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QUASIFLATS IN HADAMARD SPACES

BY URS LANG AND VIKTOR SCHROEDER

ABSTRACT. – Let X be a simply connected, complete geodesic metric space which is nonpositively curved in the sense of Alexandrov. We assume that X contains a k-flat F of maximal dimension and consider quasiisometric embeddings $f: \mathbb{R}^k \to X$ whose distance function from F satisfies a certain asymptotic growth condition. We prove that if X is locally compact and cocompact, then the Hausdorff distance between $f(\mathbb{R}^k)$ and F is uniformly bounded. This generalizes a well-known lemma of Mostow on quasiflats in symmetric spaces of noncompact type.

RÉSUMÉ. – Soit X un espace métrique géodésique complet et simplement connexe courbé de manière non-positive au sens d'Alexandrov. Nous supposons que X contient un k-plat F de dimension maximale et considérons des quasi-plats $f : \mathbb{R}^k \to X$ tels que les fonctions mesurant la distance à partir de F vérifient une certaine condition de croissance linéaire. Nous démontrons que la distance de Hausdorff entre $f(\mathbb{R}^k)$ et F est uniformément bornée lorsque X est localement compact et cocompact. Ceci généralise un lemme bien connu de Mostow sur les quasi-plats des espaces symétriques de type non-compact.

Introduction

Quasiflats, *i.e.* quasiisometrically embedded euclidean spaces, play an important role in the theory of nonpositively curved manifolds. A striking example is Mostow's proof of the rigidity theorem for locally symmetric spaces which, in the case of higher rank, largely relies on the interplay between flats and quasiflats. More precisely, let X', X be two symmetric spaces of noncompact type and rank k, and let $\rho: \Gamma' \to \Gamma$ be an isomorphism between discrete groups of isometries acting cocompactly on X' and X respectively. A k-flat F' in X' is called Γ'-compact if its stabilizer $\Gamma'_{F'}$ in Γ' acts cocompactly on F'. Given such F', there exists a unique Γ-compact k-flat F in X with stabilizer $\Gamma_F = \rho(\Gamma'_{F'})$, *cf.* [Mo, 13.1]. A key step in Mostow's argument is the following uniform estimate, *cf.* [Mo, 13.2].

THEOREM A (Mostow). – *Let $f: X' \to X$ be a quasiisometric map equivariant with respect to ρ. Then there exists a constant $D \geq 0$ such that the following holds. If F' is a Γ'-compact k-flat in X', and if F is its unique companion in X as described above, then the Hausdorff distance between $f(F')$ and F satisfies* $\mathrm{Hd}(f(F'), F) \leq D$.

This result then gives rise to a homeomorphism between the Furstenberg boundaries of X' and X, which by a theorem of Tits is induced, after suitable renormalization, by a ρ-equivariant isometry of X' and X.

In Theorem A, the assumptions on F' and F clearly imply that $\mathrm{Hd}(f(F'), F) < \infty$. One may ask the following question: If a single quasiflat $f: \mathbb{R}^k \to X$ lies within finite distance of a k-flat F in X, does this already imply that $\mathrm{Hd}(f(\mathbb{R}^k), F)$ is bounded above by some constant depending only on X and the quasiisometry constants of f? In this paper we show that this holds true in an even more general context. We consider quasiflats $f: \mathbb{R}^k \to X$, where X is a Hadamard space in the sense of Alexandrov, *i.e.* a simply connected, complete geodesic space all of whose geodesic triangles satisfy the $\mathrm{CAT}(0)$ inequality, *cf.* Section 1. Moreover, X is assumed to be locally compact and cocompact. We prove:

THEOREM B. – *Let (X, d) be a locally compact and cocompact Hadamard space containing a k-flat but no $(k+1)$-flat, where $k \geq 1$. Then for all $L > 0$ and $C \geq 0$ there exists $D \geq 0$ such that the following holds. Let $F \subset X$ be a k-flat, $f: \mathbb{R}^k \to X$ an (L, C)-quasiisometric map* (cf. 1.1), *and for $r > 0$ define $a(r) := \sup\{d(f(x), F) : |x| \leq r\}$. If $\limsup_{r \to \infty} a(r)/r < L^{-1}$, then $\mathrm{Hd}(f(\mathbb{R}^k), F) \leq D$.*

The constant L^{-1} bounding the linear growth of $a(r)$ is optimal, as is shown by the following simple example. Let F be a geodesic line in the hyperbolic plane H^2, and let $f: \mathbb{R} \to H^2$ be a geodesic of speed L^{-1} with $f(\mathbb{R}) \neq F$ (and hence $\mathrm{Hd}(f(\mathbb{R}), F) = \infty$). Then f is an $(L, 0)$-quasiisometric map as defined in 1.1, and it is easily seen that in this case, $a(r)/r \to L^{-1}$ for $r \to \infty$. By taking products $H^2 \times \mathbb{R}^{k-1}$, similar examples are obtained for all $k \geq 1$.

In the case where $f: \mathbb{R}^k \to X$ is an isometric map, and hence $F' := f(\mathbb{R}^k)$ is a k-flat in X, the upper limit considered in Theorem B is actually a limit and can be interpreted in terms of the angle metric $\angle$ on the boundary at infinity $X(\infty)$ of X, *cf.* 5.1. Let $\mathrm{Hd}_\angle$ denote the Hausdorff distance induced by $\angle$, and let $F(\infty)$ and $F'(\infty)$ be the boundaries at infinity of F and F' respectively. Then Theorem B yields the following dichotomy.

THEOREM C. – *Let X and k be given as in Theorem B, and let F, F' be two k-flats in X. Then either $\mathrm{Hd}_\angle(F(\infty), F'(\infty)) \geq \pi/2$, or $\mathrm{Hd}_\angle(F(\infty), F'(\infty)) = 0$ (and F, F' lie within uniformly bounded distance from each other).*

The paper is divided into five short sections. In the first section we fix the terminology and prove a basic approximation lemma. Sections 2 and 3 contain several results on maps between quasiflats and flats. In Section 4 we prove Theorem B. In the last section we discuss the special case where F' is itself a flat in X.

After finishing this paper we learned that Bruce Kleiner had announced a result very similar to Theorem B in summer 1995, before the completion of our work. The proofs rely on different methods. Another result in the same spirit is proved in [KL, Section 4].

1. Preliminaries

We briefly recall some basic notions and results from the theory of nonpositively curved metric spaces in the sense of Alexandrov. The general references are [Ba], [BGS], and the forthcoming book [BH].

