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FILTRATIONS AND TILTING MODULES

BY HENNING HAAHR ANDERSEN

ABSTRACT. – In this paper we consider the modular analogue of a recent theorem by Soergel on tilting modules for quantum groups at roots of 1. The modular case is the case of a semisimple algebraic group over a field of characteristic $p > 0$. A natural conjecture is that the tilting modules in this situation should have the same characters as in the quantum case as long as the highest weights belong to the lowest p^2-alcove.

The character of a tilting module Q (modular or quantized) is determined by the spaces of homomorphisms from the Weyl modulus into Q. We introduce a "Jantzen type" filtration on each such Hom-space and we prove that if these filtrations behave in the expected way with respect to translations through walls then Soergel's theorem and its modular analogue follow.

Our filtrations also exist outside the lowest p^2-alcove but it is still a wide open problem to find a conjecture for the characters of tilting modules here.

RÉSUMÉ. – Soit G un groupe algébrique semi-simple défini sur un corps de caractéristique $p > 0$. Le caractère d'un G-module basculant Q est déterminé par les espaces des homomorphismes des modules de Weyl dans Q. Nous avons ici construit certaines filtrations de « type Jantzen » pour ces espaces.

Si notre filtration respecte les foncteurs de translations alors il est possible de calculer les caractères pour tous les G-modules basculants avec poids dominants dans la p^2-alcôve principale. En fait, l'hypothèse implique que l'analogue de la conjecture de Soergel aux modules de groupes quantiques dans les racines de l'unité est vrai. Dans le cas quantique (où notre filtration existe aussi) Soergel a vérifié sa conjecture avec une méthode différente.

Introduction

In his recent preprint [So1] Soergel has come up with a conjecture about the characters of tilting modules for quantum groups at roots of 1. Even more recently [So2] he has proved this conjecture by exploring a result of Arkhipov [Ar] applied to affine Kac-Moody algebras (and then using the equivalence [KL2] between the category of finite dimensional modules for the quantum group and a certain negative level category for the corresponding affine Kac-Moody algebra).

There is an obvious analogous conjecture for the modular case when the highest weights of the tilting modules are assumed to be in the lowest p^2-alcove. Unfortunately, Soergel's proof does not carry over to this case nor does it throw any light on the mystery of what happens when we move outside this p^2-alcove (this phenomena has no counterpart for quantum groups).

Soergel's theorem and the analogous modular conjecture give the tilting characters in terms of the Kazhdan-Lusztig polynomials [KL1] attached to the affine Weyl group in question. They are in fact stronger than the Lusztig conjectures (a theorem in the quantum case and for large primes also in the modular case [AJS]) about the irreducible characters. Moreover, Mathieu has demonstrated [Ma] that the decomposition behaviour for tilting modules for the general linear groups have interesting applications to the modular representations for symmetric groups. For this however, it is necessary to have information for all dominant weights –not just for those in the lowest p^2-alcove.

In this paper we consider mainly the modular case (in the last section we outline briefly how our approach works in the quantum case). Let G be a reductive group over an algebraically closed field k of positive characteristic p. We construct filtrations of some Hom-spaces associated with tilting modules for $G_{\mathbf{Z}_p}$, the algebraic group over the p-adics corresponding to G and we study how these filtrations behave with respect to translation functors. Our main result says that if this behaviour is decent (*see* Section 3.1 for the precise meaning of this) then the modular analoque of Soergel's theorem is true.

Unfortunately, we cannot prove this behaviour and so we have to leave it as a conjecture to the reader. We have some partial results, some evidence and some applications.

In Section 1 we give the construction and first properties of our filtrations. Then in Section 2 we study the effect of translation (wall-crossing) on the filtrations. In both these sections we work with general tilting modules (*i.e.* no restriction on the weights). In fact, it would be extremely interesting to have a conjecture about the decomposition of tilting modules for G also outside the lowest p^2-alcove but so far we haven't been able to come up with any (*cf.* also Remark 1.1 ii)). The above mentioned conjecture as well as the derivation of Soergel's conjecture from it can be found in Section 3 (where we do restrict to the lowest p^2-alcove). Finally in Section 4 we treat the quantum case.

1. Filtrations

1.1. Let G denote a semi-simple algebraic group over an algebraically closed field k of characteristic $p > 0$. We shall use the standard notation T, R, R^+, S, W, $X = X(T)$, X^+, X_1, ρ, G_1 etc. (as in [Ja]) for a maximal torus, the root system attached to (G, T), a set of positive roots, the corresponding simple roots, the Weyl group, the set of weights, dominant weights, restricted weights, half the sum of the positive roots, the Frobenius kernel, etc.

For $\lambda \in X^+$ we have a Weyl module $V_k(\lambda)$, an induced module $H^0_k(\lambda)$, and a simple module $L_k(\lambda)$, all with highest weight λ.

Recall that a G-module Q is called tilting [Do] if it allows two filtrations

$$0 = F_0 \subset F_1 \subset \ldots \subset F_r = Q$$

and

$$0 = F'_0 \subset F'_1 \subset \ldots \subset F'_r = Q$$

with $F_i/F_{i-1} \simeq V_k(\lambda_i)$, resp. $F'_i/F'_{i-1} \simeq H^0_k(\nu_i)$, $i = 1, 2, \ldots, r$ for some $\lambda_i, \nu_i \in X^+$. Then for each $\nu \in X^+$ the two numbers

$$\#\{i|\lambda_i = \nu\} \qquad \text{and} \qquad \#\{i|\nu_i = \nu\}$$

coincide. We denote it $[Q : V_k(\nu)]$.

For each $\lambda \in X^+$ we have a unique indecomposable tilting module $T_k(\lambda)$ with $\dim T_k(\lambda)_\lambda = 1$. If Q is an arbitrary tilting module we can therefore write

$$Q = \bigoplus_{\lambda \in X^+} [Q : T_k(\lambda)] T_k(\lambda)$$

for some $[Q : T_k(\lambda)] \in \mathbf{N}$.

PROBLEM. – Determine $[T_k(\lambda) : V_k(\nu)]$ for all $\lambda, \nu \in X^+$ (this is of course equivalent to finding the character of $T_k(\lambda)$ for all $\lambda \in X^+$).

Remarks. – (i) If $\nu \in X^+$ is minimal (either with respect to the usual ordering $\leq$ in X^+ or with respect to the strong linkage relation) then $V_k(\nu) = T_k(\nu) = L_k(\nu) = H_k^0(\nu)$. Suppose all fundamental weights ω_i are minimal. Then the problem above is equivalent to determining $[\otimes_i V_k(\omega_i)^{\otimes a_i} : T_k(\lambda)]$ for all $a_i \in \mathbf{N}$, $\lambda \in X^+$. It also amounts to the same to determine $[T_k(\nu) \otimes V_k(\omega_i) : T_k(\lambda)]$ for all $\lambda, \nu \in X^+$, $i = 1, \ldots, r$.

