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SINGULAR STRANGE ATTRACTORS ON THE BOUNDARY OF MORSE-SMALE SYSTEMS

BY C. A. MORALES AND E. R. PUJALS

ABSTRACT. – In this paper we introduce bifurcations of Morse-Smale systems that produce strange attractors with singularities in n-manifolds, $n \geq 3$. Some of the attractors are new in the sense that they are not equivalent to any geometric Lorenz attractor. The creation through such bifurcations of hyperbolic dynamics as well as Henón and contracting Lorenz attractors is also investigated.

RÉSUMÉ. – Dans cet article nous introduisons des bifurcations sur des champs de vecteurs Morse-Smale en dimension $n \geq 3$, ces bifurcations faisant apparaître des attracteurs étranges avec singularités. Certains de ces attracteurs sont nouveaux au sens qu'ils ne sont équivalents à aucun attracteur de Lorenz géométrique. La construction de dynamiques hyperboliques ainsi que d'attracteurs de Lorenz contractants et d'attracteurs de type Hénon grâce à de telles bifurcations est également étudiée.

1. Introduction

This paper deals with the bifurcation theory of Morse-Smale dynamical systems strongly developed during the last decades, as it can be seen in papers [ACL], [AS], [NPT], [BLMP], [PR], [PT1] or in the gradient case [CP] and [PT2]. We present a bifurcation which gives rise new dynamical phenomena at the boundary of Morse-Smale systems. Let us start with a definition based on the main features of the well known *geometric Lorenz attractor* studied in [GW]. A *singular strange attractor* of a vector field X is a transitive invariant set A with the following properties:

– A contains a dense set of periodic orbits of X;

– A has at least one singularity of X;

– A exhibits a positively dense orbit with positive Lyapunov exponent;

– there is a neighborhood U of A (*isolating block*) satisfying $A = \bigcap_{t \geq 0} X[t, U]$, where $X[t, .]$ stands for the flow generated by X. The *basin* of A is the set of points whose w-limit set belongs to A.

We say that A is *persistent* if $\cap_{t \geq 0} Y[t, U]$ is a singular strange attractor for any flow Y close to X. Two singular strange attrators A and B of X and Y respectively are *equivalent* when there are isolating blocks U and W of A and B (resp.) and a homeomorphism $h : U \to W$ sending X-orbits into Y-orbits.

The geometric Lorenz attractor is the most representative example of a persistent singular strange attractor (see [DKO] for an *axiomatic* definition). It motivated in part the study of strange attractors with singularities for transformations, the *generalized hyperbolic attractors* (see [P]), which include a number of well known examples as the *Lorenz-type attractor* early studied in [AP]. For vector fields, other examples have been introduced in [R] [LV], [PRV] and so, it is natural to pursue a classification among the category of singular strange attractors for flows. The term *Lorenz-like attractor* has been used frequently by many authors and here we shall employ it, but under the following point of view: a singular strange attractor is *Lorenz-like* if it is equivalent to some geometric Lorenz attractor.

We present a bifurcation giving rise different types of singular strange attractors just across the boundary of Morse-Smale systems. It will consists of a hyperbolic saddle singularity and a saddle-node periodic orbit. For diffeomorphisms, the analyze of cycles involving a saddle-node periodic orbit, the *saddle-node cycles*, goes back to [NPT] were it was proved that *critical* saddle-node cycles displays homoclinic tangencies in their generic unfoldings. This bifurcation was studied extensively in [DRV] which proved the existence, in the parameter space, of a large set of parameter values whose corresponding systems exhibit *Henón-like* attractors (see [PT1]). For vector fields, it was showed in [ST] how the unfolding of saddle-node periodic orbits give rise solenoid-type Axiom A attractors across the boundary of Morse-Smale systems in n-manifolds, $n \geq 4$. In presence of singularities, we can mention [ACL] where it was proved that, under certain conditions, the unfolding of such cycles yield Lorenz-like attractors accumulating the boundary of Morse-Smale systems. Our results extend the ones in [DRV] for saddle-node cycles *with singularities* and improve those in [ACL].

We give now some background to be used in forthcoming definitions and statements. Let M be a closed Riemannian 3-manifold and X a vector field defined on M. An invariant compact set H of X is *hyperbolic* if there exist a X-invariant continuous splitting $T_H M = E^s_H \oplus E^X_H \oplus E^u_H$ such that E^X_H is the flow's direction in H and $DX[t,.]$ contracts (resp. expands) uniformly vectors in E^s_H (resp. E^c_H). We say that a vector field is *hyperbolic* if its nonwandering set is hyperbolic.

The invariant manifold theory (see [HPS]) asserts the existence of partially defined smooth invariant foliations $W^s(H)$, $W^u(H)$ associated to any hyperbolic set H. They are tangent to the directions $E^s_H \oplus E^X_H$, $E^X_H \oplus E^u_H$ at H and are called the *stable* and *unstable manifold* of H respectively. It is well known that $W^s(H)$ (resp. $W^u(H)$) coincides with the set of points $x \in M$ whose w-limit set (resp. α-limit set) belongs to H. In the case when H reduces to a single singularity σ of X whose eigenvalues $\{\lambda_1, -\lambda_2, -\lambda_3\}$ are real and satisfy $-\lambda_2 < -\lambda_3 < 0 < \lambda_1$, there exists also an (unique) invariant manifold passing through σ and tangent to the eigenvalue direction associated to $\{-\lambda_2\}$: the *strong stable manifold* denoted by $W^{ss}(\sigma)$ in what follows.

A vector field whose nonwandering set is finite and its invariant manifolds intersect transversaly is called *Morse-Smale*. Clearly a Morse-Smale vector field is hyperbolic and the one exhibiting a singular strange attractor are not.

The invariant manifolds exist also in some nonhyperbolic cases as, for instance, when σ_1 is a *saddle-node periodic orbit*, i.e. the derivative $D\pi(\sigma_1)$ of its corresponding Poincaré map π has one eigenvalue with absolute value not equal to one and the second eigenvalue equal to 1 and π is nondegenerate nonzero quadratic term in the center direction (see [T]). We shall be interested in the case where the first eigenvalue belongs to $(0,1)$. Let $W^s(\sigma_1)$ be the set of points whose ω-limit set is σ_1. It is well known that, in this case, $W^s(\sigma_1)$ is a 3-manifold with boundary.

DEFINITION 1.1. – *A* **saddle-node singular cycle** *of X is a set $\Gamma = \{\sigma_0, \sigma_1, \gamma_0, \gamma_1\}$ with the following properties,*

1. *σ_0 is a hyperbolic saddle singularity whose eigenvalues $\{\lambda_1, -\lambda_2, -\lambda_3\}$ satisfy $-\lambda_2 < -\lambda_3 < 0 < \lambda_1$;*
2. *σ_1 is a saddle-node periodic orbit;*
3. *$\gamma_0 = (W^s(\sigma_0) \cap W^u(\sigma_1)) \setminus W^{ss}(\sigma_0)$ is a finite set of regular orbits;*
4. *$\gamma_1 = (W^u(\sigma_1) \setminus \gamma_0) \cup W^u(\sigma_0)$ and is contained in the interior of $W^s(\sigma_1)$* (Fig. 1).

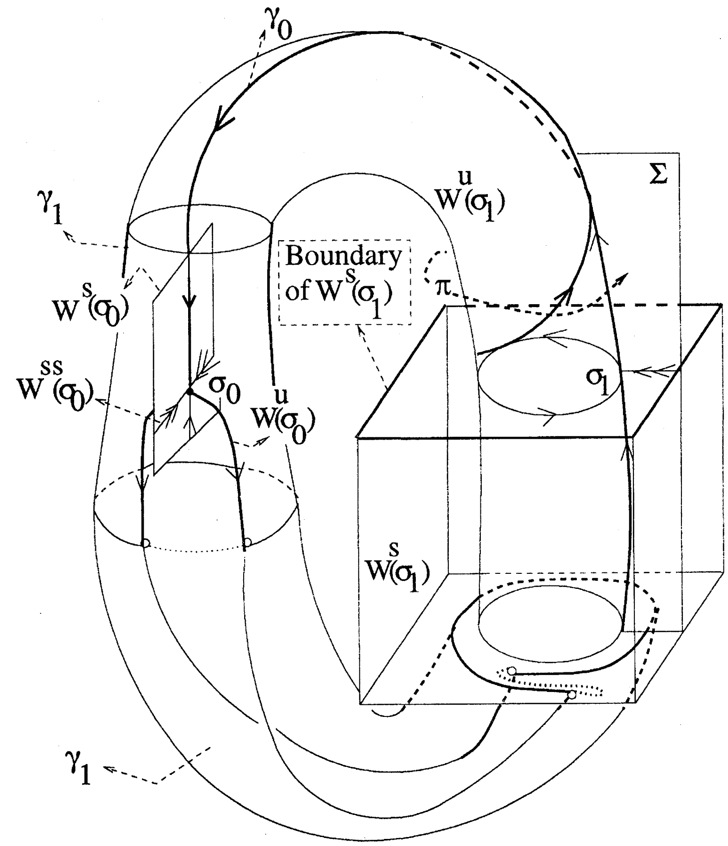

Fig. 1. – Saddle-node singular cycle.

For simplicity, we assume the existence of a neighborhood $\mathcal{U}$ of X such that if $Y \in \mathcal{U}$, then the analytic continuation of the singularity admits C^2-linearizing coordinates. For this it is necessary that the eigenvalues of the singularity satisfy certain nonresonance conditions and so $\lambda_3 \neq \lambda_1$. For these assumptions make sence, vector fields in this work will be at least C^3.

Now, it is well known the existence of a strong stable foliation $\mathcal{F}^{ss}$ for π. Saturating $\mathcal{F}^{ss}$ by X we get the foliation $\tilde{\mathcal{F}}^{ss}$ in $W^s(\sigma_1)$.

DEFINITION 1.2. – *We say that a saddle-node singular cycle* Γ *is* **transversal** *if* $W^s(\sigma_0)$ *is transversal to* $W^u(\sigma_1)$ *and* γ_1 *is transversal to* $\tilde{\mathcal{F}}^{ss}$. *The cycle is* **critical** *if* $W^s(\sigma_0)$ *is transversal to* $W^u(\sigma_1)$ *and there is a nondegenerate tangency between* $\gamma_1 \backslash W^u(\sigma_0)$ *and* $\tilde{\mathcal{F}}^{ss}$.

In section 3, we shall associate a circle map f_Γ to any saddle-node singular cycle Γ (see Definition 3.1). This enables us to define transversal saddle-node singular cycles of *k-expansive* type as the ones whose corresponding circle map f_Γ has derivative greater than $k' > k$ in modulus, $k \in I\!R^+$. We shall see (Remark 3.0 (1)) that k-expansivity of f_Γ, $k > 0$, *implies* the well known expansive condition $\lambda_3 < \lambda_1$ on the singularity's eigenvalues of Γ (see Definition 1.1.1). We then define $N_0 = N_0(f_\Gamma) \in I\!N$ by $card(f_\Gamma^{-1}(x))$ equals N_0 or N_0+1 ($\forall x \in S^1$), and $D(f_\Gamma)$ as the set of points $x \in S^1$ such that $card(f_\Gamma^{-1}(x)) = N_0+1$. Here $card(A)$ means cardinality of A. The number N_0 exists and, indeed, corresponds to the degree in the case of expanding endomorphisms of the circle.

In our first result we summarize the main dynamical features in presence of a k-expansive saddle-node singular cycle Γ, $k > 1$. It is shown how k-expansivity *characterizes* the dynamics in a suitable neighborhood of Γ. In its statement, m stands for the Lebesgue measure in $I\!R$ and $Cl(A)$ denotes the closure of A.

