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TRANSVERSELY AFFINE AND TRANSVERSELY PROJECTIVE HOLOMORPHIC FOLIATIONS

BY B. AZEVEDO SCÁRDUA

ABSTRACT. – Let $\mathcal{F}$ be a codimension one holomorphic singular foliation on M^n. $\mathcal{F}$ is *transversely affine* respectively *transversely projective* if so it is its regular foliation. We consider foliations which are transversely affine or projective in $M\backslash\Lambda$ for some analytic codimension one invariant subset $\Lambda \subset M$. Examples are *logarithmic* and *Riccati* foliations on $\mathbf{C}P(2)$. In the projective case ther is a *dual* foliation $\mathcal{F}^\perp$ generically transverse to $\mathcal{F}$. $\mathcal{F}^\perp$ is a fibration if $\mathcal{F}$ is Riccati. We prove: **1.** *Let $\mathcal{F}$ be given on $\mathbf{C}P(2)$, transversely affine outside an algebraic invariant curve Λ. Suppose that $\mathcal{F}$ has reduced non-degenerate singularities in Λ. Then $\mathcal{F}$ is logarithmic.* **2.** *Let $\mathcal{F}$ be given on $\mathbf{C}P(n)$, transversely projective non-affine, outside an invariant algebraic hypersurface Λ. Then $\mathcal{F}^\perp$ extends to $\mathbf{C}P(n)$. If this extension has a meromorphic first integral, then $\mathcal{F}$ is Riccati rational pull-back.*

Introduction

In this paper we consider holomorphic singular foliations of codimension one on a complex n-manifold M, $n \geq 2$. Let $\mathcal{F}$ be such a foliation and assume that the singular set of $\mathcal{F}$, denoted $s(\mathcal{F})$, has codimension ≥ 2. Define $M' = M\backslash s(\mathcal{F})$ and $\mathcal{F}' = \mathcal{F}/M'$ the non singular associated foliation. Thus $\mathcal{F}'$ can be defined by a covering of M' by open subsets U_i, $i \in I$, and distinguished mappings $f_i\colon U_i \to \mathbf{C}$, *i.e.* each f_i is a holomorphic submersion and the leaves of $\mathcal{F}'/U_i$ are the connected components of the level surfaces $f_i^{-1}(x)$, $x \in \mathbf{C}$. Whenever $U_i \cap U_j \neq \phi$ we have $f_i = f_{ij} \circ f_j$ for some local biholomorphism $f_{ij}\colon f_j(U_i \cap U_j) \subset \mathbf{C} \to f_i(U_i \cap U_j) \subset \mathbf{C}$. If $U_i \cap U_j \cap U_k \neq \phi$ then we have in the common domain the cocycle condition $f_{ij} \circ f_{jk} = f_{ik}$. The *transversal structure* of $\mathcal{F}$ in M is defined by the pseudogroup $\{f_{ij}\}\_i, j \in I$ so that $\mathcal{F}$ has a "simple" transversal structure if this pseudogroup is "simple" for some choice. The correct meaning of the expression "simple" above is given by the notion of transversely homogeneous foliation (Chapter II §6) where the local biholomorphisms f_{ij} are restrictions of elements of a Lie group action on an homogeneous space. In the codimension one case the remarkable examples are derived from the following ones: *transversely additive, affine* and *projective* structures; where the submersions $f_i\colon U_i \to \mathbf{C}$ are related by $f_i = f_j + b_{ij}$, $f_i = a_{ij} f_j + b_{ij}$ and $f_i = \frac{a_{ij} f_j + b_{ij}}{c_{ij} f_j + d_{ij}}$, $(a_{ij}, b_{ij}, c_{ij}, d_{ij} \in \mathbf{C})$; respectively, where in the affine case we require $a_{ij} \neq 0$ and in the projective case that $a_{ij} d_{ij} - b_{ij} c_{ij} = 1$. Of course the afffine case is a particular case of the projective case but we shall deal with the affine and the projective non-affine separately. We will investigate how often these structures appear. We remark that *the existence of an affine resp. projective transverse structure implies that the non-singular* *associated foliation is given by a holomorphic resp. meromorphic submersion in any simply connected open set.* This is a consequence of the well known notion of *development* of a transversely projective foliation (*see* [17] for instance). Using well known extension theorems for holomorphic or meromorphic functions through codimension ≥ 2 analytic subsets (in our case $s(\mathcal{F})$) we can obtain a holomorphic resp. meromorphic first integral for a transversely affine resp. projective foliation on a simply connected manifold and then conclude that *there exists no transversely affine foliation* on a compact simply connected manifold (for instance, the complex projective n-space $\mathbf{C}P(n)$), and that *the transversely projective foliations on* $\mathbf{C}P(n)$ *are the ones which have a rational first integral.* Motivated by this we will consider foliations which are transversely affine or projective in $M\backslash S$ for some analytic codimension one set $S \subset M$, invariant by the foliation $\mathcal{F}$. Well known examples of these foliations are given by linear logarithmic and Riccati foliations on $\mathbf{C}P(n)$ and its pull-backs to spaces M (*see* Chapter I, §1, Example 1.3, and Chapter II, §1, Example 1.1, for the definitions). These two families of examples play a fundamental role in our study being used as models.

In Chapter I we study transversely affine foliations proving the following (*see* Thm. 4.5):

THEOREM I. – *Let $\mathcal{F}$ be a codimension one foliation on $\mathbf{C}P(n)$ which is transversely affine outside an algebraic codimension one invariant subset $S \subset \mathbf{C}P(n)$. Suppose that $\mathcal{F}$ has reduced non-degenerate singularities in S (*see Ch. I Section 2 for the definitions*). Then $\mathcal{F}$ is a logarithmic foliation.*

For the proof of this theorem we need to study the holonomy of an irreducible component S_o of S. This goes as follows (*see* Theorem 4.1 and Proposition 5.1):

THEOREM II. – *Let $\mathcal{F}$ be a foliation on M^2 having $\Lambda \subset M$ as an analytical connected invariant curve. Suppose*

i) *all singularities of $\mathcal{F}$ in Λ are of 1$^{\text{st}}$-order*

ii) *the foliation $\widetilde{\mathcal{F}}$ obtained by the resolution of the singularities of $\mathcal{F}$ in Λ exhibits some linearizable non-resonant singularity. Then the following conditions are equivalent:*

a) *$\mathcal{F}$ is transversely affine in some neighborhood of Λ minus Λ and its local separatrices sep(Λ);*

b) *the holonomy group of the leaf $\Lambda\backslash s(\mathcal{F})$ and of any projective line in the desingularization of $\mathcal{F}$ in Λ is a solvable group and we have the solvability compatibility between them* (see Ch. I, Section 5 for definition). *This is called the property $(\mathcal{S})$ for the holonomy of Λ.*

Using this theorem an the topological invariance of the projective holonomy, for stable deformations of germs of 1-forms having a generic first jet [15] we obtain the following theorem (*see* Proposition 5.2, Ch. I).

THEOREM III. – *Let $w = Adx + Bdy$ be a germ of holomorphic 1-form in the origin of $\mathbf{C}^2$ having w_ν generic as first ν-jet, $\nu \geq 2$ and let $w' = A'dx + B'dy$ be a stable deformation of w. Suppose w has a multiform integrating factor of the form $f = \Pi f_j^{\lambda_j}$, $f_j \in \vee_2$, $\lambda_j \in \mathbf{C}^*$. Then w' has an integrating factor of the same type.*

Chapter II is devoted to the study of foliations which are transversely projective outside an invariant codimension one analytic subset. We associate to such a projective non-affine structure for $\mathcal{F}$ in M, a *dual* codimension one foliation $\mathcal{F}^\perp$ on M which is transverse to $\mathcal{F}$ almost everywhere. The *duality* between $\mathcal{F}$ and $\mathcal{F}^{\perp}$ is such that one determines the other. For example if $\mathcal{F}$ is a Riccati foliation $\mathcal{F}$: $p(x)dy - (y^2a(x) + yb(x) + c(x))dx = 0$ on $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$ then the natural dual foliation $\mathcal{F}^{\perp}$ is the fibration $x =$ Cte by vertical projective lines of $\overline{\mathbf{C}} \times \overline{\mathbf{C}}$. The existence of such a dual fibration is persistent under rational pull-backs. One central result proved in II §4 states that indeed this characterizes the existence of the pull-back from a Riccati foliation (*see* Theorem 4.1, Ch. II).

THEOREM IV. – *Let $\mathcal{F}$ be a foliation on $\mathbf{C}P(n)$ which is transversely projective but not transversely affine, outside an invariant analytic subset S of codimension one. Then the dual foliation $\mathcal{F}^{\perp}$ on $\mathbf{C}P(n)\backslash S$ extends to a foliation on $\mathbf{C}P(n)$ and if $\mathcal{F}^{\perp}$ has a meromorphic first integral then $\mathcal{F}$ is the rational pull-back of a Riccati foliation on $\mathbf{C}P(2)$.*

We also study the cases where $\mathcal{F}^{\perp}$ has an affine transverse structure in $\mathbf{C}P(n)\backslash S$ and the local case for $\mathcal{F}$. The techniques introduced here are used to give different proofs of well known results about stability of logarithmic foliations on $\mathbf{C}P(n)$, $n \geq 3$ [3] and rational foliations on $\mathbf{C}P(n)$, $n \geq 3$ having first integrals of the form f^p/g^q, $(p,q) = 1$ [18]. We also give a proof of a theorem due to A. Lins Neto and D. Cerveau on the existence of meromorphic first integrals for foliations on $\mathbf{C}P(n)$, $n \geq 3$, having a complete intersection Kupka component [11]. One important remark about the generality of the context is the following (*see* Theorem 6.1):

THEOREM V. – *Let $\mathcal{F}$ be a holomorphic singular transversely homogeneous foliation of codimension one on M^n. Then $\mathcal{F}$ is transversely projective foliation on M^n.*

These notes are derived from my doctoral thesis ([27]) held at IMPA in the year of 1994, under the advise of Prof. César Camacho to whom I am very grateful and who suggested to me the subject. I am also in debt with A. Lins Neto, P. Sad and M.Brunella for many valuable conversations and suggestions during the preparation of my thesis and of this text. I would like to thank Prof. E. Ghys for valuable discussions during the beggining of this work, for suggesting me the book of C. Godbillon "Feuilletages: Études Géométrics I" and the references on real transversely affine foliations, which where very valuable, and for suggesting me the geometric approach I use here. I am grateful to D. Cerveau for pointing out the necessity of the use of Stein's Fatorization Theorem in Chapter II. Finally I want to thank the referee for his kind interest and careful reading of the original manuscript, which has helped me to improve the paper.

Chapter I
Transversely Affine Holomorphic Foliations

1. Transversely affine foliations and differential forms

Throughout this chapter I, except for explicit mention, the 1-form Ω will be assumed to have singular set $s(\Omega)$ of codimension bigger than one.

The problem of deciding wether there exist affine transverse structures for a given foliation is equivalent to a problem on differential forms as stated below (*see* [1] for the case of real non-singular foliations):

PROPOSITION 1.1. – *Let Ω be an integrable meromorphic 1-form which defines $\mathcal{F}$ outside the polar divisor $(\Omega)_\infty$. The foliation $\mathcal{F}$ is transversely affine in M if and only if there exists a 1-form η in M satisfying: η is meromorphic, closed, $d\Omega = \eta \wedge \Omega$, $(\eta)_\infty = (\Omega)_\infty$ and $Res\,\eta = -(order\ of (\Omega)_\infty\big|_L)$ for each irreducible component L of $(\Omega)_\infty$, and $(\eta)_\infty$ has order one. Furthermore, two pairs (Ω, η) and (Ω', η') define the same affine structure for $\mathcal{F}$ in M if and only if there exists a meromorphic map $g: M \to \overline{\mathbf{C}}$ satisfying $\Omega' = g\Omega$ and $\eta' = \eta + \frac{dg}{g}$.*

Remark 1.1. – (a) For the case where M is open and Ω is holomorphic the form η is holomorphic. (b) The existence of a meromorphic 1-form Ω which defines $\mathcal{F}$ globally in M is always true if M is a complex projective space $\mathbf{C}P(n)$ or an algebraic non-singular projective variety (*see* [13] for instance), but is not really necessary (*see* Section 6 of Chapter I).

Proof of Proposition 1.1. – Let Ω be a meromorphic 1-form which defines $\mathcal{F}$ in M and suppose $\{y_i: U_i \to \mathbf{C}\}$ is a transversal affine structure for $\mathcal{F}$ in M. Since the submersions y_i define $\mathcal{F}$ locally, we can write $\Omega\big|_{U_i} = g_i\, dy_i$ for some meromorphic g_i. In $U_i \cap U_j \neq \phi$ we have: (1) $g_i\, dy_i = g_j\, dy_j$; (2) $y_i = a_{ij}\, y_j + b_{ij}$. From (2) we have $dy_i = a_{ij}\, dy_j$ and then from (1) we have $a_{ij}\, g_i = g_j$ so that $dg_i/g_i = dg_j/g_j$ and this allows us to define η in $M \backslash s(\mathcal{F})$ by $\eta\big|_{U_i} = dg_i/g_i$. The 1-form η is closed, meromorphic and satisfies $d\Omega = \eta \wedge \Omega$. Since codimension $(s(\mathcal{F})) > 1$ we can extend by Hartogs' Extension Theorem (*see* [30]) the 1-form η meromorphic to M. We also have $(\eta)_\infty = (\Omega)_\infty$ of order one and $\mathrm{Res}_L\, \eta = -$ order of $\Omega\big|_L$, for each component L of $(\Omega)_\infty$: In fact, it is clear by the construction that $(\eta)_\infty = (\Omega)_\infty$. Now given a point $p \in (\Omega)_\infty$ say, $p \in L$, L an irreducible component of $(\Omega)_\infty$, choose a holomorphic function $x: U \to \mathbf{C}$ defined in $p \in U$ such that $x^n.\Omega$ is holomorphic at p, where $n =$ order of $(\Omega)_\infty$ along L. Then $x^n.\Omega = g\, dy$ in a small neighborhood of p so that by construction we have $\Omega = x^{-n}.g\, dy$ and $\eta = \frac{d(x^{-n}.g)}{x^{-n}.g} = -\frac{n dx}{x} + \frac{dg}{g}$. Since g is holomorphic along L it follows that $\mathrm{Res}_L\, \eta = -n$. This proves the first part of the proposition.

Assume now that Ω and η are as in the statement. Since η is holomorphic and closed in $M \backslash (\Omega)_\infty$, there exists an open cover $\{U_i\}$ of $M \backslash (\Omega)_\infty$ and there are holomorphic functions $h_i \in \mathrm{Hol}(U_i)$ such that $\eta\big|_{U_i} = dh_i$. We define $g_i = \exp(h_i)$, $g_i \in \vee(U_i)^*$ to obtain $\eta\big|_{U_i} = dg_i/g_i$. Now, from condition $d\Omega = \eta \wedge \Omega$ we have $d\Big(\frac{\Omega}{g_i}\Big) = 0$, and then $\Omega = g_i\, dy_i$ for some holomorphic function $y_i \in \vee(U_i)$. This we can do in $M \backslash (\Omega)_\infty$. Now, given a point $p_i \in (\Omega)_\infty$ we can choose a local chart $(x, y) \in U_i$ such that $p_i = (0,0)$, $(\Omega)_\infty \cap U = \{y = 0\}$ and $\eta(x,y) = -n\frac{dy}{y} + \frac{df}{f}$ where $n =$ order of $(\Omega)_\infty$ and $f \in \vee(U_i)^*$. Therefore we have $\eta = \frac{d(f.y^{-n})}{f.y^{-n}} = \frac{dg_i}{g_i}$, $g_i = f.y^{-n}$. Thus the 1-form $\frac{\Omega}{g_i}$ is closed and holomorphic so that it can be writen $\frac{\Omega}{g_i} = dy_i$ for some holomorphic y_i. Thus we have covered $M \backslash s(\mathcal{F})$ with open sets U_i where we have the relations $\Omega = g_i\, dy_i$, $\eta = \frac{dg_i}{g_i}$. In each $U_i \cap U_j \neq \phi$ we have $\frac{dg_i}{g_i} = \eta = \frac{dg_j}{g_j}$ and $g_i\, dy_i = \Omega = g_j\, dy_j$. The first equality implies $g_j = a_{ij}.g_i$ for some locally constant a_{ij} and it follows from the second equality that $dy_i = a_{ij}\, dy_j$ and then $y_i = a_{ij}\, y_j + b_{ij}$ with b_{ij} locally constant in $U_i \cap U_j$. This shows that $\mathcal{F}$ is transversely affine in M.

Now we prove the last part of the proposition. Let (Ω,η) be given and let $g\colon M\to\overline{\mathbf{C}}$ be a meromorphic function. We define $\Omega'=g\Omega$ and $\eta'=\eta+\frac{dg}{g}$. Using the same notation above we have $\eta'\big|_{U_i}=\eta\big|_{U_i}+\frac{dg}{g}=\frac{dg_i}{g_i}+\frac{dg}{g}=\frac{d(g_i g)}{(g_i.g)}$ and $\Omega'\big|_{U_i}=g.\Omega\big|_{U_i}=(gg_i)dy_i$, and this shows that: $g_i'=a_{ij}\,g_j'$ and $y_i'=y_i$ so that $a_{ij}'=a_{ij}$ and $b_{ij}'=b_{ij}$. Hence, the pairs (Ω,η) and (Ω',η') define the same transversal structure for $\mathcal{F}$ in M. Finally, suppose that (Ω,η) and (Ω',η') define the same transversal structure for $\mathcal{F}$ in M. Since Ω and Ω' define $\mathcal{F}$, we have $\Omega'=g\Omega$ for some $g\colon M\to\overline{\mathbf{C}}$ meromorphic. Using the same notation above we write (locally) $\Omega=g_i\,dy_i$, $\Omega'=g_i'\,dy_i$, $\eta=dg_i/g_i$ and $\eta'=dg_i'/g_i'$; but $g_i'=gg_i$ so $\eta'=\eta+dg/g$ completing the proof. $\square$

Example 1.1. – *Transversely affine foliations on simply-connected manifolds*. Let M be simply-connected and let $\mathcal{F}$ be given by a holomorphic 1-form Ω. The transversal affine structures for $\mathcal{F}$ are given by the holomorphic maps $f\colon M\to\mathbf{C}$ which are submersions outside $s(\mathcal{F})$: In fact, it is a consequence of the well-known notion of *development* of a transversely homogeneous foliation (*see* [17] Prop. 3.3 pp.247-248), that the foliation exhibits a holomorphic first integral on $M'=M\backslash s(\mathcal{F})$ (notice that M' is also simply connected). Hartogs' theorem [20] implies that this first integral extends holomorphically to M. In particular, the existence of an affine transverse structure on a punctured neighborhood of a singularity implies that this singularity has a (local) holomorphic first integral and is therefore of first order (*see* Section 2 for the definition).

Example 1.2. – Let $\Phi\colon N\to M$ be a holomorphic map transverse to the foliation $\mathcal{F}$. If $\mathcal{F}$ is transversely affine then so it is the induced foliation $\Phi^*\mathcal{F}$. This is easily verified by taking the local submersions which define the affine transverse strcture for $\mathcal{F}$.

Example 1.3. – *Logarithmic foliations* on $\mathbf{C}P(n)$. The foliation $\mathcal{F}$ on $\mathbf{C}P(n)$ is called *logarithmic* if there is a rational map $\pi\colon\mathbf{C}P(n)\to\mathbf{C}P(m)$ such that $\mathcal{F}=\pi^*(\mathrm{L})$ where L is the linear logarithmic foliation on $\mathbf{C}P(m)$ given by $\Omega=\prod_{i=1}^n x_i\sum_{j=1}^n\lambda_j\frac{dx_j}{x_j}=0$ in some affine chart $(x_1,\dots,x_m)\in\mathbf{C}^m\hookrightarrow\mathbf{C}P(m)$. If we define the 1-form $\eta=\sum_{j=1}^m\frac{dx_j}{x_j}$ we can conclude from Proposition 1.1 that L is transversely affine in $\mathbf{C}P(m)\backslash A$ where $A\subset\mathbf{C}P(m)$ is the algebraic invariant set given by $\bigcup_{j=1}^m\overline{\{x_j=0\}}$, hence using Example 1.2 we conclude that $\mathcal{F}$ is transversely affine outside an algebraic invariant set $D=\pi^{-1}(A)\subset\mathbf{C}P(n)$. Let $\pi(x_1,\dots,x_n)=(f_1(x_1,\dots,x_n),\dots,f_m(x_1,\dots,x_n))$ in affine charts; where the f_i's are irreducible smooth polynomials; then $D=\bigcup_j\overline{\{f_j=0\}}$ and the hypersurfaces $\overline{\{f_j=0\}}$ are the compact leaves of $\mathcal{F}$; they have linearizable holonomy and any other leaf has trivial holonomy. For more information on logarithmic foliations the reader should consult [2].

Example 1.4. – *Bernoulli foliations on* $\mathbf{C}P(n+1)$. In $\mathbf{C}P(n+1)$ we consider affine coordinates $(x_1,\dots,x_n,y)\in\mathbf{C}^{n+1}\hookrightarrow\mathbf{C}P(n+1)$. Let Ω be the meromorphic 1-form given by $\Omega(x_1,\dots,x_n,y)=(\prod_{j=1}^n p_j(x_j))dy-\sum_{j=1}^n(\prod_{i\neq j}p_i(x_i))(y^k c_j(x_j)-yb_j(x_j))dx_j$; where p_j, b_j, c_j are polynomials of one variable. We say that Ω defines a *Bernoulli foliation* of order k on $\mathbf{C}P(n+1)$, if Ω satisfies the following integrability condition: $c_i(x_i).b_j(x_j)=c_j(x_j).b_i(x_i)\quad\forall i,j$. Under this hypothesis we define the 1-form $\eta:=k\frac{dy}{y}+\sum_{j=1}^n\frac{p_j'(x_j)+(k-1).b_j(x_j)}{p_j(x_j)}\,dx_j$, and we obtain a transversal affine structure for $\mathcal{F}=\mathcal{F}(\Omega)$ outside of an algebraic invariant set $\Gamma\subset\mathbf{C}P(n+1)$, which is a finite union of hyperplanes $\mathbf{C}P(n) \subset \mathbf{C}P(n+1)$. If $n = 1$ we have $\Omega(x,y) = p(x)dy - (y^k\, c(x) - y\, b(x))dx$ which is the pull-back of the particular *Riccati foliation* $(p(u)dv - (k-1)(c(u)v^2 - vb(u))du$ by a map $(u,v) = (x, y^{k-1})$. The point $p_\infty \in \mathbf{C}P(2)$ given by $x = 0$, $y = \infty$ is a dicritical singularity of $\mathcal{F}$ (*see* definitions in §2). This dicritical singularity plays a fundamental role in the study of the structure of $\mathcal{F}$ and is the responsible for the non linearization of $\mathcal{F}$. In fact in general $\mathcal{F}$ is not the pull-back of a linear logarithmic foliation because of the non-algebraic separatrices of p_∞.

