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ORDINARINESS IN GOOD REDUCTIONS OF SHIMURA VARIETIES OF PEL-TYPE

BY TORSTEN WEDHORN

ABSTRACT. – The main purpose of this paper is the definition of the μ -ordinary locus in good reductions of Shimura varieties of PEL-type and the proof that this locus is open and dense. This generalizes the well known theorem that the ordinary locus is open and dense in the Siegel case. We further give a criterion for the density of the locus where the underlying abelian variety is ordinary. For the proof we describe an easy method to construct deformations of abelian varieties. © Elsevier, Paris

RÉSUMÉ. – L'objet essentiel de cet article est la définition du lieu μ -ordinaire dans les bonnes réductions des variétés de Shimura de type PEL et la preuve que ce lieu est ouvert et dense. C'est une généralisation du théorème bien connu que le lieu ordinaire est ouvert et dense dans le cas symplectique. De plus, nous donnons un critère pour la densité du lieu où la variété abélienne sous-jacente est ordinaire. La démonstration est basée sur une méthode simple de construction des déformations des variétés abéliennes. © Elsevier, Paris

Introduction

Let $\mathbf{A}_{g,N}/\mathbb{F}_p$ be the moduli scheme of principally polarized abelian varieties of a fixed dimension $g \geq 1$ in characteristic p with level- N -structure where $N \geq 3$ is some integer prime to p , and let $\mathcal{X} \rightarrow \mathbf{A}_{g,N}$ be the universal family. It is a classical result that the ordinary locus in $\mathbf{A}_{g,N}$ is open and dense, where the ordinary locus consists of those points $s \in \mathbf{A}_{g,N}$ such that the p -divisible group of $\mathcal{X}_{\bar{s}}$ has only slopes in $\{0, 1\}$. The fact that the ordinary locus is open results from Grothendieck's specialization theorem for crystals [Gr]. The density of the ordinary locus can be proved in three different ways:

- (a) There is a proof by Koblitz [Kob] (see also [III], App. 2) who investigated by deformation theoretical methods the stratification of $\mathbf{A}_{g,N}$ by the p -rank of \mathcal{X} .
- (b) A second proof is obtained by explicitly constructing deformations by using Cartier theory which raise the p -rank (cf. Mumford [Mu], Norman and Oort [NO], Chai and Faltings [CF], chap. VII, 4).
- (c) A third method to prove the density is the construction of a smooth compactification of the moduli stack \mathbf{A}_g over \mathbb{Z} of principally polarized abelian varieties and applying Zariski's connectedness theorem to show that \mathbf{A}_g is irreducible (cf. Chai and Faltings [CF], chap. IV). Then it suffices to show that the ordinary locus is nonempty, which is trivial (take the product of ordinary elliptic curves).

The aim of this work is to generalize the above statement to good reductions of Shimura varieties of PEL-type. For a Shimura variety of PEL-type the reduction can be considered

as a moduli space of abelian varieties with additional structures. We will use a variant of method (b) above. The naive generalization, namely the density of the locus where the underlying abelian variety is ordinary, turns out to be false in general.

For the correct formulation we need some group theory: Let G be a connected reductive group and let $h: \mathbf{S} \rightarrow G_{\mathbb{R}}$ be a homomorphism such that (G, h) is a Shimura datum, and let $C \subset G(\mathbf{A}_f)$ be an open compact subgroup. Denote by $\mathrm{Sh}(G, h)_C$ the associated canonical model over the Shimura field E which is the field of definition of the conjugacy class $c = c(G, h)$ of 1-parameter subgroups associated to h . Fix a prime p and a place ν over p of E , let E_{ν} be the ν -adic completion of E , and denote by κ its residue class field. We write $C = C^p C_p$. Assume that there is a good model $\widetilde{\mathrm{Sh}}(G, h)_C$ of $\mathrm{Sh}(G, h)_C$ over $O_{E_{\nu}}$, such that every point x of $\widetilde{\mathrm{Sh}}(G, h)_C \otimes \kappa$ corresponds to an abelian variety with additional structure.