THEOREM A. – *Let* X_μ *be a generic one-parameter family of vector fields in* M *such that* X_μ *is Morse-Smale, for* $\mu < 0$, *and* X_0 *exhibits a transversal saddle-node singular cycle* Γ *of k-expansive type,* $k \in I\!R^+$. *Then,*

a. *if* $k > 1$, *there exist* $\delta > 0$ *and a neighborhood* U *of* Γ *such that* $\forall \mu \in (0, \delta)$ X_μ *exhibits a persistent singular strange attractor* $A_\mu \subset U$ *whose basin contains an open and dense set of* U. *In particular,* X_μ *is not hyperbolic* $\forall \mu \in (0, \delta)$;

b. *if* $k > \sqrt{2}$, (**a**) *holds and there exists a disjoint interval sequence* $[a_n, b_n] \subset (0, \delta)$, *such that* A_μ *is a non-Lorenz-like attractor* $\forall \mu \in I = \cup_{n>0}[a_n, b_n]$ *and*

$$\liminf_{\epsilon \to 0^+} \frac{m(I \cap (0, \epsilon))}{\epsilon} > 0;$$

c. *if* $k > \sqrt{2}$, $N_0(f_\Gamma) = 1$ *and* $Cl(D(f_\Gamma)) \neq S^1$, (**b**) *holds and there is other disjoint interval sequence* $[a'_n, b'_n] \subset (0, \delta)$ *such that* A_μ *is Lorenz-like* $\forall \mu \in I' = \cup_{n>0}[a'_n, b'_n]$, *and*

$$\liminf_{\epsilon \to 0^+} \frac{m(I' \cap (0, \epsilon))}{\epsilon} > 0;$$

d. *if* $k > 2$, (**a**) *holds and* A_μ *is a non-Lorenz-like attractor* $\forall \mu \in (0, \delta)$.

To explain why some of the attractors in this theorem are non-Lorenz-like we use the geometric model at Figure 3 in the appendix. We shall see that the unfolding of transversal saddle-node singular cycles produces a return map, as in such geometric model, whose image A' *spirals* within the annular cross section A at this figure as parameter μ varies. Constructing strong stable foliations for this return map we reduce the dynamics's description to the rotation of suitable expanding maps in the circle (Theorem 3.3). Then, we describe the behavior of A' in terms of rotations of a one-dimensional circle map f_μ *close* to f_Γ (see Definition 2.3). Sometimes the rotation of f_μ will be transitive in S^1 and, in this case, we will obtain a singular strange attractor which is not equivalent to any geometric Lorenz attractor. We give a precise description of this phenomenon in the appendix.

The following result deals with critical saddle-node singular cycles. To state it, we make first some remarks. In [NPT] it is proved that the unfonding of certain critical saddle-node cycles of diffeomorphisms involves the unfolding of homoclinic tangencies. They derive this by showing that homoclinic tangencies occur whenever one consider the family of transformations obtained by composing a given smooth endomorphism of the circle with the whole interval of rotations. Clearly, generic unfoldings of critical saddle-node singular cycles can be studied using somewhat similar techniques. However, to perform it, we require a different approach because circle maps here have not continuous extension in S^1. We define in section 2 *critical continuous extension* of certain circle maps (see Definition 2.2). Theorem below shows the main properties of critical cycles Γ such that f_Γ has a critical continuous extension.

THEOREM B. – *Let X_μ be a generic one-parameter family of vector fields in M such that X_μ is Morse-Smale, for $\mu < 0$, and X_0 exhibits a critical saddle-node singular cycle Γ whose one-dimensional map has critical continuous extension. Then, if $He(\epsilon)$ denotes the set of parameters $\mu > 0$ such that X_μ has a Henón-like attractor close to Γ,*

$$\liminf_{\epsilon\to 0^+} \frac{m(He(\epsilon))}{\epsilon} > 0.$$

We point out that conclusion of Theorem (B) is not true without the hypothesis of critical continuous extension.

It remains the question whether different types of dynamics as hyperbolicity, Henón-like or Lorenz-like attractors can occur simultaneously in the unfolding of a saddle-node singular cycle. The answer is negative in general (see for instance Theorem (A)). Despite, we shall prove that such a phenomenon happens in open examples (see Theorem (C) in the appendix). Similar questions hold for other types of nonhyperbolic behavior as, for instance, existence of *contracting Lorenz-like attractors* (see [R]). We also obtain affirmative answers in this case in open examples (see Theorem (D) the appendix).

In our point of view, the results mentioned before show a complete picture of the dynamics close to Morse-Smale systems when saddle-node singular cycles take place. New interesting questions arise and, in particular, if all the phenomena described above occur in the presence of cycles involving only singularities. In [Pu] the second author pursues this question and gives positive answers, but when the codimension of the cycle is at least two.

This paper is organized as follows. In section 2, we study one-dimensional maps. The results in this sections will be essential in our analyze. In section 3, we present some key definitions and make one-dimensional reductions on the dynamics. In section 4, we prove theorems (A) and (B) using the results in section 2 together with the one-dimensional reduction in section 3. Some final remarks will be given in section 5.

2. One-dimensional dynamics

In this section we shall study the dynamics of certain maps in S^1 or a closed real interval. The results of this section will be used in section 4. We start with a definition.

DEFINITION 2.1. – *Let J be S^1 or a real compact interval. If $k > 0$ and $c \in J$, $E_k(J, c)$ is the set of maps $f : J \setminus \{c\} \to J$ such that,*

1. *f is C^1 and $|f'(x)| > k' > k$, for some fixed k' and $\forall x \in J \setminus \{c\}$;*

2. *f has lateral limits at c, i.e. the limits $r_+ = \lim_{x \to c^-} f(x)$ and $r_- = \lim_{x\to c^+} f(x)$ exist.*

We define $N_0(f)$ as the unique positive integer such that $card(f^{-1}(x))$ is $N_0(f)$ or $N_0(f) + 1$ ($\forall x \in S^1$), and $D(f)$ as the J's subset such that $card(f^{-1}(x)) = N_0(f) + 1$ (see Figure 2 (a)). If $I \subset J$ and $f \in E_k(J, c)$, we say that f **generates the first return map** *R* **in I** *if for any $x \in I \setminus \{c\}$ there is a first positive integer $m(x)$ such that $R(x) = f^{m(x)}(x) \in I$.*

Remark 2.0. – Clearly, for $k > 1$, every $f \in E_k(J, c)$ is *expanding*, i.e. $f'(x) \geq k' > k > 1$ $\forall x \neq c$ for some constant k'. This fact, in particular, implies that $\forall I \subset J$ open there is a first integer $n \in I\!\!N$ such that $c \in f^n(I \setminus \{c\})$. This simple fact will be used in the proof of results of this section.

The following result will be used in the proof of Theorem (A-a).

THEOREM 2.1. – *For every $f \in E_k(J, c)$, $k > 1$, there exists a nontrivial compact interval I such that f generates a first return map R in I. Moreover, R is topologically transitive and $W^s(I) = \{x \in J : f^n(x) \in I, \text{ for some } n \in I\!\!N\}$ is dense in J.*

We note here that there are cases where the interval obtained in Theorem 2.1 is a proper subinterval of J. This is the case when one considers the classical Lorenz expanding one-dimensional map and extend it in order to obtain the interval map at Figure 2 (a) (notice that $I = D(f)$ in this figure).

Proof. – It will be assumed that $c = 0$ by identifying J with $[-1, 1]$. We start with

LEMMA 2.0. – *$\exists \delta > 0$ such that $\forall 0 < \epsilon < \delta$, $\exists n \in I\!\!N$ verifying $(-\delta, \delta) \subset f^n((-\epsilon, \epsilon))$.*

Proof. – Consider, for any ϵ, the intervals $U_\epsilon = (-\epsilon, \epsilon)$, $U_\epsilon^+ = (0, \epsilon)$ and $U_\epsilon^- = (-\epsilon, 0)$. Define n^+ and n^- by

$$n^{\pm}(\epsilon) = min\{k : 0 \in f^k(U_\epsilon^{\pm})\}.$$

They exist because $f^k(U_\epsilon^{\pm})$ must eventually intersect 0 as f is expanding (see Remark 2.0). It follows that:

a. $f^{n^{\pm}(\epsilon)}(U_\epsilon^{\pm})$ is open;

b. functions $\epsilon \to n^{\pm}(\epsilon)$ are decreasing;

c. $\lim_{\epsilon\to 0} n^{\pm}(\epsilon) = +\infty$.

In fact, (a) and (b) are quite obvious and (c) holds because $f^k(U_\epsilon^{\pm})$ is open for all $k \leq n^{\pm}(\epsilon)$.

Now we state

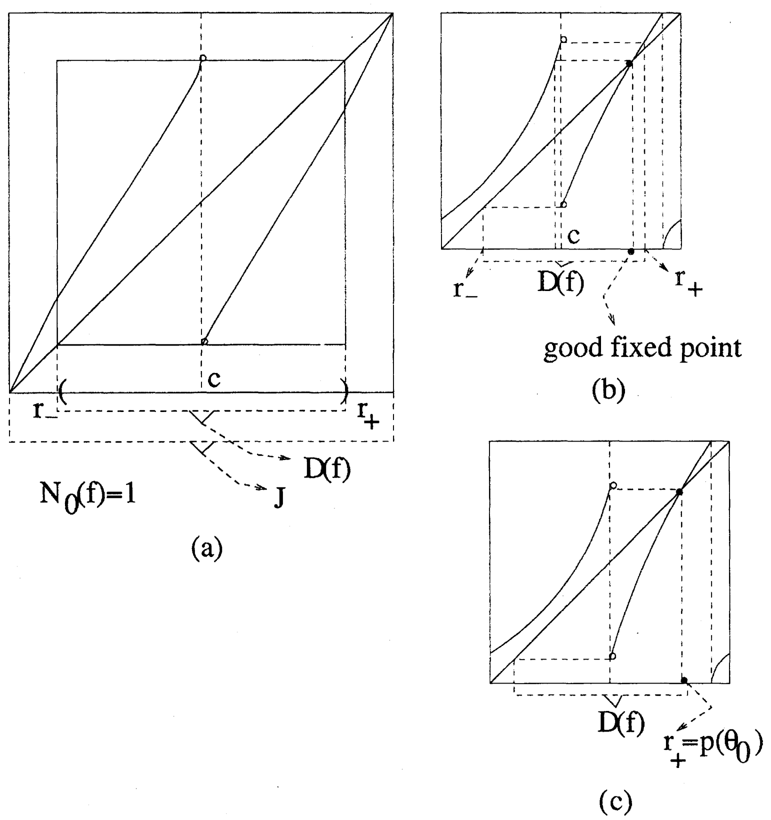

Fig. 2.

CLAIM 2.1. – *Fix $b > 2$ and ϵ_b such that $min\{(k')^{n^{\pm}(\epsilon)}\} > b$ (see Definition 2.1). Then, for all $\gamma < \epsilon_b$, the two followings assertions holds*

- $U^{+}_{\frac{b}{2}\gamma} \subset f^{n^{+}(\gamma)}(U^{+}_{\gamma})$ *or* $U^{-}_{\frac{b}{2}\gamma} \subset f^{n^{+}(\gamma)}(U^{+}_{\gamma})$;
- $U^{-}_{\frac{b}{2}\gamma} \subset f^{n^{-}(\gamma)}(U^{-}_{\gamma})$ *or* $U^{+}_{\frac{b}{2}\gamma} \subset f^{n^{-}(\gamma)}(U^{-}_{\gamma})$.

This claim follows because of $0 \in f^{n^{\pm}(\gamma)}(U^{\pm}_{\gamma})$ and $|f^{n^{\pm}(\gamma)}(U^{\pm}_{\gamma})| > (k')^{n^{\pm}(\gamma)}|U^{\pm}_{\gamma}| > \frac{b}{2}|U_{\gamma}| = |U_{\gamma\frac{b}{2}}|$, where $|.|$ denotes interval's diameter.