(ii) Suppose $p \geq 2h - 2$. Then it is known (*see* e.g. [Ja], II.11.11) that for $\lambda \in X_1$ the module $T_k((p-1)\rho + \lambda)$ is indecomposable when restricted to G_1 (as a G_1T-module we have in fact that $T_k((p-1)\rho + \lambda)$ is the projective cover of $L_k((p-1)\rho + \lambda)$). If M is an indecomposable G/G_1-module then by looking at the endomorphism rings it is easy to see that $T_k((p-1)\rho + \lambda) \otimes M$ is indecomposable as a G-module. If M is tilting (for G/G_1) so is this tensor product (because $T_k((p-1)\rho + \lambda)$ is a summand of $St_1 \otimes T_k(\lambda)$, where $St_1 = H^0((p-1)\rho)$. Recall that $St_1 \otimes H_k^0(\nu)^{(1)} \simeq H_k^0(p(\nu + \rho) - \rho), \nu \in X^+$). Hence we have (compare [Do])

$$T_k((p-1)\rho + \lambda) \otimes T_k(\mu)^{(1)} \simeq T_k(p(\mu + \rho) - \rho + \lambda)$$

for all $\lambda \in X_1$, $\mu \in X^+$.

This fact shows that (at least when $p \geq 2h - 2$) it is enough in the above problem to consider the following two sets of λ's

(*a*) $X_1 + (p-1)\rho$.

(*b*) $\{\nu \in X^+ | \langle \nu, \alpha^\vee \rangle \leq p - 2$ for some $\alpha \in S\}$.

1.2. Let $\mathbf{Z}_p$ be the ring of p-adic integers and let $G_{\mathbf{Z}_p}$ denote the algebraic group over $\mathbf{Z}_p$ corresponding to G. Then the Weyl modules and the induced modules lift to modules for $G_{\mathbf{Z}_p}$, *i.e.* for each $\lambda \in X^+$ there exist $G_{\mathbf{Z}_p}$-modules $V_{\mathbf{Z}_p}(\lambda)$ and $H^0_{\mathbf{Z}_p}(\lambda)$ with

$$V_{\mathbf{Z}_p}(\lambda) \otimes k \simeq V_k(\lambda) \qquad \text{and} \qquad H^0_{\mathbf{Z}_p}(\lambda) \otimes k \simeq H^0_k(\lambda).$$

These $G_{\mathbf{Z}_p}$-modules have the usual universal properties. Moreover, we have

$$\text{(1)} \qquad \operatorname{Ext}^i_{G_{\mathbf{Z}_p}}(V_{\mathbf{Z}_p}(\lambda), H^0_{\mathbf{Z}_p}(\mu)) \simeq \begin{cases} \mathbf{Z}_p, & \text{if } i = 0,\ \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

The tilting modules $T_k(\lambda)$ also lift to $G_{\mathbf{Z}_p}$. In fact, the Ringel construction (which even works over $\mathbf{Z}$, *see* [Do]) gives an indecomposable tilting $G_{\mathbf{Z}_p}$-module $T_{\mathbf{Z}_p}(\lambda)$ with

$\mathrm{rk}T_{\mathbf{Z}_p}(\lambda)_\lambda = 1$. It follows easily that $T_{\mathbf{Z}_p}(\lambda) \otimes k \simeq T_k(\lambda)$ (note for instance that since $\mathrm{End}_{G_{\mathbf{Z}_p}}(T_{\mathbf{Z}_p}(\lambda))$ is local the same is true about $\mathrm{End}_{G_{\mathbf{Z}_p}}(T_{\mathbf{Z}_p}(\lambda)) \otimes k \simeq \mathrm{End}_G(T_k(\lambda)))$.

Standard arguments show that $T_{\mathbf{Z}_p}(\lambda)$ is the unique tilting $G_{\mathbf{Z}_p}$-module with $T_{\mathbf{Z}_p}(\lambda)\otimes k \simeq T_k(\lambda)$ and that any tilting $G_{\mathbf{Z}_p}$-module Q may be decomposed

$$Q = \bigoplus_{\lambda \in X^+} a_\lambda T_{\mathbf{Z}_p}(\lambda)$$

for some (unique) $a_\lambda \in \mathbf{N}$ (clearly $a_\lambda = [Q \otimes k : T_k(\lambda)]$).

1.3. From now on we fix a generator c_λ for $\mathrm{Hom}_{G_{\mathbf{Z}_p}}(V_{\mathbf{Z}_p}(\lambda), H^0_{\mathbf{Z}_p}(\lambda))$ for each $\lambda \in X^+$ (*see* 1.2 (1)).

Let Q be a tilting module for G. By 1.2 there exists a unique lift $\widetilde{Q}$ of Q to $G_{\mathbf{Z}_p}$. For each $\lambda \in X^+$ we set

$$F_\lambda(Q) = \mathrm{Hom}_{G_{\mathbf{Z}_p}}(V_{\mathbf{Z}_p}(\lambda), \widetilde{Q})$$

and

$$\bar{F}_\lambda(Q) = F_\lambda(Q) \otimes k \simeq \mathrm{Hom}_G(V_k(\lambda), Q)$$

(the isomorphims comes from 1.2 (1)).

Then $F_\lambda(Q)$ is a free $\mathbf{Z}_p$-module of rank equal to $[Q : V_k(\lambda)]$ (again using 1.2 (1)). We define a filtration of $F_\lambda(Q)$ by setting

$$F_\lambda(Q)^j = \{\varphi \in F_\lambda(Q) | \, \psi \circ \varphi \in p^j \mathbf{Z}_p c_\lambda \text{ for all } \psi \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(\widetilde{Q}, H^0_{\mathbf{Z}_p}(\lambda))\}$$

and we denote by $\bar{F}_\lambda(Q)^j$ the subspace in $\bar{F}_\lambda(Q)$ spanned by the image of $F_\lambda(Q)^j$. Note that $\bar{F}_\lambda(Q)^j = 0$ for $j \gg 0$.

1.4. Let Q and $\widetilde{Q}$ be as in 1.3. For $\lambda \in X^+$ we set

$$E_\lambda(Q) = \mathrm{Hom}_{G_{\mathbf{Z}_p}}(\widetilde{Q}, H^0_{\mathbf{Z}_p}(\lambda)).$$

Then 1.2 (1) shows that $E_\lambda(Q)$ is a free $\mathbf{Z}_p$-module of rank equal to $[Q : V_k(\lambda)]$. We let $E_\lambda(Q)^*$ denote the dual module. Then we have a $\mathbf{Z}_p$-linear map

$$\phi_\lambda : F_\lambda(Q) \to E_\lambda(Q)^*$$

given by

$$\phi_\lambda(\varphi) : \psi \mapsto \psi \circ \varphi, \qquad \varphi \in F_\lambda(Q), \qquad \psi \in E_\lambda(Q).$$

Here $\psi \circ \varphi$ is identified with $a \in \mathbf{Z}_p$ if $\psi \circ \varphi = a c_\lambda$. When tensored by $\mathbf{Q}_p$ we see that ϕ_λ becomes an isomorphism (because $\widetilde{Q} \otimes \mathbf{Q}_p = \bigoplus_\lambda [Q : V_k(\lambda)] V_{\mathbf{Q}_p}(\lambda)$ where $V_{\mathbf{Q}_p}(\lambda) = V_{\mathbf{Z}_p}(\lambda) \otimes \mathbf{Q}_p$ is the irreducible $G_{\mathbf{Q}_p}$-module of highest weight λ). Hence the usual arguments (*see* e.g Section II.8 in [Ja]) give the "Jantzen type sum formula"

$$\sum_{j \geq 1} \dim \bar{F}_\lambda(Q)^j = \nu_p(\det \varphi_\lambda).$$

Here $\nu_p : \mathbf{Z}_p \to \mathbf{Z}$ denotes p-adic valuation.

