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# EXISTENCE AND UNIQUENESS OF DIFFUSIONS ON FINITELY RAMIFIED SELF-SIMILAR FRACTALS

BY C. SABOT

ABSTRACT. – We give a criterion for the existence and uniqueness or the non-existence of the diffusions on a finitely ramified self-similar fractal. In classical examples this criterion is easy to apply and in particular, it gives the uniqueness of the diffusion on nested fractals (Lindstrøm proved the existence in [19] but the problem of uniqueness remained unsolved) and completly solves the problem of existence and uniqueness in the case of the Sierpinski gasket with inhomogeneous weights.

This problem also gives a solution to a non trivial problem of fixed point for a non-linear, non-expansive map of a cone with the Hilbert's projective metric (*cf.* [23]).

RÉSUMÉ. – Nous donnons un critère de non-existence ou d'existence et d'unicité des diffusions sur un ensemble auto-similaire finiment ramifié. Dans les exemples classiques ce critère s'applique facilement et donne en particulier l'unicité dans le cas des "nested fractals" (Lindstrøm avait prouvé l'existence mais le problème de l'unicité restait ouvert) et donne une solution complète dans le cas du "Sierpinski gasket" avec des poids inhomogènes.

Ce papier donne aussi un exemple de solution à un problème non-trivial de point fixe pour une transformation non-linéaire et contractante d'un cône muni de la métrique de Hilbert (*cf.* [23]).

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## Introduction

The first construction of a diffusion on a fractal was given on the Sierpinski gasket by Goldstein [9] and Kusuoka [15] and a deep study of this diffusion was later done by Barlow-Perkins [3]. At the same time Barlow and Bass [2] constructed a diffusion on the Sierpinski carpet. The main difference between the Sierpinski gasket and the Sierpinski carpet is that the first one is finitely ramified and the second is not: roughly speaking, finitely ramified means that the fractal can be disconnected by removing a finite set of points. Kigami introduced a class of abstract fractals (*i.e.* not imbedded in $\mathbb{R}^n$), called post criticaly finite (p.c.f.) self-similar sets which well describes the notion of finite ramification.

The method to construct a diffusion on a finitely ramified fractal is now well-known (*cf.* [19], [16], [12]): one constructs a Dirichlet form on the fractal $X$ as the limit of Dirichlet forms defined on an approximating sequence of finite sets $F^{(n)}$. This construction relies on the existence of a non-degenerated eigenvector for a renormalization operator.

The problem of existence was solved by Lindstrøm for a particular class of fractals, called nested fractals, which are highly symmetric. One of the main question raised by Lindstrøm was whether this diffusion is unique or not. This amounts naturally to the uniqueness of eigenvectors of the renormalization operator. This problem was only solved by Barlow [1] in some particular cases: he proved the uniqueness for the Viscek set and the snowflake exploiting the notion of electrical networks and the symmetries of these fractals. Besides, Hattori-Hattori-Watanabe [10] gave an example of non-existence and Metz [20] one of non-uniqueness. In this paper we give (Theorem 5.1) a criterion for the non-existence or the existence and uniqueness of diffusions in the general setting of finitely ramified self-similar fractals with a symmetry group $G$. This criterion is easy to apply and in particular it gives the uniqueness in the case of nested fractals (and gives the existence in classical cases, but this was proved in general by Lindstrøm [19]). It also completly solves the problem for the Sierpinski gasket with inhomogeneous weights. It is interesting to note that a kind of critical value on the weights appears in this last example. The statement of this result was published in [26] (a proof was given in [25], unpublished).

Let $F$ denote the basic cell associated with the fractal (*cf.* Section 2). The renormalization operator, denoted by $T$, maps the cone $M$ of irreducible Dirichlet forms on the finite set $F$ to itself. The map $T$ is 1-homogeneous, non-expansive for the Hibert's metric on cones (*cf.* [21] ) and in general non-linear.

Since $M$ is the set of irreducible Dirichlet forms, it is not closed. The existence of an eigenvector of $T$ depends on the behaviour of $T$ near the boundary of $M$. To understand this behaviour we construct a compactification of the set $M$, richer than the usual one (*i.e.* the one of eventually reducible Dirichlet forms), to which the map $T$ has a continuous extension (actually, we do not construct explicitely this compactification). The Theorem 5.1 can be roughly summed-up as follows: depending on the value of the map $T$ on the boundary of $M$, two situations can occur:

- the boundary of $M$ is repulsive for $T$ (*i.e.* iterating $T$ one goes out of any small neigbourhood of the boundary of $M$). In this case we prove that $T$ has a unique eigenvector.
- The boundary is not repulsive and $T$ has no eigenvector.

(N.B.: in the sequel we say fixed point for eigenvector because all eigenvectors have the same eigenvalue (*cf.* Corollary 3.5) and because we are only interested in the invariant lines for $T$).

Actually, Theorem 5.1 does not give such a nice dichotomy and in critical cases it fails to give an answer. In [20] , Metz gave an example of non-uniqueness, in section 7.2 we shall see that this example lies in the critical case.

We apply this theorem to the Sierpinski gasket with inhomogeneous weights. In this case we find a criterion on the weights and only two situations can occur: either $T$ has a unique eigenvector, or it has no eigenvector.

In the case of nested fractals we prove that we always get the first case and thus, that $T$ has a unique eigenvector. (This answers one of the main questions raised by Linsdtrøm, whether the diffusion he constructs is unique or not). Besides, the renormalization map $T$ gives a non-trivial example of a type of maps which have been intensively studied in non-linear analysis (*cf.* [23]), namely, non-linear, non-expansive maps of a cone. It is possible that the techniques developed here could be applied in other contexts. Actually, Theorem 5.1 can be viewed as a non-linear generalization of the Perron-Frobenious theorem.

Let us now describe the organization of the paper:

In Section 1 we give a quick description of Dirichlet forms on finite sets and electrical networks.

In Section 2 we present the general framework of the paper: we give a definition of finitely ramified self-similar sets (we adopt a definition simpler than Kigami's one [12], althought all our results could be derived in his context) and of self-similar Dirichlet forms. We briefly recall how self-similar Dirichlet forms are linked with Dirichlet forms invariant by renormalization (which correspond to eigenvectors of $T$). We follow the construction of Kusuoka [16].

In Section 3 we introduce the map $T$ and give its first properties.

Section 4 contains some preliminaries to the proof of the main result: we describe the boundary of $M$ (*i.e.* the way a Dirichlet form can degenerate) and the behaviour of $T$ near this boundary. This section contains a specific introduction where we explain the underlying ideas of our procedure. We illustrate Section 4 and Section 5 by two examples, namely the Sierpinski gasket without symmetry (and inhomogeneous weights) and the snowflake.

In Section 5 we state and prove the main theorem and we apply it to the two examples. We stress the fact that the main part of the proof of uniqueness is independent of the proof of existence and of the preliminaries (it only relies on Lemma 5.7, which roughly says that the boundary of $M$ is repulsive for $T$).

In Section 6 we apply Theorem 5.1 to nested fractals and prove the uniqueness of the diffusion in this case. We also give an example where Theorem 5.1 cannot be applied.

In [27] we explicitely computed the transformation $T$ in some examples: in particular we show that $T$ can be expressed as a non-negative matrix when the fractal is a tree, *i.e.* has no loop. We also use the representations of the symmetry group $G$ to compute $T$: in that way we give an expression of $T$ in the case of the sowflake.

## 1. Dirichlet forms on finite sets and electrical networks

In this section we recall some classical facts about Dirichlet forms on finite sets and electrical networks. In [1] , Barlow first applied these technics to diffusions on fractals. A full account on the subject can be found in [5]. Because most of these results will be used in the sequel, we keep a well-ordered structure to present them and sometimes give short proofs, even if they can be found somewhere else. The only original, but not very deep, result of this section is Lemma 1.19.

### 1.1. Dirichlet forms on finite sets

Let $F$ be a finite set. Denote by $E$ the space of real functions on $F$ and by $V$ the subspace of zero mean functions.

DEFINITION 1.1. – *A positive quadratic form $A$ on $E$ is said to be a Dirichlet form on $F$ if $A$ is null on the set of constant functions and if for all $f \in E$:*

$$A(f \wedge 1, f \wedge 1) \leq A(f, f).$$

N.B.: $f \wedge 1(x)$ denotes the minimum value of $f(x)$ and 1.

REMARK 1.2. – The last assumption is called the Markovian property of Dirichlet forms.

REMARK 1.3. – Usually, the first condition is not assumed. Actually, with this definition, Dirichlet forms are supposed to be conservative.

REMARK 1.4. – Most of the time, for a quadratic form $A$, we write $A(f)$ instead of $A(f, f)$.

DEFINITION 1.5. – *A Dirichlet form $A$ is said to be irreducible if $A(f) = 0$ implies that $f$ is a constant function.*

Let $\mu$ be a strictly positive measure on $F$. For all Dirichlet form $A$ there exists a unique symmetric operator $\mathcal{A}$ such that:

$$A(f,g) = \int f \mathcal{A} g d\mu, \quad \forall f, g \in E.$$

The operator $\mathcal{A}$ is the infinitesimal generator of a unique Markov process $X_t$ with state space $F$. Since $A$ is null on constant functions, this Markov process is conservative (*i.e.* there is no killing). We say that $X_t$ is the Markov process associated with $(A, \mu)$. We describe $X_t$ in the next section.

### 1.2. Electrical networks

DEFINITION 1.6. – *A symmetric matrix $J = (j_{x,y})_{x,y \in F}$ is said to be a bond conductivity matrix if its elements are non-negative and null on the diagonal. The set $\{\{x,y\}, j_{x,y} > 0\}$ defines a graph on $F$. We say that $J$ is irreducible if this graph is connected.*

With each bond conductivity matrix $J$, we associate a positive bilinear form $A$ on $E \times E$ by:

$$A(f,g) = \frac{1}{2} \sum_{x,y \in F} j_{x,y}(f(x) - f(y))(g(x) - g(y)), \quad \forall f, g \in E.$$

The following result is well-known:

PROPOSITION 1.7. – *The quadratic form $A$ is a Dirichlet form on $F$ and is irreducible if and only if the bond conductivity matrix $J$ is irreducible.*

*The map $J \mapsto A$ is bijective from the set of bond conductivity matrices to the set of Dirichlet forms.*

Let $\mu$ be any strictly positive measure on $F$. The markov process $X_t$ associated with $(A, \mu)$ is a jump process with semi-group $\exp(-t\mathcal{A})$. The associated Markov chain has transition probabilities:

$$p_{x,y} = \frac{j_{x,y}}{\sum_{z \neq x} j_{x,z}}$$

and at a point $x$ the rate of the exponential jump law is given by: $\mu(x) \sum_{z \neq x} j_{x,z}$.

### 1.3. Restriction of a Dirichlet form

Let $F'$ be a subset of $F$ and $E'$ the space of real functions on $F'$.

DEFINITION 1.8. – *Let $A$ be a Dirichlet form on $F$, we define the restriction of $A$ to $F'$, denoted by $A_{F'}$, by the following formula:*

$$A_{F'}(f,f) = \inf\{A(g,g),\ g \in E,\ g_{|F'} = f\}, \quad \forall f \in E'.$$

PROPOSITION 1.9. – *$A_{F'}$ is a Dirichlet form.*

*If $A$ is irreducible, then $A_{F'}$ is irreducible and there exists a linear map $H_A : E' \to E$ such that $H_A(f)$ is the unique element of $E$ which realizes the minimum in Definition 1.8. The function $H_A(f)$ is called the harmonic continuation of $f$ with respect to $A$.*

REMARK 1.10. – This restriction is called the trace of a Dirichlet form on a subspace in [7], associated with the time changed process, *cf.* th 6.2.1 (*cf.* also [18] where it was called the "forme de Dirichlet balayée").

Obviously, the following decomposition holds: $A(f) = A_{F'}(f_{|F'}) + A(f - H_A(f_{|F'}))$ for all function $f$ on $F$.

REMARK 1.11. – If $A$ is an irreducible Dirichlet form then $H_A$ satisfies the maximun principle *i.e.* for all $f \in E'$:

$$\min f \leq H_A(f) \leq \max f.$$

REMARK 1.12. – If $A$ is not irreducible then the harmonic continuation is in general not unique. Precisely, the set of functions which realize the minimum is an affine space with direction: $\{g \in ker(A),\ g_{|F'} = 0\}$.

REMARK 1.13. – If $A$ is any positive quadratic form on $E$, one can define the restriction of $A$ to $F'$ in the same way. If $A$ is positive definite on the subspace $V$ (of zero mean functions), then the minimum is reached on a unique point and $H_A$ can be uniquely defined. The maximum principle is in general not satisfied.

One can give an expression of the harmonic continuation of a function $f$ on $F'$ in terms of the Markov chain associated with $A$:

PROPOSITION 1.14. – *Let $A$ be an irreducible Dirichlet form and $X_n$ be the Markov chain associated with $A$ (*cf.* Section 1.2). Let $\tau$ be the first time when $X_n$ meets the set $F'$ then for all function $f$ on $F'$:*

$$H_A(f) = E_x[f(X_\tau)].$$

N.B.: $E_x$ denotes the expectation with respect to the law of the process starting from $x$.

This notion is also connected to electrical networks.

Let $X_n$ be the Markov chain associated with the irreducible Dirichlet form $A$ and $X_n^{F'}$ be the one associated with its restriction $A_{F'}$ (defined on the state space $F'$). In [1] it is proved that $X_n^{F'}$ is the Markov chain which describes the successive visits of $F'$ by $X_n$. It easily follows that:

PROPOSITION 1.15. – *Let $A$ be an irreducible Dirichlet on $F$ associated with a bond conductivity matrix $(j_{x,y})$. Let $(\tilde{j}_{x,y})$ be the bond conductivity matrix associated with the Dirichlet form $A_{F'}$, then for $x$ and $y$ in $F'$, $\tilde{j}_{x,y} > 0$ if and only if there exists a path $z_0 = x, \ldots, z_k = y$ such that $z_i \notin F'$ for $i \in \{1 \ldots, k-1\}$ and $j_{z_i, z_{i+1}} > 0$ for $i \in \{0, \ldots, k-1\}$.*

### 1.4. Effective conductances. Distance associated with a Dirichlet form

In [13], Kigami introduced the distance associated with a Dirichlet form on $F$. We now describe this notion, useful in the sequel.

Let $A$ be an irreducible Dirichlet form on $F$.

Let $x$ and $y$ be two distinct points in $F$.

The Dirichlet form $A_{\{x,y\}}$ is associated with a unique bond conductivity that we denote by $c_A(x,y)$, *i.e.*:

$$A_{\{x,y\}}(g) = c_A(x,y)(g(x) - g(y))^2,$$

for any function $g$ on $\{x,y\}$.

The real $c_A(x,y)$ is called the effective conductance between $x$ and $y$.

PROPOSITION 1.16. – *The following equality holds:*

$$c_A(x,y) = \inf\{A(f,f),\ f \in E,\ f(x) = 0,\ f(y) = 1\}.$$

*The function* $r_A\ :\ F \times F \to \mathbb{R}_+$ *defined by:*

$$r_A(x,x) = 0,\ x \in F,$$

$$r_A(x,y) = \frac{1}{c_A(x,y)},\ x \neq y,$$

*is a distance on* $F$.

REMARK 1.17. – The real $r_A(x,y)$ will be called the effective resistance between $x$ and $y$.

*Proof.* – The first formula comes directely from Definition 1.8.

The second assertion is proved in [13], Theorem 1.6. □

We now want to estimate a Dirichlet form thanks to its effective conductances.

It leads us to introduce a kind of reference Dirichlet form: let $A^0$ be the Dirichlet form associated with conductivities equal to 1 on each bond, *i.e.*

$$A^0(f) = \frac{1}{2}\sum_{x,y}(f(x) - f(y))^2,\ \ \forall f \in E.$$

REMARK 1.18. – If $f$ has zero mean on $F$ then obviously $A^0(f) = \sum f(x)^2$, so that $A^0$ and the usual $L^2$ norm are equal on $V$.

LEMMA 1.19. – *There exists a constant* $K > 0$ *such that for all irreducible Dirichlet form* $A$ *on* $F$ *and all function* $f \in E$:

$$\frac{1}{K}\left(\min_{x\neq y} c_A(x,y)\right)A^0(f) \le A(f) \le \left(\max_{x\neq y} c_A(x,y)\right)A^0(f).$$

*Proof.* – Let $A$ be an irreducible Dirichlet form and $(j_{x,y})$ its bond conductivity matrix. If $f \in E$ is such that $f(x) = 0$ and $f(y) = 1$ then $A(f) \geq j_{x,y}$. It follows that:

$$j_{x,y} \leq c_A(x,y), \quad \forall x \neq y.$$

The upper bound thus follows from the definition of $A^0$.

Let us suppose that the left inequality is not satisfied. It means that one can find a sequence $(A_n)$ of irreducible Dirichlet forms and a sequence $(f_n)$ in $E$ such that:

$$c_{A_n}(x,y) \geq 1, \quad \forall x \neq y, \tag{1.1}$$

$$A^0(f_n) = 1, \tag{1.2}$$

$$\lim_{n \to \infty} A(f_n) = 0. \tag{1.3}$$

Let $(j^n_{x,y})$ be the bond conductivity matrix associated with $A_n$ and set $\tilde{j}^n_{x,y} = j^n_{x,y} \wedge 1$.

Since $\tilde{j}^n_{x,y}$ is bounded there exists a subsequence $n_k$ such that $(\tilde{j}^{n_k}_{x,y})$ converges to a bond conductivity matrix denoted by $J = (j_{x,y})$.

Obviously, if $J$ is irreducible then one can find a constant $C > 0$ such that $A_{n_k} \geq CA^0$, but this is incompatible with (1.2) and (1.3).

If $J$ is reducible then, by permutations of rows and columns, one can write it as a matrix with irreducible diagonal square blocks associated with a partition $\{I_1, \ldots, I_k\}$ of $F$. Take now $x$ in $I_1$ and $y$ in $I_2$. We define $f \in E$ by $f_{I_1} = 0$ and $f_{F \setminus I_1} = 1$. Obviously,

$$c_{A_{n_k}}(x,y) \leq A_{n_k}(f,f).$$

But $A_{n_k}(f)$ converges to 0, as the conductivity between any point in $I_1$ and any point out of $I_1$ converges to 0. This is incompatible with (1.1).

As a consequence, there exists a constant $K > 0$ for which the left inequality is satisfied. □

### 1.5. Calculation of some effective conductances

Our goal in this section is to give simple methods to compute some effective conductances. In fact, we give nothing more than the formula for the effective conductance of two conductors in series or in parallel (as in [5] ).

Suppose that $F$ is the union of two non-disjoint subsets, say $F_1$ and $F_2$. Let $A_1$ and $A_2$ be two irreducible Dirichlet forms on $F_1$ and $F_2$ respectively. Suppose that $A$ is given by the following formula:

$$A(f) = A_1(f_{|F_1}) + A_2(f_{|F_2}).$$

The quadratic form $A$ is clearly an irreducible Dirichlet form on $F$.

Let us denote, as in Section 1.4, by $c_{A_1}$, $c_{A_2}$ and $c_A$ the effective conductances associated with $A_1$, $A_2$ and $A$.

LEMMA 1.20. – *Let $x$ and $y$ be two distinct point in $F$.*

(i) *If $\{x,y\} \subset F_1$ and if $F_1 \cap F_2$ has a unique element then*

$$c_A(x,y) = c_{A_1}(x,y).$$

(ii) *If $F_1 \cap F_2 = \{x,y\}$ then:*

$$c_A(x,y) = c_{A_1}(x,y) + c_{A_2}(x,y).$$

(iii) *If $x \in F_1$, $y \in F_2$ and $F_1 \cap F_2 = \{z\}$ with $z \notin \{x,y\}$ then:*

$$\frac{1}{c_A(x,y)} = \frac{1}{c_{A_1}(x,z)} + \frac{1}{c_{A_2}(z,y)}.$$

*Proof.* – (i)

Let $\{z\} = F_1 \cap F_2$. Let $f_1$ be the function on $F_1$ such that $f_1(x) = 0$, $f_1(y) = 1$ and $A_1(f_1) = c_{A_1}(x,y)$. Define $f$ on $F$ by:

$$\begin{cases} f_{|F_1} = f_1 \\ f_{|F_2} = f_1(z) \end{cases}$$

We clearly have $A(f) = A_1(f_1)$ and it easily follows that $c_A(x,y) = c_{A_1}(x,y)$.

(ii) Let $f_1$ and $f_2$ be the functions on $F_1$ and $F_2$ such that $f_1(x) = f_2(x) = 0$, $f_1(y) = f_2(y) = 1$ and $c_{A_1}(x,y) = A_1(f_1)$, $c_{A_2}(x,y) = A_2(f_2)$. Let $f$ be the function on $F$ defined by $f_{|F_1} = f_1$, $f_{|F_2} = f_2$. Thus, $A(f) = A_1(f_1) + A_2(f_2)$ and the result is easy to derive.

(iii) Let $f$ be the function on $F$ such that $f(x) = 0$, $f(y) = 1$ and $A(f) = c_A(x,y)$. Set $a = f(z)$, $f_1 = f_{|F_1}$ and $f_2 = f_{|F_2} - a$. We see that $f_1(x) = 0$, $f_1(z) = a$ and $f_2(z) = 0$, $f_2(y) = 1 - a$. Moreover $f_1$ (resp. $f_2$) is harmonic on $F_1 \setminus \{x,z\}$ (resp. $F_2 \setminus \{z,y\}$) with respect to $A_1$ (resp. $A_2$). Thus, $a^2 c_{A_1}(x,z) = A_1(f_1)$ and $(1-a)^2 c_{A_2}(z,y) = A_2(f_2)$. Thus, we have:

$$c_A(x,y) = a^2 c_{A_1}(x,z) + (1-a)^2 c_{A_2}(z,y).$$

Moreover, $a$ is the value for which this expression reaches its minimum. It implies that

$$c_A(x,y) = \frac{c_{A_1}(x,z) c_{A_2}(z,y)}{c_{A_1}(x,z) + c_{A_2}(z,y)}. \quad \square$$

## 2. Construction of self-similar Dirichlet spaces

### 2.1. Finitely ramified self-similar spaces

In order to describe finitely ramified fractals, Kigami introduced a class of fractals, called p.c.f. self-similar sets (*cf.* [12]). For a matter of simplicity, in the sequel we restrict ourselves to a subclass of the p.c.f. self-similar sets, in some sense to a "constructive" subclass, in which the examples are easy to describe. All our results could be derived in the general setting of p.c.f. self-similar sets (*cf.* [25]).

Let $D$, $N$ be two integers such that $2 \leq D \leq N$.

Let $G$ be a finite group operating on the set $\{1, \ldots, N\}$ (we denote this operation by $g.i$ for $g \in G$ and $i \in \{1, \ldots, N\}$). Assume that the subset $\{1, \ldots, D\}$ is globally invariant under the operation of $G$.

Let $\mathcal{R}$ be an equivalence relation on $\{1, \ldots, N\} \times \{1, \ldots, D\}$ such that:

- for all $i \in \{1, \ldots, D\}$, the equivalence class of $(i, i)$ with respect to $\mathcal{R}$ has a unique element,
- if $(k, i)\mathcal{R}(k, i')$ for $k \in \{1, \ldots, N\}$ and $i, i' \in \{1, \ldots, D\}$, then $i = i'$,
- for any $i, i' \in \{1, \ldots, N\}$, $i \neq i'$, there exists a sequence $i_1 = i, \ldots, i_p = i'$ of $\{1, \ldots, N\}$ such that for all $k \in \{1, \ldots, p-1\}$ there exist $j$ and $j'$ in $\{1, \ldots, D\}$ such that $(i_k, j)\mathcal{R}(i_{k+1}, j')$,
- the relation $\mathcal{R}$ is invariant under the operation of $G$ on the product set $\{1, \ldots, N\} \times \{1, \ldots, D\}$ (*i.e.* $(i, j)\mathcal{R}(i', j')$ implies $(g.i, g.j)\mathcal{R}(g.i', g.j')$ for all $g$ in $G$).

We set:

$$F = \{1, \ldots, D\},$$

$$F^{(1)} = (\{1, \ldots, N\} \times F)/\mathcal{R}.$$

Thus, $F$ can naturally be regarded as a subset of $F^{(1)}$, namely $\{(i, i),\ \ i \in F\}$ (indeed, by the first assumption on $\mathcal{R}$ the map from $F$ to $F^{(1)}$ given by $i \to (i, i)$ is one-to-one).

We now construct a self-similar set over this finite structure. Set $\Omega = \{1, \ldots, N\}^{\mathbb{N}^*}$ (*i.e.* the set of sequences $(\omega_i)_{i \geq 1}$ of $\{1, \ldots, N\}$). With the distance $l$ defined by:

$$l(\omega, \omega) = 0,\ \forall \omega \in \Omega,$$

$$l(\omega, \omega') = 2^{-\inf\{k,\ \omega_k \neq \omega'_k\}},\ \forall \omega, \omega' \in \Omega,\ \omega \neq \omega',$$

the set $\Omega$ is compact.

Let us set some notations. For any $s \in \{1, \ldots, N\}$ and $\omega \in \Omega$, we denote by $< s, \omega >$ the element of $\Omega$ given by:

$$< s, \omega >_1 = s,\ \ < s, \omega >_{p+1} = \omega_p,\ \forall p \geq 1.$$

Given $s \in \{1, \ldots, N\}$, $\dot{s} \in \Omega$ denotes the infinite sequence associated with $s$.

On $\Omega$ we define the equivalence relation $\sim$ by $\omega \sim \omega'$ if and only if there exists $n \in \mathbb{N}^*$ such that

$$\omega_1 = \omega'_1, \dots \omega_{n-1} = \omega'_{n-1},$$

$$\omega_{n+1}, \omega'_{n+1} \in \{1, \dots, D\} \text{ and } (\omega_n, \omega_{n+1})\mathcal{R}(\omega'_n, \omega'_{n+1}),$$

$$\omega_{n+p} = \omega_{n+1}, \;\; \omega'_{n+p} = \omega'_{n+1}, \;\; \forall p \geq 1.$$

We set $X = \Omega/\sim$. With the topology induced by $\Omega$, $X$ is a compact set. Moreover, thanks to the third assumption on $\mathcal{R}$, $X$ is connected.

Let $\Pi$ be the canonical surjective map from $\Omega$ to $X$.

For all $s \in \{1, \dots, N\}$ we define the map $\Psi_s : X \to X$ by:

$$\Psi_s(x) = \Pi(< s, \omega >) \text{ if } x = \Pi(\omega)$$

(we easily see that $\Psi_s$ is well defined, *i.e.* that $\Psi_s(x)$ does not depend on $\omega$ choosen in $\Omega$ such that $\Pi(\omega) = x$).

The function $\Psi_i$ is one-to-one for all $i$, thanks to the second assumption on $\mathcal{R}$.

Moreover, we remark that $X$ is self-similar for the list of maps $(\Psi_1, \dots, \Psi_N)$, *i.e.* that

$$X = \cup_{i=1}^N \Psi_i(X).$$

Henceforth, $F$ will be regarded as the following subset of $X$:

$$F = \{\Pi(\dot{k}), \;\; k \in \{1, \dots, D\}\}.$$

In the same way $F^{(1)}$ will be regarded as a subset of $X$, namely

$$F^{(1)} = \{\Pi(< k, \dot{l} >), \; (k, l) \in \{1, \dots, N\} \times \{1, \dots, D\}\} = \cup_{i=1}^N \Psi_i(F).$$

We adopt the following notations:

$$F^{(0)} = F,$$

$$F^{(n)} = \cup_{(i_1, \dots, i_n) \in \{1, \dots, N\}^n} \Psi_{i_1} \circ \dots \circ \Psi_{i_n}(F).$$

The sequence $F^{(n)}$ is non-decreasing and $\cup_{n=0}^\infty F^{(n)}$ is dense in $X$.

