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LOGARITHMIC DIFFERENTIAL OPERATORS AND LOGARITHMIC DE RHAM COMPLEXES RELATIVE TO A FREE DIVISOR

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ABSTRACT. – We prove a structure theorem for differential operators in the 0-th term of the V-filtration relative to a free divisor. manifold. As an application, we give a formula for the logarithmic de Rham complex with respect to a free divisor in terms of V_0-modules, which generalizes the classical formula for the usual de Rham complex in terms of $\mathcal{D}$-modules, and the formula of Esnault-Viehweg in the case of a normal crossing divisor. We also give a sufficient algebraic condition for perversity of the logarithmic de Rham complex. © Elsevier, Paris

RÉSUMÉ. – Nous prouvons un théorème de structure pour les opérateurs différentiels dans le terme 0 de la V-filtration relative à un diviseur libre. Comme application, nous donnons une formule pour le complexe de de Rham logarithmique par rapport à un diviseur libre en termes de V_0-modules, qui généralise la formule classique pour le complexe de de Rham usuel en termes de $\mathcal{D}$-modules et celle de Esnault-Viehweg dans le cas d'un diviseur à croisements normaux. Nous donnons aussi une condition algébrique suffisante pour la perversité des tels complexes. © Elsevier, Paris

Introduction

Let X be a complex manifold and $Y \subset X$ be a divisor. We consider the $\mathcal{O}_X$-modules of the logarithmic derivations, $\mathcal{D}\mathrm{er}(\log Y)$, and logarithmic forms, $\Omega^1_X(\log Y)$, due to K. Saito; and the $\mathcal{V}-$filtration of Malgrange-Kashiwara relative to Y on the ring of differential operators on X, $\mathcal{V}^Y_\bullet(\mathcal{D}_X)$. We prove:

THEOREM 1. – *If Y is free then* $\mathcal{V}^Y_0(\mathcal{D}_X) = \mathcal{O}_X[\mathcal{D}\mathrm{er}(\log Y)]$.

As a consequence of this theorem, $\mathcal{V}^Y_0(\mathcal{D}_X)$ is a coherent sheaf. Another consequence is the equivalence between $\mathcal{V}^Y_0(\mathcal{D}_X)$-modules and $\mathcal{O}_X$-modules with logarithmic connections. Therefore, a $\mathcal{V}^Y_0(\mathcal{D}_X)$-module (or logarithmic $\mathcal{D}_X$-module) $\mathcal{M}$ defines a logarithmic de Rham complex $\Omega^\bullet_X(\log Y)(\mathcal{M})$. We also use this theorem in the proof of:

THEOREM 2. – *If Y is free and $\mathcal{M}$ is a left $\mathcal{V}^Y_0(\mathcal{D}_X)$-module, then there is a canonical isomorphism:*

$$\Omega^\bullet_X(\log Y)(\mathcal{M}) \cong \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{V}^Y_0(\mathcal{D}_X)}(\mathcal{O}_X, \mathcal{M}).$$

To show this, we first construct a resolution of $\mathcal{O}_X$ as $\mathcal{V}^Y_0(\mathcal{D}_X)$-module, which we call the logarithmic Spencer complex and denote by $\mathcal{S}p^\bullet(\log Y)$. Finally, we define the notion of Koszul free divisor (a free divisor for which the symbols of a basis of logarithmic

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derivations form a regular sequence in the graded ring associated to the filtration by the order on $\mathcal{D}_X$). Nonsingular and normal crossing divisors and all the plane curves are Koszul free divisors. We prove:

THEOREM 3. – *If Y is a Koszul free divisor, then the logarithmic de Rham complex $\Omega_X^\bullet(\log Y)$ is perverse.*

Some results of this paper have been announced in [4]. We give here the complete proofs of all of those results announced and other new results.

1. Notations and Preliminaries

Let X be a complex analytic manifold of dimension n and Y be a divisor of X defined by the ideal $\mathcal{I}$. Let $\mathcal{D}_X$ denote the sheaf of linear differential operators over X, $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$ the sheaf of derivations of $\mathcal{O}_X$, and $\mathcal{D}_X[\star Y]$ the sheaf of meromorphic differential operators with poles along Y. Given a point x of Y, we note by $I=(f)$, $\mathcal{D}$, $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O})$, $\mathcal{O}$ and $\mathcal{D}_X[\star Y]_x$ the respective stalks at x. Let $F^\bullet$ denote the filtration of $\mathcal{D}_X$ by the order of the operators and $\Omega_X^\bullet[\star Y]$ the meromorphic de Rham complex with poles along Y.

1.1. Logarithmic forms and logarithmic derivations. Free divisors

We recall some notions of [7] that we will use repeatedly:

A section δ of $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$, defined over an open set U of X, is called a *logarithmic derivation* (or vector field) if for each point x in $Y\cap U$, $\delta_x(\mathcal{I}_x)\subset\mathcal{I}_x$ (if $I=\mathcal{I}_x=(f)$, it is sufficient that $\delta_x(f)\in(f)\mathcal{O}$). The sheaf of logarithmic derivations is denoted by $\mathcal{D}\mathrm{er}(\log Y)$, and is a coherent $\mathcal{O}_X$-submodule of $\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$ and a Lie subalgebra, whose stalks are $\mathrm{Der}(\log f)=\mathcal{D}\mathrm{er}(\log Y)_x=\{\delta\in\mathrm{Der}_{\mathbb{C}}(\mathcal{O})\ /\ \delta(f)\in(f)\}$.

We say that a meromorphic q-form ω with poles along Y, defined in an open set U, is a logarithmic q-form along Y or, simply, a *logarithmic q-form*, if for every point x in U, $f\omega$ and $\mathrm{d}f\wedge\omega$ are holomorphic at x. The sheaf of logarithmic q-forms along Y in U is denoted by $\Omega_X^q(\log Y)(U)$. This definition gives rise to a coherent $\mathcal{O}_X$-module $\Omega_X^q(\log Y)$, whose stalks are $\Omega^q(\log f)=\Omega_X^q(\log Y)_x=\{\omega\in\Omega_X^q[\star Y]_x\ /\ f\omega\in\Omega^q,\ \mathrm{d}f\wedge\omega\in\Omega^{q+1}\}$.

The logarithmic q-forms along Y define a subcomplex of the meromorphic de Rham complex along Y, that we call the logarithmic de Rham complex and denote by $\Omega_X^\bullet(\log Y)$.

Contraction of forms by vector fields defines a perfect duality between the $\mathcal{O}_X$-modules $\Omega_X^1(\log Y)$ and $\mathcal{D}\mathrm{er}(\log Y)$, that we denote by $\langle\ ,\ \rangle$. Thus, both of them are reflexive. In particular, when $n=\dim_{\mathbb{C}}X=2$, $\Omega_X^1(\log Y)$ and $\mathcal{D}\mathrm{er}(\log Y)$ are locally free $\mathcal{O}_X$-modules of rank 2.

We say that Y *is free at* x, or I is a free ideal of $\mathcal{O}$, if $\mathrm{Der}(\log I)$ is free as $\mathcal{O}$-module (of rank n). If $f\in\mathcal{O}$, we say that f *is free* if the ideal $I=(f)$ is free. We say that Y is free if it is at every point x. In this case, $\mathcal{D}\mathrm{er}(\log Y)$ is a locally free $\mathcal{O}_X$-module of rank n (see [18], [7, Examples 1.1], [6]). We can use the following criterion to determine when a divisor Y is free at x:

SAITO'S CRITERION. – The $\mathcal{O}$-module $\mathrm{Der}(\log f)$ is free if and only if there exist n elements $\delta_1,\delta_2,\cdots,\delta_n$ in $\mathrm{Der}(\log f)$, with

$$\delta_i=\sum_{j=1}^n a_{ij}(z)\frac{\partial}{\partial z_j}\ (i=1,\ldots,n),$$

where $z = (z_1, z_2, \cdots, z_n)$ is a system of coordinates of X centered in x, such that the determinant $\det(a_{ij})$ is equal to af, with $a \in \mathcal{O}$ a unit. Moreover, in this case, $\{\delta_1, \delta_2, \cdots, \delta_n\}$ is a basis of Der(log f).

When Y is free, we have the equality $\Omega_X^p(\log Y) = \wedge^p \Omega_X^1(\log Y)$. Using the fact that $\Omega_X^1(\log Y) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}\text{er}(\log Y), \mathcal{O}_X)$, we can construct a natural isomorphism $\gamma^p : \Omega_X^p(\log Y) \cong \mathcal{H}om_{\mathcal{O}_X}(\wedge^p \mathcal{D}\text{er}(\log Y), \mathcal{O}_X)$, defined locally by $\gamma^p(\omega_1 \wedge \cdots \wedge \omega_p)(\delta_1 \wedge \cdots \wedge \delta_p) = \det(\langle \omega_i, \delta_j \rangle)_{1 \le i,j \le p}$.

