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ALGEBRAICALLY CONSTRUCTIBLE FUNCTIONS

BY CLINT McCRORY AND ADAM PARUSIŃSKI

ABSTRACT. – An algebraic version of Kashiwara and Schapira's calculus of constructible functions is used to describe local topological properties of real algebraic sets, including Akbulut and King's numerical conditions for a stratified set of dimension three to be algebraic. These properties, which include generalizations of the invariants modulo 4, 8, and 16 of Coste and Kurdyka, are defined using the link operator on the ring of constructible functions.

RÉSUMÉ. – On propose une version algébrique réelle du formalisme des fonctions constructibles au sens de Kashiwara et Schapira. On démontre plusieurs propriétés fondamentales de ces fonctions, en particulier le fait que les fonctions algébriquements constructibles sont préservées par la moitié de l'opérateur de link. Ce dernier résultat implique plusieurs propriétés topologiques locales des ensembles algébriques réels, en particulier les conditions d'Akbulut et King en dimension 3 et les conditions modulo 4, 8 et 16, de Coste et Kurdyka.

In 1970 Sullivan [Su] proved that if X is a real analytic set and $x \in X$, then the Euler characteristic of the link of x in X is even. Ten years later, Benedetti and Dedò [BD], and independently Akbulut and King [AK1], proved that Sullivan's condition gives a topological characterization of real algebraic sets of dimension less than or equal to two. Using their theory of resolution towers, Akbulut and King introduced a finite set of local "characteristic numbers" of a stratified space X of dimension three, such that X is homeomorphic to a real algebraic set if and only if all of these numbers vanish [AK2].

In 1992 Coste and Kurdyka [CK] proved that if Y is an irreducible algebraic subset of the algebraic set X and $x \in Y$, then the Euler characteristic of the link of Y in X at x, which is even by Sullivan's theorem, is generically constant mod 4. They also introduced invariants mod 2^k for chains of k strata, and they showed how to recover the Akbulut-King numbers from their mod 4 and mod 8 invariants. The Coste-Kurdyka invariants were generalized and given a simpler description in [MP] using complexification and monodromy.

We introduce a new approach to the Akbulut-King numbers and their generalizations which is motivated by the theory of Stiefel-Whitney homology classes, as was Sullivan's original theorem. We use the ring of constructible functions on X, which has been

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systematically developed by Kashiwara and Schapira [KS] [Sch] in the subanalytic setting and by Viro [V]. Their calculus of constructible functions includes the fundamental operations of duality and pushforward, which correspond to standard operations in sheaf theory.

Our primary object of study is the ring of algebraically constructible functions on the real algebraic set X. We say that the function $\varphi : X \to \mathbb{Z}$ and the stratification $\mathcal{S}$ of X are *compatible* if φ is constant on each stratum of $\mathcal{S}$. If X is a complex algebraic set, then φ is said to be complex algebraically constructible if there exists a complex algebraic stratification $\mathcal{S}$ of X which is compatible with φ. The pushforward of a complex algebraically constructible function by a complex algebraic map is complex algebraically constructible.

For real algebraic sets the situation is more complicated. By an algebraic stratification of the real algebraic set X, we mean a stratification $\mathcal{S}$ of X with strata of the form $Y \setminus Y'$, where Y and Y' are algebraic sets. Thus the strata are not necessarily connected. If X is a real algebraic set and $\varphi : X \to \mathbb{Z}$, let us say that φ is *strongly algebraically constructible* if there is an algebraic stratification $\mathcal{S}$ of X which is compatible with φ. The pushforward of a strongly algebraically constructible function by an algebraic map is not necessarily strongly algebraically constructible. (Consider for example $\varphi = f_* \mathbf{1}_{\mathbb{R}}$, where $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then $\varphi(x) = 2$ if $x > 0$, $\varphi(0) = 1$, and $\varphi(x) = 0$ if $x < 0$.) On the other hand, we say that φ is *semialgebraically constructible* if there is a semialgebraic stratification of X which is compatible with φ. The pushforward of a semialgebraically constructible function by a continuous semialgebraic map is semialgebraically constructible. But information about the algebraic structure of X is lost by passing to the ring of semialgebraically constructible functions.

To solve this dilemma we adopt a definition of algebraic constructibility which is not solely in terms of compatibility with a stratification. If X is a real algebraic set, we say that $\varphi : X \to \mathbb{Z}$ is *algebraically constructible* if φ is the pushforward of a strongly algebraically constructible function by an algebraic map. It follows that the pushforward of an algebraically constructible function is algebraically constructible; however, not every semialgebraically constructible function is algebraically constructible. (For example let $X = \mathbb{R}$ and let $\varphi(x) = 1$ if $x \geq 0$, $\varphi(x) = 0$ if $x < 0$.) We detect the difference between algebraically constructible functions and semialgebraically constructible functions by means of the topological *link operator* Λ on the ring of semialgebraically constructible functions. The link operator generalizes the link of a point in a space, and it is related to the Kashiwara-Schapira duality operator D by $D\varphi = \varphi - \Lambda\varphi$.

Our main results are the following. Using resolution of singularities, we prove that if φ is an algebraically constructible function then $\frac{1}{2}\Lambda\varphi$ is algebraically constructible, and in particular $\frac{1}{2}\Lambda\varphi$ is integer-valued. Another proof of this statement, which does not use the resolution of singularities, has been given recently in [PS]. Then we give a new description of the Akbulut-King numbers in terms of the operator $\widetilde{\Lambda} = \frac{1}{2}\Lambda$, and we prove that if X is a semialgebraic set of dimension less than or equal to three, then X is homeomorphic to an algebraic set if and only if all of the functions obtained from $\mathbf{1}_X$ by the arithmetic operations $+, -, *$, together with the operator $\widetilde{\Lambda}$, are integer-valued.

We prove the basic properties of (semialgebraically) constructible functions in section 1. We derive some properties of constructible functions φ which are self-dual ($D\varphi = \varphi$) or anti-self-dual ($D\varphi = -\varphi$). If φ is compatible with a stratification $\mathcal{S}$ which has only even (resp. odd) dimensional strata, then φ is self-dual (resp. anti-self-dual). If φ is self-dual (resp. anti-self-dual), then the Stiefel-Whitney homology classes [FM] satisfy $\beta w_i(\varphi) = w_{i-1}(\varphi)$ for i even (resp. odd), where β is the Bockstein homomorphism.

In section 2 we introduce algebraically constructible functions, and we give examples of functions which are constructible but not algebraically constructible, and functions which are algebraically constructible but not strongly algebraically constructible. We prove that if φ is algebraically constructible then $\widetilde{\Lambda}(\varphi)$ is algebraically constructible. Also we show that the specialization of an algebraically constructible function is algebraically constructible. We prove that if φ is a constructible function on an algebraic set of dimension d, then $2^d\varphi$ is algebraically constructible.

A constructible function φ is *Euler* if the function $\widetilde{\Lambda}(\varphi)$ is integer-valued. By a *completely Euler* function we mean a constructible function φ such that all the functions obtained from φ by means of the arithmetic operations $+, -, *$ and the operator $\widetilde{\Lambda}$ are integer-valued. In section 3 we analyze such functions in low dimensions. We give computable conditions to determine whether a constructible function φ is completely Euler, in the case that φ has support of dimension less than or equal to 2, and to determine whether $\mathbf{1}_X$ is completely Euler, in the case that X has dimension less than or equal to 3.

In section 4 we apply the preceding results to the topology of real algebraic sets. We give a new proof of our theorem [MP] concerning the iterates of the relative link operator $\Lambda_Y X$ for Y an algebraic subset of X: If $X_1, \ldots, X_k$ is an ordered collection of algebraic subsets of X, then $\varphi = \Lambda_{X_1} \cdots \Lambda_{X_k} \mathbf{1}_X$ is divisible by 2^k, and if Y is an irreducible algebraic subset of X, then φ is generically constant mod 2^{k+1} on Y. We give a new description of Akbulut and King's necessary and sufficient conditions for a compact semialgebraic set X of dimension three to be homeomorphic to an algebraic set. We prove that X satisfies the Akbulut-King conditions if and only if $\mathbf{1}_X$ is completely Euler. We give a similar description of Akbulut and King's conditions for a stratified semialgebraic set to be homeomorphic to a stratified real algebraic set, by a homeomorphism which preserves the strata.

In section 5 we introduce *Nash constructible* functions, and we show that a closed semialgebraic set S is symmetric by arcs [Ku] if and only if $\mathbf{1}_S$ is Nash constructible.

An appendix contains proofs of some elementary foundational results.

For the definitions and properties of real algebraic and semialgebraic sets and maps, and semialgebraic stratifications, we refer the reader to [BR]. We will always assume that semialgebraic maps are continuous. By a real algebraic set we mean the locus of zeros of a finite set of polynomial functions on $\mathbb{R}^n$.

1. Constructible functions

Let X be a real algebraic set. A function $\varphi : X \to \mathbf{Z}$ is called (*semialgebraically*) *constructible* if it admits a presentation as a finite sum

$$\varphi = \sum m_i \mathbf{1}_{X_i}, \tag{1.1}$$

where for each i, X_i is a semialgebraic subset of X, $\mathbf{1}_{X_i}$ is the characteristic function of X_i, and m_i is an integer. Denote by $\mathrm{F}(X)$ the ring of constructible functions on X, with the usual operations of addition and multiplication. The presentation (1.1) is not unique, but one can always find a presentation with all X_i closed in X. In what follows, unless otherwise stated, we always assume that the X_i are closed. If the support of φ is compact, then we may choose all X_i compact. Then the *Euler integral* of φ is defined as

$$\int \varphi = \sum m_i \chi(X_i).$$

By additivity of the Euler characteristic, the Euler integral does not depend on the presentation (1.1) of φ, provided all X_i are compact. Suppose Y is a semialgebraic subset of X such that the intersection of Y with the support of φ is compact. Then by $\int_Y \varphi$ we mean the Euler integral of the restriction of φ to Y.

Let $f : X \to Y$ be a (continuous) semialgebraic map of real algebraic sets. If $\psi \in \mathrm{F}(Y)$, the *inverse image*, or *pullback*, of ψ by f is defined by

$$f^*\psi(x) = \psi(f(x)),$$

and $f^*\psi$ is a constructible function on X.

Assume that $f : X \to Y$ restricted to the support of $\varphi \in \mathrm{F}(X)$ is proper. Then the *direct image*, or *pushforward*, $f_*\varphi \in \mathrm{F}(Y)$ is given by the formula

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi.$$

Suppose that X is embedded in $\mathbf{R}^n$. Then we define the *link* of φ as the constructible function on X given by

$$\Lambda\varphi(x) = \int_{S(x,\varepsilon)} \varphi,$$

where $\varepsilon > 0$ is sufficiently small, and $S(x,\varepsilon)$ denotes the ε-sphere centered at x. The function $\Lambda\varphi$ is independent of the embedding of X in $\mathbb{R}^n$. This follows from the fact that the link of a point in a semialgebraic set is well-defined up to semialgebraic homeomorphism (*cf.* the Appendix). The duality operator D on constructible functions, introduced by Kashiwara and Schapira in [KS], satisfies

$$D\varphi = \varphi - \Lambda\varphi,$$

which is equivalent to formula (2.7) of [Sch].