We call $n$-cells the subsets of $F^{(n)}$ of the type:

$$F_{i_1, \dots, i_n} = \Psi_{i_1} \circ \dots \circ \Psi_{i_n}(F).$$

For $(i_1, \dots, i_n) \neq (i'_1, \dots, i'_n)$ one clearly has:

$$\Psi_{i_1} \circ \cdots \Psi_{i_n}(X) \cap \Psi_{i'_1} \circ \cdots \Psi_{i'_n}(X) = \Psi_{i_1} \circ \cdots \Psi_{i_n}(F) \cap \Psi_{i'_1} \circ \cdots \Psi_{i'_n}(F) \subset F^{(n)}.$$

REMARK 2.1. – This last relation contains the property of finite ramification (the intersections between the copies of $X$ at level $n$ are included in the copies of the set $F$, *i.e.* the $n$-cells).

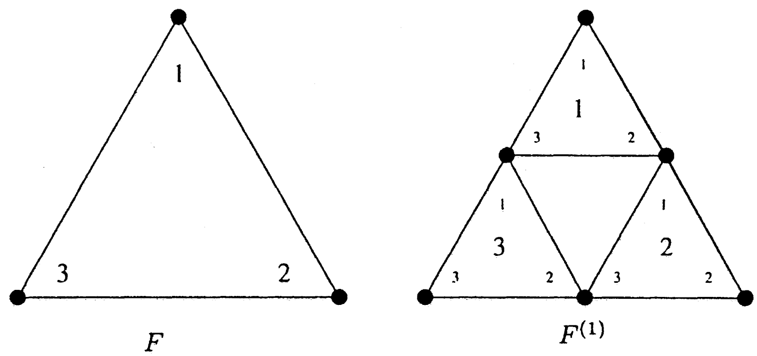
The operation of $G$ is naturally extented to the set $X$ (precisely, $G$ operates on the product space $\Omega$, and then on the quotient space $X$, thanks to the $G$-symmetry of the relation $\mathcal{R}$).

REMARK 2.2. – This method gives a canonical construction of a finitely ramified fractal from a simple finite structure.

We now give some classical examples:

*Example* 1. – (The Sierpinski gasket).

In this case N=D=3, and the relation $\mathcal{R}$ is given by the following picture:




$(1,2)\mathcal{R}(2,1)$

$(2,3)\mathcal{R}(3,2)$

$(3,1)\mathcal{R}(1,3)$

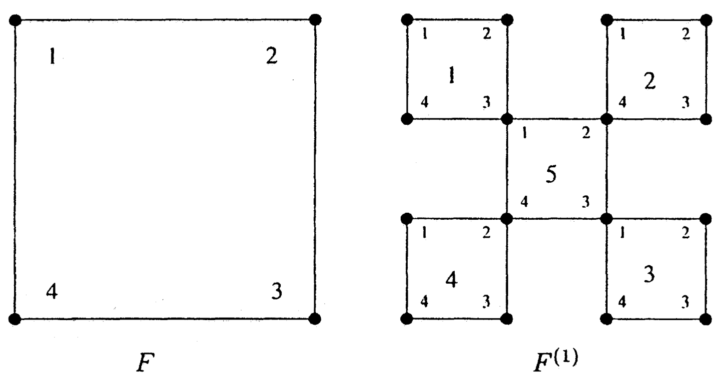
We can choose different symmetry groups. We shall study the following ones:

*Example* 1.1. – We take for $G$ the group $\sigma_3$ of permutations of $\{1,2,3\}$. We call this structure the symmetric Sierpinski gasket. It belongs, as we shall see, to the class of nested fractals.

*Example* 1.2. – We take for $G$ the trivial group. We call this structure the Sierpinski gasket without symmetry.

*Example* 2. – (The Viscek set )

Here $N = 5$ and $D = 4$. The equivalence relation $\mathcal{R}$ is given by the following picture:




$(1,3)\mathcal{R}(5,1)$

$(2,4)\mathcal{R}(5,2)$

$(3,1)\mathcal{R}(5,3)$

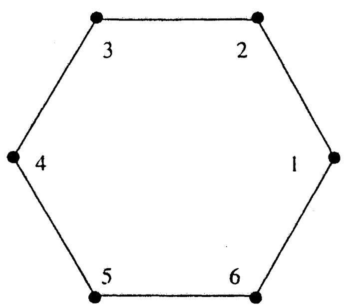
$(4,2)\mathcal{R}(5,4)$

We can take different symmetry groups. We shall study the following ones:

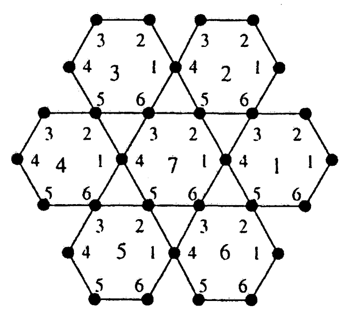
*Example* 2.1. – we take for $G$ the 4th dihedral group $\mathcal{D}_4$, operating on $\{1,\dots,4\}$ as the group of isometries of a square and leaving the point 5 invariant.

*Example* 2.2. – we take for $G$ the group $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$, operating on $\{1,\dots,4\}$ as the group generated by the orthogonal symmetries with respect to the two diagonals and leaving the point 5 invariant.

*Example* 3. – (The snowflake).
Here $N=7$ and $D=6$. The equivalence relation $\mathcal{R}$ is given by the following picture:




$F$




$F^{(1)}$

| | |
|---|---|
| $(1,4)\mathcal{R}(7,1)$ | $(1,3)\mathcal{R}(2,6)$ |
| $(2,5)\mathcal{R}(7,2)$ | $(2,4)\mathcal{R}(3,1)$ |
| $(3,6)\mathcal{R}(7,3)$ | $(3,5)\mathcal{R}(4,2)$ |
| $(4,1)\mathcal{R}(7,4)$ | $(4,6)\mathcal{R}(5,3)$ |
| $(5,2)\mathcal{R}(7,5)$ | $(5,1)\mathcal{R}(6,4)$ |
| $(6,3)\mathcal{R}(7,6)$ | $(6,2)\mathcal{R}(1,5)$ |

We take for $G$ the 6th dihedral group $\mathcal{D}_6$, operating on $\{1,\dots,6\}$ as the group of isometries of the hexagon and leaving the point 7 invariant.

*Example* 4. – All nested fractals ([19]) can be constructed in this way (actually, nested fractals are imbedded in $\mathbb{R}^D$, but are isomorphic, as self-similar sets, to a fractal constructed in this way, *cf.* Section 6). In Section 6.1 we recall the definition of nested fractals.

## 2.2. Decimation invariant Dirichlet forms and construction of self-similar diffusions on $X$.

It appears in [10], [19] and [16], that the construction of a diffusion on a finitely ramified fractal amounts to find an eigenvector of a kind of renormalization operator. Here we briefly review the construction of Kusuoka ([16]) and show how it is linked to the renormalization operator.

Notations. – Notations: we denote by $E^{(n)}$ the space of real functions on $F^{(n)}$ and by $V^{(n)}$ the subspace of the ones with zero mean value (and simply $E=E^{(0)}$, $V=V^{(0)}$ for $F=F^{(0)}$).

Let $\{\mu_1,\dots,\mu_N\}$ and $\{\alpha_1,\dots,\alpha_N\}$ be two $N$-uplets of $]0,1[^N$ invariant under the operation of $G$ on $\{1,\dots,N\}$, *i.e.*:

$$\alpha_{g.i}=\alpha_i,\ \mu_{g.i}=\mu_i,\ \forall g\in G,\ \forall i\in\{1,\dots,N\}.$$

Moreover, we assume that $\mu_1 + \ldots + \mu_N = 1$. The real $\alpha_i$ (resp. $\mu_i$) must be understood as the weight of the 1-cell $F_i = \Psi_i(F)$ for the Dirichlet form (resp. the measure) we shall construct on $X$. We define the measure $\mu$ on $X$ thanks to the following proposition:

PROPOSITION 2.3. – *There exists a unique probability measure $\mu$ on $X$ such that*

$$\int f d\mu = \sum_{i=1}^{N} \mu_i \int f \circ \Psi_i d\mu.$$

*We call $\mu$ the self-similar measure on $X$ associated with the weights $\{\mu_1, \ldots, \mu_N\}$.*

*Proof.* – this is an easy adaptation of [11], Theorem 4.4.1 and 4.4.4 (*cf.* [17], Theorem 2.18). □

We adopt the following definition:

DEFINITION 2.4. – *An irreducible Dirichlet form $A$ on $F$ is said to be decimation invariant if the Dirichlet form on $F^{(1)}$ defined by:*

$$A^{(1)}(f,f) = \sum_{i=1}^{N} \alpha_i^{-1} A(f \circ \Psi_i, f \circ \Psi_i), \quad \forall f \in E^{(1)},$$

*satisfies:*

$$A_F^{(1)} = A.$$

N.B.: $A_F^{(1)}$ denotes the restriction of $A^{(1)}$ to the subset $F$ (*cf.* Section 1.3).

Let $A$ be a $G$-symmetric decimation invariant Dirichlet form (N.B.: $G$-symmetric means that $A(g.f) = A(f)$ for all $f \in E$ and $g \in G$). We now construct a Dirichlet form $a$ on $X$ as the limit of a sequence of Dirichlet forms $A^{(n)}$ defined on the subsets $F^{(n)}$.

We define the Dirichlet form $A^{(n)}$ on $F^{(n)}$ by:

$$A^{(n)}(f,f) = \sum_{i_1,\ldots,i_n} (\alpha_{i_1} \cdots \alpha_{i_n})^{-1} A(f \circ \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}), \quad \forall f \in E^{(n)}$$

(thus, $A^{(0)} = A$).

Since $A$ is decimation invariant one has (*cf.* [16], Proposition 4.7):

PROPOSITION 2.5. – (i)$A_F^{(n)} = A$ *and* $A_{F^{(m)}}^{(n)} = A^{(m)}$ *if* $n \geq m$.

(ii) *For all $f$ in $E^{(\infty)}$, $A^{(n)}(f_{|F^{(n)}}, f_{|F^{(n)}})$ is non-decreasing.*

We define the Dirichlet domain $\mathcal{D}$ by

$$\mathcal{D} = \{f \in C(X), \ \lim_{n\to\infty} A^{(n)}(f_{|F^{(n)}}, f_{|F^{(n)}}) < \infty\}$$

(N.B.: $C(X)$ denotes the space of continuous functions on $X$) and the bilinear form $a : \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ by

$$a(f,g) = \lim_{n\to\infty} A^{(n)}(f_{|F^{(n)}}, g_{|F^{(n)}}), \quad \forall f,g \in \mathcal{D}.$$

One has the key result:

THEOREM 2.6. – *The couple $(a,\mathcal{D})$ is a regular Dirichlet form on $(X,\mu)$ and*

- $1 \in \mathcal{D}$, $a(1,1)=0$ *and $a$ is irreducible* (i.e., $a(f)=0$ *implies that $f$ is constant).*
- $(a,\mathcal{D})$ *has the spectral gap property,* i.e. *there exists $C>0$ such that $\int f d\mu \leq Ca(f,f)$ for all $f$ in $\mathcal{D}$ such that $\int f d\mu = 0$.*
- *all the points of $X$ have strictly positive capacity.*
- $(a,\mathcal{D})$ *is $G$-symmetric,* i.e.*:*

$$\forall f \in \mathcal{D},\ \forall g \in G,\ \ g.f \in \mathcal{D},$$

$$a(g.f, g.f) = a(f,f),\ \ \forall f \in \mathcal{D},\ \forall g \in G,$$

- $(a,\mathcal{D})$ *is self-similar for the weights $(\alpha_1,\ldots,\alpha_N)$,* i.e.*:*

$$\forall f \in \mathcal{D},\ \ \forall i \in \{1,\ldots,N\},\ \ f\circ\Psi_i \in \mathcal{D},$$

$$a(f,f) = \sum_{i=1}^{N}(\alpha_i)^{-1}a(f\circ\Psi_i, f\circ\Psi_i),\ \ \forall f \in \mathcal{D}.$$

REMARK 2.7. – In this section we supposed that the weights $\alpha_i$ are strictly smaller than 1 (this is called the regularity hypothesis, *cf.* [12]). In [14], Kumagaï extended the construction to the general case and proved that all the properties, except the third one on the positive capacity of points, remain valid. As suggested in Remark 3.9, it is strongly possible that in general the process avoïds some points.

*Proof.* – This result comes from [16] Theorem 4.14, except the third property which comes from [8], Theorem 2.3. □

DEFINITION 2.8. – *A Dirichlet form on $X$ satisfying all the properties of Theorem 2.6 is called a $G$-symmetric self-similar Dirichlet form (associated with the weights $(\alpha_1,\ldots,\alpha_N)$).*

Moreover, from the property of self-similarity, we can deduce (*cf.* [16], Theorem 4.14):

PROPOSITION 2.9. – *The Dirichlet form $(a,\mathcal{D})$ is local.*

Kusuoka also gives the following result (*cf.* [16], Remark 4.15):

PROPOSITION 2.10. – *The Dirichlet form $(a,\mathcal{D})$ on $L^2(X,\mu)$ is associated with a $G$-symmetric Feller diffusion process.*

The next property gives a bijective map between the set of decimation invariant Dirichlet forms on $F$ and the set of self-similar Dirichlet forms on $X$.

PROPOSITION 2.11. – *The mapping which takes $A$ to $(a,\mathcal{D})$ is bijective from the set of $G$-symmetric decimation invariant Dirichlet forms on $F$ to the set of $G$-symmetric self-similar Dirichlet forms on $X$ (of Definition 2.8).*

*Proof.* – Suppose that $(a,\mathcal{D})$ is a $G$-symmetric self-similar Dirichlet form on $X$, associated with the weights $(\alpha_1,\ldots,\alpha_N)$. We define a quadratic form $A$ on $E$ by:

$$A(f,f) = \inf\{a(g,g),\ g \in \mathcal{D},\ g_{|F} = f\} \tag{2.1}$$

(N.B.: to define $g_{|F}$ we use the fact that $g$ has a continuous version, since the points of $X$ have strictly positive capacity.)

The irreducibility of $(a, \mathcal{D})$ and the spectral gap property imply that $A$ is an irreducible Dirichlet form on $F$, and that the infimum in (2.1) is reached on a unique function of $\mathcal{D}$, called the harmonic continuation of $f$. (This essentialy comes from the fact that, thanks to the spectral gap property, the space $(L_0^2(m), a)$ (where $L_0^2(m) = \{f \in L^2(m),\ \int f dm = 0\}$) is a Hilbert space. It is proved precisely in [27] ). Moreover, the self-similarity of $a$ implies that $A$ is decimation invariant.

Let $(\tilde{a}, \tilde{\mathcal{D}})$ be the Dirichlet form on $X$ constructed from $A$ as described in this section.

We prove that $(\tilde{a}, \tilde{\mathcal{D}}) = (a, \mathcal{D})$.

The inclusion $\mathcal{D} \subset C(X)$ follows from the positive capacity of the points. Let $f$ be in $\mathcal{D}$:

$$A^{(n)}(f_{|F^{(n)}}) \leq a(f, f),$$

so that $f \in \tilde{\mathcal{D}}$. Thus, we have proved that $\mathcal{D} \subset \tilde{\mathcal{D}}$.

Let $f$ be in $\tilde{\mathcal{D}}$. For all $n$ let $f_n \in \mathcal{D}$ be the harmonic continuation of $f_{F^{(n)}}$, *i.e.* the unique continuation of $f_{F^{(n)}}$ in $\mathcal{D}$ such that:

$$A^{(n)}(f_{|F^{(n)}}) = a(f_n, f_n).$$

The functions $f_n$ and $f_m$ satisfy: $a(f_n - f_m) = a(f_n) - a(f_m)$, for all $n$ and $m$, thus $(f_n)$ is a Cauchy sequence for $a$ and, thanks to the spectral gap property, it is also a Cauchy sequence for the norm $a + ||.||$. Since $(a, \mathcal{D})$ is a closed form, the sequence $(f_n)$ admits a limit in $\mathcal{D}$ for the norm $a + ||.||$. This limit can only be $f$ (because, since $f$ is continuous, $(f_n)$ converges to $f$ for $||.||$, thanks to the maximum principle). So we have proved that $\mathcal{D} = \tilde{\mathcal{D}}$. Moreover,

$$a(f_n) = \tilde{a}(f_n),$$

$$lim_{n\to\infty} a(f_n) = a(f),$$

$$lim_{n\to\infty} \tilde{a}(f_n) = \tilde{a}(f).$$

Thus $a = \tilde{a}$ on their domain. $\square$

## 3. Definition of the map $T$ and first properties

### 3.1. Introduction

#### *3.1.1. Notations, conventions*

We have proved in Section 2.2, Proposition 2.11, that there is a bijective map between the set of $G$-symmetric self-similar Dirichlet forms on $X$ and the set of $G$-symmetric decimation invariant Dirichlet forms on $F$. Decimation invariant Dirichlet forms are clearly solutions of a problem of fixed point. We now describe the map involved in this problem.

We adopt the framework given in Section 2.1. Henceforth, we will only be interested in the finite structure $(F, F^{(1)})$. We briefly recall the notations relative to this finite structure.

We remind that $F$ denotes the finite subset $\{1,\dots,D\}$ of $\{1,\dots,N\}$ and that the set $F^{(1)}$ is defined by:

$$F^{(1)} = \{1,\dots,N\} \times F/\mathcal{R},$$

where $\mathcal{R}$ is the equivalence relation on the set $\{1,\dots,N\} \times F$ which describes the connections between the cells. The set $F$ is regarded as a subset of $F^{(1)}$ namely $\{(i,i),\quad i \in F\}$. The maps $\Psi_1,\dots,\Psi_N$ defined by:

$$\begin{aligned} \Psi_i : &F \to F^{(1)} \\ &x \to (i,x) \end{aligned}$$

are injective and obviously $F^{(1)} = \cup_{i=1}^{N} \Psi_i(F)$.

We call 1-cells the subsets $\Psi_i(F)$ of $F^{(1)}$ and we denote $F_i = \Psi_i(F)$.

The symmetry group $G$ operates on $\{1,\dots,N\}$, leaving the subset $F$ globally invariant. This induces an operation of $G$ on $F^{(1)}$.

We remind that $E$ (resp. $E^{(1)}$) denotes the space of real functions on $F$ (resp. $F^{(1)}$) and that $V$ (resp. $V^{(1)}$) denotes the subspace of the ones with zero mean.

Let us now set up few new notations. We denote by $P : E \to \mathbb{R}$ the mapping that takes $f \in E$ to its mean value on $F$ (with respect to the uniform probability measure). The map $(I - P)$ is then a projection over $V$.

The set of $G$-symmetric irreducible Dirichlet forms on $F$ is denoted by $M$ ($G$-symmetric means that $A(f,f) = A(g.f, g.f)$ for all $f \in F$ and all $g \in G$). The elements of $M$ are positive quadratic forms on $E$, null on the space of constant functions and positive definite on the subspace $V$. They can be thus defined by their restriction to $V$. The set $M$ is a cone included in the set of positive quadratic forms and is not closed since it does not contain the reducible Dirichlet forms. We denote by $PM$ the projective set associated with $M$.

If $A$ and $A'$ are two positive quadratic forms with same kernel, then we denote by $\sup(A'/A)$ and $\inf(A'/A)$ the supremum and the infimum of the ratio $A'/A$ taken on the set where these quadratic forms are strictly positive.

Besides, if $A$ and $A'$ are positive definite on a subspace $W$ then $A'$ can be diagonalized on an orthogonal basis with respect to $A$, we write:

$$A' = \lambda_0 A_{|W_0} \oplus \cdots \oplus \lambda_r A_{|W_r}, \tag{3.1}$$

with $W = W_0 \oplus \cdots \oplus W_r$ and $0 < \lambda_0 < \cdots < \lambda_r$. Relation (3.1) means that $A'_{|W_i} = \lambda_i A_{|W_i}$ and that the subspaces $W_0,\dots,W_r$ are orthogonal with respect to both quadratic forms $A'$ and $A$ (in particular, $\inf(A'/A) = \lambda_0$ and $\sup(A'/A) = \lambda_r$). The decomposition (3.1) is unique.

For the elements of $M$ are defined by their restriction to $V$, it will be understood that the simultaneous diagonalization of two Dirichlet forms of $M$ is implicitely done on the subspace $V$.

*3.1.2. Definition of $T$, examples*

Let $T : M \to M$ be the map defined as follows.

For $A$ in $M$ we denote by $A^{(1)}$ the irreducible Dirichlet form on $F^{(1)}$ given by:

$$A^{(1)}(f,f) = \sum_{i=1}^{N} \alpha_i^{-1} A(f \circ \Psi_i, f \circ \Psi_i), \ \ \forall f \in E^{(1)},$$

we defined $TA$ as the restriction of $A^{(1)}$ to $F$ (in the meaning of Section 1.3), *i.e.*:

$$TA = A_F^{(1)}.$$

Thanks to the connectivity property of the relation $\mathcal{R}$ (third assumption on $\mathcal{R}$, *cf.* Section 2.1) it is easy to see that $TA$ is irreducible. Moreover, as $A^{(1)}$ is irreducible, the harmonic continuation is uniquely defined and for $f \in E$ we denote it by $H_A(f)$ *i.e.*:

$$TA(f) = A^{(1)}(H_A(f)), \ \ \forall f \in E.$$

The map $T$ is clearly 1-homogeneous (*i.e.* $T(\lambda A) = \lambda TA$). In consequence, we shall adopt the following definition of a fixed point:

DEFINITION 3.1. – *A Dirichlet form $A$ of $M$ is called a fixed point of $T$ if there exists a strictly positive real $\lambda$ such that $TA = \lambda A$.*

*The Dirichlet form $A$ is called a regular fixed point if moreover $\lambda$ satisfies $\lambda\alpha_i < 1$ for all $i$ in $\{1, \ldots, N\}$.*

*We will say that $T$ has a unique fixed point if there is a unique line invariant by $T$ (*i.e.* the fixed point is unique up to a multiplicative constant).*

N.B.: Actually, what we call fixed points are eigenvectors of $T$ but we prefer to adopt this terminology since they are really fixed points for the map $T$ on the projective space associated with $M$ and since all the eigenvalues are equal (*cf.* forthcoming Corollary 3.5).

With this definition and Definition 2.4 of Section 2.2, we remark that a fixed point for $T$ is a $G$-symmetric decimation invariant Dirichlet form on $F$ for the weights $(\lambda\alpha_1, \ldots, \lambda\alpha_N)$. To construct the self-similar Dirichlet form on $X$ we just have to check the regularity of the fixed point (we recall that the assumption that the weights are strictly smaller than 1 is essential in the construction of the Dirichlet form on the fractal space $X$). In Corollary 3.5 we shall show that the property of regularity depends only on the choosen weights $(\alpha_1, \ldots, \alpha_N)$ (*i.e.*, with given weights, either all fixed points are regular either all are not). In Proposition 3.8 we give a sufficient condition on the weights $(\alpha_i)$ for the fixed points to be regular.

*Example* 1.1. – Here, the symmetries only allow to take equal weights $\alpha_1 = \alpha_2 = \alpha_3$. The set $M$ has dimension 1 (thanks to the symmetries), so existence and uniqueness are trivially satisfied.

*Example* 1.2. – We can take for the $\alpha_i$'s any positive 3-tuplet. In Section 5.2 we shall solve the problem of existence and uniqueness for all the possible values of $(\alpha_1, \alpha_2, \alpha_3)$.

*Example* 2.1 and *Example* 3. – If $\alpha_i = 1$ for all $i$, it is a nested fractal so the existence follows from [19]. The uniqueness was proved in these two particular cases in [1]. In [27], we explicitely compute $T$ in these two examples.

*Example* 2.2. – We recall that this example is the Viscek set with symmetry group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Considering the symmetries, we can take $\alpha_1 = \alpha_3 = \gamma_1$, $\alpha_2 = \alpha_4 = \gamma_2$ and $\alpha_5 = \beta$. Metz proved that for $\gamma_1 = \gamma_2 = \beta$ there are infinitely many fixed points (*cf.* [20] ).

In general, in [27] , we proved that when $\gamma_1 = \gamma_2$, $T$ has infinitely many fixed points and when $\gamma_1 \neq \gamma_2$ it has none. In fact, in this case, the map $T$ can be expressed as a non-negative matrix, and the problem of fixed point reduces to the Perron-Frobenius theorem (we have proved that this situation arises each time the fractal is a tree, *i.e.* has no loop).

*Example* 4. – Lindstrøm proved the existence for the class of nested fractals (*cf.* [19]). In Section 6.1, we shall prove the uniqueness (and we partly recover the existence).

### 3.2. First properties of the map $T$

PROPOSITION. – *The map $T$ is non-decreasing* i.e.*:*

$$\forall A, A' \in M, \ \ (A \leq A' \Rightarrow TA \leq TA'),$$

N.B.: the order on quadratic forms is the usual one: $A \leq A'$ if and only if $A' - A$ is positive.

*Proof.* – Let $f$ be in $E$ and $H_{A'}(f)$ be its harmonic continuation to $F^{(1)}$ with respect to $(A')^{(1)}$ then:

$$TA'(f) = (A')^{(1)}(H_{A'}(f)) \geq A^{(1)}(H_{A'}(f)) \geq TA(f). \quad \square$$

We introduce the Hilbert's pseudo-metric $d$ on $M$ by

$$d(A, A') = \ln\left(\frac{\sup(A/A')}{\inf(A/A')}\right),$$

for $(A, A')$ in $M \times M$.

In fact, $d$ is not a metric on $M$, but on the projective space $PM$ (as we easily see that $d(A, A') = d(\lambda A, \lambda' A')$ for all positive reals $\lambda, \lambda'$).

The following property of $T$ is important:

PROPOSITION 3.3. – *Let $A$ and $A'$ be in $M$ then:*

$$\sup \frac{TA}{TA'} \leq \sup \frac{A}{A'}, \quad \inf \frac{TA}{TA'} \geq \inf \frac{A}{A'}.$$

*The map $T$ is non-expansive for $d$,* i.e.*:*

$$d(TA,TA') \leq d(A,A'),\ \ \forall (A,A') \in M^2.$$

*Proof.* – Let $A$ and $A'$ be in $M$. We have $A \leq \sup(\frac{A}{A'})A'$, thus $TA \leq \sup(\frac{A}{A'})TA'$, which gives the first inequality. The proof of the second is similar and the non-expansiveness property is easily derived form these two inequalities. □

REMARK 3.4. – Proposition 3.2 and 3.3 where first remarked by Metz, *cf.* [21]. In [23], Hilbert's metric and non-expansive maps for this metric are studied in a general framework. In the terminoly of [23], Proposition 3.2 means that $T$ is order-preserving for the order induced by the cone of positive quadratic forms (but it is not order preserving for the order induced by the cone $M$). Proposition 3.3 means that $T$ is non-expansive and we can remark that this only comes from the fact that $T$ is order-preserving and 1-homogenous (as remarked in Proposition 1.15 of [23]).

