 0$, $P_1^{(0)} = 1$, $P_{>1}^{(0)} = 0$, and $P_k^{(n+1)} = DP_k^{(n)} + (k-1)P_{k-1}^{(n)} + \gamma_0(k+1)P_{k+1}^{(n)}$, for $n \geq 0$, where $D = \sum_{i\geq 0}\gamma_{i+1}\partial/\partial\gamma_i$. Then*

$$\sum \partial^n e^i \otimes e_i = \sum_{k\geq 0} P_k^{(n)}(\gamma, \partial_z\gamma, \dots)G^k, \tag{3.3}$$

$$\sum e^i \otimes \partial^n e_i = (-1)^n \sum_{k\geq 0} P_k^{(n)}((-1)^i \partial_w^i \tilde{\gamma})G^k. \tag{3.4}$$

The proof is a simple induction. Then, we have $\sum(\sum_{n\geq 1}\hbar^n\partial^{n-1}/n!)e^i \otimes e_i = \sum_{k\geq 0}(\sum_{n\geq 1}\frac{\hbar^n}{n!}P_k^{(n-1)}(\gamma,\partial_z\gamma,\dots))G^k$; let us set

$$u_k(\hbar,\gamma_0,\gamma_1,\dots) = \sum_{n\geq 1}\frac{\hbar^n}{n!}P_k^{(n-1)}(\gamma_0,\gamma_1,\dots). \tag{3.5}$$

We have, with t an auxiliary variable, and with $u(\hbar,t,\gamma_i) = \sum_{k\geq 0} t^k u_k$, the equations

$$\frac{\partial u}{\partial \hbar} = t + Du + (t^2+\gamma_0)\frac{\partial u}{\partial t}, \quad u_{|\hbar=0} = 0; \tag{3.6}$$

by Cauchy's principle, they determine u uniquely. Let $\phi, \psi \in \hbar\mathbf{C}[\gamma_i][[\hbar]]$ be the solutions to

$$\frac{\partial\psi}{\partial\hbar} = D\psi - 1 - \gamma_0\psi^2, \quad \frac{\partial\phi}{\partial\hbar} = D\phi - \gamma_0\psi, \tag{3.7}$$

then the expansions of ϕ and ψ are $\psi = -\hbar + \dots$, $\phi = \hbar^2\gamma_0 + \dots$, and ϕ, ψ have the properties

$$\phi(-\hbar,(-1)^i\gamma_i) = \phi(\hbar,\gamma_i), \quad \psi(-\hbar,(-1)^i\gamma_i) = -\psi(\hbar,\gamma_i). \tag{3.8}$$

Moreover, $\phi - \ln(1+t\psi)$ satisfies (3.6); this identifies this function with u.

We conclude from this the first part of

PROPOSITION 3. – *With ϕ and ψ the solutions of* (3.7), *we have*

$$\sum_i \frac{q^\partial - 1}{\partial} e^i \otimes e_i = \phi(\hbar, \partial_z^i\gamma) - \ln(1 + G\psi(\hbar,\partial_z^i\gamma)), \tag{3.9}$$

(3.10) $$\sum_i \frac{1-q^{-\partial}}{\partial} e^i \otimes e_i = -\phi(-\hbar, \partial_z^i\gamma) + \ln(1 + G\psi(-\hbar, \partial_z^i\gamma)),$$

(3.11) $$\sum_i e^i \otimes \frac{q^{\partial}-1}{\partial} e_i = -\phi(\hbar, \partial_w^i\widetilde{\gamma}) + \ln(1 - G\psi(\hbar, \partial_w^i\widetilde{\gamma})),$$

(3.12) $$\sum_i e^i \otimes \frac{q^{-\partial}-1}{\partial} e_i = -\phi(-\hbar, \partial_w^i\widetilde{\gamma}) + \ln(1 - G\psi(-\hbar, \partial_w^i\widetilde{\gamma})).$$

The last part is proved using the last part of prop. 2.

Note that the formal series $f(\partial_z) - f(-\partial_w)$ is divisible by $\partial_z + \partial_w$ in $\mathbf{C}[\partial_z, \partial_w][[\hbar]]$, and denote their ratio by $\frac{f(\partial_z)-f(-\partial_w)}{\partial_z+\partial_w}$.

We then have:

(3.13) $$\phi(\hbar, \partial_z^i\gamma) - \ln(1 + G\psi(\hbar, \partial_z^i\gamma)) - \phi(-\hbar, \partial_w^i\widetilde{\gamma}) + \ln(1 - G\psi(-\hbar, \partial_w^i\widetilde{\gamma})) = \frac{f(\partial_z) - f(-\partial_w)}{\partial_z + \partial_w}(\gamma - \widetilde{\gamma}),$$

with $f(x) = \frac{q^x-1}{x}$; to see it, it suffices to add (3.9) and (3.11) and to use the following lemma.

LEMMA 1. – *We have the equality*

$$(\partial_z + \partial_w)G = \gamma - \widetilde{\gamma}.$$

Proof. – We know that $G^2 - \widetilde{G}^2$ satisfies $\langle G^2 - \widetilde{G}^2, Id \otimes \alpha\rangle_k = \partial\alpha$, for any $\alpha \in k$; on the other hand, $(\partial \otimes 1)(G + \widetilde{G})$ satisfies the same identity. It follows that $G^2 - \widetilde{G}^2 = (\partial\otimes 1)(G+\widetilde{G})$, so that $(\partial\otimes 1)\widetilde{G} = -\gamma - \widetilde{G}^2$. Hence, $(1\otimes\partial)G = -\widetilde{\gamma} - G^2$. ∎

In particular, we have

(3.14)
$$\phi(\hbar, \partial_z^i\gamma) - \phi(-\hbar, \partial_z^i\gamma) + \ln(-\psi(-\hbar, \partial_z^i\gamma)/\psi(\hbar, \partial_z^i\gamma)) = \frac{f(\partial_z) - f(-\partial_w)}{\partial_z + \partial_w}(\gamma - \widetilde{\gamma}) + \nu_0,$$

$\nu_0 \in R \otimes R$, vanishing on the diagonal.