1.2. PROPOSITION

(i) $D(D\varphi) = \varphi$,

(ii) $f_*D = Df_*$,

(iii) $(g \circ f)_* = g_* \circ f_*$.

Proof. – (i)-(iii) are proved in [Sch] using the corresponding operations on constructible sheaves. For a different proof see the Appendix below. □

1.3. COROLLARY

(i) $\Lambda \circ \Lambda = 2\Lambda$,

(ii) $f_*\Lambda = \Lambda f_*$,

(iii) $\int \Lambda\varphi = 0$.

Proof. – (i)-(ii) are clear. If the support of φ is compact then (iii) follows from (ii). Indeed, let $f : X \to P$ be a constant map to a one point space P. Then

$$\int \Lambda\varphi = f_*\Lambda\varphi(P) = \Lambda f_*\varphi(P) = 0,$$

since the link of any constructible function on P vanishes. In general we need only the compactness of the support of $\Lambda\varphi$ for (iii) since this case reduces to the previous one by (i). □

A constructible function φ is called *self-dual* if $D\varphi = \varphi$, or equivalently $\Lambda\varphi = 0$. Similarly, φ is *anti-self-dual* if $D\varphi = -\varphi$, or equivalently $\Omega\varphi = 0$, where $\Omega\varphi = \varphi + D\varphi$.

We say that $\varphi \in \mathrm{F}(X)$ is *Euler* if $\Lambda\varphi(x)$ is even for all $x \in X$. Clearly every self-dual and every anti-self-dual function is Euler. On the other hand, every Euler function admits a canonical decomposition into self-dual and anti-self-dual parts,

$$\varphi = \tilde{\Omega}\varphi + \tilde{\Lambda}\varphi, \tag{1.4}$$

where $\tilde{\Lambda} = \frac{1}{2}\Lambda$ and $\tilde{\Omega} = \frac{1}{2}\Omega$. By (ii) of Proposition 1.2 the direct image of a function which is Euler (resp. self-dual, anti-self-dual) is Euler (resp. self-dual, anti-self-dual). Note that whether a constructible function is Euler it depends only on its reduction modulo 2. This is no longer true for self-dual or anti-self-dual functions.

We list here some more consequences of Proposition 1.2 which we use in the sequel:

$$\begin{aligned} &D \circ \Lambda = -\Lambda \circ D, \quad D \circ \Omega = -\Omega \circ D \\ &\tilde{\Lambda} \circ \tilde{\Lambda} = \tilde{\Lambda}, \quad \tilde{\Omega} \circ \tilde{\Omega} = \tilde{\Omega}, \quad \tilde{\Lambda} \circ \tilde{\Omega} = \tilde{\Omega} \circ \tilde{\Lambda} = 0. \end{aligned} \tag{1.5}$$

Let $\mathcal{S}$ be a semialgebraic stratification of X. We say that $\mathcal{S}$ is *locally trivial* if X as a stratified set can be topologically trivialized locally along each stratum of $\mathcal{S}$. For instance every Whitney stratification is locally trivial. Also a semialgebraic triangulation of X gives rise to a locally trivial stratification of X by taking open simplices as strata. We say that $\varphi \in \mathrm{F}(X)$ and $\mathcal{S}$ are *compatible* if φ is locally constant on strata of $\mathcal{S}$. For each constructible function $\varphi \in \mathrm{F}(X)$ there exist a Whitney stratification of X and a triangulation of X that are compatible with φ.

Although Proposition 1.2 is elementary, it carries a nontrivial information. For instance, (i) of Proposition 1.2 implies the well-known fact that the links of points in complex algebraic sets have Euler characteristic zero. Actually we can show a more general fact:

1.6. PROPOSITION. – *Let φ be a constructible function on X compatible with a locally trivial stratification $\mathcal{S}$. Then if all strata of $\mathcal{S}$ are of even (resp. odd) dimension then φ is self-dual (resp. anti-self-dual).*

In particular if all strata of $\mathcal{S}$ are of odd dimension and the support of φ is compact then $\int \varphi = 0$.

Proof. – We show the even-dimensional case. The proof is by induction on the dimension of the support $\operatorname{supp}\varphi$. First note that the proposition holds generically on $\operatorname{supp}\varphi$. Indeed the geometric links of a single stratum are odd-dimensional spheres and have zero Euler characteristic.

Next, by the assumption on local topological triviality, $\mathcal{S}$ is also compatible with $\Lambda\varphi$. But, by our previous observation, the dimension of $\operatorname{supp}\Lambda\varphi$ is strictly smaller than the dimension of $\operatorname{supp}\varphi$. Hence by inductive hypothesis the statement holds for $\Lambda\varphi$; that is, $\Lambda\Lambda\varphi = 0$, which by virtue of (i) of Corollary 1.3 implies $\Lambda\varphi = 0$. This completes the proof of the even-dimensional case.

The proof in the odd-dimensional case is similar and uses Ω instead of Λ. The last statement follows from Corollary 1.3 (iii). □

In contrast to the direct image, the inverse image does not have good functorial properties. In particular, it commutes neither with the duality operator nor with the link operator. The following proposition, which we prove in the Appendix, shows that the restriction to a generic slice and the duality operator anticommute.

1.7. PROPOSITION. – *Let $h : X \to \mathbf{R}$ be semialgebraic and let $\varphi \in \mathrm{F}(X)$. Let φ_t denote the restriction of φ to the fibre $X_t = h^{-1}(t)$. Then for generic $t \in \mathbf{R}$ we have*

$$(D\varphi)_t = -D\varphi_t, \quad (\Lambda\varphi)_t = \Omega\varphi_t, \quad (\Omega\varphi)_t = \Lambda\varphi_t. \quad \square$$

Let $f : X \to \mathbf{R}$ be semialgebraic, and let $x \in X_0 = f^{-1}(0)$. Fix a local semialgebraic embedding $(X, x) \subset (\mathbf{R}^n, 0)$. Then we define the *positive*, resp. *negative*, *Milnor fibre* of f at x by

$$F_f^+(x) = B(0,\varepsilon) \cap f^{-1}(\delta),$$
$$F_f^-(x) = B(0,\varepsilon) \cap f^{-1}(-\delta),$$

where $B(0,\varepsilon)$ is the ball of radius ε centered at 0 and $0 < \delta \ll \varepsilon \ll 1$.

Let $\varphi \in \mathrm{F}(X)$. We define the *positive* (resp. *negative*) *specialization* of φ with respect to f by

$$(\Psi_f^+\varphi)(x) = \int_{F_f^+(x)} \varphi,$$
$$(\Psi_f^-\varphi)(x) = \int_{F_f^-(x)} \varphi.$$

Both specializations are well-defined, and they are constructible functions supported in X_0.

If Y is a closed semialgebraic subset of X, then there exists, at least locally, a non-negative semialgebraic function $f : X \to \mathbf{R}$ such that $Y = f^{-1}(0)$. For instance, if X is a subset of $\mathbf{R}^n$ then we may take f to be the distance to Y. If $x \in Y$, then by the *link along Y at x*, denoted $\mathrm{lk}_x(Y;X)$, we mean the positive Milnor fibre of f at x. If φ is a constructible function on X, by the *link of φ along Y* we mean the positive specialization of φ with respect to f, denoted by $\Lambda_Y\varphi$.

The link of X at $x \in X$, which we denote by $\operatorname{lk}(x; X)$, is well-defined up to semialgebraic homeomorphism, as proven in [CK, Prop. 1] using [SY]. A similar argument using [H] shows that the link of X along Y at x is well-defined up to a semialgebraic homeomorphism. A sheaf-theoretic construction of [DS], see also [MP, Remark 2], shows that the cohomology of $\operatorname{lk}_x(Y; X)$, with coefficents in any semialgebraically constructible sheaf, is well-defined. This construction shows that the Euler characteristic of the link is a topological invariant, which we also prove by elementary means in the Appendix.

We note also the the Milnor fibres of $f : X \to \mathbf{R}$ are special cases of the link construction since $(\Psi_f^{\pm}\varphi)(x) = \operatorname{lk}_x(Y; X_{\pm})$, where $X_+ = f^{-1}[0, \infty)$, $X_- = f^{-1}(-\infty, 0]$.

1.8. Proposition. – *Let $f : X \to \mathbf{R}$ be semialgebraic and continuous. Let $\varphi \in \mathrm{F}(X)$. Then $\Psi_f^+\varphi + \Psi_f^-\varphi$ does not depend on f but only on $Y = f^{-1}(0)$ and equals*

$$\Lambda_Y\varphi = \varphi|_Y - D((D\varphi)|_Y) = \Lambda(\varphi|_Y) - \Lambda((\Lambda\varphi)|_Y) + (\Lambda\varphi)|_Y.$$

Proof. – If one replaces f by f^2 one gets $\Psi_f^+\varphi + \Psi_f^-\varphi = \Lambda_Y\varphi$. The formula is shown in the Appendix. □

1.9. Corollary. – *Let $f : X \to \mathbf{R}$ be semialgebraic and continuous and let $\varphi \in \mathrm{F}(X)$. Let $Y = f^{-1}(0)$. Then*

$$\Lambda_Y \circ D = -D \circ \Lambda_Y, \quad \Lambda_Y \circ \Lambda = \Omega \circ \Lambda_Y, \quad \Omega_Y \circ \Lambda = \Lambda \circ \Lambda_Y,$$

and similar formulas hold if we replace Λ_Y by Ψ_f^+ or Ψ_f^-.

Proof. – The formulas for Λ_Y follow immediately from Propostion 1.8. The case of the specializations Ψ_f^+ or Ψ_f^- can be reduced to the link Λ_Y by replacing X by $X_+ = f^{-1}[0, \infty)$ or $X_- = f^{-1}(-\infty, 0]$. □

Let X be an algebraic subset of the nonsingular real algebraic set M. In [FM] there is defined for each $\varphi \in \mathrm{F}(X)$ the *conormal cycle* $N^*(\varphi)$, which is a Legendrian cycle in ST^*M, the cotangent ray space of M. The Euler integral of φ can be computed from the conormal cycle by a generalization of the Gauss-Bonnet theorem, and the duality operator on constructible functions corresponds to the action of the antipodal map (multiplication by -1 in the fibres) on the conormal cycle.

Fu and McCrory show that, for each $i = 0, 1, 2, \ldots$, there exists a unique additive natural transformation w_i from $\mathbf{Z}_2$-valued compactly supported Euler constructible functions to mod 2 homology, such that if X is nonsingular and purely d-dimensional, then $w_i(\mathbf{1}_X)$ is Poincaré dual to the classical cohomology Stiefel-Whitney class $w^{d-i}(X)$. For $\varphi \in \mathrm{F}(X)$, the class $w_i(\varphi) \in H_i(X; \mathbf{Z}_2)$ is the ith *Stiefel-Whitney class* of φ. Clearly the Stiefel-Whitney classes can be defined for a $\mathbf{Z}$-valued constructible function φ by first taking the reduction mod 2 of φ. However by doing that one loses some information–for instance at the level of $\mathbf{Z}_2$ coefficients one cannot distinguish self-dual and anti-self-dual functions. Instead one may follow the construction of [FM], which uses conormal cycles. In particular one gets the following proposition, which generalizes the well-known fact that for a manifold M of pure dimension n, $w_{n-2k-1}(M)$ can be defined over $\mathbf{Z}$; this is implied by the fact that $w_{n-2k-1}(M)$ is the image of $w_{n-2k}(M)$ by the Bockstein homomorphism $\beta : H_{n-2k}(M; \mathbf{Z}_2) \to H_{n-2k-1}(M; \mathbf{Z}_2)$ (*cf.* [HT]).