From this proposition we easily deduce (*cf.* [10] Corollary 3.7):

COROLLARY 3.5. – *Let $A$ and $A'$ be two fixed points of $T$ such that $TA = \lambda A$ and $TA' = \lambda' A'$, then $\lambda = \lambda'$.*

*Proof.* – Indeed,

$$\sup \frac{TA}{TA'} = \frac{\lambda}{\lambda'} \sup \frac{A}{A'},$$

and from Proposition 3.3 we deduce that $\lambda \leq \lambda'$. In the same way we obtain $\lambda' \leq \lambda$ and thus, $\lambda = \lambda'$. □

This explains the announced fact that the regularity of a fixed point depends only on the weights $(\alpha_i)$.

### 3.3. Sufficient conditions for the regularity of the fixed points

For $A \in M$ and $x \in F$ we set:

$$L_A(x) = \{f \in E,\ A(f, 1_{\{x\}}) = 1\},$$

$$l_A(x) = \frac{1}{\sup\{A(f),\ \ f \in L_A(x)\}}. \tag{3.2}$$

REMARK 3.6. – Actually, $l_A(x)$ is the distance from $x$ to the affine space generated by the points of $F \setminus \{x\}$, for the distance $r_A$ associated with $A$ (or to be more precise, for the distance associated with the unique quadratic form defined on the affine subset generated by the points of $F$ that coincide with $r_A$ on these points. This quadratic form is the dual form of $A$, *cf.* [27]).

We first prove:

LEMMA 3.7. – *Let $x$ be in $F$. One has that:*

$$l_{TA}(x) > \alpha_x l_A(x).$$

*Proof.* – We first remind that $x$ is the fixed point of $\Psi_x$ ($x$ is a point of $F \subset \{1, \dots, N\}$).

Let $f$ be an element of $E$ which realizes the supremum in (3.2) for $TA$. Let $H_A(f)$ be the harmonic continuation of $f$ to $F^{(1)}$ with respect to $A^{(1)}$. As $F \subset F^{(1)}$, we can regard $1_{\{x\}}$ as an element of $E^{(1)}$. One has that:

$$\begin{aligned} 1 &= TA(f, 1_{\{x\}}) \\ &= A^{(1)}(H_A(f), 1_{\{x\}}) \\ &= \sum_{i=1}^{N} \alpha_i^{-1} A(H_A(f) \circ \Psi_i, 1_{\{x\}} \circ \Psi_i) \\ &= \alpha_x^{-1} A(H_A(f) \circ \Psi_x, 1_{\{x\}} \circ \Psi_x) \end{aligned}$$

because $x \in F_i = \Psi_i(F)$ if and only if $i = x$.

The last relation proves that $\alpha_x^{-1} H_A(f) \circ \Psi_x \in L_A(x)$ (because $\Psi_x(x) = x$). Thus,

$$\begin{aligned} l_{TA}(x) &= TA(f, f) \\ &= \sum_{i=1}^{N} \alpha_i^{-1} A(H_A(f) \circ \Psi_i) \\ &> \alpha_x^{-1} A(H_A(f) \circ \Psi_x) \\ &= \alpha_x A(\alpha_x^{-1} H_A(f) \circ \Psi_x) \\ &\geq \alpha_x l_A(x). \end{aligned}$$

The third inequality is strict because $H_A(f)$ is not constant out of $F_x$, with regard to Proposition 1.14. □

PROPOSITION 3.8. – *Let $A$ be a fixed point of $T$ such that $TA = \lambda A$, then for all $x$ in $F = \{1, \dots, D\}$, $\lambda \alpha_x < 1$.*

*Proof.* – This follows from Lemma 3.7 since $l_{TA} = \lambda^{-1} l_A$. □

REMARK 3.9. – This proposition gives a good answer for all the examples we shall consider, in particular it follows that if all the $\alpha_i$'s are equal then all the fixed points are regular. In [14], Kumagai extended the construction of the process to non-regular cases. It could be interesting to study the capacity of points for such processes, in particular it is likely that these processes avoïd some points, maybe the ones for which $\overline{\lim} \alpha_{\omega_1} \cdots \alpha_{\omega_n} \geq 1$.

REMARK 3.10. – Kigami has given a very similar result in [12], Theorem 4.10.

## 4. Preliminaries to Theorem 5.1. Behaviour of $T$ near the boundary of $M$.

Our goal in this section is to describe the behaviour of $T$ near the boundary of $M$. The underlying idea of our procedure is to construct a suitable closure of the set $M$ (which is not closed, since it is the set of irreducible Dirichlet forms). The natural closure of $M$, that contains the reducible Dirichlet forms is not relevant to our problem. In particular, the map $T$ has no continuous extension to this closure, when considered as a map on the projective set $PM$ (with the Hilbert's metric for example, *cf.* Section 3). A Dirichlet form of $M$ is close to a reducible Dirichlet form when it is nearly vanishing on a subspace of $V$: to understand the behaviour of $T$ near this reducible Dirichlet form we need to know what happens on this subspace. This means that the suitable closure needs to handle both the "leading" part of the Dirichlet form and its "vanishing" part. Let us now be more precise. With a reducible Dirichlet form one can naturally associate the partition of its irreducible components, say $\{I_1, \ldots, I_k\}$ and consequently the equivalence relation $\mathcal{J}$ defined by $F/\mathcal{J} = \{I_1, \ldots, I_k\}$. This means that the effective conductance is strictly positive between two points in relation with respect to $\mathcal{J}$ and null between two points that are not. We thus say that a Dirichlet form $A \in M$ is close to the reducibility associated with $\mathcal{J}$ if the effective conductance between two points in relation with respect to $\mathcal{J}$ is small compared with the one between two points not in relation.

When a Dirichlet form is near the reducibility associated with $\mathcal{J}$ we can approximate it by a couple $(A_{\mathcal{J}}, A_{F/\mathcal{J}})$ in the following way:

- Considering that the bond conductivity between two points not in relation with respect to $\mathcal{J}$ is small we replace it by 0: it leads to a Dirichlet form, denoted by $A_{\mathcal{J}}$, reducible on the partition $F/\mathcal{J}$. $A_{\mathcal{J}}$ is then null on the space of functions constant on the equivalence classes with respect to $\mathcal{J}$.
- Considering that the bond conductivity between two points in relation with respect to $\mathcal{J}$ is big, we replace it by an infinite conductivity (*i.e.* null resistivity): on the electrical point of view, this is equivalent to identifying the points in relation with respect to $\mathcal{J}$ and this leads to a Dirichlet form, denoted by $A_{F/\mathcal{J}}$, defined on the quotient set $F/\mathcal{J}$ (and irreducible). $A_{F/\mathcal{J}}$ can also be considered as a quadratic form on the subspace of functions constant on the equivalence classes (since we can identify the functions on $F/\mathcal{J}$ and the functions on $F$ constant on the equivalence classes). Thus, $A_{F/\mathcal{J}}$ is defined on the kernel of $A_{\mathcal{J}}$.

When $A$ is close to the reducibility $\mathcal{J}$ we see that $A_{F/\mathcal{J}}$ is small compared with $A_{\mathcal{J}}$, and we shall prove (Corollary 4.17, Section 4.1.3) that $A$ can be approximated by the sum of $A_{\mathcal{J}}$ and $A_{F/\mathcal{J}}$. In some sense, the Dirichlet form $A$ can be decomposed, near the reducibility $\mathcal{J}$, in its leading part $A_{\mathcal{J}}$ and its vanishing part $A_{F/\mathcal{J}}$.

In Section 4.3. we extend the map $T$ to the boundary of $M$, *i.e.* we define the value of $T$ on Dirichlet forms of the type of $A_{\mathcal{J}}$ (called Dirichlet forms reducible on $\mathcal{J}$) and $A_{F/\mathcal{J}}$ (*i.e.* irreducible Dirichlet forms on $F/\mathcal{J}$): we still get a renormalization operator of the same kind as $T$.

The results of this section are essentially used in Lemma 5.7: from the value of $T$ on $A_{\mathcal{J}}$ and $A_{F/\mathcal{J}}$ we can know if the reducibility $\mathcal{J}$ is repulsive for $T$ or not (repulsive means that iterating $T$, one goes out of the small neigbourhoods of $\mathcal{J}$). The dichotomy in

Theorem 5.1 comes from the behaviour of $T$ near the boundary: if all the reducibilities are repulsive then $T$ has a unique fixed point, if one is not then $T$ has no fixed point.

## 4.1. $G$-relations. Approximation of a Dirichlet form near a $G$-relation.

### *4.1.1. G-relations, definition, examples*

DEFINITION 4.1. – *An equivalence relation $\mathcal{J}$ on $F$ is called a $G$-relation if $\mathcal{J}$ is invariant under the operation of the group $G$,* i.e. *if:*

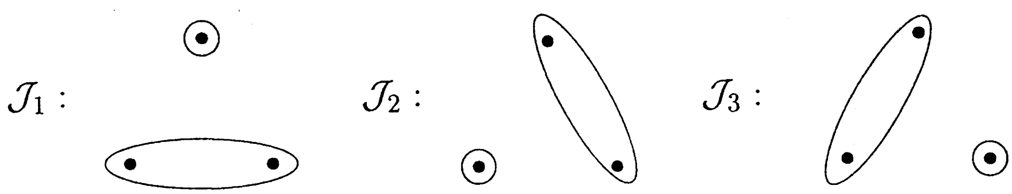
$$\forall x, y \in F,\ \forall g \in G\ \ (x\mathcal{J}y \Rightarrow g.x\mathcal{J}g.y).$$

For a $G$-relation $\mathcal{J}$ we denote by $F/\mathcal{J}$ the quotient set formed by $\mathcal{J}$. The group $G$ operates naturally on $F/\mathcal{J}$.

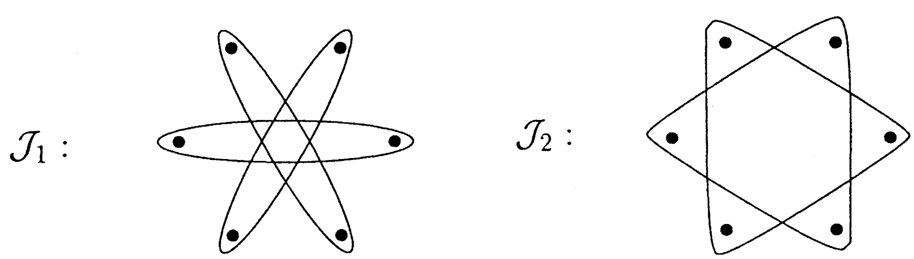
We set up some notations: 1 will be the full relation, *i.e.* $\mathcal{J} = 1$ if and only if $x\mathcal{J}y$ for all $(x, y) \in F^2$, and 0 will be the empty relation, *i.e.* $\mathcal{J} = 0$ if $x\mathcal{J}y$ implies $x = y$.

We say that $\mathcal{J}$ is non-trivial if $\mathcal{J} \neq 0$ and $\mathcal{J} \neq 1$. We order the set of $G$-relations by the inclusion relation *i.e.* $\mathcal{J}$ is smaller than $\mathcal{J}'$ if the graph of $\mathcal{J}$ on $F \times F$ is included in the graph of $\mathcal{J}'$ and we denote it by $\mathcal{J} \subset \mathcal{J}'$ (and thus, $F/\mathcal{J}$ is a sub-partition of $F/\mathcal{J}'$).

*Exemple* 1.2. – In the case of the Sierpinski gasket without symmetry, there are 5 $G$-relations, and 3 are non-trivial. They are represented by the following picture:




*Example* 3. – In the case of the snowflake the group is $G = \mathcal{D}_6$. There are 4 $G$-relations, 2 are non-trivial and are represented by the following picture:




For $A \in M$ we set:

$$\delta(A) = \frac{\inf\limits_{\substack{f \in V \\ \|f\|=1}} A(f)}{\sup\limits_{\substack{f \in V \\ \|f\|=1}} A(f)}$$

and for a non-trivial $G$-relation $\mathcal{J}$:

$$\delta_{\mathcal{J}}(A) = \frac{\sup\limits_{x \not\mathcal{J} y} c_A(x,y)}{\inf\limits_{\substack{x \mathcal{J} y \\ x \neq y}} c_A(x,y)}.$$

N.B.: We recall that $c_A(x,y)$ is the effective conductance defined in Section 1.4. and that $r_A = 1/c_A$ is a distance on $F$.

REMARK 4.2. – If $\delta_{\mathcal{J}}(A)$ is small it means that the effective conductance between two points in relation with respect to $\mathcal{J}$ is big compared with the effective conductance between two points not in relation with respect to $\mathcal{J}$. In some sense, $\delta_{\mathcal{J}}(A)$ gives the proximity of $A$ to the reducibility associated with $\mathcal{J}$.

We give the following result.

PROPOSITION 4.3. – *For all $\epsilon > 0$, there exists $\beta > 0$ such that for all $A \in M$, $\delta(A) \leq \beta$ implies that there exists a non-trivial $G$-relation $\mathcal{J}$ such that $\delta_{\mathcal{J}}(A) \leq \epsilon$.*

*Proof.* – As in Section 1.4, let $A^0$ be the element of $M$ associated with bond conductivities equal to 1.

We recall that Lemma 1.19 gives a constant $K > 0$ such that for all $A$ in $M$,

$$(1/K)\left(\inf_{\substack{x,y \in F, \\ x \neq y}} c_A(x,y)\right) A^0 \leq A \leq \left(\sup_{x,y \in F} c_A(x,y)\right) A^0. \tag{4.1}$$

If for $\epsilon > 0$ the property is not satisfied, then there exists a sequence $(A_n)$ in $M$ such that $\delta(A_n)$ converges to 0 and $\delta_{\mathcal{J}}(A_n) \geq \epsilon$ for all $n$ and all non-trivial $G$-relation $\mathcal{J}$. Let $\tilde{A}_n$ be defined by

$$\tilde{A}_n = \frac{A_n}{\inf_{x \neq y} c_{A_n}(x,y)}$$

Obviously, $\delta(A_n) = \delta(\tilde{A}_n)$ and $r_{\tilde{A}_n}(x,y) = 1/c_{\tilde{A}_n}(x,y) \leq 1$ for all $x \neq y$. Let $n_k$ be a sub-sequence such that $r_{\tilde{A}_{n_k}}(x,y)$ converges for all $x \neq y$. Let $\mathcal{J}$ be the relation given by $x\mathcal{J}y$ if and only if $r_{\tilde{A}_{n_k}}(x,y)$ converges to 0. Thanks to the triangular inequality satified by the distance $r_A$, $\mathcal{J}$ is transitive. Moreover, $\mathcal{J}$ is non-trivial: indeed, if $\mathcal{J} = 0$ then $r_{\tilde{A}_{n_k}}(x,y)$ is bounded from below, which is incompatible with (4.1) and the fact that $\delta(\tilde{A}_{n_k})$ converges to 0. Besides, for all $k$ there exist $x$ and $y$ such that $r_{\tilde{A}_{n_k}}(x,y) = 1$, thus $\mathcal{J}$ cannot be the relation 1. Finally, $\mathcal{J}$ is $G$-symmetric as the Dirichlet forms $A_n$ are $G$-symmetric and then $\mathcal{J}$ is a non-trivial $G$-relation. But $\delta_{\mathcal{J}}(A_{n_k}) = \delta_{\mathcal{J}}(\tilde{A}_{n_k})$ converges to 0, and this leads to a contradiction. □

LEMMA 4.4. – *Let $\mathcal{J}$ and $\mathcal{J}'$ be two distinct non-trivial $G$-relations. If $\mathcal{J}$ and $\mathcal{J}'$ are not ordered then for all $A \in M$ the following relation is true:*

$$\delta_{\mathcal{J}}(A) \geq \frac{1}{\delta_{\mathcal{J}'}(A)}.$$

*Proof.* – If $\mathcal{J}$ and $\mathcal{J}'$ are not ordered, then one can find $x, y$ in $F$ in relation with respect to $\mathcal{J}$ but not with respect to $\mathcal{J}'$ and $x'$, $y'$ in relation with respect to $\mathcal{J}'$ but not with respect to $\mathcal{J}$. The inequality follows from the definition of $\delta_{\mathcal{J}}(A)$ and $\delta_{\mathcal{J}'}(A)$. □

*4.1.2. Projection of $A$ on the boundary of $M$*

*a) Notations and definitions*

Let $\mathcal{J}$ be a $G$-relation and denote by $\{I_1, \cdots, I_k\} = F/\mathcal{J}$ the quotient set formed by $\mathcal{J}$. We set:

$$E_{F/\mathcal{J}} = \{f \in E, \ \ f(x) = f(y) \text{ if } x\mathcal{J}y\},$$

$$V_{F/\mathcal{J}} = V \cap E_{F/\mathcal{J}}.$$

REMARK 4.5. – $E_{F/\mathcal{J}}$ is the set of functions constant on the equivalence classes with respect to $\mathcal{J}$. We will often consider $E_{F/\mathcal{J}}$ as the set of real functions on $F/\mathcal{J}$.

For all $i \in \{1, \ldots, k\}$ we set:

$$V_{I_i} = \{f \in V, \ \ f = 0 \text{ on } F \setminus I_i\},$$

$$V_{\mathcal{J}} = V_{I_1} \oplus \cdots \oplus V_{I_k}.$$

We thus have the following decomposition:

$$V = V_{F/\mathcal{J}} \oplus V_{I_1} \oplus \cdots \oplus V_{I_k}.$$

N.B.: The space $V_{I_i}$ will often be regarded as the space of functions on $I_i$ with zero mean.

Define $P_{F/\mathcal{J}} : E \to E_{F/\mathcal{J}}$ by the following formula:

$$P_{F/\mathcal{J}} f(I_i) = \frac{1}{\# I_i} \sum_{x \in I_i} f(x).$$

The map $P_{F/\mathcal{J}}$ is clearly the projection over $E_{F/\mathcal{J}}$ with respect to the decomposition $E = E_{F/\mathcal{J}} \oplus V_{\mathcal{J}}$.

DEFINITION 4.6. – *We shall say that a Dirichlet form $A$ on $F$ is reducible on the $G$-relation $\mathcal{J}$ if $A$ satisfies the following property:*

$$(A(f, f) = 0) \Leftrightarrow (f(x) = f(y) \textit{ if } x\mathcal{J}y).$$

We denote by $M_{\mathcal{J}}$ the set of Dirichlet forms on $F$, reducible on the $G$-relation $\mathcal{J}$ and $G$-symmetric and by $M_{F/\mathcal{J}}$ the set of Dirichlet forms on $F/\mathcal{J}$, irreducible and $G$-symmetric.

N.B.: When $\mathcal{J} = 1$ we clearly have $M_{\mathcal{J}} = M$ and $M_{F/\mathcal{J}} = \{0\}$ (we recall that 1 denotes the full relation which connects any two points of $F$). In the same way, when $\mathcal{J} = 0$ one has that $M_{\mathcal{J}} = \{0\}$ and $M_{F/\mathcal{J}} = M$.

REMARK 4.7. – The subspaces $V_{I_i}$ are orthogonal with respect to any element of $M_{\mathcal{J}}$

As we did in Section 1.4, we define some reference Dirichlet forms.

We denote by $A^0_{F/\mathcal{J}} \in M_{F/\mathcal{J}}$ the Dirichlet form on $F/\mathcal{J}$ with bond conductivities equal to 1 between any two points of $F/\mathcal{J}$.

For all $i \in \{1, \ldots, k\}$, we denote by $A^0_{I_i}$ the Dirichlet form on $I_i$, such that the bond conductivity between any two points of $I_i$ is equal to 1. We denote by $A^0_{\mathcal{J}}$ the element of $M_{\mathcal{J}}$ defined by

$$A^0_{\mathcal{J}}(f, f) = \sum_{i=1}^{k} A^0_{I_i}(f_{|I_i}), \ \ \forall f \in E.$$

REMARK 4.8. – As in Remark 1.18, we clearly have if $f_i \in V_{I_i}$: $A^0_{I_i}(f_i, f_i) = \sum_{x \in I_i}(f_i(x))^2$.

*b) Projection of an irreducible Dirichlet form*

For all $A$ in $M$, we denote by $A_{F/\mathcal{J}}$ the element of $M_{F/\mathcal{J}}$ defined by:

$$A_{F/\mathcal{J}}(f, f) = A(f, f), \ \ \forall f \in E_{F/\mathcal{J}}.$$

The Dirichlet form $A_{F/\mathcal{J}}$ is called the projection of $A$ on $M_{F/\mathcal{J}}$.

N.B.: To understand this definition one has to recall Remark 4.5, since on the left side of the equality $E_{F/\mathcal{J}}$ is regarded as the set of functions on $F/\mathcal{J}$ and on the right side it is regarded as a subset of $E$.

REMARK 4.10. – If $A$ is associated with a bond conductivity matrix $(j_{x,y})$, then it is easy to check that the bond conductivity associated with $A_{F/\mathcal{J}}$ between two points $X$ and $Y$ of $F/\mathcal{J}$ is given by:

$$j_{X,Y} = \sum_{x \in X, y \in Y} j_{x,y}.$$

REMARK 4.11. – If $X_n$ is the Markov chain associated with $A$ and $\hat{X}$ is the image of $X$ by the canonical surjection over $F/\mathcal{J}$, then $\hat{X}$ is not Markovian in general. In particular $\hat{X}$ is not the Markov chain associated with $A_{F/\mathcal{J}}$.

DEFINITION 4.12. – *Let $A$ be in $M$, we denote by $A_{\mathcal{J}}$ the element of $M_{\mathcal{J}}$ defined by*

$$A_{\mathcal{J}}(f, f) = \sum_{i=1}^{k} A_{I_i}(f_{|I_i}, f_{|I_i}), \ \ \forall f \in E.$$

*We call $A_{\mathcal{J}}$ the projection of $A$ on $M_{\mathcal{J}}$.*

REMARK 4.13. – We recall that $A_{I_i}$ is the restriction of $A$ to the subset $I_i$ (*cf.* Section 1.3). In probabilistic terms, the Markov chain associated with $A_{I_i}$ describes the successive visits of $I_i$ by the Markov chain associated with $A$.

We now compare $A_{F/\mathcal{J}}$ and $A_{\mathcal{J}}$ to the reference Dirichlet forms $A^0_{F/\mathcal{J}}$ and $A^0_{\mathcal{J}}$ introduced in previous section.

LEMMA 4.14. – *There exists $K_1 > 0$ such that for all $A \in M$:*

$$\frac{1}{K_1}\left(\inf_{x \not\mathcal{J} y} c_A(x,y)\right) A^0_{F/\mathcal{J}} \leq A_{F/\mathcal{J}} \leq K_1 \left(\sup_{x \not\mathcal{J} y} c_A(x,y)\right) A^0_{F/\mathcal{J}},$$

$$\frac{1}{K_1}\left(\inf_{\substack{x\mathcal{J}y \\ x\neq y}} c_A(x,y)\right) A^0_{\mathcal{J}} \leq A_{\mathcal{J}} \leq \left(\sup_{x\mathcal{J}y} c_A(x,y)\right) A^0_{\mathcal{J}}.$$

REMARK 4.15. – In particular, it means that the ratio $A_{F/\mathcal{J}}/A_{\mathcal{J}}$ is small near the reducibility associated with $\mathcal{J}$.

*Proof.* – The second relation is a direct application of Lemma 1.19 to the Dirichlet forms $A_{I_i}$ (indeed, it is easy to check that if $x, y \in I_i$ then $c_{A_{I_i}}(x,y) = c_A(x,y)$).

To prove the first estimate, we need to compare $c_A$ and $c_{A_{F/\mathcal{J}}}$. Let $X$ and $Y$ be in $F/\mathcal{J}$. Let $f$ be the unique element of $E_{F/\mathcal{J}}$ such that $f(X) = 0$, $f(Y) = 1$ and $c_{A_{F/\mathcal{J}}}(X,Y) = A_{F/\mathcal{J}}(f,f)$. If $x, y \in F$ are representatives of $X$ and $Y$ respectively, one clearly has

$$A_{F/\mathcal{J}}(f,f) = A(f,f) \geq c_A(x,y)$$

(since $f(x) = 0$ and $f(y) = 1$).

Thus, it follows that:

$$\inf_{x \not\mathcal{J} y} c_A(x,y) \leq \inf_{X,Y\in F/\mathcal{J}, X\neq Y} c_{A_{F/\mathcal{J}}}(X,Y). \tag{4.2}$$

So the left inequality comes from the last relation and Lemma 1.19 applied to the irreducible Dirichlet form $A_{F/\mathcal{J}}$.

Let now $x$ and $y$ be in $F$, $x \neq y$, and $f \in E$ be such that $f(x) = 0$ and $f(y) = 1$, if $A \in M$ is associated with the bond conductivity matrix $(j_{x,y})$ then:

$$A(f,f) \geq j_{x,y}.$$

It implies that $j_{x,y} \leq c_A(x,y)$.

Let us denote by $\tilde{j}_{i,j}$ the bond conductivity between $I_i$ and $I_j$ associated with the Dirichlet form $A_{F/\mathcal{J}}$. From Remark 4.10 and the previous relation:

$$\tilde{j}_{i,j} \leq (\#I_i)(\#I_j) \sup_{x\in I_i, y\in I_j} c_A(x,y).$$

By definition of $A^0_{F/\mathcal{J}}$, one has that:

$$A_{F/\mathcal{J}} \leq \left(\sup_{i\neq j}(\#I_i)(\#I_j)\right)\left(\sup_{x \not\mathcal{J} y} c_A(x,y)\right) A^0_{F/\mathcal{J}}.$$

This concludes the proof of the lemma. □

*4.1.3. Estimates of $A$ near a $G$-relation*

We recall that $E$ can be decomposed as follows (*cf.* Section 4.1.2.a):

$$E = E_{F/\mathcal{J}} \oplus V_{\mathcal{J}} \tag{4.3}$$

$$= E_{F/\mathcal{J}} \oplus V_{I_1} \oplus \cdots \oplus V_{I_k}, \tag{4.4}$$

and that $P_{F/\mathcal{J}}$ denotes the projection over $E_{F/\mathcal{J}}$ with respect to Decomposition (4.3).

In general the sets involved in Decomposition (4.4) are not orthogonal with respect to a quadratic form $A$ in $M$. Nevertheless, we now prove that this tends to be true when $A$ is near the $G$-relation $\mathcal{J}$.