Then we can associate to a point x of $\widetilde{\mathrm{Sh}}(G, h)_C \otimes \kappa$ its isocrystal with G -structure. The isomorphism class of the isocrystal is uniquely determined by its Newton point $\bar{\nu}(x) \in (X_*)_{\mathbb{Q}}/\Omega_0$ (Kottwitz [Ko2], Rapoport and Richartz [RR]), where X_* is the cocharacter group of some maximal torus and Ω_0 is the Weyl group of the associated root datum. In the Siegel case, i.e. if $G = \mathrm{GSp}_{2g}$, we can take as model the moduli space of principally polarized abelian varieties with some level structure, and the Newton point $\bar{\nu}(x)$ of some point x corresponding to an abelian variety is given by the slope sequence of its p -divisible group.

On the other hand we can consider the conjugacy class c as an element of X_*/Ω_0 . Identifying $(X_*)_{\mathbb{Q}}/\Omega_0$ with some closed Weyl chamber \bar{C} we can take the arithmetic mean $\bar{\mu} \in \bar{C}$ of c with respect to the action of $\mathrm{Gal}(E_{\nu}/\mathbb{Q}_p)$. By [RR] we always have $\bar{\nu}(x) \leq \bar{\mu}$ (with respect to the order on $(X_*)_{\mathbb{Q}}/\Omega_0$ defined in loc. cit.), and the locus of points x with $\bar{\nu}(x) = \bar{\mu}$ is open. It is called the μ -ordinary locus. In the Siegel case the μ -ordinary locus equals the ordinary locus.

M. Rapoport [Ra] conjectured that if $G_{\mathbb{Q}_p}$ is unramified and C_p is hyperspecial the μ -ordinary locus is open and dense in $\widetilde{\mathrm{Sh}}(G, h)_C \otimes \kappa$. This generalizes therefore the Siegel case. The aim of this work is to prove this conjecture in the case of PEL-Shimura varieties. Note that the hypotheses “ $G_{\mathbb{Q}_p}$ unramified” and “ C_p hyperspecial” are both necessary as has been shown by the examples of Drinfeld [Dr] and of Stamm [St].

To be more precise let us fix some notations: Let B be a finite dimensional semisimple \mathbb{Q} -algebra equipped with a positive involution $*$ and let V be finitely generated B -module equipped with an alternating non-degenerate \mathbb{Q} -valued skew-hermitian pairing $\langle \ , \ \rangle$. The B -linear similitudes of $(V, \langle \ , \ \rangle)$ form an algebraic group G over \mathbb{Q} which is reductive; denote by G^0 its connected component. Fix a homomorphism $h: \mathbf{S} \rightarrow G_{\mathbb{R}}$ satisfying the usual Riemann conditions, and let X_{∞} be the symmetric space of $G(\mathbb{R})$ -conjugates of h . Let us call this collection of data \mathcal{D} a *PEL-datum*. Further let p be a prime of good reduction; this implies that $G_{\mathbb{Q}_p}$ is unramified and in particular we may choose a hyperspecial subgroup C_p of $G(\mathbb{Q}_p)$ (for the precise definition see 1.4).

Let $\mu_h: \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$ be the cocharacter associated to h and let us denote by c_0 its $G^0(\mathbb{C})$ -conjugacy class. This class can be considered as an element in X_*/Ω_0 . Now $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ acts on X_*/Ω_0 and we denote by $\bar{\mu} \in (X_*)_{\mathbb{Q}}/\Omega_0$ the arithmetic mean of the translates of c_0 with respect to this action.

On the other hand, after the choice of some compact open subgroup $C^p \subset G(\mathbf{A}_f^p)$, Kottwitz [Kol] has defined a moduli problem of abelian varieties over $O_E \otimes \mathbb{Z}_{(p)}$. If C^p is sufficiently small this moduli problem is representable [Kol] by a moduli scheme $\mathbf{A}_{\mathcal{D}}$ which is smooth over $O_E \otimes \mathbb{Z}_{(p)}$.