1.10. Proposition. – *Let X be a real algebraic set, and let $\varphi \in \mathrm{F}(X)$. If φ is self-dual, then the odd dimensional Stiefel-Whitney classes $w_{2k-1}(\varphi)$ are the images of the even dimensional Stiefel-Whitney classes $w_{2k}(\varphi)$ by the Bockstein homomorphism.*

If φ is anti-self-dual then the even dimensional Stiefel-Whitney classes are the images of the odd dimensional Stiefel-Whitney classes by the Bockstein homomorphism.

Proof. – Let φ be an Euler constructible function on X. Embed X as an algebraic subset of the smooth n-dimensional algebraic set M. Then for $i \geq 0$, the ith Stiefel-Whitney class of φ is defined in [FM 4.6] by

$$w_i(\varphi) = (\pi_X)_*([\mathbb{P}N^*(\varphi)] \frown \gamma^{n-i-1}),$$

where $\mathbb{P}N^*(\varphi)$ is the projectivized conormal cycle in $\mathbb{P}T^*M$, γ is the mod 2 Euler class of the tautological line bundle on $\mathbb{P}T^*M$, $\pi : \mathbb{P}T^*M \to M$ is the projection, $\pi_X : \pi^{-1}(X) \to X$ is its restriction, and $[\mathbb{P}N^*(\varphi)]$ is the mod 2 homology class of $\mathbb{P}N^*(\varphi)$ in $\pi^{-1}(X)$.

The proof that $\mathbb{P}N^*(\varphi)$ is a cycle mod 2 [FM, 4.5] hinges on the fact that φ is Euler if and only if $a_*N^*(\varphi) \equiv N^*(\varphi) \pmod 2$. That proof shows that if $a_*N^*(\varphi) = N^*(\varphi)$, then $\mathbb{P}N^*(\varphi)$ lifts to a cycle with integer coefficients, and hence $\beta[\mathbb{P}N^*(\varphi)] = 0$. Now

$$a_*N^*(\varphi) = (-1)^n N^*(D\varphi),$$

where $a : ST^*M \to ST^*M$ is the antipodal involution [FM, 3.12]. Suppose that the constructible function φ is self-dual ($D\varphi = \varphi$). If we choose the embedding $X \subset M$ so that $n = \dim M$ is even, then $a_*N^*(\varphi) = N^*(\varphi)$, and hence $\beta[\mathbb{P}N^*(\varphi)] = 0$. Therefore we have

$$\begin{aligned}
\beta w_i(\varphi) &= \beta(\pi_X)_*([\mathbb{P}N^*(\varphi)] \frown \gamma^{n-i-1}) \\
&= (\pi_X)_*\beta([\mathbb{P}N^*(\varphi)] \frown \gamma^{n-i-1}) \\
&= (\pi_X)_*([\mathbb{P}N^*(\varphi)] \frown \beta\gamma^{n-i-1}) \\
&= \begin{cases} (\pi_X)_*([\mathbb{P}N^*(\varphi)] \frown \gamma^{n-i}) & n-i-1 \text{ odd} \\ (\pi_X)_*([\mathbb{P}N^*(\varphi)] \frown 0) & n-i-1 \text{ even} \end{cases} \\
&= \begin{cases} w_{i-1}(\varphi) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}
\end{aligned}$$

Here we use elementary properties of the Bockstein:

$$\begin{aligned}
\beta(x \frown u) &= (\beta x \frown u) + (x \frown \beta u), \\
\beta(u \smile v) &= (\beta u \smile v) + (u \smile \beta v).
\end{aligned}$$

The second equation implies that if γ is 1-dimensional, then $\beta(\gamma^k) = \gamma^{k+1}$ if k is odd, and $\beta(\gamma^k) = 0$ if k is even. (Recall that $\beta(\gamma) = \gamma^2$.)

On the other hand, if φ is anti-self-dual ($D\varphi = -\varphi$), we choose M so that $n = \dim M$ is odd. Again $a_*N^*(\varphi) = N^*(\varphi)$, and the above computation shows that

$$\beta w_i(\varphi) = \begin{cases} w_{i-1}(\varphi) & i \text{ odd} \\ 0 & i \text{ even,} \end{cases}$$

as desired. □

2. Algebraically constructible functions

Let X be a real algebraic set. In this section we define and investigate the notion of an algebraically constructible function on X. Of course by a simple analogy to the semialgebraic case one can define an algebraically constructible function as one admitting a presentation (1.1) with all X_i algebraic subsets of X. Unfortunately this class of functions is not preserved by such elementary operations as duality or direct image by regular mappings. In this paper we propose to call a different class of functions algebraically constructible. Namely, a function $\varphi : X \to \mathbf{Z}$ will be called *algebraically constructible* if there exists a finite collection of algebraic sets Z_i and regular proper morphisms $f_i : Z_i \to X$ such that φ admits a presentation as a finite sum

$$\varphi = \sum m_i f_{i*} \mathbf{1}_{Z_i}, \tag{2.1}$$

where m_i are integers. We denote by A(X) the ring of algebraically constructible functions on X. The functions which admit a presentation (1.1) with X_i algebraic will be called *strongly algebraically constructible*. We note that these two sets of functions on X coincide if we reduce the coefficients m_i modulo 2. This follows easily from the following well-known result.

2.2. Lemma. – *Let $f : Z \to X$ be a regular morphism of real algebraic sets, and suppose that X is irreducible. Then there exists a proper algebraic subset $Y \subset X$ such that $\chi(f^{-1}(x))$ is constant modulo 2 on $X \setminus Y$.*

Proof. – See, for instance, [AK2, Proposition 2.3.2]. □

The rings $\mathrm{F}(X)$, A(X), and of strongly algebraically constructible functions are all different if $\dim(X) > 0$. Here are some examples.

2.3. Examples. –

(i) Let $X = \mathbf{R}$. The constructible function $\varphi \in \mathrm{F}(\mathbf{R})$ is strongly algebraically constructible if and only if φ is generically constant. On the other hand, $\varphi \in \mathrm{A}(\mathbf{R})$ if and only if φ is Euler or, equivalently in this case, φ is generically constant mod 2.

(ii) Let $\mathbf{P}^2 = \mathbf{P}^2_{\mathbf{R}}$ be the real projective plane with homogeneous coordinates $(x : y : z)$. Let $f : \mathbf{P}^2 \to \mathbf{R}^2$ be given by $f(x : y : z) = (\frac{x^2}{x^2+y^2+z^2}, \frac{y^2}{x^2+y^2+z^2})$. Then the image of f is the triangle Δ with vertices $(0,0)$, $(1,0)$, and $(0,1)$. The pushforward $f_*(\mathbf{1}_{\mathbf{P}^2})$ is an algebraically constructible function on $\mathbf{R}^2$ which equals 4 inside Δ, 2 on its sides, 1 at the vertices, and 0 in the complement of Δ.

(iii) Let $\varphi \in \mathrm{F}(\mathbf{R}^2)$ equal twice the characteristic function of the closed first quadrant. Since φ is even, it is Euler. We show in Remark 2.7 below that φ is not algebraically constructible.

(iv) Let f be a regular function on X. Then the sign of f is an algebraically constructible function on X. Indeed, let $\widetilde{X} = \{(x,t) \in X \times \mathbf{R} \mid f(x) = t^2\}$. Then $\mathrm{sgn} f = \pi_* \mathbf{1}_{\widetilde{X}} - \mathbf{1}_X$. Actually, the signs of regular functions generate the ring A(X), as shown in [PS].

It is clear from the definition that the ring A(X) of algebraically constructible functions is preserved by the direct image by proper regular maps. It is also easy to see that A(X) is preserved by the inverse image. We shall show below that $\mathrm{A(X)}$ is also preserved by the other standard operations on constructible functions such as duality, link, and specialization. To show this we need the following lemma, which is a consequence of resolution of singularities.

2.4. LEMMA. – *Let $\varphi \in \mathrm{A(X)}$. Then there exists a presentation* (2.1) *of φ with all Z_i nonsingular and pure-dimensional.*

Proof. – It is sufficient to find such a presentation for $\varphi = \mathbf{1}_Z$ with Z irreducible. We proceed by induction on $\dim(Z)$. By resolution of singularities there exists a proper regular morphism $\sigma : \widetilde{Z} \to Z$ with the following properties: $\widetilde{Z}$ is irreducible, nonsingular, and of pure dimension, and there is a proper algebraic subset $\Sigma \in Z$ such that σ induces an isomorphism between $\widetilde{Z} \setminus \sigma^{-1}(\Sigma)$ and $Z \setminus \Sigma$. Let $\widetilde{\Sigma} = \sigma^{-1}(\Sigma)$. Then $\dim(\Sigma) < \dim(Z)$ and $\dim(\widetilde{\Sigma}) < \dim(Z)$. Finally

$$\mathbf{1}_Z = \sigma_* \mathbf{1}_{\widetilde{Z}} + (\mathbf{1}_\Sigma - \sigma_* \mathbf{1}_{\widetilde{\Sigma}}),$$

and the second summand admits the required presentation by the inductive assumption. This ends the proof. □

The next two results are consequences of Lemma 2.4 and Proposition 1.2.

2.5. THEOREM. – *Let $\varphi \in \mathrm{A(X)}$. Then φ is Euler and $\tilde{\Lambda}\varphi \in \mathrm{A(X)}$. Hence $\tilde{\Omega}\varphi$ and $D\varphi$ are also algebraically constructible.*

Proof. – Let $\varphi = \sum m_i f_{i*} \mathbf{1}_{Z_i}$ be a presentation given by Lemma 2.4. Then each Z_i is nonsingular and of pure dimension, and either $\tilde{\Lambda}\mathbf{1}_{Z_i} = \mathbf{1}_{Z_i}$ if $\dim Z_i$ is odd or $\tilde{\Lambda}\mathbf{1}_{Z_i} = 0$ if $\dim Z_i$ is even. Hence by Corollary 1.3 (ii),

$$\tilde{\Lambda}\varphi = \tilde{\Lambda} \sum m_i f_{i*} \mathbf{1}_{Z_i} = \sum m_i f_{i*} \tilde{\Lambda} \mathbf{1}_{Z_i} = \sum{}' m_i f_{i*} \mathbf{1}_{Z_i},$$

where the latter sum is only over odd-dimensional Z_i. □

2.6. THEOREM. – *Let φ be an algebraically constructible function on X and let $f : X \to \mathbf{R}$ be a regular morphism. Then $\Psi_f^+ \varphi$ and $\Psi_f^- \varphi$ are algebraically constructible functions on X.*

Proof. – Let $\widetilde{X} \subset X \times \mathbf{R}$ be the algebraic set defined by $\widetilde{X} = \{(x, t) \mid f(x) = t^2\}$. Let $\pi : \widetilde{X} \to X$ denote the standard projection and let $\widetilde{\varphi} = \varphi \circ \pi$, $\widetilde{f} = f \circ \pi : \widetilde{X} \to \mathbf{R}$. We identify $\widetilde{f}^{-1}(0)$ with $X_0 = f^{-1}(0)$.