1.2. $\mathcal{V}$-filtration. logarithmic operators

We define the $\mathcal{V}$-filtration relative to Y on $\mathcal{D}_X$ as in the smooth case ([12], [11]): $\mathcal{V}_k^Y(\mathcal{D}_X) = \{P \in \mathcal{D}_X \ / \ P(\mathcal{I}^j) \subset \mathcal{I}^{j-k}, \forall j \in \mathbb{Z}\}$, $k \in \mathbb{Z}$, where $\mathcal{I}^p = \mathcal{O}_X$ when p is negative. Similarly, $\mathcal{V}_k^I(\mathcal{D}) = \{P \in \mathcal{D} \ / \ P(I^j) \subset I^{j-k}, \forall j \in \mathbb{Z}\}$, with k an integer, and $I^p = \mathcal{O}$ when $p \ge 0$. If $I = (f)$, we note $\mathcal{V}_k^f(\mathcal{D}) = \mathcal{V}_k^I(\mathcal{D})$. A logarithmic differential operator (or, simply, a logarithmic operator) is a differential operator of degree 0 with respect to the $\mathcal{V}$-filtration. As $F^1(\mathcal{D}_X) = \mathcal{O}_X \oplus \mathcal{D}\text{er}_{\mathbb{C}}(\mathcal{O}_X)$, we have $F^1(\mathcal{V}_0^Y(\mathcal{D}_X)) = \mathcal{O}_X \oplus \mathcal{D}\text{er}(\log Y)$. We also have $\mathcal{D}\text{er}(\log Y) = \mathcal{D}\text{er}_{\mathbb{C}}(\mathcal{O}_X) \cap \mathcal{V}_0^Y(\mathcal{D}_X) = \mathcal{G}r^1_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))$.

REMARK 1.2.1. – The inclusion $\mathcal{D}\text{er}(\log Y) \subset \mathcal{G}\text{r}_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))$ gives rise to a canonical graded morphism of graded algebras:

$$\kappa : \ \mathcal{S}\text{ym}_{\mathcal{O}_X}(\mathcal{D}\text{er}(\log Y)) \longrightarrow \mathcal{G}\text{r}_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X)).$$

Similarly, we have a canonical graded morphism of graded $\mathcal{O}$-algebras: κ_x : $\text{Sym}_{\mathcal{O}}(\text{Der}(\log I)) \to \text{Gr}_{F^\bullet}(\mathcal{V}_0^I(\mathcal{D}))$, which is the stalk of κ at x.

2. Logarithmic operators relative to a free divisor

2.1. The Structure Theorem

We denote by $\{\ ,\ \}$ the Poisson bracket defined in the graded ring $\text{Gr}_{F^\bullet}(\mathcal{D})$ (cf. [14], [10]). Given two polynomials F, G in $\text{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{O}[\xi_1, \cdots, \xi_n]$:

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial \xi_i}\frac{\partial G}{\partial x_i} - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial \xi_i}.$$

PROPOSITION 2.1.1. – *Let f be free. Consider a basis $\{\delta_1, \cdots, \delta_n\}$ of* Der(log f). *Let R_0 be a polynomial in* $\text{Gr}_{F^\bullet}(\mathcal{D})$, *homogeneous of order d, and such that there exist other polynomials R_k in* $\text{Gr}_{F^\bullet}(\mathcal{D})$, *with $k = 1, \cdots, d$, homogeneous of order $d - k$ such that:*

$$\{R_k, f\} = fR_{k+1}, \ (0 \le k < d) \tag{1}$$

(we will say that R_0 verifies the property (1) for $R_1, R_2, \cdots, R_d$). Then there exist polynomials H_j^k in $\text{Gr}_{F^\bullet}(\mathcal{D})$, *homogeneous of order $d - k - 1$, with $j = 1, \cdots, n$ and $k = 1, \cdots, d - 1$, such that:*

a) $R_k = \sum_{j=1}^n H_j^k \sigma(\delta_j)$, where $\sigma(\delta_j)$ denotes the principal symbol of δ_j.

b) $\{H_j^k, f\} = fH_j^{k+1}$ $(1 \le j \le n, \ 0 \le k < d - 1)$. This is the same as saying: H_j^k satisfies the property (1) for $H_j^{k+1}, \cdots, H_j^{d-1}$.

Proof. – Let $\{\delta_1, \cdots, \delta_n\}$ be a basis of Der(log f) and $A = (\alpha_i^j)$ such that:

$$\delta_j = \sum_{i=1}^{n} \alpha_i^j \frac{\partial}{\partial x_i} = \underline{\alpha}^j \bullet \underline{\partial}^t,$$

with $j = 1, \cdots, n$. We consider the ring $\mathcal{O}_{2n} = \mathbb{C}\{x_1, \cdots, x_n, \xi_1, \cdots, \xi_n\}$. Thanks to Saito's Criterion, we know that $\{\delta_1, \cdots, \delta_n, \frac{\partial}{\partial \xi_1}, \cdots, \frac{\partial}{\partial \xi_n}\}$ is a basis of the $\mathcal{O}_{2n}$-module $\mathrm{Der}_{\mathcal{O}_{2n}}(\log f)$. So, as we have, for $k = 1, \cdots, d$, $\{R_k, f\} \in (f)$, then there exist homogeneous polynomials G_j^k in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, of degree $d - k - 1$, or null, with $j = 1, \cdots, n$ and $k = 1, \cdots, d-1$, such that:

$$((R_k)_{\xi_1}, (R_k)_{\xi_2}, \cdots, (R_k)_{\xi_n}) = (G_1^k, G_2^k, \cdots, G_n^k)A \qquad \left((R_k)_{\xi_i} = \frac{\partial R_k}{\partial \xi_i}\right).$$

Using the Euler relation $R_k = \frac{1}{d}\sum_{i=1}^{n}(R_k)_{\xi_i}\xi_i$, and as $\sigma(\delta_i) = \underline{\alpha}^i \bullet \underline{\xi}^t$, we obtain $R_k = \frac{1}{d}\sum_{j=1}^{n} G_j^k \sigma(\delta_j)$. By Saito's Criterion, the determinant of the matrix A is equal to $g = uf$, with $u \in \mathcal{O}$ invertible. Let $B = (b_{ij}) = \mathrm{Adj}\,(A)^t$. We have:

$$g\{G_j^k, f\} = \sum_{i=1}^{n} f_{x_i} \frac{\partial(\sum_{l=1}^{n}(R_k)_{\xi_l} b_{lj})}{\partial \xi_i} \stackrel{(1)}{=} f \sum_{l=1}^{n} b_{lj}(R_{k+1})_{\xi_l} = fgG_j^{k+1}.$$

We conclude by setting $H_j^k = \frac{1}{d}G_j^k$, for $j = 1, \cdots, n$ and $k = 0, \cdots, d-1$. □

PROPOSITION 2.1.2. – *Let* $\{\delta_1, \cdots, \delta_n\}$ *be a basis of* Der(log f)*. If a polynomial* R_0 *of* $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$ *is homogeneous and satisfies the property (1) of proposition 2.1.1, we can find a differential operator* Q *in* $\mathcal{O}[\delta_1, \cdots, \delta_n]$ *such that* R_0 *is the symbol of* Q.

Proof. – We will do the proof by induction on the order of R_0. If $R_0 \in \mathcal{O}$, it is obvious. We suppose that the result holds if the order of R_0 is less than d. Now let R_0 of order d satisfying (1). By proposition 2.1.1 there exist n homogeneous polynomials H_j^0 of order $d-1$ such that: $R_0 = \sum_{j=1}^{n} H_j^0 \sigma(\delta_j)$, H_j^0 satisfies (1) $(j = 1, \ldots, n)$. By induction hypothesis, there exist $Q_j \in \mathcal{O}[\delta_1, \cdots, \delta_n]$ such that $H_j^0 = \sigma(Q_j)$. So $R_0 = \sigma(Q)$, and $Q = \sum_{i=1}^{n} Q_i \delta_i \in \mathcal{O}[\delta_1, \cdots, \delta_n]$. □

REMARK 2.1.3. – Really, the previous argument proves that if R_0 satisfies (1), then R_0 is a polynomial in $\mathcal{O}[\sigma(\delta_1), \cdots, \sigma(\delta_n)]$.