Let (X,d) be a metric space. A continuous map $\sigma: I \to X$ from a connected subset I of $\mathbb{R}$ into X is said to be a *minimizing geodesic* if it is isometric up to a constant factor, *i.e.* if there exists $v \geq 0$ such that $d(\sigma(r),\sigma(s)) = v|r-s|$ for all $r,s \in I$. Then v is called the *speed* of σ. More generally, the curve σ is said to be a *geodesic* if there is $v \geq 0$ such that σ is locally minimizing and of speed v. A *geodesic segment* is a geodesic defined on a compact interval. A *geodesic triangle* Δ in X is a triple $(\sigma_1,\sigma_2,\sigma_3)$ of geodesic segments $\sigma_i: I_i \to X$, the *sides* of Δ, whose endpoints match in the usual way. A euclidean *comparison triangle* $\overline{\Delta}$ for Δ is a corresponding triple of geodesic segments $\overline{\sigma}_i: I_i \to \mathbb{R}^2$ such that $\overline{\sigma}_i$ has the same length as σ_i, $i = 1,2,3$, and such that the endpoints of $\overline{\sigma}_1,\overline{\sigma}_2,\overline{\sigma}_3$ match in the same way as those of $\sigma_1,\sigma_2,\sigma_3$. Then Δ is said to satisfy the $\mathrm{CAT}(0)$ *inequality* if $\overline{\Delta}$ as above exists and if $d(\sigma_i(r),\sigma_j(s)) \leq |\overline{\sigma}_i(r) - \overline{\sigma}_j(s)|$ whenever $r \in I_i$, $s \in I_j$, and $i,j \in \{1,2,3\}$.

A metric space (X,d) is called a *geodesic space* if for all $x,y \in X$ there exists a minimizing geodesic $\sigma: [0,1] \to X$ with $\sigma(0) = x$ and $\sigma(1) = y$. Note that for a geodesic triangle Δ in X with minimizing sides, a euclidean comparison triangle $\overline{\Delta}$ exists (and is unique up to isometry). Throughout the paper, (X,d) is assumed to be a simply connected, complete geodesic space all of whose geodesic triangles satisfy the $\mathrm{CAT}(0)$ inequality. We will refer to X simply as a *Hadamard space*. A fundamental feature of these spaces is that for all $x,y \in X$ there is a unique geodesic from x to y (whose image we denote by $[x,y]$), and hence all geodesics in X are minimizing. Moreover, for all pairs of geodesics $\sigma_1,\sigma_2: I \to X$, the function mapping $t \in I$ to $d(\sigma_1(t),\sigma_2(t))$ is convex. We will often use the nearest point projection $\pi: X \to F$ from X onto a closed convex subset F of X. This map is well-defined and Lipschitz with constant 1, *cf.* [Ba, I.5.6]. In many of our proofs the assumptions on X are only used through these convexity properties.

Next we state the definition of quasiisometric maps between metric spaces.

DEFINITION 1.1. – *Let $L > 0$ and $C \geq 0$. A map $f: X' \to X$ between two metric spaces (X',d') and (X,d) is called (L,C)-*quasiisometric *if for all $x,y \in X'$, $L^{-1}d'(x,y) - C \leq d(f(x),f(y)) \leq Ld'(x,y) + C$.*

It is often necessary to approximate general quasiisometric maps by continuous ones. The argument proving the following lemma is well-known, but an appropriate reference does not seem to be available in the literature.

LEMMA 1.2. – *Let B be a closed convex subset of $\mathbb{R}^k$, X a Hadamard space, $L > 0$, $C \geq 0$, and $f: B \to X$ a map satisfying $d(f(x),f(y)) \leq L|x-y| + C$ for all $x,y \in B$. Then for every $\lambda > 0$ there exists a map $f': B \to X$ which is Lipschitz with constant $\sqrt{k}(L + \lambda^{-1}C)$ and satisfies $d(f'(x),f(x)) \leq \sqrt{k}\lambda L + C$ for all $x \in B$. If moreover f is (L,C)-quasiisometric, then $d(f'(x),f'(y)) \geq L^{-1}|x-y| - (2\sqrt{k}\lambda L + 3C)$ for all $x,y \in B$.*

Proof. – Let $\pi: \mathbb{R}^k \to B$ denote the nearest point projection. We replace f by $f \circ \pi: \mathbb{R}^k \to X$ and call this map f again. Then f satisfies $d(f(x),f(y)) \leq L|x-y| + C$ for all $x,y \in \mathbb{R}^k$.

For $i = 0,1,\ldots,k$ let S_i denote the closed i-skeleton of the canonical subdivision of $\mathbb{R}^k$ into cubes of edge length λ. On $S_0 = (\lambda\mathbb{Z})^k$, put $f' := f$. Then for $i = 1,\ldots,k$, assuming that f' is already defined on S_{i-1}, we extend f' to S_i as follows. Let $e_1,\ldots,e_k$

be the canonical basis of $\mathbb{R}^k$, and let $x \in S_i \backslash S_{i-1}$. Let $j = j(x) \in \{1, \ldots, k\}$ be the maximal index such that there exists $z = z(x) \in S_{i-1}$ with $x \in [z, z + \lambda e_j]$. Then $f'(x)$ is determined by defining $f'|[z, z + \lambda e_j]$ to be a geodesic.

Consider the following assertion $A(i)$: for all pairs $(j, z) = (j(x), z(x))$ with $x \in S_i \backslash S_{i-1}$, $d(f'(z), f'(z + \lambda e_j)) \leq \lambda L + C$, and for each i-plane $P_i \subset S_i$, $f'|P_i$ is Lipschitz with constant $\sqrt{i}(L + \lambda^{-1}C)$. Clearly $A(1)$ holds. We show that for $i = 2, \ldots, k$, $A(i-1)$ implies $A(i)$. Given $(j, z) = (j(x), z(x))$ with $x \in S_i \backslash S_{i-1}$, there exist $p, q \in S_{i-2}$ such that $z \in [p, q]$ and both $f'|[p, q]$ and $f'|[p + \lambda e_j, q + \lambda e_j]$ are geodesics. Moreover, $A(i-1)$ implies that $d(f'(p), f'(p + \lambda e_j)), d(f'(q), f'(q + \lambda e_j)) \leq \lambda L + C$. Thus by convexity,

$$d(f'(z), f'(z + \lambda e_j)) \leq \lambda L + C.$$

It suffices to prove the claimed Lipschitz property for $f'|Q_i$, where $Q_i \simeq [0, \lambda]^i$ is an i-cell of the given cubical subdivision of $\mathbb{R}^k$. Let $x, x' \in Q_i$, and let $[z, z + \lambda e_j]$ and $[z', z' + \lambda e_j]$ be the segments in Q_i containing x and x' respectively, where j is maximal. Let $y \in [z, z + \lambda e_j]$ be the point closest to x'. Since $f'|[z, z + \lambda e_j]$ is a geodesic of length at most $\lambda L + C$,