Take $x \in X_0$. Then the positive Milnor fibre $F_{\widetilde{f}}^+(x)$ is the disjoint union of two copies of $F_f^+(x)$ and the negative Milnor fibre $F_{\widetilde{f}}^-$ is empty. Hence, by Proposition 1.8,

$$\Psi_f^+ \varphi = \tfrac{1}{2}(\Psi_{\widetilde{f}}^+ \widetilde{\varphi} + \Psi_{\widetilde{f}}^- \widetilde{\varphi}) = \tfrac{1}{2} \Lambda_{X_0} \widetilde{\varphi} = \tilde{\Lambda}(\widetilde{\varphi}|_{X_0}) - \tilde{\Lambda}((\Lambda \widetilde{\varphi})|_{X_0}) + (\tilde{\Lambda} \widetilde{\varphi})|_{X_0}.$$

The function given by the latter expression is algebraically constructible by Theorem 2.5. □

Note that the first part of Theorem 2.5, that is φ is Euler, is equivalent to Sullivan's observation that the links of points in real algebraic sets have even Euler characteristic. On the other hand, the assertion that $\tilde{\Lambda}\varphi \in \mathrm{A(X)}$ is much stronger. It gives further restrictions for a function to be algebraically constructible that are of "greater depth". Let us consider the simplest possible example. On $X = \mathbf{R}$ the algebraically constructible functions are exactly those constructible functions that are Euler, see Example 2.3 (i). This is no longer true on $\mathbf{R}^2$. To see this let us justify the claim of Example 2.3 (iv).

2.7. *Remark.* – Let $\varphi = 2 \cdot \mathbf{1}_Q$, where $Q \subset \mathbf{R}$ is the closed first quadrant. Since φ is even, it is also Euler. Let $Y \subset \mathbf{R}^2$ denote the x-axis. Then $(\tilde{\Lambda}\varphi)|_Y(x)$ equals 1 for $x \geq 0$ and 0 for $x < 0$. Consequently $(\tilde{\Lambda}\varphi)|_Y$ is not algebraically constructible and hence, by Theorem 2.5, neither is φ.

On the other hand, all constructible functions on $\mathbf{R}^2$ which are divisible by 4 are algebraically constructible (by the following Theorem). This gives a clear limit to the depth of information carried by algebraically constructible functions. In general we have the following nontrivial fact.

2.8. Theorem. – *Let X be an algebraic set of dimension d. Then*

$$2^d\mathrm{F}(X) \subseteq \mathrm{A(X)}$$

Proof. – Let $S \subset X$ be a semialgebraic subset of X. We show by induction on $d = \dim X$ that $2^d\mathbf{1}_S$ is algebraically constructible. We suppose that X is irreducible.

Up to a set of dimension $< d$, the set S is a finite union of basic open semialgebraic sets; that is, sets of the form

$$\{x \in X \mid g_1(x) > 0, \ldots, g_k(x) > 0\},$$

where the g_i's are polynomials on $\mathbf{R}^n$ [BCR, 2.7.1]. Since a finite intersection of basic open sets is still basic, for the inductive step it suffices to consider S basic and open. Then, by [BCR, Théorème 7.7.8], there exist polynomials $f_1, \ldots, f_d$ such that

$$\mathcal{U} = \mathcal{U}(f_1, \ldots, f_d) = \{x \in X \mid f_1(x) > 0, \ldots, f_d(x) > 0\}$$

is contained in S and $\dim(S \setminus \mathcal{U}) < d$. Hence, by the inductive assumption, it suffices to show $2^d\mathbf{1}_{\mathcal{U}} \in \mathrm{A(X)}$. Let $\widetilde{X} \subset X \times \mathbf{R}^d$ be given by

$$\widetilde{X} = \{(x, t_1, \ldots t_d) \in X \times \mathbf{R}^d \mid f_1(x) = t_1^2, \ldots, f_d(x) = t_d^2\}.$$

Let $\pi : \widetilde{X} \to X$ denote the standard projection. Then $\widetilde{Y} = \{f_1(\pi(\tilde{x})) = \cdots = f_d(\pi(\tilde{x})) = 0\}$ is an algebraic subset of $\widetilde{X}$. In particular, $\mathbf{1}_{\widetilde{X}\setminus\widetilde{Y}}$ is algebraically constructible, and so is

$$\pi_*\mathbf{1}_{\widetilde{X}\setminus\widetilde{Y}} = 2^d\mathbf{1}_{\mathcal{U}}$$

as required. $\square$

2.9. Definition. – *A constructible function φ (respectively a set $\mathcal{F}$ of constructible functions) is completely Euler if all constructible functions obtained from φ (resp. from* *the functions in $\mathcal{F}$) by means of the arithmetic operations $+, -, *$, and the operator $\tilde{\Lambda}$, are integer valued.*

In particular Theorem 2.5 implies that every algebraically constructible function is completely Euler. We shall study some consequences of this fact in section 4.

The following result is an immediate consequence of Theorem 2.8. In the next section we give an alternative purely topological proof of a slightly more general statement (Proposition 3.1).

2.10. COROLLARY. – *Let $\varphi \in \mathrm{F}(X)$ be divisible by $2^{\dim X}$. Then φ is completely Euler.* □

3. Completely Euler functions

In this section we study the completely Euler functions in low dimensions. In particular we show how to decide in a systematic way whether a constructible function is completely Euler. In the next section we show that the characterization of completely Euler functions obtained in this way is equivalent to some combinatorial conditions discovered by Akbulut and King [AK2].

If X is a semialgebraic set, we denote by $\mathcal{A}_X$ the ideal of $\mathrm{F}(X)$ consisting of all φ such that for each positive integer k, $\dim \operatorname{supp}(\varphi \pmod{2^k}) < k$; that is to say, φ is divisible by 2^k in the complement of a subset of dimension $< k$. If X is an algebraic set then by Theorem 2.8 all functions in $\mathcal{A}_X$ are algebraically constructible. In particular they are also completely Euler, which is also a consequence of the following.

3.1. PROPOSITION. – *$\mathcal{A}_X$ is preserved by $\tilde{\Lambda}$.*

Proof. – We proceed by induction on $d = \dim X$. Let $\varphi \in \mathcal{A}_X$, and let $\mathcal{S}$ be a semialgebraic stratification of X compatible with φ. Denote by X^{d-1} the $(d-1)$-skeleton of $\mathcal{S}$, that is the union of strata of dimension $< d$. Then $\psi = (\mathbf{1}_X - \mathbf{1}_{X^{d-1}})\varphi$ is divisible by 2^d and hence both $\tilde{\Lambda}\psi$ and $\tilde{\Omega}\psi$ are divisible by 2^{d-1}. And either $\tilde{\Lambda}\psi$ for d even, or $\tilde{\Omega}\psi$ for d odd, has support in X^{d-1}. In both cases $\tilde{\Lambda}\psi = \psi - \tilde{\Omega}\psi \in \mathcal{A}_X$.

On the other hand $\varphi|_{X^{d-1}} = (\mathbf{1}_{X^{d-1}})\varphi$ is in $\mathcal{A}_{X^{d-1}}$ and hence satisfies the inductive assumption. Therefore $\tilde{\Lambda}\varphi = \tilde{\Lambda}\psi + \tilde{\Lambda}(\varphi|_{X^{d-1}}) \in \mathcal{A}_X$, as required. □

Fix a constructible function φ (or a set of functions $\mathcal{F}$) on X. We denote by $\tilde{\Lambda}\{\varphi\}$ (or $\tilde{\Lambda}\mathcal{F}$, respectively) the set of functions, which in general may not be integer-valued, obtained from φ (resp. $\mathcal{F}$) by means of the arithmetic operations $+, -, *$, and the operator $\tilde{\Lambda}$. Let $\mathcal{S}$ be a topologically trivial semialgebraic stratification compatible with φ. Let $\mathrm{F}_{\mathcal{S}}(X)$ be the subring of $\mathrm{F}(X)$ consisting of functions compatible with $\mathcal{S}$, and let $\mathcal{A}_{\mathcal{S}} = \mathcal{A}_X \cap \mathrm{F}_{\mathcal{S}}(X)$. By Proposition 3.1, $\mathcal{A}_{\mathcal{S}}$ is preserved by $\tilde{\Lambda}$ and hence $\mathcal{A}_{\mathcal{S}}$ is completely Euler. Consequently, in order to determine whether φ is completely Euler we may work in $\mathrm{F}_{\mathcal{S}}(X)$ modulo $\mathcal{A}_{\mathcal{S}}$; that is to say, whether φ is completely Euler is determined by its values mod 2^k on strata of dimension k. In particular, since $\mathcal{S}$ is finite there are finitely many conditions to check.

We begin with some elementary observations. First note that whether φ is Euler depends only on the reduction of φ modulo 2. Morever, since all positive powers of φ are congruent mod 2,

$$\varphi \equiv \varphi^2 \equiv \varphi^3 \equiv \cdots \pmod 2, \tag{3.2}$$

and if one of them is Euler so are all the others.

Let $\dim \operatorname{supp}\varphi \leq 1$. Then whether φ is completely Euler is determined by its values modulo 2. Assume that φ is Euler. Then by (3.2) all the powers of φ are also Euler. Moreover, by dimension assumption, $\tilde{\Omega}\varphi$ has finite support and so belongs to $\mathcal{A}_S$. Hence $\tilde{\Lambda}\{\varphi\}$ modulo $\mathcal{A}_S$ contains at most one element, namely the class of φ. This shows the following result.

3.3. Lemma. – *If* $\dim \operatorname{supp}\varphi \leq 1$ *then* φ *is completely Euler if and only if* φ *is Euler.* □

Note also that $\dim \operatorname{supp}\tilde{\Lambda}\mathbf{1}_X < \dim X$ (for $\dim X$ even) or $\dim \operatorname{supp}\tilde{\Omega}\mathbf{1}_X < \dim X$ (for $\dim X$ odd). Hence the following observation will allow a reduction of dimension.

3.4. Lemma. – $\mathbf{1}_X$ *is completely Euler if and only if* $\mathbf{1}_X$ *is Euler and* $\tilde{\Lambda}\mathbf{1}_X$ *(or equivalently* $\tilde{\Omega}\mathbf{1}_X$*) is completely Euler. In particular, if* $\dim X \leq 2$ *then* $\mathbf{1}_X$ *is completely Euler if and only if* $\mathbf{1}_X$ *is Euler.*

Proof. – The first statement is obvious since multiplication by $\mathbf{1}_X$ acts trivially on $\tilde{\Lambda}\{\mathbf{1}_X\}$. Suppose that $\dim X \leq 2$. Then $\dim \operatorname{supp}\tilde{\Lambda}\mathbf{1}_X \leq 1$. If $\mathbf{1}_X$ is Euler, so is $\tilde{\Lambda}\mathbf{1}_X$, since $\tilde{\Lambda} \circ \tilde{\Lambda} = \tilde{\Lambda}$. So the second statement follows from Lemma 3.3. □

On the other hand there exist Euler constructible functions φ with $\dim \operatorname{supp}\varphi = 2$ which are not completely Euler (see Example 3.13 below). Let us consider this case in detail. We assume φ is Euler and, as before, determine $\tilde{\Lambda}\{\varphi\}$ modulo $\mathcal{A}_S$, in particular modulo 4. The algebra of powers $\varphi, \varphi^2, \ldots$, modulo 4, is generated (additively) by $\varphi, \varphi^2, \varphi^3$. By (3.2) all these powers are Euler. The supports of $\tilde{\Lambda}\varphi^k$, $k = 1, 2, 3$, are contained in X^1, that is the union of strata of dimension ≤ 1. Hence, again modulo $\mathcal{A}_S$, $\tilde{\Lambda}\{\varphi\}$ is generated additively by the products of the following functions:

$$\varphi,\ \tilde{\Lambda}\varphi,\ \tilde{\Lambda}\varphi^2,\ \tilde{\Lambda}\varphi^3.$$

Moreover, all such products except the powers of φ are supported in X^1 and hence it suffices to consider their values mod 2. Consequently, by (3.2) only the following products matter:

$$\varphi^a(\tilde{\Lambda}\varphi)^b(\tilde{\Lambda}\varphi^2)^c(\tilde{\Lambda}\varphi^3)^d, \tag{3.5}$$

where $a, b, c, d = 0$ or 1 , and $b + c + d > 0$. Moreover, $\tilde{\Lambda}\varphi$, $\tilde{\Lambda}\varphi^2$, $\tilde{\Lambda}\varphi^3$ are automatically Euler. Thus we have the following result.