PROBLEM. – *Determine* $\nu_p(\det \varphi_\lambda)$ *for* $Q = T_k(\nu)$, $\lambda, \nu \in X^+$.

Remark. – On the category of G-modules we have the notion of 'contravariant' duality. It takes a G-module M into $M^* = \operatorname{Hom}_k(M, k)$ equipped with a G-action such that $L_k(\lambda) \simeq L_k(\lambda)^*$, $\lambda \in X^+$. Then we have $V_k(\lambda)^* \simeq H^0_k(\lambda)$ and $T_k(\lambda)^* \simeq T_k(\lambda)$, $\lambda \in X^+$. In particular, all tilting modules are self-dual. We can thus think of ϕ_λ above as coming from the non-degenerate bilinear pairing

$$(\, , \,) : F_\lambda(Q) \times F_\lambda(Q) \to \mathbf{Z}_p$$

given by

$$(\varphi, \varphi') = \psi' \circ \varphi, \qquad \varphi, \varphi' \in F_\lambda(Q)$$

where $\psi' \in E_\lambda(Q) \simeq \operatorname{Hom}_{G_{\mathbf{Z}_p}}(H^0_{\mathbf{Z}_p}(\lambda)^*, \widetilde{Q}^*) \simeq \operatorname{Hom}_{G_{\mathbf{Z}_p}}(V_{\mathbf{Z}_p}(\lambda), \widetilde{Q}) = F_\lambda(Q)$ corresponds to φ'.

1.5. Fix $\lambda \in X^+$. Let Q be a tilting module for G and let q be an indeterminate. The filtration $(\bar{F}_\lambda(Q)^j)_{j \geq 0}$ constructed in 1.3 determines a polynomial

$$f_\lambda(Q) = \sum_{j \geq 0} (\dim \bar{F}_\lambda(Q)^j / \bar{F}_\lambda(Q)^{j+1}) q^j \in \mathbf{Z}[q].$$

PROPOSITION. – $T_k(\lambda)$ *is a direct summand of* Q *iff* $f_\lambda(Q)(0) \neq 0$. *More precisely,*

$$[Q : T_k(\lambda)] = f_\lambda(Q)(0).$$

Proof. – Suppose first $Q = T_k(\lambda) \oplus P$ and let $\widetilde{P}$ be a lift of P to $\mathbf{Z}_p$. If $\varphi' : V_{\mathbf{Z}_p}(\lambda) \to T_{\mathbf{Z}_p}(\lambda)$, resp. $\psi' : T_{\mathbf{Z}_p}(\lambda) \to H^0_{\mathbf{Z}_p}(\lambda)$ is the canonical inclusion, resp. projection, then $\psi' \circ \varphi' = c_\lambda$. This clearly gives rise to $\varphi \in F_\lambda(Q)$ and $\psi \in E_\lambda(Q)$ with $\psi \circ \varphi = c_\lambda$. Hence $f_\lambda(Q)(0) \neq 0$.

Conversely, suppose $f_\lambda(Q)(0) \neq 0$ and choose $\varphi \in F_\lambda(Q)^0 \setminus F_\lambda(Q)^1$. Then there exists $\psi \in E_\lambda(Q)$ with $\psi \circ \varphi = c_\lambda$. We now observe that we have a commutative diagram

$$\begin{array}{ccc} V_{Z_p}(\lambda) & \xrightarrow{\varphi} & \widetilde{Q} \\ \downarrow & \nearrow_i & \\ T_{Z_p}(\lambda) & & \end{array}$$

where the vertical map is the natural inclusion and where the existence of the homomorphism i comes from the fact that $\operatorname{Ext}^1_{G_{\mathbf{Z}_p}}(T_{\mathbf{Z}_p}(\lambda)/V_p(\lambda), \widetilde{Q}) = 0$, *see* 1.2 (1). Similarly, we obtain a homomorphism π making the diagram

$$\begin{array}{ccc} T_{Z_p}(\lambda) & \longrightarrow & H^0_{\mathbf{Z}_p}(\lambda) \\ \uparrow^{\pi} & \nearrow_{\psi} & \\ \widetilde{Q} & & \end{array}$$

commutative (here the horizontal map is the natural projection). When we now trace a highest weight vector in $T_{\mathbf{Z}_p}(\lambda)$ we see that $\pi \circ i$ is an isomorphism (because it gives rise to an endomorphism of $T_k(\lambda)$ which is non-zero on the λ-weight space).

Repeating the above argument if necessary we get also the more precise statement.

2. Filtrations and translations

2.1. Let $C \subset X^+$ be an alcove and suppose $\lambda, \mu \in \bar{C}$. The usual translation functors T_λ^μ and T_μ^λ, [Ja], II.7, may be defined just as well on the level of $G_{\mathbf{Z}_p}$-modules (where we will denote them by the same symbols). The fact that they are adjoint to each other means that we have isomorphisms

$$adj_1 : \mathrm{Hom}_{G_{\mathbf{Z}_p}}(M, T_\mu^\lambda N) \xrightarrow{\sim} \mathrm{Hom}_{G_{\mathbf{Z}_p}}(T_\lambda^\mu M, N)$$

and

$$adj_2 : \mathrm{Hom}_{G_{\mathbf{Z}_p}}(N, T_\lambda^\mu M) \xrightarrow{\sim} \mathrm{Hom}_{G_{\mathbf{Z}_p}}(T_\mu^\lambda N, M)$$

for all $G_{\mathbf{Z}_p}$-modules M and N in the blocks for λ and μ, respectively.

2.2. Lemma. – *Let M be a $G_{\mathbf{Z}_p}$-module belonging to the block of λ. Suppose that M is free of finite rank over $\mathbf{Z}_p$. Then we have*

$$Tr(adj_2(id_{T_\lambda^\mu M}) \circ adj_1^{-1}(id_{T_\lambda^\mu M})) = \mathrm{rk}(T_\lambda^\mu M).$$

Proof. – Let E be a $G_{\mathbf{Z}_p}$-module whose highest weight is the dominant weight conjugated under W to $\mu - \lambda$. Suppose that E is free over $\mathbf{Z}_p$ (take e.g. E to be the relevant Weyl module). Pick a basis $\{e_i\}$ for E and denote by $\{e_i^*\}$ the dual basis in E^*. Then we have

$$T_\lambda^\mu M = pr_\mu(E \otimes M) \qquad \text{and} \qquad T_\mu^\lambda(T_\lambda^\mu M) = pr_\lambda(E^* \otimes T_\lambda^\mu M)$$

with pr_μ (resp. pr_λ) denoting the projection onto the block of μ (resp. λ).