$$d(f'(x), f'(y)) \leq (L + \lambda^{-1}C)|x - y|.$$

On the other hand, $A(i-1)$ asserts that $d(f'(z), f'(z')), d(f'(z + \lambda e_j), f'(z' + \lambda e_j)) \leq \sqrt{i-1}(L + \lambda^{-1}C)|z - z'|$, hence

$$d(f'(y), f'(x')) \leq \sqrt{i-1}(L + \lambda^{-1}C)|y - x'|$$

by convexity. Combining these estimates we see that

$$\begin{aligned} d(f'(x), f'(x'))^2 &\leq [d(f'(x), f'(y)) + d(f'(y), f'(x'))]^2 \\ &\leq i d(f'(x), f'(y))^2 + i(i-1)^{-1} d(f'(y), f'(x'))^2 \\ &\leq i(L + \lambda^{-1}C)^2 |x - x'|^2, \end{aligned}$$

proving $A(i)$. In particular, we have shown that f' is Lipschitz with constant $\sqrt{k}(L + \lambda^{-1}C)$.

Given $x \in \mathbb{R}^k$, let $V \subset (\lambda\mathbb{Z})^k$ be the set of vertices of a cube of edge length λ containing x. Then f maps V into the metric ball with center $f(x)$ and radius $\sqrt{k}\lambda L + C$. By the above construction, $f'(x)$ is contained in the convex hull of $f(V)$. Since metric balls in a Hadamard space are convex, it follows that $d(f'(x), f(x)) \leq \sqrt{k}\lambda L + C$. Finally, if f is (L, C)-quasiisometric on B, we see that

$$L^{-1}|x - y| - C \leq d(f(x), f(y)) \leq d(f'(x), f'(y)) + 2(\sqrt{k}\lambda L + C)$$

for all $x, y \in B$. □

2. Continuous quasiflats

In this section we prove two independent results applying to continuous quasiflats $f:\mathbb{R}^k \to X$, where X is a general Hadamard space containing a k-flat F for some $k \geq 1$. We emphasize that here k need not be maximal as in Theorem B.

We start with a simple topological lemma. For $r \geq 0$ and $z \in \mathbb{R}^k$ we define $B^k(z,r) := \{x \in \mathbb{R}^k : |x-z| \leq r\}$ and $S^{k-1}(z,r) := \{x \in \mathbb{R}^k : |x-z| = r\}$. Instead of $B^k(0,r)$ and $S^{k-1}(0,r)$ we write $B^k(r)$ and $S^{k-1}(r)$ respectively.

LEMMA 2.1. – *Let $k \geq 1$, $r > 0$, and let $f: B^k(r) \to \mathbb{R}^k$ be a continuous map satisfying $f(x) \neq f(y)$ for all $x, y \in B^k(r)$ with $|x| + |y| = |x-y| = r$. Then $f|S^{k-1}(r)$ is not contractible* (i.e. *homotopic to a constant map) in $\mathbb{R}^k \backslash \{f(0)\}$.*

Proof. – Assume the contrary. Then there exists a continuous map $\tilde{f}: B^k(r) \to \mathbb{R}^k \backslash \{f(0)\}$ with $\tilde{f}|S^{k-1}(r) = f|S^{k-1}(r)$. Define $h_0: B^k(r) \to S^{k-1}(1)$ by

$$h_0(x) := \frac{\tilde{f}(x) - f(0)}{|\tilde{f}(x) - f(0)|}\,.$$

For $x \in S^{k-1}(r)$ and $0 < t \leq 1$, the denominator of

$$h_t(x) := \frac{f((1-t/2)x) - f((-t/2)x)}{|f((1-t/2)x) - f((-t/2)x)|}$$

is nonzero. The map $h_1: S^{k-1}(r) \to S^{k-1}(1)$ defined this way is homotopic to $h_0|S^{k-1}(r)$ and can thus be extended to a continuous map $h_1: B^k(r) \to S^{k-1}(1)$. Finally, since $h_1(-x) = -h_1(x)$ for $x \in S^{k-1}(r)$, h_1 gives rise to a continuous map from $S^k(1)$ into $S^{k-1}(1)$ preserving antipodes, in contradiction to the Borsuk-Ulam theorem, *cf.* [Ma]. □

For the statement of the next lemma the following definition is useful.

DEFINITION 2.2. – *Let $k \geq 1$ and $\overline{\delta} > 0$. We call a continuous map $f:\mathbb{R}^k \to \mathbb{R}^k$* $\overline{\delta}$-expanding *if for all $0 < \delta < \overline{\delta}$, the following condition is satisfied for $r > 0$ sufficiently large (depending on δ): $|f(x) - f(0)| > \delta r$ for $x \in S^{k-1}(r)$, and $f|S^{k-1}(r)$ is not contractible in $\mathbb{R}^k \backslash \{f(0)\}$.*

The definition is independent of the choice of basepoints. Clearly 2.1 implies that every continuous map $f:\mathbb{R}^k \to \mathbb{R}^k$ satisfying $|f(x) - f(y)| \geq L^{-1}|x-y| - C$ for some constants $L > 0$, $C \geq 0$ is L^{-1}-expanding. In particular, it follows that f is surjective, a fact which was proved in [Mo, 10.1]. More generally, the following holds.

LEMMA 2.3. – *Let X be a Hadamard space containing a k-flat F for some $k \geq 1$, and let $\pi: X \to F$ be the nearest point projection. Let $L > 0$, $C \geq 0$, and let $f:\mathbb{R}^k \to X$ be a continuous map satisfying $d(f(x), f(y)) \geq L^{-1}|x-y| - C$ for all $x, y \in \mathbb{R}^k$. For $r > 0$ define $a(r) := \sup\{d(f(x), F) : |x| \leq r\}$. If $\overline{a} := \limsup_{r\to\infty} a(r)/r < L^{-1}$, then $h := \pi \circ f: \mathbb{R}^k \to F \simeq \mathbb{R}^k$ is $(L^{-1} - \overline{a})$-expanding.*

Proof. – For all $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that $a(r)/r \leq \overline{a} + \varepsilon$ for $r \geq r_\varepsilon$. Then for $x, y \in B^k(r)$ with $|x| + |y| = |x - y| = r$,

$$\begin{aligned} d(f(x), F) + d(f(y), F) + d(h(x), h(y)) &\geq d(f(x), f(y)) \\ &\geq L^{-1}r - C. \end{aligned}$$