3.6. Proposition. – *If* $\dim \operatorname{supp}\varphi \leq 2$, *then* φ *is completely Euler if and only if* φ *is Euler and all 11 functions supported in the one dimensional set* X^1 *and given by (3.5) with* $a, b, c, d = 0$ *or* 1, $a + b + c + d \geq 2$, *are Euler.* □

Suppose $\dim X \leq 3$. By Lemma 3.4, Proposition 3.6 applied to $\varphi = \tilde{\Omega}\mathbf{1}_X$ gives a criterion for $\mathbf{1}_X$ to be completely Euler. Since in this case $\tilde{\Lambda}\varphi = \tilde{\Lambda}\tilde{\Omega}\mathbf{1}_X = 0$, whether φ is completely Euler is determined by the products

$$\varphi^a(\tilde{\Lambda}\varphi^2)^b(\tilde{\Lambda}\varphi^3)^c \tag{3.7}$$

with $a, b, c = 0$ or 1. Six of these products, for $b + c > 0$, have support in X^1. The functions $\tilde{\Lambda}\varphi^2$ and $\tilde{\Lambda}\varphi^3$ are automatically Euler. Consequently we have the following.

3.8. PROPOSITION. – *If* $\dim X \leq 3$, *then* $\mathbf{1}_X$ *is completely Euler if and only if* $\mathbf{1}_X$ *is Euler and the following functions supported in* X^1 *are Euler:*

$$\varphi(\tilde{\Lambda}\varphi^2),\ \varphi(\tilde{\Lambda}\varphi^3),\ (\tilde{\Lambda}\varphi^2)(\tilde{\Lambda}\varphi^3),\ \varphi(\tilde{\Lambda}\varphi^2)(\tilde{\Lambda}\varphi^3), \tag{3.9}$$

where $\varphi = \tilde{\Omega}\mathbf{1}_X$. □

The conditions given by Proposition 3.8 can be expressed in an equivalent way in terms of characteristic sets. For every $\delta = (\delta_0, \delta_1, \delta_2) \in (\mathbf{Z}_2)^3$ define

$$X_\delta = \{x \in X \mid \varphi(x) \equiv \delta_0,\ \tilde{\Lambda}\varphi^2(x) \equiv \delta_1,\ \tilde{\Lambda}\varphi^3(x) \equiv \delta_2 \pmod 2\}$$

Note that the X_δ are disjoint, not necessarily closed, and of dimension ≤ 1 if $\delta_1 \neq 0$ or $\delta_2 \neq 0$. The supports of the functions of (3.7), considered modulo 2, are unions of the sets X_δ. In particular the six functions of (3.7) with $b + c > 0$ correspond to the six set X_δ with $\delta_1 \neq 0$ or $\delta_2 \neq 0$:

$$\begin{aligned}
&\mathrm{supp}_2\tilde{\Lambda}\varphi^2 = X_{111} \cup X_{110} \cup X_{011} \cup X_{010},\\
&\mathrm{supp}_2\tilde{\Lambda}\varphi^3 = X_{111} \cup X_{101} \cup X_{011} \cup X_{001},\\
&\mathrm{supp}_2\varphi(\tilde{\Lambda}\varphi^2) = X_{111} \cup X_{110},\\
&\mathrm{supp}_2\varphi(\tilde{\Lambda}\varphi^3) = X_{111} \cup X_{101},\\
&\mathrm{supp}_2(\tilde{\Lambda}\varphi^2)(\tilde{\Lambda}\varphi^3) = X_{111} \cup X_{011},\\
&\mathrm{supp}_2\varphi(\tilde{\Lambda}\varphi^2)(\tilde{\Lambda}\varphi^3) = X_{111},
\end{aligned} \tag{3.10}$$

where by supp_2 we mean the support modulo 2. Thus Proposition 3.8 can be reformulated as follows.

3.8′. PROPOSITION. – *If* $\dim X \leq 3$, *then* $\mathbf{1}_X$ *is completely Euler if and only if it is Euler and the subsets* $X_{111}, X_{101}, X_{011}, X_{110}$ *of* X^1 *are Euler.* □

3.11. *Remark.* – If X is Euler then $\mathrm{supp}_2\tilde{\Lambda}\varphi^2$ and $\mathrm{supp}_2\tilde{\Lambda}\varphi^3$ are Euler. Therefore we may choose in Proposition 3.8′ another family of four characteristic sets X_δ, provided that if these sets are Euler then all the sets X_δ are Euler. For instance, $X_{111}, X_{101}, X_{001}, X_{110}$ is such a family, which we use in the next section.

Recall that we have fixed a stratification $\mathcal{S}$ of X. Let X^i denote the i-skeleton of $\mathcal{S}$ and suppose, in addition, that all skeleta of $\mathcal{S}$ are Euler. We may apply the above method to obtain a stratified version of Proposition 3.8 that is a characterisation of those $\mathcal{S}$ such that the family $\{\mathbf{1}_{X^i} \mid i = 0, 1, 2, 3\}$ is completely Euler.

3.12. PROPOSITION. – *Let* $\mathcal{S}$ *be a locally topologically trivial stratification of a semialgebraic set* X, $\dim X \leq 3$. *Then the family* $\{\mathbf{1}_{X^i} \mid i = 0, 1, 2, 3\}$ *of characteristic functions of the skeleta of* $\mathcal{S}$ *is completely Euler if and only if all* $\mathbf{1}_{X^i}$ *are Euler and one of the following equivalent conditions holds:*

(i) The following 12 functions supported in X^1 are Euler:

$$\varphi \mathbf{1}_{X^1}, \quad \varphi^a (\tilde{\Lambda}\varphi^2)^b (\tilde{\Lambda}\varphi^3)^c (\tilde{\Lambda}\mathbf{1}_{X^2})^d,$$

where $\varphi = \tilde{\Omega}\mathbf{1}_X$, $a, b, c, d = 0$ or 1, and we consider only $d = 0$, $a + b + c \geq 2$, and $d = 1$, $a + b + c > 0$.

(ii) The following 12 characteristic sets contained in X^1 are Euler:

$$X_{\delta,0} = \{x \in X \mid \varphi(x) \equiv \delta_0,\ \tilde{\Lambda}\varphi^2(x) \equiv \delta_1,\ \tilde{\Lambda}\varphi^3(x) \equiv \delta_2,\ \tilde{\Lambda}\mathbf{1}_{X^2} \equiv 0 \pmod 2\},$$

for $\delta = (1,1,1),\ (1,0,1),\ (0,1,1),\ (1,1,0)$,

$$X_{\delta,1} = \{x \in X \mid \varphi(x) \equiv \delta_0,\ \tilde{\Lambda}\varphi^2(x) \equiv \delta_1,\ \tilde{\Lambda}\varphi^3(x) \equiv \delta_2,\ \tilde{\Lambda}\mathbf{1}_{X^2} \equiv 1 \pmod 2\},$$

for $\delta \neq (0,0,0)$, and

$$X' = \{x \in X_1 \mid \varphi(x) \equiv 1,\ \tilde{\Lambda}\varphi^2(x) \equiv \tilde{\Lambda}\varphi^3(x) \equiv \tilde{\Lambda}\mathbf{1}_{X^2} \equiv 0 \pmod 2\}.$$

Proof. –

First note that the family $\{\mathbf{1}_{X^i} \mid i = 0,1,2,3\}$ is completely Euler if and only if $\{\varphi, \mathbf{1}_{X^2}, \mathbf{1}_{X_1}\}$ is completely Euler. The latter family is supported in X^2, so we work modulo 4. By repeating the arguments of the proofs of Propositions 3.6 and 3.8, we see that $\{\varphi, \mathbf{1}_{X^2}, \mathbf{1}_{X_1}\}$ is completely Euler if and only if the functions

$$\varphi^a (\tilde{\Lambda}\varphi^2)^b (\tilde{\Lambda}\varphi^3)^c (\tilde{\Lambda}\mathbf{1}_{X^2})^d (\mathbf{1}_{X^1})^e,$$

$a, b, c, d, e = 0$ or 1, are Euler. The supports of $\tilde{\Lambda}\varphi^2$, $\tilde{\Lambda}\varphi^3$, and $\tilde{\Lambda}\mathbf{1}_{X^2}$ are contained in X^1, so if $b + c + d > 0$ we may forget the last factor.

Since $\tilde{\Lambda}\varphi^2$, $\tilde{\Lambda}\varphi^3$, and $\tilde{\Lambda}\mathbf{1}_{X^2}$ are automatically Euler, we are left with exactly the 12 functions of condition (i).

The equivalence of (i) and (ii) can be shown in exactly the same way as Proposition 3.8′. □

3.13. Examples. – Let X be Akbulut and King's first published example of an Euler space which is not homeomorphic to a real algebraic set [Ki, Example, p. 647]. Recall that X is the suspension of the algebraic set Y shown in Figure 3 (*loc. cit.*, p. 646). Let A be the suspension of the figure eight, with suspension points a, a'; let B be the suspension of three points, with suspension points b, b'; let C be an arc with endpoints c, c'. The space Y is obtained from the disjoint union of A, B, C, by identifying a' with b, b' with c, and c' with a. (Note that there is a mistake in the picture of Y in [BR, p. 181.]) In fact Y is homeomorphic to an algebraic set in projective 3-space, the union of the umbrella $wx^2 = yz^2$ and the circle $x = 0$, $(y-1)^2 + z^2 = w^2$. The support of $\varphi = \tilde{\Omega}\mathbf{1}_X$ is of dimension 2 and φ is Euler, but φ is not completely Euler. In fact $\varphi(\tilde{\Lambda}\varphi^2)$ is not Euler, which is exactly the reason that X is not homeomorphic to an algebraic set.

4. Topology of real algebraic sets

Let X be a triangulable topological space such that the one point compactification of X is also triangulable. By a theorem of Sullivan [Su], a necessary condition for X to be homeomorphic to a real algebraic set is that X is mod 2 Euler space; that is, the Euler characteristic of the link of every point of X is even. By [AK1], [BD] this condition is also sufficient if $\dim X \leq 2$, but this is no longer true if $\dim X = 3$. In this case necessary and sufficient topological conditions were given by Akbulut and King [AK2], and then reinterpreted by Coste and Kurdyka [C], [CK]. More restrictions on the Euler characteristic of links of real algebraic sets were given in [C], [CK], and [MP]. We show below that all these conditions are simple consequences of Theorem 2.5.