Example 1.5 (*see* [1],[16]). – We will define a transversely affine foliation on a compact 3-manifold. This will be a non-singular foliation with dense leaves which are biholomorphic to $\mathbf{C}^* \times \mathbf{C}^*$ or cylinders $\mathbf{C}^*/\mathbf{Z} \times \mathbf{C}^*$. We begin with a general construction inspired in [1] and [16]. Let M be a compact complex n- manifold. Let w be a closed 1-form on M and let $f\colon M \to M$ be a biholomorphism such that $f^*w = \lambda w$ for some $\lambda \in \mathbf{C}^*$ with $|\lambda| \neq 1$. Define Ω on $M \times \mathbf{C}^*$ by $\Omega(x,t) = t.w(x)$. Then we have $d\Omega = \eta \wedge \Omega$ where $\eta(x,t)$ is defined by $\eta(x,t) = \frac{dt}{t}$. We have $d\eta = 0$ and η holomorphic, thus Ω defines a codimension one foliation $\widetilde{\mathcal{F}}$ on $M \times \mathbf{C}^*$ which is transversely affine in the sense of Definition 1.1. Now we consider the action $\Phi\colon \mathbf{Z} \times (M \times \mathbf{C}^*) \longrightarrow M \times \mathbf{C}^*, n, (x,t) \longmapsto (f^n(x), \lambda^{-n}.t)$. This is a locally free action generated by the biholomorphism $\varphi\colon M \times \mathbf{C}^* \to M \times \mathbf{C}^*$, $\varphi(x,t) = (f(x), \lambda^{-1}t)$. We have $\varphi^*\,\Omega(x,t) = \lambda^{-1}\,t.\lambda\, w(x) = \Omega(x,t)$ and $\varphi^*\eta = \eta$. Thus, the foliation $\widetilde{\mathcal{F}}$ induces a codimension one foliation $\mathcal{F}$ on the quotient manifold $V = (M \times \mathbf{C}^*)/\mathbf{Z}$, this foliation inherits a transverse affine structure induced by the pair (Ω, η). For instance, we consider a variant of the Furness example (*see* [1]): Consider the unimodular map $U = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\colon \mathbf{C}^2 \to \mathbf{C}^2$; $\quad U(x,y) = (x+y, x+2y)$. This map induces a biholomorphism $f\colon M \to M$, where $M = \mathbf{C}^*/\mathbf{Z} \times \mathbf{C}^*/\mathbf{Z}$ and where $\mathbf{C}^* = \mathbf{C}/\mathbf{Z}$ has the coordinate obtained from the action $\mathbf{Z} \times \mathbf{C} \to \mathbf{C}(n,z) \to z+n$, and $\mathbf{C}^*/\mathbf{Z}$ is defined from the action $\mathbf{Z} \times \mathbf{C}^* \to \mathbf{C}^*$, $(n,t) \to \mu^{-n}.t$; where $\mu \in \mathbf{C}^* \backslash S^1$ is arbitrary. The biholomorphism f is induced by $F\colon \mathbf{C}^* \times \mathbf{C}^* \to \mathbf{C}^* \times \mathbf{C}^*$, $F(z,w) = (zw, zw^2)$. We consider $\tilde{\tilde{w}} = (1+\sqrt{5})dx - 2dy$ in $\mathbf{C}^2$. We have $U^*\tilde{\tilde{w}} = \lambda.\tilde{\tilde{w}}$ where $\lambda = \frac{2}{3-\sqrt{5}}$ and U is $\mathbf{Z} \times \mathbf{Z}$ invariant ($\mathbf{Z} \times \mathbf{Z}$ acts on $\mathbf{C} \times \mathbf{C}$ by the natural product action) so that it induces a 1-form $\tilde{w}$ in $\mathbf{C}^* \times \mathbf{C}^*$, this last is also $\mathbf{Z} \times \mathbf{Z}$ invariant so that it induces a closed holomorphic 1-form w in the bitorus M. The 1-form w satisfies $f^*w = \lambda.w$. The foliation induced on $V = (M \times \mathbf{C}^*)/\mathbf{Z} = ((\mathbf{C}^*/\mathbf{Z} \times \mathbf{C}^*/\mathbf{Z}) \times \mathbf{C}^*)/\mathbf{Z}$ is transversely affine, has dense leaves and its leaves are biholomorphic to $\mathbf{C}^*/\mathbf{Z} \times \mathbf{C}^*$ or $\mathbf{C}^* \times \mathbf{C}^*$.

Example 1.6. – *The Integration Lemma for closed rational 1-forms.* Let $\mathcal{F}$ be a foliation on $\mathbf{C}P(n)$ which is given by a closed meromorphic 1-form, say, w. Then $\mathcal{F}$ has a transverse structure by translations in $\mathbf{C}P(n)\backslash(w)_\infty$ where the polar divisor $(w)_\infty$ is invariant and algebraic of codimension one. The *Integration Lemma* ([12]), states that if $\tilde{w} = \pi^*w$ where $\pi\colon \mathbf{C}^{n+1}\backslash 0 \to \mathbf{C}P(n)$ is the canonical projection then we have $\tilde{w} = \sum_{j=1}^r \lambda_j \frac{df_j}{f_j} + d\big(\frac{g}{\Pi f_j^{n_j-1}}\big)$ for some $\lambda_j \in \mathbf{C}$, and some homogeneous polynomials f_j, g in $\mathbf{C}^{n+1}$. We have $n_j =$ order of $(\tilde{w})_\infty$ along the hypersurface $(f_j = 0)$ and $(\tilde{w})_\infty = \bigcup_{j=1}^r (f_j = 0)$, so that $(w)_\infty = \pi((\tilde{w})_\infty) = \bigcup_{j=1}^r \pi(f_j = 0)$. As it is easy to see $\mathcal{F}$ may not be of a logarithmic or Bernoulli type. The reason is on the type of the singularities that may arise.

As a corollary of Example 1.1 we obtain:

PROPOSITION 1.2. – *There is no transversely affine foliation on* $\mathbf{C}P(n)$.

Proof. – In fact, $\mathbf{C}P(n)$ is simply-connected and, since it is compact, it admits no nonconstant holomorphic function. □

2. Resolution of singularities

Let $\mathcal{F}$ be a holomorphic singular codimension one foliation with isolated singularities on a compact two dimensional complex manifold M^2. Let $\Lambda \subset M$ be an analytic invariant curve. A theorem of Seidenberg [28] gives a *resolution* of the singular points of $\mathcal{F}$ on Λ.

THEOREM 2.1 [28]. – *There is a finite sequence of blow-ups at the points of* $s(\mathcal{F})$ *such that their composition gives a proper holomorphic map* $\pi: \widetilde{M} \to M$ *a complex compact 2-manifold* $\widetilde{M}$ *and a foliation* $\mathcal{F}^* = \pi^*\mathcal{F}$ *with isolated singularities such that:*

i) $\pi^{-1}(s(\mathcal{F})) = \bigcup_{j=1}^{k} P_j$ *is a finite connected union of complex projective lines with normal crossings and* $\pi: \widetilde{M}\backslash \bigcup_{j=0}^{k} P_j \to M\backslash s(\mathcal{F})$ *is a biholomorhism (the union* $D = \pi^{-1}(\Lambda) = \pi^{-1}(s(\mathcal{F})) \cup \pi^{-1}(\Lambda\backslash s(\mathcal{F})) = \bigcup_{j=0}^{k} P_j$ *is called the desingularizing divisor of* $s(\mathcal{F}) \cap \Lambda$*),* P_o *is the closure of* $\pi^{-1}(\Lambda\backslash s(\mathcal{F}))$ *on* $\widetilde{M}$*);*

ii) *At any singularity* $p \in \bigcup_{j=0}^{k} P_j$ *of* $\mathcal{F}^*$ *there is a local chart* (x, y) *such that* $x(p) = y(p) = 0$ *and* $\mathcal{F}^*$ *is given by one of the Pfaff forms: (i)* $xdy - \lambda y\, dx + h.o.t$, $\lambda \notin \mathbf{Q}_+$ *(non-degenerate linear part); (ii)* $x^{p+1}\, dy + y(1+\lambda x^p)dx + (h.o.t)dx$, $p \geq 1$ *(called saddle-node).* In case (i) we say that p is *resonant* if $\lambda \in \mathbf{Q}_-$. Let $p \in s(\mathcal{F})$, be a singular point of $\mathcal{F}$, by the Separatrix Theorem [5] the foliation $\mathcal{F}$ admits at least one *separatrix* through p; if the number of these separatrices is finite the singularity is called *non-dicritical*. This fact is equivalent to the fact that all the projective lines P_j belonging to $\pi^{-1}(p)$ are tangent to $\mathcal{F}^*$. The foliation $\mathcal{F}^*$ is called the *resolution* of the foliation $\mathcal{F}$ (for more information the reader should consult [4] or [24]). We remark that if a foliation $\mathcal{F}$ has only non-dicritical singularities in an invariant irreducible hypersurface $\Lambda \subset M$ then it is well defined the analytic codimension one set $\operatorname{sep}(\Lambda)$ of the local separatrices of $\mathcal{F}$ through the points of $\Lambda \cap s(\mathcal{F})$, in a neighborhood of Λ in M.

DEFINITION 2.1. – A singularity $p \in s(\mathcal{F})$ is said to be of *first order* when it is non-dicritical and there are no saddle-nodes in its resolution (*see* [6]).

We finish this section defining what we will consider as an extended affine structure.

DEFINITION 2.2. – Let $\mathcal{F}$ be given by Ω, and let $\Lambda \subset M$ be an analytic invariant hypersurface, not containing dicritical singularities of Ω. A 1-form η defined in a neighborhood of Λ is *adapted to* Ω *along* the hypersurface Λ if: (i) η is meromorphic, closed, $d\Omega = \eta \wedge \Omega$; (ii) the polar divisor $(\eta)_\infty = \Lambda \cup \operatorname{sep}(\Lambda) \cup (\Omega)_\infty$, has order one along Λ and $(\Omega)_\infty$, and $\operatorname{Res}_L \eta = -$ (order of $(\Omega)_\infty$ along L) for each irreducible *non-invariant* component L of $(\Omega)_\infty$.

For example, if we consider $\Omega = xdy - y^k\, dx$ in affine coordinates in $\mathbf{C}P(2)$ then $\eta = k\frac{dy}{y} + \frac{dx}{x}$ is an adapted form to Ω along the algebraic leaf $\overline{\{y=0\}}$ and also along the algebraic leaf $\overline{\{x=0\}}$. The same does not hold for the singular leaf $L_\infty = \mathbf{C}P(2)\backslash\mathbf{C}^2$, because $\operatorname{Res}_{L_\infty} \eta = -(k+1)$ and (order of $(\Omega)_\infty$ along L_∞) $= k+2$.

3. Extended affine structures

Our basic tools in the study of the holonomy of transversely affine foliations are the two following lemmas.

LEMMA 3.1. – *Let $\mathcal{F}$ be given by Ω, let $\Lambda \subset M$ be an analytic non-singular invariant hypersurface and let η be an adapted form to Ω along Λ.*

(1) *Suppose $Res_\Lambda \eta = a \notin \{2,3,\ldots\}$. Then given a regular point $p \in \Lambda \backslash s(\mathcal{F})$ there exists a local chart $(x,y) \in U$ such that $p=(0,0)$, $\Lambda \cap U = \{y=0\}$, $\Omega = g\,dy$ and $\eta = a\frac{dy}{y} + \frac{dg}{g}$ where g is meromorphic in U. Furthermore if $(\tilde{x},\tilde{y}) \in \widetilde{U}$ is another such system with $U \cap \widetilde{U} \neq \phi$ and connected, then we have $\tilde{y} = c.y$ for some $c \in \mathbf{C}^*$.*

(2) *Suppose $Res_\Lambda \eta = k \in \{2,3,\ldots\}$ and suppose that we have $\Omega = \hat{g}\,d\hat{y}$, $\eta = k\frac{d\hat{y}}{\hat{y}} + \frac{d\hat{g}}{\hat{g}}$ for some local chart $(\hat{x},\hat{y}) \in \widehat{U}$ with $\widehat{U} \cap \Lambda = \{\hat{y}=0\}$ and $\hat{g}$ meromorphic in $\widehat{U}$. Then given a regular point $p \in \Lambda \backslash s(\mathcal{F})$ there exists a local chart $(x,y) \in U$ such that $p=(0,0)$, $\Lambda \cap U = \{y=0\}$, $\Omega = g\,dy$ and $\eta = k\frac{dy}{y} + \frac{dg}{g}$.*

Furthermore if $(\tilde{x},\tilde{y}) \in \widetilde{U}$ is another such chart with $U \cap \widetilde{U} \neq \phi$ and connected, then we have $\tilde{y}^{k-1} = h(y^{k-1})$ for some homography $h(z) = \frac{\lambda z}{1+az}$.

Remark 3.1. – We will show as a consequence of Lemma 3.2 that condition (2) is always satisfied if $s(\mathcal{F}) \cap \Lambda$ contains some linearizable non-resonant singularity.

Proof of Lemma 3.1. – We will assume that M is 2-dimensional. The general case is proved in the same way. First we consider the case (1) where $\mathrm{Res}_\Lambda \eta = a \notin \{2,3,\ldots\}$ and make the following claim:

CLAIM 1. – Given a holomorphic function $r(y)$ defined in a neighborhood of $y=0 \in \mathbf{C}$ with $r(0)=1$, there exists a local holomorphic non-vanishing function $u=u(y)$ defined in a neighborhood of $y=0 \in \mathbf{C}$, such that $\frac{u^a}{u+y.u'} = r(y)$.

Proof of Claim 1. – To prove the claim we consider the distinct cases $a=1$ and $a \notin \{1,2,3,\ldots,\}$.

Case 1. – $a=1$: We define $\xi(y) = \frac{1}{r(y)} - 1$. Since $\xi(0)=0$ we have $\xi(y)/y$ holomorphic in $y=0$. So it is enough to define $u(y) = \exp\left(\int \frac{\xi(y)}{y}\,dy\right)$ which is holomorphic, non-vanishing and satisfies $\frac{u'}{u}(y) = \left(\frac{1}{r(y)} - 1\right)/y$, which gives $r(y) = \frac{u(y)}{u(y)+y.u'(y)}$.

Case 2. – $a \notin \{1,2,3,\ldots\}$: In this case we solve the problem formally and then we conclude that the solution converges. First we rewrite $\frac{u^a}{u+y.u'} = r$ as $\frac{(uy)'}{(uy)^a} = \frac{1}{r.y^a}$. We can write $\frac{1}{r(y)} = 1 + a_1 y + a_2 y^2 + \cdots$ in a convergent series. Thus, we have $\frac{(uy)'}{(uy)^a} = \frac{1}{y^a} + \frac{a_1}{y^{a-1}} + \cdots + \frac{a_k}{y^{a-k}} + \cdots$ and since $a \notin \{1,2,3,\ldots\}$ we have the formal solution $\frac{1}{(uy)^{a-1}} = \frac{1}{y^{a-1}} + \frac{a_1}{y^{a-2}} + \cdots$ which gives $u^{a-1} = \frac{1}{1+\frac{a-1}{a-2}\cdot a_1 y + \frac{a-2}{a-3}\cdot a_2 y^2 + \cdots}$; this formal solution is convergent in a neighborhood of $0 \in \mathbf{C}$. In fact, since $1 + a_1 y + a_2^2 + \cdots$ is convergent in some neighborhood of the origin we have that $\lim\sup_{k\to\infty} \sqrt[k]{|a_k|} < +\infty$ and then $\lim\sup_{k\to\infty} \sqrt[k]{\left|\frac{a-k}{a-k-1}\cdot a_k\right|} < \infty$ so that the series $1 + \frac{a-1}{a-2}\cdot a_1 y + \frac{a-2}{a-3}\cdot a_2 y^2 + \cdots$ is convergent in some neighborhood of $0 \in \mathbf{C}$. This proves Claim 1. $\square$

Now given a local coordinate system $(x,y) \in U$ with $\Lambda \cap U = \{y=0\}$ and $\Omega = g\,dy$ we have $\eta = a\frac{dy}{y} + \frac{dg}{g} + \frac{dr}{r}$ for some holomorphic local function $r = r(y)$ with $r(0) = 1$. We define $\hat{y} := u(y).y$ where u is given by the Claim 1 above. Then we have $r(y)d\tilde{y} = u^a(y)dy$. Now, define $\tilde{g} := \frac{g.r(y)}{u^a(y)}$, so we have $\tilde{g}\,d\tilde{y} = g\,dy$ and since $u^a.\tilde{g} = g.r$ we have $\frac{adu}{u} + \frac{d\tilde{g}}{\tilde{g}} = \frac{dg}{g} + \frac{dr}{r}$ and then we have $\frac{ad\tilde{y}}{\tilde{y}} + \frac{d\tilde{g}}{\tilde{g}} = \frac{ady}{y} + \frac{dg}{g} + \frac{dr}{r} = \eta(x,y)$ which proves the first part of case (1). Now we make another claim:

CLAIM 2. – Let $u = u(y)$ be a holomorphic local function defined near $y = 0 \in \mathbf{C}$ with $u(0) \neq 0$. Assume that we have $r.u^a = u + y.u'$ for some $r, a \in \mathbf{C}$. Then u is locally constant provided that $a \notin \{2,3,\ldots\}$. If $a = k \in \{2,3,\ldots\}$ then we have $u^{k-1} = \frac{1}{r+\alpha.y^{k-1}}$ for some $\alpha \in \mathbf{C}$.

Proof of Claim 2. – We write $u(y) = u(0) + u'(0)y + \cdots + \frac{u^{(k)}(0)y^k}{k!} + \cdots$ in convergent power series. Assume that $a \notin \{2,3,\ldots,\}$. Derivating the expression $r.u^a = u + y.\frac{du}{dy}$ and using $r.u^{a-1}(0) \neq 0$ we obtain by induction that $u^{(k)}(0) = 0$, $\forall\, k \geq 1$ and then u is constant. Suppose now $a = k \in \{2,3,\ldots,\}$. From $r.u^k = u + y.\frac{du}{dy}$ we obtain $\frac{(u.y)'}{u^k.y^k} = \frac{r}{y^k}$ and then $\frac{1}{(uy)^{k-1}} = \frac{r}{y^{k-1}} + \alpha$ for some constant $\alpha \in \mathbf{C}$. The claim now follows easily. This is enough to finish the proof of Case 1: In fact, given $(x,y) \in U$, $(\tilde{x},\tilde{y}) \in \widetilde{U}$ such that $\Omega = g\,dy$, $\quad \eta = \frac{ady}{y} + \frac{dg}{g}$, $\quad \Omega = \tilde{g}\,d\tilde{y}$, $\quad \eta = \frac{ad\tilde{y}}{\tilde{y}} + \frac{d\tilde{g}}{\tilde{g}}$, and $U \cap \widetilde{U} \neq \phi$ then writting $\tilde{y} = u.y$ we obtain $r.u^a = u + \frac{y\,du}{dy}$ for some $r \in \mathbf{C}^*$. Using Claim 2 we conclude that: $a \notin \{2,3,\ldots\} \Rightarrow \tilde{y} = c.y$ for some $c \in \mathbf{C}^*$ $a = k \in \{2,3,\ldots\} \Rightarrow \tilde{y}^{k-1} = \frac{\lambda y^{k-1}}{1+\alpha.y^{k-1}}$ for some $\lambda, \alpha \in \mathbf{C}$. This finishes the proof of Case 1.

Case 2. – $\mathrm{Res}_\Lambda\,\eta = k \in \{2,3,\ldots\}$.

Let $(x,y) \in U$ be a local coordinate system such that $\Omega = g\,dy$, $\Lambda = \{y=0\}$ and then $\eta = \frac{kdy}{y} + \frac{dg}{g} + \frac{dr}{r}$ for some holomorphic $r = r(y)$ with $r(0) = 1$ and $U \cap \widehat{U} \neq \phi$.

In $U \cap \widehat{U} \neq \phi$ we have $\hat{g}\,d\hat{y} = g\,dy$ and $\frac{kd\hat{y}}{\hat{y}} + \frac{d\hat{g}}{\hat{g}} = \frac{kdy}{y} + \frac{dg}{g} + \frac{dr}{r}$. This gives $\frac{d\hat{y}}{\hat{y}^k} = c \cdot \frac{dy}{y^k.r(y)}$ for some constant $c \in \mathbf{C}^*$. Therefore $\mathrm{Res}_{\{y=0\}}\left(\frac{dy}{y^k.r(y)}\right) = 0$ and this allows us to use the same proof given for Claim 1 to show that there exists a new coordinate system $(\tilde{x},\tilde{y}) \in U$ such that $\eta(\tilde{x},\tilde{y}) = \frac{kd\tilde{y}}{\tilde{y}} + \frac{d\tilde{g}}{\tilde{g}}$ and $\Omega = \tilde{g}\,d\tilde{y}$. Since Λ is connected this implies that the first part of (2) is true. The last part follows from what we have observed above. $\square$

LEMMA 3.2 (Extension Lemma). – *Let $\mathcal{F}$ be given by Ω on M^2 and let $\Lambda \subset M^2$ be an analytic smooth invariant curve. Suppose:*

(1) *Given any singularity $p \in \Lambda \cap s(\mathcal{F})$ there is a local coordinate system (x,y) such that $p = (0,0)$, $\Lambda = \{y=0\}$ and $\mathcal{F}$ is given by $xdy - \lambda y\,dx = 0$, $\lambda \in \mathbf{C}^* \backslash \mathbf{Q}_+$.*

(2) *One of the singularities, say $p_o \in \Lambda \cap s(\mathcal{F})$, is non-resonant (which means that we have $\lambda \notin \mathbf{Q}$ in* (1).*)*

(3) *There exists a 1-form η defined in some neighborhood of Λ minus Λ and its local separatrices satisfying:* (i) *η is meromorphic and closed;* (ii) *$d\Omega = \eta \wedge \Omega$;* (iii) *$(\eta)_\infty = (\Omega)_\infty$ has order one and $Res_L\,\eta = -$ order of $(\Omega)_\infty$ along L, for each non-invariant component L of $(\Omega)_\infty$.*

Then η extends meromorphically to a neighborhood of Λ as an adapted form to Ω along the curve Λ.