We denote by $\alpha_A(f,g)$ the cosinus of the angle of $f$ and $g$ with respect to $A$, *i.e.*:

$$\alpha_A(f,g) = \frac{|A(f,g)|}{\sqrt{A(f)A(g)}}, \quad f,g \in V^*.$$

PROPOSITION 4.16. – *There exists $K_2 > 0$ such that for all $A$ in $M$:*

$$\alpha_A(f,g) \le K_2 \delta_{\mathcal{J}}(A)^{\frac{1}{2}}, \quad \forall f \in V_{F/\mathcal{J}}, \ \forall g \in V_{\mathcal{J}},$$

$$\alpha_A(f_i,f_j) \le K_2 \delta_{\mathcal{J}}(A), \quad \forall f_i \in V_{I_i}, \ \forall f_j \in V_{I_j}, \ i \neq j.$$

From this proposition we deduce the following important corollary:

COROLLARY 4.17. – *There exist some constants $K_3 > 0$ and $\epsilon_3 > 0$ such that for all $A$ in $M$ satisfying $\delta_{\mathcal{J}}(A) \le \epsilon_3$:*

$$(1 - K_3\delta_{\mathcal{J}}(A)^{\frac{1}{2}}) \le \frac{A(f)}{A_{F/\mathcal{J}}(P_{F/\mathcal{J}}(f)) + A_{\mathcal{J}}(f)} \le (1 + K_3\delta_{\mathcal{J}}(A)^{\frac{1}{2}}), \quad \forall f \in V^*.$$

Proof of Proposition 4.16: Set:

$$C_{F/\mathcal{J}} = \sup_{x \not\mathcal{J} y} c_A(x,y),$$

$$C_{\mathcal{J}} = \inf_{\substack{x \mathcal{J} y \\ x \neq y}} c_A(x,y).$$

At the beginning of the proof of Lemma 4.14 we remarked that $j_{x,y} \le c_A(x,y)$ so it implies that:

$$j_{x,y} \le C_{F/\mathcal{J}}, \quad \forall x,y, \ x \not\mathcal{J} y. \tag{4.6}$$

Using Remark 4.8 and Lemma 4.14, we have for all $p \in \{1,\ldots,k\}$ and $g_p \in V_{I_p}$:

$$\sum_{x\in I_p}(g_p(x))^2 = A^0_{\mathcal{J}}(g_p) \leq K_1\frac{1}{C_{\mathcal{J}}}A_{\mathcal{J}}(g_p) \leq K_1\frac{1}{C_{\mathcal{J}}}A(g_p). \tag{4.6}$$

Let us now prove the first inequality.
Take $f \in E_{F/\mathcal{J}}$, $p \in \{1,\ldots,k\}$ and $g_p \in V_{I_p}$. The function $f$ can be writen as

$$f = \sum_{i=1}^{k} c_i 1_{I_i},$$

for some real numbers $c_1,\ldots,c_k$ ($1_X$ denotes the characteristic function of the set $X$). Using that $g_p = 0$ on $F \setminus I_p$, (4.5) and the fact that $f$ is constant on the equivalence classes with respect to $\mathcal{J}$, one gets that:

$$\begin{aligned}
|A(f,g_p)| &\leq \frac{1}{2}\sum_{x\neq y} j_{x,y}|(f(x)-f(y))(g_p(x)-g_p(y))| \\
&= \sum_{j\neq p}\sum_{x\in I_p}\sum_{y\in I_j} j_{x,y}|c_p - c_j||g_p(x)| \\
&\leq \sqrt{C_{F/\mathcal{J}}}\sum_{x\in I_p}\left[|g_p(x)|\left(\sum_{j\neq p}\sum_{y\in I_j}\sqrt{j_{x,y}}|c_p-c_j|\right)\right]
\end{aligned}$$

Now, using two times the Cauchy-Schwartz inequality:

$$\begin{aligned}
|A(f,g_p)| &\leq \sqrt{C_{F/\mathcal{J}}}\left[\sum_{x\in I_p} g_p(x)^2\right]^{\frac{1}{2}}\left[\sum_{x\in I_p}\left(\sum_{j\neq p}\sum_{y\in I_j}\sqrt{j_{x,y}}|c_p-c_j|\right)^2\right]^{\frac{1}{2}} \\
&\leq \sqrt{C_{F/\mathcal{J}}}\left[\sum_{x\in I_p} g_p(x)^2\right]^{\frac{1}{2}}\left[\sum_{x\in I_p}\#(F\setminus I_p)\sum_{j\neq p}\sum_{y\in I_j}(c_p-c_j)^2 j_{x,y}\right]^{\frac{1}{2}} \\
&\leq (\#F)^{\frac{1}{2}}\sqrt{C_{F/\mathcal{J}}}\left[\sum_{x\in I_p} g_p(x)^2\right]^{\frac{1}{2}}\left[\frac{1}{2}\sum_{i\neq j}\sum_{x\in I_i}\sum_{y\in I_j} j_{x,y}(c_i-c_j)^2\right]^{\frac{1}{2}}
\end{aligned}$$

(N.B.: $\#A$ denotes the cardinal of the set $A$).
And we conclude, thanks to (4.6) that:

$$|A(f,g_p)| \leq (\#F)^{\frac{1}{2}} K_1 \delta_{\mathcal{J}}(A)^{\frac{1}{2}} A(g_p)^{\frac{1}{2}} A_{F/\mathcal{J}}(f)^{\frac{1}{2}}.$$

The first inequality for $g \in V_{\mathcal{J}}$ is easily deduced from this result by the Cauchy-Schwartz inequality.
Let us now prove the second inequality.

Let $p$ and $q$ be two distinct points of $\{1,\dots,k\}$. Let $g_p$ be in $V_{I_p}$ and $g_q$ be in $V_{I_q}$. Using (4.5) and the Cauchy-Schwartz inequality, one has that:

$$\begin{aligned}|A(g_p,g_q)| &\le \sum_{x\in I_p}\sum_{y\in I_q} j_{x,y}|g_p(x)g_q(y)|\\ &\le C_{F/\mathcal{J}}\sum_{x\in I_p}\sum_{y\in I_q}|g_p(x)g_q(y)|\\ &= C_{F/\mathcal{J}}\left(\sum_{x\in I_p}|g_p(x)|\right)\left(\sum_{y\in I_q}|g_q(y)|\right)\\ &\le C_{F/\mathcal{J}}(\#I_p)^{\frac12}(\#I_q)^{\frac12}\left(\sum_{x\in I_p}g_p(x)^2\right)^{\frac12}\left(\sum_{y\in I_q}g_q(y)^2\right)^{\frac12}\end{aligned}$$

We now can conclude, using (4.6), that:

$$|A(g_p,g_q)|\le \delta_{\mathcal{J}}(A)A(g_p)^{\frac12}A(g_q)^{\frac12}.\quad\square$$

*Proof of Corollary* 4.17. – Let $A$ be in $M$, $p$ in $\{1,\dots,k\}$ and $g_p$ in $V_{I_p}$. We know that

$$A(g_p)\ge A_{I_p}(g_p).$$

Let $\overline{g}_p$ be the harmonic continuation of $g_p$ from $I_p$ to $F$ with respect to the Dirichlet form $A$. Set $\phi_p=\overline{g}_p-g_p$. One has that:

$$\begin{aligned}A_{I_p}(g_p)=A(\overline{g}_p)&=A(g_p+\phi_p)\\ &=A(g_p)+A(\phi_p)+2A(g_p,\phi_p).\end{aligned}$$

But, since $g_p\in V_{I_p}$ and $\phi_p\in\oplus_{j\neq p}V_{I_j}$, one has from Proposition 4.16 that:

$$\begin{aligned}A_{I_p}(g_p)&\ge A(g_p)+A(\phi_p)-2K_2\delta_{\mathcal{J}}(A)\sqrt{A(g_p)A(\phi_p)}\\ &=A(g_p)\left(1+(\sqrt{\frac{A(\phi_p)}{A(g_p)}}-K_2\delta_{\mathcal{J}}(A))^2-K_2^2\delta_{\mathcal{J}}^2(A)\right)\\ &\ge A(g_p)(1-K_2^2\delta_{\mathcal{J}}^2(A))\end{aligned}$$

So, setting $C=K_2^2$, we have proved that:

$$A_{I_p}(g_p)\le A(g_p)\le(1-C\delta_{\mathcal{J}}^2(A))^{-1}A_{I_p}(g_p).\tag{4.7}$$

Let $g$ be in $V$. We decompose $g$ on $V=V_{F/\mathcal{J}}\oplus V_{I_1}\oplus\ldots\oplus V_{I_k}$, as:

$$g=f+g_1+\cdots+g_k,$$

with $f \in V_{F/\mathcal{J}}$, $g_i \in V_{I_i}$. We have:

$$A(g) = A_{F/\mathcal{J}}(f) + \sum_{i=1}^{k} A(g_i) + 2A(f, \sum_{i=1}^{k} g_i) + \sum_{i \neq j} A(g_i, g_j). \tag{4.8}$$

Using (4.6), Proposition 4.16, and (4.5) one obtains that:

$$\begin{aligned} A(g) &\geq A_{F/\mathcal{J}}(f) + \sum_{i=1}^{k} A_{I_i}(g_i) - 2K_2 \delta_{\mathcal{J}}(A)^{\frac{1}{2}} A_{F/\mathcal{J}}(f)^{\frac{1}{2}} (\sum_{i=1}^{k} A(g_i)^{\frac{1}{2}}) \\ &- K_2 \delta_{\mathcal{J}}(A) (\sum_{i=1}^{k} A(g_i)^{\frac{1}{2}}) (\sum_{j=1}^{k} A(g_j)^{\frac{1}{2}}) \\ &\geq (A_{F/\mathcal{J}}(f) + A_{\mathcal{J}}(g)) \left( 1 - 2K_2 k \left( \frac{\delta_{\mathcal{J}}(A)}{1 - C\delta_{\mathcal{J}}^2(A)} \right)^{\frac{1}{2}} - K_2 k \frac{\delta_{\mathcal{J}}(A)}{(1 - C\delta_{\mathcal{J}}^2(A))} \right) \end{aligned}$$

So, for $\delta_{\mathcal{J}}(A)$ small enough, one can find a constant $K_3$ such that the left inequality is true.

The proof of the right inequality is similar and left to the reader. □

### 4.2. Preserved $G$-relations, extension of $T$ to $M_{\mathcal{J}}$ and $M_{F/\mathcal{J}}$ and behaviour of $T$ near a $G$-relation

#### *4.2.1. Preserved G-relations*

Let $\mathcal{J}$ be a $G$-relation, and denote by $\{I_1, \cdots, I_k\} = F/\mathcal{J}$ the quotient set formed by $\mathcal{J}$. We recall that $F^{(1)}$ is defined by

$$F^{(1)} = \{1, \ldots, N\} \times F/\mathcal{R},$$

where $\mathcal{R}$ is the equivalence relation which describes the connections between the 1-cells.

With $\mathcal{J}$ we naturally associate an equivalence relation $\mathcal{J}^{(1)}$ on $F^{(1)}$ in the following way: in the same 1-cell, two points are in relation with respect to $\mathcal{J}^{(1)}$ if they are with respect to $\mathcal{J}$. Formally $\mathcal{J}^{(1)}$ is defined as the smallest equivalence relation on $F^{(1)}$ such that:

$$\forall x, y \in F, \ \forall i \in \{1, \ldots, N\}, \ \ (x\mathcal{J}y) \Rightarrow (\Psi_i(x)\mathcal{J}^{(1)}\Psi_i(y)).$$

The relation $\mathcal{J}^{(1)}$ is clearly $G$-symmetric for the operation of $G$ on $F^{(1)}$.

The maps $\Psi_i$ can be regarded as maps from $F/\mathcal{J}$ to $F^{(1)}/\mathcal{J}^{(1)}$.

The set $F^{(1)}/\mathcal{J}^{(1)}$ can be writen as:

$$F^{(1)}/\mathcal{J}^{(1)} = \{1, \ldots, N\} \times (F/\mathcal{J})/\mathcal{R},$$

where we still denote by $\mathcal{R}$ the relation on $\{1, \ldots, N\} \times (F/\mathcal{J})$, which is the image of the relation $\mathcal{R}$ on $\{1, \ldots, N\} \times F$ by the canonical projection (*i.e.* $(i, X)\mathcal{R}(j, Y)$ in $\{1, \ldots, N\} \times (F/\mathcal{J})$ if and only if there exist $x$ and $y$ in $F$, which are representatives of $X$ and $Y$ with respect to $\mathcal{J}$ and such that $(i, x)\mathcal{R}(j, y)$).

We denote by $(F/\mathcal{J})_i$ the 1-cell of $F^{(1)}/\mathcal{J}^{(1)}$ defined by $(F/\mathcal{J})_i = \Psi_i(F/\mathcal{J})$.

REMARK 4.18. – It is possible that two points of a same 1-cell $F_i$ are in relation with respect to $\mathcal{J}^{(1)}$ and are not the images by $\Psi_i$ of two points in relation with respect to $\mathcal{J}$. In this case the map $\Psi_i$ is not one-to-one on $F/\mathcal{J}$.

DEFINITION 4.19. – *With $\mathcal{J}$ we associate the $G$-relation $T(\mathcal{J})$ on $F$ in the following way:*

$$xT(\mathcal{J})y \Leftrightarrow x\mathcal{J}^{(1)}y \text{ in } F^{(1)}.$$

*The set $F/T(\mathcal{J})$ is then naturally included in $F^{(1)}/\mathcal{J}^{(1)}$ (as $F$ is included in $F^{(1)}$).*

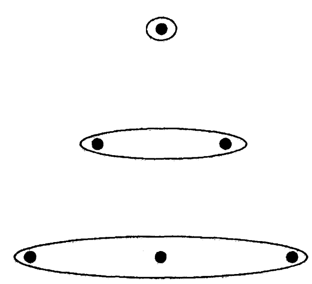
*We will say that the $G$-relation $\mathcal{J}$ is preserved if $T(\mathcal{J}) = \mathcal{J}$.*

N.B.: The relations 1 and 0 are always preserved.

REMARK 4.20. – By definition, a function $f \in E$ admits a continuation to $F^{(1)}$ which is constant on the equivalence classes with respect to $\mathcal{J}^{(1)}$ if and only if it is constant on equivalence classes with respect to $T(\mathcal{J})$ (*i.e.* $f \in E_{F/T(\mathcal{J})}$).

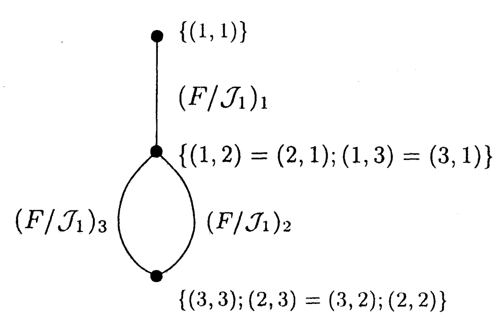
*Examples* 1.2. – In the case of the Sierpinski gasket we draw for $\mathcal{J}_1$, the relation $\mathcal{J}_1^{(1)}$

$\mathcal{J}_1^{(1)}$ :



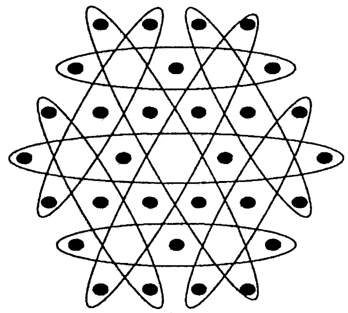
This $G$-relation is obviously preserved, and so are the two other non-trivial $G$-relations. We draw the set $F^{(1)}/\mathcal{J}^{(1)}$ by the following picture:

$F^{(1)}/\mathcal{J}_1^{(1)}$ :



*Example* 3. – In the case of the snowflake, $\mathcal{J}_1^{(1)}$ is drawn by the following picture:

$\mathcal{J}_1^{(1)}$ :

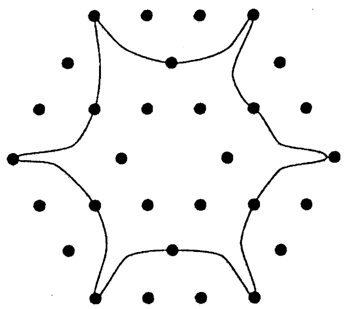


The relation $\mathcal{J}_1$ is then obviously preserved.

We do not describe the set $F^{(1)}/\mathcal{J}_1^{(1)}$. Nevertheless, it will be usefull to remark that all the points of $F$ are in relation with respect to $\mathcal{J}_1^{(1)}$ with a point of the central 1-cell $\Psi_7(F)$. The set $F/\mathcal{J}_1$, regarded as a subset of $F^{(1)}/\mathcal{J}_1^{(1)}$, will be thus contained in the central 1-cell $\Psi_7(F/\mathcal{J}_1)$.

The relation $\mathcal{J}_2^{(1)}$ is represented on the following picture:

$\mathcal{J}_2^{(1)}$ :



(we only represent on this picture the equivalence class of the points of $F$ : it contains the points linked on the picture)

One clearly has $T(\mathcal{J}_2) = 1$, so $\mathcal{J}_2$ is not preserved.

*4.2.2. Extension of $T$ to $M_{\mathcal{J}}$ and $M_{F/\mathcal{J}}$*

Let $\mathcal{J}$ be a non-trivial preserved $G$-relation and $\{I_1, \cdots, I_k\} = F/\mathcal{J}$ be the quotient set formed by $\mathcal{J}$.

Denote by $E_{F/\mathcal{J}}^{(1)}$ the space of real functions on $F^{(1)}/\mathcal{J}^{(1)}$ ($E_{F/\mathcal{J}}^{(1)}$ can naturally be considered as the subspace $E^{(1)}$ of functions constant on the equivalence classes with respect to $\mathcal{J}^{(1)}$).

We now define the extension of $T$ to $M_{\mathcal{J}}$.

Let $A$ be in $M_{\mathcal{J}}$, one defines $A^{(1)}$ on $F^{(1)}$ by:

$$A^{(1)}(f, f) = \sum_{i=1}^{N} \alpha_i^{-1} A(f \circ \Psi_i, f \circ \Psi_i), \ \ \forall f \in E^{(1)}.$$

The Dirichlet form $A^{(1)}$ is reducible on the relation $\mathcal{J}^{(1)}$ and then $A_F^{(1)}$, the restriction of $A^{(1)}$ to $F$ (in the meaning of Section 1.3), is reducible on the $G$-relation $T(\mathcal{J})$ (because $A_F^{(1)}(f) = 0$ if and only if $f$ admits a continuation in $E_{F/\mathcal{J}}^{(1)}$ and, from Remark 4.20, it does if and only if $f \in E_{F/T(\mathcal{J})}$)). We denote by $T_{\mathcal{J}}$ the following map:

$$\begin{aligned} T_{\mathcal{J}} \ : \ M_{\mathcal{J}} &\to M_{T(\mathcal{J})} \\ A &\to T_{\mathcal{J}} A = A_F^{(1)}. \end{aligned}$$

N.B: The definition of $T_{\mathcal{J}}$ is thus the same as the one of $T$ but on the set of reducible Dirichlet forms on $\mathcal{J}$.

N.B.: If $T(\mathcal{J}) = 0$ then $T_{\mathcal{J}} A$ is null for all $A$ in $M_{\mathcal{J}}$. If $T(\mathcal{J}) = 1$ then $T_{\mathcal{J}} A$ is irreducible for all $A \in M_{\mathcal{J}}$.

REMARK 4.21. – We recall, from Remark 1.12, that the harmonic continuation with respect to $A^{(1)}$ is not unique in general, but defined up to a function of $E_{F/\mathcal{J}}^{(1)} \cap \{f \in E,\ f_{|F} = 0\}$ (because $ker(A^{(1)}) = E_{F/\mathcal{J}}^{(1)}$).

We now define the map on the set $M_{F/\mathcal{J}}$.

If $A \in M_{F/\mathcal{J}}$ one defines $A^{(1)}$ on $F^{(1)}/\mathcal{J}^{(1)}$ by:

$$A^{(1)}(f,f) = \sum_{i=1}^{N} \alpha_i^{-1} A(f \circ \Psi_i, f \circ \Psi_i), \ \ \forall f \in E_{F/\mathcal{J}}^{(1)}.$$

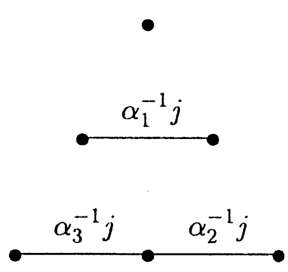
The set $F/T(\mathcal{J})$ is naturally included in $F^{(1)}/\mathcal{J}^{(1)}$ (because $F$ is included in $F^{(1)}$) and the restriction of $A^{(1)}$ to $F/T(\mathcal{J})$ is irreducible on $F/T(\mathcal{J})$, since $A^{(1)}$ is irreducible on $F^{(1)}/\mathcal{J}^{(1)}$. So, one defines:

$$\begin{aligned} T_{F/\mathcal{J}} : M_{F/\mathcal{J}} &\to M_{F/T(\mathcal{J})} \\ A &\to T_{F/\mathcal{J}} A = A_{F/T(\mathcal{J})}^{(1)}. \end{aligned}$$

REMARK 4.22. – We easily see that Proposition 3.2 and 3.3 remain true for $T_{\mathcal{J}}$ and $T_{F/\mathcal{J}}$.

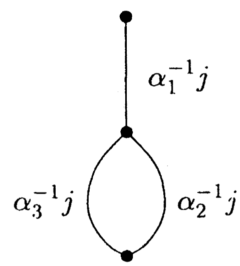
We now explicitely calculate the maps $T_{\mathcal{J}}$ and $T_{F/\mathcal{J}}$ for our two examples, *i.e.* the Sierpinski gasket and the snowflake.

*Examples* 1.2. – *The Sierpinski gasket.* We calculate $T_{\mathcal{J}}$ for $\mathcal{J} = \mathcal{J}_1$ (the other ones are clearly obtained by a permutation of indices). A Dirichlet form $A \in M_{\mathcal{J}_1}$ can be determinated by the bond conductivity between the two points in relation with respect to $\mathcal{J}_1$. We denote it by $j$. The electrical network on $F^{(1)}$ associated with $A^{(1)}$ is drawn by the following picture:



We easily check that if $\tilde{j}$ is the conductivity associated with $T_{\mathcal{J}}A$, then $\tilde{j} = (\alpha_2+\alpha_3)^{-1}j$ (it comes from the calculation of the effective conductance between the points 2 and 3, *cf.* Lemma 1.20).

A Dirichlet form $A \in M_{F/\mathcal{J}_1}$ is determinated by the conductivity between the two points of $F/\mathcal{J}_1$. We denote it by $j$. The electrical network associated with $A^{(1)}$ is represented by the following picture:



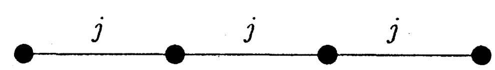

We easily check that if $\tilde{j}$ is the conductivity associated with $T_{F/\mathcal{J}_1}A$ then:

$$\tilde{j} = \left(\alpha_1 + \frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3}\right)^{-1} j,$$

(it comes from the calculation of effective conductance between the two points of $F/\mathcal{J}_1$, *cf.* Lemma 1.20).

*Example* 3. – *The snowflake*. We calculate $T_{\mathcal{J}}$ for $\mathcal{J} = \mathcal{J}_1$, the unique non-trivial preserved $G$-relation. We remind that we have choosen $\alpha_1 = \cdots = \alpha_7 = 1$.

In this case, using the symmetries induced by $G$, an element $A$ of $M_{\mathcal{J}_1}$ is determinated by the conductivity between two opposite points of $F$. We denote it by $j$. On a diagonal of $F^{(1)}$ the electrical network associated with $A^{(1)}$ is represented on the following picture (the other diagonals are similar by symmetry):




So, if $\tilde{j}$ is the conductivity associated with $T_{\mathcal{J}_1}A$ then $\tilde{j} = \frac{1}{3}j$.

We do not calculate $T_{F/\mathcal{J}_1}$, as we shall not need such precise information to apply Theorem 5.1.

### 4.2.3 *Behaviour of $T$ near a $G$-relation*

The following proposition is the main result of these preliminaries. Associated with Corollary 4.17, and with forthcoming Proposition 4.26 it gives an estimate of $TA$ in terms of $T_{\mathcal{J}}A_{\mathcal{J}}$ and $T_{F/\mathcal{J}}A_{F/\mathcal{J}}$ when $A$ is near the $G$-relation $\mathcal{J}$. It is a key result in the proof of Theorem 5.1 as it gives the behaviour of $T$ near the boundary of $M$. (It will be used in Lemma 5.7 to prove the repulsivity of the $G$-relations in case (ii) of Theorem 5.1).

PROPOSITION 4.23. – *There exist some constants $K_4 > 0$ and $\epsilon_4 > 0$ such that*

(i) *if $\mathcal{J}$ is a non-trivial G-relation and $A$ an element of $M_{F/\mathcal{J}}$, such that $T(\mathcal{J}) \neq 1$ and $\delta_{\mathcal{J}}(A) \leq \epsilon_4$, then:*

$$1 - K_4\delta_{\mathcal{J}}(A)^{\frac{1}{2}} \leq \inf\left(\frac{(TA)_{F/T(\mathcal{J})}}{T_{F/\mathcal{J}}A_{F/\mathcal{J}}}\right) \leq \sup\left(\frac{(TA)_{F/T(\mathcal{J})}}{T_{F/\mathcal{J}}A_{F/\mathcal{J}}}\right) \leq 1.$$

(ii) *if $\mathcal{J}$ is a non-trivial G-relation and $A$ an element of $M$, such that $T(\mathcal{J}) \neq 0$ and $\delta_{\mathcal{J}}(A) \leq \epsilon_4$, then:*

$$1 - K_4\delta_{\mathcal{J}}(A)^{\frac{1}{2}} \leq \inf\left(\frac{(TA)_{T(\mathcal{J})}}{T_{\mathcal{J}}A_{\mathcal{J}}}\right) \leq \sup\left(\frac{(TA)_{T(\mathcal{J})}}{T_{\mathcal{J}}A_{\mathcal{J}}}\right) \leq 1 + K_4\delta_{\mathcal{J}}(A)^{\frac{1}{2}}.$$

*Proof.* – We first construct a decomposition of the space $E^{(1)}$ that we shall use for both first and second estimates.

LEMMA 4.24. – *There exists a complementary subspace of $E^{(1)}_{F/\mathcal{J}}$ in $E^{(1)}$, denoted by $H$, such that all $f \in H$ satisfies $f_{|F} \in V_{T(\mathcal{J})}$.*

N.B.: We remind that $V_{T(\mathcal{J})}$ is the set of functions with zero mean on each equivalence class with respect to $T(\mathcal{J})$.

*Proof.* – Set

$$H' = \{f \in E^{(1)},\ \ f_{|F} \in V_{T(\mathcal{J})}\}.$$

One clearly has: $E^{(1)}_{F/\mathcal{J}} + H' = E^{(1)}$.

Let $H$ be a supplementary subspace of $E^{(1)}_{F/\mathcal{J}} \cap H'$ in $H'$. The subspace $H$ is then a supplementar of $E^{(1)}_{F/\mathcal{J}}$ in $E^{(1)}$, *i.e.*:

$$E^{(1)}_{F/\mathcal{J}} \oplus H = E^{(1)}.$$

Moreover, since $H \subset H'$, if $f \in H$ then $f_{|F} \in V_{T(\mathcal{J})}$ so $H$ satisfies the hypotheses of the lemma. □

LEMMA 4.25. – *There exists a constant $C > 0$ such that for all $f$ in $H$:*

$$\sum_{i=1}^{N} \alpha_i^{-1} A^0_{F/\mathcal{J}}(P_{F/\mathcal{J}}(f \circ \Psi_i)) \leq C \sum_{i=1}^{N} \alpha_i^{-1} A^0_{\mathcal{J}}(f \circ \Psi_i).$$

*Proof.* – It is clearly enough to prove that $\sum \alpha_i^{-1} A^0_{\mathcal{J}}(f \circ \Psi_i)$ is positive definite on $H$ and this comes from the fact that if $f \in H$ satisfies $A^0_{\mathcal{J}}(f \circ \Psi_i) = 0$ for all $i$ in $\{1, \ldots, N\}$ then $f \in E^{(1)}_{F/\mathcal{J}}$ and thus $f = 0$. □

We now give some notations we will use for both (i) and (ii).

Take a non-trivial $G$-relation $\mathcal{J}$, and denote by $\{I_1, \cdots, I_k\} = F/\mathcal{J}$ its quotient set.