In case $|x|, |y| \geq r_\varepsilon$, we have $d(f(x), F) + d(f(y), F) \leq (\overline{a} + \varepsilon)(|x| + |y|) = (\overline{a} + \varepsilon)r$. In case $|x| < r_\varepsilon$ or $|y| < r_\varepsilon$, we see that

$$d(f(x), F) + d(f(y), F) \leq (\overline{a} + \varepsilon)(r + r_\varepsilon) \leq (\overline{a} + \varepsilon)(1 + \varepsilon)r$$

for $r \geq r_\varepsilon/\varepsilon$. Thus $d(h(x), h(y)) \geq [L^{-1} - (\overline{a} + \varepsilon)(1 + \varepsilon)]r - C$, and the claim follows from 2.1. □

The definition of the Hausdorff distance of two subsets A, A' of X reflects the fact that there are two ways of measuring distances between these sets: $\mathrm{Hd}(A, A') := \sup\{a, a'\}$, where $a := \sup_{x' \in A'} d(x', A)$ and $a' := \sup_{x \in A} d(x, A')$. The following lemma shows, in particular, that if $A = F$ and $A' = f(\mathbb{R}^k)$ are (quasi-)flats in X, then a is bounded above in terms of a'. More generally, the statement is adapted to the hypotheses of Theorem B.

LEMMA 2.4. – *Let X be a Hadamard space containing a k-flat F for some $k \geq 1$, and let $z \in F$ be some basepoint. Let $L, L' > 0$, $C \geq 0$, and let $f: \mathbb{R}^k \to X$ be a map satisfying $L^{-1}|x - y| - C \leq d(f(x), f(y)) \leq L'|x - y|$ for all $x, y \in \mathbb{R}^k$. For $r > 0$ define $a(r) := \sup\{d(f(x), F) : |x| \leq r\}$ and $a'(r) := \sup\{d(y, f(\mathbb{R}^k)) : y \in F,\ d(y, z) \leq r\}$. If $\overline{a} := \limsup_{r \to \infty} a(r)/r < L^{-1}$, then for all $0 < \beta < (1 - \overline{a}L)/L'$ there exists $r_\beta > 0$ such that $a(\beta r) \leq \overline{L}a'(r) + \overline{C}$ for $r \geq r_\beta$, where $\overline{L} := 5LL' + 1$ and $\overline{C} := LL'C$.*

Proof. – Let $\pi: X \to F$ denote the projection, and let $h := \pi \circ f$. Without loss of generality we may assume that $z = h(0)$.

For fixed $r > 0$ we define a map $g: B^k(r) \to \mathbb{R}^k$ as follows. First, by the properties of f, we can associate to each $x \in B^k(r)$ a point $\overline{x} \in \mathbb{R}^k$ with

$$d(f(\overline{x}), h(x)) = d(h(x), f(\mathbb{R}^k)) =: b(x)\,.$$

Note that $b(x) \leq a(r)$ and, since $d(h(x), z) \leq d(f(x), f(0)) \leq L'|x| \leq L'r$, $b(x) \leq a'(L'r)$. Moreover,

$$\begin{aligned} |\overline{x} - x| &\leq L[d(f(\overline{x}), f(x)) + C] \\ &\leq L[d(f(\overline{x}), h(x)) + d(h(x), f(x)) + C] \\ &\leq L[b(x) + a(r) + C]. \end{aligned}$$

We define $g(x)$ to be the point in $B^k(x, L[a(r) + C])$ closest to $\overline{x}$; then $|g(x) - \overline{x}| \leq Lb(x)$. Hence, for $x, y \in B^k(r)$,

$$\begin{aligned} |g(x) - g(y)| &\leq L[b(x) + b(y)] + |\overline{x} - \overline{y}| \\ &\leq L[b(x) + b(y) + d(f(\overline{x}), f(\overline{y})) + C] \\ &\leq L[2b(x) + 2b(y) + d(h(x), h(y)) + C] \\ &\leq L[4a'(L'r) + L'|x - y| + C]. \end{aligned}$$

Let $\lambda > 0$. The procedure carried out in the proof of 1.2 yields a continuous map $g': B^k(r) \to \mathbb{R}^k$ which satisfies

$$|g'(x) - g(x)| \le L[\sqrt{k}\lambda L' + 4a'(L'r) + C]$$

for all $x \in B^k(r)$ and coincides with g on $(\lambda\mathbb{Z})^k \cap B^k(r)$.

We claim that the image of g' contains $B^k(r - L[a(r) + C] - 2\sqrt{k}\lambda) =: B'$. Let $y \in B^k(r - \sqrt{k}\lambda)$, and let $V \subset (\lambda\mathbb{Z})^k \cap B^k(r)$ denote the set of vertices of a cube of edge length λ containing y. For $x \in V$, $|g(x) - y| \le |g(x) - x| + |x - y| \le L[a(r) + C] + \sqrt{k}\lambda$. Hence, by the definition of g',

$$|g'(y) - y| \le L[a(r) + C] + \sqrt{k}\lambda$$

for all $y \in B^k(r - \sqrt{k}\lambda)$. Now the claim follows easily from the fact that there is no retraction $B^k(1) \to S^{k-1}(1)$.

Let $y \in B'$. Then there exists $x \in B^k(r)$ with $g'(x) = y$, hence

$$\begin{aligned} d(f(y), F) &\le d(f(y), f(\overline{x})) + d(f(\overline{x}), h(x)) \\ &\le L'[|g'(x) - g(x)| + |g(x) - \overline{x}|] + b(x) \\ &\le L'L[\sqrt{k}\lambda L' + 5a'(L'r) + C] + a'(L'r)\,. \end{aligned}$$

Since this holds for all $\lambda > 0$, using the continuity of f we obtain

$$d(f(x), F) \le \overline{L}a'(L'r) + \overline{C}$$

for all $x \in B^k(r - L[a(r) + C])$, where $\overline{L} := 5L'L + 1$ and $\overline{C} := L'LC$. Let $\varepsilon > 0$. Then for r sufficiently large, $a(r) \le (\overline{a} + \varepsilon)r$ and $C \le \varepsilon r$. Thus $a(r - L[\overline{a} + 2\varepsilon]r) \le \overline{L}a'(L'r) + \overline{C}$, and replacing r by r/L' we get the desired result. □

3. A basic area estimate

In [Mo, 6.4], Mostow proved an estimate for the k-dimensional Jacobian of the normal projection $\pi: X \to F$ onto a k-flat $F \subset X$, where X is a nonpositively curved symmetric space of rank k. Our aim is to deduce a non-infinitesimal analogue of this result applying to flats of maximal dimension in a Hadamard space X. Here now we assume X to be locally compact and cocompact, where the latter condition means that there exists a compact subset K of X such that the translates of K by the isometry group of X cover X.