Proof. – Using Hartogs' Extension Theorem (*see* [30]) we conclude that it is enough to prove that η extends as a meromorphic 1-form to a neighborhood of an arbitrary singularity $p \in s(\mathcal{F}) \cap \Lambda$. First we consider the case $p = p_o$ given in (2). Choosing local coordinates (x, y) such that $p_o = (0,0)$, $\Lambda = \{y = 0\}$, $\Omega(x,y) = g(xdy - \lambda y\, dx)$, $\lambda \notin \mathbf{Q}$ we can write $\eta(x,y) = \lambda_1 \frac{dy}{y} + \lambda_2 \frac{dx}{x} + \frac{dg}{g} + df$ with $f \in \vee^*(\{xy \neq 0\})$. From condition $d\Omega = \eta \wedge \Omega$ we conclude that $df \wedge (xdy - \lambda y\, dx) = 0$ and then $xf_x + \lambda y\, f_y = 0$. Using Laurent Series $f = \sum_{i,j \in \mathbf{Z}} f_{ij}\, x^i y^j$ we obtain $(i + \lambda j) \cdot f_{ij} = 0$, $\forall\, (i,j) \in \mathbf{Z}^2$ and since $\lambda \notin \mathbf{Q}$ we obtain $f_{ij} = 0$, $\forall\, (i,j) \neq (0,0)$, so f is constant and $\eta = \lambda_1 \frac{dy}{y} + \lambda_2 \frac{dx}{x} + \frac{dg}{g}$. (It is now easy to check that we also have $1 + \lambda = \lambda_1 \lambda + \lambda_2$. This fact will be used later).

Therefore η extends meromorphically to a neighborhood of p_o having poles of order one. Now, this implies that η extends meromorphically to all $\Lambda \backslash (s(\mathcal{F}) \cap \Lambda)$ having order one polar divisor (Hartogs' Extension Theorem).

Now we fix an arbitrary singularity $p \in \Lambda \cap s(\mathcal{F})$ $p \neq p_o$, and choose (x,y) as in (1). Again we have $\eta = \lambda_1 \frac{dy}{y} + \lambda_2 \frac{dx}{x} + \frac{dg}{g} + df$ and $xf_x + \lambda y\, f_y = 0$ and then $(i + \lambda j) \cdot f_{ij} = 0$, $\forall\, (i,j) \in \mathbf{Z}^2$. Since $(\eta)_\infty$ has order one along $\Lambda \backslash (\Lambda \cap s(\mathcal{F}))$ we have f holomorphic along Λ and then $f_{ij} = 0$, $\forall\, (i,j) \in \mathbf{Z} \times \mathbf{Z}_-$. Now, fixed $j \in \mathbf{Z}_+$ since $\lambda \notin \mathbf{Q}_+$ we have $f_{ij} = 0$, $\forall\, i \in \mathbf{Z}_-$. Thus we have $f_{ij} = 0$, $\forall\, (i,j) \notin \mathbf{Z}_+^2$, so f is holomorphic and this proves the Lemma 3.2. $\square$

We finish this section with a lemma that we will use to linearize some singularities in the proof of the main theorems. Consider $\underline{\underline{\mathcal{F}}}$ a germ of singular foliation on $(\mathbf{C}^2, 0)$ with non-degenerate linear part and in the *Siegel domain*; *i.e.*, $xdy - \lambda y\, dx +$h.o.t., $\lambda \in \mathbf{C}^* \backslash \mathbf{R}_-$. We can assume that $\{y = 0\}$ is a separatrix of $\underline{\underline{\mathcal{F}}}$.

LEMMA 3.3. – *Let $\underline{\underline{\mathcal{F}}}$ be as above and let $p \in (\mathbf{C}^2, 0) \backslash 0$. Suppose that there exists a local transversal section Σ, $\Sigma \cap \{y = 0\} = \{p\}$ and coordinate system $y \in \Sigma$, $y(p) = 0$, such that the holonomy of the local separatrix $\{y = 0\}$ is given by $h(y)^{k-1} = \frac{\mu\, y^{k-1}}{1 + ay^{k-1}}$ for some $k \in \{2, 3, \ldots\}, \mu, a \in \mathbf{C}, \mu^{k-1} \neq 1$. Then $\underline{\underline{\mathcal{F}}}$ can be made linear in some system of coordinates $Z = T(y)$ for some homography T.*

Proof. – It is enough to show that $h \colon (\Sigma, p) \hookleftarrow$ can be made linear in some coordinate system $z \in \Sigma$, $z(p) = 0$ [24]. Let $H \colon \mathbf{P}_1(\mathbf{C}) \to \mathbf{P}_1(\mathbf{C})$ be the homography $H(z) = \frac{\mu^{k-1} z}{1 + az}$. Since $\mu^{k-1} \neq 1$ there exists an other homography T such that if $Z = T(y) \in (\mathbf{C}, 0)$ then $H(Z) = \mu Z$. By the hypothesis we have $h(y)^{k-1} = H(y^{k-1})$ and therefore $h(Z)^{k-1} = \mu^{k-1} \cdot Z^{k-1}$ so that $h(Z) = \mu \cdot Z$. $\square$

4. Statement and proof of the main results

THEOREM 4.1. – *Let $\mathcal{F}$ be a foliation on M^2 having $\Lambda \subset M$ as an analytical connected invariant curve, and given by a meromorphic 1-form Ω. Suppose:*

(i) *all singularities of $\mathcal{F}$ in Λ are of 1^{st}-order;*

(ii) *the foliation $\widetilde{\mathcal{F}}$ obtained as the resolution of the singularities of $\mathcal{F}$ in Λ, has one linearizable non-resonant singularity.*

Then the following conditions are equivalent:

(a) *$\mathcal{F}$ is transversely affine in some neighborhood of Λ minus Λ and its local separatrices.*

(b) *The form Ω admits an adapted form along Λ.*

Moreover, if one of these conditions holds then the holonomy group of Λ *and of any component* P_j *of the desingularization divisor* D *of* $s(\mathcal{F}) \cap \Lambda$ *is either linearizable or is a finite covering of a group of homographies. In the linearizable case there exists a closed meromorphic* 1*-form* $\widetilde{w}_j$ *defined in a neighborhood* $\widetilde{U}_j$ *of* P_j*, with* $(\widetilde{w}_j)_\infty = P_j \cup sep\,(P_j)$*, such that* $\widetilde{\mathcal{F}}\big|_{\widetilde{U}_j}$ *is given by* $\widetilde{w}_j$ *outside the polar divisor* $(\widetilde{w}_j)_\infty$.

We remark that the hypothesis i) above comes from the difficult exhibited by our approach in dealing with the dicritical case (the complementar of the divisor D is not necessarily a Stein manifold so that Levi's Extension Theorem does not apply), and from the fact that we do not know wether an affine transverse structure defined in the complementar of the separatrices of a germ of saddle-node extends to these separatrices in the sense of section 3 above. With respect to hypothesis ii) above, it seems that actually it is possible to construct examples of germs of *resonant* non degenerate singularities, which admit affine transverse structures on the complementar of the two local separatrices, but do not exhibit a *Liouvillian first integrals i.e.*, extended affine structures. The construction of these examples is based on the techniques of [25].

Proof. – The implication (b) $\Rightarrow$ (a) is a straighforward consequence of Proposition 1.1. Now we prove that (a) $\Rightarrow$ (b). Let $\pi\colon \widetilde{M} \to M$, $\widetilde{\mathcal{F}} = \pi^*(\mathcal{F})$ be the resolution of the singular points of $\mathcal{F}$ in Λ and let $D = \pi^{-1}(\Lambda) = \bigcup_j \mathbf{P}_j$, the *desingularizing divisor* of $s(\mathcal{F}) \cap \Lambda$ given by Theorem 2.1. Let $q_{j_o} \in \mathbf{P}_{j_o}$ be a linearizable non-resonant singularity of $\widetilde{\mathcal{F}}$. The foliation $\widetilde{\mathcal{F}}$ is transversely affine in $\widetilde{V}\backslash[(\bigcup_j \mathbf{P}_j) \cup \pi^{-1}(\mathrm{sep}\,(\Lambda))]$ for some neighborhood $\widetilde{V}$ of D in $\widetilde{M}$. Therefore there exists a pair $(\widetilde{\Omega}, \widetilde{\eta})$ with $\widetilde{\Omega} = \pi^*(\Omega)$, $\widetilde{\eta} = \pi^*(\eta)$, and $\widetilde{\eta}$ meromorphic closed in $\widetilde{V}\backslash[(\bigcup_j \mathbf{P}_j) \cup \pi^{-1}(\mathrm{sep}\,(\Lambda))]$ satisfying the conditions stated in Proposition 1.1. Our objective is to show that $\widetilde{\eta}$ extends meromorphically to $\widetilde{V}$. Using Lemma 3.2 we show that $\widetilde{\eta}$ extends meromorphically to $\mathbf{P}_{j_o}$ minus the other singular points of $\widetilde{\mathcal{F}}$ in $\mathbf{P}_{j_o}$. But this extension already allows us to calculate the holonomy of the leaf $\mathbf{P}_{j_o}$ of $\widetilde{\mathcal{F}}$. According to Lemma 3.1 this holonomy is either linearizable or is a finite covering of a group of homographies and in particular given any singular point $q'_{j_o} \in \mathbf{P}_{j_o} \cap s(\widetilde{\mathcal{F}})$ there is a local coordinate $y \in \Sigma$ in any local transversal Σ, such that the holonomy of the separatrix $\mathbf{P}_{j_o}$ in this singularity is of one of the following forms: (A) $h(y) = a.y$, $a \in \mathbf{C}^*$; (B) $h(y)^k = \frac{ay^k}{1+b.y^k}$ $a \in \mathbf{C}^*$, $b \in \mathbf{C}$ (in this case we have $\mathrm{Res}_{\mathbf{P}_{j_o}}\widetilde{\eta} = k+1$).

In case (B) we have two possibilities:

(1) If $a^k \neq 1$: In this case the homography $\left(z \mapsto \frac{a^k z}{1+bz}\right)$ can be made linear in some local coordinate $\widetilde{y} \in \Sigma$ which is obtained by an homography from y and therefore we can assume that $h(y) = \mu y$ as in (A) (*see* Lemma 3.3). Therefore in this case and in case (A) the singularity is linearizable and we use Lemma 3.2 to extend $\widetilde{\eta}$ to the singularity q'_{j_o}.

(2) $a^k = 1$: In this case we have that q'_{j_o} is a singularity of the form $\widetilde{\omega}\big|_{U_\alpha} = g(x\,dy - \lambda\,y\,dx + h.o.t.)$, $\lambda = -\frac{m}{n} \in \mathbf{Q}_-$, $(m,n) = 1$, and we assume that it is nonlinearizable with $(y=0) \subset \mathbf{P}_{j_o}$. The local holonomy h of q'_{j_o} in $\mathbf{P}_{j_o}$ satisfies $h(y)^k = \frac{y^k}{1+ay^k}$. This implies that $mk = ln$ for some $l \in \mathbf{N}$. Furthermore we can assume that $a = \frac{i}{2\pi}$ and so this holonomy is conjugated to the holonomy of $(y=0)$ of the germ of foliation $\omega_{k,l} = l\,x\,dy + k\,y(1+\frac{i}{2\pi}x^k y^l)dx = 0$. Thus by [25] the foliation $\widetilde{\mathcal{F}}$ is conjugated near q'_{j_o} to the germ of foliation $\omega_{k,l}$. Thus there are local coordinates (x_α, y_α) centered at

$q'_{j_o} \in \mathbf{P}_{j_o}$ such that for some meromorphic function $\widetilde{g}_\alpha$, $\widetilde{\omega}\big|_{U_\alpha} = \widetilde{g}_\alpha \omega_{k,l}$ (we observe that $\omega_{k,l} = \widehat{g}_{k,l}\, d\widehat{y}_{k,l}$ where $\widehat{y}_{k,l} = \frac{y_\alpha^l x_\alpha^k}{\frac{ik}{2\pi} x_\alpha^k y_\alpha^l \log x_\alpha - 1}$ and $\widehat{g}_{k,l} = -\frac{(\frac{ik}{2\pi} x_\alpha^k y_\alpha^l \log x_\alpha - 1)^2}{x_\alpha^{k-1} y_\alpha^{l-1}}$). We define $\eta_{j_o,\alpha} := (k+1)\frac{dy_\alpha}{y_\alpha} + (l+1)\frac{dx_\alpha}{x_\alpha} + \frac{d\widetilde{g}_\alpha}{\widetilde{g}_\alpha}$. The form $\widetilde{\eta}$ extends to q'_{j_o} as $\eta_{j_o,\alpha}$ because both define affine transverse structures outside the axis $(x = 0), (y = 0)$ and have the same residue $(k+1)$ along the axis $(y = 0)$.

Thus we have showed that the form $\widetilde{\eta}$ extends meromorphically to all $\mathbf{P}_{j_o}$ and given any projective $\mathbf{P}_j$ with $\mathbf{P}_j \cap \mathbf{P}_{j_o} \neq \phi$, then $\widetilde{\eta}$ extends meromorphically to $\mathbf{P}_j$ minus the other singularities of $\widetilde{\mathcal{F}}$ in $\mathbf{P}_j$. Using these arguments we can show that $\widetilde{\eta}$ extends meromorphically to all the divisor $D = \bigcup_j \mathbf{P}_j$ and the local separatrices $\pi^{-1}(\operatorname{sep}(\Lambda))$. This shows that (a) $\Rightarrow$ (b). The last part of Theorem 4.1 is a consequence of the following remarks:

(I): Let $(x, y) \in U$ be a local chart around any linearizable singularity of $\widetilde{\mathcal{F}}$ in $\mathbf{P}_j$, say, a singular point $p \in \mathbf{P}_j \cap s(\mathcal{F})$, such that we have $\widetilde{\Omega}(x, y) = g(x dy - \lambda y\, dx)$, $\lambda \in \mathbf{C} \backslash \mathbf{Q}_+$. Let $(q, 0) \in U \backslash \{p\}$ be any regular point of $\widetilde{\mathcal{F}}$ and choose V a small neighborhood of $(q, 0)$ such that $V \cap \{x = 0\} = \phi$ and $V \cap \mathbf{P}_j$ is simply-connected. Let $\hat{x} = x - q$, $\hat{y} = yx^{-\lambda}$, $\hat{g} = gx^{1+\lambda}$. Then $(\hat{x}, \hat{y})$ define new coordinates in V such that $\mathbf{P}_j \cap V = \{\hat{y} = 0\}$ and $\Omega = \hat{g}\, d\hat{y}$. Now, since the singularity is linearizable it follows (as remarked in the proof of Lemma 3.2) that we have $\widetilde{\eta}(x, y) = a\frac{dy}{y} + b\frac{dx}{x} + \frac{dg}{g}$, where $1 + \lambda = a \cdot \lambda + b$. Therefore we have $\widetilde{\eta}(x, y) = a\frac{dy}{y} + (1 + \lambda)\frac{dx}{x} - a\lambda\frac{dx}{x} + \frac{dg}{g} = a\frac{d(yx^{-\lambda})}{yx^{-\lambda}} + \frac{d(gx^{1+\lambda})}{gx^{1+\lambda}} = a\frac{d\hat{y}}{\hat{y}} + \frac{d\hat{g}}{\hat{g}}$ in V.

(II): For the linearizable case we assume that $a \notin \{2, 3, \ldots\}$. Notice that since the holonomy is linearizable all the singularities are linearizable ([24] and [25]). Using now Lemma 3.1 we can conclude that there exists a family of local charts $(x_\alpha, y_\alpha) \in U_\alpha$ with the U_α's covering a neighborhood of $\mathbf{P}_j$, such that: (1) $\mathbf{P}_j \cap U_\alpha = \{y_\alpha = 0\}$, $\forall \alpha$; (2) If $\widetilde{\mathcal{F}}\big|_{U_\alpha}$ is regular it is given by $dy_\alpha = 0$, and if $\widetilde{\mathcal{F}}\big|_{U_\alpha}$ is singular it is given by $x_\alpha\, dy_\alpha - \lambda_\alpha\, y_\alpha\, dx_\alpha = 0$, $\lambda_\alpha \in \mathbf{C}\backslash\mathbf{Q}$; (3) If $U_\alpha \cap U_\beta \neq \phi$; and $(U_\alpha \cup U_\beta) \cap s(\mathcal{F}) = \phi$ then we have $y_\alpha = c_{\alpha\beta} . y_\beta$ for some $c_{\alpha\beta} \in \mathbf{C}^*$ and if $U_\alpha \cap s(\mathcal{F}) \neq \phi$ then $U_\beta \cap s(\mathcal{F}) = \phi$ and we have $y_\alpha x_\alpha^{-\lambda_\alpha} = c_{\alpha\beta} . y_\beta$ for some $c_{\alpha\beta} \in \mathbf{C}^*$.

(III) Finally, we proceed as in [13] and [6]. We define local meromorphic forms w_α in the U_α's by: $w_\alpha(x_\alpha, y_\alpha) := \frac{dy_\alpha}{y_\alpha}$ if $\mathcal{F}\big|_{U_\alpha}$ is regular; $w_\alpha(x_\alpha, y_\alpha) := \frac{dy_\alpha}{y_\alpha} - \lambda_\alpha\frac{dx_\alpha}{x_\alpha}$ if $\mathcal{F}\big|_{U_\alpha}$ is singular. Using condition (3) above we have $w_\alpha = w_\beta$ in each $U_\alpha \cap U_\beta \neq \phi$ and thus we have defined a closed meromorphic 1-form $\widetilde{w}_j$, (which defines $\widetilde{\mathcal{F}}$) in a neighborhood of $\mathbf{P}_j$, having order one polar divisor $(\widetilde{w}_j)_\infty = \mathbf{P}_j \cup \operatorname{sep}(\mathbf{P}_j)$. $\square$

Remark 4.1. *Generalized Levi's Extension Theorem.*

Let M be a compact complex manifold (of dimesion ≥ 2), and let $\Lambda \subset M$ be an analytic subset of codimension one, such that $M \backslash \Lambda$ is a Stein manifold. Then any meromorphic differential q-form defined in a neighborhood of Λ extends meromorphically to M.

This is a consequence of Levi's extension Theorem [30] (*see* [6] Lemma 5 Section 3).

In particular if $\Lambda \subset \mathbf{C}P(n)$ is an algebraic hypersurface then $\mathbf{C}P(n)\backslash\Lambda$ is a Stein manifold [30] *and any meromorphic differential q-form ω defined in a neighborhood of Λ extends meromorphically to $\mathbf{C}P(n)$.*

Remark 4.2. – Let $\mathcal{F}$ be a foliation on $\mathbf{C}P(2)$. The foliation $\mathcal{F}$ has *degree* n if and only if in affine coordinates $(x, y) \in \mathbf{C}^2 \hookrightarrow \mathbf{C}P(2)$, $\mathcal{F}$ is given by $\Omega = P\, dy - Q\, dx = 0$

where $P = \sum_{j=0}^{n} P_j + x.g$, $Q = \sum_{j=0}^{n} Q_j + y.g$ where P_j, Q_j are homogeneous polynomials of degree j and g is an homogeneous polynomial of degree $n+1$. Geometrically the degree of $\mathcal{F}$ is the number of tangencies of its leaves with a generic projective line $\mathbf{C}P(1) \subset \mathbf{C}P(2)$ (*see* [22]). If Ω is like above and the line $L_\infty = \mathbf{C}P(2)\backslash\mathbf{C}^2$ is not invariant then $(\Omega)_\infty = L_\infty$ has order $= \deg\mathcal{F} + 2$. The *Poincare Problem* for foliations on $\mathbf{C}P(2)$ is to bound the degree of a projective foliation $\mathcal{F}$ in terms of the degree of an algebraic solution $S \subset \mathbf{C}P(2)$ of $\mathcal{F}$ (*see* [22] and [9]). In the non-dicritical case it is proved that $\deg\mathcal{F} \leq \deg S + 2$ [9].

The next theorem proves that we have an equality in the "Poincare Problem" for a foliation under our assumptions. We refer to [9], [13] and [22] for any further information on this subject.

THEOREM 4.2. – *Let $\mathcal{F}$ be a foliation on $\mathbf{C}P(2)$ and let Λ be a smooth algebraic curve invariant by $\mathcal{F}$. Suppose: (i) all singularities of $\mathcal{F}$ in Λ are of first order;* (ii) *the foliation $\widetilde{\mathcal{F}}$ obtained by the resolution of $s(\mathcal{F})\cap\Lambda$ has at least one linearizable non-resonant singularity;* (iii) *$\mathcal{F}$ is transversely affine in some neighborhood of Λ minus Λ and its local separatrices.*

Then: (a) *$\mathcal{F}$ has a finite number of algebraic leaves; let* Sep$(\mathcal{F})$ *denote this set:* (b) $\deg\,\mathcal{F} + 2 =$ *degree of* Sep$(\mathcal{F})$.

Proof of Theorem 4.2. – The proof is based on the Index Theorem [5] and the Residue Theorem. Let $\pi\colon \widetilde{M} \to \mathbf{C}P(2)$, $\widetilde{\mathcal{F}} = \pi^*(\mathcal{F})$, be the desingularization of the singularities of $\mathcal{F}$ in Λ. Let $\pi^{-1}(s(\mathcal{F})) = \bigcup_j \mathbf{P}_j = D$ denote the divisor D of the desingularization and let $\widetilde{\Lambda}$ denote the curve $\overline{\pi^{-1}(\Lambda\backslash(\Lambda \cap s(\mathcal{F})))}$. It follows from Theorem 4.1 that $\mathcal{F}$ can be given by a meromorphic 1-form Ω which admits an adapted form along Λ, say η, defined in all $\mathbf{C}P(2)$. We have $(\eta)_\infty =$ Sep$(\mathcal{F})$ which proves (a).