PROPOSITION 4. – *Recall that* $T = \frac{q^{\partial}-q^{-\partial}}{2\hbar\partial}$; *let us set*

$$\psi_0 = \frac{1}{2\hbar}(\phi(\hbar, \partial_z^i\gamma) - \phi(-\hbar, \partial_z^i\gamma)), \quad \psi_+(\gamma_i) = \psi(-\hbar, \gamma_i), \quad \psi_-(\gamma_i) = \psi(\hbar, \gamma_i),$$

then

(3.15) $$\sum Te^i \otimes e_i = \psi_0(\gamma, \partial_z\gamma, ...) + \frac{1}{2\hbar}\ln\frac{1 + G\psi_+(\gamma, \partial_z\gamma, ...)}{1 + G\psi_-(\gamma, \partial_z\gamma, ...)},$$

and

$$\sum e^i \otimes Te_i = -\psi_0(\partial_w^i \widetilde{\gamma}) + \frac{1}{2\hbar} \ln \frac{1 - G\psi_-(\partial_w^i \widetilde{\gamma})}{1 - G\psi_+(\partial_w^i \widetilde{\gamma})}. \tag{3.16}$$

From (3.13) follows

$$\begin{aligned}\psi_0(\partial_z^i \gamma) + \frac{1}{2\hbar} \ln \frac{1 + G\psi_+(\partial_z^i \gamma)}{1 + G\psi_-(\partial_z^i \gamma)} + \psi_0(\partial_w^i \widetilde{\gamma}) - \frac{1}{2\hbar} \ln \frac{1 - G\psi_-(\partial_w^i \widetilde{\gamma})}{1 - G\psi_+(\partial_w^i \widetilde{\gamma})} \\ = \frac{T_z - T_w}{\partial_z + \partial_w}(\gamma - \widetilde{\gamma}).\end{aligned} \tag{3.17}$$

Remark that

$$\sum Te^i \otimes e_i - e^i \otimes Te_i = \frac{T_z - T_w}{\partial_z + \partial_w}(\gamma - \widetilde{\gamma}); \tag{3.18}$$

recall that $\tau = \sum Ue_i \otimes e^i$ satisfies

$$\tau + \widetilde{\tau} = \frac{T_z - T_w}{\partial_z + \partial_w}(\gamma - \widetilde{\gamma}). \tag{3.19}$$

Let us precise now the $e - e$ and $f - f$ relations. Let

$$A = \sum e^i \otimes (T + U)e_i, \tag{3.20}$$

then $A = \sum Te^i \otimes e_i + e^i \otimes [(Te_i)_R + Ue_i]$; so

$$\begin{aligned}A &= -\tau + \psi_0(\gamma, \partial_z \gamma, ...) + \frac{1}{2\hbar} \ln \frac{1 + G\psi_+(\gamma, \partial_z \gamma, ...)}{1 + G\psi_-(\gamma, \partial_z \gamma, ...)} \\ &= \widetilde{\tau} - \psi_0(\widetilde{\gamma}, \partial_w \widetilde{\gamma}, ...) + \frac{1}{2\hbar} \ln \frac{1 - G\psi_-(\widetilde{\gamma}, \partial_w \widetilde{\gamma}, ...)}{1 - G\psi_+(\widetilde{\gamma}, \partial_w \widetilde{\gamma}, ...)}.\end{aligned} \tag{3.21}$$

The relations are then written

$$\begin{aligned}e^{2\hbar\psi_0(\gamma, \partial_z \gamma, ...)}[z - w + \alpha(z, w)\psi_+(\gamma, \partial_z \gamma, ...)]e(z)e(w) = \\ e^{2\hbar\tau(z,w)}[z - w + \alpha(z, w)\psi_-(\gamma, \partial_z \gamma, ...)]e(w)e(z)\end{aligned} \tag{3.22}$$

$$\begin{aligned}q^{K(\partial_z + \partial_w)}\{e^{2\hbar\tau(z,w)}[z - w + \alpha(z, w)\psi_-(\gamma, \partial_z \gamma, ...)]\}f(z)f(w) = \\ q^{K(\partial_z + \partial_w)}\{e^{2\hbar\psi_0(\gamma, \partial_z \gamma, ...)}[z - w + \alpha(z, w)\psi_+(\gamma, \partial_z \gamma, ...)]\}f(w)f(z);\end{aligned} \tag{3.23}$$

recall that $\alpha(z, w) = (z - w)G(z, w)$ belongs to $(\oplus_i k_{z_i})^{\hat{\otimes} 2}$.

Relation (3.22) and (3.23) should be understood as follows. Expand for example $e(z)$ as $\sum_{k \in \mathbf{Z}} e_k z^{-k}$. Then (3.23) gives a formula for $[e_k, e_{l+1}] - [e_{k+1}, e_l]$ in terms of a formal series with positive powers in $\hbar$, with coefficients of the form $\sum_{i,j \geq p} a_{ij} e_i e_j$. Summing up such relations for (k, l), $(k + 1, l - 1)$, etc., and using either $[e_m, e_m] = 0$ (it $k - l$ is odd) are $[e_{m+1}, e_m] + [e_m, e_{m+1}] = 0$ (if it is even), we arrive at a formula expressing $[e_k, e_l]$

as a formal series with positive powers in $\hbar$, with coefficients of the form $\sum_{i,j\geq p} a_{ij}e_ie_j$. This means that these relations define the structure of a topological Hopf algebra on $U_\hbar\mathfrak{g}$ (the topology of this algebra being defined by the basis of neighborhoods of zero defined by the two-sided ideals generated by $\hbar^N$ and the $x[\varepsilon]$, $x = e, f, h^+$, $\varepsilon \in z^M\mathcal{O}$).

Let us show that the $e-e$ and $f-f$ relations define a flat deformation of the symmetric algebras in the $e[\varepsilon]$ and $f[\varepsilon]$, $\varepsilon \in k$. The reason why this deformation could be non-flat is that in the procedure described above, we could have used relation (3.22) with z and w exchanged instead. This could lead to other formulas for $[e_k, e_l]$. To show that this is not the case, we will prove that one can pass from one of these identities to the other (with both sides exchanged) by multiplying it by an element ξ of $1+\hbar\prod_{i,j}\mathbf{C}((z_i,w_j))[[\hbar]]$.

Because of the identity $(z-w)(G+\widetilde{G})=0$, it is enough for this element to satisfy both

$$\xi\cdot e^{2\hbar\psi_0}[1+G\psi_+] = e^{2\hbar\widetilde{\tau}}[1-G\widetilde{\psi}_-]$$

and

$$\xi\cdot e^{2\hbar\tau}[1+G\psi_-] = e^{2\hbar\widetilde{\psi}_0}[1-G\widetilde{\psi}_+],$$

where $\psi_i = \psi_i(\gamma,\partial_z\gamma,\cdots)$.

Using (3.9) and (3.11), the first identity becomes

$$\xi = e^{2\hbar\widetilde{\tau}}e^{-2\hbar\psi_0}e^{\widetilde{\phi(-\hbar)}-\phi}e^{\sum_i \frac{q^\partial-1}{\partial}e^i\otimes e_i - e^i\otimes\frac{q^{-\partial}-1}{\partial}e_i},$$

and using (3.10) and (3.12) the second one becomes

$$\xi = e^{-2\hbar\tau}e^{2\hbar\widetilde{\psi}_0}e^{-\phi(-\hbar)+\widetilde{\phi}}e^{\sum_i -e_i\otimes\frac{q^\partial-1}{\partial}e^i+\frac{q^{-\partial}-1}{\partial}e_i\otimes e^i}.$$

Note that since $\sum_i \frac{q^\partial-1}{\partial}e^i\otimes e_i - e^i\otimes\frac{q^{-\partial}-1}{\partial}e_i$ and $-\sum_i e_i\otimes\frac{q^\partial-1}{\partial}e^i+\frac{q^{-\partial}-1}{\partial}e_i\otimes e^i$ both belong to $\hbar(R\otimes R)[[\hbar]]$, the r.h.s. of both equations belong to $1+\hbar(R\otimes R)[[\hbar]]$, as desired.