Let $I = (-\delta, \delta)$ be the interval such that $I \subset f^{n^{+}(\epsilon_b)}(U^{+}_{\epsilon_b}) \cap f^{n^{+}(\epsilon_b)}(U^{+}_{\epsilon_b})$. It will be proved that for each $0 < \epsilon < \delta$ $\exists n \in I\!N$ such that

$$(-\delta, \delta) \subset f^{n}((-\epsilon, \epsilon)) \tag{1}$$

Indeed, by Claim 2.1, we can find $k \in I\!N$ such that either $U^{+}_{\frac{b}{2}\epsilon} \subset f^{k}((-\epsilon, \epsilon))$ or $U^{-}_{\frac{b}{2}\epsilon} \subset f^{k}((-\epsilon, \epsilon))$. Repeating this argument while i satisfies $(\frac{b}{2})^{i}\epsilon \leq \epsilon_b$, it follows that there is n such that either $U^{+}_{\epsilon_b} \subset f^{n}((-\epsilon, \epsilon))$ or $U^{-}_{\epsilon_b} \subset f^{n}((-\epsilon, \epsilon))$ and, hence

$$\text{either} \quad (-\delta, \delta) \subset f^{n+n^{+}(\epsilon_b)}((-\epsilon, \epsilon)) \quad \text{or} \quad (-\delta, \delta) \subset f^{n+n^{-}(\epsilon_b)}((-\epsilon, \epsilon)). \tag{$*$}$$

This proves (1). The proof of Lemma 2.0 is complete.

Now we prove Theorem 2.1. We show the existence of an interval I with a well defined transitive return map. To do this choose δ as in Lemma 2.0. Let $H = \cup_{0\leq k<\infty} f^k((-\delta,\delta))$ and $C_0(H)$ be the connected component of H containing zero. Thus it follows that:

• A first return map (on $C_0(H)$) is defined, because if $x \in C_0(H)$ then there exist k such that $x \in U \subset f^k((-\delta,\delta))$, for some some open U. A similar argument implies the existence of an integer m such that $f^m(U)$ is open and contains 0 (see Remark 2.0). So $f^m(U) \subset C_0(H)$ and it follows that there exist a positive integer $r = m + k$ such that $f^r(x) \in C_0(H)$.

• The return map with domain $C_0(H)\setminus\{0\}$ obtained above is transitive because $(\star)$.

We choose $I = Cl(C_0(H))$ and so f generates a transitive first return map in I. Now if $f \in E_k(J,c)$ generates a first return map in some interval I containing c, $k > 1$, then $W^s(I)$ is dense in J by Remark 2.0. Recall that by definition $W^s(I) = \{x \in J : f^n(x) \in I, \text{ for some } n \in I\!N\}$ (see statement of Theorem 2.1). This finishes the proof of Theorem 2.1.

If $f \in E_{\sqrt{2}}(S^1,c)$ and $\theta \in [0,2\pi)$, f_θ is $R_\theta \circ f$, where R_θ is the rotation on S^1 with angle θ.

THEOREM 2.2. – *Let $f \in E_k(S^1,c)$, $k > \sqrt{2}$. Then, there exist an interval $I \subset [0,2\pi)$ such that f_θ is transitive in S^1 $\forall\theta \in I$.*

Remark 2.1.

1. Observe that $f'(x) > 0$ or else $f'(x) < 0$, $\forall x \in S^1\setminus\{c\}$, $f \in E_k(S^1,c)$ and $k > 0$. As usual, the first case is called *orientation preserving* and the remainer *orientation reversing*.

2. Suppose that $f \in E_k(S^1,c)$ and $N_0(f) = 1$ with $k > 1$. If f is orientation preserving (resp. reversing), f has at most one fixed point (resp. two fixed points).

Proof of Theorem 2.2. – We start with a lemma whose proof follows as in [W].

LEMMA 2.3. – *Let $f \in E_k(S^1,c)$, $k > \sqrt{2}$. Then, for each nontrivial interval $I \subset S^1$, there exists a positive integer n such that $f^n(I)$ contains either an interval with end points $\{r_-,c\}$ or $\{r_+,c\}$.*

For now on we fix $k > \sqrt{2}$. Given $f \in E_k(S^1,c)$, we say that a fixed point p of f is a *good* if for each connected component J of $S^1\setminus\{r_-,c\}$ or $S^1\setminus\{r_+,c\}$, there exist $x \in int(J)$ such that $f^n(x) = p$ for some $n \in I\!N$ (see Figure 2 (b)). Here int means interior. By Lemma 2.3, it follows that $f \in E_k(S^1,c)$ is transitive if it has good fixed points. The strategy will be to prove that f_θ has a good fixed point for suitable $\theta \in [0,2\pi]$ as it is an open property. We can assume that f has a fixed point $p \in S^1\setminus\{c\}$.

Consider $N_0 = N_0(f)$, $D = D(f)$, $r_\pm = r_\pm(f)$ and the "discontinuity point" $c = c(f)$ as in Definition 2.1. It follows that $f \in E_k(S^1,c)$ implies $f_\theta \in E_k(S^1,c)$, $N_0(f_\theta) = N_0$ and $D_\theta = D(f_\theta) = R_\theta(D)$ $\forall\theta \in [0,2\pi)$. There is a continuation of the fixed point p of f for f_θ. This is a smooth map $p : \theta \to p(\theta)$ such that $p(0) = p$ and $f_\theta(p(\theta)) = p(\theta)$. In the orientation preserving case (see Remark 2.1 (1)) we have

$$p'(\theta) = \frac{1}{1 - f'(p(\theta))} < 0,$$

and so D_θ and $p(\theta)$ are moving in opposite directions when θ does. Then there exists $\theta_0 \in [0, 2\pi)$ such that $r_+(\theta_0)$ (say) equals $p(\theta_0)$ where $r_\pm(\theta) = r_\pm(f_\theta)$ (see Figure 2 (c)). This is because the end points of D_θ are $r_\pm(\theta)$. In the orientation reversing case we do not have $p'(\theta) < 0$, but now D_θ is moving more quickly than $p(\theta)$ so we have $p(\theta_0)$ equals $r_+(\theta_0)$ (or $r_-(\theta_0)$) for some θ_0 even in this case.

On the other hand, by expansivity of f, it follows that $\exists \epsilon_0 > 0$ such that $\forall \theta \in (\theta_0 - \epsilon_0, \theta_0 + \epsilon_0)$ and for each connected component $J(\theta)$ of $S^1 \setminus \{r_+(\theta), c\}$ or $S^1 \setminus \{r_-(\theta), c\}$ there is a first $n(J(\theta)) \in I\!N$ such that $c \in int(f_\theta^{n(J(\theta))}(J(\theta)))$. Define

$$J^*(\theta) = \cap_{J(\theta)} int(f_\theta^{n(J(\theta))}(J(\theta))),$$

and choose $\delta > 0$ so small such that $(c - 2\delta, c + 2\delta) \subset J^*(\theta_0)$. Then, $\exists 0 < \epsilon_1 < \epsilon_0$ such that $(c - \delta, c + \delta) \subset J^*(\theta)$, $\forall \theta \in (\theta_0 - \epsilon_1, \theta_0 + \epsilon_1)$.

Now choose $\theta_1 \in (\theta_0 - \epsilon_1, \theta_0 + \epsilon_1)$ close to θ_0 such that $p(\theta_1) \in int(f_{\theta_1}(c - \delta, c + \delta))$ ($p(\theta_1) \in D(f_{\theta_1})$ works). Then $p(\theta_1) \in int(f_{\theta_1}(c - \delta, c + \delta))$ and so $p(\theta_1) \in int(f_{\theta_1}^{(n(J(\theta_1))+1)}(J(\theta_1)))$ $\forall J(\theta_1)$. This implies that $p(\theta_1)$ is a good fixed point of $f(\theta_1)$. The proof is complete.

Notice that in Theorem 2.2 we do not make any assumption on $N_0(f)$ and $D(f)$.

PROPOSITION 2.4. – *Let f be a map as the one in Theorem 2.2 with $N_0(f) = 1$ and $Cl(D(f)) \neq S^1$. Then there exist nontrivial disjoint intervals I_1 and I_2 in $[0, 2\pi]$ such that,*

1. *If $\theta \in I_1$, f_θ is a transitive in S^1;*

2. *If $\theta \in I_2$ and f is orientation preserving, then f_θ is transitive in an invariant proper closed subinterval of S^1 containing c;*

3. *If $\theta \in I_2$ and f is orientation reserving, then f_θ^2 is transitive in an invariant proper closed subinterval of S^1 containing c.*

Proof. – The existence of I_1 follows from Theorem 2.2. Suppose that f is orientation preserving (recall Remark 2.1 (2)) and consider θ_0 as in the proof of Theorem 2.2. Choose θ_1 close to θ_0 such that $p(\theta_1) \notin D(f_{\theta_1})$. We claim that $c \in D(f_{\theta_1})$ and $f(Cl(D(f_{\theta_1})) \setminus \{c\}) \subset Cl(D(f_{\theta_1}))$. This claim follows by considering f as an interval map cutting S^1 at $p(\theta_1)$ (here we use $N_0(f) = 1$ and $Cl(D(f)) \neq S^1$). Choose a small interval I_2 so that $f(Cl(D(f_\theta)) \setminus \{c\}) \subset Cl(D(f_\theta))$ and $f_\theta / Cl(D(f_\theta) \in E_k(Cl(D(f_\theta), c)$ $\forall \theta \in I_2$. Then $f_\theta / Cl(D(f_\theta)$ is transitive by Lemma 2.3 and the proof is complete. Now suppose that f is orientation reversing. Then f has just two fixed points and so does f_θ, $\forall \theta$ (recall Remark 2.1 (2)). Choose θ_1 such that $c \notin D(f_{\theta_1})$. Consider the fixed points p_1, p_2 of f_{θ_1}. Then one of the connected component of $S^1 \setminus \{p_1, p_2\}$ (J_1 say) satisfies $c \in int(J_1)$. Then $f^2/J_1 \in E_{k^2}(J_1, c)$, it is orientation preserving and the end points of J_1 are fixed points of f^2/J_1. The proof follows as in the orientation preserving case, but now in an interval instead of S^1.

Proposition below shows that functions in $E_k(S^1, c)$ are transitive in S^1, $k > 2$.

PROPOSITION 2.5 – *Every $f \in E_k(S^1, c)$, $k > 2$, is transitive in S^1.*

A classical Willians's result implies that every $f \in E_k(J, c)$ is transitive when $k > \sqrt{2}$, J is a compact real interval and the the end points of J are $\{r_+, r_-\}$ (see [W]). We note that this is not longer true in general. A simple example is the map depicted at Figure 2 (a).

Proof. – Let I an open interval in $S^1 \setminus \{c\}$ and consider the first $n(I) \in I\!N$ such that $c \in f^{n(I)}(I)$ (of course $n(I) \geq 1$). Then, it follows that the largest component of $f^{n(I)}(I) \setminus \{c\}$ has diameter greater than $(\frac{k'}{2})^{n(I)}|I|$ (where k' is the infimum of the derivative of f which is bigger than 2 by hypothesis). Repeating an argument in [W], we reach n such that $f^n(I) = S^1 \setminus \{c\}$. The proof is completed.

Next we study a special class of circle maps. They will come from critical saddle-node singular cycles (see section 1). Let us give its definition.

DEFINITION 2.2. – *Let $f : S^1 \setminus \{c_1, c_2, ..., c_r\} \to S^1$ satisfying*

1. *f is C^1 and has a criticality in S^1, i.e. $\exists p \in S^1$ such that $f'(p) = 0$;*
2. *the lateral limits of f at c_i exist $\forall i = 1, ..., r$;*
3. *f admits a degree one lifting (see [Mi] for definition).*

Under such conditions, we say that f has a **critical continuous extension** *F (related to p) if*

- *$F : S^1 \to S^1$ is continuous and onto S^1;*
- *there exist a finite interval's collection $\{I_i\}$ such that F/I_i is constant ($\forall i$), $F/(S^1 \setminus \cup I_i) = f$ and $p \in int(S^1 \setminus \cup I_i)$.*

Remark 2.2.