It will be convenient for us to proceed using the language of semialgebraic geometry. Alternatively, one could use Euclidean simplicial complexes or subanalytic sets.

Let X be an algebraic subset of $\mathbf{R}^n$. Let $Y \subset X$ be closed and semialgebraic, and let $Z \subset X$ be semialgebraic. Choose a nonnegative continuous semialgebraic function $f : X \to \mathbf{R}$ defining Y; that is, $Y = f^{-1}(0)$. Recall from section 1 that by the link $\mathrm{lk}_p(Y; Z)$ of Y in Z at $p \in Y$ we mean the positive Milnor fibre of $f|_Z$ at p. Such a link can be understood as a generalization of the link considered in [C], [CK], which was only defined at generic points of Y. In particular the Coste-Kurdyka link has the same homotopy type as ours; see [MP, §2.3] for details. In what follows we use only the Euler characteristic of the link, that is, the operator Λ_Y introduced in section 1. In [C] Michel Coste made important observations on the behaviour modulo 4, 8, and 16, of the Euler characteristic of links of real algebraic subsets. These results are special cases of the following general statement.

4.1. THEOREM [MP, Theorem 2]. – *Let $X_1, \ldots, X_k$ be algebraic subsets of X. Then $\varphi = \Lambda_{X_1} \ldots \Lambda_{X_k} \mathbf{1}_X$ is always divisible by 2^k. Moreover, let Y be an irreducible algebraic subset of X. Then there exists a proper algebraic subset $Y' \subset Y$ such that for all $x, x' \in Y \setminus Y'$*

$$\varphi(x) \equiv \varphi(x') \pmod{2^{k+1}}.$$

Proof. – $\varphi/2^k$ is algebraically constructible–and, in particular, integer-valued–by Proposition 1.8 and Theorem 2.5. The second part of the statement follows from Lemma 2.2. □

In [C], Theorem 4.1 was shown only for $k = 1, 2, 3$, and under special assumptions. In particular, it was assumed that $X_1 \subset \cdots \subset X_k$ and $\dim X = \dim X_k + 1 = \cdots = \dim X_1 + k$. This dimensional assumption was first dropped in [CK, Theorem 1′] for $k = 1$. The proof of Theorem 4.1 presented here is different from the proof in [MP], which was based on the relation between complex monodromy and complex conjugation.

In [C] and [CK] the authors show how to use Theorem 4.1 to recover Akbulut and King's combinatorial conditions [AK2, 7.1.1] characterizing real algebraic sets of dimension ≤ 3. We show below that it is even more natural to look at these conditions as consequences of Theorem 2.5.

4.2. THEOREM. – *Let X be a semialgebraic subset of $\mathbf{R}^n$ with $\dim X \leq 3$. Then X satisfies the Akbulut-King conditions if and only if $\mathbf{1}_X$ is completely Euler.*

Thus the main result of [AK2] can be rephrased as follows: If $\dim X \leq 3$, X is homeomorphic to a real algebraic set if and only if $\mathbf{1}_X$ is completely Euler. In particular, Theorem 2.5 shows the necessity of the Akbulut-King conditions (*cf.* Remark A.7 of the Appendix).

To prove Theorem 4.2, we first recall the Akbulut-King conditions, using the approach of [C], [CK]. Then we apply the results of section 3.

Let X be a semialgebraic subset of $\mathbf{R}^n$, $\dim X \leq 3$. We suppose that X is Euler, and we fix a locally trivial semialgebraic stratification $\mathcal{S}$ of X. Let $C_0(X)$ be the union of the 1-skeleton X^1 and those strata T of dimension 2 such that for $x \in T$,

$$\chi(\mathrm{lk}_x(T;X)) = \Lambda_T \mathbf{1}_X(x) \equiv 0 \pmod 4.$$

Equivalently we may say that we include in $C_0(X)$ those two-dimensional strata T such that $\tilde{\Omega}\mathbf{1}_X \equiv 1 \pmod 2$ on T. Let $\varphi = \tilde{\Omega}\mathbf{1}_X$. Then in the complement of X^1,

$$\varphi \equiv \mathbf{1}_{C_0(X)} \pmod 2, \qquad \varphi^2 \equiv \mathbf{1}_{C_0(X)} \pmod 4. \tag{4.3}$$

4.4. LEMMA. – *$C_0(X)$ is Euler in the complement of X^0. Moreover, for each stratum S of dimension 1 and $p \in S$,*

$$\chi(\mathrm{lk}_p(S;C_0(X))) = \Lambda_S \mathbf{1}_{C_0(X)}(p) \equiv \Omega\varphi^2(p) \pmod 4.$$

Proof. – Fix a stratum S of dimension 1 and let p be a generic point of S. Let N be a transverse slice to S at p. Denote by X' a small neighbourhood of p in $N \cap X$, and set $C' = X' \cap C_0(X)$. Then by Proposition 1.7,

$$\varphi|_{X'} = \tilde{\Lambda}\mathbf{1}_{X'}.$$

This shows $\varphi|_{X'}$ is Euler near p and hence, by (4.3), so is $\mathbf{1}_{C'}$. Hence, again by Proposition 1.7, $C_0(X)$ is Euler near p. If we apply the same arguments to φ^2 we get the last equality of the statement. $\square$

Given S and $p \in S$ as above, following [C] and [CK] we consider the number

$$\Delta_p(S, C_0(X), X) = \chi(\mathrm{lk}_p(S;X) \setminus \mathrm{lk}_p(S;C_0(X))) - \chi(\mathrm{lk}_p(S;X)) + \chi(\mathrm{lk}_p(S;C_0(X))).$$

Note that, as follows from Lemma 4.5 below, $\Delta_p(S, C_0(X), X)$ does not depend on p but only on S (actually in the notation of [C] it equals $-\Delta(S, C_0(X), X)$). Here we follow the notation of [MP], where it is shown that the number $\Delta_p(S, C_0(X), X)$ has the following geometric interpretation. Let S, resp. $C_0(X)$, be given in a neighbourhood of p as the zero set of a continuous nonnegative semialgebraic function f, resp. g. Then, following [MP], we define the *iterated link* $\mathrm{lk}_p(S, C_0(X); X)$ as the iterated Milnor fibre

$$\mathrm{lk}_p(S, C_0(X); X) = B(p,\varepsilon) \cap f^{-1}(\delta_1) \cap g^{-1}(\delta_2),$$

where $0 < \delta_2 \ll \delta_1 \ll \varepsilon$. As shown in [MP, §3.4], $\Delta_p(S, C_0(X), X)$ is the Euler characteristic of $\mathrm{lk}_p(S, C_0(X); X)$. Hence [MP, §3.5] shows that

$$\Delta_p(S, C_0(X), X) = \chi(\mathrm{lk}_p(S, C_0(X); X)) = \Lambda_S(\Lambda_{C_0(X)}\mathbf{1}_X)(p).$$

4.5. Lemma.

$$\Delta_p(S, C_0(X), X) = \Omega(\mathbf{1}_{C_0(X)}\Lambda\mathbf{1}_X)(p).$$

In particular,

$$\Delta_p(S, C_0(X), X) \equiv 4\tilde{\Lambda}(\varphi^2 + \varphi^3)(p) \pmod 8$$

Proof. – We use again a transverse slice N to S at p and Proposition 1.7. Let $X' = N \cap X$, $C' = N \cap C_0(X)$, as before. If p is a generic point of S, for instance S is a stratum of a Whitney stratification of X near p, then

$$\Lambda_S(\Lambda_{C_0(X)}\mathbf{1}_X)(p) = \Lambda(\Lambda_{C'}\mathbf{1}_{X'})(p).$$

By Proposition 1.8 the expression above can be written in terms of the link operator Λ and the characteristic functions of $\mathbf{1}_{C'}$ and $\mathbf{1}_{X'}$, after simplification $\Lambda(\Lambda_{C'}\mathbf{1}_{X'}) = \Lambda(\mathbf{1}_{C'}\Omega\mathbf{1}_{X'})$. Hence the first formula of the lemma follows again from Proposition 1.7, since we have to exchange Λ and Ω when taking the slice. To show the second formula we use (4.3):

$$\Omega(\mathbf{1}_{C_0(X)}\Lambda\mathbf{1}_X) \equiv 4\tilde{\Omega}(\varphi^2 - \varphi^3) \equiv 4\tilde{\Lambda}(\varphi^2 + \varphi^3) + 4(\varphi^2 + \varphi^3) \pmod 8,$$

which gives the formula since $\varphi^2 + \varphi^3$ is even. $\square$

Proof of Theorem 4.2. – Given a 1-dimesional stratum S of X, the Akbulut-King invariant

$$(\varepsilon_0(S), \varepsilon_1(S), \varepsilon_2(S)) \in (\mathbf{Z}_2)^3$$

is defined as follows (see [C], [CK]):

$$\begin{aligned} \varepsilon_0(S) &= \tfrac{1}{4}\chi(\mathrm{lk}_p(S, C_0(X); X)) \pmod 2 \\ \varepsilon_0(S) + \varepsilon_1(S) + \varepsilon_2(S) &= \tfrac{1}{2}\chi(\mathrm{lk}_p(S; X)) \pmod 2 \\ \varepsilon_2(S) &= \tfrac{1}{2}\chi(\mathrm{lk}_p(S; C_0(X))) \pmod 2, \end{aligned}$$

where p can be any point of S. Given $(a, b, c) \in (\mathbf{Z}_2)^3$, define the characteristic set $\mathcal{E}_{abc}(\mathcal{X})$ as the union of the 0-skeleton X^0 and those one-dimensional strata S such that $(\varepsilon_0(S), \varepsilon_1(S), \varepsilon_2(S)) = (a, b, c)$.

Now for every $x \in X$ we define

$$\begin{aligned} \varepsilon_0(x) &= \tilde{\Lambda}(\varphi^2 + \varphi^3)(x) \pmod 2 \\ \varepsilon_1(x) &= \tilde{\Lambda}\varphi^3(x) \pmod 2 \\ \varepsilon_2(x) &= \varphi(x) + \tilde{\Lambda}\varphi^2(x) \pmod 2, \end{aligned} \tag{4.6}$$

where $\varphi = \tilde{\Omega}\mathbf{1}_X$. If $(a,b,c) \neq (0,0,0)$, $(0,0,1)$ then $\{x \in X \mid (\varepsilon_0(x), \varepsilon_1(x), \varepsilon_2(x)) = (a,b,c)\}$ is of dimension ≤ 1 and hence by Lemmas 4.4 and 4.5,

$$\mathcal{E}_{abc}(\mathcal{X}) = \mathcal{X}^0 \cup \{x \in \mathcal{X} \mid (\varepsilon_0(x), \varepsilon_1(x), \varepsilon_2(x)) = (a,b,c)\}.$$

In particular, for these (a,b,c) the set $\mathcal{E}_{abc}(\mathcal{X})$ is independent of the choice of stratification (up to a finite set, since we may always add some point strata).