The fact that they coincide is a consequence of

$$e^{2\hbar(-\psi_0-\widetilde{\psi}_0)}e^{2\hbar\frac{T_z-T_w}{\partial_z+\partial_w}(\gamma-\widetilde{\gamma})}\frac{1+G\psi_-(\gamma,\partial_z\gamma,\ldots)}{1+G\psi_+(\gamma,\partial_z\gamma,\ldots)}\frac{1-G\psi_-(\widetilde{\gamma},\partial_w\widetilde{\gamma},\ldots)}{1-G\psi_+(\widetilde{\gamma},\partial_w\widetilde{\gamma},\ldots)}=1,$$

which amounts to the statement (3.17) above.

To summarize, we have:

THEOREM 5. – *Let $\tau \in R\otimes R[[\hbar]]$ satisfy (2.0.1). The algebra $U_{\hbar,\Lambda,\tau}\hat{\mathfrak{g}}$ defined by generators K, D, $h^+[r]$, $h^-[\lambda]$, $e[\varepsilon]$, $f[\varepsilon]$, $\lambda\in\Lambda$, $r\in R$, $\varepsilon\in k$, subject to relations* (1.4.1-4) *organized in generating series* (1.4.5), *subject to relations*

$$[h^+(z), e(w)] = 2(\sum_i e_i\otimes e^i)e(w),$$

$$[h^-(z), e(w)] = 2\big(\sum_i q^{-K\partial}e^i\otimes(T+U)e_i\big)e(w),$$

$$[h^+(z), f(w)] = -2(\sum_i e_i\otimes q^{K\partial}e^i)f(w),$$

$$[h^-(z), f(w)] = -2\big(\sum_i q^{K\partial} e^i \otimes q^{K\partial}(T+U)e_i\big) f(w),$$

$$[h^+(z), h^+(w)] = 0, \quad [h^+(z), h^-(w)] = \frac{2}{\hbar} \sum_i e_i \otimes (q^{K\partial} - q^{-K\partial}) e^i,$$

$$[h^-(z), h^-(w)] = \frac{1}{\hbar}(q^{K\partial} \otimes q^{K\partial} - q^{-K\partial} \otimes q^{-K\partial}) \sum_i e^i \otimes (T+U)e_i,$$

$$[e(z), f(w)] = (q^{K\partial_w}\delta(z,w)) q^{(T+U)h^+(z)} - (q^{-K\partial_w}\delta(z,w)) q^{-h^-(w)},$$

where the variables z *and* w *are attached respectively to the first and second factor of the tensor products;* K *is central,* (3.22), (3.23); (2.1.3), (2.1.4), (2.4.6), (2.4.7); *with coproduct defined by* (2.1.5), (2.1.6), *and to be opposite to* (2.2.4), (2.2.5), *counit defined to be zero on all generators, and skew antipode* S' *defined by to coincide with* S'_+ *given by* (2.1.7), (2.1.8), *and*

$$S'(K) = -K, \quad S'(h^-[\lambda]) = -h^-[\lambda], \quad S'(f(z)) = -f(z)\exp(\hbar h^-(z)),$$

is a Hopf algebra, quantizing the Manin triple of section 1.4.

4. Dependence in τ and Λ

1. Dependence in τ

Let us study the dependence of the algebra $U_{\hbar,\Lambda,\tau}\hat{\mathfrak{g}}$ defined in thm. 5, with respect to τ. Let $\tau' = \tau + \upsilon$, $\upsilon \in \wedge^2 R[[\hbar]]$. Let us denote with a prime all quantities corresponding to the algebra $U_{\hbar,\Lambda,\tau'}\hat{\mathfrak{g}}$. Let us denote by $u : \Lambda \to R[[\hbar]]$ the linear map defined by $u(\lambda) = \langle \upsilon, 1 \otimes \lambda \rangle_k$. We have

$$u = U' - U, \quad \tau' - \tau = \upsilon = \sum_i (U' - U)e_i \otimes e^i. \tag{4.1}$$

Then:

PROPOSITION 6. – *The formulae*

$$i(e'(z)) = e^{\frac{1}{2}\hbar u(h^+)(z)} e(z), \quad i(f'(z)) = f(z) e^{\frac{\hbar}{2}(q^{K\partial_z}) u(h^+)(z)}, \tag{4.2}$$

$$i(h^{+\prime}(z)) = h^+(z), \quad i(h^{-\prime}(z)) = h^-(z) - \left(\frac{q^{K\partial} + q^{-K\partial}}{2} u h^+\right)(z), \tag{4.3}$$

$i(K') = K$, $i(D') = D$, *define an algebra isomorphism* $i : U_{\hbar,\Lambda,\tau'}\hat{\mathfrak{g}} \to U_{\hbar,\Lambda,\tau}\hat{\mathfrak{g}}$. *Moreover, we have*

$$\Delta(i(x)) = \mathrm{Ad}\exp\left(\frac{\hbar}{4}\sum_i h^+[\upsilon_i] \otimes h^+[\upsilon_i']\right)\{(i \otimes i)\Delta'(x)\}, \quad \forall x \in U_{\hbar,\Lambda,\tau'}\hat{\mathfrak{g}}, \tag{4.4}$$

with $\upsilon = \sum_i \upsilon_i \otimes \upsilon_i'$, *so that both Hopf algebra structures are isomorphic up to a twist operation.*

Proof. – i is well-defined, because $uh^+(z)$ is expressed as $\sum_{i\geq 0}\hbar^i\sum_j h^+(r_j^{(i)})r_j^{\prime(i)}(z)$, $r_j^{(i)}$, $r_j^{\prime(i)}\in R$, the sums $\sum_j h^+(r_j^{(i)})r_j^{\prime(i)}(z)$ being finite. To prove e.g. that i is an algebra morphism, we make use (while checking the $e-f$ relations) of the following sequence of identities:

$$
\begin{aligned}
&(q^{-K\partial_w}\delta(z,w))e^{\frac{\hbar}{2}(uh^+)(z)}q^{-h^-(w)}e^{\frac{\hbar}{2}(q^{K\partial}uh^+)(w)} = \\
&= (q^{-K\partial_w}\delta(z,w))e^{\frac{\hbar}{2}(uh^+)(z)}q^{-h^-(w)}e^{\frac{\hbar}{2}(q^{2K\partial}uh^+)(z)} \\
&= (q^{-K\partial_w}\delta(z,w))e^{-\frac{\hbar}{2}[((1-q^{2K\partial})uh^+)(z),h^-(w)]}q^{-h^-(w)+\frac{1}{2}((1+q^{-2K\partial})uh^+)(z)} \\
&= (q^{-K\partial_w}\delta(z,w))e^{-\frac{\hbar}{2}\{(1-q^{2K\partial})\otimes\frac{2}{\hbar}(q^{K\partial}-q^{-K\partial})\}(\sum ue_i\otimes e^i)}q^{-h^-(w)+\frac{1}{2}((1+q^{-2K\partial})uh^+)(z)} \\
&= q^{-K\partial}\{\delta(z,w)e^{(q^{2K\partial}-1)\otimes(q^{2K\partial}-1)v}\}q^{-h^-(w)+\frac{1}{2}((q^{K\partial}+q^{-K\partial})uh^+)(w)} \\
&= (q^{-K\partial_w}\delta(z,w))q^{-h^-(w)+\frac{1}{2}((q^{K\partial}+q^{-K\partial})uh^+)(w)}
\end{aligned}
$$