1. Any onto map $f : S^1 \setminus \{c_1, c_2, ..., c_r\} \to S^1$ with $r = 1$ and satisfying (1), (2) and (3) of Definition 2.2 has critical continuous extension.

2. Besides degree one liftings for circle maps, it was introduced in [Mi] the concept of *old heavy maps*. It can be shown that the class of old heavy maps equals to the class of maps $f : S^1 \setminus \{c_1, c_2, ..., c_r\} \to S^1$ with $r = 1$ and having critical continuous extension. It is not true in general when $r > 1$.

The following result extends one given in [NPT] to circle maps with critical continuous extension. It will be used in section 5 for the proof of Theorem (B).

THEOREM 2.6. – *If $f : S^1 \setminus \{c_1, c_2, ..., c_r\} \to S^1$ has critical continuous extension, then there exists $\theta_0 \in [0, 2\pi]$ such that the one-parameter family $R_\theta \circ f$ unfolds a homoclinic tangency at θ_0.*

Proof. – Let F be a critical continuous extension of f (it is not necessary unique). It is immediatly from the definition that $R_\theta \circ F$ is a continuous critical extension of $R_\theta \circ f$. It is well known (see [NPT]) that for $F_\theta = R_\theta \circ F$, there exists $\theta \in [0, 2\pi]$ such that it exhibits a tangency, i.e. there exists a critical point q and a periodic point p, such that $F_\theta(q) = p$ and $W^u(F_\theta, q) = S^1$. This last means that, for all open set U such that $q \in U$, $\cup_{n\geq 0} F_\theta^n(U) = S^1$. From this it follows that $q \notin \cup I_i$ and $F_\theta^n(U) \nsubseteq \cup I_i$. Hence:

– q is a periodic point for f_θ;

– $\cup_{n\geq 0} f_\theta^n(U) = S^1$ for all open set U such that $q \in U$;

– $f_\theta(p) = q$,

completing the proof.

A topology for discomtinuous maps.

To finish this section, we introduce a topology in $E_k(J,c)$ ($k \in I\!R^+$) to be used later on. We shall use a notion of closeness adapted to our specific situation. In what follows, $dist_J$ will be the usual metric in J (recall that $J = S^1$ or a compact real interval) and $dist_i$ denotes the C^0 and C^1 metric for $i = 0, 1$ respectively. If $f \in E(J) = \cup_{c \in J, k \in I\!R^+} E_k(J,c)$, then $k(f) = \inf_{x \in J \setminus \{c\}} |f'(x)|$. Clearly $k(f)$ is well defined and satisfies $k(f) \geq k$ if $f \in E_k(J,c)$.

DEFINITION 2.3. – *We say that f, g are δ-**close** ($\delta > 0$) if $dist_J(c(f), c(g)) < \delta$, $|k(f) - k(g)| < \delta$, $dist_0(f,g) < \delta$ and $dist_1(f,g) < \delta$ outside a compact 2δ-neighborhood of $\{c(f), c(g)\}$. If I is a compact real interval and $\{f_\theta\}_{\theta \in I}, \{g_\theta\}_{\theta \in I} \subset E(J)$ are parametrized families, we say that $\{f_\theta\}_{\theta \in I}$ and $\{g_\theta\}_{\theta \in I}$ are δ-close if f_θ and g_θ does $\forall \theta \in I$.*

We state a proposition in which are summarized the main properties of the notion of closeness introduced in Definition 2.3. It will be used in the proof of Theorem (A) at section 4.

PROPOSITION 2.7. – *Let $I = [0, 2\pi]$ (say) and $\{f_\theta\}_{\theta \in I}, \{g_\theta\}_{\theta \in I} \subset E(S^1)$ parametrized families. Define $k(\{h_\theta\}_{\theta \in I}) = \inf_{\theta \in I} k(h_\theta)$ for $h = f, g$. Then,*

1. *$\forall \epsilon > 0$, $\exists \delta > 0$ such that $|k(\{f_\theta\}_{\theta \in I}) - k(\{g_\theta\}_{\theta \in I})| < \epsilon$ if $\{f_\theta\}_{\theta \in I}$ and $\{g_\theta\}_{\theta \in I}$ are δ-close;*

2. *suppose $f_\theta = R_\theta \circ f$ for some f as in Theorem 2.2 and let I_f the interval obtained in that theorem. Then $\exists \delta > 0$ such that if $\{f_\theta\}_{\theta \in I}$ and $\{g_\theta\}_{\theta \in I}$ are δ-close, there is an interval I_g (close to I_f) such that conclusion of Theorem 2.2 holds changing f, I_f by g, I_g (resp.);*

3. *suppose $f_\theta = R_\theta \circ f$ for some f as in Proposition 2.4 and let I_{1f} and I_{2f} the intervals obtained in that proposition. Then $\exists \delta > 0$ such that if $\{f_\theta\}_{\theta \in I}$ and $\{g_\theta\}_{\theta \in I}$ are δ-close, there are intervals I_{1g}, I_{2g} (close to I_{1f}, I_{2f} resp.) such that conclusion of Proposition 2.4.1 holds changing f, I_{1f} by g, I_{1g} (resp.) and g_θ or g_θ^2 is transitive in some proper closed subinterval of S^1 $\forall \theta \in I_{2g}$.*

Proof. – Note that in the proof of theorems 2.1, 2.2 and Proposition 2.4 all construction involved are open with respect to the notion of closeness in Definition 2.3. Now use compactness of I.

Remark 2.3. – We will be interested in study the particular case when the parametrized families $\{g_\theta\}_{\theta \in I}$ and $\{f_\theta\}_{\theta \in I}$ satisfy $f_\theta = R_\theta \circ f$ and $g_\theta = R_\theta \circ g$ for fixed $f, g \in E(S^1)$. It is easy to see that if f and g are δ-close, then $\{g_\theta\}_{\theta \in I}$ and $\{f_\theta\}_{\theta \in I}$ are δ-close. This will be used in the proof of Claim 3.5 at section 3.

3. One-dimensional reductions

In this section we prove some preliminary results concerning the dynamics after unfold saddle-node singular cycles. The main result of this section is Theorem 3.3 which tells about existence of stable foliations in the case when the cycle is transversal. This will permit one-dimensional reductions to be used in section 4 for the proof of Theorem (A). Since criticalities are clearly an obstruction for existence of stable foliations, Theorem (B) will require a different approach. In this case, instead, we shall use renormalization techniques as in [DRV]. Such techniques will be use as well in the proof of theorems (C) and (D) in the appendix.

To start, we associate a circle map f_Γ to any saddle-node singular cycle Γ. This idea was introduced in [NPT] for the analyse of saddle-node cycles in absence of singularities (see also [DRV]). To construct f_Γ we require some background (see [NPT] pp 13 for details). Suppose that $\Gamma = \{\sigma_0, \sigma_1, \gamma_0, \gamma_1\}$ is a saddle-node singular cycle of a vector field X. Then we have the following facts.

Fact 1. There exist a transversal section Σ of X and a first return map π on Σ such that σ_1 is a saddle-node fixed point of π. The following *structures* are well defined,

• A centre-unstable manifold W^{cu} passing through σ_1.

• A strong stable manifold W^{ss}, containig σ_1, whose tangent space (at σ_1) is the one corresponding to the hyperbolic eigenvalue of $D\pi(\sigma_1)$. Also, W^{ss} is tranversal to W^{cu} at σ_1 and divides W^{cu} in two branches denoted by $W^{cu,+}$ and $W^{cu,-}$ in the sequel.

• A strong stable foliation $\mathcal{F}^{ss}$ defined in Σ. This foliation induces a coordinate system (x, y) in Γ such that π takes the form $\pi(x, y) = (\varphi_0(x), \psi_0(x, y))$. Here $\psi(x, y)$ satisfies $\psi_0(x, 0) = 0$ and $|\partial_y \psi_0(x, y)| \le \lambda < 1$ for some fixed constant λ. The right and left boundaries of Σ as well as W^{ss} are leaves of $\mathcal{F}^{ss}$.

Fact 2. Identifying points in the same orbit of π when restricted to W^{cu}, both $W^{cu,+}$ and $W^{cu,-}$ are *circles*, i.e. they are copies of S^1. We shall use this identification without explicit mention. There is a *tangent vector field* Z associated to π, i.e. $\varphi_0 = \pi/W^{cu}$ is just the time-one map of Z (see [DRV]). If $a \in W^{cu,+}$, we define a coordinate system $\varphi_a : W^{cu,+} \to S^1$ by $\varphi_a(x) = e^{2\pi i t_a(x)}$, where $t_a(x)$ solves $Z[t, x] = a$. Similar constructions can be done for any $b \in W^{cu,-}$.

Fact 3. There is a return map $H_0 : ([\varphi_0^{-1}(a), \varphi_0(a)] \times [-\Delta, \Delta]) \setminus D_0 \to [b, \varphi_0(b)] \times [-\Delta, \Delta]$, induced by the flow of X, where Δ is just the diameter of Σ and $D_0 = W^s(\sigma_0) \cap [\varphi_0^{-1}(a), \varphi_0(a)] \times [-\Delta, \Delta]$. It turns out that D_0 is a finite set of almost vertical curves and it reduces to a single one when Γ is transversal (see Definition 1.1). In that case, D can be chosen to be the vertical line $\{c_0\} \times [-\Delta, \Delta]$. The restriction of H_0 to $[\varphi_0^{-1}(a), \varphi_0(a)] \times \{0\}$ will be denoted by h.

It is well known that structures (Σ, W^{cu}, a, b) satisfying facts (1), (2) and (3) are not unique. However, we shall see in Proposition 3.1 below they satisfy some sort of uniqueness. Now we are ready to define f_Γ.

DEFINITION 3.1. – *Let Γ be a saddle-node singular circle of X. Then $f_\Gamma : S^1 \setminus D \to S^1$ is defined by the formula*

$$f_\Gamma = \varphi_b \circ \pi^{ss} \circ h \circ \varphi_a^{-1},$$

*where π^{ss} denotes projection along $\mathcal{F}^{ss}$. When Γ is transversal, we say that it is k-***expansive** *($k > 0$) if $f_\Gamma = f_\Gamma(\Sigma, W^{cu}, a, b) \in E_k(S^1, c_0)$ for some structure (Σ, W^{cu}, a, b).*

We note that if D_0 is a single point $\{c_0\}$ and $f_\Gamma = f_\Gamma(\Sigma, W^{cu}, a, b) \in E_k(S^1, c_0)$ for some structure (Σ, W^{cu}, a, b) and $k > 0$, then it is not true that Γ is transversal. The notion of expansivity in Definition 3.1 is based on the following result whose proof will be given in the appendix.

PROPOSITION 3.1. – *Suppose that* Γ *is a saddle-node singular cycle of a vector field and consider structures* (Σ, W^{cu}, a, b) *and* $(\hat{\Sigma}, \hat{W}^{cu}, \hat{a}, \hat{b})$ *according (1), (2) and (3) above. Then* $f_\Gamma = f_\Gamma(\Sigma, W^{cu}, a, b)$ *and* $\hat{f}_\Gamma = \hat{f}_\Gamma(\hat{\Sigma}, \hat{W}^{cu}, \hat{a}, \hat{b})$ *differ by rotation in* S^1*, i.e.* $\exists \theta \in [0, 2\pi)$ *such that* $f_\Gamma = R_\theta \circ \hat{f}_\Gamma$.