For $A$ in $M$, we denote by $\overline{A}$, the quadratic form on $E$ defined by the following formula:

$$\overline{A}(f) = A_{F/\mathcal{J}}(P_{F/\mathcal{J}}(f)) + A_{\mathcal{J}}(f), \quad \forall f \in E.$$

The quadratic form $\overline{A}$ is positive definite on $V$ and null on constant functions. The subspaces $E_{F/\mathcal{J}}$, and $V_{I_i}$ are orthogonal with respect to $\overline{A}$. The Dirichlet form $\overline{A}$ is not always in $M$ as it may not satisfy the Markov property. Nevertheless, $\overline{A}^{(1)}$ can be defined as usual and we define $T\overline{A}$ to be the restriction of $\overline{A}^{(1)}$ to $F$ (*cf.* Remark 1.13). Since $\overline{A}$ is definite positive on $V$, the harmonic continuation of a function $f \in E$ with respect to $\overline{A}^{(1)}$ is unique, and still denoted by $H_{\overline{A}}$. It is also easy to see that Proposition 3.2 and Proposition 3.3 remain true for $\overline{A}$, even if it is not in $M$. Finally, we remind that $\overline{A}$ approximates $A$ near the $G$-relation $\mathcal{J}$ (*cf.* Corollary 4.17). In the proof of this proposition it is sometimes more convenient to use $\overline{A}$ instead of $A$, since the decomposition of $\overline{A}$ in terms of $A_{\mathcal{J}}$ and $A_{F/\mathcal{J}}$ is independant of $\overline{A}$ (the subspaces $E_{F/\mathcal{J}}$ and $V_{I_1}, \ldots, V_{I_k}$ are always orthogonal with respect to $\overline{A}$).

For $A$ in $M$ we set:

$$C_{F/\mathcal{J}}(A) = \sup_{x \not\mathcal{J} y} c_A(x, y),$$

$$C_{\mathcal{J}}(A) = \inf_{\substack{x \mathcal{J} y \\ x \neq y}} c_A(x, y)$$

(we write $C_{\mathcal{J}}$ and $C_{F/\mathcal{J}}$ when no ambiguity is possible).

From Lemma 4.14, we have the following two relations:

$$A_{F/\mathcal{J}}(f) \leq K_1 C_{F/\mathcal{J}} A^0_{F/\mathcal{J}}(f), \quad \forall f \in E_{F/\mathcal{J}}, \tag{4.9}$$

$$A_{\mathcal{J}}(f) \geq \frac{1}{K_1} C_{\mathcal{J}} A^0_{\mathcal{J}}(f), \quad \forall f \in E. \tag{4.10}$$

Let us now prove (i).

We suppose that $T(\mathcal{J}) \neq 1$. Let $A$ be in $M$. Let $f$ be in $E_{F/T(\mathcal{J})}$ and $\overline{f}$ in $E^{(1)}_{F/\mathcal{J}}$ be its harmonic continuation to $F^{(1)}/\mathcal{J}^{(1)}$ with respect to $A^{(1)}_{F/\mathcal{J}}$. Considering $\overline{f}$ as an element of $E^{(1)}$, $\overline{f}_{|F} = f$ and so

$$\begin{aligned} TA(f) &\leq \sum_{i=1}^{N} \alpha_i^{-1} A(\overline{f} \circ \Psi_i) \\ &= \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(\overline{f} \circ \Psi_i) \\ &= T_{F/\mathcal{J}} A_{F/\mathcal{J}}(f). \end{aligned}$$

So, we have proved the right inequality of (i), namely:

$$(TA)_{F/T(\mathcal{J})} \leq T_{F/\mathcal{J}} A_{F/\mathcal{J}}.$$

We first prove the left inequality for the quadratic form $\overline{A}$. Let $f$ be in $E_{F/T(\mathcal{J})}$ and $\overline{f} = H_{\overline{A}}(f) \in E^{(1)}$ be the harmonic continuation of $f$ with respect to $\overline{A}^{(1)}$.

Using Lemma 4.24, we write $\overline{f} = \overline{f}' + \overline{f}''$, with $\overline{f}' \in E^{(1)}_{F/\mathcal{J}}$, $\overline{f}'' \in H$. One has that:

$$T\overline{A}(f) = \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(P_{F/\mathcal{J}}((\overline{f}' + \overline{f}'') \circ \Psi_i)) + \sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(\overline{f}'' \circ \Psi_i). \tag{4.11}$$

(because $A_{\mathcal{J}}(\overline{f}' \circ \Psi_i) = 0$).

But, since $f \in E_{F/T(\mathcal{J})}$, using Lemma 4.24 we know that $f = P_{F/T(\mathcal{J})}(\overline{f}_{|F}) = P_{F/T(\mathcal{J})}(\overline{f}'_{|F}) = \overline{f}'_{|F}$, so that $\overline{f}'$ is a continuation of $f$. It follows that:

$$T\overline{A}(f) \le \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(\overline{f}' \circ \Psi_i), \tag{4.11}$$

Moreover $\overline{f}'$ is in $E^{(1)}_{F/\mathcal{J}}$, thus:

$$T_{F/\mathcal{J}} A_{F/\mathcal{J}}(f) \le \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(\overline{f}' \circ \Psi_i). \tag{4.13}$$

From (4.11) and (4.12) we deduce that:

$$\sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(\overline{f}'' \circ \Psi_i) \le \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(\overline{f}' \circ \Psi_i). \tag{4.14}$$

From (4.11) we also deduce that:

$$T\overline{A}(f) \ge \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(P_{F/\mathcal{J}}((\overline{f}' + \overline{f}'') \circ \Psi_i))) \tag{4.15}$$

We want to prove that the last right term is not far from the same one with only $\overline{f}'$. It remains to prove that $\overline{f}''$ is small compared to $\overline{f}'$: when $\delta_{\mathcal{J}}(A)$ is small then $A_{\mathcal{J}}$ is big compared to $A_{F/\mathcal{J}}$ (*cf.* Lemma 4.14 and Remark 4.15), so (4.14) implies that $\overline{f}''$ is small compared to $\overline{f}'$.

Precisely, using (4.9), Lemma 4.25, and (4.10) we obtain that:

$$\begin{aligned}
\sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(P_{F/\mathcal{J}}((\overline{f}'') \circ \Psi_i)) &\le K_1 C_{F/\mathcal{J}} \sum_{i=1}^{N} \alpha_i^{-1} A^0_{F/\mathcal{J}}(P_{F/\mathcal{J}}((\overline{f}'') \circ \Psi_i)) \\
&\le C K_1 C_{F/\mathcal{J}} \sum_{i=1}^{N} \alpha_i^{-1} A^0_{\mathcal{J}}(\overline{f}'' \circ \Psi_i) \\
&\le C K_1^2 \delta_{\mathcal{J}}(A) \sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(\overline{f}'' \circ \Psi_i).
\end{aligned}$$

Then, using (4.14), one has that:

$$\sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(P_{F/\mathcal{J}}((\overline{f}'') \circ \Psi_i)) \leq CK_1^2 \delta_{\mathcal{J}}(A) \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(\overline{f}' \circ \Psi_i). \tag{4.16}$$

Using (4.15), (4.16) and (4.14):

$$\begin{aligned} T\overline{A}(f) &\geq \left( \left[ \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(\overline{f}' \circ \Psi_i) \right]^{\frac{1}{2}} - \left[ \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(P_{F/\mathcal{J}}(\overline{f}'' \circ \Psi_i)) \right]^{\frac{1}{2}} \right)^2 \\ &\geq (1 - 2(CK_1^2 \delta_{\mathcal{J}}(A))^{\frac{1}{2}}) \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(\overline{f}' \circ \Psi_i) \\ &\geq (1 - 2(CK_1^2 \delta_{\mathcal{J}}(A))^{\frac{1}{2}}) T_{F/\mathcal{J}} A_{F/\mathcal{J}}(f) \end{aligned}$$

This proves the left inequality with $\overline{A}$ instead of $A$. The inequality with $A$ follows easily from Corollary 4.17 and Proposition 3.3 applied to $\overline{A}$ and $A$.

(ii) Here we use $\overline{A}$. More precisely we first prove:

$$1 \leq \frac{(T\overline{A})}{T_{\mathcal{J}} A_{\mathcal{J}}} \leq (1 + CK_1^2 \delta_{\mathcal{J}}(A)). \tag{4.17}$$

Denote by $\{I_1', \ldots, I_{k'}'\} = F/T(\mathcal{J})$, the quotient set formed by $T(\mathcal{J})$. Each set $I_i'$ is included in an equivalence classe with respect to $\mathcal{J}^{(1)}$ that we denote by $(I_i')^{(1)}$ (because $T(\mathcal{J})$ is the restriction of $\mathcal{J}^{(1)}$ to $F$).

Let $f$ be in $V_{I_i'}$ (*i.e.* $f$ has zero mean on $F$ and is null out of $I_i'$). Let $\tilde{f} \in E$ be the harmonic continuation of $f_{|I_i'}$ with respect to $T\overline{A}$. Let $\tilde{f}^{(1)} = H_{\overline{A}}(\tilde{f})$ be the harmonic continuation of $\tilde{f}$ with respect to $\overline{A}^{(1)}$. We define $f^{(1)}$ by:

$$f^{(1)} = \begin{cases} \tilde{f}^{(1)}(x) & \text{if } x \in (I_i')^{(1)}, \\ 0 & \text{if } x \notin (I_i')^{(1)}. \end{cases}$$

The function $f^{(1)}$ is naturally a continuation of $f$ (since the set $I_i'$ is included in $(I_i')^{(1)}$). So, one has that:

$$\begin{aligned} (T\overline{A})_{\mathcal{J}}(f) &= T\overline{A}(\tilde{f}) \\ &= \sum_{i=1}^{N} \alpha_i^{-1} \overline{A}(\tilde{f}^{(1)} \circ \Psi_i) \\ &= \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}((P_{F/\mathcal{J}}(\tilde{f}^{(1)} \circ \Psi_i))) + \sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(\tilde{f}^{(1)} \circ \Psi_i) \\ &\geq \sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(\tilde{f}^{(1)} \circ \Psi_i). \end{aligned}$$

But the subspaces $V_{I_i}$ are orthogonal with respect to $A_{\mathcal{J}}$, so for all $i \in \{1,\ldots,N\}$ one has $A_{\mathcal{J}}(\tilde{f}^{(1)} \circ \Psi_i) \geq A_{\mathcal{J}}(f^{(1)} \circ \Psi_i)$, and the last inequality becomes:

$$\begin{aligned}(T\overline{A})_{\mathcal{J}}(f) &\geq \sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(f^{(1)} \circ \Psi_i)\\ &\geq T_{\mathcal{J}} A_{\mathcal{J}}(f).\end{aligned}$$

Since the subspaces $V_{I'_i}$ are orthogonal for both quadratic forms $(T\overline{A})_{\mathcal{J}}$ and $T_{\mathcal{J}}A_{\mathcal{J}}$, we have proved that:

$$(T\overline{A})_{\mathcal{J}} \geq T_{\mathcal{J}} A_{\mathcal{J}}.$$

REMARK 4.26. – It is not true in general that $(TA)_{\mathcal{J}} \geq T_{\mathcal{J}}A_{\mathcal{J}}$. Nevertheless, if we suppose that there do not exist two points in a same 1-cell in relation with respect to $\mathcal{J}^{(1)}$ and not with respect to $\mathcal{J}$ (*cf.* Remark 4.18) then we can prove the above relation. Indeed, in this case, the previous proof remains true for $A$ instead of $\overline{A}$ since we can prove that $A(\tilde{f}^{(1)} \circ \Psi_i) \geq A_{\mathcal{J}}(f^{(1)} \circ \Psi_i)$ for all $i$ because, either $f^{(1)} \circ \Psi_i$ is null on $F$, or it is null out of an equivalence class $I_p \in F/\mathcal{J}$ and then $\tilde{f}^{(1)} \circ \Psi_i$ is a continuation of $f^{(1)} \circ \Psi_i$ from $I_p$ to $F$. This remark will allow us to improve a little Theorem 5.1 in this case.

We now prove the right inequality of (4.17).

Let again $i$ be in $\{1,\ldots,k'\}$ and $f$ in $V_{I'_i}$.

Since $A^{(1)}_{F/\mathcal{J}}$ is null on the space $E^{(1)}_{F/\mathcal{J}}$, using Lemma 4.24, one can find $f^{(1)}$ in $H$ such that $f^{(1)}_{|F} = f$ and $T_{\mathcal{J}}A_{\mathcal{J}}(f) = A^{(1)}_{\mathcal{J}}(f^{(1)})$ (*i.e.* $f^{(1)}$ is a harmonic continuation of $f$ with respect to $A^{(1)}_{\mathcal{J}}$, *cf.* Remark 4.21).

It follows that:

$$T_{\mathcal{J}}A_{\mathcal{J}}(f) = \sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(f^{(1)} \circ \Psi_i). \tag{4.18}$$

Besides, one has that:

$$\begin{aligned}(T\overline{A})_{\mathcal{J}}(f) &\leq T\overline{A}(f)\\ &\leq \overline{A}^{(1)}(f^{(1)})\\ &= \sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(P_{F/\mathcal{J}}(f^{(1)} \circ \Psi_i)) + \sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(f^{(1)} \circ \Psi_i).\end{aligned} \tag{4.19}$$

We now want to prove that the contribution of the term $A_{F/\mathcal{J}}$ is small compared to the one of the term $A_{\mathcal{J}}$. But (4.9), Lemma 4.25 and (4.10) imply that:

$$\begin{aligned}\sum_{i=1}^{N} \alpha_i^{-1} A_{F/\mathcal{J}}(P_{F/\mathcal{J}}(f^{(1)} \circ \Psi_i)) &\leq K_1 C_{F/\mathcal{J}} \sum_{i=1}^{N} \alpha_i^{-1} A^0_{F/\mathcal{J}}(P_{F/\mathcal{J}}(\overline{f}^{(1)} \circ \Psi_i))\\ &\leq CK_1 C_{F/\mathcal{J}} \sum_{i=1}^{N} \alpha_i^{-1} A^0_{\mathcal{J}}(f^{(1)} \circ \Psi_i)\\ &\leq CK_1^2 \delta_{\mathcal{J}}(A) \sum_{i=1}^{N} \alpha_i^{-1} A_{\mathcal{J}}(f^{(1)} \circ \Psi_i)\end{aligned}$$

Thanks to (4.18), the previous inequality applied to (4.19) gives that:

$$\begin{aligned}(T\overline{A})_{\mathcal{J}}(f) &\leq (1 + CK_1^2\delta_{\mathcal{J}}(A))\sum_{i=1}^{N}\alpha_i^{-1}A_{\mathcal{J}}(f^{(1)}\circ\Psi_i)\\ &= (1 + CK_1^2\delta_{\mathcal{J}}(A))T_{\mathcal{J}}A_{\mathcal{J}}(f)\end{aligned}$$

And, since the subspaces $V_{I_i'}$ are orthogonal with respect to both $(T\overline{A})_{\mathcal{J}}$ and $T_{\mathcal{J}}A_{\mathcal{J}}$, the inequality is true for any $f \in E$.

Finally, we apply Corollary 4.17 and Proposition 3.3 to the quadratic forms $\overline{A}$ and $A$ and we deduce (ii) from (4.17). □

The following corollary means that if $A$ is close to the reducibility $\mathcal{J}$ then so is $TA$.

COROLLARY 4.27. – *There exist constants $K_5 > 0$ and $\epsilon_5 > 0$ such that for all non-trivial $G$-relations $\mathcal{J}$ and all $A \in M$ such that $T(\mathcal{J})$ is non-trivial and $\delta_{\mathcal{J}}(A) \leq \epsilon_5$, one has that:*

$$\delta_{T(\mathcal{J})}(TA) \leq K_5\delta_{\mathcal{J}}(A).$$

*Proof.* – It is an easy application of Proposition 4.23. Indeed, let $\mathcal{J}$ be a non-trivial $G$-relation, applying Proposition 4.23 and Lemma 4.14 one obtains that for all $A \in M$ such that $\delta_{\mathcal{J}}(A) \leq \epsilon_4 \wedge 1/4K_4^2$:

$$(TA)_{T(\mathcal{J})}(f') \geq (1 - \frac{1}{2})T_{\mathcal{J}}A_{\mathcal{J}}(f') \geq \frac{1}{2}\frac{1}{K_1}\left(\inf_{\substack{x\mathcal{J}y\\ x\neq y}} c_A(x,y)\right)T_{\mathcal{J}}A_{\mathcal{J}}^0(f'),\ \ \forall f' \in E,$$

$$(TA)_{F/T(\mathcal{J})}(f'') \leq T_{F/\mathcal{J}}A_{F/\mathcal{J}}(f'') \leq K_1\left(\sup_{x \not{\mathcal{J}} y} c_A(x,y)\right)T_{F/\mathcal{J}}A_{F/\mathcal{J}}^0(f''),\ \ \forall f'' \in E_{F/\mathcal{J}}.$$

Now, using that $c_A(x,y) \leq c_{A_{F/\mathcal{J}}}(X,Y)$ for any $x$ and $y$ representatives of $X$ and $Y$ in $F/T(\mathcal{J})$ (*cf.* the beginning of the proof of Lemma 4.14) one has that for all $A \in M$ satisfying $\delta_{\mathcal{J}}(A) \leq \frac{1}{4K_4^2} \wedge \epsilon_4$:

$$\begin{aligned}\delta_{T(\mathcal{J})}(TA) &\leq \frac{\sup_{X,Y\in F/T(\mathcal{J})} c_{(TA)_{F/T(\mathcal{J})}}(X,Y)}{\inf_{\substack{xT(\mathcal{J})y\\ x\neq y}} c_{(TA)_{T(\mathcal{J})}}(x,y)}\\ &\leq 2K_1^2\left(\frac{\sup_{X,Y\in F/T(\mathcal{J})} c_{T_{F/\mathcal{J}}A^0_{F/\mathcal{J}}}(X,Y)}{\inf_{\substack{xT(\mathcal{J})y\\ x\neq y}} c_{T_{\mathcal{J}}A^0_{\mathcal{J}}}(x,y)}\right)\delta_{\mathcal{J}}(A),\end{aligned}$$

and this concludes the proof. □

### 4.3. Maximum and minimum value of the ratio $TA/A$

*4.3.1. Notation*

For $A \in M$ one defines:

$$\underline{\rho}(A) = \inf \frac{TA}{A},$$

$$\overline{\rho}(A) = \sup \frac{TA}{A}.$$

REMARK 4.28. – $A$ is a fixed point of $T$ if and only if $\underline{\rho}(A) = \overline{\rho}(A)$.

For a non-trivial preserved $G$-relation $\mathcal{J}$, and for $A \in M_{\mathcal{J}}$, one denotes the same values associated with the map $T_{\mathcal{J}}$ by $\underline{\rho}_{\mathcal{J}}(A)$ and $\overline{\rho}_{\mathcal{J}}(A)$.

For $A \in M_{F/\mathcal{J}}$, we also denote the same values associated with the map $T_{F/\mathcal{J}}$ by $\underline{\rho}_{F/\mathcal{J}}(A)$ and $\overline{\rho}_{F/\mathcal{J}}(A)$.

Set:

$$\underline{\rho} = \sup_{A \in M} \underline{\rho}(A),$$

$$\overline{\rho} = \inf_{A \in M} \overline{\rho}(A),$$

and $\underline{\rho}_{\mathcal{J}}$, $\overline{\rho}_{\mathcal{J}}$ associated with $T_{\mathcal{J}}$ and $\underline{\rho}_{F/\mathcal{J}}$, $\overline{\rho}_{F/\mathcal{J}}$ associated with $T_{F/\mathcal{J}}$.

The following relations are simple:

PROPOSITION 4.29. – (i) *For all $A, A' \in M$, one has: $\underline{\rho}(A) \leq \overline{\rho}(A')$.*

(ii) $\underline{\rho} \leq \overline{\rho}$.

*Proof.* – (i) One has that $TA \geq \underline{\rho}(A)A$ and $TA' \leq \overline{\rho}(A')A'$, thus,

$$\sup \frac{TA}{TA'} \geq \frac{\underline{\rho}(A)}{\overline{\rho}(A')} \sup \frac{A}{A'}.$$

Proposition 3.3 implies that $\underline{\rho}(A) \leq \overline{\rho}(A')$.

(ii) comes from (i). □

PROPOSITION 4.30. – *For all $A \in M$:*

$$\underline{\rho}(TA) \geq \underline{\rho}(A),$$

$$\overline{\rho}(TA) \leq \overline{\rho}(A).$$

REMARK 4.31. – Propositions 4.29 and 4.30 are also true for the expressions associated with $T_{\mathcal{J}}$ and $T_{F/\mathcal{J}}$ for a non-trivial preserved $G$-relation $\mathcal{J}$.

*Proof.* – By definition $TA \geq \underline{\rho}(A)A$, so $T(TA) \geq \underline{\rho}(A)TA$, since $T$ is non-decreasing, and it follows that $\underline{\rho}(TA) \geq \underline{\rho}(A)$. The proof of the second inequality is strictly similar. □

*4.3.2. Behaviour near a G-relation*

The first proposition will be usefull in the proof of non-existence ((i) of Theorem 5.1).

PROPOSITION 4.32. – *Let $\mathcal{J}$ be a non-trivial preserved G-relation. For all $A \in M$ one has that:*

$$(i) \quad \underline{\rho}(A) \leq \underline{\rho}_{F/\mathcal{J}}(A_{F/\mathcal{J}}),$$
$$(ii) \quad \overline{\rho}(A) \geq \underline{\rho}_{\mathcal{J}}.$$

REMARK 4.33. – The result (ii) is weaker than (i): this comes from the already mentioned fact that the relation $T_{\mathcal{J}} A_{\mathcal{J}} \leq (TA)_{\mathcal{J}}$ is not true in general since the subspaces $V_{I_i}$ are not orthogonal with respect to $A$ (*cf.* the proof of Proposition 4.23 and Remark 4.26). Nevertheless, as we mentionned in Remark 4.26, this inequality is true if a certain condition is satisfied and in this case it leads to a stronger result in (ii), namely that $\overline{\rho}(A) \geq \overline{\rho}_{\mathcal{J}}(A_{\mathcal{J}})$.

*Proof.* – Let $\mathcal{J}$ be a non-trivial preserved $G$-relation and $A$ be in $M$.

Let us first prove (i). We recall from Proposition 4.23 the following relation:

$$T_{F/\mathcal{J}} A_{F/\mathcal{J}} \geq (TA)_{F/\mathcal{J}}.$$

Thus we have that:

$$T_{F/\mathcal{J}} A_{F/\mathcal{J}} \geq \underline{\rho}(A) A_{F/\mathcal{J}}$$

and then $\underline{\rho}_{F/\mathcal{J}}(A_{F/\mathcal{J}}) \geq \underline{\rho}(A)$.

(ii) In the proof of (ii) some technicalities are induced by the fact that the relation $T_{\mathcal{J}} A_{\mathcal{J}} \leq (TA)_{\mathcal{J}}$ is not true in general. When this relation is true the proof of the stronger result (*cf.* Remark 4.33) is the natural counterpart of the proof of (i).

Let $\overline{P}_{F/\mathcal{J}}$ be the orthogonal projection over $E_{F/\mathcal{J}}$ with respect to the quadratic form $A$ (actually, since $A$ is positive definite on $V$ we can define $\overline{P}_{F/\mathcal{J}}$ on $V$ as the projection over $V_{F/\mathcal{J}} = E_{F/\mathcal{J}} \cap V$, and for $f \in E$ we write $f = c + g$, with $c \in \mathbb{R}$ and $g \in V$, and set $\overline{P}_{F/\mathcal{J}} f = c + \overline{P}_{F/\mathcal{J}} g$). In general we have $\overline{P}_{F/\mathcal{J}} \neq P_{F/\mathcal{J}}$.

Let us define the quadratic form $A_1$ by:

$$A_1(f) = A(f - \overline{P}_{F/\mathcal{J}}(f)), \quad f \in E.$$

It follows that:

$$A(f) = A_{F/\mathcal{J}}(\overline{P}_{F/\mathcal{J}} f) + A_1(f), \quad \forall f \in E,$$

and thus,

$$A_1(f) \leq A(f), \quad \forall f \in E. \tag{4.20}$$

The quadratic form $A_1$ is null on $E_{F/\mathcal{J}}$ and positive definite on $V_{\mathcal{J}}$ (which is a supplementary subspace of $E_{F/\mathcal{J}}$). In general $A_1 \notin M_{\mathcal{J}}$ since $A_1$ may be non-markovian. Nevertheless, we can define $T_{\mathcal{J}} A_1$ and $\underline{\rho}_{\mathcal{J}}(A)$ as we would do if $A_1$ was in $M_{\mathcal{J}}$. The quadratic form $T_{\mathcal{J}}A_1$ is then null on $E_{F/\mathcal{J}}$ and definite on $V_{\mathcal{J}}$ (we remind that $\mathcal{J}$ is preserved).

From (4.20) and the definitions of $T$ and $T_{\mathcal{J}}$, we easily deduce that:

$$T_{\mathcal{J}}A_1 \leq TA \leq \overline{\rho}(A)A,$$

But, for all $f \in E$:

$$\begin{aligned} T_{\mathcal{J}}A_1(f) &= T_{\mathcal{J}}A_1(f - \overline{P}_{F/\mathcal{J}}f) \\ &\leq \overline{\rho}(A)A(f - \overline{P}_{F/\mathcal{J}}f) \\ &= \overline{\rho}(A)A_1(f) \end{aligned}$$

This implies that:

$$\overline{\rho}_{\mathcal{J}}(A_1) \leq \overline{\rho}(A). \tag{4.21}$$

If $B$ is in $M_{\mathcal{J}}$ then one has that:

$$\inf \frac{T_{\mathcal{J}}A_1}{T_{\mathcal{J}}B} \leq \left(\frac{\overline{\rho}_{\mathcal{J}}(A_1)}{\underline{\rho}_{\mathcal{J}}(B)}\right) \inf \frac{A_1}{B}.$$

Thus, Proposition 3.3 adapted to $T_{\mathcal{J}}$ and applied to $A_1$ and $B$ (obviously, Proposition 3.3 remains true even if $A_1$ is not markovian) gives that $\underline{\rho}_{\mathcal{J}}(B) \leq \overline{\rho}_{\mathcal{J}}(A_1)$. This is true for all $B \in M_{\mathcal{J}}$, thus $\overline{\rho}_{\mathcal{J}}(A_1) \geq \underline{\rho}_{\mathcal{J}}$. This proves the lemma with (4.21). □

The next lemma gives the behaviour of $\underline{\rho}(A)$ and $\overline{\rho}(A)$ near a non-preserved $G$-relation.

PROPOSITION 4.34. – *There exist some constants $K_6 > 0$ and $\epsilon_6 > 0$ such that for all non-trivial G-relation $\mathcal{J}$*

(i) *If $T(\mathcal{J}) \not\subset \mathcal{J}$, then for all $A \in M$ such that $\delta_{\mathcal{J}}(A) \leq \epsilon_6$:*

$$\overline{\rho}(A) \geq K_6(\delta_{\mathcal{J}}(A))^{-1},$$

(ii) *If $T(\mathcal{J}) \not\supset \mathcal{J}$, then for all $A \in M$ such that $\delta_{\mathcal{J}}(A) \leq \epsilon_6$:*

$$\underline{\rho}(A) \leq \frac{1}{K_6}\delta_{\mathcal{J}}(A).$$

REMARK 4.35. – In particular, it implies that $\theta(A) = \overline{\rho}(A)/\underline{\rho}(A)$ goes to infinity near a non-preserved $G$-relation.