(the first identity follows from $\delta(z,w)f(z)=\delta(z,w)f(w)$, the second from $e^ae^b = e^be^ae^{[a,b]}$ if $[a,b]$ is scalar, the last one from the fact that v is antisymmetric, so that $\{(q^{2K\partial}-1)\otimes(q^{2K\partial}-1)\}v$ vanishes on the diagonal). The other identities are easily checked. While checking the twist identity for $f'(z)$, we use also the fact that

$$
\begin{aligned}
[((q^{K\partial}-q^{-K\partial})uh^+)(z),h^-(z)] &= \sum_i (q^{K\partial}-q^{-\partial})v_i(z)[h^+(v_i'),h^-(z)] \\
&= \sum_i (q^{K\partial}-q^{-K\partial})v_i(z)\frac{2}{\hbar}(q^{K\partial}-q^{-K\partial})v_i'(z) \\
&= 0,
\end{aligned}
$$

with $v=\sum_i v_i\otimes v_i'$, because $\{(q^{K\partial}-q^{-K\partial})\otimes(q^{K\partial}-q^{-K\partial})\}v\in\wedge^2R[[\hbar]]$. ∎

PROPOSITION 7. – *The formulae*

$$i'(e'(z)) = e(z)e^{\frac{1}{2}\hbar u(h^+)(z)},\quad i'(f'(z)) = e^{\frac{\hbar}{2}(q^{K\partial_z})u(h^+)(z)}f(z), \tag{4.5}$$

$$i'(h^{+\prime}(z)) = h^+(z),\quad i'(h^{-\prime}(z)) = h^-(z) - \left(\frac{q^{K\partial}+q^{-K\partial}}{2}uh^+\right)(z), \tag{4.6}$$

$i'(K')=K$, $i'(D')=D$, *also define an algebra isomorphism* $i' : U_{\hbar,\Lambda,\tau'}\hat{\mathfrak{g}} \to U_{\hbar,\Lambda,\tau}\hat{\mathfrak{g}}$, *satisfying*

$$\Delta(i'(x)) = \operatorname{Ad}\exp\left(\frac{\hbar}{4}(h^+\otimes h^+)v\right)\{(i'\otimes i')\Delta'(x)\},\quad \forall x\in U_{\hbar,\Lambda,\tau'}\hat{\mathfrak{g}}. \tag{4.7}$$

It follows that $i'^{-1}\circ i$ *is a Hopf algebra automorphism of* $U_{\hbar,\Lambda,\tau'}\hat{\mathfrak{g}}$.

2. Dependence in Λ

Let Λ and $\bar{\Lambda}$ be two Lagrangean supplementaries to R. Then we have, $\bar{\Lambda}=(1+r)\Lambda$, with $r:\Lambda\to R$, given by

$$r(\lambda) = \langle r_0, 1\otimes\lambda\rangle,\quad r_0\in\wedge^2R. \tag{4.8}$$

Dual bases for R and $\bar{\Lambda}$ are then (e^i) and $(\bar{e}_i)$, with $\bar{e}_i = (1+r)e_i$. Let us set

$$\bar{\tau} = \sum_i U\bar{e}_i \otimes e^i = \tau - \sum_i Tre_i \otimes e^i; \tag{4.9}$$

we have then

$$\sum_i (T+\bar{U})\bar{e}_i \otimes e^i = \sum_i (T+U)e_i \otimes e^i. \tag{4.10}$$

Let us consider the Hopf algebras $U_{\hbar,\Lambda,\tau}\hat{\mathfrak{g}}$, $U_{\hbar,\bar{\Lambda},\bar{\tau}}\hat{\mathfrak{g}}$ and let us denote with a bar the quantities occurring in the second.

PROPOSITION 8. – *The mapping*

$$j : U_{\hbar,\Lambda,\tau}\hat{\mathfrak{g}} \to U_{\hbar,\bar{\Lambda},\bar{\tau}}\hat{\mathfrak{g}}$$

defined by $j(\bar{e}(z)) = e(z)$, $j(\bar{f}(z)) = f(z)$, $j(\bar{h}^+(e^i)) = h^+(e^i)$, $j(\bar{h}^-(\bar{e}_i)) = h^-(e_i)$, $j(\bar{D}) = D$, $j(\bar{K}) = K$, *defines a Hopf algebras isomorphism between* $U_{\hbar,\Lambda,\tau}\hat{\mathfrak{g}}$ *and* $U_{\hbar,\bar{\Lambda},\bar{\tau}}\hat{\mathfrak{g}}$.

5. Finite dimensional representations

Let us fix Λ and τ, and denote by $U_\hbar\hat{\mathfrak{g}}_{|K=0,\mathrm{no}D}$ the algebra defined in thm. 5, without generator D and with K specialized to zero. We construct a morphism of algebras

$$\pi : U_\hbar\hat{\mathfrak{g}}_{|K=0,\mathrm{no}D} \to \mathrm{End}(\mathbf{C}^2) \otimes k[[\hbar]], \tag{5.1}$$

as follows: let us denote by $\zeta = (\zeta_i)$ the system of coordinates (z_i), occurring in the r.h.s.; we define

$$\pi(h^+[r]) = r(\zeta)h + \rho^+(r)(\zeta)\mathrm{Id}_{\mathbf{C}^2}, \pi(h^-[\lambda]) = (T+U)(\lambda)(\zeta)h + \rho^-(\lambda)\mathrm{Id}_{\mathbf{C}^2}, \tag{5.2}$$

$$\pi(e(z)) = F(\zeta)\delta(z,\zeta)e, \quad \pi(f(z)) = \delta(z,\zeta)f, \tag{5.3}$$

where the e, f, h occurring in the r.h.s. are the matrices with nonzero coefficients $e_{12} = f_{21} = h_{11} = -h_{22} = 1$, and

$$\rho^+ : R \to R[[\hbar]], \rho^- : \Lambda \to (k)[[\hbar]], F(\zeta) \in k[[\hbar]]$$

are subject to the following conditions: recall that $A(\zeta, z) = \sum e^i(\zeta)(T+U)(e_i)(z)$, and let

$$\beta(\zeta,z) = \sum \rho^+(e^i)(\zeta)(T+U)(e_i)(z), \gamma(\zeta,z) = \sum \rho^-(e_i)(\zeta)e^i(z); \tag{5.4}$$

then

$$q^{A+\beta} - q^{-\widetilde{A}-\gamma} = F(z)\delta(z,\zeta), q^{-A+\beta} - q^{\widetilde{A}-\gamma} = -F(z)\delta(z,\zeta). \tag{5.5}$$