Thus, expansivity actually does not depent on structures. On the other hand, it is a natural to ask which circle maps are f_Γ for some cycle Γ. We give conditions to guarantee it. Consider $r \in \mathbb{N}$ and let $\tilde{E}^r(S^1, \tilde{c})$ be the set of function $f_\Gamma : S^1 \setminus \{\tilde{c}\} \to S^1$ such that,

- f is C^r and the lateral limits of f at $\tilde{c}$ exist;
- The derivatives $f^{(i)}(x)$ for $1 \leq i \leq r$ satisfy

$$(**) \qquad \lim_{x \to \tilde{c}} \frac{f^{(i)}(x)}{(x - \tilde{c})^{\alpha - i}} = k_i,$$

for some fixed constant k_i and $\alpha \in (0, 1)$.

PROPOSITION 3.2. – *For any* $r \in \mathbb{N}$ *and* $f \in \tilde{E}^r(S^1, \tilde{c})$ *there is a* C^r*-vector field* X *in* M *with a saddle-node singular* Γ *such that* $f = f_\Gamma$.

This proposition is not used in the proof of results in section 1. Instead, it will be used to prove Theorems (C) and (D) in the appendix. Its proof is straighforward.

Remark 3.0

1. Observe that for any cycle Γ, property $(\star\star)$ holds with $f = f_\Gamma$, $i = 0, 1$ and $\tilde{c} \in D$. Here $\alpha = \frac{\lambda_3}{\lambda_1}$ (recall Definition 1.1). One has $\alpha < 1$ when Γ is k-expansive for some $k > 0$. This is because the singularity in Γ must satisfies some nonresonance conditions to guarantee existence of C^2-linearizing coordinates and so $\lambda_3 \neq \lambda_1$ (see section 1). In what follows we use the notation $\beta = \frac{\lambda_2}{\lambda_1}$ so $\beta - \alpha > 0$ (recall Definition 1.1 (1)).

2. We can make some reductions on H_0 when Γ is transversal (see Fact (3)). Choosing $b < 0 < a$ sufficiently close to 0, H_0 takes the form $H_0(x, y) = (H_{10}(x, y), H_{20}(x, y))$ with respect to the coordinate system (x, y) in Σ with

$$\left| \frac{\partial_x H_{20}(x, y)}{\partial_x H_{10}(x, y)} \right| \quad \text{and} \quad \left| \frac{\partial_y H_{10}(x, y)}{\partial_x H_{10}(x, y)} \right|$$

uniformly small for $(x, y) \in ([\varphi_0^{-1}(a), \varphi_0(a)] \times [-\epsilon^*, \epsilon^*]) \setminus \{c_0\}$ and $\epsilon^* > 0$ small. In the particular case when Γ is k-expansive for some $k > 0$, such reductions can be made and by Proposition 3.1 they do not affect the expansivity of Γ.

Next we recall well known facts. If X_0 is a vector field and $\Gamma = \{\sigma_0, \sigma_1, \gamma_0, \gamma_1\}$ is a saddle-node singular cycle of X, then vector fields close to X_0 and having a saddle-node periodic orbit nearby σ_1 belong to a codimension-one submanifold S in the space of all vector fields. Now, S separates a small neighborhood $\mathcal{U}$ of X_0 in two open regions. One of them ($\mathcal{U}^-$ say) consists of vector fields whose nonwandering set close to σ_1 is formed by two hyperbolic periodic orbits. We say that a one-parameter family X_μ, $\mu \in \mathbb{R}$, *unfolds* Γ *positively* whenever X is transversal to S at $\mu = 0$ and $X_\mu \in \mathcal{U}^-$ if $\mu < 0$. An *annulus* A will be a 2-manifold diffeomorphic to $\{z \in \mathbb{C} : |z| \in [r, R]\}$, $R > r > 0$. A smooth curve $l \subset A$ is *radial* if it is transversal to any circle $\{z \in \mathbb{C} : |z| = t\}$ $(t \in [r, R])$. Following

[PT1], *analytical continuation* of hyperbolic critical elements in Γ will be indicated as $\sigma_0(\mu)$, for instance, when one-parameter families are considered. The remainer of this section will be devoted to prove the following key result.

THEOREM 3.3. – *Let X_μ be a one-parameter family of vector fields such that X_0 exhibits a transversal saddle-node singular cycle $\Gamma = \{\sigma_0, \sigma_1, \gamma_0, \gamma_1\}$. Suppose that X_μ unfolds Γ positively. Then $\exists \delta > 0$ such that for $0 < \mu < \delta$ the following hold,*

a. *there exist an annular cross section A_μ of X_μ and a radial curve $l_\mu \subset A_\mu \cap W^s(\sigma_0(\mu))$ such that $\forall p \in A_\mu \backslash l_\mu$ there is a first time $t(p) > 0$ such that $F_\mu(p) = X_\mu[t(p), p] \in int(A_\mu)$;*

b. *$F_\mu : A_\mu \setminus l_\mu \to A_\mu$ is C^1-conjugate to a map $\hat{F}_\mu : S^1 \times [-1, 1] \setminus \hat{l}_\mu \to S^1 \times [-1, 1]$ satisfying the properties below,*

b.1 *$\hat{l}_\mu$ is a radial curve in $A = S^1 \times [-1, 1]$;*

b.2 *$\hat{F}_\mu$ admits an invariant strong stable C^1-foliation $\mathcal{L}_\mu^{ss}$ in A, consisting of radial curves, such that $\hat{l}_\mu$ is a leaf of $\mathcal{L}_\mu^{ss}$;*

c. *let f_μ be the $\hat{F}_\mu$-induced map on $S^1 = A/\mathcal{L}_\mu^{ss}$ (so $f_\mu \in E(S^1)$, recall Definition 2.3). Then, there are decreasing sequences $\delta_n, \mu_n^* > 0$ (converging to 0) and an orientation reversing onto difeomorphisms sequence $\mu_n : [0, 2\pi] \to [\mu_n^*, \mu_{n-1}^*]$ (with inverse θ_n) such that,*

c.1. *$\{f_{\mu_n(\theta)}\}_{\theta \in [0,2\pi]}$ and $\{R_\theta \circ f_\Gamma\}_{\theta \in [0,2\pi]}$ are δ_n-close (recall Definition 2.3);*

c.2. *distorsion property holds for sequence θ_n, i.e. given $\epsilon > 0$ and every Borel set $A \subset [0, 2\pi]$ we have*

$$(1 - \epsilon)m(A) \leq \frac{m(\theta_n(A))}{m([\mu_n^*, \mu_{n-1}^*])} \leq (1 + \epsilon)m(A).$$

Proof. – The proof goes through the following steps.

Step 1: Discontinuous dynamical systems on annular cross sections.

In this step we present the main ingredients for the proof of Theorem 3.3. For the analyze return maps, we shall introduce the coordinate systems Φ_μ^+ and Φ_μ^- which permit to transport the dynamic on the cross section to a discontinuous map in $S^1 \times [-1, 1]$. The induced dynamic will resemble the one exhibit by the geometric Lorenz attractor, but now in the annulus $S^1 \times [-1, 1]$ instead of the square $[-1, 1]^2 \subset I\!R^2$ (see Figure 3). We follow closely [NPT] (pp 13) and [DRV] in this step.

To start, let us consider the cycle Γ and its corresponding circle map f_Γ depending on structure (Σ, W^{cu}, a, b) according Definition 3.1. Using analytic continuation of $\mathcal{F}^{ss}$ (see Fact (3)) we obtain a μ-dependend coordinate system (x, y) in Σ, such that continuation π_μ of π (see (1)) has the following form,

$$\pi_\mu(x, y) = (\varphi_\mu(x), \psi_\mu(x, y)),$$

where φ_μ is a *saddle-node arc* and ψ_μ satisfies $\psi_\mu(x, 0) = 0$, $|\partial_y \psi_\mu(x, y)| \leq \lambda < 1$ for some fixed constant λ. Such a coordinate system can be chosen in a way that $W^s(\sigma_0(\mu)) \cap \Sigma$ is the vertical $\{(c_\mu, y) : y \in [-\Delta, \Delta]\}$. Recall Δ is almost the diameter of Σ.

Let Z_μ be the tangent vector field associated to the saddle-node arc φ_μ. Recall it is a μ-dependent vector field such that φ_μ is the time one map of Z_μ. The following sets will be considered in the sequel. Take A_μ^+ as the Σ's subset with vertical boundaries $\{(\varphi_\mu^{-1}(a), y) : y \in [-\Delta, \Delta]\}$ and $\pi_\mu(\{(\varphi_\mu^{-1}(a), y) : y \in [-\Delta, \Delta]\})$. We also consider the set A_μ^-, with vertical boundaries $\{(b, y) : y \in [-\Delta, \Delta]\}$ and $\pi_\mu(\{(b, y) : y \in [-\Delta, \Delta]\})$.

Now we shall introduce two coordinate systems $\Phi_\mu^+ : A_\mu^+ \cup \pi_\mu(A_\mu^+) \to S^1 \times [-1, 1]$ and $\Phi_\mu^- : A_\mu^- \to S^1 \times [-1, 1]$ which will play an important role. To simplify notation, we write $C_\mu = A_\mu^+ \cup \pi_\mu(A_\mu^+)$. Consider the solutions $t_\mu^+(x)$ and $t_\mu^-(x)$ of the time equations $Z_\mu[t, x] = a$ and $Z_\mu[t, x] = b$ for x in $[\varphi_\mu^{-1}(a), \varphi_\mu(a)]$ and $[b, \varphi_\mu(b)]$ respectively. Such solutions will be angular coordinates in the contruction below. We define now Φ_μ^+ and Φ_μ^- by the formulae

$$\Phi_\mu^+(x, y) = (e^{2\pi t_\mu^+(x)i}, \Theta_\mu^+(x, y))$$

and

$$\Phi_\mu^-(x, y) = (e^{2\pi t_\mu^-(x)i}, \Theta_\mu^-(x, y)),$$

for (x, y) in C_μ and A_μ^- respectively. Here $\Theta_\mu^+ : C_\mu \to S^1 \times [-1, 1]$ and $\Theta_\mu^- : A_\mu^- \to S^1 \times [-1, 1]$ are smooth and satisfy,

– $\Theta_\mu^\pm(\pi_\mu(x, y)) = \Theta_\mu^\pm(x, y)$;

– There exist positive constants c_1, c_2 such that $c_1 \leq ||D\Theta_\mu^\pm(x, y)|| \leq c_2$ and $\Theta_\mu^\pm(x, 0) = 0$, where D denotes derivation in (x, y).

Once defined Φ_μ^+ and Φ_μ^-, we observe that there is a return map

$$H_\mu : C_\mu \setminus \{(r_\mu, y) : r_\mu \in \{\varphi_\mu^{-1}(c_\mu), c_\mu\} \text{ and } y \in [-\Delta, \Delta]\} \to A_\mu^-$$

induced by the X_μ's flow such that H_0 is the one in Fact (3). Here c_μ is such that $\{(c_\mu, y) : y \in [-\Delta, \Delta]\} = C_\mu \cap W^s(\sigma_0(\mu))$.

We define the *transition map* T_μ as follows. Consider $k_\mu \in \mathbb{N}$ and $s_\mu \in \mathbb{R}^+$ (for $\mu > 0$ small) satisfying $k_\mu < s_\mu \leq k_\mu + 1$, $\pi^{k_\mu}(A_\mu^-) \subset C_\mu$ and $Z_\mu[s_\mu, b] = a_\mu$. Define

$$T_\mu : (x, y) \in A_\mu^- \to \pi^{k_\mu}(x, y) = (\varphi_\mu^{k_\mu}(x), \Delta_\mu^{k_\mu}(x, y)),$$

where Δ_μ^n are *cocycle-like* maps defined inductively by $\Delta_\mu^{n+1}(x, y) = \psi_\mu(\varphi_\mu^n(x), \Delta_\mu^n(x, y))$. We separate the following properties of the transition map (see [DRV]).

LEMMA 3.4. – *The limit of T_μ when μ goes to 0^+, in the C^2- topology, exists and takes the form $(T\infty(x), 0))$. The map $T\infty(x)$ is smooth and has derivative bounded away from* 0.