First we establish a diameter estimate for certain subsets of X.

LEMMA 3.1. – *Let X be a locally compact and cocompact Hadamard space containing a k-flat but no $(k + 1)$-flat, where $k \ge 1$. Then there exist $s, m > 0$ such that the following holds. Let $F \subset X$ be a k-flat, $\pi: X \to F$ the projection, and let $Q \subset F$ be a closed k-dimensional euclidean cube of edge length $3s$. Let $q_1, \ldots, q_v$ be the vertices of Q, where $v := 2^k$. For $i = 1, \ldots, v$ let $Q_i \subset Q$ denote the k-cube of edge length s containing q_i.*

If V is a subset of X such that $\pi(V)$ meets each Q_i, and if $d(V,F) \geq s$, then the diameter of V satisfies $\mathrm{diam}V \geq s + md(V,F)$.

Proof. – By the assumptions on X there exists $s > 0$ such that X does not contain a closed convex set isometric to a $(k+1)$-dimensional euclidean cube of edge length s, *cf.* [Br, 3.1]. For $i = 1, \ldots, v$ pick $x_i \in V$ with $\pi(x_i) \in Q_i$, and let $x_i' \in [x_i, \pi(x_i)]$ be the point with $d(x_i', F) = s$. First we show that there are $i, j \in \{1, \ldots, v\}$ such that $d(x_i', x_j') > d(\pi(x_i), \pi(x_j))$. If not, then $d(x_i', x_j') = d(\pi(x_i), \pi(x_j))$ for all i, j since π is Lipschitz with constant 1. The convexity of the distance function on $X \times X$ then implies that the segments $[x_i', \pi(x_i)]$ and $[x_j', \pi(x_j)]$ are at constant distance from each other. As in [Ba, I.5.9] it follows that the convex hull of $\bigcup_{i=1}^{v} [x_i', \pi(x_i)]$ is isometric to the product of the convex hull of $\{\pi(x_1), \ldots, \pi(x_v)\}$ and $[0, s]$. Thus X contains an isometric copy of a $(k+1)$-dimensional euclidean cube of edge length s, contrary to the choice of s. Using the cocompactness of X again, we see that in fact $d(x_i', x_j') \geq \lambda d(\pi(x_i), \pi(x_j))$ for some $i, j \in \{1, \ldots, v\}$ and for some $\lambda > 1$ depending only on X and s (but not on the particular choice of F, Q and V). Hence by convexity,

$$\begin{aligned} d(x_i, x_j) &\geq [1 + (\lambda - 1)s^{-1} d(V,F)] d(\pi(x_i), \pi(x_j)) \\ &\geq s + (\lambda - 1) d(V,F). \end{aligned}$$

Thus the lemma holds for $m = \lambda - 1$. □

Together with the coarea formula, 3.1 leads to the desired analogue of [Mo, 6.4]. For $0 \leq i \leq k$ we denote by $\mathcal{H}^i$ the i-dimensional Hausdorff measure on $\mathbb{R}^k$.

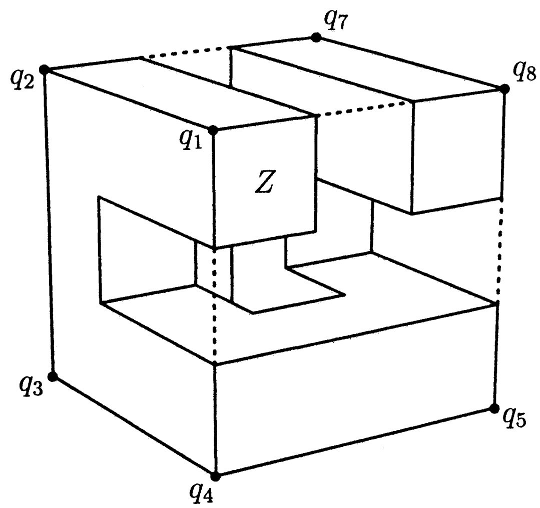
PROPOSITION 3.2. – *Let X, k, s, and m be given as in* 3.1. *Let $F \subset X$ be a k-flat, $\pi: X \to F$ the projection, and let $Q \subset F$ be a closed k-dimensional cube of edge length* $3s$. *Let f be an L-Lipschitz map from a compact ball $B \subset \mathbb{R}^k$ into X, and let $h = \pi \circ f$. Assume that $h(\partial B) \cap Q = \emptyset$ and that $h|\partial B$ is not contractible in $F \backslash Q$. If $d := d(f[h^{-1}(Q)], F) \geq s$, then $\mathcal{H}^k(h^{-1}(Q)) \geq s^{k-1}(s + md)/L^k$.*

Proof. – Let $v = 2^k$. Using induction on k one shows that the vertices $q_1, \ldots, q_v$ of Q can always be ordered in such a way that for $i = 1, \ldots, v-1$, $[q_i, q_{i+1}]$ is an edge of Q. Then let $P = \bigcup_{i=1}^{v-1} [q_i, q_{i+1}]$, and let R denote the union of all k-cubes $Q' \subset Q$ of edge length s meeting P. Define the cubes $Q_i \subset R$ as in 3.1, and denote by Z the common $(k-1)$-face of R and Q_1 isometric to $[0, s]^{k-1}$. For $k = 3$ the situation is illustrated in Figure. We define a 1-Lipschitz retraction $\phi: R \to Z$ such that each fiber $P_z := \phi^{-1}\{z\}$ is a connected polygonal arc lying at constant distance from P. Let $h = \pi \circ f$; then $\phi \circ h$ is Lipschitz with constant L. Now the coarea formula [Fe, 3.2.22] yields

$$\int_Z \mathcal{H}^1(h^{-1}(P_z))\, d\mathcal{H}^{k-1}(z) \leq L^{k-1} \mathcal{H}^k(h^{-1}(R)).$$

We claim that for each z in the relative interior Z' of Z there exists a connected component C_z of $h^{-1}(P_z)$ such that $h(C_z)$ meets each Q_i. With this at hand one can proceed as follows. By 3.1, $\mathrm{diam} f(C_z) \geq s + md$. Since C_z is connected, and since f is L-Lipschitz, we get

$$\mathcal{H}^1(h^{-1}(P_z)) \geq \mathcal{H}^1(C_z) \geq \mathrm{diam} C_z \geq (s + md)/L.$$



The proof of 3.2.

Together with the above integral estimate and the fact that $R \subset Q$ this yields the desired result.