Let

$$F(z)\delta(z,\zeta) = e^{\sigma}\hbar(G+\widetilde{G})(z,\zeta), \tag{5.6}$$

with $\sigma \in \hbar(R\otimes R)[[\hbar]]$, then we have for some $\rho_{1,2} \in \hbar^{-1} + (R\otimes R)[[\hbar]]$,

$$A = \frac{1}{2\hbar}\ln\frac{\rho_1+G}{\rho_2-G}, \beta = \frac{1}{2\hbar}\ln\hbar^2(\rho_1+G)(\rho_2-G)+\sigma, \tag{5.7}$$

$$\widetilde{A} = \frac{1}{2\hbar}\ln\frac{\rho_2+\widetilde{G}}{\rho_1-\widetilde{G}}, \gamma = -\frac{1}{2\hbar}\ln\hbar^2(\rho_1-\widetilde{G})(\rho_2+\widetilde{G})-\sigma. \tag{5.8}$$

Let us determine the possible $\rho_{1,2}$ satisfying the first equations of (5.7) and (5.8) (which we will call (5.7.a), (5.8.a)). Comparing (5.7.a) and the second line of (3.21), it is enough to have

$$\ln\hbar(\rho_1+G) = \ln(1-G\psi_-(\partial_w^i\widetilde{\gamma}))+2\hbar\lambda \tag{5.9}$$

$$\ln\hbar(\rho_2-G) = \ln(1-G\psi_+(\partial_w^i\widetilde{\gamma}))+2\hbar\bar{\lambda} \tag{5.10}$$

with $\lambda,\bar{\lambda} \in (R\otimes R)[[\hbar]]$, $\lambda-\bar{\lambda} = \widetilde{\tau}-\psi_0(\partial_w^i\widetilde{\gamma})$; and comparing (5.8.a) and the first line of (3.21), it is enough to have

$$\ln\hbar(\widetilde{\rho}_2+G) = \ln(1+G\psi_+(\partial_z^i\gamma))+2\hbar\mu \tag{5.11}$$

$$\ln\hbar(\widetilde{\rho}_1-G) = \ln(1+G\psi_-(\partial_z^i\gamma))+2\hbar\bar{\mu} \tag{5.12}$$

with $\mu,\bar{\mu} \in R\otimes R$, and $\mu-\bar{\mu} = \psi_0(\partial_z^i\gamma)-\tau$.

(5.10-12) are equivalent to the fact that for certain $\nu,\nu' \in R\otimes R$, equal to 0 on the diagonal,

$$\bar{\lambda} = \widetilde{\mu} = \nu + \frac{1}{2\hbar}\ln(\hbar/\psi_+(\partial_w^i\widetilde{\gamma})), \rho_2 = \frac{e^{\nu}}{\psi_+(\partial_w^i\widetilde{\gamma})}+G(1-e^{\nu}), \tag{5.13}$$

and

$$\lambda = \widetilde{\bar{\mu}} = \nu' + \frac{1}{2\hbar}\ln(-\hbar/\psi_-(\partial_w^i\widetilde{\gamma})), \rho_1 = -\frac{e^{\nu'}}{\psi_-(\partial_w^i\widetilde{\gamma})}+G(e^{\nu'}-1), \tag{5.14}$$

with the conditions on ν and ν'

$$\nu'-\nu+\frac{1}{2\hbar}\ln(\psi_+(\partial_w^i\widetilde{\gamma})/-\psi_-(\partial_w^i\widetilde{\gamma})) = \widetilde{\tau}-\psi_0(\partial_w^i\widetilde{\gamma}). \tag{5.15}$$

Let us see now, how $\rho^{\pm}$ can be deduced from these equalities. The conditions on them are

$$\ln(1-G\psi_-(\partial_w^i\widetilde{\gamma}))+2\hbar\lambda = \hbar(1+\rho^+)(e^i)\otimes(T+U)(e_i)-\hbar\sigma, \tag{5.16}$$

$$\ln(1 - G\psi_+(\partial_w^i\widetilde{\gamma})) + 2\hbar\bar{\lambda} = \hbar(\rho^+ - 1)(e^i) \otimes (T+U)(e_i) - \hbar\sigma, \tag{5.17}$$

$$\ln(1 + G\psi_+(\partial_z^i\gamma)) + 2\hbar\mu = \hbar\{e^i \otimes (T+U)(e_i) - e^i \otimes \rho^-(e_i)\} - \hbar\widetilde{\sigma}, \tag{5.18}$$

$$\ln(1 + G\psi_-(\partial_z^i\gamma)) + 2\hbar\bar{\mu} = -\hbar\{e^i \otimes (T+U)(e_i) + e^i \otimes \rho^-(e_i)\} - \hbar\widetilde{\sigma}. \tag{5.19}$$

Let T_+, T_- be the endomorphisms of R, defined by

$$T_\pm(r) = \langle \ln(1 - G\psi_\mp(\partial_w^i\widetilde{\gamma})), 1 \otimes r\rangle; \tag{5.20}$$

we have $T_+ = \frac{1-q^{-\partial}}{\partial}$, $T_- = \frac{1-q^{\partial}}{\partial}$. Since $T_{>pm} = \hbar(\rho^+ \pm 1)T$, (recall that $T = \frac{q^\partial - q^{-\partial}}{2\hbar\partial}$),

$$\rho^+ = \frac{1-q^\partial}{1+q^\partial}. \tag{5.21}$$

Due to (5.13) (resp. (5.14)), (5.16) and (5.17) (resp. (5.18) and (5.19)) are equivalent. (2.0.1) and (5.15) can be solved by posing

$$\tau = \frac{1}{2\hbar}\frac{f(\partial_z) - f(-\partial_w)}{\partial_z + \partial_w}(\gamma - \widetilde{\gamma}), \quad \nu' = 0, \nu = \widetilde{\nu}_0; \tag{5.22}$$

this follows directly from Lemma 1.

Let us explain the meaning of (5.22). The formal series $f(\partial_z) - f(-\partial_w)$ is divisible by $\partial_z + \partial_w$; we denote the corresponding quotient by $\frac{f(\partial_z)-f(-\partial_w)}{\partial_z+\partial_w}$. This is an element of $\mathbf{C}[\partial_z, \partial_w][[\hbar]]$. In (5.22) the operators ∂_z and ∂_w the act as usual as partial derivatives in z and w. We will indicate after Prop. 9 how to modify our result in the case of a general solution τ of (2.0.1).