By the Integration Lemma (*see* Example 1.6) we have $\eta = \sum_j \lambda_j \frac{df_j}{f_j} - n.\frac{dg}{g}$ in some affine chart $(x, y) \in \mathbf{C}^2 \hookrightarrow \mathbf{C}P(2)$, where f_j and g are polynomials transverse to $\mathbf{C}P(2)\backslash\mathbf{C}^2$, Sep$(\mathcal{F}) = \bigcup_j \overline{(f_j = 0)}$, $(\Omega)_\infty = (g = 0)$, g has degree one, $n =$ order of $(\Omega)_\infty = \deg\,\mathcal{F}+2$. The Residue Theorem shows that (1) $\sum_j \lambda_j \deg\, f_j = n = \deg\,\mathcal{F}+2$. Let $\Lambda_1 = \Lambda = \overline{(f_1 = 0)}$ and $\Lambda_j = \overline{(f_j = 0)}; \forall j \neq 1$. and $\widetilde{\Lambda}_j = \overline{\pi^{-1}(\Lambda_j\backslash(\Lambda_j \cap s(\mathcal{F})))}$. Now we fix a singularity $p \in \widetilde{\Lambda}_j \cap D$, say, $p \in \widetilde{\Lambda}_j \cap \mathbf{P}_\nu$. We know that the residue λ_j and the index ind$(p; \mathbf{P}_\nu)$ are related by the formula: $\lambda_j = 1 + (1 - a_\nu).\mathrm{ind}(p; \mathbf{P}_\nu)$ where $a_\nu =$ $\mathrm{Res}_{\mathbf{P}_\nu}(\pi^*\eta)$. (In fact as in the first part of the proof of Lemma 3.2 we have that $1 + \mathrm{ind}(p; \mathbf{P}_\nu) = \mathrm{ind}(p; \mathbf{P}_\nu).a_\nu + \mathrm{Res}_{\widetilde{\Lambda}_j}\eta)$. Hence we have $\mathrm{ind}(p; \mathbf{P}_\nu) = -\frac{\lambda_j - 1}{a_\nu - 1}$ for each singularity $p \in \mathbf{P}_\nu \cap \widetilde{\Lambda}_j$. Let $w(\mathbf{P}_\nu)$ denote the *weight* of the projective $\mathbf{P}_\nu$ in the desingularization process, that is, the number of times that we have blowed-up points over $\mathbf{P}_\nu$ plus one; we have $-w(\mathbf{P}_\nu) :=$ first Chern class of $\mathbf{P}_\nu$ in $\widetilde{M}$. Using Camacho-Sad Index Theorem [5] we obtain $-w(\mathbf{P}_\nu) = \sum_{p\in s(\widetilde{\mathcal{F}})\cap\mathbf{P}_\nu} \mathrm{ind}(p; \mathbf{P}_\nu)$. Now we have

$$\sum_{p\in s((\widetilde{\mathcal{F}})\cap\mathbf{P}_\nu} = \sum_{\substack{p\in s(\widetilde{\mathcal{F}})\cap\mathbf{P}_\nu \\ p\in\mathbf{P}_\nu\cap\bigcup_{j\neq 1}\widetilde{\Lambda}_j}} + \sum_{\substack{p\in s(\widetilde{\mathcal{F}})\cap\mathbf{P}_\nu \\ p\in\mathbf{P}_\mu, \mu\neq\nu}} + \sum_{\substack{p\in\mathbf{P}_\nu\cap s(\widetilde{\mathcal{F}}) \\ p\in\widetilde{\Lambda}}} .$$

Therefore we obtain

$$-w(\mathbf{P}_\nu) = \sum_{j\neq 1} -\#(\mathbf{P}_\nu \cap \widetilde{\Lambda}_j).\frac{\lambda_j - 1}{a_\nu - 1}$$
$$- \#(\mathbf{P}_\nu \cap \widetilde{\Lambda})\frac{(\lambda_1 - 1)}{a_\nu - 1} - \sum_{\substack{\mathbf{P}_\mu \cap \mathbf{P}_\nu \neq \phi \\ \mu\neq\nu}} \#(\mathbf{P}_\mu \cap \mathbf{P}_\nu).\frac{a_\mu - 1}{a_\nu - 1}$$

and then

$$(2)\; w(\mathbf{P}_\nu).(a_\nu - 1) = \sum_{j\neq 1} \#(\mathbf{P}_\nu \cap \widetilde{\Lambda}_j)(\lambda_j - 1)$$
$$+ \#(\mathbf{P}_\nu \cap \widetilde{\Lambda}).(\lambda_1 - 1) + \sum_{\mu\neq\nu} \#(\mathbf{P}_\nu \cap \mathbf{P}_\mu).(a_\mu - 1).$$

Now we sum over all $\mathbf{P}_\nu$ obtaining

$$\sum_\nu w(\mathbf{P}_\nu).(a_\nu - 1) = \sum_{j\neq 1}(\sum_\nu \#(\mathbf{P}_\nu \cap \widetilde{\Lambda}_j))(\lambda_j - 1)$$
$$+ (\lambda_1 - 1).\sum_\nu \#(\mathbf{P}_\nu \cap \widetilde{\Lambda}) + \sum_{\mu,\nu\, \mu\neq\nu} \#(\mathbf{P}_\nu \cap \mathbf{P}_\mu).(a_\mu - 1).$$

We observe that:

(a) $\sum_\nu \#(\mathbf{P}_\nu \cap \widetilde{\Lambda}_1))(\lambda_1 - 1) = \#(s(\mathcal{F}) \cap \Lambda)(\lambda_1 - 1)$

(b) $\sum_{j\neq 1}(\sum_\nu \#(\mathbf{P}_\nu \cap \widetilde{\Lambda}_j))(\lambda_j - 1)$

$$= \sum_{j\neq 1}(\lambda_j - 1).\deg\Lambda_j.\deg f_1 = \deg f_1.[\sum_{j\neq 1}\lambda_j.\deg f_j - \sum_{j\neq 1}\deg f_j]$$
$$= \deg f_1.[\deg\mathcal{F} + 2 - \lambda_1.\deg f_1 - \sum_{j\neq 1}\deg f_j].$$

(c) $\sum_{\substack{\mu,\nu \\ \mu\neq\nu}} \#(\mathbf{P}_\nu \cap \mathbf{P}_\mu).(a_\mu - 1)$

$$= \sum_{\substack{\mu,\nu \\ \mu\neq\nu}} \#(\mathbf{P}_\nu \cap \mathbf{P}_\mu).(a_\nu - 1)$$
$$= \sum_{\substack{\nu\neq\mu \\ \mathbf{P}_\nu\cap\widetilde{\Lambda}\neq\phi}} (a_\nu - 1)\#(\mathbf{P}_\nu \cap \mathbf{P}_\mu) + \sum_{\substack{\nu\neq\mu \\ \mathbf{P}_\nu\cap\widetilde{\Lambda}=\phi}} (a_\nu - 1).\#(\mathbf{P}_\nu \cap \mathbf{P}_\mu).$$
$$= \sum_{\mathbf{P}_\nu\cap\widetilde{\Lambda}=\phi} w(\mathbf{P}_\nu).(a_\nu - 1) + \sum_{\mathbf{P}_\nu\cap\widetilde{\Lambda}\neq\phi} (w(\mathbf{P}_\nu) - 1).(a_\nu - 1)$$
$$= \sum_\nu w(\mathbf{P}_\nu).(a_\nu - 1) - \sum_{\mathbf{P}_\nu\cap\widetilde{\Lambda}\neq\phi} (a_\nu - 1).$$

Now, using (1), (2), (a), (b) and (c) we obtain (*) $0 = \deg f_1.[\deg\mathcal{F} + 2 - \lambda_1.\deg f_1 - \sum_{j\neq 1}\deg f_j] + (\lambda_1 - 1).\#(s(\mathcal{F}) \cap \Lambda) - \sum_{\mathbf{P}_\nu\cap\widetilde{\Lambda}\neq\phi}(a_\nu - 1)$. Now applying the Index Theorem for the curve $\widetilde{\Lambda}$ we obtain: $(\deg f_1)^2 - \#(s(\mathcal{F}) \cap \Lambda) = \sum_{\mathbf{P}_\nu\cap\widetilde{\Lambda}\neq\phi}\operatorname{ind}(p_\nu, \widetilde{\Lambda})$, where $\mathbf{P}_\nu \cap \widetilde{\Lambda} = \{p_\nu\}$ and $\operatorname{ind}(p_\nu, \widetilde{\Lambda}) = -\frac{a_\nu - 1}{\lambda_1 - 1}$. Thus we have $\sum_{\mathbf{P}_\nu\cap\widetilde{\Lambda}\neq\phi}(a_\nu - 1) =$

$(\lambda_1 - 1).[(\deg f_1)^2 - \#(s(\mathcal{F}) \cap \Lambda)]$. Using this last equation and (*) we obtain $0 = \deg f_1.[\deg \mathcal{F} + 2 - \sum_{j\geq 1} \deg f_j]$ and then $\deg \mathcal{F} + 2 = \sum_{j\geq 1} \deg f_j = \deg \mathrm{Sep}\,(\mathcal{F})$. $\square$

In the following theorem we make hypothesis on all the singularities of $\mathcal{F}$ lying over algebraic leaves.

THEOREM 4.3. – *Let $\mathcal{F}$, Λ be as in Theorem 4.2. Suppose: (i) all singularities of $\mathcal{F}$ lying over algebraic leaves of $\mathcal{F}$, are non-degenerate of the form $xdy - \lambda ydx + h.o.t. = 0$, $\lambda \in \mathbf{C}\backslash\mathbf{Q}_+$; (ii) at least one of the singularities of $\mathcal{F}$ in Λ is linearizable non-resonant; (iii) $\mathcal{F}$ is transversely affine in some neighborhood of Λ minus Λ and its local separatrices.*

Then $\mathcal{F}$ is a logarithmic foliation and $\deg \mathcal{F} + 2 = \deg Sep\,(\mathcal{F})$.

Proof. – As in the proof of Theorem 4.2, given any affine chart $(x,y) \in \mathbf{C}^2 \hookrightarrow \mathbf{C}P(2)$ such that the line $\mathbf{C}P(2)\backslash\mathbf{C}^2$ is not invariant and given a polynomial 1-form $\Omega = P\,dy - Q\,dx$ which defines $\mathcal{F}$ in $\mathbf{C}^2$, we can find a meromorphic 1-form η defined in a neighborhood of Λ in $\mathbf{C}P(2)$ and adapted to Ω along this curve. Since $\mathbf{C}P(2)\backslash\Lambda$ is a Stein manifold, η extends meromorphically to all $\mathbf{C}P(2)$ (Remark 4.1). As in the proof of Theorem 4.2 we have $\eta = \sum_j \lambda_j \frac{df_j}{f_j}$ where $\mathrm{Sep}\,(\mathcal{F}) \cap \mathbf{C}^2 = \bigcup(f_j = 0)$ and $\sum_j \lambda_j \cdot \deg(f_j) = \deg \mathcal{F} + 2$ as a consequence of the Residue Theorem. Now according to Theorem 4.2 we have $\Sigma \deg f_j = \deg \mathcal{F} + 2$ and then $\sum_j (\lambda_j - 1) \cdot \deg f_j = 0$ and this shows that $\lambda_{j_o} \notin \{2, 3, \ldots\}$ for some j_o. Using now Theorem 4.1 we conclude that the algebraic leaf $\Lambda_{j_o} = \overline{(f_{j_o} = 0)}$ of $\mathcal{F}$ has a linearizable holonomy in the same way that in the proof of Theorem 4.1.

Therefore, (since the singularities of $\mathcal{F}$ on Λ are already reduced) according to Theorem 4.1 and to Remark 4.1, $\mathcal{F}$ is defined in $\mathbf{C}P(2)$ by a closed meromorphic 1-form w having *order one* polar divisor $(w)_\infty = \mathrm{Sep}\,(\mathcal{F})$. By the Integration Lemma w is a logarithmic 1-form. $\square$

Remark 4.3. – We remark that Theorem 4.3 still holds (and with the same proof) if we replace condition i) by: (i')*all the singularities of $\mathcal{F}$ lying on some algebraic leaf of $\mathcal{F}$ are of first order and exhibit local meromorphic integrating factors* (that is, the foliation is given by a closed meromorphic local 1-form in a neighborhood of a singularity): In fact using the abelian holonomy of a leaf $\Lambda_{j_o}\backslash s(\mathcal{F})$ as in the proof above we can glue the local closed meromorphic 1-forms given by the local integrating factors around the singularities, in order to obtain a closed meromorphic 1-form ω which describes the foliation $\mathcal{F}$ in a neighborhood of the algebraic curve Λ_{j_o} (*see* [13] or [6] for a similar procedure). Thus we obtain:

THEOREM 4.3$'$. – *Let $\mathcal{F}$, Λ be as in Theorem 4.2. Suppose: (i) all singularities of $\mathcal{F}$ lying over algebraic leaves of $\mathcal{F}$, are of first order and admit local meromorphic integrating factors; (ii) at least one of the singularities of $\mathcal{F}$ in Λ is linearizable non-resonant; (iii) $\mathcal{F}$ is transversely affine in some neighborhood of Λ minus Λ and its local separatrices.*

Then $\mathcal{F}$ is given by a closed rational 1-form ω on $\mathbf{C}P(2)$ and $\deg \mathcal{F} + 2 = \deg Sep\,(\mathcal{F})$.

Finally, we remark that in the next results we do *not* require that $\mathcal{F}$ exhibits a linearizable singularity in its desingularization. However we suppose that $\mathcal{F}$ is transversely affine in all $\mathbf{C}P(n)$ minus the algebraic invariant set S of codimension one.

THEOREM 4.4. – *Let $\mathcal{F}$ be a codimension one foliation on $\mathbf{C}P(n)$ which is transversely affine outside an algebraic codimension one invariant set $S \subset \mathbf{C}P(n)$. Suppose that $\mathcal{F}$ has only 1^{st}-order singularities in some component S_o of S. Then* $\deg \mathcal{F} + 2 = \deg S$.

THEOREM 4.5. – *Let $\mathcal{F}$, S be as in Theorem 4.4 above. Suppose that $\mathcal{F}$ has only non-degenerate singularities in S. Then $\mathcal{F}$ is given by a closed rational 1-form on $\mathbf{C}P(2)$ and* $\deg \mathcal{F} + 2 = \deg S$. *The foliation $\mathcal{F}$ is a logarithmic foliation on $\mathbf{C}P(n)$ provided that it exhibits only non-resonant singularities on S.*

We would like to call the reader's attention to the fact that both Theorems 4.4 and 4.5 above are stated for codimension one foliations on $\mathbf{C}P(n)$. We recall that according to [6] a codimension one foliation $\mathcal{F}$ on $\mathbf{C}P(n)$ is said to have only *non-dicritical singularites* in some algebraic codimension one invariant set $S \subset \mathbf{C}P(n)$ if there exists a linearly embedded $\mathbf{E} = \mathbf{C}P(2) \hookrightarrow \mathbf{C}P(n)$ in *general position with respect to $\mathcal{F}$ (see* [6] for a definition), such that the induced foliation $\mathcal{F}^* = \mathcal{F}\big|_{\mathbf{C}P(2)}$ (by the inclusion $i: \mathbf{E} \to \mathbf{C}P(n)$) has codimension ≥ 2 singular set in $\mathbf{C}P(2)$ and has only non-dicritical singularities in $S^* = i^{-1}(S) \subset \mathbf{C}P(2)$. Proceeding the same way we say that $\mathcal{F}$ *has only* 1^{st} *(non-resonant) singularities in* S if $\mathcal{F}^*$ has only 1^{st} (non-resonant) singularities in S^*.

Proof of Theorem 4.4. – We can assume that $n = 2$: In fact if $\mathcal{F}$ is a codimension one foliation on $\mathbf{C}P(n)$ then given a generic linearly embedded $\mathbf{C}P(2) \hookrightarrow \mathbf{C}P(n)$ the induced foliation $\mathcal{F}^* = \mathcal{F}\big|_{\mathbf{C}P(2)}$ has the same degree that $\mathcal{F}$. Moreover the singular set of $\mathcal{F}^*$ consists of the intersection $s(\mathcal{F}) \cap \mathbf{C}P(2)$ and of the tangencies of $\mathcal{F}$ with $\mathbf{C}P(2)$. The tangencies of $\mathcal{F}$ with $\mathbf{C}P(2)$ originate singularities wich have a local holomorphic first integral (in fact if $p \in \mathbf{C}P(n)\backslash s(\mathcal{F})$ then $\mathcal{F}$ has a local holomorphic first integral at p) and thus these are non-dicritical singularities. This shows that $\mathcal{F}^*$ has only non-dicritical singularities in $S \cap \mathbf{C}P(2)$. Thus we assume $n = 2$. Let $\Omega = P\,dy - Q\,dx$ be a polynomial 1-form which defines $\mathcal{F}$ in affine coordinates $(x, y) \in \mathbf{C}^2$ as in the proof of Theorem 4.2, with S transverse to the line $\mathbf{C}P(2)\backslash\mathbf{C}^2$. Write $S \cap \mathbf{C}^2 = \bigcup_j (f_j = 0)$ f_j irreducible polynomial relatively prime with f_i for $i \neq j$. Since $\mathcal{F}$ is transversely affine in $\mathbf{C}P(2)\backslash S$ we have a 1-form η defined in $\mathbf{C}P(2)\backslash S$, closed and meromorphic with polar divisor $(\eta)_\infty = (\Omega)_\infty = (\mathbf{C}P(2)\backslash\mathbf{C}^2)$ and satisfying the conditions stated in Proposition 1.1. By the Integration Lemma we have $\eta = \sum_j \lambda_j \frac{df_j}{f_j} + \frac{dF}{F}$ for some holomorphic $F: \mathbf{C}^2\backslash S \to \mathbf{C}^*$. By the Residue Theorem we have $(*)$ $\sum_j \lambda_j \deg f_j = \deg \mathcal{F} + 2$. Now we remark that the arguments used in the proof of Theorem 4.2 can be repeated in this case using equation (*) above even in the non-linearizable case (notice that we suppose the singularities to be of 1^{st}-order). Thus, we leave the rest of the proof to the reader. $\square$

Proof of Theorem 4.5. – According to [6] if a codimension one foliation $\mathcal{F}$ on $\mathbf{C}P(n)$ is such that $\mathcal{F}\big|_{\mathbf{C}P(2)}$ is (given by a closed rational 1-form) a logarithmic foliation for some linearly embedded $\mathbf{C}P(2) \hookrightarrow \mathbf{C}P(n)$, in general position with respect to $\mathcal{F}$, then $\mathcal{F}$ is (given by a closed rational 1-form) a logarithmic foliation on $\mathbf{C}P(n)$. Therefore we will assume, as in the proof of Theorem 4.4, that $n = 2$. Let $\Omega = P\,dy - Q\,dx$, $\eta = \sum \lambda_j \frac{df_j}{f_j} + \frac{dF}{F}$ be as in the proof of Theorem 4.3 above. Since $\sum \lambda_j \deg f_j = \deg \mathcal{F} + 2$ and $\sum \deg f_j = \deg \mathcal{F} + 2$ we have $\sum(\lambda_j - 1) \deg f_j = 0$ and then there exists $\lambda_{j_o} \notin \{2, 3, \ldots\}$. Now we put $\Omega' = F.\Omega$ and $\eta' = \Sigma \lambda_j \frac{df_j}{f_j} = \eta - \frac{dF}{F}$. Then, according to

Proposition 1.1, the pair (Ω', η') defines the same affine structure for $\mathcal{F}$ in $\mathbf{C}P(2)\backslash S$ and in this case η' is meromorphic in $\mathbf{C}P(2)$.

CLAIM. – For each regular point $p \in \Lambda_{j_o}\backslash s(\mathcal{F})$ there exists a local chart $(x, y) \in U$ such that $p = (0,0)$, $\Lambda_{j_o} \cap U = \{y = 0\}$, $\Omega' = F.g\,dy$ and $\eta' = \lambda_{j_o} \cdot \frac{dy}{y} + \frac{dg}{g}$. Furthermore if $(\widetilde{x}, \widetilde{y}) \in \widetilde{U}$ is another such chart with $\widetilde{x}(\widetilde{p}) = \widetilde{y}(\widetilde{p}) = 0$, $U \cap \widetilde{U} \neq \phi$ then we have $\widetilde{y} = c.y$ for some $c \in \mathbf{C}^*$.

This claim is proved as Lemma 3.1 (1) because $\lambda_{j_0} \notin \{2, 3, \ldots\}$. Using the claim we prove that the holonomy of the algebraic leaf $\overline{(f_{j_o} = 0)} = \Lambda_{j_o}$ is linearizable in the sense of Theorem 4.1. Proceeding as in Theorem 4.3 we prove that $\mathcal{F}$ is a logarithmic foliation. □

The same way we prove Theorem 4.3′ we can prove:

THEOREM 4.5′. – *Let $\mathcal{F}$, S be as in Theorem* 4.4. *Suppose that all singularities of $\mathcal{F}$ lying over S are of first order and admit local meromorphic integrating factors. Then $\mathcal{F}$ is given by a closed rational 1-form ω on* $\mathbf{C}P(2)$ *and* $\deg \mathcal{F} + 2 = \deg Sep\,(\mathcal{F})$.

5. Solvable holonomy groups and transversely affine foliations

A subgroup $G \subset \mathrm{Bih}(\mathbf{C}, 0)$ is *solvable* if the group of commutators $[G, G]$ is an abelian group. In particular any abelian subgroup $G \subset \mathrm{Bih}(\mathbf{C}, 0)$ is a solvable group. A less trivial example of solvable groups is given by the subgroups $G \subset \mathbf{H}_k$ where $\mathbf{H}_k = \{g \in \mathrm{Bih}(\mathbf{C}, 0)/g(z) = \frac{\lambda z}{\sqrt[k]{1+az^k}}\,; \lambda, a \in \mathbf{C}\}$, $k \in \mathbf{N}$. A theorem of Cerveau-Moussu ([14]) states that except for some exceptional cases these are the only non-commutative solvable groups. Let $\mathcal{F}$ be a foliation on M^2 and let $\Lambda \subset M^2$ be an analytical invariant curve. Under generic hypothesis on $s(\mathcal{F}) \cap \Lambda$, $\mathcal{F}$ is transversely affine in some neighborhood of Λ minus Λ and its local separatrices if and only if the holonomy of Λ is a solvable group in a strong way which we define below:

DEFINITION 5.1. – Assume that $s(\mathcal{F}) \cap \Lambda$ is non-dicritical. We say that *the holonomy of Λ has the property* $(\mathcal{S})$ if:

(i) the holonomy group G_i of each component P_i of the divisor obtained in the desingularization $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ on Λ; is either an abelian *analytically normalizable* group (that is, the group embedds in the flow of a holomorphic vector field on $(\mathbf{C}, 0)$), or a solvable normalizable group $G_i \hookrightarrow \mathbf{H}_{k_i}$ as above.

(ii) We have the following compatibility condition: Given any *corner* $\{q\} = P_i \cap P_j$, such that $\widetilde{\mathcal{F}}$ has a holomorphic first integral in a neighborhood of q, say $x^q y^p$ with $P_i = (x = 0)$ and $P_j = (y = 0)$; then, if the holonomy group G_j of P_j is nonabelian $G_j \subset \mathbf{H}_{k_j}$, we have $p\big|(k_j q)$ in $\mathbf{N}$. In the case both groups are nonabelian, if we take normalizing coordinates z and w such that the holonomy groups of G_i, and G_j are of the form $z \mapsto \frac{\lambda z}{\sqrt[k_i]{1+az^{k_i}}}$ and $w \mapsto \frac{\lambda w}{\sqrt[k_j]{1+aw^{k_j}}}$, respectively then (via the *Dulac correspondence* which is defined by the local first integral) we have $z^{k_i} = \frac{\alpha w^{k_j}}{1+\beta w^{k_j}}$ for some homography $x \mapsto \frac{\alpha x}{1+\beta x}$.