Finally we introduce two useful functions $\hat{H}_\mu : A^* \to A$ and $\hat{T}_\mu : A \to A$, where $A = S^1 \times [-1, 1]$ and $A^* = A \setminus \{(r_\mu, y) : r_\mu \in \{\varphi_\mu^{-1}(c_\mu), c_\mu\}, y \in [-\epsilon, \epsilon]\}$. They are given by the relations

$$\hat{H}_\mu(s, r) = (\Phi_\mu^- \circ H_\mu)(x, y)$$

and

$$\hat{T}_\mu(s, r) = (\Phi_\mu^+ \circ T_\mu)(x, y),$$

where (x, y) satisfies $\Phi_\mu^+(x, y) = (s, r)$ and $\Phi_\mu^-(x, y) = (s, r)$ respectively.

Remark 3.1.

1. Observe that $F_\mu = T_\mu \circ H_\mu$ is the lift of $\hat{F}_\mu = \hat{T}_\mu \circ \hat{H}_\mu$ to D_μ via $\Phi_\mu^+ : C_\mu \to A$. In particular, F_μ and $\hat{F}_\mu$ are C^1-conjugates. This fact will be used in Step 3.

2. $\hat{T}_\mu$ is a rotation when restricted to $S^1 \times \{0\}$. Indeed, one has $\hat{T}_\mu(s,0) = (e^{2\pi i(s_\mu - k_\mu)}s, 0)\ \forall s \in S^1$.

3. If P_{S^1} denotes projection on S^1, $s \in S^1 \to (P_{S^1} \circ \hat{H}_\mu)(s,0)$ is δ-close to f_Γ for some $\delta > 0$ (see Definition 2.3). Notice that both maps have $c = e^{2\pi i t_\mu^+(c_\mu)}$ as common "discontinuity" point.

4. By Lemma 3.4, the cocycle-like maps sequence Δ_μ^n converges to 0 in the C^2-topology. This fact enables us to reduce the analyze of the H_μ's dynamics for $\mu > 0$ small in the region $\{(x,y) : |y| \leq \epsilon^*\}$, where ϵ^* comes from Remark 3.0 (2).

This finishes step 1.

Step 2: Stable foliations.

In this setp, we show existence of strong stable foliation for $F_\mu = T_\mu \circ H_\mu$ when $\mu > 0$ is small. We use graph transformed techniques in this part (see [HPS], [BLMP] and [M] for details). The main result in this step is the following.

LEMMA 3.5. – *Let T_μ and H_μ be as in Step 1 and $F_\mu = T_\mu \circ H_\mu$. Then there exists a C^1-foliation $\hat{\mathcal{L}}_\mu^{ss}$ in C_μ with the following properties:*

1. $\pi_\mu(\hat{\mathcal{L}}_\mu^{ss}(q)) \subset \hat{\mathcal{L}}_\mu^{ss}(\pi_\mu(q))$ *for all* $q \in A_\mu^+$;

2. $F_\mu(\hat{\mathcal{L}}_\mu^{ss}(q)) \subset \hat{\mathcal{L}}_\mu^{ss}(F_\mu(q))$ *for all q in $C_\mu \setminus (l_{1\mu} \cup l_{2\mu})$, Here $l_{i\mu}$ $i = 1,2$ stand for* $\{(\varphi_\mu^{-1}(c_\mu), y) : y \in [-\epsilon, \epsilon]\} \cap C_\mu$ *and* $\{(c_\mu, y) : y \in [-\epsilon, \epsilon]\} \cap C_\mu$ *respectively;*

3. *The curves $l_{i\mu}$ $i = 1,2$ are leaves of $\hat{\mathcal{L}}_\mu^{ss}$.*

4. *There exist $L \in (0,1)$ such that if $n \in I\!N$, then dist$((T_\mu \circ H_\mu)^n(p), (T_\mu \circ H_\mu)^n(q)) \leq L^n$, for all $p, q \in C_\mu$, $q \in \hat{\mathcal{L}}_\mu^{ss}(p)$. Here $\hat{\mathcal{L}}_\mu^{ss}(p)$ is the lift of $\hat{\mathcal{L}}_\mu^{ss}$ passing through p.*

Proof. – We start with

CLAIM 3.1. – *There exists $B > 0$ such that if $H_\mu = (H_{1\mu}, H_{2\mu})$,*

$$0 < \sup \left\{ \frac{|\partial_y H_{1\mu}|}{|\partial_x H_{1\mu}|}, \frac{|\partial_x H_{2\mu}|}{|\partial_x H_{1\mu}|}, \frac{|\partial_y H_{2\mu}|}{|\partial_x H_{1\mu}|}, \frac{|\det DH_\mu|}{|\partial_x H_{1\mu}|} \right\} \leq B.$$

Here D denotes derivative with respect to (x,y) and *det* means determinat. This claim follows by Remark 3.0 (1) and the eigenvalue conditions in Definition 1.1 (1) (see also [BLMP]).

Now write

$$DF_\mu = \begin{pmatrix} \tilde{A}_\mu & \tilde{B}_\mu \\ \tilde{C}_\mu & \tilde{D}_\mu \end{pmatrix}$$

with

a^*. $\tilde{A}_\mu = [(\varphi_\mu^{k_\mu})' \circ H_{1\mu}]\partial_x H_{1\mu}$; $\tilde{B}_\mu = [(\varphi_\mu^{k_\mu})' \circ H_{1\mu}]\partial_y H_{1\mu}$;
b^*. $\tilde{C}_\mu = [(\partial_x \Delta_\mu^{k_\mu}) \circ H_\mu]\partial_x H_{1\mu} + [(\partial_y \Delta_\mu^{k_\mu}) \circ H_\mu]\partial_x H_{2\mu}$;

c^*. $\tilde{D}_\mu = [(\partial_x \Delta_\mu^{k_\mu}) \circ H_\mu]\partial_y H_{1\mu} + [(\partial_y \Delta_\mu^{k_\mu}) \circ H_\mu]\partial_y H_{2\mu}$.

Consider the functional space

$$\mathcal{A} = \{\phi : C_\mu \to [-1,1] : \phi \text{ is continuous}, \phi(r_\mu, y) = 0, r_\mu = c_\mu, \varphi_\mu^{-1}(c_\mu), \forall y\}$$

endowed with the supremum norm $||.||$ and the graph operator

$$G(\phi) = \frac{(F_\mu \circ \phi)\tilde{D}_\mu - \tilde{B}_\mu}{\tilde{A}_\mu - (F_\mu \circ \phi)\tilde{C}_\mu}$$

defined in $\mathcal{A}$. Now it follows that

$$|G(\phi)(q) - G(\phi')(q)| \leq \frac{|detDF_\mu(q)|.|\phi(q) - \phi'(q)|}{|\tilde{A}_\mu(q) - (F_\mu(\phi(q)))\tilde{C}_\mu(q)||\tilde{A}_\mu(q) - (F_\mu(\phi'(q)))\tilde{C}_\mu(q)|},$$

$\forall q = (x,y) \in C_\mu \setminus \{(r_\mu, y) : r_\mu \in \{\varphi_\mu^{-1}(c_\mu), c_\mu\}$ and $y \in [-\Delta, \Delta]\}$.

CLAIM 3.2. – *There are fixed constants $K_0, K_1 > 0$ such that if $O(L)$ denotes order L for (x,y) close to $(r_\mu, 0)$, then*

1. $O(\tilde{A}_\mu) \geq K_1|x - r_\mu|^{(1-\alpha)}$;
2. $O(\tilde{B}_\mu), O(\tilde{D}_\mu) \leq K_1|x - r_\mu|^{(\beta-\alpha+1)}$;
3. $|\tilde{C}_\mu/\tilde{A}_\mu| \leq K_0$;
4. $O(detDF_\mu) \geq K_1|x - r_\mu|^{(\beta+\alpha-1)}$,

recall Remark 3.0 (1). The constant K_0 can be chosen small.

The proof of this claim follows using the linearization assumption in section 1 and Remark 3.0 (2).

Now, by Lemma 3.4, $|\partial_y \Delta_\mu^{k_\mu}|$ and $|\partial_y \Delta_\mu^{k_\mu}|$ are small for $\mu > 0$ close to 0 and there is a fixed constant $K > 0$ such that $|(\varphi_\mu^{k_\mu})'(x)| > K$. Then, by Remark 3.0 (2), (a^*), (b^*), (c^*) and claims 3.1 and 3.2, we conclude that $G(\phi)$ extends continuously to $\{(r_\mu, y) : r_\mu \in \{\varphi_\mu^{-1}(c_\mu), c_\mu\}, \forall y\}$, $G(\phi) \in \mathcal{A}$, $\forall \phi \in \mathcal{A}$, and G is a contraction in $\mathcal{A}$. Thus, G has a fixed point $\phi*$ and $(\phi^*, 1)$ gives rise a continuous F_μ-invariant set of lines $\{E_q\}_{q \in C_\mu}$ with the properties below.

– $DF_\mu^{-1}(F_\mu(q))(E_{F_\mu(q)}) = E_q$.

– $||DF_\mu^{-1}/E|| \geq M$, for some fixed constant $M > 1$.

– If $\hat{\mathcal{L}}_\mu^{ss}$ is the foliation (in C_μ) obtained by integrating $\{E_q\}$, then both $l_{1\mu}$ and $l_{2\mu}$ are leaves $\hat{\mathcal{L}}_\mu^{ss}$.

– $\hat{\mathcal{L}}_\mu^{ss}$ is π_μ–invariant, i.e. if $q \in A_\mu^+$, then $\pi_\mu(\hat{\mathcal{L}}_\mu^{ss}) \subset \hat{\mathcal{L}}_\mu^{ss}(\pi_\mu(q))$. This follows because $H_\mu(\pi_\mu(q)) = H_\mu(q)$ for all $q \in A_\mu^+$.

To complete the proof of Lemma 3.5, we must prove that $\hat{\mathcal{L}}_\mu^{ss}$ is C^1. For this we introduce the functional space

$$\mathcal{A}_1 = \{A : C_\mu \longrightarrow \mathcal{L}(I\!R^2, I\!R) : A \text{ is continuous}, A(r_\mu, y) = 0, r_\mu = c_\mu, \varphi_\mu^{-1}(c_\mu), \forall y\}$$

and the operator

$$S(\phi, A) = \frac{(F_\mu \circ \phi)\nabla \tilde{D}_\mu - \nabla \tilde{B}_\mu}{\tilde{A}_\mu - (F_\mu \circ \phi)\tilde{C}_\mu}$$

$$-G(\phi)\frac{\nabla \tilde{A}_\mu - (F_\mu \circ \phi)\nabla \tilde{C}_\mu}{\tilde{A}_\mu - (F_\mu \circ \phi)\tilde{C}_\mu} + \frac{detDF_\mu}{(\tilde{A}_\mu - (F_\mu \circ \phi)\tilde{C}_\mu)^2}(A \circ F_\mu)DF_\mu,$$

for fixed $\phi \in \mathcal{A}$.

We use the following straighforward

CLAIM 3.3. – *For some fixed constant* $K_2 > 0$ *the following hold,*

1. $O(\nabla \tilde{D}_\mu / \tilde{A}_\mu), O(\nabla \tilde{D}_\mu / \tilde{A}_\mu) \leq K_2 |x - r_\mu|^{(\beta - \alpha)}$;
2. $O(\nabla \tilde{A}_\mu)/\tilde{A}_\mu), O(\nabla \tilde{C}_\mu / \tilde{A}_\mu) \leq K_2 |x - r_\mu|^{-1}$;

Now we finish the proof of Lemma 3.5 as follows. First we observe that Claim 3.1 (a^*),(b^*) and (c^*) imply that $|detDF_\mu|/\tilde{A}_\mu| \leq K_4 ||\partial_y \Delta_\mu^{k_\mu}|$ for some fixed constant K_4. This implies that $A \to S(\phi, A)$ is a contraction with contracting constant independent on ϕ for $\mu > 0$ small (recall Lemma 3.4). By claims 3.2 and 3.3, we can extend $S(\phi, A)$ to $\{(r_\mu, y) : r_\mu \in \{\varphi_\mu^{-1}(c_\mu), c_\mu\}, \forall y\}$ and so we get $S(\phi, A) \in \mathcal{A}_1$ $\forall (\phi, A) \in \mathcal{A} \times \mathcal{A}_1$. Thus $(\phi, A) \to (G(\phi), S(\phi, A))$ has a unique fixed point (ϕ^*, A^*) (recall ϕ^* is the fixed point of G). Thus ϕ^* is C^1 with derivative equals A^* proving Lemma 3.5.