(5.16) then gives us

$$\begin{aligned}\hbar\sigma = &\frac{1}{\partial_z + \partial_w}\left[\frac{1}{1+q^{\partial_z}}(f(-\partial_z) + f(-\partial_w)) - f(\partial_w)\right](\gamma - \widetilde{\gamma}) \\ &+ \phi(\hbar, \partial_w^i\widetilde{\gamma}) - \ln(-\hbar/\psi_-(\partial_w^i\widetilde{\gamma})),\end{aligned} \tag{5.23}$$

and (5.19) gives then

$$\begin{aligned}\sum e^i \otimes \rho^-(e_i) = \sum e^i \otimes \frac{(q^\partial - 1)(1 - q^{-\partial})}{2\hbar\partial} e_i - \frac{1}{\hbar}[\phi(-\hbar, \partial_z^i\gamma) + \phi(\hbar, \partial_z^i\gamma)] \\ - \frac{1}{\hbar}\frac{1}{\partial_z + \partial_w}\left[\frac{1}{2}(f(\partial_w) - f(-\partial_z)) - \frac{1}{1+q^{\partial_w}}(f(-\partial_w) + f(-\partial_z)) + f(\partial_z)\right](\gamma - \widetilde{\gamma})\end{aligned}$$

and so

$$\begin{aligned}\rho^-(\lambda) = &\frac{(q^\partial - 1)(1 - q^{-\partial})}{2\hbar\partial}\lambda - \frac{1}{\hbar}\Big\langle [\phi(-\hbar, \partial_z^i\gamma) + \phi(\hbar, \partial_z^i\gamma)] \\ &+ \frac{1}{\partial_z + \partial_w}\left[\frac{1}{2}(f(\partial_w) - f(-\partial_z)) - \frac{1}{1+q^{\partial_w}}(f(-\partial_w)\right. \\ &\left.+ f(-\partial_z)) + f(\partial_z)\right](\gamma - \widetilde{\gamma}), \lambda \otimes 1\Big\rangle\end{aligned} \tag{5.24}$$

for $\lambda \in \Lambda$. So we have:

PROPOSITION 9. – *The formulae* (5.2), (5.3) *define a morphism of algebras*

$$\pi : U_\hbar \hat{\mathfrak{g}}_{|K=0,\mathrm{no}D} \to \mathrm{End}(\mathbf{C}^2) \otimes k[[\hbar]],$$

provided τ is chosen according to (5.22), *with F and $\rho^\pm$ given by* (5.6), (5.21), (5.23) *and* (5.24).

Let us indicate how the formulae giving $\rho^\pm$ and σ would be altered in the case of an arbitrary τ (satisfying (2.0.1)). Let us denote with an exponent $^{(0)}$ the quantities implied in prop. 9. The general form of a solution of (2.0.1) is $\tau = \tau^{(0)} + \alpha$, $\alpha \in \wedge^2 R[[\hbar]]$; we have then $\sigma = \sigma^{(0)} - ((1+\rho^+) \otimes 1)\alpha$, $\rho^+ = \rho^{+(0)}$, $\rho^-(\lambda) = \rho^{-(0)}(\lambda) - \rho^+(\langle \lambda \otimes 1, \alpha \rangle)$.

6. Examples

1. Trigonometric case

Let $X = \mathbf{C}P^1$, let z be a coordinate on X, and let $\omega = dz/z$. The set of marked points is $\{0, \infty\}$. Let us pose

$$\Lambda = \{(\lambda_0, \lambda_\infty) \in \mathbf{C}[[z]] \times \mathbf{C}[[z^{-1}]] | \lambda_0(0) + \lambda_\infty(\infty) = 0\}.$$

Dual bases for R and Λ are $e^i = z^i$ for $i \in \mathbf{Z}$, and $e_i = (z^{-i}, 0)$ for $i < 0$, $-(0, z^{-i})$ for $i > 0$, $\frac{1}{2}(1, -1)$ for $i = 0$. We compute then

$$\sum_i Te^i \otimes e_i = \frac{1}{2\hbar}\left(\ln \frac{qz - w}{z - qw}, 0\right) - \frac{1}{2\hbar}\left(0, \ln \frac{qw - z}{w - qz}\right),$$

so that we can take $U = 0$; $\exp(2\hbar \sum_i Te^i \otimes e_i) = (\frac{qz-w}{z-qw}, \frac{qz-w}{z-qw})$ and the $e - e$ relation is

$$(z - qw)e(z)e(w) = (qz - w)e(w)e(z),$$

as it appeared first in [4].

2. Elliptic case

Let X be the elliptic curve $\mathbf{C}/\mathbf{Z} + \tau_0 \mathbf{Z}$; let z be the coordinate on $\mathbf{C}$, and let $\omega = dz$. Let us consider the case $\{x_i\} = \{0\}$. We choose Λ to be spanned by $z^{-1}, z, z^2, z^3, \ldots$.

We define $t = e^{2i\pi z}$, $q_0 = e^{2i\pi\tau_0}$ (we assume $|q_0| < 1$);

$$\theta(t) = \prod_{n \geq 0} (1 - q_0^n t) \prod_{n > 0} (1 - q_0^n t^{-1}), \quad \zeta = \frac{d}{dz}(\ln \theta).$$

Let us compute the kernel of T. We have $G(z, w) = \zeta(z - w) - \zeta(z) + \zeta(w)$, so that for $r \in R$,

$$r(z) = \mathrm{res}_{w=0}(\zeta(z - w) - \zeta(z) + \zeta(w))r(w)dw,$$

and

$$[(q^{\partial}-q^{-\partial})r](z)=\mathrm{res}_{w=0}(\zeta(z-w+\hbar)-\zeta(z+\hbar)-\zeta(z-w-\hbar)+\zeta(z-\hbar))r(w)dw,$$

so

$$[\partial^{-1}(q^{\partial}-q^{-\partial})r](z)=\mathrm{res}_{w=0}\ln\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}\frac{\theta(z-\hbar)}{\theta(z+\hbar)}\frac{\theta(-w-\hbar)}{\theta(-w+\hbar)}r(w)dw.$$

We have then,

$$\sum\partial^{-1}(q^{\partial}-q^{-\partial})e^i\otimes e_i\in\ln\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}\frac{\theta(z-\hbar)}{\theta(z+\hbar)}\frac{\theta(-w-\hbar)}{\theta(-w+\hbar)}+R\otimes R[[\hbar]]$$

and so

$$\sum e^i\otimes[-\partial^{-1}(q^{\partial}-q^{-\partial})+U]e_i\in 2i\pi\hbar+\ln\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}\frac{\theta(z-\hbar)}{\theta(z+\hbar)}\frac{\theta(-w-\hbar)}{\theta(-w+\hbar)}+(\wedge^2R)[[\hbar]];$$

so that in the present case, the $e-e$ relation takes the form

$$e(z)e(w)=e^{2i\pi\hbar}\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}\frac{\theta(z-\hbar)}{\theta(z+\hbar)}\frac{\theta(-w-\hbar)}{\theta(-w+\hbar)}e(w)e(z);\tag{6.1}$$

this relation is analogous to the relation (7.3) occurring in [8].