THEOREM 2.1.4. – *If* f *is free and* $\{\delta_1, \cdots, \delta_n\}$ *is a basis of* Der(log f)*, each logarithmic operator* P *can be written in a unique way as a polynomial* $P = \sum \beta_{i_1 \cdots i_n} \delta_1^{i_1} \cdots \delta_n^{i_n}$, $\beta_{i_1 \cdots i_n} \in \mathcal{O}$*. In other words, the ring of logarithmic operators is the* $\mathcal{O}$*-subalgebra of* $\mathcal{D}$ *generated by logarithmic derivations:*

$$\mathcal{V}_0^I(\mathcal{D}) = \mathcal{O}[\delta_1, \cdots, \delta_n] = \mathcal{O}[\mathrm{Der}(\log f)].$$

Proof. – The inclusion $\mathcal{O}[\delta_1, \cdots, \delta_n] \subseteq \mathcal{V}_0^I(\mathcal{D})$ is clear. We prove the other inclusion by induction on the order of $P_0 \in \mathcal{V}_0^I(\mathcal{D})$. If the order of P_0 is zero, then it is a holomorphic function and the result is obvious. We suppose that the result is true for every logarithmic operator Q whose order is strictly less than d. Let P_0 be a logarithmic operator of order d. We know that: $[P_0, f] = fP_1$, with $P_1 \in \mathcal{V}_0^I(\mathcal{D})$. So, there exist some P_k, with

$k = 0, \cdots, d$, such that $[P_k, f] = fP_{k+1}$. If we set $R_k = \sigma(P_k)$, in the case that P_k has order $d-k$, and $R_k = 0$ otherwise, we obtain $\{R_k, f\} = fR_{k+1}$. By the proposition 2.1.2, there exists Q in $\mathcal{O}[\delta_1, \cdots, \delta_n]$ of order d and such that $\sigma(P_0) = \sigma(Q)$. We apply the induction hypothesis to $P_0 - Q$ and obtain $P_0 = P_0 - Q + Q \in \mathcal{O}[\delta_1, \cdots, \delta_n]$.

On the other hand, using the structure of Lie algebra it is clear that we can write a logarithmic operator as a $\mathcal{O}$-linear combination of the monomials $\{\delta_1^{i_1} \cdots \delta_n^{i_n}\}$. The uniqueness of this expression follows from the fact that these monomials are linearly independent over $\mathcal{O}$. □

REMARK 2.1.5. – As a immediate consequence of the theorem (see remark 2.1.3), we obtain an isomorphism $\alpha : \mathrm{Gr}_{F^\bullet}(\mathcal{V}_0^I(\mathcal{D})) \cong \mathcal{O}[\sigma(\delta_1), \cdots, \sigma(\delta_n)]$.

COROLLARY 2.1.6. – *If Y is free at x, the morphism κ_x from the symmetric algebra* $\mathrm{Sym}_{\mathcal{O}}(\mathrm{Der}(\log f))$ *to* $\mathrm{Gr}_{F^\bullet}(\mathcal{V}_0^f(\mathcal{D}))$ *(see remark 1.2.1) is an isomorphism of graded $\mathcal{O}$-algebras. As a consequence, if Y is a free divisor, the canonical morphism* $\kappa : \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{D}\mathrm{er}(\log Y)) \to \mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))$ *is an isomorphism.*

Proof. – Let x be in X and $f \in \mathcal{O}$ a reduced local equation of Y. Let $\{\delta_1, \cdots, \delta_n\}$ be a basis of Der(log f). The symmetric algebra of the $\mathcal{O}$-module Der(log f) is isomorphic to a polynomial ring:

$$\beta : \mathrm{Sym}_{\mathcal{O}}(\mathrm{Der}(\log f)) \cong \mathcal{O}[\sigma(\delta_1), \cdots, \sigma(\delta_n)].$$

Also $\oplus_{i=1}^n \mathcal{O}\sigma(\delta_i) = \mathrm{Gr}^1_{F^\bullet}(\mathcal{V}_0^I(\mathcal{D})) \subset \mathrm{Gr}_{F^\bullet}(\mathcal{V}_0^I(\mathcal{D}))$, where $\sigma(\delta_i)$ is the image of δ_i by the morphism κ_x. Therefore we conclude that the morphism $\kappa_x = \alpha^{-1}\beta$ is an isomorphism (see remark 2.1.5). On the other hand, the inclusion $\mathcal{D}\mathrm{er}(\log Y) = \mathcal{G}\mathrm{r}^1_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X)) \subset \mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))$ gives rise to a canonical graded morphism of graded $\mathcal{O}_X$-algebras (see remark 1.2.1) $\kappa : \mathcal{S}\mathrm{ym}_{\mathcal{O}_X}(\mathcal{D}\mathrm{er}(\log Y)) \to \mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))$, whose stalk at each point x of Y is the canonical graded isomorphism κ_x. □

COROLLARY 2.1.7. – $\mathcal{V}_0^Y(\mathcal{D}_X)$ *is a coherent sheaf of rings.*

Proof. – By [1, theorem 9.16, p. 83], we have to prove that $\mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))$ is coherent, but this sheaf is locally isomorphic to a polynomial ring which is coherent ([3, lemma 3.2, VI, p. 205]). □

2.2. Equivalence between $\mathcal{O}_X$-modules with a logarithmic connection and left $\mathcal{V}_0^Y(\mathcal{D}_X)$-modules

DEFINITION 2.2.1. – (cf. [8]) *Let $\mathcal{M}$ be a $\mathcal{O}_X$-module. A connection on $\mathcal{M}$, with logarithmic poles along Y, (or logarithmic connection on $\mathcal{M}$), is a $\mathbb{C}$-homomorphism $\nabla : \mathcal{M} \to \Omega^1_X(\log Y) \otimes \mathcal{M}$, that satisfies Leibniz's identity: $\nabla(hm) = \mathrm{d}h \cdot m + h \cdot \nabla(m)$, (d is the exterior derivative over $\mathcal{O}_X$). We will note $\Omega^q_X(\log Y)(\mathcal{M}) = \Omega^q_X(\log Y) \otimes \mathcal{M}$. If δ is a logarithmic derivation along Y, it defines a $\mathbb{C}$-morphism $\nabla_\delta : \mathcal{D}\mathrm{er}(\log Y) \to \mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{M})$, with $\nabla_\delta(m) = \langle \delta, \nabla(m) \rangle$.*

REMARK 2.2.2. – A logarithmic connection ∇ on $\mathcal{M}$ gives rise to a $\mathcal{O}_X$-morphism $\nabla' : \mathcal{D}\mathrm{er}(\log Y) \to \mathcal{H}\mathrm{om}_{\mathbb{C}}(\mathcal{M}, \mathcal{M})$, which satisfies Leibniz's condition ($\nabla'_\delta(fm) = \delta(f) \cdot m + f \cdot \nabla'_\delta(m)$). Conversely, given ∇' satisfying this condition, we define $\nabla : \mathcal{M} \to \Omega^1_X(\log Y)(\mathcal{M})$, with $\nabla(m)(\delta) = \nabla'_\delta(m)$.

DEFINITION 2.2.3. – *A logarithmic connection* ∇ *is integrable if, for each pair* δ *and* δ' *of logarithmic derivations, it satisfies* $\nabla_{[\delta,\delta']} = [\nabla_\delta, \nabla_{\delta'}]$, *([,] is the Lie bracket in* $\mathcal{D}\mathrm{er}(\log Y)$ *and the commutator in* $\mathcal{H}\mathrm{om}_{\mathbb{C}}(\mathcal{M},\mathcal{M})$*).*

Given a logarithmic connection ∇, *we can define, for* $q = 1,\cdots,n$, *a morphism* $\nabla^q : \Omega^q_X(\log Y)(\mathcal{M}) \to \Omega^{q+1}_X(\log Y)(\mathcal{M})$, $\nabla^q(\omega\otimes m) = d\omega\otimes m + (-1)^q\omega\wedge\nabla(m)$. *The integrability condition is equivalent to* $\nabla^q\circ\nabla^{q-1} = 0$, *for every* q (cf. [8]).

DEFINITION 2.2.4. – *Let* $\mathcal{M}$ *be a* $\mathcal{O}_X$*-module, and* ∇ *an integrable logarithmic connection along* Y *on* $\mathcal{M}$. *With the above notation, we call the logarithmic de Rham complex of* $\mathcal{M}$, *and we denote by* $\Omega^\bullet_X(\log\ Y)(\mathcal{M})$, *the complex (of sheaves of* $\mathbb{C}$*-vector spaces)* $(\Omega^\bullet_X(\log\ Y)(\mathcal{M}), \nabla^\bullet)$.