When G is connected, the generic fibre of $\mathbf{A}_{\mathcal{D}}$ consists of isomorphic copies of $\mathrm{Sh}(G, h)_C$, and in this case our main result will be:

Density Theorem: *The μ -ordinary locus is open and dense in $\mathbf{A}_{\mathcal{D}} \otimes \kappa(O_{E_\nu})$.*

The situation is more complicated when G is disconnected, i.e. essentially when we are dealing with an orthogonal group. In this case the $G(\mathbb{C})$ -conjugacy class of μ_h is the disjoint union of $G^0(\mathbb{C})$ -conjugacy classes $c_0^{(1)} = c_0, \dots, c_0^{(m)}$ of homomorphisms $\mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$. For every $i = 1, \dots, m$ we can form $\bar{\mu}^{(i)}$, and we will prove that the points x with $\bar{\nu}(x) = \bar{\mu}^{(i)}$ for some i is open and dense.

The proof of the density theorem will also show that if G is connected the ordinary locus (i.e. the locus of points in the moduli scheme where the underlying abelian variety is ordinary) is non-empty if and only if E_ν equals \mathbb{Q}_p . This criterion has also been anticipated (in the more general case of Shimura varieties of Hodge type) by Vasu in [Va]. Further I am thankful to O. Bültel for pointing out that the necessity of this condition follows from Noot [No] (again more generally). Further O. Bültel has also shown this criterion generically, i.e. that it holds for all but finitely many primes p (see [Bü] for details).

I will now give an overview of the structure of this work. In the first chapter the main result is stated: After fixing some notations, there is a preliminary section about the abstract based root datum of a reductive group together with its Galois action, its Weyl group, and behaviour with respect to change of base fields. In sections 1.3–1.5 the objective is to define PEL-data and the conjugacy class c_0 , to define primes of good reduction and the rational class of cocharacters $\bar{\mu}$, and to recall the definition of the associated moduli problem. Finally in 1.6 the main result, the density theorem, and the criterion for the non-emptiness of the ordinary locus (theorem (1.6.3)) are stated.

The second chapter starts with a section where the proof of the density theorem is reduced to a deformation problem which implies in particular that the density theorem and theorem (1.6.3) depend only on the “ p -adic completion” of the PEL-datum. This fact will be used in section 2.2 for a further reduction to four special cases according to the Dynkin type of the algebraic group G . Finally in 2.3, $\bar{\mu}$ will be calculated in these four cases and this finishes the proof of theorem (1.6.3) (under the assumption of the density theorem).

In chapter 3 an easy way for constructing equicharacteristic deformations of abelian varieties is developed. For this Zink’s theory of displays [Z1] is used. In the first section there is a brief review of this theory together with some easy lemmas. In 3.2 displays are used to associate deformations to certain endomorphisms of the (covariant) Dieudonné module of a p -divisible group X with additional structures. This technique, which is basic for all of chapter 4, was inspired by Norman and Oort ([NO]) and by Chai and Faltings ([CF] chap. VII.4). We will use its form given in (3.2.9).

Finally, in chapter 4, we use the methods of chapter 3 to attack our four deformation problems. This is the heart of the proof of the density theorem. In a first section the technical notion of a *deformation sequence* is defined and it is shown how deformation sequences define deformations where the p -rank rises. In sections 4.2–4.5 the deformations

into the μ -ordinary locus in the four special cases are constructed via the method described in (2.1.7).

It remains for me to thank all those who helped me during the period of gestation of this work. I thank H. Hötte, R. Huber, M. Reineke, and M. Richartz who endured patiently a lot of questions. I am grateful to T. Fimmel who gave fruitful advice when I got stuck with combinatorial problems and to O. Bültel for helpful conversations on Shimura varieties. Thanks also go to S. Orlik for his numerous remarks on this work. I thank R. Kottwitz for pointing out a mistake in my calculations. Further I am thankful to B. Wehmeyer who typed most of this manuscript and helped with \TeX nical questions. I am very grateful to T. Zink for giving me generous advice on Cartier and Display theory. Finally I owe special thanks to M. Rapoport who initiated this work and taught me a great deal about mathematics.