The full set of relations is in addition to (6.1) (for more symmetry in the relations, we replace K and $f(z)$ by $2K$ and $(q^{K\partial}f)(z)$, and introduce $K^+(z)=q^{((T+U)h^+)(z)}$ and $K^-(z)=q^{K\partial}(q^{h^-})(z))$:

$$f(w)f(z)=e^{2i\pi\hbar}\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}\frac{\theta(z-\hbar)}{\theta(z+\hbar)}\frac{\theta(-w-\hbar)}{\theta(-w+\hbar)}f(z)f(w),\tag{6.2}$$

(6.3)

$$[K^{\pm}(z),K^{\pm}(w)]=0,\ K^+(z)K^-(w)=\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}\frac{\theta(z-w+K-\hbar)}{\theta(z-w+K+\hbar)}K^-(w)K^+(z),$$

$$K^+(z)e(w)=e^{2i\pi\hbar}\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}\frac{\theta(z-\hbar)}{\theta(z+\hbar)}\frac{\theta(-w-\hbar)}{\theta(-w+\hbar)}e(w)K^+(z),\tag{6.4}$$

$$K^-(z)e(w)=e^{2i\pi\hbar}\frac{\theta(w-z+\hbar+K)}{\theta(w-z-\hbar+K)}\frac{\theta(w-\hbar)}{\theta(w+\hbar)}\frac{\theta(-z-\hbar+K)}{\theta(-z+\hbar+K)}e(w)K^-(z),\tag{6.5}$$

$$K^+(z)f(w)=e^{2i\pi\hbar}\frac{\theta(w-z+\hbar)}{\theta(w-z-\hbar)}\frac{\theta(w-\hbar)}{\theta(w+\hbar)}\frac{\theta(-z-\hbar)}{\theta(-z+\hbar)}f(w)K^+(z),\tag{6.6}$$

$$(6.7)\qquad K^-(z)f(w) = e^{2i\pi\hbar}\frac{\theta(z-w+\hbar)}{\theta(z-w-\hbar)}\frac{\theta(z-\hbar)}{\theta(z+\hbar)}\frac{\theta(-w-\hbar)}{\theta(-w+\hbar)}f(w)K^-(z),$$

$$(6.8)\qquad [e(z), f(w)] = \delta(z-w)K^+(z) - \delta(z-w+K)K^-(w)^{-1}.$$

3. Double extensions and infinite twists of the Reyman-Semenov triples

Let as above X be an elliptic curve, and X_n be the set of its n-division points. We fix an isomorphism of X_n with $(\mathbf{Z}/n\mathbf{Z})^2$, and denote by $a \mapsto I_a$ the projective representation of $(\mathbf{Z}/n\mathbf{Z})^2$ on $\oplus_{i\in\mathbf{Z}/n\mathbf{Z}}\mathbf{C}\epsilon_i$, defined by $I_{(1,0)}\epsilon_i = \zeta^i\epsilon_i$ $I_{(0,1)}\epsilon_i = \epsilon_{i+1}$, ζ being a primitive n-th root of 1.

The following Manin triple was introduced in [12]. Let k_0, $\mathcal{O}_0$ be the local field and ring at $0 \in X$, and let us define in $\mathfrak{g} = \mathfrak{sl}_n(k_0)$ the scalar product $\langle x, y\rangle_{\mathfrak{g}} = \mathrm{res}_0\,\mathrm{tr}(xy)(z)dz$. Let $\mathfrak{g}_+ = \mathfrak{sl}_n(\mathcal{O}_0)$ and $\mathfrak{g}_-$ be the set of the expansions at 0, of the regular maps $\sigma : X - X_n \to \mathfrak{sl}_n(\mathbf{C})$, such that $\sigma(x+a) = \mathrm{Ad}(I_a)(\sigma(x))$, for $a \in X_n$. Then $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ forms a Manin triple. Its quantization was treated in [16] in the $\mathfrak{sl}_2$ case, and is connected with Sklyanin algebras ([15]).

We propose the following double extension for this triple. Let $\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbf{C}K \oplus \mathbf{C}D$, and let us denote with an index 0 the Lie bracket in $\mathfrak{g}$. We endow $\hat{\mathfrak{g}}$ with the bracket $[x, y] = [x, y]_0 + \mathrm{res}_0\,\mathrm{tr}(xdy)K$, $[D, x] = \frac{dx}{dz}$, for $x, y \in \mathfrak{g}$, K is central. Let us also define a scalar product $\langle,\rangle_{\hat{\mathfrak{g}}}$ on $\hat{\mathfrak{g}}$ by $\langle x, y\rangle_{\hat{\mathfrak{g}}} = \langle x, y\rangle_{\mathfrak{g}}$ for $x, y \in \mathfrak{g}$, $\langle K, x\rangle_{\hat{\mathfrak{g}}} = \langle D, x\rangle_{\hat{\mathfrak{g}}} = 0$ for $x \in \mathfrak{g}$, $\langle K, D\rangle_{\hat{\mathfrak{g}}} = 2$. Then $\widetilde{\mathfrak{g}}_+ = \mathfrak{g}_+ \oplus \mathbf{C}D$ and $\hat{\mathfrak{g}}_- = \mathfrak{g}_- \oplus \mathbf{C}K$ are Lagrangean subalgebras of $\hat{\mathfrak{g}}$, so that $(\hat{\mathfrak{g}}, \widetilde{\mathfrak{g}}_+, \hat{\mathfrak{g}}_-)$ forms a Manin triple.

We also propose the following twist for this triple. Let $\mathfrak{h}$ and $\mathfrak{n}_\pm$ be the diagonal and upper (resp. lower) triangular subalgebras of $\mathfrak{sl}_n$, and let $\bar{\mathfrak{g}}_+ = \mathfrak{h}(\mathcal{O}_0) \oplus \mathfrak{n}_+(k_0) \oplus \mathbf{C}D$,

$$\bar{\mathfrak{g}}_- = \{\sigma : X - X_n \to \mathfrak{h}(\mathbf{C}) | \sigma(x+a) = \mathrm{Ad}(I_a)(\sigma(x)),\ \text{for}\ \ a \in X_n\} \oplus \mathfrak{n}_-(k_0) \oplus \mathbf{C}K.$$

Then $(\hat{\mathfrak{g}}, \bar{\mathfrak{g}}_+, \bar{\mathfrak{g}}_-)$ is a twist of the previous Manin triple.

4. Quantization for twists of Reyman-Semenov triples

Let us restrict ourselves to the case $n = 2$. Identify as above X with a quotient $\mathbf{C}/\mathbf{C} + \tau_0\mathbf{C}$. To apply the techniques developed above to the quantization of $(\hat{\mathfrak{g}}, \bar{\mathfrak{g}}_+, \bar{\mathfrak{g}}_-)$, we have to take $R = \mathcal{O}_0 k_0$, and for Λ the space of regular functions f on $X - X_2$, such that $f(x + 1/2) = f(x)$, and $f(x + \tau/2) = -f(x)$.