We consider the rings $R_0 = \mathcal{O}_X \subset R_1$ *and* $\mathbf{R} = \mathcal{V}_0^Y(\mathcal{D}_X) = \bigcup_{k\geq 0} R_k$ $(1\in R_0\subset\mathbf{R})$, *with* $R_k = F^k(\mathcal{V}_0^Y(\mathcal{D}_X))$. *They satisfy the following properties:*

(1) $\mathcal{G}r_{F^\bullet}(\mathbf{R})$ *is commutative;*

(2) the canonical morphism $\alpha : \mathrm{Sym}_{R_0}(\mathcal{G}r^1_{F^\bullet}(\mathbf{R})) \to \mathcal{G}r_{F^\bullet}(\mathbf{R})$, *defined by* $\alpha(s_1\otimes\cdots\otimes s_t) = s_1\cdots s_t$, *is an isomorphism (see Corollary 2.1.6);*

(3) R_1 *is an* (R_0,R_0)*-bimodule, and a Lie algebra (*$\mathcal{G}r_{F^\bullet}(\mathbf{R})$ *is conmutative);*

(4) R_0 *is a sub-*(R_0,R_0)*-bimodule of* R_1 *such that the two induced structures of* R_0*-module over the quotient* R_1/R_0 *are the same.*

Let $\mathbf{T}_{R_0}(R_1) = R_0\oplus R_1\oplus(R_1\otimes_{R_0}R_1)\oplus\cdots$ be the tensor algebra of the (R_0,R_0)-bimodule R_1, and let $\psi : \mathbf{T}_{R_0}(R_1)\to\mathbf{R}$ be the canonical morphism defined by the inclusion $R_1\subset\mathbf{R}$. L. Narváez proposed the following result and its proof. It can be considered as the reciprocal of a Poincaré-Birkhoff-Witt theorem [15, theorem 3.1, p. 198].

PROPOSITION 2.2.5. – *The morphism* ψ *induces an isomorphism:*

$$\phi : \mathbf{S} = \frac{\mathbf{T}_{R_0}(R_1)}{J} \xrightarrow{\sim} \mathbf{R}, \quad \phi((i(x_1)\otimes\cdots\otimes i(x_t)) + J) = x_1x_2\cdots x_t,$$

where i *is the inclusion of* R_1 *in the tensor algebra, and* J *is the two sided ideal generated by the elements:*

a) $a - i(a),\ a\in R_0\subset R_1$, b) $i(x)\otimes i(y) - i(y)\otimes i(x) - i([x,y]),\ x,y\in R_1$.

Proof. – The morphism ϕ is well defined. The algebra $\mathbf{T}_{R_0}(R_1)$ is graded, so it is filtered, and induces a filtration $G^\bullet$ on the quotient. The induced morphism $\phi : \mathbf{S}\to\mathbf{R}$ is filtered ($\psi(a) = a\in R_0$, $\psi(i(x_1)\otimes\cdots\otimes i(x_t)) = x_1x_2\cdots x_t\in R_t$). So, we can define a graded morphism of R_0-rings:

$$\pi : \mathcal{G}r_{G^\bullet}(\mathbf{S}) \to \mathcal{G}r_{F^\bullet}(\mathbf{R}),$$

$$\pi(\sigma_t(i(x_1)\otimes\cdots\otimes i(x_t) + J)) = \sigma'_t(x_1\cdots x_t) = \overline{x_1}\cdots\overline{x_t},$$

where $x_i\in R_1$, $\overline{x_i} = \sigma'_1(x_1)$ is the class of x_i in R_1/R_0, $\sigma_t(P)$ is the class of $P\in\mathbf{S}$ in $\mathcal{G}r^t_{G^\bullet}(\mathbf{S})$, and $\sigma'_t(Q)$ the class of $Q\in R_t$ in $\mathcal{G}r^t_{F^\bullet}(\mathbf{R})$. (Note that $\mathcal{G}r_{G^\bullet}(\mathbf{S})$ is conmutative).

On the other hand, the image of $R_0\subset R_1$ in $\mathbf{S}$ is exactly the part of degree zero of $\mathbf{S}$, and then we obtain a morphism of R_0-modules from $\mathcal{G}r^1_{F^\bullet}(\mathbf{R}) = R_1/R_0$ to $\mathcal{G}r^1_{G^\bullet}(\mathbf{S})$ which induces a morphism of R_0-algebras:

$$\rho : \mathcal{S}\mathrm{ym}_{R_0}\left(\frac{R_1}{R_0}\right) \to \mathcal{G}r_{G^\bullet}(\mathbf{S}),\quad \rho(\overline{x_1}\otimes\cdots\otimes\overline{x_t}) = \sigma_t(i(x_1)\otimes\cdots\otimes i(x_t) + J),$$

which is obviously surjective. The composition $\pi\rho$ is equal to α, and, by property (2) of $\mathbf{R}$, we deduce that ρ is injective. As ρ and $\pi\rho$ are isomorphisms, π is also an isomorphism, as we wanted to prove. □

COROLLARY 2.2.6. – *Let Y be a free divisor. Let $\mathcal{M}$ be a $\mathcal{O}_X$-module. An integrable logarithmic connection on $\mathcal{M}$ gives rise to a left $\mathcal{V}_0^Y(\mathcal{D}_X)$-structure on $\mathcal{M}$, and vice versa.*

Proof. – A $\mathcal{O}_X$-module $\mathcal{M}$ with an integrable logarithmic connection ∇ has a natural structure of left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module defined by its structure as $\mathcal{O}_X$-module. Let μ be the morphism of $(\mathcal{O}_X, \mathcal{O}_X)$-bimodules $\mu : R_1 = \mathcal{O}_X \oplus \mathcal{D}\mathrm{er}(\log Y) \to \mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{M})$, $\mu(a+\delta)(m) = am + \nabla_\delta(m)$. μ induces a morphism ν from $\mathbf{T}_{R_0}(R_1)$ and, as $\nu(J)=0$, a morphism from $\mathcal{V}_0^Y(\mathcal{D}_X) \simeq \mathbf{T}_{R_0}(R_1)/J$ to $\mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{M})$, which defines an structure of $\mathcal{V}_0^Y(\mathcal{D}_X)$-module on $\mathcal{M}$.

On the other hand, a left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module structure on $\mathcal{M}$ defines an integrable logarithmic connection ∇ on $\mathcal{M}$: $\nabla_\delta(m) = \delta \cdot m$. □

REMARK 2.2.7. – A left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module structure on $\mathcal{M}$ defines a logarithmic de Rham complex. If $\{\delta_1, \cdots, \delta_n\}$ is a local basis of $\mathcal{D}\mathrm{er}(\log Y)$ and $\{\omega_1, \cdots, \omega_n\}$ its dual basis, the differential of the complex is defined by: $\nabla^p(U)(\omega \otimes m) = \mathrm{d}\omega \otimes m + \sum_{i=1}^n ((\omega_i \wedge \omega) \otimes \delta_i \cdot m)$.

3. The logarithmic de Rham complex

3.1. The logarithmic Spencer complex

DEFINITION 3.1.1. – *We call the logarithmic Spencer complex, and denote by $\mathcal{S}p^\bullet(\log Y)$, the complex:*

$$0 \to \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}} \wedge^n \mathcal{D}\mathrm{er}(\log Y) \stackrel{\varepsilon_{-n}}{\to} \cdots \stackrel{\varepsilon_{-2}}{\to} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}} \mathcal{D}\mathrm{er}(\log Y) \stackrel{\varepsilon_{-1}}{\to} \mathcal{V}_0^Y(\mathcal{D}_X),$$

$$\varepsilon_{-1}(P \otimes \delta) = P\delta; \quad \varepsilon_{-p}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = \sum_{i=1}^p (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \widehat{\delta_i} \cdots \wedge \delta_p)$$
$$+ \sum_{1 \le i < j \le p} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \widehat{\delta_i} \cdots \widehat{\delta_j} \cdots \wedge \delta_p), \quad (2 \le p \le n).$$

We can augment this complex of left $\mathcal{V}_0^Y(\mathcal{D}_X)$-modules by another morphism ε_0 : $\mathcal{V}_0^Y(\mathcal{D}_X) \to \mathcal{O}_X$, $\varepsilon_0(P) = P(1)$. We call the new complex $\widetilde{\mathcal{S}p}^\bullet(\log Y)$.

This definition is essentially the same as the definition of the usual Spencer complex $\mathcal{S}p^\bullet$ of $\mathcal{O}_X$ (cf. [13, 2.1]) and generalizes the definition given by Esnault and Viehweg [9, App. A] in the case of a normal crossing divisor. We denote by $\mathcal{S}p^\bullet[\star Y] = \mathcal{D}_X[\star Y] \otimes_{\mathcal{D}_X} \mathcal{S}p^\bullet$ the meromorphic Spencer complex of $\mathcal{O}_X[\star Y]$.