We have

$$adj_1^{-1}(id_{T_\lambda^\mu M})(m) = \sum_i e_i^* \otimes pr_\mu(e_i \otimes m), \qquad m \in M$$

and

$$adj_2(id_{T_\lambda^\mu M})(e_i^* \otimes n) = n_i \qquad \text{if} \quad n = \sum_i e_i \otimes n_i \in T_\lambda^\mu M.$$

The lemma is now an immediate computation.

2.3. The functoriality of the isomorphisms adj_1 and adj_2 from Section 2.1 means that have

$$(1) \qquad adj_1(f \circ g) = adj_1(f) \circ T_\lambda^\mu g, \qquad adj_2(f \circ g) = adj_2(f) \circ T_\mu^\lambda g$$

$$(2) \qquad adj_1(T_\mu^\lambda f \circ g) = f \circ adj_1(g), \qquad adj_2(T_\lambda^\mu f \circ g) = f \circ adj_2(g)$$

for all f and g in the relevant Hom-spaces.

Taking inverses we also get

$$(3) \qquad adj_1^{-1}(f \circ g) = T_\mu^\lambda f \circ adj_1^{-1}(g), \qquad adj_2^{-1}(f \circ g) = T_\lambda^\mu f \circ adj_2^{-1}(g)$$

$$(4) \qquad adj_1^{-1}(f \circ T_\lambda^\mu g) = adj_1^{-1}(f) \circ g, \qquad adj_2^{-1}(f \circ T_\mu^\lambda g) = adj_2^{-1}(f) \circ g.$$

2.4. Suppose from now on that λ is regular and μ is semi-regular. Let s denote the reflection belonging to the wall containing μ. Assume $\lambda s < \lambda$ (as in [So1] we consider the right action of the affine Weyl group on X).

Then we have the two short exact sequences

$$0 \to V_{\mathbf{Z}_p}(\lambda) \xrightarrow{i} T_\mu^\lambda V_{\mathbf{Z}_p}(\mu) \xrightarrow{\pi} V_{\mathbf{Z}_p}(\lambda s) \to 0 \tag{1}$$

$$0 \to H^0_{\mathbf{Z}_p}(\lambda s) \xrightarrow{i'} T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu) \xrightarrow{\pi'} H^0_{\mathbf{Z}_p}(\lambda) \to 0 \tag{2}$$

where

$$i = adj_1^{-1}(id_{V_{\mathbf{Z}_p}(\mu)}), \qquad i' = adj_1^{-1}(id_{H^0_{\mathbf{Z}_p}(\mu)}),$$

and

$$\pi = adj_2(id_{V_{\mathbf{Z}_p}(\mu)}), \qquad \pi' = adj_2(id_{H^0_{\mathbf{Z}_p}(\mu)}).$$

(We have used here the facts that $T_\lambda^\mu V_{\mathbf{Z}_p}(\lambda) \simeq V_{\mathbf{Z}_p}(\mu) \simeq T_\lambda^\mu V_{\mathbf{Z}_p}(\lambda s)$ and $T_\lambda^\mu H^0_{\mathbf{Z}_p}(\lambda) \simeq H^0(\mu) \simeq T_\lambda^\mu H^0_{\mathbf{Z}_p}(\lambda s)$).

Set now

$$r = adj_2(id_{V_{\mathbf{Z}_p}(\mu)}) : T_\mu^\lambda V_{\mathbf{Z}_p}(\mu) \to V_{\mathbf{Z}_p}(\lambda),$$

$$r' = adj_2(id_{H^0_{\mathbf{Z}_p}(\mu)}) : T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu) \to H^0_{\mathbf{Z}_p}(\lambda s),$$

$$s = adj_1^{-1}(id_{V_{\mathbf{Z}_p}(\mu)}) : V_{\mathbf{Z}_p}(\lambda s) \to T_\mu^\lambda V_{\mathbf{Z}_p}(\mu),$$

and

$$s' = adj_1^{-1}(id_{H^0_{\mathbf{Z}_p}(\mu)}) : H^0_{\mathbf{Z}_p}(\lambda) \to T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu).$$

LEMMA. – *Let $n = \nu_p(\mathrm{dim} V_k(\mu))$. Then up to units in $\mathbf{Z}_p$ we have*

(i) $$r \circ i = p^n id_{V_{\mathbf{Z}_p}(\lambda)}, \qquad r' \circ i' = p^n id_{H^0_{\mathbf{Z}_p}(\lambda s)},$$

(ii) $$\pi \circ s = p^n id_{V_{\mathbf{Z}_p}(\lambda s)}, \qquad \pi' \circ s' = p^n id_{H^0_{\mathbf{Z}_p}(\lambda)}.$$

Proof. – Since $\mathrm{End}_{G_{\mathbf{Z}_p}}(V_{\mathbf{Z}_p}(\lambda)) \simeq \mathbf{Z}_p$ we have $r \circ i = c \cdot id_{V_{\mathbf{Z}_p}(\lambda)}$ for some $c \in \mathbf{Z}_p$. The fact that $\nu_p(c) = n$ then follows from Lemma 2.2 (note that $\nu_p(\mathrm{dim} V_k(\lambda)) = 0$ because λ is regular).

The other relations are proved in the same way.

Remark. – From Weyl's dimension formula we see that the integer n appearing in the lemma equals $\nu_p(\langle \mu + \rho, \alpha^\vee \rangle)$ where $\alpha \in R^+$ is the unique positive root for which this valuation is non-zero.

2.5. Let the assumptions be as in 2.4. Set

$$\alpha = i' \circ c_{\lambda s} \circ \pi \qquad \text{and} \qquad \beta = T_\mu^\lambda c_\mu.$$

LEMMA. – *Up to units in* $\mathbf{Z}_p$ *we have*

(i) $$\pi' \circ \beta \circ i = c_\lambda$$

(ii) $$r' \circ \beta = c_{\lambda s} \circ \pi.$$

Proof. – (i) follows by noticing that λ is the highest weight in $T_\mu^\lambda V_{\mathbf{Z}_p}(\mu)$ and $\pi' \circ \beta \circ i$ is an isomorphism on the λ-weight space.