We find the dual bases $(\sigma_0^{(k)}(z))_{k\geq 0}$ and $(w^k/k!)_{k\geq 0}$ of Λ and R, where

$$\sigma_0(z) = \zeta(z) + \zeta\left(z + \frac{1}{2}\right) - \zeta\left(z + \frac{\tau_0}{2}\right) - \zeta\left(z + \frac{1+\tau_0}{2}\right),$$

with the same conventions as above. Then $\sum_i Te^i \otimes e_i$ is equal to $\frac{1}{2\hbar}[\exp((\hbar - w)\partial_z) - \exp((-\hbar - w)\partial_z)]\partial_z^{-1}\sigma_0(z)$, or

$$\frac{1}{2\hbar}\ln\frac{\theta(z-w+\hbar)}{\theta(z-w+\hbar+\frac{\tau_0}{2})}\frac{\theta(z-w+\frac{1}{2}+\hbar)}{\theta(z-w+\hbar+\frac{1+\tau_0}{2})}\frac{\theta(z-w-\hbar+\frac{\tau_0}{2})}{\theta(z-w-\hbar)}\frac{\theta(z-w-\hbar+\frac{1+\tau_0}{2})}{\theta(z-w+\frac{1}{2}-\hbar)}.$$

We can again take for $\sum_i Ue_i \otimes e^i$ a constant function, so that $e^{2\hbar \sum_i (T+U)e_i(z)e^i(w)}$ is equal to

$$(q_0 e^{2i\pi\hbar})^2 \frac{\theta(w-z+\hbar)}{\theta(w-z+\hbar+\frac{\tau_0}{2})} \frac{\theta(w-z+\hbar+\frac{1}{2})}{\theta(w-z+\hbar+\frac{1+\tau_0}{2})} : \frac{\theta(w-z-\hbar)}{\theta(w-z-\hbar+\frac{\tau_0}{2})} \frac{\theta(w-z-\hbar+\frac{1}{2})}{\theta(w-z-\hbar+\frac{1+\tau_0}{2})}.$$

Let $K^{\pm}(z), e(z)$ and $f(z)$ be the generators analogous to those introduced above. We obtain the relations

$$[K^{\pm}(z), K^{\pm}(w)] = 0,$$

$$\rho(z-w)K^+(z)K^-(w) = \rho(z-w+K)K^-(w)K^+(z),$$

with $\rho(x) = \frac{\vartheta(x+\hbar)}{\vartheta(x-\hbar)}$, and $\vartheta(x) = \frac{\theta(x)}{\theta(x+\frac{\tau_0}{2})} \frac{\theta(x+\frac{1}{2})}{\theta(x+\frac{1+\tau_0}{2})}$,

$$(q_0 e^{2i\pi\hbar})^2 \vartheta(z-w+\hbar)e(z)e(w) = \vartheta(z-w-\hbar)e(w)e(z),$$

$$(q_0 e^{2i\pi\hbar})^2 \vartheta(z-w+\hbar)f(w)f(z) = \vartheta(z-w-\hbar)f(z)f(w),$$

$$(q_0 e^{2i\pi\hbar})^2 \vartheta(z-w+\hbar)K^+(z)e(w) = \vartheta(z-w-\hbar)e(w)K^+(z),$$

$$\vartheta(z-w-K-\hbar)K^-(z)e(w) = (q_0 e^{2i\pi\hbar})^2 \vartheta(z-w-K+\hbar)e(w)K^-(z),$$

$$\vartheta(z-w-\hbar)K^+(z)f(w) = (q_0 e^{2i\pi\hbar})^2 \vartheta(z-w+\hbar)f(w)K^+(z),$$

$$(q_0 e^{2i\pi\hbar})^2 \vartheta(z-w+\hbar)K^-(z)f(w) = \vartheta(z-w-\hbar)f(w)K^-(z),$$

$$[e(z), f(w)] = \delta(z-w)K^+(z) - \delta(z-w+K)K^-(w)^{-1};$$

$K^+(z)$ is subject to

$$K^+\left(z+\frac{1}{2}\right) = K^+(z), \quad K^+\left(z+\frac{tau_0}{2}\right) = K^+(z)^{-1}.$$

We expect that the algebra presented here is isomorphic to the $\mathfrak{sl}_2$ version of the Sklyanin algebra.

REFERENCES

[1] J. Beck, *Braid group action and quantum affine algebras* (*Commun. Math. Phys.*, Vol. 165, 1994, pp. 555-68).

[2] V. Chari and A. Pressley, *Quantum affine algebras* (*Commun. Math. Phys.*, Vol. 142, 1991, pp. 261-83).

[3] J. Ding and I. B. Frenkel, *Isomorphism of two realizations of quantum affine algebras* $U_q(\widehat{gl}_n)$ (*Commun. Math. Phys.*, Vol. 156, 1993, pp. 277-300).

[4] V. G. Drinfeld, *A new realization of Yangians and quantized affine algebras* (*Sov. Math. Dokl.*, Vol. 36, 1988).

[5] V. G. Drinfeld, *Quasi-Hopf algebras* (*Leningrad Math. J.*, Vol. 1:6, 1990, pp. 1419-57).

[6] B. Enriquez and G. Felder, in preparation.

[7] P. Etingof and D. Kazhdan, *Quantization of Lie bialgebras, I* (Selecta Math. 2 (1996), No. 1, pp. 1-41), q-alg/9506005.

[8] B. L. FEIGIN and E. V. FRENKEL, *Quantum $\mathcal{W}$-algebras and elliptic algebras* (*Commun. Math. Phys.*, Vol. 178, 1996, pp. 653-678), q-alg/9508009.

[9] I. B. FRENKEL and N. JING, *Vertex representations of quantum affine algebras* (*Proc. Natl. Acad. Sci. USA*, Vol. 85, 1988, pp. 9373-7).

[10] S. M. KHOROSHKIN and V. N. TOLSTOY, *On Drinfeld's realization of quantum affine algebras* (*J. Geom. Phys.*, Vol. 11, 1993, pp. 445-52).

[11] N. Yu. RESHETIKHIN and M. A. SEMENOV-TIAN-SHANSKY, *Central extensions of quantum current groups* (*Lett. Math. Phys.*, Vol. 19, 1990, pp. 133-42).

[12] A. G. REYMAN and M. A. SEMENOV-TIAN-SHANSKY, *Integrable systems II, ch. 11,* (*Encycl. Sov. Math.*, Vol. 16, "Dynamical systems, 7", Springer-Verlag, 1993, pp. 188-225).

[13] M. A. SEMENOV-TIAN-SHANSKY, *Poisson-Lie groups, quantum duality principle, and the quantum double* (*Theor. Math. Phys.*, Vol. 93, 1992, pp. 1292-307).

[14] J.-P. SERRE, *Groupes algébriques et corps de classes*, Hermann, Paris, 1959.

[15] E. K. SKLYANIN, *Some algebraic structures connected with the Yang-Baxter equation* (*Funct. An. Appl.*, Vol. 16, 1982, pp. 263-70).

[16] D. B. UGLOV, *The quantum bialgebra associated with the eight-vertex R-matrix* (*Lett. Math. Phys.*, Vol. 28, 1993, pp. 139-42).

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