It remains to show that for each $z \in Z'$ there exists a connected component C_z of $h^{-1}(P_z)$ such that $h(C_z)$ meets each Q_i. Suppose this is false for some $z \in Z'$. Every connected component C of $h^{-1}(P_z)$ is mapped by h to a connected subarc of P_z. Denote by G the union of those components C with $h(C) \cap Q_1 \neq \emptyset$. Then $h(G) \cap Q_v = \emptyset$. Note also that both G and $P_z \backslash G$ are compact. For $\varepsilon > 0$ let $U_\varepsilon := \{x \in h^{-1}(R) : d(x, G) < \varepsilon\}$. We can choose ε such that $U_\varepsilon \cap h^{-1}(P_z) = G$ and, since h is L-Lipschitz, such that $h(U_\varepsilon)$ intersects the boundary of R (relative to F) only in Z. Our aim is to construct a continuous map $\tilde{h}: B \to F$ that coincides with h on $B \backslash U_\varepsilon$ and maps G to $\{z\}$. For $y \in Z$ and $x \in P_y$ let $\alpha(x)$ denote the length of the subarc of P_y with endpoints y and x. Then for $t \geq 0$ let $\phi_t(x)$ be the point on P_y with $\alpha(\phi_t(x)) = \max\{0, \alpha(x) - t\}$. For $x \in U_\varepsilon$ define

$$t(x) := l\,(1 - \varepsilon^{-1} d(x, G)),$$

where l is the length of P_z. Finally, set $\tilde{h}(x) := \phi_{t(x)}(h(x))$ for $x \in U_\varepsilon$ and $\tilde{h}(x) := h(x)$ for $x \in B \backslash U_\varepsilon$. Clearly $\tilde{h}$ is a continuous map with $\tilde{h}|\partial B = h|\partial B$, and it is not difficult to see that $\tilde{h}$ omits the set $(P_z \backslash \{z\}) \cap Q_1$. It follows that $h|\partial B$ is contractible in $F \backslash Q$, contrary to the assumption. □

4. Main result

In this section we prove our main result. We use the following elementary fact.

LEMMA 4.1. – *Let* $a: (0, \infty) \to [0, \infty)$ *be a function with* $\limsup_{r \to \infty} a(r)/r < \infty$. *Then for all* $0 < \varepsilon' < \varepsilon < 1$ *there exists an unbounded sequence* $0 < r_1 < r_2 < \ldots$ *such that* $\varepsilon' a(r_i) \leq a(\varepsilon r_i)$ *for all* i.

Proof. – Assume first that for all $r_0 > 0$ there exists $r \geq r_0$ with $a(r)/r > \overline{a} := \limsup_{r\to\infty} a(r)/r$. Then one easily finds $0 < r_1 < r_2 < \ldots$ such that $a(r_i)/r_i < a(\varepsilon r_i)/(\varepsilon r_i)$ for all i. Thus $\varepsilon' a(r_i) \leq \varepsilon a(r_i) < a(\varepsilon r_i)$.

On the other hand, if $a(r)/r \leq \overline{a}$ for $r \geq r_0 > 0$ say, then one may pick $r_0 \leq r_1 < r_2 < \ldots$ such that $a(\varepsilon r_i)/(\varepsilon r_i) \geq (\varepsilon'/\varepsilon)\overline{a}$ for $i \geq 1$, hence $\varepsilon' a(r_i) \leq \varepsilon' \overline{a} r_i \leq a(\varepsilon r_i)$.

We restate Theorem B for convenience.

THEOREM 4.2. – *Let (X, d) be a locally compact and cocompact Hadamard space containing a k-flat but no $(k+1)$-flat, where $k \geq 1$. Then for all $L > 0$ and $C \geq 0$ there exists $D \geq 0$ such that the following holds. Let $F \subset X$ be a k-flat, $f: \mathbb{R}^k \to X$ an (L, C)-quasiisometric map, and for $r > 0$ define $a(r) := \sup\{d(f(x), F) : |x| \leq r\}$. If $\limsup_{r\to\infty} a(r)/r < L^{-1}$, then* $\mathrm{Hd}(f(\mathbb{R}^k), F) \leq D$.

Proof. – According to 1.2 there exists a map $f': \mathbb{R}^k \to X$ satisfying

$$L^{-1}|x-y| - C' \leq d(f'(x), f'(y)) \leq L'|x-y|$$

and $d(f'(x), f(x)) \leq C''$ for all $x, y \in \mathbb{R}^k$, where $L' := \sqrt{k}(L + C)$, $C' := 2\sqrt{k}L + 3C$, and $C'' := \sqrt{k}L + C$. Redefining $a(r) := \sup\{d(f'(x), F) : |x| \leq r\}$, we still have

$$\overline{a} := \limsup_{r\to\infty} a(r)/r < L^{-1}.$$

Let $\pi: X \to F$ denote the projection, $h := \pi \circ f'$, and $F' := f'(\mathbb{R}^k)$. For $z \in F$ and $r > 0$ define $B_F(z, r) := \{y \in F : d(y, z) \leq r\}$ and $a'(r) := \sup\{d(y, F') : y \in B_F(h(0), r)\}$. Fix $0 < \delta < L^{-1} - \overline{a}$, and let $\beta := \delta L/L'$. From 2.4 we know that for r sufficiently large, $a(\beta r) \leq \overline{L}a'(r) + \overline{C}$, where $\overline{L} := 5LL' + 1$ and $\overline{C} := LL'C'$. Replacing r by δr we see that $a(\varepsilon r) \leq \overline{L}a'(\delta r) + \overline{C}$ for $\varepsilon := \delta^2 L/L'$. Finally, by 4.1 and 2.3, we get the existence of an unbounded sequence $0 < r_1 < r_2 < \ldots$ such that for all i,

$$a(r_i) \leq 2\varepsilon^{-1}[\overline{L}a'(\delta r_i) + \overline{C}]$$

(say) and $h(B^k(r_i))$ contains $B_F(h(0), \delta r_i)$. We pick $y_i \in B_F(h(0), \delta r_i)$ with $d(y_i, F') = a'(\delta r_i)$. Then we choose $z_i \in B^k(r_i)$ such that

$$y_i \in D_i := B_F\left(h(z_i), \frac{1}{4}a'(\delta r_i)\right) \subset B_F(h(0), \delta r_i)$$

(since $a'(\delta r_i) \leq \delta r_i + d(h(0), F')$ we may assume that $a'(\delta r_i) \leq 4\delta r_i$ for all i). For $x \in K_i := h^{-1}(D_i) \cap B^k(r_i)$ we have

$$\begin{aligned} L^{-1}|x - z_i| - C' &\leq d(f'(x), f'(z_i)) \\ &\leq d(h(x), h(z_i)) + 2a(r_i) \\ &\leq \left(\frac{1}{4} + 4\varepsilon^{-1}\overline{L}\right)a'(\delta r_i) + 4\varepsilon^{-1}\overline{C}. \end{aligned}$$