(ii) By 2.3 (1) we get

$$r' \circ \beta = adj_2(id_{H^0_{\mathbf{Z}_p}(\mu)}) \circ T_\mu^\lambda c_\mu = adj_2(c_\mu).$$

Now $c_\mu = T_\lambda^\mu c_{\lambda s}$ (this is again seen by tracing a highest weight vector) and 2.3 (2) gives

$$adj_2(c_\mu) = c_{\lambda s} \circ adj_2(id_{V_{\mathbf{Z}_p}(\mu)}) = c_{\lambda s} \circ \pi.$$

2.6. PROPOSITION. – *With* α *and* β *as in* 2.5 *we have*

$$\mathrm{Hom}_{G_{\mathbf{Z}_p}}(T_\mu^\lambda V_{\mathbf{Z}_p}(\mu), T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu)) \simeq \mathbf{Z}_p\alpha \oplus \mathbf{Z}_p\beta.$$

Proof. – Note that $\alpha \circ i = 0$. Lemma 2.5 (i) therefore implies that α and β are linearly independent.

Take $\varphi \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(T_\mu^\lambda V_{\mathbf{Z}_p}(\mu), T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu))$. Then there exists $c_1 \in \mathbf{Z}_p$ such that $\pi' \circ \varphi \circ i = c_1 c_\lambda$. By Lemma 2.5 (ii) we get $\pi' \circ (\varphi - c_1\beta) \circ i = 0$, *i.e.* $(\varphi - c_1\beta) \circ i = 0$ (because $\mathrm{Hom}_{G_{\mathbf{Z}_p}}(V_{\mathbf{Z}_p}(\lambda), H^0_{\mathbf{Z}_p}(\lambda s)) = 0$). But this implies $\varphi - c_1\beta = c_2\alpha$ for some $c_2 \in \mathbf{Z}_p$.

2.7. Recall that $\lambda > \lambda s$ and $n = \nu_p(\dim V_k(\mu))$.

PROPOSITION. – *Let* Q *be any tilting module belonging to the* μ*-block. Then the isomorphism* $adj_1^{-1} : F_\mu(Q) \xrightarrow{\sim} F_\lambda(T_\mu^\lambda Q)$, resp. $F_\mu(Q) \xrightarrow{\sim} F_{\lambda s}(T_\mu^\lambda Q)$ *gives for each* $j \geq 0$ *an isomorphism*

$$F_\mu(Q)^j \simeq F_\lambda(T_\mu^\lambda Q)^j,$$

resp.

$$F_\mu(Q)^j \simeq F_{\lambda s}(T_\mu^\lambda Q)^{j+n}.$$

Proof. – Let $\varphi \in F_\mu(Q)$ and $\psi \in E_\mu(Q)$. Then we get from 2.3 (1) and (3)

$$\begin{aligned} adj_2(\psi) \circ adj_1^{-1}(\varphi) &= (adj_2(id_{H^0_{\mathbf{Z}_p}(\mu)}) \circ T_\mu^\lambda \psi) \circ (T_\mu^\lambda \varphi \circ adj_1^{-1}(id_{V_{\mathbf{Z}_p}(\mu)}) \\ &= \pi' \circ T_\mu^\lambda(\psi \circ \varphi) \circ i. \end{aligned}$$

Now $\psi \circ \varphi = ac_\mu$ for some $a \in \mathbf{Z}_p$ and using Lemma 2.5 we get

$$\pi' \circ T_\mu^\lambda(ac_\mu) \circ i = \pi' \circ (a\beta) \circ i = ac_\lambda.$$

This gives the isomorphisms involving λ. The analogous arguments for λs show that the relevant composite in that case is equal to $a(r' \circ \beta \circ s)$. By Lemma 2.5 (ii) and 2.4 (ii) we have $r' \circ \beta \circ s = p^n c_{\lambda s}$ and we are done.

2.8. LEMMA. – *Set* $\gamma = adj_2^{-1}(id_{T_\mu^\lambda V_{\mathbf{Z}_p}(\mu)}) : V_{\mathbf{Z}_p}(\mu) \to T_\lambda^\mu T_\mu^\lambda V_{\mathbf{Z}_p}(\mu)$. *Then we have (up to units in* $\mathbf{Z}_p$*)*

$$adj_1(\alpha) \circ \gamma = c_\mu = adj_1(\beta) \circ \gamma.$$

Proof. – Using first 2.3 (1) and then 2.3 (3) we get

$$adj_1(\alpha) \circ \gamma = adj_1(i') \circ T_\lambda^\mu(c_{\lambda s}) \circ \pi) \circ \gamma$$
$$= id_{H^0_{\mathbf{Z}_p}((\mu)} \circ T_\lambda^\mu(c_{\lambda s}) \circ (T_\lambda^\mu \pi \circ \gamma) = c_\mu \circ adj_2^{-1}(\pi) = c_\mu$$

(as in 2.5 we have used $T_\lambda^\mu(c_{\lambda s}) = c_\mu$).

For the second equality we use 2.3 (2) to see

$$adj_1(\beta) \circ \gamma = c_\mu \circ adj_1((id_{T_\mu^\lambda V_{\mathbf{Z}_p}(\mu)}) \circ \gamma.$$

By Lemma 2.2 the composite of the last two maps here has trace equal to $\mathrm{rk}(T_\mu^\lambda V_{\mathbf{Z}_p}(\mu))$ (note that the roles of λ and μ (and hence of adj_1 and adj_2) are interchanged in the present situation). Hence $adj_1(\beta) \circ \gamma = c_\mu \cdot c$ where

$$c = \mathrm{rk} T_\mu^\lambda V_{\mathbf{Z}_p}(\mu)/\mathrm{rk} V_{\mathbf{Z}_p}(\mu) = (\mathrm{dim} V_k(\lambda) + \mathrm{dim} V_k(\lambda s))/\mathrm{dim} V_k(\mu).$$

Now Weyl's dimension formula gives $\nu_p(c) = 0$.

2.9. Let again Q denote a tilting module belonging to the λ-block. Pick a lift $\widetilde{Q}$ of Q to $\mathbf{Z}_p$. We now want to consider the filtration $F_\mu(T_\lambda^\mu Q)^j, j \geq 0$. Recall that we have isomorphisms

$$adj_2 : F_\mu(T_\lambda^\mu Q) \xrightarrow{\sim} \mathrm{Hom}_{G_{\mathbf{Z}_p}}(T_\mu^\lambda V_{\mathbf{Z}_p}(\mu), \widetilde{Q})$$

and

$$adj_1^{-1} : E_\mu(T_\lambda^\mu Q) \xrightarrow{\sim} \mathrm{Hom}_{G_{\mathbf{Z}_p}}(\widetilde{Q}, T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu)).$$

If $\varphi \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(T_\mu^\lambda V_{\mathbf{Z}_p}(\mu), \widetilde{Q})$ and $\psi \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(\widetilde{Q}, T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu))$ then by Proposition 2.6 we can write

$$\psi \circ \varphi = a_{\psi,\varphi}\alpha + b_{\psi,\varphi}\beta$$

for some unique $a_{\psi,\varphi}, b_{\psi,\varphi} \in \mathbf{Z}_p$.