PROPOSITION 5.1. – *Let $\mathcal{F}$, M and Λ be as in Theorem* 4.1. *Assume that each component D_j of the desingularizing divisor D of $s(\mathcal{F}) \cap \Lambda$ exhibits some non-resonant linearizable singularity. Then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is transversely affine in some neighborhood of Λ minus Λ and its local separatrices.*
(ii) *The holonomy of Λ has the property $(\mathcal{S})$.*

In particular if $M\backslash\Lambda$ is a Stein manifold with M compact then any local separatrix of $\mathcal{F}$ through some singular point in $s(\mathcal{F})\cap\Lambda$ is the germ of a global analytic separatrix of $\mathcal{F}$ in M, provided that (i) *or* (ii) *holds.*

The proof of the Proposition 5.1 is based on the refered characterization of Cerveau-Moussu (*see* [14]) and in the following lemma whose proof is a straighforward calculation left to the reader.

LEMMA 5.1. – *Let $G \subset Bih(\mathbf{C},0)$ be a subgroup such that:*
(i) *There exists a holomorphic coordinate $y \in (\mathbf{C},0)$, $y(0)=0$ such that each element $g \in G$ is of the form $g(y) = \frac{\lambda_g . y}{\sqrt[k]{1+a_g . y^k}}$; $a_g \in \mathbf{C}, \lambda_g \in \mathbf{C}^*$, where $k \in \{1,2,\ldots\}$ is independent of g;* (ii) *G contains a non-periodic linearizable element, say, $g_o \in G$, $g_o(z) = \lambda_o \cdot z + h.o.t., \lambda_o^n \neq 1, \forall n \in \mathbf{N}^*$. Then there exists a holomorphic coordinate $z \in (\mathbf{C},0)$, $z(0)=0$, such that $g_o(z)=\lambda_o . z$, and each $g \in G$ is of the form $g(z) = \frac{\lambda_g . z}{\sqrt[k]{1+b_g . z^k}}$; indeed this holds for any holomorphic coordinate z which linearizes g_o.*

Proof of Proposition 5.1. – According to (the proof of) Lemma 3.1, (i) $\Rightarrow$ (ii), except for the compatibility condition (ii). This condition is easily proved using the local expression $\widetilde{\Omega} = g(pxdy + qydx), \widetilde{\eta} = a\frac{dx}{x} + b\frac{dy}{y} + \frac{dg}{g}$, in suitable coordinates around the corner q, which admits a local holomorphic first integral $x^q y^p$ (see the proof of Lemma 3.1). Now we proceed to prove that (ii) $\Rightarrow$ (i): Let G_i denote the holonomy group of a component P_i of the divisor D.

1^{st} *case.* – G_i is a commutative group. In this case since G_i contains a non-periodic linearizable element, G_i is linearizable in some coordinate system and therefore $\widetilde{\mathcal{F}}$ is given by a closed meromorphic 1-form $\widetilde{w}_i$ defined in a neighborhood of P_i in $\widetilde{M}$, with $(\widetilde{w}_i)_\infty = P_i \cup \text{sep}\,(P_i)$ (see the last part of the proof of Theorem 4.1).

2^{nd} *case.* – G is a solvable non-commutative group. In this case since G_i contains a linearizable non-periodic element G_i is holomorphically conjugated to a subgroup of $\mathbf{H}_{k_i}$; for some unique $k_i \in \{1,2,\ldots\}$ [14].

CLAIM 1. – There exists a collection of charts $(x_\alpha, y_\alpha) \in U_\alpha$, $\alpha \in A$, such that: (i) $\bigcup_{\alpha\in A} U_\alpha = V\backslash\text{sep}\,(P_i)$, $V =$ some neighborhood of P_i in $\widetilde{M}$; (ii) $U_\alpha \cap P_i = \{y_\alpha = 0\}$ and $U_\alpha \cap s(\widetilde{\mathcal{F}}) = \phi$, $\forall\, \alpha \in A$; (iii) $\widetilde{\mathcal{F}}\big|_{U_\alpha}$ is given by $dy_\alpha = 0$; (iv) If $U_\alpha \cap U_\beta \neq \phi$ then $y_\alpha^k = h_{\alpha\beta}(y_\beta^k)$ for some homography $h_{\alpha\beta} \in \mathbf{H}_1$.

Proof of Claim 1. – The claim is proved using the embedding $G_i \hookrightarrow \mathbf{H}_{k_i}$, Lemma 5.1 and a procedure similar to that used in [6]. $\square$

Now, for each $\alpha \in A$ there exists a holomorphic function $g_\alpha \in \vee(U_\alpha)$ such that $\widetilde{\Omega}(x_\alpha, y_\alpha) = g_\alpha\ dy_\alpha$ in U_α. We therefore define the local model $\widetilde{\eta}_\alpha(x_\alpha, y_\alpha) = (k_i+1)\frac{dy_\alpha}{y_\alpha} + \frac{dg_\alpha}{g_\alpha}$ in U_α.

CLAIM 2. – In each $U_\alpha \cap U_\beta \neq \phi$ we have $\widetilde{\eta}_\alpha = \widetilde{\eta}_\beta$.

Proof of Claim 2. – In fact in $U_\alpha \cap U_\beta$ we have $\widetilde{\Omega} = g_\alpha \, dy_\alpha = g_\beta \, dy_\beta$ and $y_\alpha^{k_i} = \frac{\lambda_{\alpha\beta} \cdot y_\beta^{k_i}}{1+a_{\alpha\beta} \cdot y_\beta^{k_i}}$ so that $\frac{dy_\alpha}{y_\alpha^{k_i+1}} = \frac{1}{\lambda_{\alpha\beta}} \cdot \frac{dy_\beta}{y_\beta^{k_i+1}}$ and that $g_\alpha y_\alpha^{k_i+1} = \lambda_{\alpha\beta} . g_\beta y_\beta^{k_i+1}$ and thus $(k_i+1)\frac{dy_\alpha}{y_\alpha} + \frac{dg_\alpha}{g_\alpha} = (k_i+1)\frac{dy_\beta}{y_\beta} + \frac{dg_\beta}{g_\beta}$. □

It follows from the claim that there exists a meromorphic 1-form $\widetilde{\eta}_i$ in $V \backslash \mathrm{sep}\,(P_i)$ with, $(\widetilde{\eta}_i)_\infty = (P_i \cup (\widetilde{\Omega})_\infty) \cap (V \backslash \mathrm{sep}\,(P_i))$ wich defines a transverse affine structure to $\widetilde{\mathcal{F}}$ in $V \backslash (P_i \cup \mathrm{sep}\,(P_i))$. This form $\widetilde{\eta}_i$ extends to the singularities $s(\widetilde{\mathcal{F}}) \cap P_i$ as in Lemma 3.1 and part (2) of case (B) in the proof of Theorem 4.1. Now, using condition (iii) in Definition 5.1 we can glue $\widetilde{\eta}_i$ to the analogous ones constructed in a neighborhood of the D_i's and obtain $\widetilde{\eta}$ in a neighborhood of D in $\widetilde{M}$. This form blows down and extends (by Hartogs' Theorem) to a closed meromorphic 1-form η in a neighborhood of $\Lambda = \pi(D)$ as required to define an affine transverse structure to $\mathcal{F}$ in this neighborhood minus $\Lambda \cup \mathrm{sep}(\Lambda)$ as stated. □

We recall that according to [15] a germ $w = A\,dx + B\,dy$ has its ν-jet w_ν called *generic* if $w_\nu(x,y) = a_\nu(x,y)dx + b_\nu(x,y)dy$ where a_ν, b_ν are homogeneous polynomials of degree ν having $P_{\nu+1}(x,y) = xa_\nu(x,y) + yb_\nu(x,y)$ as the tangent cone, satisfying:

(i) w_ν is non-dicritic, *i.e.*, $P_{\nu+1} \not\equiv 0$

(ii) the residues $\lambda_j = \frac{1}{2\pi i}\int_{\gamma_j} \frac{w_\nu}{P_{\nu+1}}$ where the γ_j's are generators of $H_1(\mathbf{C}^2 - (P_{\nu+1} = 0))$, are non-real and $P_{\nu+1}(x,y) = c.\prod_{i=1}^{\nu+1}(y - t_i x)$, $t_i \neq t_j \ \forall i \neq j$, $c \in \mathbf{C}$ so that $w_\nu = c.\prod_{i=1}^{\nu+1}(y-t_i x) \cdot \sum_{i=1}^{\nu+1} \lambda_i \frac{dy - t_i dx}{y - t_i x}$. In particular ω is desingularized with one blow-up.

One basic tool here is the following consideration: Let α, β be germs of holomorphic 1-forms in $(\mathbf{C}^2, 0)$. The 1-form β is a *stable deformation* of α if there exists a family $t \mapsto \alpha_t$, continuous in $t \in [0,1]$ such that $\alpha_o = \alpha$, $\alpha_1 = \beta$ and $\{\alpha_t\}$ is topologically trivial in the sense that there exists a continuous family of germs of homeomorphisms $\{h_t : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)\}$ such that $h_o =$ Id and h_t is a topological equivalence between α_t and α_o, for all t. According to [15] any stable deformation of a germ of holomorphic 1-form $w = A\,dx + B\,dy$ in $(\mathbf{C}^2, 0)$ having ν-jet, w_ν generic $\nu \geq 2$, has projective holonomy topologically conjugated to the projective holonomy of w.

The main result of this section is the following proposition:

PROPOSITION 5.2. – *Let $w = A\,dx + B\,dy$ be a germ of holomorphic 1-form in the origin of $\mathbf{C}^2$ having w_ν generic as ν-jet, $\nu \geq 2$ and let w' be a stable deformation of w. Suppose that w has a multiform integrating factor of the form $f = \prod_{j=1}^r f_j^{\lambda_j}$, $f_j \in \vee_2$, $\lambda_j \in \mathbf{C}^*$. Then w' has a multiform integrating factor of the same type.*

The proposition follows from what we have remarked above, from Proposition 5.1 and from the two following remarks:

(a) Let G and G' be subgroups of $\mathrm{Bih}(\mathbf{C}, 0)$ topologically conjugated. Then G is solvable if and only if G' is solvable.

(b) Let $w = A\,dx + B\,dy$ where w is as in Proposition 5.1. Then w has an integrating factor of the form $f = \Pi_j f_j^{\lambda_j}$, $f_j \in \vee_2$, $\lambda_j \in \mathbf{C}^*$ if and only if the projective holonomy $\mathcal{H}_w$ of w is a solvable group.

We supply a proof for (b): Assume that ω has such an integrating factor f. Then $\eta = \frac{df}{f} = \sum_j \frac{df_j}{f_j}$ is an adapted form to ω along the separatrices set $\bigcup_j \{f_j = 0\}$. Therefore it follows that the holonomy of the projective $\mathbf{P}^1$ arising in the desingularization of ω is solvable (*see* Theorem 4.1). This proves the first part. Conversely if the projective holonomy is solvable, then since it contains non-periodic linearizable elements it is nonexceptional [14] and therefore it is, either abelian analytically linearizable or it is conjugated to a subgroup of $\mathbf{H}_k$ as above. In the abelian case ω admits a meromorphic integrating factor f [13]. In the nonabelian linearizable case we can construct η as in the proof of Proposition 5.1. According to the local version of the Integration Lemma [13] we can write $\eta = \sum_j \frac{df_j}{f_j} = \frac{df}{f}, f = \Pi_j f_j^{\lambda_j}$, as stated (recall that by construction η has simple poles). □

6. Transversely affine foliations on complex manifolds

A holomorphic singular codimension one foliation $\mathcal{F}$ on a complex manifold M is defined by a colection (Ω_i, U_i) of holomorphic integrable 1-forms Ω_i in open sets U_i such that $\bigcup_i U_i = M$ and in each $U_i \cap U_j \neq \phi$ we have $\Omega_i = f_{ij}\ \Omega_j$ for some $f_{ij} \in \vee(U_i \cap U_j)^*$. If M is a complex projective space then we can describe $\mathcal{F}$ by global integrable meromorphic 1-forms, but this may not be possible if M is not projective. For this general case we have in the place of Proposition 1.1 the following:

PROPOSITION 6.1. – *Let $\mathcal{F}$, M be as above. The possible transverse affine structures for $\mathcal{F}$ in M are classified by the collections (Ω_i, η_i) of differential 1-forms defined in the open sets $U_i \subset M$ such that:* (i) *(Ω_i, U_i) is like above;* (ii) *η_i is holomorphic, closed and $d\Omega_i = \eta_i \wedge \Omega_i$;* (iii) *In each $U_i \cap U_j \neq \phi$ we have $\eta_i = \eta_j + \frac{df_{ij}}{f_{ij}}$.*

Furthermore two such collections (Ω_i, η_i) and (Ω'_i, η'_i) define the same transverse affine structure for $\mathcal{F}$ in M if and only if $\Omega'_i = f_i \Omega_i$ and $\eta'_i = \eta_i + \frac{df_i}{f_i}$ for some $f_i \in \vee(U_i)^*$.

Now in the place of Theorem 4.1 or Proposition 5.1 we have:

PROPOSITION 6.2. – *Let $\mathcal{F}$, M^2 be as above with M of dimension 2. Let $\Lambda \cap M$ be an analytic invariant curve and assume that $s(\mathcal{F}) \cap \Lambda$ contains only first order singularities and that the foliation $\widetilde{\mathcal{F}}$ obtained as the desingularization of the singularities of $\mathcal{F}$ in Λ exhibits some linearizable non-resonant singularity. Then the following conditions are equivalent:*

(i) *$\mathcal{F}$ is transversely affine in some neighborhood of Λ minus $\Lambda \cup Sep(\Lambda)$*

(ii) *The holonomy of Λ has the property $(\mathcal{S})$.*

(iii) *There is a collection (Ω_i, η_i, U_i) with $\bigcup_i U_i =$ some neighborhood of Λ in M such that η_i is meromorphic in U_i, $(\eta_i)_{-\infty} = (\Lambda \cup Sep(\Lambda)) \cap U_i$ has order one and (Ω_i, η_i, U_i) defines (in the sense of Proposition* 6.1*) a transverse affine structure for $\mathcal{F}$ in $\bigcup_i U_i \backslash (\Lambda \cup sep(\Lambda))$.*

7. Generalizations for algebraic projective manifolds

Most of the results present in this chapter established for projective spaces extend naturally to algebraic non-singular projective varieties ([1]). We pay special attention to the so called Poincare Problem application (Theorem 4.2). We show how to extend Theorems 4.4 and 4.5 to any non-singular algebraic projective surface: Let $\mathcal{F}$ be a foliation by curves on M^2, a non-singular algebraic projective surface. Since we can define in a natural way, polynomial and rational functions on M^2 we can define in a natural

([1]) I thank M. Sebastiani for suggesting this.

way *algebraic leaves of* $\mathcal{F}$. Now we show how to extend the notions of *degree of the foliation* $\mathcal{F}$ and of *degree of the set* $\mathrm{Sep}(\mathcal{F})$ of the algebraic leaves of $\mathcal{F}$ (according to [8]). Denote by $S = \mathrm{Sep}(\mathcal{F})$ and denote by L the holomorphic line bundle in M that extends canonically the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$. We will denote by $S.S$ the intersection index of S with itself in M (if $M = \mathbf{C}P(2)$ then we have $S.S = \deg(S)^2$). Let us denote by $c_1(L)$ the first Chern class of L in M and by $[\mathbf{E}]$ the fundamental class of any projective line $\mathbf{E} \hookrightarrow M$. We know that if $M = \mathbf{C}P(2)$ then using the geometric meaning of the degree of $\mathcal{F}$ as the number of tangencies of $\mathcal{F}$ with a generic $\mathbf{C}P(1) \hookrightarrow \mathbf{C}P(2)$, then $\lg c_1(L), [\mathbf{E}]\rangle = 1 - \deg(\mathcal{F})$ and this shows us how to extend the notion of degree of $\mathcal{F}$. According to the proof of Theorems 4.4 and 4.5 we have (under the hypothesis of these theorems): $(\deg \mathcal{F} + 2) \cdot \deg \mathrm{Sep}(\mathcal{F}) = (\deg \mathrm{Sep}(\mathcal{F}))^2$. So for our present context we would have: (1) $(\deg \mathcal{F} + 2) \cdot \deg S = S.S$. On the other hand we have $\lg c_1(L), [S]) \rangle = (1 - \deg(\mathcal{F})) \cdot \deg S$; and if we denote by K the canonical divisor of $\mathbf{C}P(2)$ then we have, for the case $M = \mathbf{C}P(2)$, $K = -3\mathbf{E}$ so that $K.S = -3 \deg S$. Therefore for the case $M = \mathbf{C}P(2)$ we have: (2) $\lg c_1(L), [S]\rangle + K.S = -(2 + \deg(\mathcal{F})) \cdot \deg S.$. Using (1) and (2) we can the equivalent formulation which generalizes immediately: $S.S + \lg c_1(L), [S]\rangle + K.S = 0$. Thus we can state:

THEOREM 7.1. – *Let $\mathcal{F}$ be a foliation by curves on a non-singular algebraic projective surface M^2 and suppose $\mathcal{F}$ is transversely affine outside and algebraic codimension one invariant set $S \subset M$. Assume that $S \subset M$ is such that $M \backslash S$ is an affine variety, and that $\mathcal{F}$ has only non-dicritical singularities in some component S_o of S. Then, we have $S.S + \lg c_1(L), [S]\rangle + K.S = 0$; where L is the holomorphic line bundle that extends canonically $T\mathcal{F}$ and K is the canonical divisor of M.*

The hypothesis that $M \backslash S$ is affine is equivalent to say that it is a Stein manifold. This does not hold in general (for instance if $M = \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ and S is a "vertical" projective line. But holds for example if $S \subset \overline{\mathbf{C}} \times \overline{\mathbf{C}}$ is the "diagonal").

THEOREM 7.2. – *Let $\mathcal{F}$, M and S be as in Theorem* 7.1 *above. Assume that $\mathcal{F}$ has only 1^{st}-order singularities in S and which admit local meromorphic integrating factors. Then $\mathcal{F}$ is given by a closed meromorphic 1-form on M. This form has only simple poles provided that the singularities of $\mathcal{F}$ on S are desingularized into non-resonant singularities.*

Chapter II
Transversely Projective Holomorphic Foliations

1. Transversely projective foliations and differential forms

Throughout this chapter II, except for explicit mention, the 1-form Ω will be assumed to have singular set of codimension bigger than one.

Analogously to the affine case, the problem of deciding if there exist projective structures for a given foliation is equivalent to a problem on differential 1-forms as stated below (*see* also [17] Proposition 3.20 pp-262):

PROPOSITION 1.1. – *Let $\mathcal{F}$ be a singular codimension one foliation on M which is defined by a holomorphic integrable 1-form Ω and suppose that there exists a holomorphic 1-form η in M such that $d\Omega = \eta \wedge \Omega$. The foliation $\mathcal{F}$ is transversely projective in M if and only if there exists holomorphic 1-form ξ in M satisfying:*(i) $d\eta = \Omega \wedge \xi$; (ii)$d\xi = \xi \wedge \eta$.

Furthermore, two such triples (Ω, η, ξ) and (Ω', η', ξ') define the same projective structure if and only if we have: $\Omega' = f\Omega; \eta' = \eta + \frac{df}{f} + 2g\Omega; \xi' = \frac{1}{f}(\xi - 2dg - 2g\eta - 2g^2\Omega)$; for some holomorphic functions $f, g\colon M \to \mathbf{C}^*, \mathbf{C}$. In particular (Ω, η, ξ) and $(f\Omega, \eta + \frac{df}{f}, \frac{1}{f}\xi)$ define the same projective transverse structure for $\mathcal{F}$.

Now, if Ω, η are meromorphic then we have: If $\mathcal{F}$ is transversely projective in M then there exists ξ meromorphic on M.

We give some remarkable examples of transversely projective foliations.

Example 1.0. – *Transversely projective foliations on simply-connected manifolds.* Let $\mathcal{F}$ be defined by a meromorphic function $f\colon M \to \overline{\mathbf{C}}$ then $\mathcal{F}$ is transversely projective on M. Conversely *any transversely projective foliation defined on a simply-connected manifold admits a meromorphic first integral*: In fact, as in I Example 1.1 the foliation $\mathcal{F}$ admits a meromorphic first integral on $M' = M \backslash s(\mathcal{F})$ (which is simply-connected), and this extends by Hartogs's Theorem to a meromorphic first integral on M because codim.$s(\mathcal{F}) \geq 2$.

Example 1.1. – A *Riccati foliation* $\mathcal{F}$ on $\mathbf{C}P(2)$ is given in some affine chart $(x, y) \in \mathbf{C}^2 \hookrightarrow \mathbf{C}P(2)$ by a polynomial 1-form $\Omega = p(x)dy - (y^2\,c(x) - yb(x) - a(x))dx$ where p, a, b and c are polynomials. Motivated by the affine case (*see* I Proposition 1.1) we define $\eta = 2\frac{dy}{y} + \frac{p'+b}{p}\,dx + \frac{2a}{yp}\,dx$ and $\xi = \frac{-2a}{y^2p^2}\,dx$. Then (Ω, η, ξ) satisfies the relations stated in Proposition 1.1. This shows that $\mathcal{F}$ is transversely projective in $\mathbf{C}P(2)$ minus the algebraic subset $\overline{\{x \in \mathbf{C} \mid p(x) = 0\} \times \mathbf{C} \cup \mathbf{C} \times \{y = 0\}}$. But since in the case $a(x) \not\equiv 0$, only the subset $S = \{p(x) = 0\} \times \overline{\mathbf{C}}$ is $\mathcal{F}$ invariant it follows that the transverse projective structure extends to $\mathbf{C}P(2)\backslash S$. Indeed according to Proposition 1.1 if we define $g = \frac{-1}{p(x)y}$ then $\eta' = \eta + 2g\Omega = \frac{p'-b+2yc}{p}\,dx$ and $\xi' = \xi - 2dg - 2g\eta - 2g^2\Omega = \frac{2c}{p^2}\,dx$; define a triple (Ω, η', ξ') holomorphic in $\mathbf{C}P(2)\backslash S$ which gives a projective structure for $\mathcal{F}$ in $\mathbf{C}P(2)\backslash S$ and this projective structure coincides with the one given in $\mathbf{C}P(2)\backslash(S \cup \overline{\mathbf{C} \times \{y = 0\}})$ by (Ω, η, ξ). The 1-form η is closed if and only if $a \equiv 0$. Therefore $\mathcal{F}$ is transversely affine in $\mathbf{C}P(2)\backslash(S \cup \overline{\mathbf{C}} \times \overline{\{y = 0\}})$ if the projective line $\{y = 0\}$ is invariant (*see* I Proposition 1.1). A Riccati foliation can also be seen as the suspension of a Kleinian group (*see* Example 1.5 below and [23]).