Note that $(\varepsilon_0(x), \varepsilon_1(x), \varepsilon_2(x))$ equals $(0,0,0)$, resp. $(0,0,1)$, for nonsingular points of X, resp. nonsingular points of $C_0(X)$. In [AK2], the characteristic sets corresponding to $(1,1,1)$, $(0,1,0)$, $(1,0,0)$, $(1,1,0)$ are denoted by $Z_0(X)$, $Z_1(X)$, $Z_2(X)$, $Z_3(X)$ respectively. It is shown in [AK2, 7.1.1] that X is homeomorphic to an algebraic set if and only if X is Euler and $Z_0(X)$, $Z_1(X)$, $Z_2(X)$, $Z_3(X)$ are Euler. Now the theorem follows from Proposition 3.8′ and Remark 3.11 since, in the notation of section 3, $\delta_0(x) = \varepsilon_0(x) + \varepsilon_1(x) + \varepsilon_2(x)$, $\delta_1(x) = \varepsilon_0(x) + \varepsilon_1(x)$, and $\delta_2(x) = \varepsilon_1(x)$. Hence $Z_0(X) = X_{101}$, $Z_1(X) = X_{111}$, $Z_2(X) = X_{110}$, and $Z_3(X) = X_{001}$. $\square$

In [AK2, Theorem 7.1.2] the authors also give a stratified version of their characterization of real algebraic sets of dimension ≤ 3 which involves 12 characteristic sets $Z_i(X)$, $i = 0, \ldots, 11$. Again we show that these combinatorial conditions follow from Theorem 2.5 and section 3.

4.7. THEOREM. – *Let $\mathcal{S}$ be a locally topologically trivial semmialgebraic stratification of the semialgebraic set X, with $\dim X \leq 3$, such that all the skeletons X^i of $\mathcal{S}$ are Euler. Then the characteristic sets $Z_i(X)$, $i = 0, \ldots, 11$, are Euler if and only if the family $\{\mathbf{1}_{X^i} \mid i = 0,1,2,3\}$ is completely Euler.*

The proof is similar to that of Theorem 4.2. We just sketch the main points.

First we recall briefly the construction of $Z_i(X)$, $i = 0, \ldots, 11$, again following ideas of [C] and [CK]. It is important to note that this time the characteristic sets will depend on the stratification $\mathcal{S}$ of X. Let $C_0(X)$ be defined as above and let $C_1(X)$ be the union of X^1 and the remaining 2-dimensional strata T; that is, those strata T such that $\tilde{\Omega}\mathbf{1}_X \equiv 0 \pmod 2$ on T. Given a 1-dimensional stratum S we define

$$\varepsilon_3(S) = \tfrac{1}{2}\chi(\mathrm{lk}_p(S; C_1(X))) \pmod 2,$$

where p can be any point of S. The following lemma shows that $\varepsilon_3(S)$ is well-defined.

4.8. LEMMA. – *The set $C_1(X)$ is Euler in the complement of X^0. Moreover, for each stratum S of dimension 1 and $p \in S$,*

$$\varepsilon_2(S) + \varepsilon_3(S) = \tilde{\Omega}(\mathbf{1}_{X^2})(p) \pmod 2.$$

Proof. – The set $C_1(X)$ is Euler because so are $C_0(X)$, $X^2 = C_0(X) \cup C_1(X)$, and $X^1 = C_0(X) \cap C_1(X)$. The last statement of the lemma follows from

$$\varepsilon_2(S) + \varepsilon_3(S) = \tfrac{1}{2}\chi(\mathrm{lk}_p(S; X^2)) = \tilde{\Omega}(\mathbf{1}_{X^2})(p) \pmod 2. \quad \square$$

Proof of Theorem 4.7. – Given $(a,b,c,d) \in (\mathbf{Z}_2)^4$, define the characteristic set $\mathcal{E}_{abcd}(\mathcal{X})$ as the union of X^0 and those 1-dimensional strata S such that

$(\varepsilon_0(S), \varepsilon_1(S), \varepsilon_2(S), \varepsilon_3(S)) = (a, b, c, d)$. (We follow here the notation of [AK2, §7.1].) It is easy to check that

$$\mathcal{E}_{abcd}(\mathcal{X}) = \mathcal{X}^0 \cup \{x \in \mathcal{X}^1 \mid (\varepsilon_0(x), \varepsilon_1(x), \varepsilon_2(x), \varepsilon_3(x) = (a, b, c, d)\},$$

where $\varepsilon_0(x)$, $\varepsilon_1(x)$, $\varepsilon_2(x)$ are given by (4.6), and $\varepsilon_3(x) = \varepsilon_2(x) + \tilde{\Lambda}(\mathbf{1}_{X^2})(x) \pmod 2$. The characteristic sets $Z_i(X)$, $i = 0, \dots, 11$, are unions of some of the sets $\mathcal{E}_{abcd}(\mathcal{X})$. The interested reader may consult [AK2, §7.1] for their definitions. The important property of the Z_i's is that they are Euler if and only if all the sets $\mathcal{E}_{abcd}(\mathcal{X})$ are Euler, as follows from Lemma 7.1.6 *loc. cit.* On the other hand, it is easy to see by Proposition 3.12 that all the sets $\mathcal{E}_{abcd}(\mathcal{X})$ are Euler if and only if the family $\{\mathbf{1}_{X^i} \mid i = 0, 1, 2, 3\}$ is completely Euler. This completes the proof.

Note also that Proposition 3.12 explains why we need only 12 conditions out of 16. □

5. Nash constructible functions and Arc-symmetric sets

We present a variation of our notion of algebraically constructible functions.

Let X be a real algebraic set. A constructible function $\varphi \in \mathrm{F}(X)$ is called *Nash constructible* if it admits a presentation as a finite sum

$$\varphi = \sum m_i f_{i*} \mathbf{1}_{T_i},$$

where for each i, m_i is an integer, T_i is a connected component of an algebraic set Z_i, and $f_i : Z_i \to X$ is proper and regular. By the same arguments as in Section 2, one shows that the family of Nash constructible functions is preserved by the inverse image by a regular map, the direct image by a proper regular map, duality, and $\tilde{\Lambda}$. Hence not all constructible functions are Nash constructible. On the other hand, there are Nash constructible functions which are not algebraically constructible. Consider for instance the following classical example. Let $X \subset \mathbf{R}^2$ be the curve defined by $y^2 = (x-1)x(x+1)$. Then X is irreducible and nonsingular and consists of two connected components X_i, $i = 1, 2$. Moreover, the Zariski closure of either of these components is X itself. Hence by Lemma 2.2 the characteristic functions $\mathbf{1}_{X_i}$ are not algebraically constructible, though they are clearly Nash constructible.

Note that Lemma 2.2 does not hold any longer if we merely assume that Z is a component of a real algebraic set, so this lemma cannot be applied to study Nash constructible functions. Instead one can use the following general statement.

5.1. PROPOSITION. – *Let $f : Z \to X$ be a proper analytic mapping of real analytic spaces, and suppose that X is connected and nonsingular. Then $\chi(f^{-1}(x))$ is generically constant* mod 2*;* i.e., *there exists a subanalytic subset $Y \subset X$ such that* $\dim Y < \dim X$ *and, for all $x, x' \in X \setminus Y$,*

$$\chi(f^{-1}(x)) \equiv \chi(f^{-1}(x')) \pmod 2.$$

Moreover, if Z, X and f are semialgebraic, then Y can be chosen to be semialgebraic.

Proof. – By [Su] Z is an Euler space. Let $\varphi = f_* \mathbf{1}_Z$. Then φ is a subanalytically constructible function in the sense of [KS] and [Sch]. Since, by *loc. cit.*, $f_* D = D f_*$, it follows that φ is an Euler function; that is, $\Lambda\varphi = f_* \Lambda \mathbf{1}_Z$ attains only even values. Now the proposition follows from the following lemma.

5.2. LEMMA. – *Let X be a connected real analytic manifold and let φ be a subanalytically constructible Euler function on X. Then φ is generically constant* mod 2.

Proof. – X admits a subanalytic triangulation such that φ is constant on open simplices. Let Δ_1, Δ_2 be two simplices of dimension $n = \dim X$ such that $\Delta_{12} = \Delta_1 \cap \Delta_2$ is a simplex of dimension $n-1$. Let p be a point in the interior of Δ_{12} and denote by a_1, a_2, a_{12} the values of φ on the interiors of Δ_1, Δ_2, and Δ_{12} respectively. Then by definition of the link operator, $\Lambda\varphi(p) = a_1 + a_2 + (1 + (-1)^n)(a_{12} - a_1 - a_2)$. Thus if $\Lambda\varphi(p)$ is even then $a_1 \equiv a_2 \pmod 2$. □

5.3. DEFINITION. – *Let X be a real algebraic set. A semialgebraic subset S of X is called arc-symmetric (or symmetric by arcs) if, for every analytic arc $\gamma : (-1,1) \to X$, if $\gamma((-1,0)) \subset S$ then $\gamma((-1,1)) \subset S$.*

Every arc-symmetric semialgebraic set is closed in X. The notion of arc-symmetric sets was introduced by Kurdyka; in many ways these sets resemble algebraic subsets, but they form a much wider class (*cf.* [Ku]). It is interesting to note that they can be studied using the techniques introduced in this paper.

5.4. PROPOSITION. – *Let S be a closed semialgebraic subset of an algebraic set X. Then $\mathbf{1}_S$ is Nash constructible if and only if S is arc-symmetric.*

5.5. COROLLARY. – *Every arc-symmetric semialgebraic set S is Euler and $\mathbf{1}_S$ is completely Euler.* □

5.6. COROLLARY. – *Every arc-symmetric semialgebraic set of dimension ≤ 3 is homeomorphic to an algebraic set.* □

Proof of 5.4. – Let $\mathbf{1}_S = \sum m_i f_{i*} \mathbf{1}_{T_i}$ be Nash constructible. Let $\gamma : (-1,1) \to X$ be an analytic arc in X such that $\gamma((-1,0)) \subset S$. Then by Lemma 5.1, $\chi(f_i^{-1}(\gamma(t)))$ is generically constant mod 2 on $(-1,1)$. Hence so is $\mathbf{1}_S$. This gives, for S closed, $\gamma((-1,1)) \subset S$, as required.

Conversely, let S be an arc-symmetric semialgebraic subset of X. We show by induction on $n = \dim S$ that $\mathbf{1}_S$ is Nash constructible. We may assume that X is the smallest algebraic set containing S; that is, X is the Zariski closure of S. Then $\dim S = \dim X$. Let $\sigma : \widetilde{X} \to X$ be a resolution of singularities of X. Then σ is an isomorphism over $X \setminus \Sigma$, where Σ is an algebraic subset of X, and both Σ and $\widetilde{\Sigma} = \sigma^{-1}(\Sigma)$ are of dimension smaller than n. Let $\widetilde{X}_1, \ldots, \widetilde{X}_s$ be the connected components of $\widetilde{X}$ of dimension n, and let $\widetilde{S}$ be the union of those $\widetilde{X}_i$ such that

$$\sigma(\widetilde{X}_i) \cap (S \setminus \Sigma) \neq \emptyset.$$

Then, since S is arc-symmetric, by an argument of Kurdyka [Ku, Théorème 2.6],

$$\sigma(\widetilde{S}) \subset S.$$

Hence

$$\mathbf{1}_S = \mathbf{1}_{S\setminus\Sigma} + \mathbf{1}_{S\cap\Sigma} = \sigma_*\mathbf{1}_{\tilde{S}} - \sigma_*\mathbf{1}_{\tilde{S}\cap\widetilde{\Sigma}} + \mathbf{1}_{S\cap\Sigma}.$$

The first two summands are Nash constructible by definition, and the latter is Nash constructible by the inductive hypothesis, since clearly $S \cap \Sigma$ is arc-symmetric. □

APPENDIX

Proofs of some properties of constructible functions

In this section we present elementary proofs of some basic properties of semialgebraically constructible functions. These proofs use either stratifications or triangulations of semialgebraic sets (see [Ł] for references). We believe our arguments are well-known to specialists, and we do not claim any originality (*cf.* [Sch, Remark 3.5]).