PROPOSITION. – *With the above notation we have*

$$F_\mu(T_\lambda^\mu Q)^j = \{adj_2^{-1}(\varphi) |\, a_{\psi,\varphi}, b_{\psi,\varphi} \in \mathbf{Z}_p^j \ for\ all\ \psi \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(\widetilde{Q}, T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu))\}.$$

Proof. – By 2.3 (1) and (3) we get (with γ as in 2.7)

$$adj_1(\psi) \circ adj_2^{-1}(\varphi) = adj_1(\psi) \circ T_\lambda^\mu \varphi \circ \gamma = adj_1(\psi \circ \varphi) \circ \gamma = (a_{\psi,\varphi} + b_{\psi,\varphi})c_\mu.$$

Here the last equality comes from Lemma 2.8. This proves one inclusion and also shows that if $adj_2^{-1}(\varphi) \in F_\mu(T_\lambda^\mu Q)^j$ then $a_{\psi,\varphi} + b_{\psi,\varphi} \in p^j\mathbf{Z}_p$ for all ψ.

Set now $\psi' = i' \circ r' \circ \psi$. From Lemma 2.4 we get $i' \circ r' \circ \alpha = p^n \alpha$ and Lemma 2.5 shows $i' \circ r' \circ \beta = \alpha$. Hence

$$a_{\psi',\varphi} = p^n a_{\psi,\varphi} + b_{\psi,\varphi} \qquad (\text{and } b_{\psi',\varphi} = 0).$$

So if $\varphi \in F_\mu(T_\lambda^\mu Q)^j$ then

$$a_{\psi,\varphi} + b_{\psi,\varphi},\ p^n a_{\psi,\varphi} + b_{\psi,\varphi} \in \mathbf{Z}_p$$

for all ψ. Hence also $a_{\psi,\varphi}, b_{\psi,\varphi} \in p^j\mathbf{Z}_p$ for all ψ and the other inclusion follows.

3. Conjectures

3.1. Preserve the notation from Section 2. In particular, recall that λ is a p-regular weight and μ is semi-regular. In this section we assume moreover that $\nu_p(\dim V_k(\mu)) = 1$.

Let Q denote a tilting module belonging to the λ-block. Then the exact sequences 2.4 (1) and (2) give (via 1.2 (1)) rise to the following exact sequences of free $\mathbf{Z}_p$-modules (recall that $\lambda > \lambda s$)

$$0 \to F_{\lambda s}(Q) \xrightarrow{\tilde{\pi}} F_\mu(T_\lambda^\mu Q) \xrightarrow{\tilde{i}} F_\lambda(Q) \to 0, \tag{1}$$

$$0 \to E_{\lambda s}(Q) \to E_\mu(T_\lambda^\mu Q) \to E_\lambda(Q) \to 0. \tag{2}$$

We have analogous sequences of k-spaces (equip the terms in (1) and (2) with bars). The construction in 1.3 gives filtrations of all the terms in (1) and we conjecture that they behave as follows.

CONJECTURE. – *Assume Q has no summands $T_k(\nu)$ with $\nu > \nu s$. Then for each $j \geq 0$ we have*

$$\widetilde{i}(\bar{F}_\mu(T_\lambda^\mu Q)^j) = \bar{F}_\lambda(Q)^{j+1}, \tag{i}$$

$$\widetilde{\pi}^{-1}(\bar{F}_\mu(T_\lambda^\mu Q)^j) = \bar{F}_{\lambda s}(Q)^j. \tag{ii}$$

In the following subsections we shall give some evidence, some partial proofs and some consequences of this conjecture.

3.2. LEMMA. – *With notation as in* 3.1 *we have for all $j \geq 0$*

$$\widetilde{i}(F_\mu(T_\lambda^\mu Q)^j) \subset F_\lambda(Q)^j, \tag{i}$$

$$pF_\lambda(Q)^j \subset \widetilde{i}(F_\mu(T_\lambda^\mu Q)^j). \tag{ii}$$

Proof. – (i) Let $\varphi \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(T_\mu^\lambda V_{\mathbf{Z}_p}(\mu), \widetilde{Q})$ such that $adj_2^{-1}(\varphi) \in F_\mu(T_\lambda^\mu Q)^j$. According to Proposition 2.8 this means that $a_{\psi,\varphi}, b_{\psi,\varphi} \in p^j\mathbf{Z}_p$ for all $\psi \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(\widetilde{Q}, T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu))$.

Consider now $\varphi \circ i \in F_\lambda(Q)$. For each $|\psi_1 \in E_\lambda(Q)$ there exists (by 1.2 (1)) $\psi \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(\widetilde{Q}, T_\mu^\lambda H^0_{\mathbf{Z}_p}(\mu))$ such that $\psi_1 = \pi' \circ \psi$. Then

$$\psi_1 \circ (\varphi \circ i) = \pi' \circ (\psi \circ \varphi) \circ i = a_{\psi,\varphi}(\pi' \circ \alpha \circ i) + b_{\psi,\varphi}(\pi' \circ \beta \circ i).$$

But $\alpha \circ i = 0$ and $\pi' \circ \beta \circ i = c_\lambda$ (Lemma 2.5) and we see that $\psi_1 \circ (\varphi \circ i) = b_{\psi,\varphi} c_\lambda \in p^j\mathbf{Z}_p c_\lambda$.

(ii) Let $\varphi_1 \in F_\lambda(Q)^j$. Pick first $\varphi' \in \mathrm{Hom}_{G_{\mathbf{Z}_p}}(T_\mu^\lambda V_{\mathbf{Z}_p}(\mu), \widetilde{Q})$ such that $\varphi' \circ i = \varphi_1$ and set then $\varphi = p\varphi' - \varphi' \circ s \circ \pi$. Then $\varphi \circ i = p\varphi_1$.

By Lemma 2.4 (ii) we have $\alpha \circ s \circ \pi = p\alpha$. Moreover, arguing as in 2.5 we see that $\beta \circ s = i' \circ c_{\lambda s}$ so that $\beta \circ s \circ \pi = \alpha$. It follows that

$$a_{\psi,\varphi} = pa_{\psi,\varphi'} - (pa_{\psi,\varphi'} + b_{\psi,\varphi'}) = -b_{\psi,\varphi'}$$

and

$$b_{\psi,\varphi} = pb_{\psi,\varphi'}$$

for all ψ. By the arguments in (i) above we have $b_{\psi,\varphi'} \in p^j\mathbf{Z}_p$ and hence $adj_2^{-1}(\varphi) \in F_\mu(T_\lambda^\mu Q)^j$.

3.3. LEMMA.

(i) $$\widetilde{\pi}^{-1}(F_\mu(T_\lambda^\mu Q)^j) \subset F_{\lambda s}(Q)^j.$$

(ii) $$\widetilde{\pi}(F_{\lambda s}(Q)^{(j+1)}) \subset F_\mu(T_\lambda^\mu Q)^j.$$

Proof. – (i) Suppose $\varphi_2 \in F_{\lambda s}(Q)$ satisfies $adj_2^{-1}(\varphi_2 \circ \pi) \in F_\mu(T_\lambda^\mu Q)^j$, *i.e.* $a_{\psi,\varphi_2\circ\pi}, b_{\psi,\varphi_2\circ\pi} \in p^j\mathbf{Z}_p$ for all ψ. But since $\pi' \circ \psi \circ \varphi_2 = 0$ and $\pi' \circ \alpha = 0$ we must have $b_{\psi,\varphi_2\circ\pi} = 0$.