Example 1.2. – Let $\mathcal{F}$ be a transversely projective foliation on M as in Proposition 1.1. Let $\pi\colon N \to M$ be a holomorphic map transverse to $\mathcal{F}$, then the foliation $\pi^*(\mathcal{F})$ is transversely projective in N (*see* I Example 1.2).

Example 1.3. – Let α be a closed meromorphic 1-form on M and let $f\colon M \to \overline{\mathbf{C}}$ be a meromorphic function. Define (Ω, η, ξ) by: $\Omega = df - f^2\alpha, \quad \eta = 2f\alpha \quad \text{and} \quad \xi = 2\alpha$. Then (Ω, η, ξ) is a projective triple and therefore Ω defines a holomorphic foliation on M, transversely projective in the complement of the analytic invariant codimension one set $S \subset M$, $S = (\alpha)_\infty \cup (f)_\infty$. The same conclusion holds for $\Omega_\lambda = \Omega + \lambda\alpha$, where $\lambda \in \mathbf{C}$. The foliation $\mathcal{F}(\Omega_\lambda)$ is also transversely affine in some smaller open set of the form $M\backslash S'$ where $S' \supset S$, $S' = S \cup (f^2 - \lambda = 0)$. (In fact $\frac{\Omega_\lambda}{f^2-\lambda} = \frac{df}{f^2-\lambda} - \alpha$ is closed and holomorphic in $M\backslash S'$).

Example 1.4. – Let $h: M \to \mathbf{C}^*$ be holomorphic such that $d\xi = -\frac{dh}{2h} \wedge \xi$ where ξ is holomorphic. (We can write this condition as $d(\sqrt{h}.\xi) = 0$). Let F be any holomorphic function and write (for $\lambda \in \mathbf{C}$) $\Omega = F \cdot \left(\frac{dF}{F} - \frac{1}{2}\frac{dh}{h}\right) - \left(\frac{F^2}{2} - \frac{\lambda}{2}h\right).\xi$, $\eta = \frac{1}{2}\frac{dh}{h} + F \cdot \xi$. The triple (Ω, η, ξ) satisfies the conditions of Proposition 1.1 and then $\mathcal{F} = \mathcal{F}(\Omega)$ is a transversely projective foliation on M.

Example 1.5. – *Suspension of a foliation by a group of biholomorphisms.* A well known way of constructing transversely homogeneous foliations on fibered spaces, having a prescribed holonomy group is the *suspension* of a foliation by a group of biholomorphisms. This construction is briefly described below: Let G be a group of biholomorphisms of a complex manifold N. We can regard G as the image of a representation $h: \pi_1(M) \to \text{Bih}(N)$ of the fundamental group of a complex (connected) manifold M. Considering the universal holomorphic covering of M, $\pi: \widetilde{M} \to M$ we have a natural free action $\pi_1: \pi_1(M) \times \widetilde{M} \to \widetilde{M}$, *i.e.*, $\pi_1(M) \subset \text{Bih}(\widetilde{M})$ in a natural way. Using this we define an action $H: \pi_1(M) \times \widetilde{M} \times N \to \widetilde{M} \times N$ in the natural way: $H = (\pi_1, h)$. The quotient manifold $\frac{\widetilde{M} \times N}{H} = M_h$ is called the *suspension manifold* of the representation h. The group G appears as the *global holonomy* of a natural foliation $\mathcal{F}_h$ on M_h (*see* [17]).

Proposition 1.1 is stated (for the real non-singular case) with an idea of its proof, in [17] (*see* Prop. 3.20, pp. 262). However, it seems that the suggested proof uses some triviality hypothesis on principal fiber-bundles of structural group Aff($\mathbf{C}$), over the manifold M (*see* [17] Prop. 3.6 pp. 249-250). In our case this is replaced by the existence of the form η in the statement. On the other hand, since some of its elements will be useful later, we supply a proof for Proposition 1.1. We will use the two following lemmas whose proofs are straighforward calculations left to the reader:

LEMMA 1.1. – *Let* $x, y, \widetilde{x}, \widetilde{y}: U \subset \mathbf{C}^n \to \overline{\mathbf{C}}$ *be meromorphic functions satisfying: (i)* $y dx - x dy = \widetilde{y} d\widetilde{x} - \widetilde{x} d\widetilde{y}$; *(ii)* $\frac{\widetilde{x}}{\widetilde{y}} = \frac{ax+by}{cx+dy}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (2, \mathbf{C})$.
Then $\widetilde{x} = \varepsilon.(ax + by)$ *and* $\widetilde{y} = \varepsilon.(cx + dy)$ *for some* $\varepsilon \in \mathbf{C}$, $\varepsilon^2 = 1$.

LEMMA 1.2. – *Let* $x, y, \widetilde{x}, \widetilde{y}: U \subset \mathbf{C}^n \to \overline{\mathbf{C}}$ *be meromorphic functions satisfying* $\widetilde{x} = ax + by$, $\widetilde{y} = cx + dy$ *for some* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in (2, \mathbf{C})$. *Then* $x dy - y dx = \widetilde{x} d\widetilde{y} - \widetilde{y} d\widetilde{x}$.

Proof of Proposition 1.1. – Suppose $\mathcal{F}$ is transversely projective in M^n, say, $\{f_i: U_i \to \mathbf{C}\}$ is a projective transverse structure for $\mathcal{F}$ in $M\backslash s(\mathcal{F})$. In each U_i we have $\Omega = -g_i\, df_i$ for some holomorphic $g_i \in \vee(U_i)^*$. In each $U_i \cap U_j \neq \phi$ we have: $g_i\, df_i = g_j\, df_j$ and (1) $f_i = \frac{a_{ij} f_j + b_{ij}}{c_{ij} f_j + d_{ij}}$ as in Definition 1.1. Since $d\Omega = d(-g_i\, df_i) = \frac{dg_i}{g_i} \wedge \Omega$ we have $\eta = \frac{dg_i}{g_i} - h_i \Omega$ for some holomorphic h_i in U_i. We define $x_i, y_i, u_i, v_i: U_i \to \mathbf{C}$ in the following way: (2) $y_i^2 = g_i$, $\frac{x_i}{y_i} = f_i$, $h_i = \frac{2v_i}{y_i}$ and $x_i v_i - y_i u_i = 1$. Thus we have: $\Omega = x_i\, dy_i - y_i\, dx_i$ and (3) $\eta = 2(v_i\, dx_i - u_i\, dy_i)$. This motivates us to define local models (*see* [17] Section 3.18 pp. 261): $\xi_i = 2(v_i\, du_i - u_i\, dv_i)$ in U_i. It is easy to check that we have $d\xi_i = \xi_i \wedge \eta$, $d\eta = \Omega \wedge \xi_i$ in U_i. We can assume that dx_i and dy_i are independent for all $i \in I$. In fact $dx_i \wedge dy_i = 0 \Rightarrow d\Omega\big|_{U_i} = 2\, dx_i \wedge dy_i = 0 \Rightarrow d\Omega = 0$

in M (we can assume M to be connected) $\Rightarrow$ we have $0 = d\Omega = \eta \wedge \Omega$ so that $\eta = h\Omega$ for some holomorphic function $h\colon M \to \mathbf{C} \Rightarrow$ we can choose $\xi = \frac{h^2\Omega}{2} + h\eta + dh$ which satisfies the relations $d\eta = \Omega \wedge \xi$ and $d\xi = \xi \wedge \eta$.

Claim 1. – $\xi_i = \xi_j$ in each $U_i \cap U_j \neq \phi$ and therefore the ξ_i's can be glued into a holomorphic 1-form ξ in $M \backslash s(\mathcal{F})$ satisfying the conditions of the statement.

Proof. – From (1) and (2) we obtain $\frac{x_i}{y_i} = \frac{a_{ij}x_j + b_{ij}y_j}{c_{ij}x_j + d_{ij}y_j}$. Therefore according to Lemma 1.1 we have (4) $x_i = \varepsilon.(a_{ij}x_j + b_{ij}x_j)$, $y_i = \varepsilon.(c_{ij}x_j + d_{ij}y_j)\, \varepsilon^2 = 1$. Using (3) and (4) we obtain: $(a_{ij}v_i - c_{ij}u_i)dx_j + (b_{ij}v_i - d_{ij}u_i)dy_j = \varepsilon.(v_j\, dx_j - u_j\, dy_j)$ and therefore: (5) $v_j = \epsilon(a_{ij}\, v_i - c_{ij}\, u_i)$, $u_j = \epsilon(-b_{ij}\, v_i + d_{ij}\, u_j)$. It follows form (5) and Lemma 1.2 that $v_i\, du_i - u_i\, dv_i = v_j\, du_j - u_j\, dv_j$ which proves the claim.

Claim 2. – We have $\xi = \xi_i = h_i^2 \frac{\Omega}{2} + h_i\eta + dh_i$ in each U_i.

Proof. – We have $h_i^2\Omega = \frac{4v_i^2}{y_i^2}\,(x_i\, dy_i - y_i\, dx_i)$, $h_i\eta = \frac{4v_i}{y_i}\,(v_i\, dx_i - u_i\, dy_i)$, $dh_i = 2d\Big(\frac{v_i}{y_i}\Big)$. Hence $\frac{h_i^2\Omega}{4} + \frac{h_i\eta}{2} + \frac{dh_i}{2} = \frac{v_i^2}{y_i}\, dx_i - \frac{v_i}{y_i^2}\,(x_iv_i - 1)dy_i + \frac{dv_i}{y_i}$.

On the other hand a straightforward calculation shows that $\frac{\xi_i}{2} = v_i\, du_i - u_i\, dv_i = \frac{v_i^2}{y_i}\, dx_i - \frac{v_i}{y_i}(x_iv_i - 1)dy_i + \frac{dv_i}{y_i}$. And thus Claim 2 is proved. Since $\operatorname{cod} s(\mathcal{F}) \geq 2$ it follows that ξ extends holomorphically to M. This proves the first part. Now we assume that (Ω, η, ξ) is *holomorphic* as in the statement of the proposition:

Claim 3. – Given any $p \in M \backslash s(\mathcal{F})$ there exist holomorphic $x, y, u, v\colon U \to \mathbf{C}$ defined in an open neighborhood $U \ni p$ such that: $\Omega = xdy - ydx$, $\eta = 2(vdx - udy)$ and $\xi = 2(vdu - udv)$.

Proof. – This claim is a consequence of Darboux's Theorem (*see* [17] pp. 230), but we can give an alternative proof as follows: We write locally $\Omega = -gdf = xdy - ydx$ and $\eta = \frac{dg}{g} - h\Omega = 2(vdx - udy)$ as in the proof of the first part. Using Claim 2 and the last part of Proposition 2.1 below we obtain locally $\xi = \frac{h^2\Omega}{2} + h\eta + dh + \ell.\Omega$; for some holomorphic function ℓ satisfying $\frac{d\ell}{-2\ell} \wedge \Omega = d\Omega$. This last equality implies that $d(\sqrt{\ell}.\Omega) = 0$ and then $\ell = \frac{r(f)}{g^2}$ for some holomorphic function $r(z)$. Now we look for holomorphic functions $\widetilde{f}$, $\widetilde{g}$ and $\widetilde{h}$ satisfying: $\Omega = -\widetilde{g}d\widetilde{f}$, $\eta = \frac{d\widetilde{g}}{g} - \widetilde{h}\Omega$ and $\xi = \frac{\widetilde{h}^2\Omega}{2} + \widetilde{h}\eta + d\widetilde{h}$. We try $\widetilde{f} = U(f)$ for some holomorphic non-vanishing $U(z)$. Using $\Omega = gdf = -\widetilde{g}d\widetilde{f}$ we get $\widetilde{g} = \frac{g}{U'(f)}$. Using $\eta = \frac{dg}{g} - d\Omega = \frac{d\widetilde{g}}{g} - \widetilde{h}\Omega$ we get $\widetilde{h} = h - \frac{U''}{gU'}$. Using $\xi = \frac{h^2\Omega}{2} + h\eta + dh + \ell\Omega = \frac{\widetilde{h}^2\Omega}{2} + \widetilde{h}\eta + d\widetilde{h}$ we get $d\big(\frac{U''(f)}{U'(f)}\big) = r(f)df$.

Therefore it is possible to write Ω, η and ξ as in the statement of the claim: define $x = \widetilde{f}y$, $y = \sqrt{\widetilde{g}}$, $v = \frac{\widetilde{h}y}{2}$ and $u = \frac{xv-1}{y}$ as in the first part of the proof. This proves Claim 3.

Using Claim 3 we prove that $\mathcal{F}$ is transversely projective in $M\backslash s(\mathcal{F})$, that is in M. The last part of Proposition 1.1 can be proved using the relation stated above between the projective structure and the local trivializations for Ω, η and ξ. For instance we prove the following.

Claim 4. – (Ω, η, ξ) and $(f\Omega, \eta + \frac{df}{f}, \frac{1}{f}\,\xi)$ define the same projective structure for $\mathcal{F}$, for any holomorphic $f\colon M \to \mathbf{C}^*$.

Proof. – Using the notation of the first part we define $\hat{x}_i = \sqrt{f}.x_i$, $\hat{y}_i = \sqrt{f}.y_i$, $\hat{u}_i = \frac{1}{\sqrt{f}}.u_i$ and $\hat{v}_i = \frac{1}{\sqrt{f}}.v_i$. Then: $f\Omega = \hat{x}_i\, d\hat{y}_i - \hat{y}_i\, d\hat{x}_i$, $\eta + \frac{df}{f} = 2(\hat{v}_i\, d\hat{x}_i - \hat{u}_i\, d\hat{y}_i)$ and $\frac{1}{f}\,\xi = 2(\hat{v}_i\, d\hat{u}_i - \hat{u}_i\, d\hat{v}_i)$. Furthermore we have $\frac{\hat{x}_i}{\hat{y}_i} = \frac{x_i}{y_i} = \frac{a_{ij}x_j + b_{ij}y_j}{c_{ij}x_j + d_{ij}y_j} = \frac{a_{ij}\hat{x}_j + b_{ij}\hat{y}_j}{c_{ij}\hat{x}_j + d_{ij}\hat{y}_j}$, and this proves the claim and finishes the holomorphic part of the proof. Now we only have to observe that if (Ω, η) is a pair of meromorphic 1-forms and if $\mathcal{F}$ is transversely projective in M, then the same steps of the first part of the proof apply to construct a meromorphic 1-form ξ satisfying the relations of the statement. $\square$

2. Meromorphic projective triples

Motivated by Proposition 1.1 we make the following definition:

DEFINITION 2.1. – Let $\mathcal{F}$ be a codimension one foliation on M. A meromorphic triple (Ω, η, ξ) of *meromorphic* 1-forms in M is called a *projective triple* if it satisfies the *projective relations*: $d\Omega = \eta \wedge \Omega$, $d\eta = \Omega \wedge \xi$, $d\xi = \xi \wedge \eta$. We say that this is a projective triple *for* $\mathcal{F}$ if $\mathcal{F}$ is given by Ω outside $(\Omega)_\infty$.

In the following proposition we investigate the relation between two projective triples for the same foliation:

PROPOSITION 2.1. – *Let* (Ω, η, ξ) *and* (Ω', η', ξ') *be (meromorphic) projective triples for* $\mathcal{F}$ *in* M. *Then we have* $\Omega' = f\Omega$, $\eta' = \eta + \frac{df}{f} + 2g\Omega'$, $\xi' = \frac{1}{f}\big(\xi - 2dg - 2g.\big(\eta + \frac{df}{f}\big) - 2g^2\Omega'\big) + \ell\Omega$ *for some meromorphic* f, g *and* ℓ *satisfying* $d\Omega' = \frac{d\ell}{-2\ell} \wedge \Omega'$. *In particular if* (Ω, η, ξ) *and* (Ω, η, ξ') *define projective triples for* $\mathcal{F}$ *then* $\xi' = \xi + \ell.\Omega$ *for some meromorphic* ℓ *with* $d\Omega = \frac{-d\ell}{2\ell} \wedge \Omega$.

Proof. – First we consider the case $\Omega' = \Omega$, $\eta' = \eta$, that is, (Ω, η, ξ) and (Ω, η, ξ') are projective triples for $\mathcal{F}$ in M.

CLAIM 1. – We have $\xi' = \xi + \ell.\Omega$ for some meromorphic $\ell\colon M \to \mathbf{C}$ satisfying $d\Omega = -\frac{d\ell}{2\ell} \wedge \Omega$.

Proof. – We have $(\xi - \xi') \wedge \Omega = -d\eta - (-d\eta) = 0$ and therefore $\xi' = \xi + \ell.\Omega$ for some meromorphic ℓ. Using $d\xi = \xi \wedge \eta$ and $d\xi' = \xi' \wedge \eta$ we obtain $d\xi + d\ell \wedge \Omega + \ell d\Omega = d\xi' = (\xi + \ell.\Omega) \wedge \eta = \xi \wedge \eta + \ell\Omega \wedge \eta = d\xi + \ell\Omega \wedge \eta$ and thus $d\ell \wedge \Omega + \ell d\Omega = \ell\Omega \wedge \eta = -\ell d\Omega$ and therefore $2\ell d\Omega = -d\ell \wedge \Omega$ which proves the claim.

Now we prove the general case. Since Ω and Ω' define the same foliation we have $\Omega' = f.\Omega$ for some meromorphic f. Since $d(f\Omega) = \big(\frac{df}{f} + \eta\big) \wedge f\Omega$, we have $\big[\eta' - \big(\eta + \frac{df}{f}\big)\big] \wedge \Omega' = 0$ and therefore $\eta' = \eta + \frac{df}{f} + 2g\Omega'$ for some meromorphic g. Now, substituting (Ω', η', ξ') by $\left(\frac{1}{f}\Omega', \eta' - \frac{df}{f}, f.\xi'\right)$ we can assume that $f \equiv 1$ so that $\Omega' = \Omega$ and $\eta' = \eta + 2g\Omega$. In this case we observe that if we define $\widetilde{\xi} = \xi - 2dg - 2g\eta - 2g^2\Omega$ then we have $d\eta' = \Omega' \wedge \widetilde{\xi}$, $d\widetilde{\xi} = \widetilde{\xi} \wedge \eta'$. Using the first part of the proof we conclude that $\xi' = \widetilde{\xi} + \ell.\Omega'$ for some holomorphic ℓ satisfying $d\Omega' = -\frac{d\ell}{2\ell} \wedge \Omega'$. Therefore we have $\Omega' = f\Omega, \eta' = \eta + \frac{df}{f} + 2g\Omega', \xi' = \frac{1}{f}.\big(\xi - 2dg - 2g\big(\eta + \frac{df}{f}\big) - 2g^2\Omega'\big) + \ell\Omega$ as stated. $\square$

Remark 2.1. – In the situation above if $\mathcal{F}$ is supposed to be non-affine in any $M \backslash S$, $S \subset M$ a codimension one analytic invariant set, then ℓ is identically null (in fact since $d\Omega' = -\frac{d\ell}{2\ell} \wedge \Omega'$ it follows from Proposition 1.1 of I that $\mathcal{F}$ is transversely affine outside

$S = (\ell = 0) \cup (\ell = \infty))$ and thus we have $\xi' = \frac{1}{f}.\big(\xi - 2dg - 2g\big(\eta + \frac{df}{f}\big) - 2g^2 f\Omega\big)$. Therefore using Proposition 2.1 above we conclude that, in this case, $\mathcal{F}$ has at most one projective structure. Also motivated by the statement of Proposition 1.1 we make the following definition:

DEFINITION 2.2. – Let Ω be a meromorphic integrable 1-form on M with singular set $s(\Omega)$ (possibly of codimension one). A meromorphic 1-form η is called a *logarithmic derivative* for Ω if $d\Omega = \eta \wedge \Omega$ in M.

Two logarithmic derivatives for Ω are related by $\eta' - \eta = h\Omega$ for some meromorphic function h in M. The next proposition assures the existence of logarithmic derivatives in complex projective spaces.

PROPOSITION 2.2. – *A codimension one foliation on* $\mathbf{C}P(n)$, $n \geq 2$, *can be described in any affine chart* $(x_1, \ldots, x_n) \in \mathbf{C}^n \hookrightarrow \mathbf{C}P(n)$ *by a polynomial integrable* 1*-form* Ω *which admits a rational logarithmic derivative.*

Proof. – Suppose $n = 2$. In this case we have $\Omega = P\,dy - Q\,dx$ for some polynomials P, Q in $\mathbf{C}^2$. Define $\eta = \frac{P_x}{P}\,dx + \frac{Q_y}{Q}\,dy$. Now we assume that $n = 3$. Write $\Omega = A\,dx + B\,dy + C\,dz$ for polynomials A, B, C in $\mathbf{C}^3$. The integrability condition $\Omega \wedge d\Omega = 0$ implies (*) $\frac{C_y - B_z}{BC} + \frac{A_z - C_x}{AC} + \frac{B_x - A_y}{AB} = 0$. Choose any rational functions R, S and T such that $\frac{R}{A} - \frac{S}{B} = \frac{B_x - A_y}{AB}$ and $\frac{R}{A} - \frac{T}{C} = \frac{C_x - A_z}{AC}$. Then we obtain $\frac{S}{B} - \frac{T}{C} = \frac{C_y - B_z}{BC}$ as a consequence of (*). Now we define $\eta = R\,dx + S\,dy + T\,dz$ to obtain $d\Omega = \eta \wedge \Omega$. The case $n > 3$ is proved in the same way that the case $n = 3$. $\square$

We can assure the existence of *holomorphic* logarithmic derivatives in the following case:

PROPOSITION 2.3. – *Suppose that the additive (or first) Cousin Problem has always a solution on* M [1]. *Let* Ω *be an integrable holomorphic singular* 1*-form in* M *defining a foliation* $\mathcal{F}$ *satisfying:* i) *The singular set of* $\mathcal{F}$, $s(\mathcal{F})$, *has codimension* ≥ 2; ii) *any singularity* $p \in s(\mathcal{F})$ *admits a holomorphic first integral.*

Then Ω *admits a holomorphic logarithmic derivative* η *in* M.