Hence, denoting by α_k the volume of the unit ball in $\mathbb{R}^k$, we see that

$$\mathcal{H}^k(K_i) \le \alpha_k L^k \left[\left(\frac{1}{4} + 4\varepsilon^{-1}\overline{L} \right) a'(\delta r_i) + 4\varepsilon^{-1}\overline{C} + C' \right]^k .$$

On the other hand, for $x \in K_i$,

$$\begin{aligned} d(f'(x), F) &\ge d(f'(x), y_i) - d(h(x), y_i) \\ &\ge a'(\delta r_i) - \frac{1}{2} a'(\delta r_i) \\ &= \frac{1}{2} a'(\delta r_i). \end{aligned}$$

Let $n(i)$ be the maximal number of pairwise disjoint k-cubes of edge length $3s$ fitting into D_i, where $s > 0$ is the constant given by 3.1. We may assume that $\frac{1}{4}a'(\delta r_i) \ge 3\sqrt{k}s$, then clearly $n(i) \ge \alpha_k[\frac{1}{4}a'(\delta r_i) - 3\sqrt{k}s]^k/(3s)^k$. According to 2.3 we may apply 3.2 to each of these cubes, thus we deduce that

$$\begin{aligned} \mathcal{H}^k(K_i) &\ge n(i) s^{k-1} \left(s + \frac{1}{2} m a'(\delta r_i) \right) / L'^k \\ &\ge \alpha_k \left[\frac{1}{12} a'(\delta r_i) - \sqrt{k} s \right]^k \left(1 + \frac{1}{2} s^{-1} m a'(\delta r_i) \right) / L'^k . \end{aligned}$$

Combining the two estimates for $\mathcal{H}^k(K_i)$, we obtain an upper barrier for $a'(\delta r_i)$. Since $a(r_i) \le 2\varepsilon^{-1}[\overline{L}a'(\delta r_i) + \overline{C}]$, it follows that $\mathrm{Hd}(F', F)$ is bounded above by some constant which, however, still depends on the choice of $\delta \in (0, L^{-1} - \overline{a})$ and hence on $\overline{a}$. But now the argument can be repeated with $\overline{a} = 0$, which then yields a estimate in terms of L, C, k, s, and m. Finally, since $d(f'(x), f(x)) \le C'' = \sqrt{k}L + C$ for all $x \in \mathbb{R}^k$, the theorem follows. □

As mentioned in the introduction, the constant L^{-1} given in the theorem is optimal. Moreover, if for a concrete Hadamard space X a pair of numbers s, m satisfying 3.1 is known, then the present method allows, in principle, to determine D explicitly as a function of the quasiisometry constants L and C.

5. Pairs of flats

In this section we specialize the statement of Theorem 4.2 to the case where $f: \mathbb{R}^k \to X$ is an isometric map, *i.e.* $F' := f(\mathbb{R}^k)$ is a k-flat in X. In this case, the upper limit $\overline{a} := \limsup_{r\to\infty} a(r)/r$ considered in the theorem is actually a limit and can be interpreted in terms of the angle metric $\angle$ on the boundary at infinity $X(\infty)$ of X, as explained below. (*See* [Ba, Ch. II] for the definition of the metric space $(X(\infty), \angle)$.) For illustration, consider two intersecting straight lines F, F' in $\mathbb{R}^2$. Then clearly a correspondingly defined $\overline{a}$ would just be equal to the sine of the angle between F and F'. More generally, for two flats F, F' of possibly distinct dimension in a Hadamard space X, define an asymptotic angle by

$$\angle_{F'} F := \sup_{\xi \in F'(\infty)} \angle(\xi, F(\infty)),$$

where $F(\infty)$ and $F'(\infty)$ denote the boundaries at infinity of F and F', respectively. By compactness, the supremum is attained at some point $\xi \in F'(\infty)$. The Hausdorff distance between $F(\infty)$ and $F'(\infty)$ with respect to $\angle$ is defined by

$$\mathrm{Hd}_{\angle}(F(\infty), F'(\infty)) := \max\{\angle_{F'}F, \angle_F F'\}.$$

Now Theorem C in the introduction is a direct consequence of Theorem B and the following proposition. Note that everything stated below remains true if F is replaced by a locally compact, closed convex subset of X containing a geodesic ray.

PROPOSITION 5.1. – *Let (X,d) be a Hadamard space containing two flats F, F' (of possibly distinct dimensions). Pick $z \in F'$, and for $r \geq 0$ define $a(r) := \sup\{d(x,F) : x \in F', d(x,z) \leq r\}$. Then the limit $\overline{a} := \lim_{r\to\infty} a(r)/r$ exists (and is independent of the choice of z). Moreover $\overline{a} \in [0,1]$, and $\overline{a} = 1$ if and only if $\angle_{F'}F \geq \pi/2$. If $\angle_{F'}F \leq \pi/2$, then $\sin(\angle_{F'}F) = \overline{a}$.*

We need several lemmas.

LEMMA 5.2. – *The function defined by $a_0(r) := [a(r)-a(0)]/r$ for $r > 0$ is nondecreasing, and $0 \leq a_0 \leq 1$. In particular the limit $\overline{a} := \lim_{r\to\infty} a(r)/r$ exists and $\overline{a} \in [0,1]$.*

Proof. – Clearly $a(0) \leq a(r) \leq r + a(0)$, thus $0 \leq a_0 \leq 1$. Let $0 < r_1 \leq r_2$. Since $x \mapsto d(x,F)$ is a convex function on F', $a(r_1) = d(x_1,F)$ for some $x_1 \in F'$ with $d(x_1,z) = r_1$. Let $\sigma: [0,\infty) \to F'$ be the unit speed ray with $\sigma(0) = z$ and $\sigma(r_1) = x_1$, and let $b: [0,\infty) \to \mathbb{R}$ be the convex function defined by $b(r) := d(\sigma(r),F)$. Then $b(0) = a(0)$, $b(r_1) = a(r_1)$, and $b(r) \leq a(r)$ for all $r \geq 0$. By convexity, $[b(r_2) - b(0)]/r_2 \geq [b(r_1) - b(0)]/r_1$, which implies the result.

LEMMA 5.3. – *Let $\sigma: [0,\infty) \to X$ be a unit speed ray with $\sigma(\infty) =: \xi$, and for $r \geq 0$ define $b(r) := d(\sigma(r),F)$. Then $\overline{b} := \lim_{r\to\infty} b(r)/r$ exists, $\overline{b} \in [0,1]$, and $\angle(\xi, F(\infty)) \geq \arcsin \overline{b}$.*

Proof. – The existence of $\overline{b} \in [0,1]$ follows as in 5.2.