THEOREM 3.1.2. – *Let Y be a free divisor. The complex $\mathcal{S}p^\bullet(\log Y)$ is a locally free resolution of $\mathcal{O}_X$ as left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module.*

Proof. – To see the exactness of $\widetilde{\mathcal{S}p}^\bullet(\log Y)$ we define a discrete filtration $G^\bullet$ such that it induces an exact graded complex (cf. [1, ch. 2, lemma 3.13]):

$$G^k(\mathcal{V}_0^Y(\mathcal{D}_X) \otimes \wedge^p \mathcal{D}\mathrm{er}(\log Y)) = F^{k-p}(\mathcal{V}_0^Y(\mathcal{D}_X)) \otimes \wedge^p \mathcal{D}\mathrm{er}(\log Y),$$

$$\mathcal{G}r_{G^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X) \otimes \wedge^p \mathcal{D}er(\log Y)) = \mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))[-p] \otimes \wedge^p \mathcal{D}er(\log Y),$$

and $G^k(\mathcal{O}_X) = \mathcal{O}_X$, $\mathcal{G}r_{G^\bullet}(\mathcal{O}_X) = \mathcal{O}_X$. As the above filtrations are compatible with the differential of $\widetilde{\mathcal{S}p}^\bullet(\log Y)$, we can consider the graduated complex $\mathcal{G}r_{G^\bullet}(\widetilde{\mathcal{S}p}^\bullet(\log Y))$:

$$0 \to \mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))[-n] \otimes \overset{n}{\wedge} \mathcal{D}er(\log Y) \overset{\psi_{-n}}{\to} \cdots \overset{\psi_{-1}}{\to} \mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X)) \overset{\psi_0}{\to} \mathcal{O}_X \to 0,$$

$$\psi_{-p}(G \otimes \delta_{j_1} \wedge \cdots \wedge \delta_{j_p}) = \sum_{i=1}^{p} (-1)^{i-1} G\sigma(\delta_{j_i}) \otimes \delta_{j_1} \wedge \cdots \widehat{\delta_{j_i}} \cdots \wedge \delta_{j_p}, \quad (2 \le p \le n),$$

$\psi_{-1}(G \otimes \delta_i) = G\sigma(\delta_i)$, $\psi_0(G) = G_0$, with $\{\delta_1, \cdots, \delta_n\}$ a basis of $\mathcal{D}er(\log Y)$. This complex is the Koszul complex of the graduated ring $\mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X)) \cong \mathcal{S}ym_{\mathcal{O}_X}(\mathcal{D}er(\log Y))$ with respect to the $\mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))$-regular sequence $\sigma(\delta_1), \cdots, \sigma(\delta_n)$ in the ring $\mathcal{G}r_{F^\bullet}(\mathcal{V}_0^Y(\mathcal{D}_X))$. So, it is exact. □

LEMMA 3.1.3. – *For every logarithmic operator $P \in \mathcal{V}_0^f(\mathcal{D})$, there exists, for each positive integer p, a logarithmic operator $Q \in \mathcal{V}_0^f(\mathcal{D})$ and an integer k such that $f^{-p}P = Qf^{-k}$.*

Proof. – We will prove the lemma by induction on the order of the logarithmic operator. If P has order 0, it is in $\mathcal{O}$, and it is clear that $f^{-p}P = Pf^{-p}$. Let P be of order d, and consider the logarithmic operator $[P, f^p] = -f^p[P, f^{-p}]f^p$, of order $d-1$. By induction hypothesis, there exists an integer m such that $[P, f^{-p}]f^m \in \mathcal{V}_0^f(\mathcal{D})$. Let k be the greatest of the integers m and p. It is clear that: $f^{-p}Pf^k = Pf^{k-p} - [P, f^{-p}]f^k \in \mathcal{V}_0^f(\mathcal{D})$. □

REMARK 3.1.4. – For every operator Q in $\mathcal{D}_X[\star Y]_x$, we can always find a positive integer m such that $f^m Q \in \mathcal{V}_0^f(\mathcal{D})$. Equivalently, for each meromorphic differential operator Q, there exists a positive integer p and a logarithmic operator Q' such that we can write: $Q = f^{-p}Q'$.

LEMMA 3.1.5. – *We have the following isomorphisms:*

1. $\mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \overset{\sim}{\hookrightarrow} \mathcal{D}_X[\star Y] \overset{\sim}{\hookleftarrow} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{O}_X[\star Y]$;
2. $\alpha : \mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{O}_X \cong \mathcal{O}_X[\star Y]$, $\alpha(P \otimes g) = P(g)$;
3. $\rho : \mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{D}_X[\star Y] \cong \mathcal{D}_X[\star Y]$, $\rho(P \otimes Q) = PQ$.

Proof. – 1. The inclusions $\mathcal{V}_0^Y(\mathcal{D}_X), \mathcal{O}_X[\star Y] \subset \mathcal{D}_X[\star Y]$ give rise to the previous isomorphisms of $(\mathcal{V}_0^Y(\mathcal{D}_X), \mathcal{O}_X[\star Y])$-modules. Locally, $af^{-k} \otimes P = af^{-k}P = aQ \otimes f^{-p}$, with P and Q logarithmic operators such that $f^{-k}P = Qf^{-p}$. We have seen how to obtain Q from P (lemma 3.1.3), and we can obtain P from Q in the same way. On the other hand, we saw in remark 3.1.4 how to express a meromorphic differential operator as a product of a meromorphic function and a logarithmic operator.

2. We compose the following isomorphisms of left $\mathcal{D}_X[\star Y]$-modules:

$$\mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{V}_0^Y(\mathcal{D}_X) \otimes_{\mathcal{V}_0^Y(\mathcal{D}_X)} \mathcal{O}_X \cong \mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{O}_X[\star Y].$$

3. We obtain this isomorphism of $\mathcal{D}_X[\star Y]$-bimodules from 1. and the isomorphism $\mathcal{O}_X[\star Y] \otimes_{\mathcal{O}_X} \mathcal{O}_X[\star Y] \cong \mathcal{O}_X[\star Y]$, $g_1 \otimes g_2 \mapsto g_1 g_2$. □

PROPOSITION 3.1.6. – *Let Y be a free divisor. We have the following isomorphisms of complexes of $\mathcal{D}_X[\star Y]$-modules:*

$$\mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y} \mathcal{S}p^\bullet \cong \mathcal{S}p^\bullet[\star Y], \quad \mathcal{D}_X[\star Y] \otimes_{\mathcal{V}_0^Y} \mathcal{S}p^\bullet(\log Y) \cong \mathcal{S}p^\bullet[\star Y].$$

Proof. – As $\mathcal{S}p^\bullet$ is a subcomplex of $\mathcal{D}_X$-modules of $\mathcal{S}p^\bullet[\star Y]$, and $\mathcal{D}_X[\star Y]$ is flat over $\mathcal{V}_0^Y(\mathcal{D}_X)$, the complex $\mathcal{D}_X[\star Y]\otimes_{\mathcal{V}_0^Y}\mathcal{S}p^\bullet$ is a subcomplex of $\mathcal{D}_X[\star Y]\otimes_{\mathcal{V}_0^Y}\mathcal{S}p^\bullet[\star Y]$, (see lemma 3.1.5, 1.). But, by the third isomorphism of lemma 3.1.5, this complex is the same as $\mathcal{S}p^\bullet[\star Y]$. Hence, we have an injective morphism of complexes from $\mathcal{D}_X[\star Y]\otimes_{\mathcal{V}_0^Y}\mathcal{S}p^\bullet$ to $\mathcal{S}p^\bullet[\star Y]$, defined locally in each degree by: $P\otimes Q\otimes\delta_1\wedge\cdots\wedge\delta_p\mapsto PQ\otimes(\delta_1\wedge\cdots\wedge\delta_p)$. This morphism is clearly surjective and, consequently, an isomorphism.