If $\psi_2 \in E_{\lambda s}(Q)$ then $(i' \circ \psi_2) \circ (\varphi_2 \circ \pi) = a_{i'\circ\psi_2,\varphi_2\circ\pi} \cdot \alpha$ and hence $\psi_2 \circ \varphi_2 = a_{i'\circ\psi_2,\varphi_2\circ\pi} \cdot c_{\lambda s}$.

(ii) Suppose $\varphi_2 \in F_{\lambda s}(Q)^{(j+1)}$ and consider $\varphi = \varphi_2 \circ \pi$. As above we get $b_{\psi,\varphi} = 0$ for all ψ. Moreover, we have $r' \circ \psi \circ \varphi = a_{\psi,\varphi}(r' \circ \alpha) = a_{\psi,\varphi}p \cdot (c_{\lambda s} \circ \pi)$, *i.e.* $(r' \circ \psi) \circ \varphi_2 = a_{\psi,\varphi}p \cdot c_{\lambda s}$. By assumption on φ_2 we have $a_{\psi,\varphi}p \in p^{j+1}\mathbf{Z}_p$, hence $a_{\psi,\varphi} \in p^j\mathbf{Z}_p$.

3.4. LEMMA. – (i) *If* $F_{\lambda s}(Q) = 0$ *then Conjecture* 3.1 (i) *holds.*

(ii) *If* $F_\lambda(Q) = 0$ *then Conjecture* 3.1 (ii) *holds.*

Proof. – Just as in the proof of Lemma 3.3 (i) we have for $adj_2^{-1}(\varphi) \in F_\mu(T_\lambda^\mu Q)^j$ that $\psi_1 \circ (\varphi \circ i) = b_{\psi,\varphi}c_\lambda$ for all ψ with $\psi_1 = \pi' \circ \psi$. We claim that $b_{\psi,\varphi} \in p^{j+1}\mathbf{Z}_p$. To see this note that our assumption implies that $\varphi \circ s = 0$. Since $\alpha \circ s = p(i' \circ c_{\lambda s})$ and $\beta \circ s = i' \circ c_{\lambda s}$ we get

$$0 = a_{\psi,\varphi}p + b_{\psi,\varphi}$$

and the claim follows.

Now our assumption also implies that $\widetilde{i}$ is an isomorphism. Hence Lemma 3.2 (ii) implies $F_\lambda(Q)^{j+1} \subset \widetilde{i}(F_\mu(T_\lambda^\mu Q)^j)$ and (i) follows.

To prove (ii) note that in the proof of Lemma 3.3 (ii) we may (because of the assumption $F_\lambda(Q) = 0$) write any ψ as $\psi = i' \circ \psi_2$ with $\psi_2 \in E_{\lambda s}(Q)$. Then

$$\psi \circ \varphi = i' \circ \psi_2 \circ \varphi_2 \circ \pi = a\alpha$$

where $a \in \mathbf{Z}_p$ is determined by $\psi_2 \circ \varphi_2 = a \cdot c_{\lambda s}$. Hence $a_{\psi,\varphi_2\circ\pi} \in p^j\mathbf{Z}_p$ for all $\varphi_2 \in F_{\lambda s}(Q)^j$ and all ψ. This means that $\widetilde{\pi}(F_{\lambda s}(Q)^j) \subset F_\mu(T_\lambda^\mu Q)^j$ and (ii) follows by comparing with Lemma 3.3 (i).

3.5. Recall that in 1.5 we have defined a polynomial $f_\lambda(Q) \in \mathbf{Z}[q]$ associated to the filtration $\bar{F}_\lambda(Q)^j$.

PROPOSITION. – *Suppose Q is a tilting module belonging to the λ-block such that Q has no summands of the form $T_k(\nu)$ with $\nu > \nu s, \nu \in X^+$ linked to λ. Assume that Conjecture* 3.1 *holds for Q.* Then

$$f_\nu(T_\mu^\lambda T_\lambda^\mu Q) = \begin{cases} f_{\nu s}(Q) + q^{-1} f_\nu(Q), & \text{if } \nu > \nu s, \\ f_{\nu s}(Q) + q f_\nu(Q), & \text{if } \nu < \nu s. \end{cases}$$

Proof. – Suppose first that $\nu > \nu s$. Denote by $\bar{\nu}$ the weight on the common wall of the alcoves containing ν and νs which is linked to μ. By Proposition 2.6 we have $F_\nu(T_\mu^\lambda T_\lambda^\mu Q)^j = F_{\bar{\nu}}(T_\lambda^\mu Q)^j$ and Conjecture 3.1 gives for each $j \geq 0$ a short exact sequence

$$0 \to \bar{F}_{\nu s}(Q)^j \to \bar{F}_{\bar{\nu}}(T_\lambda^\mu Q)^j \to \bar{F}_\nu(Q)^{j+1} \to 0.$$

These two properties imply the first relation. The case $\nu < \nu s$ is proved in the same way using this time the second isomorphism in Proposition 2.7.

3.6. For $\nu \in X^+$ linked to λ we define

$$f_{\nu,\lambda} = f_\nu(T_k(\lambda)).$$

and we let $f'_{\nu,\lambda}$ denote its derived polynomial.

COROLLARY. – *Assume Conjecture* 3.1 *holds for $Q = T_k(\lambda s)$ (where $\lambda > \lambda s$). Then*

$$T_\mu^\lambda T_\lambda^\mu(T_k(\lambda s)) = T_k(\lambda) \oplus (\bigoplus_{\nu > \nu s} f'_{\nu,\lambda s}(0) T_k(\nu)).$$

Proof. – By Proposition 3.5 we see that $f_\nu(T_\mu^\lambda T_\lambda^\mu(T_k(\lambda s)))(0) = 0$ for all ν with $\nu < \nu s$ whereas if $\nu > \nu s$ then

$$f_\nu(T_\mu^\lambda T_\lambda^\mu(T_k(\lambda s)))(0) = \begin{cases} f_{\lambda s,\lambda s}(0) = 1 & \text{if } \nu = \lambda\ , \\ (q^{-1} f_{\nu,\lambda s})(0) = f'_{\nu,\lambda s}(0) & \text{if } \nu < \lambda s. \end{cases}$$

The corollary now follows from Proposition 1.5.

Remarks. – (i) Assume now that Conjecture 3.1 holds for all tilting modules with highest weights in the lowest p^2-alcove. Then the above corollary clearly gives an algorithm for determining the characters of all indecomposable tilting modules with such highest weights. It starts out by the observation from Remark 1.1 (i) that $T_k(\lambda) = V_k(\lambda)$ when λ belongs to the bottom alcove in X^+. A comparison with [So1] shows that this is exactly the algorithm predicted by Soergel's conjecture (Vermutung 7.2 in [So1]).