Proof of Theorem 3.3 (a). – Consider the annulus A_μ obtained identifying π_μ-orbits in C_μ. Another way to see A_μ is considering a plane cross section of $W^u(\sigma_0)$ as $W^u(\sigma_0)$ is locally a cilinder (recall Definition 1.1 (1)). Let l_μ equals $\{(r_\mu, y) : r_\mu \in \{\varphi_\mu^{-1}(c_\mu), c_\mu\}, y \in [-\Delta, \Delta]\}$ up to quotient identification in C_μ. Then A_μ,l_μ and $F_\mu = T_\mu \circ H_\mu : A_\mu \setminus l_\mu \to A_\mu$ satisfy Theorem 3.3 (a).

Step 3: Proof Theorem 3.3 (b)-(c).

We introduce the foliations $\mathcal{L}_{\mu,-}^{ss}$ and $\mathcal{L}_{\mu,+}^{ss}$ in A as the pullback of $\hat{\mathcal{L}}_\mu^{ss}$ by $\Phi_\mu^- \circ T_\mu^{-1}$ and Φ_μ^+ respectively (recall $A = S^1 \times [-1, 1]$). Consider $\hat{F}_\mu = \hat{T}_\mu \circ \hat{H}_\mu$, where $\hat{H}_\mu$ and $\hat{T}_\mu$ are as in Remark 3.1 (1). Then it follows that $\mathcal{L}_{\mu,-}^{ss}$ is C^1 and

$$(\star\star\star) \qquad \lim_{\mu \to 0^+} \mathcal{L}_{\mu,-}^{ss} = P_{S^1}$$

in the C^1-topology, where P_{S^1} is the projection of $A = S^1 \times [-1, 1]$ onto S^1 (recall Remark 3.1). Now $H_\mu(\hat{\mathcal{L}}_\mu^{ss}) \subset T_\mu^{-1}(\hat{\mathcal{L}}_\mu^{ss})$, i.e. H_μ sends leaves of $\hat{\mathcal{L}}_\mu^{ss}$ into leaves of the form $T_\mu^{-1}(\hat{\mathcal{L}}_\mu^{ss}(q))$. This implies that $(s, 0) \to (\hat{T}_\mu \circ P_{\mathcal{L}_{\mu,-}^{ss}} \circ \hat{H}_\mu)(s, 0)$ is just the one-dimensional map f_μ induced by $\hat{F}_\mu$ in the quotient space $A/\mathcal{L}_{\mu,+}^{ss}$, where $P_{\mathcal{L}_{\mu,-}^{ss}}$ denotes projection along $\mathcal{L}_{\mu,-}^{ss}$.

CLAIM 3.4. – *Let* $L_\mu(s) = (P_{\mathcal{L}_{\mu,-}^{ss}} \circ \hat{H}_\mu)(s, 0)$ *for* $s \in S^1$. *Then* $\forall \delta > 0$ $\exists \mu_\delta > 0$ *such that* L_μ *and* f_Γ *are* δ*-close,* $\forall \mu \in (0, \mu_\delta)$ *(recall Definition 2.3).*

Proof. – Use (***) and Definition 3.1.

CLAIM 3.5. – *For all* $\delta > 0$ $\exists \mu_\delta > 0$ *such that* $\hat{T}_\mu \circ L_\mu$ *and* $R_{\theta(\mu)} \circ f_\Gamma$ *are* δ*-close,* $\forall \mu \in (0, \mu_\delta)$*, where* $\theta(\mu) = 2\pi(s_\mu - k_\mu)$ *(see Remark 3.1 (2)).*

Proof. – By Remark 3.1 (2) $\hat{T}_\mu/(S^1 \times \{0\})$ is $R_{\theta(\mu)}$. Now use Claim 3.4 (see Remark 2.3).

Now we state a lemma whose proof can be found in [DRV].

LEMMA 3.6. – *There is a decreasing sequence* $\mu_n^* > 0$ *converging to* 0 *such that if* $\theta : \mu \to \theta(\mu)$*, then* $\theta_n = \theta/[\mu_n^*, \mu_{n-1}^*] : [\mu_n^*, \mu_{n-1}^*] \to [0, 2\pi]$ *satisfies*

1. θ_n *is an onto diffeomorphism* $\forall n \in \mathbb{N}$*;*
2. *given* $\epsilon > 0$ $\exists n_0 \in \mathbb{N}$ *such that* $\forall n > n_0$ *and every Borel set* $A \subset [0, 2\pi]$ *we have*

$$(1 - \epsilon)m(A) \leq \frac{m(\theta_n(A))}{m([\mu_n^*, \mu_{n-1}^*])} \leq (1 + \epsilon)m(A).$$

The proof of Theorem 3.3 (b)-(c) follows from Claim 3.5 (to get sequence δ_n) and Lemma 3.6 because now we choose $\hat{F}_\mu = \hat{T}_\mu \circ \hat{H}_\mu$ (recall Remark 3.1.1), $\mathcal{L}_\mu^{ss} = \mathcal{L}_{\mu,+}^{ss}$, $\hat{l}_\mu = \Phi_\mu^+(l_\mu)$ and μ_n as the inverse of θ_n $\forall n \in \mathbb{N}$.

This finishes Step 3 and completes the proof of Theorem 3.3.

4. Proof of theorems A and B

Proof of Theorem (A). – First we must construct the neighborhood U required in that theorem. For this we use Remark 3.1 (4) in order to reduce the dynamics to a tubular neighborhood $\{(x, y) : |y| \leq \epsilon^*\}$. Now observing that Δ_μ^n converges to 0 in the C^2-topology, we have that our tubular neighborhood generates an open U satisfying $X_\mu[t, U] \subset U$ $\forall t > 0$ and $\mu > 0$ small. Indeed, U is obtained by saturating the neighborhood by the flow of X_0. Clearly, by Theorem 3.3, U gives rise an annular cross section A_μ plus the return map F_μ on it. The conjugacy at Theorem 3.3 (b) reduces the dynamics of the return map to the one-dimensional map f_μ. By Theorem 3.3 (c), it turns out that f_μ is δ_n-close to a suitable rotation of f_Γ with δ_n converging to 0^+ as n goes to infinity. Now we apply the results of section 2 (in particular Proposition 2.7) taking into account that $\cap_{t>0} X_\mu[t, U]$ is a non-Lorenz-like if and only if f_μ is transitive in S^1. We come back to this point later in the appendix. Under such conditions, (a) follows by Proposition 2.7 (1), Theorem 2.1 and Theorem 3.3. To prove (d) use Proposition 2.7 (1), Proposition 2.5 and Theorem 3.3. To prove (b) and (c) use Proposition 2.7 (2)-(3) and Theorem 3.3 (b)-(c).

Proof of Theorem (B). – Although Theorem 3.3 does not work here, we can apply the renormalization scheme in [DRV] to get a somewhat convergence to a circle map in this case. Indeed, by Theorem 2.6, there exists a $\theta_0 \in [0, 2\pi]$ such that $R_{\theta_0} \circ f_\Gamma$ exhibits a homoclinic tangency associated to a hyperbolic periodic point. Now to obtain Henón-like attractors we must make renormalizations nearby the tangency. This can be made since the preimage of the critical point involved in this tangency does not intersect a small compact set around the discontinuity point. The renormalization converges to the quadratic family for some subset J of values σ in $[0, 2\pi]$. Thus we conclude (see [MV]) that the renormalization has Henon-like strange attractors for a set J of values of $\sigma \in [0, 2\pi]$ with positive Lebesgue measure. Moreover, since the limit family itself undergoes quadratic tangencies, we can apply the renormalization scheme in [MV] uniformly to the renormalization family, to conclude that $m(J_k)$ is uniformly bounded away from zero. The proof of Theorem (B) is completed because of a distorsion property similar to the one in Theorem 3.3 (see [DRV]).

5. Conclusions and final remarks

The results in this paper show new complicated dynamics in presence of a saddle-node singular cycle. In particular, Theorem (A-d) shows that Morse-Smale systems and systems with persistent singular strange attractors have commom boundary in some regions in the space of all vector fields in dimension $n \geq 3$. A similar result, but now involving Axiom A solenoid-type attractors instead of singular ones, was proved in dimension $n \geq 4$ (see [ST]). This suggest several interesting questions as, for instance, if contructions in [ST] can be done in 3-manifolds. Our results can be considered as partial answers to this question.

We observe that in [GS] it was proved Theorem 2.1, but when J is a real interval, f is orientation preserving and the lateral limits of f at the discontinuity point are just the end points of J. We belive that Theorem 2.1 still valid in more general situations as, for instance, when J is a branched 1-manifold (see [W]).

Finally, we want to point out that the bifurcations showed here, althought nonhyperbolic, are far from homoclinic tangencies. Instead, saddle-node singular cycles and the strange attractors arising from its generic unfoldings are accumulated by *singular cycles* (see [BLMP]). This suggests that singular cycles should be considered in the search of *good dense set* in the farmework of vector fields. Therefore, the well known Palis's conjeture about density of diffeomorphisms with either hyperbolic-like dynamic or homoclinic tangencies in surfaces could be formulated, for vector fields in 3-manifolds, as: *systems with either hyperbolic dynamic, homoclinic tangencies or singular cycles are dense in the space of all vector fields in closed 3-manifolds*.

Appendix

Here we prove Proposition 3.1. Also we exhibit examples of saddle-node singular cycles whose unfoldings genrerate different types of dynamics. We finish explaining why some of attractors obtained in Theorem (A) are non-Lorenz-like.

Proof of Proposition 3.1. – We require first some notation and facts. Recall that a *structure* is (Σ, W^{cu}, a, b) where Σ, W^{cu}, a and b are as in facts (1), (2) and (3) in section 3. In that section we associate the maps φ_a, φ_b and h as well as the projection π^{ss} along the strong stable manifold in Σ. The corresponding maps, for some other structure $(\hat{\Sigma}, \hat{W}^{cu}, \hat{a}, \hat{b})$, will be denoted by $\hat{\varphi}_{\hat{a}}$, $\hat{\varphi}_{\hat{b}}$ and $\hat{h}$. In this case the projection will be denoted by $\hat{\pi}^{ss}$. It was also associated a tangent vector field Z to W^{cu}. The corresponding vector field for $\hat{W}^{cu}$ will be denoted by $\hat{Z}$.

Now, if (Σ, W^{cu}, a, b) and $(\hat{\Sigma}, \hat{W}^{cu}, \hat{a}, \hat{b})$ are structures, then the following relations take place. Recall that R_θ denotes the rotation in S^1 with angle $\theta \in [0, 2\pi)$.

R1. Suppose $(\Sigma, W^{cu}) = (\hat{\Sigma}, \hat{W}^{cu})$. If $(a', b') \in W^{cu,+} \times W^{cu,-}$, $\exists(\theta, \theta') \in [0, 2\pi)^2$ such that $\varphi_a = R_\theta \circ \varphi_{a'}$ and $\varphi_b = R_{\theta'} \circ \varphi_{b'}$.