*Proof.* – (i) Let $\mathcal{J}$ be such that $T(\mathcal{J}) \not\subset \mathcal{J}$. Denote by $\{I'_1, \ldots, I'_{k'}\} = F/T(\mathcal{J})$ the quotient set formed by $T(\mathcal{J})$.

Let $x$ and $y$ be two points of $F$ such that $xT(\mathcal{J})y$ and $x \not{\mathcal{J}} y$.

Set $f = 1_{\{x\}} - 1_{\{y\}}$. We can suppose that $x$ and $y$ are in $I'_1$, so that $f \in V_{I'_1}$ and $f \in E_{F/\mathcal{J}}$. Using Proposition 4.23 and Lemma 4.14, one has that for all $A \in M$ such that $\delta_{\mathcal{J}}(A) \leq \epsilon_4 \wedge \frac{1}{4K_4^2}$:

$$\begin{aligned}\overline{\rho}(A) \geq \frac{TA(f)}{A(f)} &\geq \frac{(TA)_{T(\mathcal{J})}(f)}{A_{F/\mathcal{J}}(f)}\\ &\geq \left(1-\frac{1}{2}\right)\frac{T_{\mathcal{J}}A_{\mathcal{J}}(f)}{A_{F/\mathcal{J}}(f)}\\ &\geq \frac{1}{2}K_1^{-2}(\delta_{\mathcal{J}}(A))^{-1}\left(\frac{T_{\mathcal{J}}A^0_{\mathcal{J}}(f)}{A^0_{F/\mathcal{J}}(f)}\right)\end{aligned}$$

This proves the lemma, as $f$ depends only on $\mathcal{J}$.

(ii) the proof is similar and left to the reader. □

## 5. Statement and proof of the main theorem. Application to the Sierpinski gasket without symmetry and to the snowflake.

### 5.1. Statement of Theorem 5.1

Denote the following assumption by (H):

(H) There do not exist two strictly ordered non-trivial preserved $G$-relations (*i.e.* $\mathcal{J} \neq \mathcal{J}'$ such that $\mathcal{J} \subset \mathcal{J}'$).

The following theorem is the main result of the paper:

THEOREM 5.1. – (i) *If there exist two non-trivial preserved $G$-relations $\mathcal{J}$ and $\mathcal{J}'$ such that $\underline{\rho}_{F/\mathcal{J}} < \underline{\rho}_{\mathcal{J}'}$, then $T$ has no fixed point.*

(ii) *If for all non-trivial preserved $G$-relation $\mathcal{J}$, the inequality $\overline{\rho}_{\mathcal{J}} < \underline{\rho}_{F/\mathcal{J}}$ is satisfied then $T$ has at most one fixed point (up to a multiplicative constant). If moreover (H) is satisfied then $T$ has exactly one fixed point (up to a multiplicative constant).*

N.B.: in (i) $\mathcal{J}$ and $\mathcal{J}'$ can be equal.

REMARK 5.2. – Condition (H) is a simplifying condition, and there are some reasons to think that it is not necessary. It is fullfilled in all the classical examples.

REMARK 5.3. – If the condition of Remark 4.26 is satisfied then we can replace in (i) the inequality $\underline{\rho}_{F/\mathcal{J}} < \underline{\rho}_{\mathcal{J}'}$ by $\underline{\rho}_{F/\mathcal{J}} < \overline{\rho}_{\mathcal{J}'}$: indeed the proof of (i) is a direct application of Proposition 4.32 and the improvement proposed in Remark 4.33 leads to this stronger result. It is interesting to remark that in this case, Theorem 5.1 gives a solution to the problem of existence and uniqueness as soon as we are not in the critical case $\underline{\rho}_{F/\mathcal{J}} = \overline{\rho}_{\mathcal{J}}$ for one preserved $G$-relation $\mathcal{J}$.

### 5.2. Application to the Sierpinski gasket without symmetry and to the snowflake

We now apply this theorem to the Sierpinski gasket without symmetry and to the snowflake (the two examples we follow since the beginning of Section 4).

*Example* 1.2. – *The Sierpinski gasket.* In this case we have calculated $T_{\mathcal{J}_1}$ and $T_{F/\mathcal{J}_1}$.

We found $T_{\mathcal{J}_1} = (\alpha_2 + \alpha_3)^{-1} Id$ (where $Id$ is identity), thus it follows that $\underline{\rho}_{\mathcal{J}_1} = \overline{\rho}_{\mathcal{J}_1} = (\alpha_2 + \alpha_3)^{-1}$. Denote by $\rho_{\mathcal{J}_1}$ this last value.

We found $T_{F/\mathcal{J}_1} = (\alpha_1 + \frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3})^{-1} Id$, thus $\underline{\rho}_{F/\mathcal{J}_1} = \overline{\rho}_{F/\mathcal{J}_1} = (\alpha_1 + \frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3})^{-1}$. Denote by $\rho_{F/\mathcal{J}_1}$ this last value.

The expressions for $\mathcal{J}_2$ and $\mathcal{J}_3$ are deduced by a permutation of indices.

It is easy to see that $\rho_{F/\mathcal{J}_i} \leq \rho_{\mathcal{J}_i}$ implies that $\alpha_i > \sup_{j \neq i} \alpha_j$.

So, if we suppose that $\alpha_1$ is bigger than $\alpha_2$ and $\alpha_3$, then $\rho_{\mathcal{J}_i} < \rho_{F/\mathcal{J}_i}$ for all $i$ is equivalent to $\rho_{\mathcal{J}_1} < \rho_{F/\mathcal{J}_1}$, *i.e.* to $\alpha_1 < \frac{\alpha_2^2+\alpha_3^2+\alpha_2\alpha_3}{\alpha_2+\alpha_3}$. Consequently:

If $\alpha_1 > \frac{\alpha_2^2+\alpha_3^2+\alpha_2\alpha_3}{\alpha_2+\alpha_3}$ then we are in the case (i) of Theorem 5.1.

If $\alpha_1 < \frac{\alpha_2^2+\alpha_3^2+\alpha_2\alpha_3}{\alpha_2+\alpha_3}$ then we are in the case (ii) of Theorem 5.1.

In this example it is possible to solve the problem in the case of equality (*i.e.* when $\rho_{\mathcal{J}_1} = \rho_{F/\mathcal{J}_1}$), we prove that $T$ has no fixed point.

This is based on the fact that the inequality in (i) of Proposition 4.32 can be proved to be a strict inequality. It implies that if $\rho_{\mathcal{J}_1} = \rho_{F/\mathcal{J}_1}$ and if $A$ is a fixed point of $T$ then

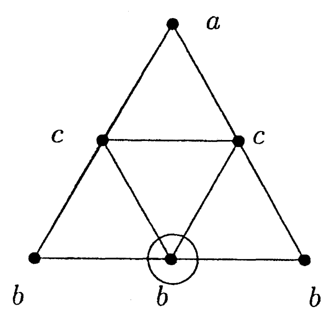
$$\underline{\rho}(A) < \rho_{F/\mathcal{J}_1} = \rho_{\mathcal{J}_1} \leq \overline{\rho}(A),$$

but this is impossible since $\underline{\rho}(A) = \overline{\rho}(A)$ for a fixed point.

We now prove that (i) of Proposition 4.32 is actually a strict inequality in this example.

Let $A$ be in $M$ and $f$ be a non-constant function of $E_{F/\mathcal{J}_1}$. The function $f$ takes two distinct values, say $a$ on the point 1 and $b$ on the points 2 and 3.

Let $f^{(1)}$ be the harmonic continuation of $f$ to $F^{(1)}/\mathcal{J}_1^{(1)}$ with respect to $A^{(1)}_{F/\mathcal{J}_1}$. As $f^{(1)}$ is constant on the equivalence classes with respect to $\mathcal{J}_1^{(1)}$ (*cf.* the picture of $\mathcal{J}_1^{(1)}$ in Section 4.1) it takes 3 values $a$, $b$ and $c$ as on the following picture:




One has that $T_{F/\mathcal{J}_1} A_{F/\mathcal{J}_1}(f) = A^{(1)}_{F/\mathcal{J}_1}(f^{(1)})$ and $T_{F/\mathcal{J}_1} A_{F/\mathcal{J}_1}(f) \geq TA(f)$, and the last inequality is strict if and only if $f^{(1)}$ is not the harmonic continuation of $f$ with respect to $A^{(1)}$.

But, since $A$ is irreducible, it has at least two strictly positive bond conductivities. So, if $X_n$ is the random walk associated with $A$ as in Section 1.2, then there exists a path, with strictly positive probability, going from the circled point to the point 1 (with value

$a$) without passing through the points 2 or 3. It follows from Proposition 1.15, that the harmonic continuation of $f$ with respect to $A$ cannot take the value $b$ on the circled point.

Thus, we have proved that $TA(f) < T_{F/\mathcal{J}_1} A_{F/\mathcal{J}_1}(f) = \rho_{F/\mathcal{J}_1} A_{F/\mathcal{J}_1}(f)$ for all non-constant function $f \in E_{F/\mathcal{J}_1}$. It implies that $\overline{\rho}(A) < \overline{\rho}_{F/\mathcal{J}_1} = \rho_{F/\mathcal{J}_1}$.

Moreover, from Proposition 3.8, all the fixed points of $T$ are regular, we thus can conclude from Proposition 2.11 by the following corollary:

COROLLARY 5.4. – *Suppose that $\alpha_1$ is bigger than $\alpha_2$ and $\alpha_3$:*

- *If $\alpha_1 < \frac{\alpha_2^2+\alpha_3^2+\alpha_2\alpha_3}{\alpha_2+\alpha_3}$ then there exists a unique (up to a multiplicative constant) self-similar Dirichlet form on $X$ associated with some weights proportional to $(\alpha_1, \alpha_2, \alpha_3)$.*
- *If $\alpha_1 \geq \frac{\alpha_2^2+\alpha_3^2+\alpha_2\alpha_3}{\alpha_2+\alpha_3}$, then there does not exist any self-similar Dirichlet form associated with some weights proportional to $(\alpha_1, \alpha_2, \alpha_3)$.*

REMARK 5.5. – When $\alpha_1 = \alpha_2 = \alpha_3 = 1$ the unique self-similar Dirichlet form is the symmetric one (*i.e.* the one initially constructed in [15], [3]).

REMARK 5.6 – When $\alpha_2 = \alpha_3 = \beta$ and $\alpha_1 = \alpha$ the critical value is $\alpha = \frac{3}{2}\beta$. When $\alpha$ is strictly smaller there exists a unique self-similar Dirichlet form and when $\alpha$ is bigger there does not exist any. Roughly speaking, if the bottom cells are too attractive the process can only run on horizontal lines (it is then degenerated) and the fractal becomes well-balanced down a critical value.

*Example* 3. – *The snowflake.* We recall that in this case we chose $\alpha_i = 1$ for all $i$. $\mathcal{J}_1$ is the only non-trivial preserved $G$-relation and we obtained $T_{\mathcal{J}_1} = 1/3Id$, thus, $\underline{\rho}_{\mathcal{J}_1} = \overline{\rho}_{\mathcal{J}_1} = 1/3$.

The set $F/\mathcal{J}_1$ contains 3 points (*cf.* Section 4.1.1) and, considered as a subset of $F^{(1)}/\mathcal{J}_1^{(1)}$, it is included in the central 1-cell $\Psi_7(F/\mathcal{J}_1)$ (*cf.* Section 4.2.1). Actually, when viewed as a map from $F/\mathcal{J}_1$ to $F^{(1)}/\mathcal{J}_1^{(1)}$, $\Psi_7$ is the identity.

Let $A$ be in $M_{F/\mathcal{J}_1}$:

$$T_{F/\mathcal{J}_1} A(f) \geq A(f \circ \Psi_7, f \circ \Psi_7) = A(f, f),\ \forall f \in E_{F/\mathcal{J}_1};$$

This implies that $\overline{\rho}_{F/\mathcal{J}_1} \geq 1$.

We can apply (ii) of Theorem 5.1, $T$ thus has a unique fixed point, which is regular thanks to Proposition 3.8. It follows that there exists a unique (up to a multiplicative constant) self-similar Dirichlet form associated with equal weights.

## 5.3. Proof of existence

(i) If $A \in M$ is a fixed point of $T$ then $\underline{\rho}(A) = \overline{\rho}(A)$ but it is impossible because, from Proposition 4.32, one has that:

$$\underline{\rho}(A) \leq \underline{\rho}_{F/\mathcal{J}} < \underline{\rho}_{\mathcal{J}'} \leq \overline{\rho}(A).$$

(ii) We recall that for $A$ in $M$, $\theta(A)$ is defined by:

$$\theta(A) = \frac{\overline{\rho}(A)}{\underline{\rho}(A)}.$$

Thus, $A$ is a fixed point of $T$ if and only if $\theta(A) = 1$. We set:

$$\theta = \inf_{A\in M} \theta(A).$$

One clearly has: $\theta \geq \overline{\rho}/\underline{\rho} \geq 1$.

To prove the existence we proceed in two steps:

- Step 1 : We prove that $\theta(A)$ reaches its infimum on $M$.
- Step 2 We prove that $\theta = 1$. :

Step 1 and Step 2 obviously imply that $T$ has a fixed point in $M$.

**Step 1**

We first give the following lemma. We stress the fact that this lemma is true even without Assumption (H).

LEMMA 5.7. – *There exists $\gamma > 0$ such that for all non-trivial preserved $G$-relation $\mathcal{J}$ and for all $A \in M$, there exists an integer $p \geq 0$ such that $\delta_{\mathcal{J}}(T^p A) > \gamma$.*

N.B.: It means that, iterating $T$, one cannot stay near a preserved $G$-relation. It is a kind of repulsivity property of the $G$-relations. Unfortunately, this lemma does not prove that one can not stay in the neighbourhood of all preserved $G$-relations: we use Assumption (H) to avoid this difficulty.

*Proof.* – Let $\mathcal{J}$ be a non-trivial preserved $G$-relation.

As $\underline{\rho}_{F/\mathcal{J}}/\overline{\rho}_{\mathcal{J}} > 1$, we choose $\gamma > 0$ such that $\gamma \leq \epsilon_4$ and:

$$\frac{\underline{\rho}_{F/\mathcal{J}}}{\overline{\rho}_{\mathcal{J}}}\left(\frac{1-K_4\gamma^{\frac{1}{2}}}{1+K_4\gamma^{\frac{1}{2}}}\right) > 1.$$

Let us call $\beta$ this value: $\beta > 1$.

Suppose that $A \in M$ satisfies $\delta_{\mathcal{J}}(A) \leq \gamma$ and

$$\delta_{\mathcal{J}}(T^s A) \leq \gamma, \ \ \forall s \leq p,$$

for an integer $p > 0$.

As $\gamma \leq \epsilon_4$, Proposition 4.23 applied to $T^s A$ gives:

$$\frac{(T^p A)_{F/\mathcal{J}}(f')}{(T^p A)_{\mathcal{J}}(f)} \geq \left(\frac{1+K_4\gamma^{\frac{1}{2}}}{1-K_4\gamma^{\frac{1}{2}}}\right)^p \frac{T^p_{F/\mathcal{J}} A_{F/\mathcal{J}}(f')}{T^p_{\mathcal{J}} A_{\mathcal{J}}(f)}, \ \ \forall f \in V^*_{\mathcal{J}}, \ \forall f' \in V^*_{F/\mathcal{J}}. \tag{5.1}$$

It is easy to prove that there exists $g \in V_{\mathcal{J}}$, $g \neq 0$, such that

$$T^p_{\mathcal{J}} A_{\mathcal{J}}(g) \leq \overline{\rho}^p_{\mathcal{J}} A_{\mathcal{J}}(g). \tag{5.2}$$

Indeed, if it does not, it means that one can find $\rho > \overline{\rho}_{\mathcal{J}}$ such that $T^p_{\mathcal{J}} A_{\mathcal{J}} \geq \rho^p A_{\mathcal{J}}$, and choosing $A' \in M_{\mathcal{J}}$ such that $\overline{\rho}_{\mathcal{J}}(A') < \rho$ one has that:

$$\inf \frac{T^p_{\mathcal{J}} A'}{T^p_{\mathcal{J}} A_{\mathcal{J}}} \leq \left(\frac{\overline{\rho}_{\mathcal{J}}(A')}{\rho}\right)^p \inf \frac{A'}{A_{\mathcal{J}}} < \inf \frac{A'}{A_{\mathcal{J}}},$$

but this is incompatible with Proposition 3.3 adapted to $T_{\mathcal{J}}$.

In the same way, we can easily prove that there exists $g' \in V_{F/\mathcal{J}}$, $g' \neq 0$, such that:

$$T^p_{F/\mathcal{J}} A_{F/\mathcal{J}}(g') \geq \underline{\rho}^p_{F/\mathcal{J}} A_{F/\mathcal{J}}(g'). \tag{5.3}$$

So, thanks to (5.2) and (5.3), (5.1) applied to $g$ and $g'$ gives:

$$\frac{(T^p A)_{F/\mathcal{J}}(g')}{(T^p A)_{\mathcal{J}}(g)} \geq \beta^p \frac{A_{F/\mathcal{J}}(g')}{A_{\mathcal{J}}(g)}$$

Set:

$$b = \left(\sup \frac{A_{\mathcal{J}}}{A^0_{\mathcal{J}}}\right)\left(\inf \frac{A_{F/\mathcal{J}}}{A^0_{F/\mathcal{J}}}\right),$$

one has that:

$$\frac{(T^p A)_{F/\mathcal{J}}(g')}{(T^p A)_{\mathcal{J}}(g)} \geq b\beta^p \frac{A^0_{F/\mathcal{J}}(g')}{A^0_{\mathcal{J}}(g)}. \tag{5.4}$$

But, using Lemma 4.14, $\delta_{\mathcal{J}}(T^p A) \leq \gamma$ implies that:

$$\frac{(T^p A)_{F/\mathcal{J}}(g')}{(T^p A)_{\mathcal{J}}(g)} \leq K_1^2 \gamma \frac{A^0_{F/\mathcal{J}}(g')}{A^0_{\mathcal{J}}(g)}. \tag{5.5}$$

Thus, from (5.4) and (5.5) we deduce that:

$$\beta^p \leq \frac{\gamma K_1^2}{b},$$

and $\beta > 1$ implies that $p$ is bounded from above. So, there exists $p > 0$ such that $\delta_{\mathcal{J}}(T^p A) > \gamma$. $\square$

We now prove Step 1.

Suppose that $\theta$ is not reached on $M$. For $\epsilon > 0$ we denote by $Z_\epsilon$ the following subset of $M$:

$$Z_\epsilon = \{A \in M, \ \ \theta(A) \leq \theta + \epsilon\}.$$

Proposition 4.30 implies that $Z_\epsilon$ is $T$-invariant (*i.e.* that $T(Z_\epsilon) \subset Z_\epsilon$).

We recall that Corollary 4.27 says that for all preserved $G$-relation $\mathcal{J}$ and for all $A$ in $M$ such that $\delta_{\mathcal{J}}(A) \leq \epsilon_5$, one has that:

$$\delta_{\mathcal{J}}(TA) \leq K_5 \delta_{\mathcal{J}}(A).$$

Set $\eta = \gamma \wedge \frac{1}{2K_5} \wedge \epsilon_5 \wedge 1$. As $\theta$ is not reached on $M$, we know from Proposition 4.34 (or Remark 4.35) and Proposition 4.3 that $\theta(A)$ approaches $\theta$ in the neighbourhood of the preserved $G$-relations, so one can find $\epsilon > 0$ such that:

$$(A \in Z_\epsilon) \Rightarrow (\text{ there exists a preserved } G\text{-relation } \mathcal{J} \text{ such that } \delta_{\mathcal{J}}(A) \leq \eta).$$

Let $A$ be in $Z_\epsilon$ and $\mathcal{J}$ be a preserved $G$-relation such that $\delta_{\mathcal{J}}(A) \leq \eta$. Let $p$ be the first integer such that $\delta_{\mathcal{J}}(T^p A) > \eta$ (this integer exists from Lemma 5.7, since $\eta \leq \gamma$).

As $\delta_{\mathcal{J}}(T^{p-1}A) \leq \eta$ one has that:

$$\delta_{\mathcal{J}}(T^p A) \leq K_5 \eta \leq \frac{1}{2}.$$

Thus, using Assumption (H) and Lemma 4.4, we know that for all preserved $G$-relation $\mathcal{J}' \neq \mathcal{J}$, one has that $\delta_{\mathcal{J}'}(T^p A) \geq 2 > \eta$. Moreover, since $\delta_{\mathcal{J}}(T^p A) > \eta$, $T^p A$ cannot be in $Z_\epsilon$. This leads to a contradiction as $Z_\epsilon$ is $T$-invariant.

So we have proved that $\theta$ is reached on $M$.

**Step 2**

Suppose that $\theta > 1$.

For all $A$ in $M$, we set:

$$\underline{W}(A) = \{f \in V,\ \ TA(f) = \underline{\rho}(A)A(f)\},$$

$$\overline{W}(A) = \{f \in V,\ \ TA(f) = \overline{\rho}(A)A(f)\}.$$

The subspaces $\underline{W}(A)$ and $\overline{W}(A)$ are orthogonal with respect to both quadratic forms $A$ and $TA$. They are the eigenspaces associated with the smallest eigenvalue $\underline{\rho}(A)$ and the largest eigenvalue $\overline{\rho}(A)$ for the diagonalization of $TA$ on an orthogonal basis with respect to $A$ ($TA$ and $A$ are simultaneously diagonalizable).

Using Step 1, we choose $A$ in $M$ such that:

• $\theta(A) = \theta$,

• $(\underline{W}(A), \overline{W}(A))$ is minimal in the following sense: there does not exist $A'$ in $M$ such that $\theta(A') = \theta$ and $\underline{W}(A') \subset \underline{W}(A)$, $\overline{W}(A') \subset \overline{W}(A)$ with a strict inclusion between the two.

We set $W_0 = \underline{W}(A)$ and $W_1 = \overline{W}(A)$.

We recall that for $f \in E$, $Pf$ is the constant function equal to the mean value of $f$. The map $I - P$ is then a projection on $V$.

We first prove:

LEMMA 5.8. – *If $f$ is in $W_0$ (resp. $W_1$) then:*

$$(I-P)(H_A(f) \circ \Psi_i) \in W_0 \text{ (resp. } W_1), \quad \forall i \in \{1, \ldots, N\}.$$

N.B.: We recall that $H_A(f)$ denotes the harmonic continuation of $f$ with respect to $A^{(1)}$.

*Proof.* – We diagonalize $TA$ on an orthogonal basis with respect to $A$:

$$TA = \underline{\rho}(A)A_{|W_0} \oplus \overline{\rho}(A)A_{|W_1} \oplus \lambda_2 A_{|W_2} \oplus \cdots \oplus \lambda_r A_{|W_r}. \tag{5.6}$$

with $V = W_0 \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_r$ and $\underline{\rho}(A) < \lambda_2 < \cdots < \lambda_r < \overline{\rho}(A)$.

Choose $t$ in $]0,1[$. We set:

$$A_t = tA + (1-t)TA.$$

From (5.6) one has that:

$$A_t = (t + (1-t)\underline{\rho}(A))A_{|W_0} \oplus (t + (1-t)\overline{\rho}(A))A_{|W_1} \\ \oplus (t + (1-t)\lambda_2)A_{|W_2} \oplus \cdots \oplus (t + (1-t)\lambda_r)A_{|W_r}.$$

In particular, from (5.7) we deduce that:

$$A_t \geq (t + (1-t)\underline{\rho}(A))A,$$

thus,

$$TA_t \geq (t + (1-t)\underline{\rho}(A))TA.$$

From the previous relation and (5.6) we deduce that:

$$TA_t \geq \underline{\rho}(A)(t + (1-t)\underline{\rho}(A))A_{|W_0} \oplus \overline{\rho}(A)(t + (1-t)\underline{\rho}(A))A_{|W_1} \\ \oplus \lambda_2(t + (1-t)\underline{\rho}(A))A_{|W_2} \oplus \cdots \\ \oplus \lambda_r(t + (1-t)\underline{\rho}(A))A_{|W_r}. \tag{5.8}$$

But, we remark that if $\lambda > \underline{\rho}(A)$ one has that:

$$\lambda(t + (1-t)\underline{\rho}(A)) = \lambda\underline{\rho}(A)\left(\frac{t}{\underline{\rho}(A)} + (1-t)\right) \\ > \lambda\underline{\rho}(A)\left(\frac{t}{\lambda} + (1-t)\right) = \underline{\rho}(A)(t + (1-t)\lambda).$$

So (5.7) and (5.8) give:

$$TA_t \geq \underline{\rho}(A)A_t \tag{5.9}$$

$$TA_t(f) > \underline{\rho}(A)A_t(f), \quad \forall f \in V \setminus W_0. \tag{5.10}$$

Proceeding in the same way we also obtain:

$$TA_t \leq \overline{\rho}(A)A_t \tag{5.11}$$

$$TA_t(f) < \overline{\rho}(A)A_t(f), \quad \forall f \in V \setminus W_1. \tag{5.12}$$

From (5.9) and (5.11) we deduce that $\underline{\rho}(A_t) \geq \underline{\rho}(A)$, $\overline{\rho}(A_t) \leq \overline{\rho}(A)$ so, as $\theta(A)$ is minimal, $\underline{\rho}(A_t) = \underline{\rho}(A)$ and $\overline{\rho}(A_t) = \overline{\rho}(A)$. From (5.10) and (5.12) we deduce that $\underline{W}(A_t) \subset W_0$, $\overline{W}(A_t) \subset W_1$. Finally, using the property of minimality of $(W_0, W_1)$, one obtains that:

$$\underline{\rho}(A_t) = \underline{\rho}(A) \text{ and } \underline{W}(A_t) = W_0, \tag{5.13}$$

$$\overline{\rho}(A_t) = \overline{\rho}(A) \text{ and } \overline{W}(A_t) = W_1. \tag{5.14}$$

Let now $f$ be in $W_0$, $f \neq 0$:

$$\begin{aligned}
\underline{\rho}(A)A(f) &= \underline{\rho}(A)\frac{1}{t+(1-t)\underline{\rho}(A)}A_t(f)\\
&= \frac{1}{t+(1-t)\underline{\rho}(A)}TA_t(f)\\
&\leq \sum_{i=1}^{N}\alpha_i^{-1}\frac{1}{t+(1-t)\underline{\rho}(A)}A_t(H_A(f)\circ\Psi_i)\\
&\overset{(1)}{\leq} \sum_{i=1}^{N}\alpha_i^{-1}A(H_A(f)\circ\Psi_i)\\
&= TA(f)\\
&= \underline{\rho}(A)A(f),
\end{aligned}$$

and Inequality (1) is strict as soon as $(I-P)(H_A(f)\circ\Psi_i)$ is not in $W_0$ for one $i$ in $\{1,\ldots,N\}$. Thus, it implies that $(I-P)(H_A(f)\circ\Psi_i)$ belongs to $W_0$ for all $i\in\{1,\ldots,N\}$.

If now $f\in W_1$ and $f\neq 0$:

$$\begin{aligned}
\overline{\rho}(A)A_t(f) &= \overline{\rho}(A)(t+(1-t)\overline{\rho}(A))A(f)\\
&= (t+(1-t)\overline{\rho}(A))TA(f)\\
&= \sum_{i=1}^{N}\alpha_i^{-1}(t+(1-t)\overline{\rho}(A))A(H_A(f)\circ\Psi_i)\\
&\overset{(2)}{\geq} \sum_{i=1}^{N}\alpha_i^{-1}A_t(H_A(f)\circ\Psi_i)\\
&\geq TA_t(f)\\
&= \overline{\rho}(A)A_t(f)
\end{aligned}$$

and Inequality (2) is strict as soon as $(I-P)(H_A(f)\circ\Psi_i)$ is not in $W_1$ for one $i$ in $\{1,\ldots,N\}$. Thus, it implies that $(I-P)(H_A(f)\circ\Psi_i)$ belongs to $W_1$ for all $i\in\{1,\ldots,N\}$.