Let X be a closed semialgebraic set and let $x \in X$. By [CK, Prop. 1] the link $\mathrm{lk}(x;X)$ is well-defined up to semialgebraic homeomorphism. The Euler characteristic of the link is a topological invariant of the germ (X,x). Indeed, X is locally contractible and hence

$$\chi(\mathrm{lk}(x;X)) = 1 - \chi(X, X\setminus\{x\}). \tag{A.1}$$

Similarly let Y be a compact semialgebraic subset of X. Then the quotient space X/Y has a natural structure as a semialgebraic set and we may define the *link of* Y *in* X, by $\mathrm{lk}(Y;X) = \mathrm{lk}(*;X/Y)$, where $*$ denotes the class of Y in X/Y. If $Y \subset \mathbf{R}^n$ is compact and semialgebraic, and not necessarily contained in X, then by $\mathrm{lk}(Y;X)$ we mean $\mathrm{lk}(Y\cap X;X)$. By the above, $\chi(\mathrm{lk}(Y;X))$ is also a topological invariant of the pair (X,Y); this also follows from the following corollary of [MP, Lemma 1]:

$$\chi(\mathrm{lk}(Y;X)) = \chi(Y\cap X) + \chi(X\setminus Y) - \chi(X) = \chi(Y\cap X) - \chi(X, X\setminus Y). \tag{A.2}$$

Let $f: Z \to X$ be a proper semialgebraic map and let Y be a compact subset of X. In what follows we often use the following consequence of the definition of the link:

$$f^{-1}(\mathrm{lk}(Y;X)) = \mathrm{lk}(f^{-1}(Y);Z).$$

Recall that the additivity of Euler characteristic,

$$\chi(X\cup X') = \chi(X) + \chi(X') - \chi(X\cap X'), \tag{A.3}$$

allowed us to define in section 1 the Euler integral of the semialgebraically constructible function $\varphi \in \mathrm{F}(X)$, provided φ has compact support. Here are some other elementary consequences of (A.3).

A.4. LEMMA. – *Let X and Y be closed semialgebraic subsets of $\mathbf{R}^n$ and suppose that Y is compact. Then*

$$\chi(\mathrm{lk}(Y;X)) = \int_Y \Lambda\mathbf{1}_X. \tag{A.4.1}$$

Let $f : Z \to X$ be a semialgebraic map, and let $\varphi \in \mathrm{F}(Z)$ have compact support. Then

$$\int f_*\varphi = \int \varphi. \tag{A.4.2}$$

Let $x \in X$ and let $\varphi \in \mathrm{F}(X)$. Then for $\epsilon > 0$ sufficiently small,

$$\int_{B_\varepsilon} \varphi = \varphi(x), \tag{A.4.3}$$

where B_ε is the closed ball of radius ϵ centered at x.

Proof. – The right hand side of (A.2) is additive with respect to X for Y fixed and additive with respect to Y for X fixed. By (A.2) so is $\chi(\mathrm{lk}(Y;X))$. On the other hand, the right hand side of (A.4.1) is also additive with respect to both X and Y. Hence, by the triangulability of semialgebraic sets, it suffices to verify (A.4.1) for X and Y simplices such that $X \cap Y$ is their common face. In this case the verification is straightforward. This shows (A.4.1).

To show (A.4.2) we may assume that Z is compact and $\varphi = \mathbf{1}_Z$. We may assume also that, up to a semialgebraic homeomorphism, X is equal to a simplex Δ and f is topologically trivial with fibre F over the interior of Δ. Let $Y = \partial\Delta$, and let $Z' = f^{-1}(Y)$. Since f is topologically trivial over $\mathrm{lk}(Y;X)$,

$$\chi(\mathrm{lk}(Z';Z)) = \chi(F)\chi(\mathrm{lk}(Y;X)),$$

which, by (A.2) and the inductive assumption on $\dim X$, gives

$$\begin{aligned}
\int \varphi &= \chi(Z) \\
&= \chi(Z') + \chi(Z \setminus Z') - \chi(\mathrm{lk}(Z', Z)) \\
&= \int_Y f_*\varphi + \chi(F)(\chi(X \setminus Y) - \chi(\mathrm{lk}(Y;X))) \\
&= \int_Y f_*\varphi + \chi(F)(\chi(X) - \chi(Y)) \\
&= \int \mathbf{1}_Y f_*\varphi + \int \mathbf{1}_{X\setminus Y} f_*\varphi \\
&= \int f_*\varphi,
\end{aligned}$$

as required. This shows (A.4.2).

It suffices to show (A.4.3) for $\varphi = \mathbf{1}_X$ and in this case (A.4.3) follows from the local contractibilty of X. $\square$

Proof of Proposition 1.2. – To show (i) we note that D is additive, *i.e.* $D(\varphi + \psi) = D\varphi + D\psi$. Therefore it suffices to verify (i) for $\varphi = \mathbf{1}_\Delta$, where Δ is a simplex. In this case the verification is straightforward.

Let $f : X \to X'$ be proper semialgebraic, $x \in X'$, and $Y = f^{-1}(x)$. Then by (A.4.1),

$$\begin{aligned} f_*(\Lambda \mathbf{1}_X)(x) &= \int_Y \Lambda \mathbf{1}_X = \chi(\mathrm{lk}(Y; X)) = \chi(f^{-1}(\mathrm{lk}(x; X'))) \\ &= \int_{\mathrm{lk}(x, X')} f_* \mathbf{1}_X = \Lambda(f_* \mathbf{1}_X)(x). \end{aligned}$$

Hence $f_* \Lambda = \Lambda f_*$, which implies (ii) of Propostion 1.2.

Statement (iii) follows from (A.4.2). Indeed, let $Z \xrightarrow{f} X \xrightarrow{g} Y$ and let $y \in Y$. Then by (A.4.2),

$$(g \circ f)_* \varphi(y) = \int_{(g \circ f)^{-1}(y)} \varphi = \int_{g^{-1}(y)} f_* \varphi = g_*(f_* \varphi)(y). \quad \square$$

Proof of Proposition 1.7. – Let $h : X \to \mathbf{R}$ be semialgebraic. Then h is locally topologically trivial over the complement of a finite subset of $\mathbf{R}$. By [H] we may assume that this trivialization is semialgebraic. In particular for $x \in X$ in a generic fibre X_t of h, the link $\mathrm{lk}(x; X)$ is the suspension of $\mathrm{lk}(x; X_t)$. This gives

$$\chi(\mathrm{lk}(x; X)) = 2 - \chi(\mathrm{lk}(x; X_t)). \tag{A.5}$$

This, in particular, shows Propostion 1.7. Note also that, by virtue of (A.1), we may use any topological trivialization of h (not necessaily semialgebraic) to establish (A.5). $\square$

Suppose, in addition, that $h : X \to \mathbf{R}$ is proper and let $c_0 < c_1$ be generic values of h. Let $X_{c_0, c_1} = h^{-1}[c_0, c_1]$. Then by (A.5)

$$\Lambda \mathbf{1}_{X_{c_0, c_1}} = (\Lambda \mathbf{1}_X)|_{X_{c_0, c_1}} + (\mathbf{1}_{X_{c_0}} - (\Lambda \mathbf{1}_X)|_{X_{c_0}}) + (\mathbf{1}_{X_{c_1}} - (\Lambda \mathbf{1}_X)|_{X_{c_1}}). \tag{A.6}$$

Proof of Proposition 1.8. – Let $Y \subset X \subset \mathbf{R}^n$ and fix $x \in X$. Let $\varphi \in \mathrm{F}(X)$. Let B_ε denote a small closed ball centered at x and $S_\varepsilon = \partial B_\varepsilon$. By (A.6)

$$\Lambda(\varphi|_{B_\varepsilon}) = (\Lambda \varphi)|_{B_\varepsilon} - (\Lambda \varphi)|_{S_\varepsilon} + \varphi|_{S_\varepsilon},$$

and hence by (A.4.1) and (A.4.3)

$$\begin{aligned} \Lambda_Y \varphi(x) &= \int_{Y \cap B_\varepsilon} \Lambda(\varphi|_{B_\varepsilon}) \\ &= \int_{Y \cap B_\varepsilon} [(\Lambda \varphi)|_{B_\varepsilon} - (\Lambda \varphi)|_{S_\varepsilon} + \varphi|_{S_\varepsilon}] \\ &= \Lambda \varphi(x) - \Lambda((\Lambda \varphi)|_Y)(x) + \Lambda(\varphi|_Y)(x) \end{aligned}$$

if $x \in Y$, and $\Lambda_Y \varphi(x) = 0$ otherwise. This shows Proposition 1.8. $\square$

A.7. *Remark (Topological invariance of the link operator and the Euler integral).* – Let $h : X' \to X$ be a homeomorphism (not necessarily semialgebraic) of semialgebraic sets.

Let $\varphi \in \mathrm{F}(X)$ be such that $\varphi' = \varphi \circ h \in \mathrm{F}(X')$. Let $Y \subset X$ be a compact semialgebraic subset such that $Y' = h^{-1}(Y)$ is also semialgebraic. Then

$$(\Lambda\varphi) \circ h = \Lambda(\varphi'), \qquad \int_Y \varphi = \int_{Y'} \varphi'.$$

Indeed, it suffices to show that there exist closed semialgebraic sets $X_i \subset X$ such that $h^{-1}(X_i)$ are semialgebraic subsets of X' and

$$\varphi = \sum m_i \mathbf{1}_{X_i}.$$

Here is a canonical construction of such sets X_i. First we note that φ is semialgebraically constructible if and only if all the sets $\varphi^{-1}(m)$, $m \in \mathbf{Z}$, are semialgebraic and all but finitely many of them are empty. Let $\varphi_m = \varphi|_{\varphi^{-1}(m)}$. Then clearly

$$\varphi = \sum_m \varphi_m. \tag{A.8}$$

Let $Y = \varphi^{-1}(m)$. Then $\mathbf{1}_Y$ can be canonically decomposed

$$\mathbf{1}_Y = \mathbf{1}_{F_1} - \mathbf{1}_{F_2} + \mathbf{1}_{F_3} - \cdots \pm \mathbf{1}_{F_d}, \tag{A.9}$$

where $F_1 \supset F_2 \supset \cdots \supset F_d$ are closed semialgebraic in $\bar{Y}$. The sequence of F_i's is constructed recursively as follows (*cf.* [K, §12]): $Y_0 = Y$, $F_i = \bar{Y}_{i-1}$, $Y_i = F_i \setminus Y_{i-1}$, $i = 1, 2, \ldots$. Clearly all the F_i are closed and semialgebraic, and the sequence terminates since $\dim F_i < \dim F_{i-1}$. Now (A.8), together with (A.9) applied to each set $\varphi^{-1}(m)$, gives the required canonical decomposition of φ.

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