Assume first that $\overline{b} < 1$. We may replace σ by an asymptotic ray and therefore assume without loss of generality that $\sigma(0) \in F$. Let $\eta \in F(\infty)$ be an arbitrary point, and let $\rho: [0,\infty) \to F$ be the ray of speed $(1-\overline{b}^2)^{1/2}$ from $\sigma(0)$ to η. Consider the convex function defined by $c(r) := d(\sigma(r),\rho(r))$ for $r \geq 0$. Clearly $b(r) \leq c(r) \leq 2r$, thus $\overline{c} := \lim_{r\to\infty} c(r)/r$ exists and $\overline{c} \geq \overline{b}$. By [Ba, II.4.4], $\angle(\xi,\eta)$ equals the angle opposite to the side of length $\overline{c}$ in a euclidean triangle with sides of length 1, $(1-\overline{b}^2)^{1/2}$, and $\overline{c}$. Since $\overline{c} \geq \overline{b}$ it follows that $\angle(\xi,\eta) \geq \arcsin \overline{b}$.

An obvious limit argument yields the claim for $\overline{b} = 1$. □

LEMMA 5.4. – *Let σ, ξ, b, and $\overline{b}$ be given as in 5.3. If $\overline{b} < 1$ then there exists a point $\eta \in F(\infty)$ with $\angle(\xi,\eta) = \arcsin \overline{b} < \pi/2$.*

Proof. – As in the proof of 5.2 we may assume that $\sigma(0) \in F$. For $r > 0$ we consider the geodesic triangle with vertices $\sigma(0), \sigma(r), q(r)$, where $q(r)$ is the point in F closest to $\sigma(r)$. We may further assume that $b(r) = d(\sigma(r), q(r)) > 0$, and since $b(r) \leq r\overline{b} < r$, we also have $d(r) := d(\sigma(0), q(r)) > 0$. Then it is easily seen that $\angle_{q(r)}(\sigma(0), \sigma(r)) \geq \pi/2$, thus $\phi(r) := \angle_{\sigma(r)}(\sigma(0), q(r)) \leq \pi/2$.

We claim furthermore that $\cos\phi(r) \leq \overline{b}$ for all $r > 0$. To see this, consider for small $s > 0$ the geodesic triangle Δ_s in X with vertices $\sigma(r), \sigma(r - s^2), p_s$, where p_s is the point on $[\sigma(r), q(r)]$ with $d(p_s, \sigma(r)) = s$. Since b is convex, $b(r) - b(r - s^2) \leq s^2\overline{b}$, thus

$$d(\sigma(r - s^2), p_s) \geq b(r - s^2) - b(r) + s \geq s - s^2\overline{b}\,.$$

Let $\overline{\Delta}_s$ be the euclidean comparison triangle for Δ_s. The angle at the vertex of $\overline{\Delta}_s$ corresponding to $\sigma(r)$ is at least as large as the angle subtended by two sides of length 1 and s^{-1} in a euclidean triangle with third side of length $s^{-1} - \overline{b}$. For $s \to 0$ this angle converges to $\arccos\overline{b}$. This proves the claim.

By the cosine inequality [Ba, I.5.2(ii)] it follows that $r^2 \geq d(r)^2 + b(r)^2$ and $d(r)^2 \geq r^2 + b(r)^2 - 2rb(r)\overline{b}$, hence

$$1 - \frac{b(r)^2}{r^2} \geq \frac{d(r)^2}{r^2} \geq 1 + \frac{b(r)^2}{r^2} - 2\frac{b(r)}{r}\overline{b}\,.$$

Since $\lim_{r\to\infty} b(r)/r = \overline{b}$ we see that $\lim_{r\to\infty} d(r)/r = (1 - \overline{b}^2)^{1/2}$.

Let $\rho_r\colon [0, r] \to F$ be the geodesic of speed $d(r)/r$ with $\rho_r(0) = \sigma(0)$ and $\rho_r(r) = q(r)$. Consider the convex function defined by $c_r(t) := d(\sigma(t), \rho_r(t))$ for $t \in [0, r]$. We have $c_r(0) = 0$ and $c_r(r) = b(r)$, thus $c_r(t) \leq tb(r)/r$ for all $t \in [0, r]$. Let $\rho\colon [0, \infty) \to F$ be an accumulation ray of the family $(\rho_r)_{r>0}$. This is a ray of speed $(1 - \overline{b}^2)^{1/2}$, and $d(\sigma(t), \rho(t)) \leq t\overline{b}$ for all $t \geq 0$. Let $\overline{c} := \lim_{t\to\infty} d(\sigma(t), \rho(t))/t$ and $\eta := \rho(\infty)$. As in the proof of 5.3 we apply [Ba, II.4.4] which, since $\overline{c} \leq \overline{b}$, this time yields $\angle(\xi, \eta) \leq \arcsin\overline{b} < \pi/2$. Together with 5.3 this gives the result. □

Now we are ready to prove Proposition 5.1.

Proof of Proposition 5.1. – Lemma 5.2 shows that $\overline{a} \in [0, 1]$ exists.

It is not difficult to see that there exists a sequence of rays $\sigma_i\colon [0, \infty) \to F'$ with $\lim_{i\to\infty} \overline{b}_i = \overline{a}$, where $\overline{b}_i := \lim_{r\to\infty} d(\sigma_i(r), F)/r$. By 5.3, $\angle_{F'}F \geq \angle(\sigma_i(\infty), F(\infty)) \geq \arcsin\overline{b}_i$, hence $\angle_{F'}F \geq \arcsin\overline{a}$. In particular, $\overline{a} = 1$ implies that $\angle_{F'}F \geq \pi/2$.

Now assume that $\overline{a} < 1$. Pick $\xi \in F'(\infty)$ with $\angle(\xi, F(\infty)) = \angle_{F'}F$, and let $\sigma\colon [0, \infty) \to F'$ be the unit speed ray from z to ξ. Then clearly $\overline{b} := \lim_{r\to\infty} d(\sigma(r), F)/r \leq \overline{a} < 1$. By 5.4 there exists $\eta \in F(\infty)$ with $\angle(\xi, \eta) = \arcsin\overline{b}$, thus $\angle_{F'}F = \angle(\xi, F(\infty)) \leq \arcsin\overline{b} \leq \arcsin\overline{a} < \pi/2$.

These arguments also show that $\sin(\angle_{F'}F) = \overline{a}$ if $\angle_{F'}F \leq \pi/2$. □

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