This concludes the proof of the lemma. □

Set $W' = W_2\oplus\cdots\oplus W_r$. The subspaces $W_0$, $W_1$ and $W'$ are orthogonal with respect to both $A$ and $TA$ and moreover $V = W_0\oplus W_1\oplus W'$. We now need the following result:

LEMMA 5.9. – *If $B\in M$ satisfies:*

- *$W_0$, $W_1$, $W'$ are orthogonal with respect to $B$,*
- *there exist $\beta>0$ and $\beta'>0$ such that $\beta B_{|W_0} = A_{|W_0}$ and $\beta' B_{|W_1} = A_{|W_1}$, then $TB$ satisfies:*
- *$W_0$, $W_1$, $W'$ are orthogonal with respect to $TB$,*
- *$\beta TB_{|W_0} = TA_{|W_0}$ and $\beta' TB_{|W_1} = TA_{|W_1}$.*

REMARK 5.10. – As this lemma will be used in the proof of uniqueness we stress the fact that the only hypotheses we use are that $W_0$, $W_1$ and $W'$ are orthogonal with respect to both $A$ and $TA$ and that for all $f$ in $W_0$ (resp. $W_1$), $(I-P)(H_A(f)\circ\Psi_i)$ is in $W_0$ (resp. $W_1$) for all $i\in\{1,\dots,N\}$.

*Proof.* – We first prove that for $f$ in $W_0$ or in $W_1$, $H_A(f)=H_B(f)$.

Let $f$ be in $W_0$, $f\neq 0$ (the proof for $f\in W_1$ is naturally the same).

Set $g=H_B(f)-H_A(f)$ and $g_i=(I-P)(g\circ\Psi_i)$ so that $g_i\in V$. We decompose $g_i$ on $W_0\oplus W_0^\perp$ (with $W_0^\perp=W_1\oplus W'$):

$$g_i=g_i'+g_i'',$$

with $g_i'\in W_0$ and $g_i''\in W_0^\perp$. One has that:

$$\begin{aligned}
B^{(1)}((H_A-H_B)(f)) &= B^{(1)}(H_A(f),(H_A-H_B)(f))\\
&=\sum_{i=1}^N \alpha_i^{-1}B(H_A(f)\circ\Psi_i, g_i'+g_i'')\\
&=\sum_{i=1}^N \alpha_i^{-1}B(H_A(f)\circ\Psi_i, g_i')\\
&=\frac{1}{\beta}\sum_{i=1}^N \alpha_i^{-1}A(H_A(f)\circ\Psi_i, g_i')\\
&=\frac{1}{\beta}\sum_{i=1}^N \alpha_i^{-1}A(H_A(f)\circ\Psi_i, g_i'+g_i'')\\
&=\frac{1}{\beta}A^{(1)}(H_A(f),(H_A-H_B)(f))\\
&=0
\end{aligned}$$

N.B.: For the first equality (resp. the last one) we use the fact that $H_B(f)$ (resp. $H_A(f)$) is harmonic on $F^{(1)}\setminus F$ with respect to $B^{(1)}$ (resp. $A^{(1)}$). The other equalities come from the last lemma and from the fact that $W_0$ and $W_0^\perp$ are orthogonal with respect to both quadratic forms $A$ and $B$.

It follows that $H_A(f)=H_B(f)$. Since $(I-P)(H_A(f)\circ\Psi_i)$ is in $W_0$ (resp. $W_1$) for $f\in W_0$ (resp. $W_1$), we easily deduce that:

$$\beta TB_{|W_0}=TA_{|W_0},$$

$$\beta' TB_{|W_1}=TA_{|W_1}.$$

We now prove that $W_0$, $W_1$ and $W'$ are orthogonal with respect to $TB$.

Let $f$ be in $W_0$ and $f$ in $W_1$:

$$\begin{aligned}
TB(f,f') &= B^{(1)}(H_B(f),H_B(f'))\\
&= B^{(1)}(H_A(f),H_A(f'))\\
&=\sum_{i=1}^N \alpha_i^{-1}B(H_A(f)\circ\Psi_i, H_A(f')\circ\Psi_i)=0,
\end{aligned}$$

since $W_0$ and $W_1$ are orthogonal with respect to $B$. So $W_0$ and $W_1$ are orthogonal with respect to $TB$.

Let $f$ be in $W_0$ and $g$ in $W'$.

We set $g_i = (I-P)(H_B(g)\circ\Psi_i)$ so that $g_i \in V$. We decompose $g_i$ as $g_i = g_i' + g_i''$ with $g_i' \in W_0$ and $g_i'' \in W_0^{\perp}$ (we recall that $W_0^{\perp} = W_1 \oplus W'$). One has that:

$$\begin{aligned}
TB(f,g) &= B^{(1)}(H_B(f), H_B(g))\\
&= \sum_{i=1}^{N} \alpha_i^{-1} B(H_A(f)\circ\Psi_i, g_i)\\
&= \sum_{i=1}^{N} \alpha_i^{-1} B(H_A(f)\circ\Psi_i, g_i')\\
&= \sum_{i=1}^{N} \alpha_i^{-1} \frac{1}{\beta} A(H_A(f)\circ\Psi_i, g_i')\\
&= \sum_{i=1}^{N} \alpha_i^{-1} \frac{1}{\beta} A(H_A(f)\circ\Psi_i, g_i)\\
&= \frac{1}{\beta} A^{(1)}(H_A(f), H_B(g))\\
&= \frac{1}{\beta} A^{(1)}(H_A(f), H_A(g))\\
&= \frac{1}{\beta} TA(f,g) = 0
\end{aligned}$$

since $W_0$ and $W'$ are orthogonal with respect to $TA$. So $W_0$ and $W'$ are orthogonal with respect to $TB$, and in the same way we can prove that $W_1$ and $W'$ are orthogonal. This concludes the proof. $\square$

By a simple recurrence we deduce from the previous lemma that for all $n \in \mathbb{N}$, the subspaces $W_0$, $W_1$ and $W'$ are orthogonal with respect to $T^nA$, and that:

$$T^n A_{|W_0} = \underline{\rho}(A)^n A_{|W_0} \ ; \ T^n A_{|W_1} = \overline{\rho}(A)^n A_{|W_1}.$$

In particular $\delta(T^nA)$ converges to 0 (because we supposed that $\theta(A) > 1$). From Proposition 4.30, one has that $\theta(T^nA) = \theta$, and then, using Proposition 4.34 and Proposition 4.3, we see that $T^nA$ is near a preserved $G$-relation when $n$ goes to infinity.

More precisely, if we set as in Step 1: $\eta = \gamma \wedge \frac{1}{2K_5} \wedge \epsilon_5 \wedge 1$, we obtain that there exists an integer $p$ such that for $n \geq p$, $\delta_{\mathcal{J}}(T^nA) \leq \eta$ for a preserved $G$-relation $\mathcal{J}$. But as in Step 1, using Assumption (H) and Lemma 4.4, we prove that there is an integer $k$ for which $\delta_{\mathcal{J}}(T^{p+k}A) > \eta$ for all preserved $G$-relation $\mathcal{J}$: this is in contradiction with the first statement. So $\theta$ is equal to 1. We have proved Step 2.

REMARK 5.11. – *I recently learnt by V. Metz that an adaptation of Theorem 4.2 of [23] could replace Step 2. This theorem uses some results on the structure of $\omega$-limit sets of non-expansive maps* (cf. *[4], [24]). I kept the original proof for a sake of completness and because it is very different from the one of [23] and, althought technical, very elementary.*

### 5.4. Proof of uniqueness

Suppose that $T$ has two non-proportional fixed points denoted by $A$ and $A'$. In order to simplify the notations we renormalize $A'$ such that $\inf A'/A = 1$. One can diagonalize $A'$ on an orthogonal basis for $A$:

$$A' = A_{|W_0} \oplus \lambda A_{|W_1} \oplus \lambda_2 A_{|W_2} \oplus \cdots \oplus \lambda_r A_{|W_r}, \tag{5.15}$$

with $V = W_0 \oplus W_1 \oplus \cdots \oplus W_r$ and $1 < \lambda_2 < \cdots < \lambda_r < \lambda$ (in particular $\lambda = \sup(A'/A)$).

We set $W' = W_2 \oplus \cdots \oplus W_r$.

For all $\alpha \geq 1$ we define:

$$\begin{aligned} Q^\alpha(A,A') = \{B \in M,\ \ & W_0,\ W_1, \text{ and } W' \text{ are orthogonal with respect to } B, \\ & \text{there exists } \beta > 0 \text{ such that } \beta B_{|W_0} = A_{|W_0},\ \beta B_{|W_1} = \alpha A_{|W_1} \\ & \text{and } A \leq \beta B \leq \alpha A.\} \end{aligned}$$

REMARK 5.12. – $Q^1(A,A') = \mathbb{R}_+^* A$ and $A'$ is in $Q^\lambda(A,A')$ ($\lambda$ appears in (5.15)).

The following lemma is the key result of the proof of uniqueness.

LEMMA 5.13. – *For all $\alpha \geq 1$, $Q^\alpha(A,A')$ is $T$-invariant* (i.e. $T(Q^\alpha(A,A')) \subset Q^\alpha(A,A')$).

*Proof.* – Since $A$ and $A'$ satisfy $TA = \rho A$ and $TA' = \rho A'$ for the same real $\rho$ (*cf.* Corollary 3.5), we get:

$$\frac{TA}{TA'} = \frac{A}{A'}.$$

Let $f$ be in $W_0$, $f \neq 0$. Relation (5.15) implies that:

$$\begin{aligned} 1 = \frac{TA'(f)}{TA(f)} &\overset{(1)}{\geq} \frac{\sum \alpha_i^{-1} A'(H_{A'}(f)\circ\Psi_i)}{\sum \alpha_i^{-1} A(H_{A'}(f)\circ\Psi_i)} \\ &= \sum_{i=1}^N \left(\frac{\alpha_i^{-1} A(H_{A'}(f)\circ\Psi_i)}{\sum_j \alpha_j^{-1} A(H_{A'}(f)\circ\Psi_j)}\right)\left(\frac{A'(H_{A'}(f)\circ\Psi_i)}{A(H_{A'}(f)\circ\Psi_i)}\right) \\ &\overset{(2)}{\geq} 1. \end{aligned}$$

Inequality (1) is strict as soon as $H_A(f) \neq H_{A'}(f)$, and Inequality (2) is strict as soon as $(I-P)(H_{A'}(f)\circ\Psi_i)$ is not in $W_0$ for one $i$ in $\{1,\dots,N\}$. It implies that $H_A(f) = H_{A'}(f)$ and that $(I-P)(H_A(f)\circ\Psi_i) \in W_0$ for all $i$ in $\{1,\dots,N\}$.

We can prove the similar result for $f$ in $W_1$. So, with regard to Remark 5.10 (as $TA = \rho A$), we can apply Lemma 5.9 to $A$. It implies that if $B$ is in $Q^\alpha(A,A')$ (for a constant $\beta$) then $W_0$, $W_1$, $W'$ are orthogonal with respect to $TB$ and $\beta TB_{|W_0} = \rho A_{|W_0}$, $\beta TB_{|W_1} = \rho A_{|W_1}$. Moreover, since $T$ is non-decreasing $\rho A \leq \beta TB \leq \alpha\rho A$ and then $TB$ is in $Q^\alpha(A,A')$. □

Thanks to the estimate $A \leq \beta B \leq \alpha A$ we know that the projective set associated with $Q^\alpha(A,A')$ is compact. Moreover $Q^\alpha(A,A')$ is clearly convex, so if it is non-empty, then it contains a fixed point of $T$ (thanks to Brouwer theorem).

We set:

$$\Lambda(A,A') = \{\alpha \geq 1,\ \ Q^\alpha(A,A') \neq \emptyset\}.$$

We now give the following lemma:

LEMMA 5.14. – *$\Lambda(A, A')$ is a closed subset of $\mathbb{R}$ and is bounded from above.*

*Proof.* – We first prove that $\Lambda(A, A')$ is closed.

Let $(\alpha_n)$ be a sequence in $\Lambda(A, A')$ such that $\alpha_n$ converges to $\alpha \in \mathbb{R}$.

We choose $c > 0$ such that $\alpha_n \leq c$ for all $n$. For each $n$ we can choose $B_n$ in $Q^{\alpha_n}(A, A')$ such that $A \leq B_n \leq \alpha_n A \leq cA$.

One can find a sub-sequence $n_k$ and $B \in M$ such that $B_{n_k}$ converges to $B$ in $M$. It is then easy to check that $B \in Q^{\alpha}(A, A')$ so that $\alpha \in \Lambda(A, A')$.

We now prove that $\Lambda(A, A')$ is bounded from above.

There exists $C > 0$ such that $\delta(B) \leq C\frac{1}{\alpha}$ for all $B$ in $Q^{\alpha}(A, A')$. Thus, using Proposition 4.3 and Proposition 4.34 we can show that there exists $\alpha' \geq 1$ such that for all $\alpha \geq \alpha'$ and for all $B \in Q^{\alpha}(A, A')$ such that $\theta(B) = 1$ one can find a preserved $G$-relation such that $\delta_{\mathcal{J}}(B) \leq \gamma$, where $\gamma$ is the constant of Lemma 5.7.

So, if $\Lambda(A, A')$ is not bounded from above, choosing $\alpha \in \Lambda(A, A')$ such that $\alpha \geq \alpha'$ and a fixed point $B$ in $Q^{\alpha}(A, A')$, one has that $\delta_{\mathcal{J}}(B) \leq \gamma$ for a preserved $G$-relation $\mathcal{J}$. But Lemma 5.7 applied to $B$ is incompatible with the fact that $B$ is a fixed point of $T$ (we stress the fact that Lemma 5.7 does not need Assumption (H)).□

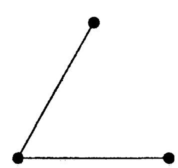
As the proof of uniqueness in general is quite technical we first give it in a simple case:

SIMPLE CASE. – We suppose that $T$ satisfies the following hypothesis:

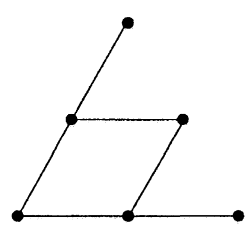
(S) For all Dirichlet form $A \in M$, $TA$ is associated with a strictly positive bond conductivity matrix (*i.e.* $j_{x,y} > 0$ for $x \neq y$).

We first remark that (S) is satisfied for the two examples we follow from the beginning of Section 4.

In the case of the Sierpinski gasket without symmetry if $A$ is irreducible then it is associated with at least two positive bond conductivities. By the following picture we draw two sides of the triangle we suppose to have positive bond conductivity.



By the following picture we draw the positive conductivities of the electrical network associated with $A^{(1)}$ on $F^{(1)}$:



So, one can go from any point of $F$ straight to any other (*i.e.* without passing through the third one), using only positive bond conductivities.
From Proposition 1.14 of Section 1.3, we can conclude that $TA$ is associated with a network with strictly positive bond conductivities.
In the same way, using the symmetries of the picture, one can see on the example of the snowflake that (S) is satisfied. More generaly, we could prove that (S) is satisfied by all nested fractals.
We denote by $\overset{\circ}{M}$ the interior of $M$.
We first remark that $\overset{\circ}{M}$ is the set of Dirichlet forms with strictly positive bond conductivities (it is easy to check it from Definition 1.6 and Proposition 1.7), (S) thus implies that $T(M) \subset \overset{\circ}{M}$.
We now prove the uniqueness in this case. We suppose that there exist two non-proportional fixed points $A$ and $A'$.
Set $\overline{\alpha} = \sup \Lambda(A, A')$. One has that $\overline{\alpha} > 1$ from Remark 5.12. The subset $Q^{\overline{\alpha}}(A, A')$ is non-empty since $\Lambda(A, A')$ is closed. We first prove that $Q^{\overline{\alpha}}(A, A')$ is included in $M \setminus \overset{\circ}{M}$.
Indeed, take $B$ in $Q^{\overline{\alpha}}(A, A')$ normalized such that $\inf(B/A) = 1$. We can write $B$ as:

$$B = A_{|W_0} \oplus \overline{\alpha} A_{|W_1} \oplus A_{|W'}.$$

Set $B' = B - A$, $B' \geq 0$.
If we suppose that $B \in \overset{\circ}{M}$ then for $\epsilon > 0$ small enough, $B + \epsilon B'$ is in $M$ and then $B + \epsilon B'$ is in $Q^{\overline{\alpha} + \epsilon(\overline{\alpha} - 1)}(A, A')$, so $\overline{\alpha} + \epsilon(\overline{\alpha} - 1) \in \Lambda(A, A')$, and this leads to a contradiction. Thus, the following inclusion is true: $Q^{\overline{\alpha}}(A, A') \subset M \setminus \overset{\circ}{M}$.
But (S) implies that $T(M) \subset \overset{\circ}{M}$, this is in contradiction with Lemma 5.13. This proves the uniqueness of the fixed point.

GENERAL CASE. – We study more precisely the set $M \setminus \overset{\circ}{M}$.

*Notation.* – Let $\mathcal{L}$ be the set of $G$-symmetric, connected graphs on $F$ (we consider a graph as a set of subsets with 2 elements of $F$). Denote by $\overline{L}$ the graph which connects any two points of $F$. We order $\mathcal{L}$ with the inclusion relation.

For any $L \in \mathcal{L}$ we denote by $M_L$ the subset of $M$ of Dirichlet forms whose bond conductivity matrix denoted by $(j_{x,y})$ satisfies:

$$\{x, y\} \in L \Leftrightarrow j_{x,y} > 0.$$

We easily check:

$$\overset{\circ}{M} = M_{\overline{L}} \ , \quad M \setminus \overset{\circ}{M} = \cup_{\substack{L \in \mathcal{L} \\ L \neq \overline{L}}} M_L,$$

$$\overline{M_L} = \cup_{\substack{L' \in \mathcal{L} \\ L' \subset L}} M_{L'}, \quad \forall L \in \mathcal{L}.$$

N.B.: $\overline{M_L}$ denotes the closure of $M_L$ in $M$.

With any graph $L$ in $\mathcal{L}$ we naturally associate the graph $L^{(1)}$ on $F^{(1)}$ that connects in each 1-cell $F_i = \Psi_i(F)$ the points connected by $\Psi_i(L)$. Formally, we define $L^{(1)}$ by:

$$L^{(1)} = \cup_{i=1}^{N}\{\{\Psi_i(x), \Psi_i(y)\},\ \{x,y\} \in L\}.$$

We then define $T : \mathcal{L} \to \mathcal{L}$ in the following way: for $x$ and $y$ in $F$, $x \neq y$, $\{x,y\}$ is in $T(L)$ if and only if there exists a path on the graph defined by $L^{(1)}$ going from $x$ to $y$ without passing trough any other point of $F$.

From Proposition 1.15 one has that:

$$T(M_L) \subset M_{T(L)}.$$

We adopt the following definition:

DEFINITION 5.15. – *We say that a graph $L$ of $\mathcal{L}$ is preserved if $T(L) = L$.*
*We denote by $\mathcal{L}_T$ the set of preserved graphs.*

REMARK 5.16. – If $M_L$ contains a fixed point of $T$ then $L$ is preserved.

We now define a kind of minimal graph: let $L_0$ be the graph on $F$ defined by $\{x,y\} \in L_0$ if and only if there exists a path $z_0 = x, \ldots, z_k = y$ on $F^{(1)}$ such that for all $i \in \{0, \ldots, k-1\}$, $z_i$ and $z_{i+1}$ belong to a same 1-cell and for all $i \in \{1, \ldots, k-2\}$, they belong to a same 1-cell which does not contain any point of $F$.

We have the following lemma:

LEMMA 5.17. – *The graph $L_0$ is in $\mathcal{L}$.*
*There exists an integer $p$ such that $T^p(L_0)$ is preserved and such that:*

$$T^p(L_0) \subset T^{p+1}(L), \ \ \forall L \in \mathcal{L}.$$

*We set $L_1 = T^p(L_0)$.*

REMARK 5.18. – The graph $L_1$ is the minimum of the set $\mathcal{L}_T$.

REMARK 5.19. – In the simple case we first studied, we had $L_1 = \overline{L}$.

*Proof.* – We first prove that $L_0$ is in $\mathcal{L}$. By construction, $L_0$ is $G$-symmetric, so we only need to show that $L_0$ is connected. It is based on the fact that each 1-cell contains at most one point of $F$. Suppose that $L_0$ is not connected. Let us choose $x$ and $y$ in two different connected components of $L_0$. We construct a path $z_0 = x, \ldots z_k = y$ on $F^{(1)}$ such that for $j \in \{0, \ldots, k-1\}$, $z_j$ and $z_{j+1}$ are both in a same 1-cell denoted by $F_{i_j}$. Denote by $x_0 = x, x_1, \ldots, x_{k'} = y$ the sequence of successive points of $F$ which appear in the cells $F_{i_0}, \ldots, F_{i_{k-1}}$. We choose two succesive points in this sequence, say $x_p$ and $x_{p+1}$, which are not in the same connected component for $L_0$. By construction, the part of the path from $x_p$ to $x_{p+1}$ takes its values on cells which do not contain any point of $F$ (except for the first and the last cell). This means that $\{x_p, x_{p+1}\}$ is in $L_0$, and it leads to a contradiction.

Now, we easily check that if $L \in \mathcal{L}$ then $L_0 \subset T(L)$. The sequence of graphs $T^k(L_0)$ is then non-decreasing (since $T$ is non-decreasing) and there exists an integer $p$ such that $T^{p+1}(L_0) = T^p(L_0)$. We set $L_1 = T^p(L_0)$. The graph $L_1$ is then preserved and moreover one has that $L_1 = T^p(L_0) \subset T^p(T(L))$ for all $L$ in $\mathcal{L}$. This concludes the proof of the lemma. $\square$

We now prove the uniqueness in the general case.

We first prove that $M_{L_1}$ contains a fixed point. Let $A$ be a fixed point of $T$. If $A$ is not in $M_{L_1}$ then

$$\inf_{B \in \overline{M}_{L_1}} d(A,B)$$

is strictly positive and is reached on $\overline{M}_{L_1}$ since the subsets of $PM$ with finite diameter for $d$ are relatively compact for the projective topology (N.B.: we recall that $d$ is the projective distance introduced in Section 3.2 and that $T$ is a non-expansive map for $d$). We denote by $d_1$ this infimum. Set $\Omega = \{B \in \overline{M}_{L_1},\ \ d(A,B) = d_1\}$. The subset $\Omega$ is clearly convex, compact for the projective topology and $T$-invariant since $M_{L_1}$ is $T$-invariant and $T$ non-expansive. Brouwer theorem implies that $\Omega$ contains a fixed point. It implies that $\overline{M}_{L_1}$ contains a fixed point, but, as $L_1$ is minimal in the set of preserved graph $\mathcal{L}_T$, any fixed point in $\overline{M}_{L_1}$ is in $M_{L_1}$ (thanks to Remark 5.16). Thus, $M_{L_1}$ contains a fixed point.

We now prove that $M_{L_1}$ contains exactly one fixed point and finally that there is not any fixed point out of $M_{L_1}$.

Suppose that $M_{L_1}$ has two non-proportionnal fixed point $A$ and $A'$, we set:

$$Q^{\alpha}_{L_1}(A,A') = Q^{\alpha}(A,A') \cap \overline{M_{L_1}}.$$

This set is clearly $T$-invariant since $L_1$ is preserved.

Set $\overline{\alpha} = \sup\{\alpha \geq 1,\ Q^{\alpha}_{L_1}(A,A') \neq \emptyset\}$.

Proceeding in the same way as for the simple case, we prove that $Q^{\overline{\alpha}}_{L_1}(A,A')$ is not empty and that $Q^{\overline{\alpha}}_{L_1}(A,A') \subset \overline{M_{L_1}} \setminus M_{L_1}$. But $T^{p+1}(\overline{M}_{L_1}) \subset M_{L_1}$ (*cf.* Lemma 5.17), and this is in contradiction with the fact that $Q^{\overline{\alpha}}_{L_1}(A,A')$ is $T$-invariant. Suppose that there exists a fixed point out of $M_{L_1}$, and choose a minimal element $\tilde{L}$ in the set $\tilde{\mathcal{L}}_T = \{L \in \mathcal{L}_T \setminus L_1,\ \ M_L \text{ contains a fixed point}\}$. The graph $L_1$ is stricly included in $\tilde{L}$ since $L_1$ is the minimum of $\mathcal{L}_T$.

Let $A$ be a fixed point in $M_{L_1}$ and $A'$ be one in $M_{\tilde{L}}$.

For all $\alpha \geq 1$ we set $Q^{\alpha}_{\tilde{L}}(A,A') = Q^{\alpha}(A,A') \cap \overline{M_{\tilde{L}}}$.

Let $\overline{\alpha} = \sup\{\alpha \geq 1,\ Q^{\alpha}_{\tilde{L}}(A,A') \neq \emptyset\}$. One has that $\overline{\alpha} > 1$ because $A'$ is in $Q^{\beta}_{\tilde{L}}(A,A')$ for a real $\beta > 1$ (*cf.* Remark 5.12).

In the same way as for the simple case, one has that $Q^{\overline{\alpha}}_{\tilde{L}}(A,A')$ is not empty and included in $\overline{M_{\tilde{L}}} \setminus M_{\tilde{L}}$.

Let $B$ be a fixed point in $Q^{\overline{\alpha}}_{\tilde{L}}(A,A')$. If $B$ is not in $M_{L_1}$ then it is in $M_{L'}$ for a preserved graph $L'$ strictly included in $\tilde{L}$ and this is in contradiction with the minimality of $\tilde{L}$. If $B$ is in $M_{L_1}$ then, since $B$ and $A$ cannot be proportional (because $\overline{\alpha} > 1$), $M_{L_1}$ has two fixed points and it leads us back to the previous case.

We thus conclude that $T$ has a unique fixed point (moreover this fixed point belongs to $M_{L_1}$). $\square$

## 6. Examples

### 6.1. Application to nested fractals. Uniqueness of the diffusion

#### *6.1.1. Definition*

Nested fractals are self-similar fractals, imbedded in $\mathbb{R}^D$, finitely ramified and highly symmetric. They have been introduced by Lindstrøm (*cf.* [19]). One can find the definition in [19] (or in [16]). We recall it briefly.

Let $D \geq 2$ be an integer and $r > 1$ a real. For $x$ and $y$ in $\mathbb{R}^D$, we denote by $|x - y|$ the Euclidian distance.