Proof. – Since $\Omega \wedge d\Omega = 0$ we can obtain an open cover $\bigcup U_i$ of $M \backslash s(\mathcal{F})$ such that in each U_i we have $\Omega = g_i\,dy_i$ for some holomorphic $g_i, y_i : U_i \to \mathbf{C}$. By hypothesis we can extend this open cover and the local trivializations above to M. Define now $\eta_i = \frac{dg_i}{g_i}$ in each U_i. Clearly η_i is holomorphic and satisfies $d\Omega = \eta_i \wedge \Omega$. In each $U_i \cap U_j \neq \phi$ we have $\eta_i - \eta_j = a_{ij}\,\Omega$ for some holomorphic $a_{ij} : U_i \cap U_j \to \mathbf{C}$. Clearly the a_{ij}'s satisfy the additive cocycle condition: $a_{ij} + a_{jk} = a_{ik}$ in each $U_i \cap U_j \cap U_k \neq \phi$. By the hypothesis this cocycle is trivial, *i.e.*, we can find holomorphic $a_i : U_i \to \mathbf{C}$ such that $a_{ij} = a_i - a_j$ and therefore $\eta_i - a_i\Omega = \eta_j - a_j\Omega$ if $U_i \cap U_j \neq \phi$. Thus we define η in M by $\eta|_{U_i} = \eta_i - a_i\Omega$. $\square$

COROLLARY 2.1. – *Let* $\mathcal{F}$ *be codimension one foliation on* $\mathbf{C}P(n)$, $n \geq 2$, *which is transversely projective and has non-dicritical singularities outside an algebraic codimension one invariant set* $S \subset \mathbf{C}P(n)$. *Then any polynomial* 1*-form* Ω *which defines* $\mathcal{F}$ *in some affine space* $\mathbf{C}^n \hookrightarrow \mathbf{C}P(n)$ *admits a holomorphic logarithmic derivative* η *defined in* $\mathbf{C}^n \backslash S$.

([1]) We write $H^{0,1}(M) = 0$ in the language of the Dolbeault cohomology [20].

Proof. – In fact put $M = \mathbf{C}^n \backslash S = \mathbf{C}P(n) \backslash (S \cup \mathbf{C}P(n-1)_\infty)$, then it is well-known that M is a Stein manifold and *a fortiori* we can always solve the additive Cousin problem in M. Furthermore since $\mathcal{F}$ is transversely projective and non-dicritical in M it follows (Example 1.0) that given any singularity $p \in M \cap s(\mathcal{F})$ we choose an open polydisc $\Delta \ni p$ contained in M such that there exists a holomorphic first integral for $\mathcal{F}\big|_\Delta$ in Δ. Thus we have showed that we are under the hypothesis of Proposition 2.3. Therefore any polynomial 1-form Ω which defines $\mathcal{F}$ in $\mathbf{C}^n$ admits a holomorphic logarithmic derivative η in M. □

Using Proposition 1.1 and Proposition 2.2 we obtain:

PROPOSITION 2.4. – *Let $\mathcal{F}$, S be as in Corollary* 2.1. *Then there exists a triple (Ω, η, ξ) of meromorphic* 1*-forms in $\mathbf{C}P(n) \backslash S$ satisfying:* (i) *Ω and η are rational on $\mathbf{C}P(n)$,* (ii) *Ω defines $\mathcal{F}$ in $\mathbf{C}P(n) \backslash (\Omega)_\infty$,* (iii) *$d\Omega = \eta \wedge \Omega$, $d\eta = \Omega \wedge \xi$, $d\xi = \xi \wedge \eta$. Furthermore given any affine space $\mathbf{C}^n \hookrightarrow \mathbf{C}P(n)$ we can choose Ω polynomial in $\mathbf{C}^n$.*

Thus it only remains to extend ξ meromorphically to $\mathbf{C}P(n)$. This is what we are concerned with in the next section. In the following remark we introduce the *transverse foliation* associated to a projective triple (Ω, η, ξ) where Ω is not transversely affine as in I (that is outside some invariant S).

Remark 2.2. – *The transverse foliation associated to a transversely projective foliation*: Suppose (Ω, η, ξ) is a projective triple for $\mathcal{F}$ in M. We assume M to be connected. We have two cases:

Case 1. – $d\eta \equiv 0$. In this case we have that $\mathcal{F}$ is transversely affine in $M \backslash S, S = (\eta)_\infty \cup (\Omega)_\infty$, notice that $(\eta)_\infty \backslash (\Omega)_\infty$ is invariant (*see* I Proposition 1.1).

Case 2. – $d\eta \not\equiv 0$. In this case $\xi \not\equiv 0$ and since $\xi \wedge d\xi = \xi \wedge \eta \wedge \xi = 0$ the 1-form ξ defines a holomorphic codimension one foliation, say $\mathcal{F}^\perp$, in M. The foliation $\mathcal{F}$ is transverse to $\mathcal{F}$ in $M \backslash \{p \in M \mid d\eta(p) = 0\}$. We can assume that $\mathcal{F}^\perp$ has singular set of codimension ≥ 2: In fact according to Proposition 1.1 we can replace (locally) if necessary ξ by $\frac{1}{f}\,\xi$ where f is a function so that $\frac{1}{f}\,\xi$ has (locally) a codimension ≥ 2 singular set. Finally we observe that clearly $(\xi, -\eta, \Omega)$ is also a projective triple, so that $\mathcal{F}^\perp$ is also transversely projective in $M \backslash S$ as $\mathcal{F}$. This also shows the existence of a *duality* between $\mathcal{F}$ and $\mathcal{F}^\perp$ so that we can suppose, if necessary, that $\mathcal{F}$ is defined by ξ and $\mathcal{F}^\perp$ by Ω. According to Proposition 1.1 this transverse foliation may not be uniquely determined by the projective transverse structure.

3. Extending a transverse projective structure to an analytic invariant set

In this section we will consider a holomorphic foliation $\mathcal{F}$ on M of codimension one and $S \subset M$ an analytic invariant set of codimension one. Our main tool in the problem of extending "meromorphically" a projective structure for $\mathcal{F}$ in $M \backslash S$ to S is the following proposition:

PROPOSITION 3.1. – *Suppose $\mathcal{F}$ is defined in $M \backslash (\Omega)_\infty$ by the meromorphic* 1*-form Ω having η as a logarithmic derivative. Assume that $\mathcal{F}$ is transversely projective in $M \backslash S$. Then there exists a meromorphic* 1*-form ξ in some neighborhood V of S such that (Ω, η, ξ) is a projective triple for $\mathcal{F}$ in $V \backslash S$.*

Proof. – According to Proposition 2.4 there exists a meromorphic 1-form ξ defined on $M\backslash S$, such that (Ω, η, ξ) is a projective triple.

CLAIM. – *There exists a 1-form ξ' meromorphic in a neighborhood V of S such that (Ω, η, ξ') is a projective triple for $\mathcal{F}$ in V and such that $\xi = \xi' + \ell \cdot \Omega$ for some meromorphic function ℓ in $V\backslash S$ satisfying $\frac{d\ell}{-2\ell} \wedge \Omega = d\Omega$.* The claim clearly proves the proposition.

Proof of the Claim. – Given any point $p \in M\backslash s(\mathcal{F})$ we can find an open set $U \ni p$ and meromorphic functions $f, g, h: U \to \overline{\mathbf{C}}$ such that in U we have: $\Omega = -gdf$, $\eta = \frac{dg}{g} - h\Omega$. Using these functions we define $\xi_U = \frac{h^2\Omega}{2} + h\eta + dh$ in U and obtain a meromorphic projective triple (Ω, η, ξ_U) in U (see the proof of Proposition 1.1). Using Proposition 2.1 we conclude that we have $\xi = \xi_U + \ell_U \cdot \Omega$ for some meromorphic function $\ell = \ell_U$ defined in $U\backslash(S \cap U)$ satisfying $\frac{d\ell}{-2\ell} \wedge \Omega = d\Omega$.

Suppose now that $\widetilde{p} \in M\backslash s(\mathcal{F})$ is another chosen point and choose $\widetilde{U}$, $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}$, $\xi_{\widetilde{U}}$ and $\widetilde{\ell} = \ell_{\widetilde{U}}$ in the same notation. Suppose also that we have $U \cap \widetilde{U} \neq \phi$. Then in $U \cap \widetilde{U}$ we have $\xi_U + \ell \cdot \Omega = \xi = \xi_{\widetilde{U}} + \widetilde{\ell} \cdot \Omega$ so that $\xi_{\widetilde{U}} = \xi_U + (\ell - \widetilde{\ell}) \cdot \Omega$ which implies, by the use of Proposition 2.1, that $\frac{d(\ell - \widetilde{\ell})}{-2(\ell - \widetilde{\ell})} \wedge \Omega = d\Omega$. We can rewrite the properties of ℓ, $\widetilde{\ell}$ and $\ell - \widetilde{\ell}$ as: $d(\sqrt{\ell} \cdot \Omega) = 0$, $d(\sqrt{\widetilde{\ell}} \cdot \Omega) = 0$, $d(\sqrt{\ell - \widetilde{\ell}} \cdot \Omega) = 0$. If $U \cap S \neq \phi \neq \widetilde{U} \cap S$, this implies that $\ell/\widetilde{\ell}$ is constant equal to 1: in fact, define $\theta = \sqrt{\ell}/\sqrt{\widetilde{\ell}}$ in a multiform way. Since $\sqrt{\ell}$ and $\sqrt{\widetilde{\ell}}$ are integrating factors for Ω it follows that θ is a multiform first integral for Ω, that is, $d\theta \wedge \Omega = 0$. But $\sqrt{\ell - \widetilde{\ell}} = \sqrt{\widetilde{\ell}} \cdot \sqrt{\theta^2 - 1}$ so that $0 = d(\sqrt{\ell - \widetilde{\ell}} \cdot \Omega) = d(\sqrt{\widetilde{\ell}} \cdot \sqrt{\theta^2 - 1} \cdot \Omega) = \sqrt{\theta^2 - 1} \cdot d(\sqrt{\widetilde{\ell}}\Omega) + \sqrt{\widetilde{\ell}} d(\sqrt{\theta^2 - 1} \cdot \Omega) = \sqrt{\widetilde{\ell}} \cdot d(\sqrt{\theta^2 - 1} \cdot \Omega)$. Moreover $d(\sqrt{\theta^2 - a} \cdot \Omega) = \frac{1}{2\sqrt{\theta^2 - 1}} \cdot 2\theta\, d\theta \wedge \Omega + \sqrt{\theta^2 - 1}\, d\Omega$ therefore $\sqrt{\widetilde{\ell}}\sqrt{\theta^2 - 1} \cdot d\Omega = 0$. If $\widetilde{\ell} \equiv 0$ and $\widetilde{U} \cap S \neq \phi$ it follows that $\xi|_{\widetilde{U}} = \xi_{\widetilde{U}} + \widetilde{\ell} \cdot \Omega = \xi_{\widetilde{U}}$ so that ξ extends meromorphically to $\widetilde{U}$ and therefore to any irreducible component of S that intersects $\widetilde{U}$. Thus we can assume that $\sqrt{\theta^2 - 1} \cdot d\Omega = 0$. Since the case $d\Omega = 0$ is trivial we can assume that Ω is not closed and it follows that $\sqrt{\theta^2 - 1} \equiv 0$ and therefore $\theta^2 \equiv 1$ so that $\frac{\ell}{\widetilde{\ell}} \equiv 1$ and $\ell \equiv \widetilde{\ell}$ in $U \cap \widetilde{U}$. Thus we have $\xi_{\widetilde{U}} \equiv \xi_U$ in $U \cap \widetilde{U}$ and this allows us to define ξ' in $\bigcup_{U \cap S \neq \phi} U = V\backslash(S \cap s(\mathcal{F}))$ by $\xi'|_U = \xi_U$. The meromorphic 1-form ξ' extends meromorphically to V and satisfies the required conditions. $\square$

Since $M = \mathbf{C}P(n)$ satisfies the hypothesis of Proposition 3.1 we can use the Generalized Levi's Extension Theorem (*see* I Remark 4.1) to obtain:

PROPOSITION 3.2. – *Let $\mathcal{F}$ be a foliation on $\mathbf{C}P(n)$, $n \geq 2$, which is transversely projective in $\mathbf{C}P(n)\backslash S$ for some algebraic codimension one set $S \subset \mathbf{C}P(n)$; which is invariant. Then there exists a rational projective triple (Ω, η, ξ) in $\mathbf{C}P(n)$.*

4. Partial classification of the transversely projective foliations on $\mathbf{C}P(n)$

In this section we give a partial classification of the foliations on $\mathbf{C}P(n)$ which are transversely projective on $\mathbf{C}P(n)\backslash S$ for some codimension one algebraic invariant set $S \subset \mathbf{C}P(n)$. Since a Riccati foliation (and therefore its rational pull-backs) always admits a transverse foliation which is a foliation by level curves, this is a necessary condition for a projective foliation as above to be a rational pull-back of a Riccati foliation (*see* Example 1.1). We will show that this condition is in fact enough to assure the pull-back existence. We also study some other cases.

PROPOSITION 4.1. – *Let $\mathcal{F}$ be a holomorphic foliation on $\mathbf{C}P(n)$, $n \geq 2$, having singular set of codimension ≥ 2. Then $\mathcal{F}$ is transversely projective in $\mathbf{C}P(n)$ if and only if $\mathcal{F}$ admits a rational first integral.*

Proof. – This is a straighforward consequence of Example 1.0 and of the fact that $\mathbf{C}P(n)$ is simply-connected and any meromorphic function $f\colon \mathbf{C}P(n) \to \overline{\mathbf{C}}$ is a rational function (Liouville-Weierstrass Theorem). $\square$

Now we consider a foliation $\mathcal{F}$ on $\mathbf{C}P(n)$, $n \geq 2$, having singular set $s(\mathcal{F})$ of codimension ≥ 2. Let $S \subset \mathbf{C}P(n)$ be an algebraic codimension one invariant set which is a finite union of algebraic hypersurfaces. We will assume that: (1) $\mathcal{F}$ is transversely projective in $\mathbf{C}P(n)\backslash S$, (2) $\mathcal{F}$ is not transversely affine in $\mathbf{C}P(n)\backslash S$.

Using Proposition 3.2 we obtain a rational projective triple (Ω, η, ξ) in $\mathbf{C}P(n)$. Let us denote by $\mathcal{F}^{\perp}$ the transverse foliation defined by ξ on $\mathbf{C}P(n)$ (*see* Remark 2.3). Using this notation we can state:

THEOREM 4.1. – *Let $\mathcal{F}$, $\mathcal{F}^{\perp}$, (Ω, η, ξ) and S be as above. Then:*

(i) *if $\mathcal{F}^{\perp}$ has a meromorphic first integral then $\mathcal{F}$ is a rational pull-back of a Riccati foliation on $\mathbf{C}P(2)$,*

(ii) *if $\mathcal{F}^{\perp}$ as a meromorphic integrating factor, say, $\xi = h.\alpha$ for some meromorphic h, α, $d\alpha = 0$, then we have* (i) *or $\mathcal{F}$ is given by $w = df - (f^2 - \lambda)\alpha$ for some meromorphic f and $\lambda \in \mathbf{C}$,*

(iii) *if $\mathcal{F}^{\perp}$ is transversely affine on $\mathbf{C}P(n)\backslash S$ then we have $d(\sqrt{h}.\xi) = 0$ for some meromorphic h and therefore $\mathcal{F}$ is given by (i), (ii) or $w = \frac{df}{g} - (f - \lambda)\xi$ for some meromorphic f and g and $\lambda \in \mathbf{C}$ such that $h = g^2/f$.*

Proof. – (i): Since $\mathcal{F}^{\perp}$ has a meromorphic first integral we can assume that $\xi = gdf$ for some rational functions g and f. But if we replace (Ω, η, ξ) by $(g\Omega, \eta + \frac{dg}{g}, \frac{1}{g}\xi)$, then we can assume that $g \equiv 1$ and therefore $\xi = df$. Since $0 = d\xi = \xi \wedge \eta$ we have $\eta = hdf$ for some meromorphic h. Now we define Ω' by $\Omega' = \frac{h^2\xi}{2} + h\eta + dh$. Then (Ω', η, ξ) is a projective triple in $\mathbf{C}P(n)\backslash S$ and therefore it follows from Proposition 2.1 that $\Omega = \Omega' + \ell\xi$ for some rational function ℓ where $0 = d\xi = -\frac{d\ell}{2\ell} \wedge \xi$ and then $d\ell \wedge df = 0$. Now, since the leaves of $\mathcal{F}^{\perp}$ are connected we can assume that f has connected fibers using Stein's Fatorization Theorem ([19]) and the remark that we can replace the triple (Ω, η, ξ) by triples $(g\Omega, \eta + \frac{dg}{g}, \frac{1}{g}\xi)$ as in the beggining. Now the relation $d\ell \wedge df = 0$, says that ℓ is constant along the fibers of f, which is *primitive*, therefore by Stein's Fatorization Theorem once again, we conclude that we have $\ell = R(f) = \frac{P(f)}{Q(f)}$ for some rational function $R(z) = \frac{P(z)}{Q(z)}$, P and Q polynomials. Therefore $\xi = \frac{h^2 df}{2} - h^2 df + dh + \frac{P(f)}{Q(f)}\,df = -\frac{1}{2}h^2 df + dh + \frac{P(f)}{Q(f)}\,df = \pi^*\big(-\frac{1}{2}y^2 dx + dy + \frac{P(x)}{Q(x)}\,dx\big)$, where $\pi\colon \mathbf{C}P(n) \to \mathbf{C}P(2)$ is the rational map $\pi(x_1, \ldots, x_n) = (f(x_1, \ldots, x_n), h(x_1, \ldots, x_n))$. This proves (i) in Theorem 4.1.

(ii): Let (Ω, η, ξ) be a meromorphic triple, projective for $\mathcal{F}$ in $\mathbf{C}P(n)\backslash S$. We can assume that $\xi = 2\alpha$ for some closed meromorphic 1-form α. Since $0 = d\xi = \xi \wedge \eta$ we obtain

$\eta = 2f\alpha$ for some meromorphic f. Using Proposition 2.1 and Example 1.3 we conclude that $\Omega = df - f^2\alpha + \ell\alpha$ for some meromorphic ℓ satisfying $\frac{d\ell}{-2\ell} \wedge \xi = d\xi = 0$. And then ℓ is a meromorphic first integral for $\mathcal{F}^\perp$. If ℓ is non-constant then we have (i). If ℓ is constant, say $\ell = \lambda \in \mathbf{C}$, then we have (ii).

(iii): Let (Ω, η, ξ) be as in (ii). Since $\mathcal{F}^\perp$ is transversely affine in $\mathbf{C}P(n)\backslash S$, there exists a meromorphic closed 1-form η_o in $\mathbf{C}P(n)\backslash S$ such that $d\xi = \xi \wedge \eta_o$ (*see* I Proposition 1.1). Since $\xi \wedge (\eta - \eta_o) = d\xi - d\xi = 0$ we have $\eta = \eta_o + f\xi$ for some meromorphic f. We have $d\eta_o = 0$, so that $\Omega \wedge \xi = d\eta = d(f\xi) = (df - f\eta) \wedge \xi$ and then $\Omega = df - f\eta + g\xi$ for some meromorphic g.

CLAIM. – We have $d\xi = -\frac{1}{2}\frac{dh}{h} \wedge \xi$ where $h = f^2 - 2g$.

Proof of the claim. – We have $d\Omega = d(df - f\eta + g\xi) = -df \wedge \eta - f d\eta + dg \wedge \xi + g d\xi$ and we have $\eta \wedge \Omega = \eta \wedge (df - f\eta + g\xi) = \eta \wedge df + g\eta \wedge \xi$. Therefore, $-f d\eta + dg \wedge \xi = -2g d\xi$ and since $d\eta = d(f\xi)$ we get $d\xi = -\frac{1}{2}\frac{d(f^2-2g)}{f^2-2g} \wedge \xi$ which proves the claim. $\square$

Observe that the claim means that $d(\sqrt{h}.\xi) = 0$ always that $\sqrt{h}$ is well defined.

Since $\left(\eta - \frac{1}{2}\frac{dh}{h}\right) \wedge \xi = -d\xi + d\xi = 0$ it follows that $\eta = \frac{1}{2}\frac{dh}{h} + F.\xi$ for some meromorphic F. Define now $\Omega' = F\left(\frac{dF}{F} - \frac{1}{2}\frac{dh}{h}\right) - \frac{F^2.\xi}{2}$, then it is easy to prove that (Ω', η, ξ) is a projective triple (*see* Example 1.4). Using Proposition 2.1 we conclude that $\Omega = \Omega' + \ell.\xi$ for meromorphic ℓ with $d\xi = -\frac{d\ell}{2\ell} \wedge \xi$. Since $\sqrt{\ell}$ and $\sqrt{h}$ are integrating factors for ξ it follows that $\frac{h}{\ell}$ is a meromorphic first integral for ξ and therefore we have two cases:

Case 1. – $\frac{h}{\ell}$ is non-constant. In this case we have (i).

Case 2. – $\frac{\ell}{h} \equiv \frac{\lambda}{2} \in \mathbf{C}$ for some constant $\lambda \in \mathbf{C}$. In this case we have $\Omega = F\left(\frac{dF}{F} - \frac{1}{2}\frac{dh}{h}\right) - \left(\frac{F^2}{2} - \frac{\lambda}{2}.h\right)\xi - \frac{1}{2}\left\{\frac{h}{F} \cdot d\left(\frac{F^2}{h}\right) - h\left(\frac{F^2}{h} - \lambda\right).\xi\right\}$. Therefore $\mathcal{F}$ can be given by $w = \frac{dx}{y} - (x - \lambda).\xi$ where $x = \frac{F^2}{h}$, $h = f$ are meromorphic and we have $d\left(\frac{y}{\sqrt{x}}\xi\right) = 0$. $\square$

5. Applications

A. Irreducible components of spaces of foliations

In this section we are concerned with the following problem.

PROBLEM. – *Describe the irreducible components of the space $\mathcal{F}(k,n)$ of foliations of degree k in $\mathbf{C}P(n)$, $n \geq 3$ (see [12]).*

Next we describe some known irreducible components of $\mathcal{F}(k,n)$, $n \geq 3$.

Example 5.1. – *Logarithmic components*: Let $f_1, \dots, f_m$ be homogeneous polynomials in $\mathbf{C}^{n+1}$, $m \geq 3$, $\lambda_1, \dots, \lambda_m \in \mathbf{C}^*$. The form $w = f_1 \dots f_m \sum_{j=1}^m \lambda_j \frac{df_j}{f_j}$ is integrable. If $\sum_{j=1}, \lambda_j \deg(f_j) = 0$ then w is well defined in $\mathbf{C}P(n)$ and defines a logarithmic foliation $\mathcal{F} = \mathcal{F}(w)$ on $\mathbf{C}P(n)$. Define $\mathrm{Log}(d_1, \dots, d_m) \subset \mathcal{F}(k,n)$ as the set of logarithmic foliations $\mathcal{F}(w)$ where w is as above, $d_j = \deg(f_j)$, $k = \deg(w) = \sum_{j=1}^m d_j - 2$ and $f_1, \dots, f_m$ are irreducible, relatively prime and $\lambda_i/\lambda_j \notin \mathbf{R}$, $\forall\, i \neq j$.