R2. Suppose $\Sigma = \hat{\Sigma}$ (thus $\pi^{ss} = \hat{\pi}^{ss}$). Denote by π^{cu} and $\hat{\pi}^{cu}$ the projection on W^{cu} and $\hat{W}^{cu}$ induced by π^{ss} respectively. Define $\pi^- : W^{cu,-} \to \hat{W}^{cu,-}$ by projecting along π^{ss}. Then,

a. $Z[t, x] = b$ implies $\hat{Z}[t, \pi^-(x)] = \pi^-(b)$, $\forall t \in I\!R$;

b. $\varphi_b(x) = \hat{\varphi}_{\pi^-(b)}(\pi^-(x))$, $\forall x \in W^{cu,-}$;

c. $\pi^- \circ \pi^{cu} = \hat{\pi}^{cu}$.

R3. There exists a return X-induced map $L : \Sigma \to \hat{\Sigma}$. Suppose that $\hat{W}^{cu} = L(W^{cu})$, $\hat{a} = L(a)$ and $\hat{b} = L(b)$. Then,

d. $\hat{\pi} = L \circ \pi \circ L^{-1}$;

e. $\hat{\varphi}_{\hat{a}}(L(x)) = \varphi_a(x)$;

f. $\hat{\varphi}_{\hat{b}}(L(x)) = \varphi_b(x)$.

Now Proposition 3.1 is consequence of the following lemmas.

LEMMA A. – *Suppose that (Σ, W^{cu}, a, b) is a structure and $(a', b') \in W^{cu,+} \times W^{cu,-}$. Then $f(\Sigma, W^{cu}, a, b)$ and $f(\Sigma, W^{cu}, a', b')$ differ by rotation in S^1.*

Proof. – Apply (R1).

LEMMA B. – *Suppose that (Σ, W^{cu}) and $(\hat{\Sigma}, \hat{W}^{cu})$ satisfy (1) in section 3 with $\Sigma = \hat{\Sigma}$. Then $\forall(a, b) \in W^{cu,+} \times W^{cu,-}$ $\exists(\hat{a}, \hat{b}) \in \hat{W}^{cu,+} \times \hat{W}^{cu,-}$ such that $f(\Sigma, W^{cu}, a, b)$ and $f(\hat{\Sigma}, \hat{W}^{cu}, \hat{a}, \hat{b})$ differ by rotation in S^1.*

Proof. – Notice that in this case $\pi^{ss} = \hat{\pi}^{ss}$. Choose $\hat{a} = a$, $\hat{b} = \pi^-(b)$ and apply (R2).

LEMMA C. – *Let (Σ, W^{cu}, a, b) be a structure and consider a cross section $\hat{\Sigma}$ satisfying (1) in section 3. Then there exists $(\hat{W}^{cu}, \hat{a}, \hat{b})$ such that $(\hat{\Sigma}, \hat{W}^{cu}, \hat{a}, \hat{b})$ is a structure such that $f(\Sigma, W^{cu}, a, b)$ and $f(\hat{\Sigma}, \hat{W}^{cu}, \hat{a}, \hat{b})$ differ by rotation in S^1.*

Proof. – Here we use the return map L in (R3). Define $\hat{W}^{cu} = L(W^{cu})$, $\hat{a} = L(a)$ and $\hat{b} = L(b)$. Now apply (R3).

This finishes the proof of Proposition 3.1.

Now we present some examples of cycles in which several types of dynamics arise after their unfoldings. For this purpose we use Proposition 3.2. In Theorem (C) we show open examples of critical saddle-node singular cycles generating hyperbolic dynamic, Lorenz and Henón like attractors after unfold. In Theorem (D) we exhibit open examples of transversal cycles generating *contracting Lorenz-like attractors* after unfold. We use the following notation. If X_μ is a one-parameter family of vector fields and Γ is a saddle-node singular cycle of X_0 then,

– $L(\epsilon) = \{\mu \in [0, \epsilon]; X_\mu$ exhibits a Lorenz-like attractor close to $\Gamma\}$;

– $He(\epsilon) = \{\mu \in [0, \epsilon]; X_\mu$ exhibits a Henon-like attractor close to $\Gamma\}$;

– $Hy(\epsilon) = \{\mu \in [0, \epsilon]; X_\mu$ is hyperbolic close to $\Gamma\}$;

– $Ro(\epsilon) = \{\mu \in [0, \epsilon]; X_\mu$ exhibits a contracting Lorenz attractor close to $\Gamma\}$.

THEOREM C. – *There are open sets of one-parameter families of vector fields X_μ unfolding Morse-Smale systems through a critical saddle-node singular cycle such that*

$$\liminf_{\epsilon\to 0^+} \frac{m(\mathcal{H}(\epsilon))}{\epsilon} > 0,$$

where $\mathcal{H}(\epsilon)$ is $L(\epsilon), He(\epsilon)$ or $Hy(\epsilon)$.

THEOREM D. – *There are open sets of one-parameter families of vector fields X_μ unfolding Morse-Smale systems through a transversal saddle-node singular cycle such that*

$$\liminf_{\epsilon\to 0^+} \frac{m(\mathcal{H}(\epsilon))}{\epsilon} > 0,$$

where $\mathcal{H}(\epsilon)$ is $Hy(\epsilon)$ or $Ro(\epsilon)$.

Proof of Theorem C. – We use Proposition 3.2 to construct a saddle-node singular cycle Γ from a suitable one-dimensional discontinuous circle map f. We obtain f by extending the classical Lorenz expansive map in a compact interval $I \subset S^1$ to S^1. The extension is done in a way that it also presents an attracting fixed point $s \in S^1 \setminus I$ and it has critical continuous extension in S^1 according Definition 2.2 (see also Remark 2.2 (1)). Clearly we have $f_\Gamma = f$ for the cycle Γ introduced above. Now it follows that rotations of f produce geometric Lorenz attractors and, by Theorem (B), Henón-like ones. The hyperbolicity is obtained by seeing that for suitable $\theta \in (0, 2\pi)$, $R_\theta \circ f$ will have an interval I_μ where the function looks like the one-dimensional map g of an expansive singular cycle (see [BLMP]). In this reference it was proved that the limit capacity of the set of parameters where the image of the singularity remains into the domain of g goes to zero when parameters approach 0^+. The same argument can be applied in this situation as well and so we get that the image of the singularity under f_θ / I_μ and the image of the other critical point go to a hyperbolic sink most of the time. Im particular, hyperbolicity holds for such cases. This completes the proof.

Proof of Theorem D. – We first construct a vector field X_0 with a trasversal cycle Γ satisfying $\lambda_3 > \lambda_1$ and $\lambda_2 > \lambda_3 + 3\lambda$ (see [R]). We do this using Proposition 3.2 to construct a transversal saddle-node singular cycle from a suitable circle map h. We choose h extending the interval map studied in [R] to the whole circle S^1 in a way that the images of the discontinuity point of h, denoted by r_+ and r_-, go to two repelling periodic orbits ξ_1 and ξ_2 of h respectively. Let define $h_\theta = R_\theta \circ h$ and $\xi_i(\theta)$ as the analytical continuation of ξ_i for h_θ $i = 1, 2$ Moreover, consider the analytical continuations $r_+(\theta)$ and $r_-(\theta)$ of r_+ and r_- respectively for the map h_θ. It follows that these elements are well defined, at least for θ in an interval of the form $(-\epsilon, \epsilon)$ with $\epsilon > 0$ small, and further the derivatives $\frac{d}{d\theta}(r_+ - \xi_1)$ and $\frac{d}{d\theta}(r_- - \xi_2)$ at $\theta = 0$ are not 0. We can construct strong stable C^3 foliations associated to the correspoinding return map as we did in Theorem 3.3 for transversal cycles (see [R]). It turns out that the one-dimensional map $\hat{h}_\mu$ associated to such an invariant foliation is in much related to h_μ. Indeed, it follows that for those maps $\hat{h}_\mu$ ($\mu > 0$) which are close to h_θ with $\theta \in (-\epsilon, \epsilon)$, there is a $\hat{h}_\mu$-invariant nontrivial interval (containing the discontinuity) where $\hat{h}_\mu$ is transitive with positive Lyapunov exponent at the discontinuity. Then proof follows using arguments in [DRV] as was done before.

Singular attractors in annular cross sections

In this part we explain why some of the attractors obtained in Theorems (A) are non-Lorenz-like. We start our exposition with the geometric model $\mathcal{M}$ depicted in Figure 3. Let us describe the main elements involved in $\mathcal{M}$.

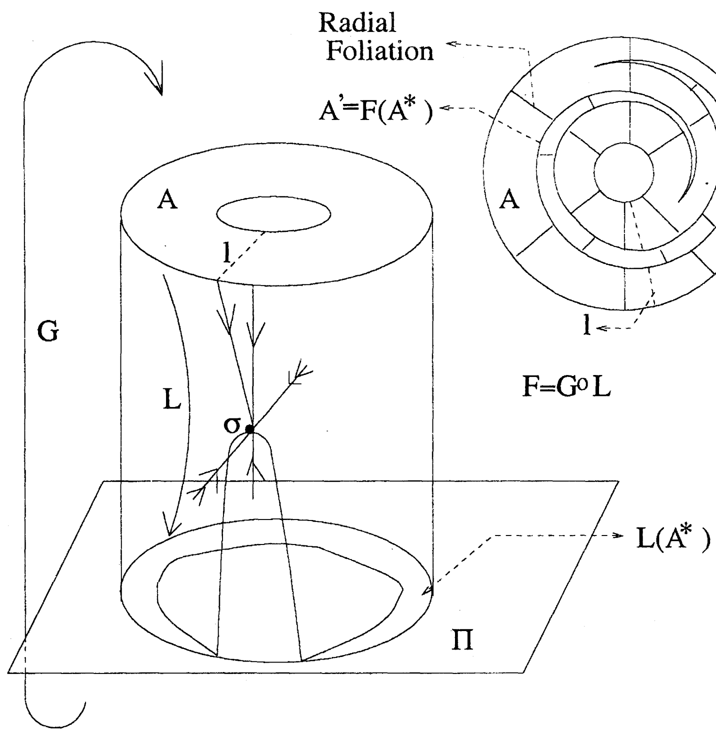

Fig. 3. – Singular attractor in annular cross section.

M1. $A \subset M$ is an annulus and Π is a plane which is parallel to A. The curve l belongs to $A \cap W^s(\sigma)$. It it just l_μ in Theorem 3.3.

M2. σ is a hyperbolic singularity. It is the one associated to Γ according Definition 1.1.

M3. $L : A^* \to \Pi$ and $G : \Pi \to A$ are return maps and $F = G \circ L$. A' denotes the image $A' = F(A^*)$ where $A^* = A \setminus l$.

M4. The right hand figure indicates the relative position of A' in A. The radial foliation in A, which is marked in this picture, is preserved by F. It represents $\mathcal{L}^{ss}_\mu$ in Theorem 3.3. The one-dimensional map induced by F in the radial foliation is denoted by $f : S^1 \to S^1$. It represents f_μ in Theorem 3.3.

Now, the geometric Lorenz attractor is contructed by means of a top square S with a return map $T : S^* \to S$ satisfying certain *quasihyperbolic* properties (see [GW] for details). A simple, but significative fact here is that the image $T(S^*) \subset S$ is *disconnected* in S. Here S^* is just S minus the midle line. The fashion of the geometric model in Figure 3 resembles the Lorenz construction *except* that, in this case, A' is *connected* in A. Now, it follows that any Lorenz-like attractor displays all the properties of the Lorenz construction mentioned before. Indeed, we can carry all those contructions using the homeomorphisms h which gives the equivalence (see section 1).

In particular, Lorenz-like attractors should exhibit global *disconnected* cross section. Recall that by global cross section it is meat a cross section $\mathcal{G}$ of the flow such that all positive orbit in the attractor meet $\mathcal{G}$. It follows that if f in (M4) is transitive, $\mathcal{M}$ can not displays disconnected global cross sections. Thus $\mathcal{M}$ can not be equivalent to the geometric Lorenz attractor in [GW] when f is transitive. Such cases are precisely the ones in which the singular strange attractors obtained in Theorems (A) are non-Lorenz-like.

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