DEFINITION 6.1. – *A map $\Psi : \mathbb{R}^D \to \mathbb{R}^D$ is called a $r$-similitude if $|\Psi(x) - \Psi(y)| = r^{-1}|x - y|$, for all $x$ and $y$ in $\mathbb{R}^D$.*

Let $N$ be an integer bigger than 2. Let $(\Psi_1, \ldots, \Psi_N)$ be $N$ $r$-similitudes. We set $\Omega = \{1, \ldots, N\}^{\mathbb{N}}$. One can prove that for all $\omega$ in $\Omega$ the limit

$$\lim_{n\to\infty} \Psi_{\omega(1)} \circ \cdots \circ \Psi_{\omega(n)}(x)$$

exists for all $x$ in $\mathbb{R}^D$ and does not depend on $x$. We denote by $\Pi(\omega)$ the value of this limit. $\Pi$ maps $\Omega$ to $\mathbb{R}^D$. We set

$$X = \Pi(\Omega).$$

One can prove that $X$ is a compact subset of $\mathbb{R}^D$ and that it is the unique subset of $X$ to satisfy $X = \cup_{i=1}^N \Psi_i(X)$. The set $X$ is said to be self-similar with respect to the family of $r$-similitudes $(\Psi_1, \ldots, \Psi_N)$ (these results are classical and can be found in [6], or [19]).

We will say that the family $(\Psi_i)$ satisfies the Moran open set condition (*cf.* [22]) if:

(A-0) there exists a non-empty open subset $U$ of $\mathbb{R}^D$ such that $\cup_{i=1}^N \Psi_i(U) \subset U$ and such that $\Psi_i(U) \cap \Psi_j(U) = \emptyset$ for $i \neq j$.

Each map $\Psi_i$ has a unique fixed point and when condition (A-0) is satisfied these fixed points are distinct (*cf.* Corollary IV-14 of [19]). We denote by $F_0$ the set of fixed points of the maps $\Psi_i$. By definition, $F_0$ is included in $X$.

DEFINITION 6.2. – *We say that $x \in F_0$ is an essential fixed point if there exist $y$ in $F_0$ and two distinct elements $i$ and $j$ of $\{1, \ldots, N\}$ such that $\Psi_i(x) = \Psi_j(y)$. We denote by $F$ the set of essential fixed points.*

Lindstrøm introduces the following conditions:

*(A-1) (Connectivity)* for all $i$ and $j$ in $\{1, \ldots, N\}$ there exists a sequence $i_1, \ldots, i_n$ of $\{1, \ldots, N\}$ such that $i_1 = i$, $i_n = j$ and $\Psi_{i_k}(X) \cap \Psi_{i_{k+1}}(X) \neq \emptyset$ for all $k \leq n - 1$.

*(A-2) (finite ramification)* for all distinct $n$-uplets $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_n)$ of $\{1, \ldots, N\}^n$

$$\Psi_{i_1} \circ \cdots \circ \Psi_{i_n}(X) \cap \Psi_{j_1} \circ \cdots \circ \Psi_{j_n}(X) = \Psi_{i_1} \circ \cdots \circ \Psi_{i_n}(F) \cap \Psi_{j_1} \circ \cdots \circ \Psi_{j_n}(F).$$

It is easy to see that with the definition of Section 2.1, $X$ is a finitely ramified fractal with $N$, $D = \#F$ and, if we suppose that $\Psi_1, \ldots \Psi_D$ are the $r$-similitudes whose fixed points are essential, $\mathcal{R}$ defined by: $(i,j)\mathcal{R}(i',j')$ if and only if $\Psi_i(x_j) = \Psi_{i'}(x_{j'})$, where $x_j$ and $x_{j'}$ are the fixed points of the $r$-similitudes $\Psi_j$ and $\Psi_{j'}$.

For all $x$ and $y$ in $\mathbb{R}^D$ denote by $H_{x,y}$ the hyperplane $H_{x,y} = \{z \in \mathbb{R}^D, \ \ |x-z| = |y-z|\}$. Let $U_{x,y}$ denote the orthogonal reflexion with respect to $H_{x,y}$.

We call $n$-cell a subset of $X$ of the type $\Psi_{i_1} \circ \cdots \circ \Psi_{i_n}(F)$ for an element $(i_1, \ldots, i_n)$ of $\{1, \ldots, N\}^n$ (N.B: $F$ is the unique 0-cell).

*(A-3) (symmetry):* For any distinct points $x$ and $y$ of $F$, $U_{x,y}$ maps a $n$-cell to a $n$-cell and if a $n$-cell has points in both open half spaces created by $H_{x,y}$ then it is globally invariant by $U_{x,y}$.

Lindstrøm gives the following definition:

DEFINITION 6.3. – *A nested fractal is a self-similar set $X$ associated with a family of $r$-similitudes satisfying conditions (A-0),. . .,(A-3) and such that $D = \#F \geq 2$.*

Let $G$ be the group generated by all the reflexions $U_{x,y}$ for $x$ and $y$ in $F$. It is easy to see that there exists an operation of $G$ on $\{1, \ldots, N\}$ such that

$$g.\Psi_i(x) = \Psi_{g.i}(g.x),$$

for all $g \in G$, $x \in \mathbb{R}^D$ and $i \in \{1, \ldots, N\}$.

Linstrøm has studied these fractals for equal weights $(\alpha_1, \ldots, \alpha_N) = (1, \ldots, 1)$.

In our framework, a nested fractal will be a finitely ramified fractal constructed as in Section 2.1 from the relation $\mathcal{R}$ defined above, the symmetry group $G$, and the weights $\alpha_1 = \cdots = \alpha_N = 1$.

The existence of the diffusion was proved by Linsdtrøm [19] (he used probabilistic methods to contruct the diffusion ; Kusuoka constructed the Dirichlet form [16]). Barlow proved the uniqueness in some particular cases (*cf.* [1] ). We will see that Theorem 5.1 proves the uniqueness in general (and also proves existence in all classical cases).

*Examples. – The Sierpinski gasket.* Here $D = 2$ and the similitudes $\Psi_1, \Psi_2, \Psi_3$ are homotheties with ratio $\frac{1}{2}$. In $\mathbb{C}$ these are given by:

$$\Psi_1(x) = \frac{x}{2} \ \ ; \ \ \Psi_3(x) = \frac{1}{2}(x-1) + 1 \ \ ;$$

$$\Psi_3(x) = \frac{1}{2}\left(x - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right) + \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Here $F = F_0 = \{0, 1, \frac{1}{2} + i\frac{\sqrt{3}}{2}\}$.

The group $G$ is the group of isometries of the triangle $F$, thus the group of permutations of $F$. It corresponds to Example 1.1 of Section 2.1.

*The Viscek set.* Here $D = 2$ and $N = 5$. The similitudes $\Psi_1, \ldots, \Psi_4$ are defined to be the homotheties with ratio $\frac{1}{3}$ and with centers the vertices of the square $\{(\pm 1, \pm 1)\}$. The similitude $\Psi_5$ is the homothety with center $(0,0)$ and with same ratio. $F$ is the set of fixed points of $\Psi_1, \ldots, \Psi_4$. $G$ is the group of isometries of the square $F$, *i.e.* $G$ is the fourth dihedral group $\mathcal{D}_4$. This example corresponds to Example 2.1 of Section 2.1.

*The snowflake.* Here $D = 2$ and $N = 7$ ; $\Psi_1, \ldots, \Psi_6$ are the homotheties with ratio $\frac{1}{3}$ and centers the vertices of a regular hexagon ; $\Psi_7$ is the homothety with same ratio and with center the center of the hexagon. $F$ is the set of fixed points of $\Psi_1, \ldots, \Psi_6$, *i.e.* the set of vertices of the regular hexagon. $G$ is the group of isometries of the regular hexagon $F$, *i.e.* the sixth dihedral group $\mathcal{D}_6$. It corresponds to Example 3 of Section 2.1.

*6.1.2. Application of Theorem 5.1*

We now consider a nested fractal. We first prove that:

LEMMA 6.4. – *Let $\mathcal{J}$ be a non-trivial $G$-relation and $x$, $y$ be two distinct points of $F$, then any equivalence class $I \in F/\mathcal{J}$ has points in both closed half-spaces created by $H_{x,y}$.*

*Proof.* – Let $I$ be in $F/\mathcal{J}$. The set $I$ has at least two elements since $G$ operates transitively on $F$, and $\mathcal{J} \neq 0$. Let $x$ and $y$ be two distinct points of $F$. Suppose that $I$ is contained in one open half-space delimited by $H_{x,y}$. Set $I' = U_{x,y}(I)$. The subset $I'$ is an equivalence class for $\mathcal{J}$ (*i.e.* $I' \in F/\mathcal{J}$) and $I' \neq I$. Let $d(z, H)$ denote the distance between a point $z$ and an hyperplane $H$.

Set $d = \min_{z \in I} d(z, H_{x,y})$ and let $t$ be an element of $I$ which realises this minimum. Let $v$ be in $I$, $v \neq t$. We set $t' = U_{x,y}(t)$ and $t'' = U_{v,t'}(t)$.

One has that:

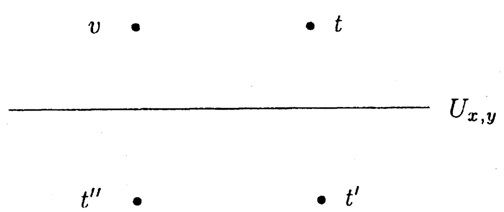
$$|t - t'| = |U_{v,t'}(t) - U_{v,t'}(t')| = |t'' - v|.$$

But $v\mathcal{J}t$ implies that $t'\mathcal{J}t''$ and thus that $t'' \in I'$. Finally

$$2d = |t - t'| = |v - t''| \geq d(v, H_{x,y}) + d(t'', H_{x,y}),$$

because $v$ and $t''$ are on both sides of $H_{x,y}$. Moreover $d(v, H_{x,y}) \geq d$ and $d(t'', H_{x,y}) \geq d$. This implies that $d(v, H_{x,y}) = d(t'', H_{x,y}) = d$ and that $v$ and $t''$ are symmetric with respect to $H_{x,y}$, *i.e.* that $U_{x,y}(v) = t''$.

Hence, the points $v, t, t', t''$ are in the same plane and are the vertices of a rectangle:




It is then easy to check that the relations $U_{x,y}(v) = t''$ and $U_{v,t'}(t) = t''$ imply that $v = t$ (because as $t$ and $t''$ are on both sides of the diagonal $(v, t')$, the relation $U_{v,t'}(t) = t''$ implies that they are on this diagonal and then $d(v, H_{x,y}) = d(t, H_{x,y})$ leads to the equality $v = t$). This concludes the lemma since we chose $v \neq t$. $\square$

For a preserved $G$-relation $\mathcal{J}$ we recall that the maps $\Psi_i$ can be viewed as maps from $F/\mathcal{J}$ to $F^{(1)}/\mathcal{J}^{(1)}$ (*cf.* Section 4.2.1). We still call 1-cells the subsets of $F^{(1)}/\mathcal{J}^{(1)}$ of the form $\Psi_i(F/\mathcal{J})$ (a 1-cell of $F^{(1)}/\mathcal{J}^{(1)}$ is then the image of a 1-cell of $F^{(1)}$ by the canonical surjection from $F^{(1)}$ over $F^{(1)}/\mathcal{J}^{(1)}$). The key result is the following lemma (one can remark that the argument used in the proof is very similar to the one of Lemma 3.5 of [19]).

LEMMA 6.5. – *Let $\mathcal{J}$ be a non-trivial preserved G-relation. If $I$ and $I'$ are two elements of $F/\mathcal{J}$ then there is a 1-cell of $F^{(1)}/\mathcal{J}^{(1)}$ which contains both $I$ and $I'$.*

N.B: In this statement $F/\mathcal{J}$ is regarded as a subset of $F^{(1)}/\mathcal{J}^{(1)}$.

REMARK 6.6. – In the case of the snowflake, for $\mathcal{J} = \mathcal{J}_1$, we remarked in Section 4.2.1 that the central 1-cell contains all the points of $F/\mathcal{J}_1$.

*Proof.* – Let $I$ and $I'$ be two distinct elements of $F/\mathcal{J}$. Let $x$ be in $I$ and $y$ be in $I'$. We set $H = H_{x,y}$, $U = U_{x,y}$. The hyperplane $H$ creates two open half-spaces, one contains $x$ the other one $y$.

Thanks to Lemma 6.4, there exists $x'$ in $I$ such that $x$ and $x'$ are on both sides of $H$. Moreover $x'$ cannot be in $H$ because if it was then we would have $U(x') = x'$, and $x\mathcal{J}x'$ would imply $y\mathcal{J}x'$, which is supposed not to be true. So $x'$ is in the same open half-space as $y$.

Since $x\mathcal{J}^{(1)}x'$ we can find a path $z_0 = x', \ldots, z_p = x$ in $F^{(1)}$ such that $z_k$ and $z_{k+1}$ are in a same 1-cell and such that $z_k\mathcal{J}^{(1)}z_{k+1}$ for $k \leq p-1$. We set:

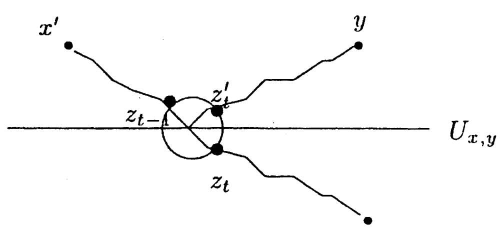
$$t = \inf\{k \in \{1, \ldots p\},\ z_k \text{ and } x \text{ are in the same open half-space}\}$$

We then define $z'_0, \ldots, z'_p$ by:

$$z'_k = z_k, \quad \forall k \leq t-1,$$

$$z'_k = U(z_k), \quad \forall k \geq t.$$

$z'_0, \ldots, z'_p$ is a walk from $x'$ to $y$ in $F^{(1)}$.




If $k \neq t-1$ then $z'_k$ and $z'_{k+1}$ are in a same 1-cell and $z'_k\mathcal{J}^{(1)}z'_{k+1}$.

If $z_{t-1} \in H$ then $Uz_{t-1} = z_{t-1}$ and $z'_{t-1}\mathcal{J}^{(1)}z'_t$, so, by transitivity $x'\mathcal{J}^{(1)}y$ which is not true since $x'\not{\mathcal{J}}y$. The points $z_{t-1}$ and $z_t$ are thus in both open half-spaces. Now, using Assumption (A-3), the cell which contains both $z_{t-1}$ and $z_t$ is globally invariant by $U$, so it implies that $z'_{t-1}$ and $z'_t$ are in a same 1-cell.

By construction $x'\mathcal{J}^{(1)}z'_{t-1}$ and $y\mathcal{J}^{(1)}z'_t$, so $z'_{t-1}$ (resp. $z'_t$) is in relation with the points of $I$ (resp. $I'$) with respect to $\mathcal{J}^{(1)}$. This concludes the proof of the lemma. □

We deduce the following proposition from the previous lemma:

PROPOSITION 6.7. – *Let $\mathcal{J}$ be a non-trivial preserved G-relation, then $\overline{\rho}_{\mathcal{J}} < 1$ and $\underline{\rho}_{F/\mathcal{J}} \geq 1$.*

*Proof.* – Let us first prove $\underline{\rho}_{F/\mathcal{J}} \geq 1$.

We recall that $A^0_{F/\mathcal{J}} \in M_{F/\mathcal{J}}$ is the Dirichlet form on $F/\mathcal{J}$ with conductivities equal to 1 on each bond, i.e:

$$A^0_{F/\mathcal{J}}(f,f) = \frac{1}{2} \sum_{X,Y \in F/\mathcal{J}} (f(X) - f(Y))^2, \ \ \forall f \in E_{F/\mathcal{J}}.$$

Let $(A^0_{F/\mathcal{J}})^{(1)}$ be the Dirichlet form on $F^{(1)}/\mathcal{J}^{(1)}$ associated with $A^0_{F/\mathcal{J}}$:

$$(A^0_{F/\mathcal{J}})^{(1)}(g,g) = \sum_{i=1}^{N} A^0_{F/\mathcal{J}}(g \circ \Psi_i, g \circ \Psi_i), \ \ \forall g \in E^{(1)}_{F/\mathcal{J}}.$$

N.B.: we recall that $\alpha_i = 1$ for all $i$.

Let $f$ be in $E_{F/\mathcal{J}}$ and $f^{(1)} \in E^{(1)}_{F/\mathcal{J}}$ be its harmonic continuation with respect to the Dirichlet form $(A^0_{F/\mathcal{J}})^{(1)}$. One has that:

$$\begin{aligned}
T_{F/\mathcal{J}} A^0_{F/\mathcal{J}}(f) &= \sum_{i=1}^{N} A^0_{F/\mathcal{J}}(f^{(1)} \circ \Psi_i) \\
&= \sum_{i=1}^{N} \frac{1}{2} \sum_{X,Y \in F/\mathcal{J},\, X \neq Y} (f^{(1)} \circ \Psi_i(X) - f^{(1)} \circ \Psi_i(Y))^2 \\
&\geq \frac{1}{2} \sum_{X,Y \in F/\mathcal{J},\, X \neq Y} (f(X) - f(Y))^2 \\
&= A^0_{F/\mathcal{J}}(f,f).
\end{aligned}$$

N.B.: The inequality (line 3) comes from the fact that $f^{(1)}_{|F/\mathcal{J}} = f$ and from Lemma 6.5.

Thus $\underline{\rho}_{F/\mathcal{J}}(A^0_{F/\mathcal{J}}) \geq 1$, so $\underline{\rho}_{F/\mathcal{J}} \geq 1$.

Let us now prove that $\overline{\rho}_{\mathcal{J}} < 1$.

We recall that $A^0_{\mathcal{J}} \in M_{\mathcal{J}}$ is the Dirichlet form on $F$, such that the bond conductivity between two points $x$ and $y$ is 1 if $x\mathcal{J}y$ and 0 if $x \not\mathcal{J} y$.

Let $I$ be in $F/\mathcal{J}$, and $f \in E$ be a zero mean function null out of $I$ (*i.e.* $f \in V_I$). Let $f^{(1)}$ be the function on $F^{(1)}$ defined by:

$$f^{(1)} = \begin{cases} f(x) & \text{for } x \text{ in } F, \\ 0 & \text{for } x \in F^{(1)} \setminus F. \end{cases}$$

We have $(A^0_{\mathcal{J}})^{(1)}(f^{(1)}) \geq T_{\mathcal{J}} A^0_{\mathcal{J}}(f)$ by definition. Moreover:

$$(A^0_{\mathcal{J}})^{(1)}(f^{(1)}) = \sum_{x \in F} A^0_{\mathcal{J}}(f^{(1)} \circ \Psi_x)$$

N.B.: Here $F$ is regarded as the subset $\{1, \ldots, D\}$ of $\{1, \ldots, N\}$ (*i.e.* any element $x \in F$ is the fixed point of the unique similitude $\Psi_x$).

So, as $\Psi_x(x) = x$, one has that:

$$\begin{aligned}(A^0_{\mathcal{J}})^{(1)}(f^{(1)}) &= \sum_{x\in F} A^0_{\mathcal{J}}(f(x)1_{\{x\}}) \\ &= \sum_{x\in I} A(f(x)1_{\{x\}}) \\ &= A^0_{\mathcal{J}}(f,f) - \sum_{\substack{x,y\in I\\ x\neq y}} f(x)f(y)A^0_{\mathcal{J}}(1_{\{x\}},1_{\{y\}}).\end{aligned}$$

But, since the bond conductivity is 1 between $x$ and $y$ in $I$, one has that $A^0_{\mathcal{J}}(1_{\{x\}},1_{\{y\}}) = -1$ and:

$$\begin{aligned}(A^0_{\mathcal{J}})^{(1)}(f^{(1)}) &= A^0_{\mathcal{J}}(f) + \sum_{\substack{x,y\in I\\ x\neq y}} f(x)f(y) \\ &= A^0_{\mathcal{J}}(f) - \sum_{x\in I} f^2(x).\end{aligned}$$

N.B.: The last relation comes from the fact that $f$ has zero mean value.

Thus, we have proved that $T_{\mathcal{J}}A^0_{\mathcal{J}}(f) < A^0_{\mathcal{J}}(f,f)$. It implies that $\overline{\rho}_{\mathcal{J}}(A^0_{\mathcal{J}}) < 1$ and thus that $\overline{\rho}_{\mathcal{J}} < 1$. □

From Proposition 6.7 and (ii) of Theorem 5.1 (with Remark 5.2) we get:

- the existence of a fixed point if (H) is satisfied
- the uniqueness of the fixed point in any case.

N.B.: I do not know any nested fractal which does not satisfy (H), although there probably exist some.

However, as Linsdtrøm proved the existence in any case, and since the fixed points are regular thanks to Proposition 3.8, one has that:

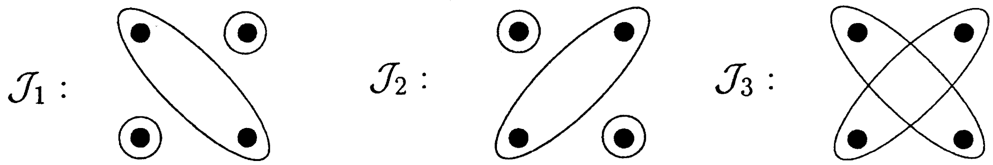
THEOREM 6.8. – *On all nested fractals there exists a unique (up to a multiplicative constant) non-degenerate self-similar Dirichlet form associated with equal weights.*

### 6.2. An example where Theorem 5.1 cannot be applied

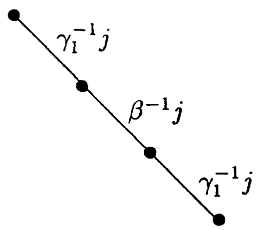
We take Example 2.2 of Section 2.1, *i.e.* the Viscek set with the symmetry group $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by the orthogonal reflexions with respect to the diagonals of the square $F$. The symmetries allow us to choose $\alpha_1 = \alpha_3 = \gamma_1$, $\alpha_2 = \alpha_4 = \gamma_2$ and $\alpha_5 = \beta$.

When $\gamma_1 = \gamma_2 = \beta$, Metz proved that $T$ has infinitely many fixed points (*cf.* [20] ). In general, in [27] , we proved that if $\gamma_1 = \gamma_2$ then $T$ has infinitely many fixed points and if $\gamma_1 \neq \gamma_2$ then $T$ has no fixed point (in fact, the Viscek set is a fractal tree, *i.e.* has no loop, and consequently $T$ can be expressed as a non-negative matrix that we explicitely compute).

In this case there are 3 non-trivial $G$-relations that we draw on the following picture. We easily see that they are all preserved:

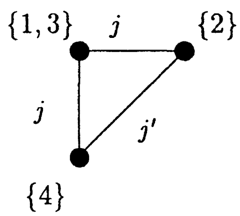



We first study the map $T_{\mathcal{J}_1}$. The set $M_{\mathcal{J}_1}$ has dimension 1. An element $A$ of $M_{F/\mathcal{J}_1}$ is defined by the bond conductivity between the two points in relation with respect to $\mathcal{J}_1$. We denote it by $j$. The electrical network associated with $A^{(1)}$ is drawn on the following picture:




If $\tilde{j}$ is the conductivity associated with $T_{\mathcal{J}_1}A$ we easily check that $\tilde{j} = (2\gamma_1 + \beta)^{-1}j$ (*cf.* Lemma 1.20). Thus, $\underline{\rho}_{\mathcal{J}_1} = \overline{\rho}_{\mathcal{J}_1} = (2\gamma_1 + \beta)^{-1}$.

The set $F/\mathcal{J}_1$ contains 3 points. Precisely, one has that $F/\mathcal{J}_1 = \{\{1,3\},\{2\},\{4\}\}$. An element $A$ of $M_{F/\mathcal{J}_1}$ is determinated by 2 conductivities $j$ and $j'$ as on the following picture:

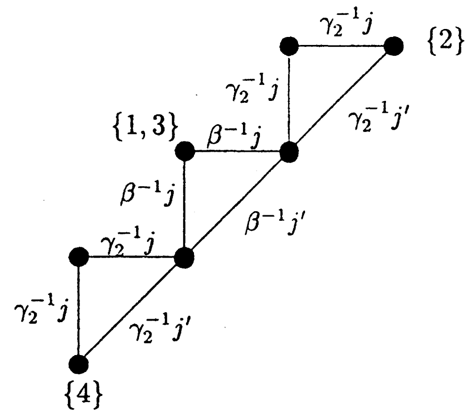



Denote by $c$ and $c'$ the effective conductances associated with $A$ (*i.e.* $c = c_A(\{2\},\{1,3\}) = c_A(\{4\},\{1,3\})$ and $c' = c_A(\{2\},\{4\})$). An easy calculation gives (*cf.* Lemma 1.20):

$$c = \frac{j(j+2j')}{j+j'}, \tag{6.1}$$

$$c' = j' + \frac{1}{2}j. \tag{6.2}$$

The electrical network associated with $A^{(1)}$ is represented on the following picture:




Let $\tilde{j}$, $\tilde{j}'$, $\tilde{c}$ and $\tilde{c}'$ be the values associated with the Dirichlet form $T_{F/\mathcal{J}_1} A$ ($\tilde{c}$ and $\tilde{c}'$ are obtained from $\tilde{j}$ and $\tilde{j}'$ by (6.1) and (6.2)). Thanks to Lemma 1.20 we get:

$$\tilde{c} = \frac{cc'}{\gamma_2 c' + \beta c}, \tag{6.3}$$

$$\tilde{c}' = (2\gamma_2 + \beta)^{-1} c'. \tag{6.4}$$

Thanks to the symmetries we know that $c' = A(f)$ and $\tilde{c}' = T_{F/\mathcal{J}_1} A(f)$ for the same function $f$ defined by $f(\{2\}) = 0$, $f(\{4\}) = 1$, $f(\{1,3\}) = \frac{1}{2}$. Thus,

$$\underline{\rho}_{F/\mathcal{J}_1}(A) \leq \frac{\tilde{c}'}{c'} \leq (2\gamma_2 + \beta)^{-1}.$$

We are now going to prove that $\underline{\rho}_{F/\mathcal{J}_1} = (2\gamma_2 + \beta)^{-1}$. In order to do that we let $\frac{j}{j'}$ converge to 0. But if $\frac{j}{j'}$ converges to 0, so does $\frac{\tilde{j}}{\tilde{j}'}$ and (6.1) and (6.2) give:

$$c \sim 2j \qquad c' \sim j'$$
$$\tilde{c} \sim 2\tilde{j} \qquad \tilde{c}' \sim \tilde{j}'$$

(N.B.: it means that these values are equivalent when $\frac{j}{j'}$ converges to 0.)
Thus, from (6.3) and (6.4) we deduce:

$$\tilde{j}' \sim (2\gamma_2 + \beta)^{-1} j,$$
$$\tilde{j} \sim \gamma_2^{-1} j.$$

Since $\gamma_2^{-1} > (2\gamma_2 + \beta)^{-1}$, it implies that:

$$T_{F/\mathcal{J}_1} A \geq ((2\gamma_2 + \beta)^{-1} + o(1))A$$

($o(1)$ is a function converging to 0 with $\frac{j}{j'}$).

Thus,

$$\underline{\rho}_{F/\mathcal{J}_1}(A) \geq (2\gamma_2 + \beta)^{-1} + o(1).$$

This proves that $\underline{\rho}_{F/\mathcal{J}_1} = (2\gamma_2 + \beta)^{-1}$.

We obtain the values for $\mathcal{J}_2$ by permuting the indices.

In conclusion, we have

- if $\gamma_1 \neq \gamma_2$ then either $\overline{\rho}_{\mathcal{J}_1} > \underline{\rho}_{F/\mathcal{J}_1}$ or $\overline{\rho}_{\mathcal{J}_2} > \underline{\rho}_{F/\mathcal{J}_2}$ so (i) of Theorem 5.1 can be applied. $T$ has no fixed point.
- if $\gamma_1 = \gamma_2$ then we are in the critical case and we cannot apply Theorem 5.1. We recall that in this case Metz [20] proved that $T$ has infinitely many fixed points.

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