THEOREM 5.1 ([2], [12]). – *If $n \geq 3$, $m \geq 3$ then $\overline{\mathrm{Log}(d_1, \dots, d_m)}$ is an irreducible component of $\mathcal{F}(k,n)$ where $k = \sum_{j=1}^m d_j - 1$.*

Example 5.2. – *Rational components*: Let f and g be homogeneous polynomials in $\mathbf{C}^{n+1}$ such that: (a) $\deg(f) = m$, $\deg(g) = \ell$ and $\frac{m}{\ell} = \frac{p}{q}$ where $(p,q) = 1$. (b) The hypersurfaces $\{f = 0\}$ and $\{g = 0\}$ meet transversely in $\mathbf{C}^{n+1}\backslash\{0\}$. (c) The hypersurfaces $\pi(\{f = 0\})$ and $\pi(\{g = 0\})$ are smooth in $\mathbf{C}P(n)$. Define $w = q\,gdf - pf\,dg$. Then the foliation $\mathcal{F}(w)$ has the first integral $\varphi = f^q/g^p$ (considered as a function in $\mathbf{C}P(n)$). The foliation $\mathcal{F}(w)$ has degree $k = m + \ell - 2$.

Let $R(m,\ell)$ denote the set of all foliations in $\mathcal{F}(k,n)$ of this type.

THEOREM 5.2 [18]. – *The closure* $\overline{R(m,\ell)}$ *is an irreducible component of* $\mathcal{F}(k,n)$, *if* $n \geq 3$.

In order to study the irreducible components of $\mathcal{F}(k,n)$ we need to study the stability of a generic type of singularities. Given any integrable polynomial homogeneous 1-form ω on $\mathbf{C}^{n+1}$ with singular set of codimension ≥ 2. We define the *Kupka singular set of* ω as $K(\omega) = \{p \in \mathbf{C}^{n+1}\backslash 0 \mid \omega(p) = 0, d\omega(p) \neq 0\}$. The *Kupka singular set of* the foliation $\mathcal{F} = \mathcal{F}(\omega)$ is $K(\mathcal{F}) = \pi(K(\omega))$. The main properties of the Kupka set are summarized in the following result:

THEOREM 5.3 ([18],[21],[26]). – *Let* $n \geq 3$, $\mathcal{F}$, w, $K(\mathcal{F})$ *be as above:*

(i) *The Kupka set* $K(\mathcal{F})$ *is a locally closed codimension 2 smooth submanifold of* $\mathbf{C}P(n)$.

(ii) *The Kupka set has the local product structure: Given a connected component* $K \subset K(\mathcal{F})$ *there exist a holomorphic* 1*-form* η, *called* the transversal type of K, *defined on a neighborhood of* $0 \in \mathbf{C}^2$ *and vanishing only at 0, a covering* $\{U_\alpha\}$ *of a neighborhood of* K *in* $\mathbf{C}P(n)$ *and a family of holomorphic submersions* $\varphi_\alpha : U_\alpha \to \mathbf{C}^2$ *satisfying:* $\varphi_\alpha^{-1}(0) = K \cap U_\alpha$, $\varphi_\alpha^* \eta$ *defines* $\mathcal{F}$ *in* U_α.

(iii) $K(\mathcal{F})$ *is persistent under small perturbations of* $\mathcal{F}$, *namely, fixed any* $p \in K(\mathcal{F})$ *with defining* 1*-form* $\varphi^*\eta$ *as above, and for any foliation* $\mathcal{F}'$ *sufficiently close to* $\mathcal{F}$, *there is a holomorphic* 1*-form* η' *close to* η *and a submersion* φ' *close to* φ, *such that* $\mathcal{F}'$ *is defined by* $(\varphi')^*\eta'$ *near the point* p.

(iv) *Let* $K \subset K(\mathcal{F})$ *be a connected compact component such that the first Chern class of the normal bundle of* K *in* $\mathbf{C}P(n)$ *is non-zero, then the transversal type of* K *is* $\eta(x,y) = pxdy - qydx$, $p,q \in \mathbf{Z}$ *and this transversal type is constant through deformations.*

When the transversal type of a component $K \subset K(\mathcal{F})$ is linearizable, there exists a transverse structure for $\mathcal{F}$ in a neighborhood of K in the ambient minus eventually the local separatrices of $\mathcal{F}$. This is stated in the following proposition:

PROPOSITION 5.1. – *Let* $n \geq 3$, $\mathcal{F}$, $K(\mathcal{F})$ *be as above. Let* $K \subset K(\mathcal{F})$ *be a connected component with linear transversal type* $\eta = \lambda xdy - \mu ydx$, $\lambda.\mu \neq 0$. *Then:* (i) $\lambda/\mu = p/q \in \mathbf{Q}$ *implies that* $\mathcal{F}$ *is transversely projective in a neighborhood of* K *in* $\mathbf{C}P(n)$; (ii) $\lambda/\mu \notin \mathbf{Q}$ *implies that* $\mathcal{F}$ *is transversely affine in some neighborhood of* K *in* $\mathbf{C}P(n)$, *minus the local set of separatrices sep*$(\mathcal{F},K)$ *through* K.

Proof. – We prove (i): It is enough to prove the following assertion.

ASSERTION 1. – Let $(f,g),(\tilde{f},\tilde{g}) : U \to \mathbf{C}^2$ be holomorphic submersions such that $pfdg - qgdf$ and $p\tilde{f}d\tilde{g} - q\tilde{g}d\tilde{f}$ define the same foliation $\mathcal{F}$ on U. Then we have $\tilde{f}^q/\tilde{g}^p = S(f^q/g^p)$ for some Möbius transformation $S(z) = \frac{az+b}{cz+d}$.

Proof of Assertion 1. – The foliation $\mathcal{F}$ has g^p/f^q and $\widetilde{g}^p/\widetilde{f}^q$ as meromorphic first integrals and has leaves of the form $\lambda.g^p - \mu.f^q = 0$ and $\lambda.\widetilde{g}^p - \mu.\widetilde{f}^q = 0$, $\lambda, \mu \in \mathbf{C}$. In particular $\{g = 0\}$, $\{\widetilde{g} - 0\}$, $\{f = 0\}$ and $\{\widetilde{f} = 0\}$ are leaves of $\mathcal{F}$. Therefore it is easy to see that there exists a Möbius transformation $S(z) = \frac{az+b}{cz+d}$, $a, d, b, c \in \mathbf{C}$, $\quad ad - bc = 1$, such that $\frac{\hat{g}^p}{\hat{f}^q} = S\left(\frac{\widetilde{g}^p}{\widetilde{f}^q}\right) = \frac{a\widetilde{g}^p + b\widetilde{f}^q}{c\widetilde{g}^p + d\widetilde{f}^q}$ defines a meromorphic first integral for $\mathcal{F}$ and the leaves $\{\hat{f} = 0\}$ and $\{f = 0\}$ coincide, the same holding for the leaves $\{\hat{g} = 0\}$ and $\{g = 0\}$. Now we only have to prove that $\hat{g}^p/\hat{f}^q = \lambda.\, g^p/f^q$ for some locally constant $\lambda \in \mathbf{C}^*$. In fact we have $\hat{f} = u.f$ and $\hat{g} = v.g$ for some holomorphic non-vanishing u, v in U. This implies that $\frac{v^p}{u^q} = \frac{(\hat{g}^p/\hat{f}^q)}{(g^p/f^q)}$ is a quotient of first integrals and therefore v^p/u^q is a holomorphic first integral for $\mathcal{F}$ in U. Since v^p/u^q is holomorphic we conclude that v^p/u^q is locally constant in U. This proves the assertion and thus (i). $\square$

Now we prove (ii): In fact it is possible to prove the following:

ASSERTION 2. – $\mathcal{F}$ is given in a neighborhood V of K by a closed meromorphic 1-form w with $(w)_\infty = K \cup \operatorname{sep}(\mathcal{F}, K)$ having order one.

Proof of Assertion 2. – Let us assume that $n = 3$ (this only simplifies the notation). Given any point $p \in K$ we can choose an open set $U \ni p$ and local coordinates $(x, y, z) \in U$ centered at p such that $\mathcal{F}\big|_U$ is given by the closed meromorphic 1-form $w_U = \frac{\lambda\, dx}{x} - \frac{dy}{y}$ and $K \cap U = \{x = y = 0\}$. Suppose now tht $\widetilde{p} \in K$ is another point, $(\widetilde{x}, \widetilde{y}, \widetilde{z}) \in \widetilde{U}$ $w_{\widetilde{U}} = \frac{\lambda\, d\widetilde{x}}{\widetilde{x}} - \frac{d\widetilde{y}}{\widetilde{y}}$ are chosen in the same way and $U \cap \widetilde{U} \neq \phi$. We can also assume that $(x = 0)$ and $(\widetilde{x} = 0)$ coincide in $U \cap \widetilde{U}$ the same holding for $(y = 0)$ and $(\widetilde{y} = 0)$. Then in $U \cap \widetilde{U}$ we have $w_{\widetilde{U}} = f.w_U$ for some meromorphic function f. Since w_U and $w_{\widetilde{U}}$ have order one polar divisors coinciding in $U \cap \widetilde{U}$ it follows that f is holomorphic in $U \cap \widetilde{U}$ and since $0 = dw_U = dw_{\widetilde{U}}$ it follows that f is a holomorphic first integral for $\mathcal{F}\big|_{U\cap\widetilde{U}}$ and since the transversal type of K does not admit a holomorphic first integral ($\lambda \notin \mathbf{Q}$) it follows that $f = f(z)$. But since w_U and $w_{\widetilde{U}}$ do not depend on z and $\widetilde{z}$ it follows that f is locally constant in $U \cap \widetilde{U}$. Finally since $w_{\widetilde{U}}$ and w_U have residue equal to 1 along $\{x = 0\} \cap U \cap \widetilde{U} = \{\widetilde{x} = 0\} \cap U \cap \widetilde{U}$ it follows that $f \equiv 1$ and therefore $w_U \equiv w_{\widetilde{U}}$ in $U \cap \widetilde{U}$. This shows the assertion and finishes the proof of the proposition. $\square$

We may also use an extension lemma.

PROPOSITION 5.2 [11]. – *Let $K \subset \mathbf{C}P(n)$, $n \geq 3$, be an algebraic codimension $-(n-1)$ smooth submanifold wich is a complete intersection. Then any meromorphic object defined in some neighborhood of K in $\mathbf{C}P(n)$, extends meromorphically to $\mathbf{C}P(n)$.*

The proposition above is a consequence of the general form of Levi's Theorem (*see* [11]) for 2-complete manifolds. Now we state a first consequence of our approach ([1]).

PROPOSITION 5.3. – *Let $\mathcal{F} \in \mathcal{F}(k, n)$, $n \geq 3$ be a foliation with Kupka set $K(\mathcal{F})$. Suppose that there exists a compact component $K \subset K(\mathcal{F})$ which is a complete intersection. Then $\mathcal{F}$ has a rational first integral.*

([1]) I am grateful to A. Lins Neto for suggesting this application.

Proof. – Let η be the transversal type of K. Using Theorem 5.3 (iv), we can assume that η is $pxdy - qydx$, $p, q \in \mathbf{N}$. According to Proposition 5.1 this implies that $\mathcal{F}$ is transversely projective in V^n for some open neighborhood V^n of K in $\mathbf{C}P(n)$. Using Proposition 5.2 we prove that $\mathcal{F}$ is transversely projective in $\mathbf{C}P(n)$. The proposition now follows from Corollary 4.1. □

This proposition is originally found, with a different proof, in [11].

In order to prove Theorems 5.1 and 5.2 we will need the following result:

PROPOSITION 5.4. – *Let $\mathcal{F} \in \mathcal{F}(k, n)$, $n \geq 3$ be a foliation with Kupka set $K(\mathcal{F})$. Suppose that there exists a compact component $K \subset K(\mathcal{F})$ which is a complete intersection. Then for any small perturbation $\mathcal{F}'$ of $\mathcal{F}$ there exists a compact component $K' \subset K(\mathcal{F}')$ which is also a complete intersection.*

Proof. – Immediate consequence of Theorem 5.3 and of a Theorem of Sernesi ([29]) assuring that holomorphic perturbations of complete intersections on $\mathbf{C}P^n$ are still complete intersections on $\mathbf{C}P(n)$. □

Proof of Theorem 5.2. – Consider $w = pgdf - qfdg$, then w defines $\mathcal{F}$ and we have that $K(\mathcal{F}) = \{p \mid w(p) = 0, dw(p) \neq 0\}$ constains the compact components of the transversal intersection $K = \{f = 0\} \cap \{g = 0\}$. Any component K_o of K is a complete intersection and has transversal type $pydx - qxdy$. It follows from Proposition 5.4 that any small deformation $\mathcal{F}'$ of $\mathcal{F}$ has a complete intersection Kupka component K_o' with the same transversal type, this component K_o' has a transverse projective structure in a neighborhood of K_o'.

Using now the fact that K_o' is a complete intersection one can conclude (as in the proof of Proposition 5.3) that $\mathcal{F}'$ has a projective transverse structure in all $\mathbf{C}P(n)$ and according to Proposition 5.3 above $\mathcal{F}'$ has a meromorphic first integral. Using the fact that any such K_o' has transversal type $pydx - qydx$ we conclude that $\mathcal{F}'$ has a first integral of the type f'^p/g'^q. □

Proof of Theorem 5.1. – Let $\mathcal{F} = \mathcal{F}(w)$ be given, where w is generic and as in Example 5. Then it is easy to see that $K(\mathcal{F}) = \bigcup{}^{i \neq j \neq k}_{\quad i \neq k} \rightarrow (K_{ij} \backslash K_{ik})$ where $K_{ij} = \{f_i = f_j = 0\}$. Therefore any component $K \subset K(\mathcal{F})$ is a complete intersection curve minus a finite number of points which are *regular singularities* of $\mathcal{F}$ and has transversal type $\lambda xdy - \mu ydx = 0$, $\lambda/\mu \in \mathbf{C}\backslash\mathbf{R}$. Using Theorem 5.3 (iv), we conclude that any foliation $\mathcal{F}'$ close enough to $\mathcal{F}$ has a Kupka component K' near to K which also has a transversal type $\lambda' xdy - \mu' ydx = 0$ with $\frac{\lambda'}{\mu'}$ close to $\frac{\lambda}{\mu}$ so that $\frac{\lambda'}{\mu'} \in \mathbf{C}\backslash\mathbf{R}$. Using arguments similar to the ones used in the proof of Theorem 5.2 (Proposition 5.4 is replaced by an analogous which is based on Sernesi's Theorem, Theorem 5.3 and in the notion of regular singularity studied in [7]) one can show that if $\mathcal{F}'$ is close enough to $\mathcal{F}$ then $\mathcal{F}'$ exhibits K' as a holomorphic perturbation of K, and also a complete intersection on $\mathbf{C}P(n)$. The foliation $\mathcal{F}'$ is transversely affine in a neighborhood of K' minus the local set of separatrices $\operatorname{sep}(K')$ as it follows from Proposition 5.1 and from Theorem 5.2 above. Therefore it follows from Proposition 5.2 above and from Proposition 1.1 of I that $\mathcal{F}'$ is transversely affine in $\mathbf{C}P(n)\backslash S'$ for some algebraic set $S' \subset \mathbf{C}P(n)$ which is a finite union of $\mathcal{F}'$-leaves. Since $\mathcal{F}'$ has only generic singularities it follows from I Theorem 4.3 that $\mathcal{F}'$ is a logarithmic foliation. □

B. The local case

Let $\mathcal{F}$ be a singular foliation on U, $U \subset \mathbf{C}^n$ an open neighborhood of the origin $0 \in \mathbf{C}^n$, having an isolated singularity at $0 \in \mathbf{C}^n$. Let $S \subset U$ be a finite union of separatrices of $\mathcal{F}$. The foliation $\mathcal{F}$ can be supposed to be defined by a holomorphic integrable 1-form Ω in U such that $\Omega(p) = 0 \Leftrightarrow p = 0$. It is easy to see that Ω always admits a meromorphic logarithmic derivative, say η, in U. (The same proof of Proposition 2.2 works).

The arguments used in the proof of Proposition 3.1 apply here in this local version:

PROPOSITION 5.5. – *Suppose $\mathcal{F}$ is transversely projective in $U \backslash S$. Then there exists an open set $0 \in V \subset U$ and meromorphic projective triple (Ω, η, ξ) in V.*

Assume that $\mathcal{F}$ is not transversely affine in $V \backslash S$ for any analytic codimension one subset S in $(\mathbf{C}^n, 0)$. Denote by $\mathcal{F}^\perp = \mathcal{F}^\perp(\xi)$ the transverse foliation defined by ξ, ξ belonging to a projective triple (Ω, η, ξ) as above. As in Theorem 4.1 we can prove:

THEOREM 5.4. – *Let $\mathcal{F}$, S, (Ω, η, ξ), $\mathcal{F}^\perp$ be as above. Then $\mathcal{F}$ induces a germ of Riccati foliation $(y^2 c(x) - a(x))dx - p(x)dy = 0$ where x, y are holomorphic and $p(z)$, $a(z)$, $c(z)$ are holomorphic; provided that $\mathcal{F}^\perp$ admits a meromorphic first integral at the origin. This occurs in particular if $\mathcal{F}^\perp$ is transversely affine in $U \backslash S$.*

Proof of Theorem 5.4. – The first part of Theorem 5.4 is proved just like in Theorem 5.1 (Stein's Fatorization Theorem is replaced by the local version stated in [24]). As in Theorem 4.1 it can be proved that if $\mathcal{F}^\perp$ is transversely affine in $U \backslash S$ then we have $d(\sqrt{x}.\xi) = 0$ for some holomorphic function x. We state the basic tool of the current proof.

CLAIM. – Let (x, y) be holomorphic coordinates in U and let ξ be a meromorphic 1-form in U such that $d\xi = -\frac{1}{2}\frac{dx}{x} \wedge \xi$. Then $\xi = \frac{d(\sqrt{x}.g(x,y))}{\sqrt{x}}$ for some meromorphic $g(x, y)$ in U. In particular $x.g^2(x, y)$ is a meromorphic first integral for ξ in U. Suppose that $d\xi = -\frac{1}{2}\left(\frac{cx}{x} + \frac{dy}{y}\right) \wedge \xi$. Then $\xi = \frac{d(\sqrt{xy}\cdot g(x,y))}{\sqrt{xy}}$ for some meromorphic $g(x, y)$ in U. In particular $xy \cdot g^2(x, y)$ is a meromorphic first integral for ξ in U. $\square$

Using now the claim above one can finish the proof of Theorem 5.4: In fact, if $dx(0) \neq 0$ then we are under the hypothesis of the claim above. Suppose now $dx(0) = 0$. We choose local coordinates (u, v) centered at 0 such that $x = u^n \cdot v^m f(u, v)$ where $n, m \in \mathbf{N}$ and $f(u, v)$ is holomorphic non-vanishing at the origin 0.

We write $n = 2k + \ell$, $m = 2r + s$ where $k, \ell, r, s \in \mathbf{N}$ with $\ell, s \in \{0, 1\}$. Thus we have $\sqrt{x}\xi = u^k \cdot v^r \cdot \sqrt{u^\ell \cdot v^s} \cdot \xi$. Replacing ξ by the meromorphic 1-form $u^k \cdot v^r \xi$ we can assume that $k = r = 0$ so that $\sqrt{x}\xi = \sqrt{u^\ell \cdot v^s} \cdot \xi$. If $\ell = s = 0$ then $\sqrt{x}\xi$ is also meromorphic and closed so that we can suppose that ℓ or s is equal to 1. In this case we can apply the assertion to conclude that $\mathcal{F}^\perp$ has a meromorphic first integral at 0. Using now the same arguments given for the proof of Theorem 4.1 one can conclude the proof. $\square$

6. Transversely homogeneous foliations

In this section we prove that a holomorphic transversely homogeneous foliation of codimension one is a transversely projective foliation ([1]).

([1]) I am grateful to E. Ghys for suggesting me this result.

The following definition is found in [17] pp. 245 in the non-singular case. Here we have a natural generalization for the case the foliation admits singularities:

DEFINITION 6.1. – Let $\mathcal{F}$ be a holomorphic singular foliation on a complex manifold P. Let G be a simply-connected Lie group and $H \subset G$ be a connected closed subgroup of G. We say that $\mathcal{F}$ is *transversely homogeneous* in P of model G/H if $P \backslash s(\mathcal{F})$ admits an open cover $\bigcup U_i = P \backslash s(\mathcal{F})$ with holomorphic submersions $y_i: U_i \to G/H$ satisfying: (i) $\mathcal{F}\big|_{U_i}$ is defined by y_i, (ii) In each $U_i \cap U_j \neq \phi$ we have $y_i = g_{ij} \circ y_j$ for some locally constant $g_{ij}: U_i \cap U_j \to G$.

THEOREM 6.1. – *A holomorphic singular transversely homogeneous foliation of codimension one is a transversely projective foliation.*

Proof. – In fact, G/H is a simply-connected complex manifold of dimension one. Thus by the Uniformization Theorem of Koebe-Riemann we have either $G/H \equiv \overline{\mathbf{C}}, \mathbf{C}$ or $\mathbf{D}$ the unitary disc. This implies that either $G \subset \text{Aut}(\overline{\mathbf{C}}) = \mathbf{P}SL(2, \mathbf{C}), G \subset \text{Aut}(\mathbf{C}) = \text{Aff}(\mathbf{C})$ or $G \subset \text{Aut}(\mathbf{D}) \cong \mathbf{P}SL(2, \mathbf{R})$. This proves the theorem. □

7. Foliations which are transverse to a compact Riemann surface

In this section we give a geometric condition that assures the existence of projective transverse structure for a codimension one singular foliation. Roughly speaking it is enough to have the foliation transversal to some fiber of a holomorphic fibration by compact Riemann surfaces. Using the Uniformization Theorem of Koebe-Riemann for Riemann surfaces we conclude existence of projective structures for any Riemann surface. Let $\mathcal{F}$ be a foliation on M and let $R \subset M$ be a holomorphically embedded Riemann surface.

DEFINITION 8.1. – We say that R is (*totally*) *transverse* to $\mathcal{F}$ if: i) $s(\mathcal{F}) \cap R = \phi$. ii) Given any point $p \in R$ the leaf L_p of $\mathcal{F}$ through p is transverse to R at p.

PROPOSITION 7.1. – *Let $\mathcal{F}$, R and M be as above and suppose that R is (totally) transverse to $\mathcal{F}$. Then there exists a neighborhood V of R in M such that $\mathcal{F}\big|_V$ is transversely affine or projective.*

Proof. – The proof is a straighforward consequence of the total transversality and of the existence of affine or projective structures for R. We leave the details to the reader. □

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