For the second isomorphism, we consider $\mathcal{V}_0^Y(\mathcal{D}_X)$ as a subsheaf of $\mathcal{O}_X$-modules of $\mathcal{D}_X$. As $\mathcal{D}\mathrm{er}(\log Y)$ is $\mathcal{O}_X$-free, we have the inclusions $\mathcal{V}_0^Y(\mathcal{D}_X)\otimes_{\mathcal{O}}\wedge^p\mathcal{D}\mathrm{er}(\log Y)\hookrightarrow\mathcal{D}_X\otimes_{\mathcal{O}}\wedge^p\mathcal{D}\mathrm{er}(\log Y)$, and $\wedge^p\mathcal{D}\mathrm{er}(\log Y)\hookrightarrow\wedge^p\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$ (cf. [2, AIII 88, Cor.]). As $\mathcal{D}_X$ is flat over $\mathcal{O}_X$, we have other inclusion $\mathcal{D}_X\otimes_{\mathcal{O}}\wedge^p\mathcal{D}\mathrm{er}(\log Y)\hookrightarrow\mathcal{D}_X\otimes_{\mathcal{O}}\wedge^p\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$ $(p\geq 0)$. So, we obtain a new inclusion $\mathcal{V}_0^Y(\mathcal{D}_X)\otimes_{\mathcal{O}}\wedge^p\mathcal{D}\mathrm{er}(\log Y)\hookrightarrow\mathcal{D}_X\otimes_{\mathcal{O}}\wedge^p\mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_X)$, for $p=0,\cdots,n$. These inclusions give rise to an injective morphism of complexes of $\mathcal{V}_0^Y(\mathcal{D}_X)$-modules $\mathcal{S}p^\bullet(\log Y)\hookrightarrow\mathcal{S}p^\bullet$. As $\mathcal{D}_X[\star Y]$ is flat over $\mathcal{V}_0^Y(\mathcal{D}_X)$ (see lemma 3.1.5, 1.) we have an injective morphism of complexes of $\mathcal{D}_X[\star Y]$-modules $\theta':\mathcal{D}_X[\star Y]\otimes_{\mathcal{V}_0^Y}\mathcal{S}p^\bullet(\log Y)\hookrightarrow\mathcal{D}_X[\star Y]\otimes_{\mathcal{V}_0^Y}\mathcal{S}p^\bullet$, that is surjective: $P\otimes Q\otimes(\delta_1\wedge\cdots\wedge\delta_p)=\theta'((Pf^{-k})\otimes Q'\otimes(f\delta_1\wedge\cdots\wedge f\delta_p))$, where $f^kQ=Q'f^p$ (using lemma 3.1.3). Composing θ' with the first isomorphism, we obtain the result. □

3.2. The logarithmic de Rham complex

For each divisor Y, we have a standard canonical isomorphism:

$$\lambda^p:\mathcal{H}om_{\mathcal{O}_X}(\wedge^p\mathcal{D}\mathrm{er}(\log Y),\mathcal{O}_X)\cong\mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{V}_0^Y(\mathcal{D}_X)\otimes_{\mathcal{O}_X}\wedge^p\mathcal{D}\mathrm{er}(\log Y),\mathcal{O}_X),$$

defined by: $\lambda^p(\alpha)(P\otimes\delta_1\wedge\cdots\wedge\delta_p)=P(\alpha(\delta_1\wedge\cdots\wedge\delta_p))$. Composing this isomorphism with the isomorphism γ^p defined in section 11. We can construct a natural morphism ψ^p, for $p=0,\cdots,n$:

$$\psi^p:\Omega_X^p(\log Y)\cong\mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{V}_0^Y(\mathcal{D}_X)\otimes\wedge^p\mathcal{D}\mathrm{er}(\log Y),\mathcal{O}_X),$$

$$\psi^p(\omega_1\wedge\cdots\wedge\omega_p)(P\otimes\delta_1\wedge\cdots\wedge\delta_p)=P(\det(\langle\omega_i,\delta_j\rangle)_{1\leq i,j\leq p}).$$

Similarly, if $\mathcal{M}$ is a left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module, and $p\in\{1,\cdots,n\}$, there exist

$$\psi^p_{\mathcal{M}}=\lambda^p_{\mathcal{M}}\circ\gamma^p_{\mathcal{M}}:\ \Omega_X^p(\log Y)(\mathcal{M})\xrightarrow{\sim}\mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{V}_0^Y(\mathcal{D}_X)\otimes\wedge^p\mathcal{D}\mathrm{er}(\log Y),\mathcal{M}).$$

$$\psi^p_{\mathcal{M}}(\omega_1\wedge\cdots\wedge\omega_p\otimes m)(P\otimes\delta_1\wedge\cdots\wedge\delta_p)=P\cdot\det(\langle\omega_i,\delta_j\rangle)_{1\leq i,j\leq p}\cdot m.$$

THEOREM 3.2.1. – *If $\mathcal{M}$ is a left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module (or, equivalently, is a $\mathcal{O}_X$-module with an integrable logarithmic connection), the complexes of sheaves of $\mathbb{C}$-vector spaces $\Omega_X^\bullet(\log Y)(\mathcal{M})$ and $\mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{S}p^\bullet(\log Y),\mathcal{M})$ are canonically isomorphic.*

Proof. – The general case is solved if we prove the case $\mathcal{M}=\mathcal{V}_0^Y(\mathcal{D}_X)$, using the isomorphisms $\Omega_X^\bullet(\log Y)(\mathcal{M})\cong\Omega_X^\bullet(\log Y)(\mathcal{V}_0^Y(\mathcal{D}_X))\otimes_{\mathcal{V}_0^Y}\mathcal{M}$, and

$$\mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{S}p^\bullet(\log Y),\mathcal{M})\cong\mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{S}p^\bullet(\log Y),\mathcal{V}_0^Y(\mathcal{D}_X))\otimes_{\mathcal{V}_0^Y}\mathcal{M}.$$

For $\mathcal{M}=\mathcal{V}_0^Y(\mathcal{D}_X)$, we obtain the right $\mathcal{V}_0^Y(\mathcal{D}_X)$-isomorphisms

$$\phi^p=\psi^p_{\mathcal{V}_0^Y(\mathcal{D}_X)}:\ \Omega_X^p(\log Y)(\mathcal{V}_0^Y(\mathcal{D}_X))\to\mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{S}p^{-p}(\log Y),\mathcal{V}_0^Y(\mathcal{D}_X)),$$

$$\phi^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = P \cdot \det(\langle \omega_i, \delta_j \rangle) \cdot Q.$$

To prove that these isomorphisms produce an *isomorphism of complexes* we have to check that they commute with the differential of the complex. By the second isomorphism of proposition 3.1.6, we obtain a natural morphism $\tau^\bullet$ of complexes of sheaves of right $\mathcal{V}_0^Y(\mathcal{D}_X)$-modules. These morphisms τ^i are clearly injective:

$$\tau^\bullet : \mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{S}p^\bullet(\log Y), \mathcal{V}_0^Y(\mathcal{D}_X)) \longrightarrow \mathcal{H}om_{\mathcal{D}_X[\star Y]}(\mathcal{S}p^\bullet[\star Y], \mathcal{D}_X[\star Y]),$$

locally defined by $\tau^p(\alpha)(R \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = f^{-k}\alpha(P \otimes (f\delta_1 \wedge \cdots \wedge f\delta_p))$, where P is a local section of $\mathcal{V}_0^Y(\mathcal{D}_X)$ such that $Rf^{-p} = f^{-k}P$ (see lemma 3.1.3). Using lemma 3.1.3 it is easy to check that the following diagram commutes for each $p \geq 0$, where the Φ^p are isomorphisms.

$$\begin{array}{ccc}
\Omega_X^p(\log Y)(\mathcal{V}_0^Y(\mathcal{D}_X)) & \xrightarrow{j^p} & \Omega_X^p[\star Y](\mathcal{D}_X[\star Y]) \\
\downarrow \phi^p & \# & \downarrow \Phi^p \\
\mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{S}p^p(\log Y), \mathcal{V}_0^Y(\mathcal{D}_X)) & \xrightarrow{\tau^p} & \mathcal{H}om_{\mathcal{D}_X[\star Y]}(\mathcal{S}p^p[\star Y], \mathcal{D}_X[\star Y])
\end{array}$$

$$\Phi^p : \Omega_X^p[\star Y](\mathcal{D}_X[\star Y]) \longrightarrow \mathcal{H}om_{\mathcal{D}_X[\star Y]}(\mathcal{D}_X[\star Y] \otimes {}^p\wedge \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X), \mathcal{D}_X[\star Y]),$$
$$\Phi^p((\omega_1 \wedge \cdots \wedge \omega_p) \otimes Q)(P \otimes (\delta_1 \wedge \cdots \wedge \delta_p)) = P \cdot \det(\langle \omega_i \cdot \delta_j \rangle_{1 \leq i,j \leq p}) \cdot Q.$$

But $\Phi^\bullet$, $j^\bullet$ and $\tau^\bullet$ are morphisms of complexes, and $\tau^\bullet$ is injective, hence we deduce that the ϕ^p commute with the differential and so define a isomorphism of complexes. □

COROLLARY 3.2.2. – *There exists a canonical isomorphism in the derived category:*

$$\Omega_X^\bullet(\log Y)(\mathcal{M}) \cong \mathbf{R}\mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{O}_X, \mathcal{M}).$$

Proof. – By theorem 3.1.2, the complex $\mathcal{S}p^\bullet(\log Y)$ is a locally free resolution of $\mathcal{O}_X$ as left $\mathcal{V}_0^Y(\mathcal{D}_X)$-module and we apply theorem 3.2.1. □