1. Statement of the main result

1.1. Notations and conventions

(1.1.1) By a reductive group G over a field k we mean an affine smooth group scheme G over k such that the radical of G (i.e. the largest invariant solvable connected smooth subgroup of G) is a torus. In particular a reductive group is not necessarily connected. We denote by G^0 its connected component.

If G is an algebraic group and if H is a connected algebraic subgroup we denote by $N_G(H)$ (resp. $Z_G(H)$) the normalizer (resp. centralizer) of H in G .

1.2. Based root data of reductive groups

(1.2.1) Notation: In this section k will denote a field, \bar{k} will be an algebraic closure of k , $\Gamma = \text{Gal}(\bar{k}/k)$ will be the Galois group of k , and G will be a reductive group over k .

(1.2.2) A *based root datum* is a root datum (X^*, R^*, X_*, R_*) together with a root base $\Delta \subset R^*$. A *morphism*

$$f: (X^*, R^*, X_*, R_*, \Delta) \longrightarrow (X'^*, R'^*, X'_*, R'_*, \Delta')$$

of *based root data* is a homomorphism of abelian groups $f: X^* \rightarrow X'^*$ such that f induces a bijection of R^* onto R'^* and of Δ onto Δ' and such that the transposed homomorphism ${}^t f: X'_* \rightarrow X_*$ induces a bijection of R'_* onto R_* . Thus we get the category of based root data.

Let Λ be a topological group. A *based root datum with Λ -action* is a pair (\mathcal{BR}, α) where $\mathcal{BR} = (X^*, R^*, X_*, R_*, \Delta)$ is a based root datum and $\alpha: \Lambda \rightarrow \text{Aut}(\mathcal{BR})$ is a group homomorphism such that the induced operation $\Lambda \times X^* \rightarrow X^*$ is a continuous map if we provide X^* with the discrete topology. We have the obvious notion of a *morphism of based root data with Λ -action*.

(1.2.3) Let k be algebraically closed. Consider Borel pairs (T, B) of G , i.e. T is a maximal torus of G and B is a Borel group containing T . Two Borel pairs (T_1, B_1) and (T_2, B_2) are conjugate by an element of $G^0(k)$ and every inner automorphism α of G^0 with $\alpha(T_1) = T_2$, $\alpha(B_1) = B_2$ induces the same isomorphism $T_1 \xrightarrow{\sim} T_2$. We get

an isomorphism of the based root datum associated to (T_1, B_1) onto the one associated to (T_2, B_2) .

We denote by

$$\mathcal{BR}(G) = (X^*, R^*, X_*, R_*, \Delta) = ({}_G X^*, {}_G R^*, {}_G X_*, {}_G R_*, {}_G \Delta)$$

the projective limit of all $(X^*(T), R^*(T), X_*(T), R_*(T), \Delta(T, B))$ where (T, B) runs through the set of Borel pairs of G . We call $\mathcal{BR}(G)$ the *based root datum of G* . By forgetting the root base we get a root datum $\mathcal{R}(G) = (X^*, R^*, X_*, R_*) = ({}_G X^*, {}_G R^*, {}_G X_*, {}_G R_*)$. Note that we have $\mathcal{BR}(G^0) = \mathcal{BR}(G)$.

Finally we denote by $\Omega_0 = {}_G \Omega_0$ the Weyl group of $\mathcal{R}(G)$ and call it the *root Weyl group of G* . It acts on $\mathcal{R}(G)$. Note that we have an injective homomorphism of ${}_G \Omega_0$ in the projective limit of the groups $N_G(T)/Z_G(T)$, but if G is not connected this is in general not an isomorphism.