(ii) It is clear from the results in Section 2 that we cannot expect Conjecture 3.1 to hold outside the lowest p^2-alcove (note the appearance of n e.g. in Lemma 2.4). As mentioned in the introduction it remains a challenge to come up with a conjecture which describes the algorithm in higher alcoves.

4. The quantum case

4.1. Let U denote the quantum group (or quantized enveloping algebra) associated with our root system R. This is a $\mathbf{Q}(v)$-algebra (v being an indeterminate) defined by a wellknown set of generators and relations, *see* e.g. [Lu]. By U_q we denote the corresponding quantum group at a complex primitive ℓ'th root of ± 1. We assume ℓ is odd. This is the specialization of the $\mathbf{Z}[v, v^{-1}]$-lattice $U_{\mathbf{Z}[v,v^{-1}]}$ in U generated by the divided powers of the generators, *see* [Lu].

The representation theory of U_q resembles to a great extent the one for G, *see* e.g. [APW]. In particular, we have for each $\lambda \in X^+$ a simple module $L_q(\lambda)$, an induced module $H_q^0(\lambda)$ and a Weyl module $V_q(\lambda)$ all with highest weight λ. Exactly as for G we have also a unique indecomposable tilting module $T_q(\lambda)$ with highest weight λ, *see* [An]. Hence Problem 1.1 has the following direct analogue

PROBLEM. – *Determine the numbers* $[T_q(\lambda) : V_q(\nu)]$ *for all* $\lambda, \nu \in X^+$.

4.2. To construct filtrations of the Hom-spaces in the quantum case we consider the ring $\mathbf{Q}[v, v^{-1}]$. The induced modules, the Weyl modules and the tilting modules lift to $U_{\mathbf{Q}[v,v^{-1}]}$, *i.e.* there exist for each $\lambda \in X^+$ unique (up to isomorphisms) $U_{\mathbf{Q}[v,v^{-1}]}$-modules $H^0_{\mathbf{Q}[v,v^{-1}]}(\lambda)$, $V_{\mathbf{Q}[v,v^{-1}]}(\lambda)$, and $T_{\mathbf{Q}[v,v^{-1}]}(\lambda)$ which specialize to the corresponding U_q-modules:

$$H^0_{\mathbf{Q}[v,v^{-1}]}(\lambda) \otimes \mathbf{C} \simeq H_q^0(\lambda),$$

$$V_{\mathbf{Q}[v,v^{-1}]}(\lambda) \otimes \mathbf{C} \simeq V_q(\lambda),$$

and

$$T_{\mathbf{Q}[v,v^{-1}]}(\lambda) \otimes \mathbf{C} \simeq T_q(\lambda).$$

Here $\mathbf{C}$ is made into a $\mathbf{Q}[v, v^{-1}]$-algebra by specializing v to q.

The results in Sections 1.2-5 now carry over. In particular, if Q denotes a tilting module for U_q and $\widetilde{Q}$ denotes its lift to $U_{\mathbf{Q}[v,v^{-1}]}$ then we define for each $\lambda \in X^+$ the filtration $(F_\lambda(Q)^j)_{j \geq 0}$ of $F_\lambda(Q) = \mathrm{Hom}_{U_{\mathbf{Q}[v,v^{-1}]}}(V_{\mathbf{Q}[v,v^{-1}]}(\lambda), \widetilde{Q})$ as follows:

Let $\phi_\ell \in \mathbf{Q}[v]$ denote the ℓ'th cyclotomic polynomial. Then $F_\lambda(Q)^j$ consists of those $\varphi \in F_\lambda(Q)$ which satisfy

$$\psi \circ \varphi \in (\phi_\ell)^j \mathbf{Q}[v, v^{-1}] c_\lambda \text{ for all } \psi \in \mathrm{Hom}_{U_{\mathbf{Q}[v,v^{-1}]}}(V_{\mathbf{Q}[v,v^{-1}]}(\lambda), \widetilde{Q}).$$

We set $\bar{F}_\lambda(Q)^j$ equal to the corresponding image in $\bar{F}_\lambda(Q) = \mathrm{Hom}_{U_q}(V_q(\lambda), Q)$.

4.3. The results in Section 2 also carry over with the important difference that the number n appearing in 2.4, 2.7 and 2.9 is always replaced by 1 (because $\nu_{\phi_\ell}(\dim V_q(\mu)) = 1$ for all semi-regular weights μ, ν_{ϕ_ℓ} denoting valuation with respect to ϕ_ℓ). In other words: all dominant weights are contained in the lowest ℓ^2-alcove.

This observation means that the general assumption in Section 3 (namely that $n = 1$) is automatically satisfied. Hence the analogue of Conjecture 3.1 is expected to hold for $Q = T_q(\lambda)$ for *all* $\lambda \in X^+$ with $\lambda > \lambda s$.

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REFERENCES

[APW] H. H. ANDERSEN, P. POLO and W. KEXIN, *Representations of quantum algebras* (*Invent. Math.*, Vol. 104, 1991, pp. 1-59).

[AJS] H. H. ANDERSEN, J. C. JANTZEN and W. SOERGEL, *Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: Independence of p* (*Astérisque*, Vol. 220, 1994, pp. 1-320).

[An] H. H. ANDERSEN, *Tensor products of quantized tilting modules* (*Commun. Math. Phys.*, Vol. 149, 1992, pp. 149-159).

[Ar] S.M. ARKHIPOV, *Semiinfinite cohomology of associative algebras and bar duality* q-alg/9602013.

[Do] S. DONKIN, *On tilting modules for algebraic groups* (*Math. Z.*, Vol. 212, 1993, pp. 39-60).

[Ja] J. C. JANTZEN, *Representations of algebraic groups*, Academic Press 1987.

[KL1] KAZHDAN and G. LUSZTIG, *Representations of Coxeter groups and Hecke algebras* (*Inventiones*, Vol. 53, 1979, pp. 165-184).

[KL2] D. KAZHDAN and G. LUSZTIG, *Tensor structures arising from affine Lie algebras I-IV* (*J. Amer. Math. Soc.*, Vol. 6, 1994, pp. 905-947, 949-1011 and Vol. 7, 1994, pp. 335-381, 383-453).

[Lu] G. LUSZTIG, *Quantum groups at roots of* 1 (*Geom. Ded.*, Vol. 35, 1993, pp. 89-114).

[Ma] O. MATHIEU, *On the dimension of some modular irreducible representations of the symmetric group* (*Lett. Math. Phys.*, Vol. 38, 1996, pp. 23-32).

[So1] W. SOERGEL, *Kazhdan-Lusztig-Polynome und eine Kombinatorik für Kipp-Moduln*, Represent. Theory 1, 1997 (to appear).

[So2] W. SOERGEL, *Characterformels für Kipp-Modules über Kac-Moody-Algebren*, Preprint, Universität Freiburg, December 1996.

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