REMARK 3.2.3. – In the specific case that $\mathcal{M} = \mathcal{O}_X$, we have that the complexes $\Omega_X^\bullet(\log Y)$ and $\mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{S}p^\bullet(\log Y), \mathcal{O}_X)$ are canonically isomorphic and so, there exists a canonical isomorphism:

$$\Omega_X^\bullet(\log Y) \cong \mathbf{R}\mathcal{H}om_{\mathcal{V}_0^Y(\mathcal{D}_X)}(\mathcal{O}_X, \mathcal{O}_X).$$

REMARK 3.2.4. – A classical problem is the comparison between the logarithmic and the meromorphic de Rham complexes relative to a divisor Y,

$$\Omega_X^\bullet[\star Y] \cong \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X[\star Y]) \cong \mathbf{R}\mathcal{H}om_{\mathcal{V}_0^Y}(\mathcal{O}_X, \mathcal{O}_X[\star Y]).$$

If Y is a normal crossing divisor, an easy calculation shows that they are quasi-isomorphic (cf. [8]). The same result is true if Y is a strongly weighted homogeneous free divisor [6]. As a consequence of theorem 2.1.4, if Y is an arbitrary free divisor, the meromorphic de Rham complex and the logarithmic de Rham complex are quasi-isomorphic if and only if:

$$0 = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}\left(\mathcal{D}_X \otimes^{\mathbf{L}}_{\mathcal{V}_0^Y} \mathcal{O}_X, \frac{\mathcal{O}_X[\star Y]}{\mathcal{O}_X}\right)\left(= \mathbf{R}\mathcal{H}om_{\mathcal{V}_0^Y}\left(\mathcal{O}_X, \frac{\mathcal{O}_X[\star Y]}{\mathcal{O}_X}\right)\right).$$

4. Perversity of the logarithmic complex

4.1. Koszul Free Divisors

DEFINITION 4.1.1. – *Let $Y \subset X$ a divisor. We say that Y is a Koszul free divisor at x if it is free at x and there exists a basis $\{\delta_1, \cdots, \delta_n\}$ of* Der(log f) $= \mathcal{D}\text{er}(\log Y)_x$ *such that the sequence of symbols $\{\sigma(\delta_1), \cdots, \sigma(\delta_n)\}$ is regular in* $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$. *If Y is a Koszul free divisor at every point, we simply say that it is a Koszul free divisor.*

It is clear that if a basis of $\mathcal{D}\text{er}(\log Y)_x$ satisfies the condition above, then every basis does. By coherence, if a divisor is a Koszul free divisor at a point, then it is a Koszul free divisor near that point. Exemples of Koszul free divisors are nonsingular and normal crossing divisors, and plane curves (see corollary 4.2.2 and remarks 4.2.3 and 4.2.4).

PROPOSITION 4.1.2. – *If $\{\delta_1, \cdots, \delta_n\}$ is a basis of* Der(log f), *and the sequence $\{\sigma(\delta_1), \cdots, \sigma(\delta_n)\}$ is* $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-*regular, then*

$$\sigma(\mathcal{D}(\delta_1, \cdots, \delta_n)) = \mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma(\delta_1), \cdots, \sigma(\delta_n)).$$

Proof. – The inclusion $\mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma(\delta_1), \cdots, \sigma(\delta_n)) \subset \sigma(\mathcal{D}(\delta_1, \cdots, \delta_n))$ is clear. Let G be the symbol of an operator P of order d, with $P = \sum_{i=1}^n P_i \delta_i \in \mathcal{D}(\delta_1, \cdots, \delta_n)$. We will prove by induction that $G = \sigma(P)$ belongs to the ideal $\mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma_1, \cdots, \sigma_n)$, with $\sigma_i = \sigma(\delta_i)$. We will do the induction on the maximum order of the P_i $(i = 1, \cdots, n)$, which we will denote by k_0. As P has order d, k_0 is greater or equal to $d - 1$. If $k_0 = d - 1$, we have $\sigma(P) = \sum_{i \in K} \sigma(P_i)\sigma_i$, with K the set of subindices j such that P_j has order k_0 in $\mathcal{D}$. We suppose that the result holds when $d - 1 \leq k_0 < m$. Let $G = \sigma(P)$, with $P = \sum_{i=1}^n P_i \delta_i$ and $k_0 = m$. If $\sum_{i \in K} \sigma(P_i)\sigma_i \neq 0$, then $G = \sigma(P) = \sum_{i \in K} \sigma(P_i)\sigma_i \in \mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma_1, \cdots, \sigma_n)$. If $\sum_{i \in K} \sigma(P_i)\sigma_i = 0$, we define G_i by $G_i = \sigma(P_i)$ if $i \in K$ and 0 otherwise. Then, as $\{\sigma_1, \cdots, \sigma_n\}$ is a $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular sequence, we have:

$$(G_1, \cdots, G_n) = \sum_{i<j} G_{ij}(\sigma_j e_i - \sigma_i e_j), \qquad (e_i = (0, \cdots, 0, \overset{i}{\smile}{1}, 0, \cdots, 0)),$$

with $G_{ij} \in \mathrm{Gr}_{F^\bullet}(\mathcal{D})$ homogeneous polynomials of order $m - 1$. We choose, for $1 \leq i < j \leq n$, operators Q_{ij}, of order $m - 1$ in $\mathcal{D}$, such that $\sigma(Q_{ij}) = G_{ij}$, and define $(Q_1, \cdots, Q_n) = (P_1, \cdots, P_n) - \sum_{i<j} Q_{ij}(\delta_j e_i - \delta_i e_j - \underline{\alpha}_{ij})$, where $\underline{\alpha}_{ij}$ are the vectors with n coordinates in $\mathcal{O}$ defined by the relations:

$$[\delta_i, \delta_j] = \sum_{k=1}^n a_{ij}^k \delta_k = \underline{\alpha}_{ij}(\delta_1, \cdots, \delta_n)^t.$$

These Q_i, of order m in $\mathcal{D}$, verify $(\sigma_m(Q_1), \cdots, \sigma_m(Q_n)) = (G_1, \cdots, G_n) - \sum_{i<j} G_{ij}(\sigma_j e_i - \sigma_i e_j) = 0$. So, Q_i has order $m - 1$ in $\mathcal{D}$. Moreover,

$$\sum_{i=1}^n Q_i \delta_i = \sum_{i=1}^n P_i \delta_i - \sum_{i<j} Q_{ij}(\delta_i \delta_j - \delta_j \delta_i - [\delta_i, \delta_j]) = \sum_{i=1}^n P_i \delta_i = P.$$

We apply the induction hypothesis to $G = \sigma(P)$, with $P = \sum_{i=1}^n Q_i \delta_i$, and obtain $\sigma(P) \in \mathrm{Gr}_{F^\bullet}(\mathcal{D})(\sigma_1, \cdots, \sigma_n)$.

Now we consider the complex $\mathcal{D}_X \otimes_{\mathcal{V}_0^Y} \mathcal{S}p^\bullet(\log Y)$:

$$0 \to \mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^n \mathcal{D}\mathrm{er}(\log Y) \overset{\varepsilon_{-n}}{\to} \cdots\cdots \overset{\varepsilon_{-2}}{\to} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}\mathrm{er}(\log Y) \overset{\varepsilon_{-1}}{\to} \mathcal{D}_X,$$

where the local expressions of the morphisms are defined in 3.1.1. We can augment this complex of $\mathcal{D}$-modules by another morphism

$$\varepsilon_0 : \mathcal{D}_X \to \frac{\mathcal{D}_X}{\mathcal{D}_X(\mathcal{D}\mathrm{er}(\log Y))}, \quad \varepsilon_0(P) = P + \mathcal{D}_X(\mathcal{D}\mathrm{er}(\log Y)).$$

We denote by $\mathcal{D}_X \otimes_{\mathcal{V}_0^Y} \widetilde{\mathcal{S}p}^\bullet(\log Y)$ the new complex.