(1.2.4) Now let k be an arbitrary field. Then we set $\mathcal{BR}(G) = \mathcal{BR}(G_{\bar{k}})$. The Galois group Γ of k acts on $\mathcal{BR}(G)$ as follows (we describe the action of Γ on X_*): Take $\gamma \in \Gamma$ and $\lambda \in X_*$. Choose some Borel pair (T, B) of $G_{\bar{k}}$; this induces an isomorphism $X_* \cong X_*(T)$. Let $\lambda_{(T,B)}$ be the image of λ in $X_*(T)$. Its twist $\lambda_{(T,B)}^\gamma$ under γ is an element of $X_*(T^\gamma)$. The Borel pair (T^γ, B^γ) induces an isomorphism $X_*(T^\gamma) \xrightarrow{\sim} X_*$ and we denote by $\gamma(\lambda) \in X_*$ the image of $\lambda_{(T,B)}^\gamma$ under this isomorphism. This map $(\gamma, \lambda) \mapsto \gamma(\lambda)$ gives the operation of Γ on X_* , and this defines an action on $\mathcal{BR}(G)$. We denote by $\mathcal{BR}_{\bar{k}/k}(G)$ the based root datum $\mathcal{BR}(G)$ together with its Γ -action.

If G splits over some Galois extension k' of k then the action of Γ factorizes over $\text{Gal}(k'/k)$.

(1.2.5) Let K be an extension of k and let \bar{K} be some algebraic closure of K . By choosing an embedding $\iota: \bar{k} \hookrightarrow \bar{K}$ making the diagram

$$\begin{array}{ccc} \bar{K} & \xleftarrow{\iota} & \bar{k} \\ \uparrow & & \uparrow \\ K & \longleftarrow & k \end{array}$$

commutative we get an isomorphism of based root data

$$\varphi: \mathcal{BR}(G) \xrightarrow{\sim} \mathcal{BR}(G_K).$$

Via ι we get a homomorphism $\alpha: \Gamma_K = \text{Gal}(\bar{K}/K) \rightarrow \Gamma = \text{Gal}(\bar{k}/k)$, and if we identify $\mathcal{BR}(G)$ and $\mathcal{BR}(G_K)$ via φ , Γ_K acts via $\alpha(\Gamma_K)$ on $\mathcal{BR}(G)$. Finally φ induces an isomorphism ${}_G \Omega_0 \xrightarrow{\sim} ({}_{G_K}) \Omega_0$.

(1.2.6) Let $\mathcal{R}(G) = (X^*, R^*, X_*, R_*)$ be the root datum of G and let Ω_0 be the root Weyl group of G . Then we have a canonical identification of the set $\mathcal{M}_G(\bar{k})$ of $G^0(\bar{k})$ -conjugacy classes of 1-parameter subgroups $\mathbb{G}_{m,\bar{k}} \rightarrow G_{\bar{k}}$ and the set X_*/Ω_0 . In particular (1.2.5) we have $\mathcal{M}_G(\bar{K}) = \mathcal{M}_G(\bar{k})$ for every algebraically closed extension \bar{K} and we denote it simply by \mathcal{M}_G .

A homomorphism $\varphi: G \rightarrow G'$ of reductive k -groups induces a map $\mathcal{M}_G \rightarrow \mathcal{M}_{G'}$ and therefore a map

$$\gamma_\varphi: {}_G X_*/{}_G \Omega_0 \longrightarrow {}_{G'} X_*/{}_{G'} \Omega_0.$$

Note that even if φ is a closed immersion γ_φ need not be injective.

(1.2.7) Let G' be a smooth normal subgroup of G , such that G/G' is commutative. By [SGA 3, Exp. XXII, 6.6.3, 6.3.4, 6.2.8] we have:

- (1) G' is reductive (in the sense of (1.1.1)).
- (2) $(G'^0)^{der} = (G^0)^{der}$.
- (3) The map $T \mapsto T' = T \cap G' = T \cap G'^0$ induces an isomorphism

$$\mathbf{Tor}(G) \xrightarrow{\sim} \mathbf{Tor}(G')$$

whose inverse is given by $T' \mapsto T = \text{rad}(G) \cdot T' = Z_{G^0}(T')$. Here for an affine algebraic k -group H we denote by $\mathbf{Tor}(H)$ the k -scheme of maximal tori of H .