PROPOSITION 4.1.3. – *Let Y be a Koszul free divisor. Then the complex $\mathcal{D}_X \otimes_{\mathcal{V}_0^Y} \widetilde{\mathcal{S}p}^\bullet(\log Y)$ is exact.*

Proof. – We can work locally. Fix a point $x \in Y$ and a reduced local equation f. To prove that the complex $\mathcal{D} \otimes_{\mathcal{V}_0^f} \widetilde{\mathcal{S}p}^\bullet(\log f)$ is exact, we define a discrete filtration $G^\bullet$ such that the graded complex is exact (cf. [1, ch. 2, lemma 3.13]):

$$G^k\Big(\mathcal{D} \otimes_{\mathcal{O}} \overset{p}{\wedge} \mathrm{Der}(\log f)\Big) = F^{k-p}(\mathcal{D}) \otimes_{\mathcal{O}} \overset{p}{\wedge} \mathrm{Der}(\log f),$$

$$G^k\left(\frac{\mathcal{D}}{\mathcal{D}(\delta_1, \cdots, \delta_n)}\right) = \frac{F^k(\mathcal{D}) + \mathcal{D} \cdot (\delta_1, \cdots, \delta_n)}{\mathcal{D}(\delta_1, \cdots, \delta_n)},$$

with $\{\delta_1, \cdots, \delta_n\}$ a basis of $\mathcal{D}\mathrm{er}(\log Y)$. Clearly the filtration is compatible with the differential of the complex. Moreover, $\mathrm{Gr}_{G^\bullet}(\mathcal{D} \otimes \wedge^p \mathrm{Der}(\log f)) = \mathrm{Gr}_{F^\bullet}(\mathcal{D})[-p] \otimes \wedge^p \mathrm{Der}(\log f)$, and, by the previous proposition,

$$\mathrm{Gr}_{G^\bullet}\left(\frac{\mathcal{D}}{\mathcal{D}(\delta_1, \cdots, \delta_n)}\right) = \frac{\mathrm{Gr}_{F^\bullet}(\mathcal{D})}{\sigma(\mathcal{D} \cdot (\delta_1, \cdots, \delta_n))} = \frac{\mathrm{Gr}_{F^\bullet}(\mathcal{D})}{\mathrm{Gr}_{F^\bullet}(\mathcal{D}) \cdot (\sigma(\delta_1), \cdots, \sigma(\delta_n))}.$$

We consider the complex $\mathrm{Gr}_{G^\bullet}(\mathcal{D} \otimes_{\mathcal{V}_0^f(\mathcal{D})} \widetilde{\mathcal{S}p}^\bullet(\log\ f))$:

$$0 \to \mathrm{Gr}_{F^\bullet}(\mathcal{D})[-n] \otimes \wedge^n \mathrm{Der}(\log\ f) \overset{\psi_{-n}}{\to} \cdots \overset{\psi_{-2}}{\to} \mathrm{Gr}_{F^\bullet}(\mathcal{D})[-1] \otimes \mathrm{Der}(\log\ f) \overset{\psi_{-1}}{\to}$$

$$\mathrm{Gr}_{F^\bullet}(\mathcal{D}) \overset{\psi_0}{\to} \frac{\mathrm{Gr}_{F^\bullet}(\mathcal{D})}{\mathrm{Gr}_{F^\bullet}(\mathcal{D}) \cdot (\sigma(\delta_1), \cdots, \sigma(\delta_n))} \to 0,$$

$$\psi_{-p}(G \otimes \delta_{j_1} \wedge \cdots \wedge \delta_{j_p}) = \sum_{i=1}^{p} (-1)^{i-1} G\sigma(\delta_{j_i}) \otimes \delta_{j_1} \wedge \cdots \widehat{\delta_{j_i}} \cdots \wedge \delta_{j_p}, \quad (2 \le p \le n),$$

$$\psi_{-1}(G \otimes \delta_i) = G\sigma(\delta_i), \quad \psi_0(G) = G + \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \cdot (\sigma(\delta_1), \cdots, \sigma(\delta_n)).$$

This complex is the Koszul complex of the ring $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$ with respect to the sequence $\sigma(\delta_1), \cdots, \sigma(\delta_n)$. So, as this sequence is $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, the complex $\mathrm{Gr}_{G^\bullet}(\mathcal{D} \otimes_{\mathcal{V}_0^f} \widetilde{\mathcal{S}p}^\bullet(\log f))$ is exact. So $\mathcal{D} \otimes_{\mathcal{V}_0^f} \widetilde{\mathcal{S}p}^\bullet(\log f)$ is exact and $\mathcal{D} \otimes_{\mathcal{V}_0^f} \mathcal{S}p^\bullet(\log\ f)$ is a resolution of $\mathcal{D}/\mathcal{D}(\delta_1, \cdots, \delta_n)$. □

4.2. Perversity of the logarithmic complex

THEOREM 4.2.1. – *Let Y be a Koszul free divisor. Then the logarithmic de Rham complex $\Omega_X^\bullet(\log Y)$ is a perverse sheaf.*

Proof. – By proposition 4.1.3 the homology of the complex $\mathcal{D}_X \otimes_{\mathcal{V}_0^Y} \mathcal{S}p^\bullet(\log Y)$ is concentrated in degree 0. Its homology group in degree 0 is:

$$h^0(\mathcal{D}_X \otimes_{\mathcal{V}_0^Y} \mathcal{S}p^\bullet(\log Y)) = \frac{\mathcal{D}_X}{\mathcal{D}_X \cdot \mathcal{D}\mathrm{er}(\log Y)} = \mathcal{E}.$$

But $\mathcal{E}$ is a holonomic $\mathcal{D}_X$-module because, if $\{\delta_1, \cdots, \delta_n\}$ is a local basis of $\mathcal{D}\mathrm{er}(\log Y)$ at x, $\mathcal{G}r_F(\mathcal{E})_x = \mathrm{Gr}_{F^\bullet}(\mathcal{D})/(\sigma(\delta_1), \cdots, \sigma(\delta_n))$ has dimension n (using the fact that $\sigma(\delta_1), \cdots, \sigma(\delta_n)$ is a $\mathcal{G}r_{F^\bullet}(\mathcal{D}_X)$-regular sequence). So, using remark 3.2.3 for the first equality and theorem 3.1.2 for the last equality, we have:

$$\Omega_X^\bullet(\log Y) = \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{D}_X \otimes^{\mathbf{L}}_{\mathcal{V}_0^Y} \mathcal{O}_X, \mathcal{O}_X) = \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{E}, \mathcal{O}_X)$$

and then the logarithmic de Rham complex is a perverse sheaf, as solution of a holonomic $\mathcal{D}_X$-module, (cf. [13]). □

COROLLARY 4.2.2. – *Let Y be any divisor in X, with $\dim_{\mathbb{C}} X = 2$. Then the logarithmic de Rham complex $\Omega_X^\bullet(\log Y)$ is a perverse sheaf.*

Proof. – We know that, if $\dim_{\mathbb{C}} X = 2$, any divisor Y in X is free [17]. So, we have only to check that the other condition of definition 4.1.1 holds. We consider the symbols $\{\sigma_1, \sigma_2\}$ of a basis $\{\delta_1, \delta_2\}$ of Der(log f), where f is a reduced equation of Y. We have to see that they form a $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular sequence. If they do not, they have a common factor $g \in \mathcal{O}$, because they are symbols of operators of order 1. If g is a unit, we divide one of them by g and eliminate the common factor. If g is not a unit, it would be in contradiction with Saito's Criterion, because the determinant of the coefficients of the basis $\{\delta_1, \delta_2\}$ would have as factor g^2, with g not invertible, and this determinant has to be equal to f multiplied by a unit. □

REMARK 4.2.3. – There are Koszul free divisors Y in higher dimensions, and not necessarily normal crossing divisor. For example ([16]), $X = \mathbb{C}^3$ and $Y \equiv \{f = 0\}$, with $f = 2^8z^3 - 2^7x^2z^2 + 2^4x^4z + 2^43^2xy^2z - 2^2x^3y^2 - 3^3y^4$. A basis of Der(log f) is $\{\delta_1, \delta_2, \delta_3\}$, with $\delta_1 = 6y\partial_x + (8z - 2x^2)\partial_y - xy\partial_z$, $\delta_2 = (4x^2 - 48z)\partial_x + 12xy\partial_y + (9y^2 - 16xz)\partial_z$, $\delta_3 = 2x\partial_x + 3y\partial_y + 4z\partial_z$, and the sequence $\{\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)\}$ is $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular.

REMARK 4.2.4. – The Koszul condition over free divisor is not necessary for the perversity of the logarithmic de Rham complex. For example ([5]), if $X = \mathbb{C}^3$ and $Y \equiv \{f = 0\}$, with $f = xy(x + y)(y + tx)$, a basis of Der(log f) is $\{x\partial_x + y\partial_y, x^2\partial_x - y^2\partial_y - t(x + y)\partial_t, (xt + y)\partial_t\}$ and the graded complex $\mathcal{G}r_{G^\bullet}(\mathcal{D}_X \otimes_{\mathcal{V}_0^Y} \mathcal{S}p^\bullet(\log Y)) = K(\sigma(\delta_1), \sigma(\delta_1), \sigma(\delta_3); \mathcal{G}r_{F^\bullet}(\mathcal{D}_X))$ is not concentrated in degree 0, but the complex $\mathcal{D}_X \otimes_{\mathcal{V}_0^Y} \mathcal{S}p^\bullet(\log Y)$ is. Moreover, in this case the dimension of $\mathcal{D}_X/\mathcal{D}_X(\delta_1, \delta_2, \delta_3)$ is 3 and so, $\Omega_X^\bullet(\log Y)$ is a perverse sheaf.

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