- (4) The map $B \mapsto B' = B \cap G' = B \cap G'^0$ induces an isomorphism

$$\mathbf{Bor}(G) \xrightarrow{\sim} \mathbf{Bor}(G')$$

whose inverse is given by $B' \mapsto B = \text{rad}(G) \cdot B' = N_{G^0}(B')$. Here for an affine algebraic k -group H we denote by $\mathbf{Bor}(H)$ the k -scheme of Borel subgroups of H .

From (3) and (4) we deduce that the inclusion $G' \hookrightarrow G$ induces a morphism of based root data

$$\mathcal{BR}_{\bar{k}/k}(G) \longrightarrow \mathcal{BR}_{\bar{k}/k}(G')$$

which is clearly compatible with the action of Γ . In particular we get an isomorphism ${}_G\Omega_0 \xrightarrow{\sim} {}_{G'}\Omega_0$.

1.3. PEL-data

(1.3.1) Notations: Let B denote a finite-dimensional semi-simple \mathbb{Q} -algebra, let $*$ be a positive involution on B , let $V \neq \{0\}$ be a finitely generated left B -module, and let $\langle \cdot, \cdot \rangle$ be a perfect alternating bilinear form $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{Q}$ of the underlying \mathbb{Q} -vector space of V which is skew-hermitian, i.e. $\langle bv, w \rangle = \langle v, b^*w \rangle$ for all $v, w \in V, b \in B$.

From these data we get a reductive group $G = G(B, *, V, \langle \cdot, \cdot \rangle)$ over \mathbb{Q} whose R -valued points for some commutative \mathbb{Q} -algebra R are given by

$$G(R) = \{g \in GL_{B \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid \langle gv, gw \rangle = \eta(g)\langle v, w \rangle, \eta(g) \in R^\times\}.$$

The mapping $g \mapsto \eta(g)$ defines a homomorphism $\eta: G \rightarrow \mathbb{G}_{m, \mathbb{Q}}$ of algebraic \mathbb{Q} -groups. We denote by $G' = G'(B, *, V, \langle \cdot, \cdot \rangle)$ its kernel, i.e. for every commutative \mathbb{Q} -algebra R we have

$$G'(R) = \{g \in GL_{B \otimes_{\mathbb{Q}} R}(V \otimes_{\mathbb{Q}} R) \mid \langle gv, gw \rangle = \langle v, w \rangle\}.$$

We can apply (1.2.7) to G and G' .

Let $h: \mathbf{S} := \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m, \mathbb{C}}) \rightarrow G_{\mathbb{R}}$ be a homomorphism which defines on $V_{\mathbb{R}}$ a Hodge structure of type $\{(-1, 0), (0, -1)\}$ (i.e. h defines a decomposition

$$V_{\mathbb{C}} = V^{0, -1} \oplus V^{-1, 0}$$

where $V^{0,-1} = \{v \in V_{\mathbb{C}} | h(\mathbb{C})(1 \otimes z)v = \bar{z}v\}$ and $V^{-1,0} = \overline{V^{0,-1}}$. We further suppose that $V_{\mathbb{R}} \times V_{\mathbb{R}} \rightarrow \mathbb{R}$, $(v, w) \mapsto \langle v, h(\mathbb{R})(\sqrt{-1})w \rangle$ is a symmetric positive bilinear form on $V_{\mathbb{R}}$, where $\sqrt{-1} \in \mathbb{C}$ is a square root of -1 which we fix once and for all.

(1.3.2) Definition: A tuple $\mathcal{D} = (B, *, V, \langle \cdot, \cdot \rangle, h)$ satisfying the above conditions is called PEL-datum. The group $G = G(B, *, V, \langle \cdot, \cdot \rangle) = G(\mathcal{D})$ is called the associated algebraic group.

(1.3.3) Let $\mathcal{D} = (B, *, V, \langle \cdot, \cdot \rangle, h)$ be a PEL-datum and let G be its associated algebraic group. We denote by $\mu(\mathcal{D})$ the composition

$$\mu(\mathcal{D}) := h_{\mathbb{C}} \circ \mu: \mathbb{G}_{m, \mathbb{C}} \longrightarrow G_{\mathbb{C}},$$

where

$$\mu: \mathbb{G}_{m, \mathbb{C}} \longrightarrow \mathbf{S}_{\mathbb{C}} = \prod_{\tau \in \text{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{G}_{m, \mathbb{C}}$$

is the embedding whose image is the factor of $\mathbf{S}_{\mathbb{C}}$ corresponding to $\tau = \text{id}$.

Let $c_0(\mathcal{D})$ be the $G^0(\mathbb{C})$ -conjugacy class of the 1-parameter subgroup $\mu(\mathcal{D})$ of $G_{\mathbb{C}}$. If $(X^*, R^*, X_*, R_*, \Delta)$ is the based root datum of G (1.2.3) and Ω_0 is the root Weyl group of G (1.2.3), we can consider $c_0(\mathcal{D})$ as an element

$$c_0(\mathcal{D}) \in X_*/\Omega_0$$

by (1.2.6).

Let $\bar{\mathbb{Q}}$ be the algebraic closure of \mathbb{Q} in \mathbb{C} . Then $\Gamma := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on X_*/Ω_0 . Let $E(\mathcal{D})$ be the reflex field, i.e. $E(\mathcal{D})$ is the fixed field of the stabilizer of $c_0(\mathcal{D})$ in Γ . This is a finite extension of \mathbb{Q} .

(1.3.4) Let $\mathcal{D} = (B, *, V, \langle \cdot, \cdot \rangle, h)$ be a PEL-datum with associated group G , let F be the center of B , and let F_0 be the ring of elements in F which are fixed by $*$. Let C be the \mathbb{Q} -algebra $\text{End}_B(V)$. The adjoint map for $\langle \cdot, \cdot \rangle$ gives an involution $*$ on C . We have

$$G(\mathbb{Q}) = \{x \in C | xx^* \in \mathbb{Q}\},$$

and the group G' is the restriction of scalars of the algebraic group G'_0 over F_0 given by

$$G'_0(R) = \{x \in C \otimes_F R | xx^* = 1\}$$

for a commutative F_0 -algebra R .

Now assume that B is simple. Then C is also simple and has the same center F as B . The involutions on F induced by the involution on B and by the involution on C coincide, and this is a positive involution on F . Therefore its fixed field F_0 is totally real, and if $*$ is an involution of the second kind, F is a totally complex quadratic extension of F_0 . Set $n = 1/2[F : F_0](\dim_F C)^{1/2}$; by the existence of h , this is an integer. Now there are three possibilities for the form of $C_{\mathbb{R}}$:

- (A) $C_{\mathbb{R}}$ is a product of $[F_0 : \mathbb{Q}]$ copies of $M_n(\mathbb{C})$, and in this case G'_0 is group of type A_{n-1} .
- (C) $C_{\mathbb{R}}$ is a product of $[F_0 : \mathbb{Q}]$ copies of $M_{2n}(\mathbb{R})$, and in this case G'_0 is group of type C_n .

(D) $C_{\mathbb{R}}$ is a product of $[F_0 : \mathbb{Q}]$ copies of $M_n(\mathbb{H})$, and in this case G'_0 is group of type D_n . Here we define $C_1 := A_1$, $D_1 := \emptyset$, $D_2 := A_1 + A_1$.

(1.3.5) (Functoriality of PEL-data): Let $\mathcal{D} = (B, *, V, \langle \cdot, \cdot \rangle, h)$ be a PEL-datum and let $B_0 \subset B$ be a semisimple \mathbb{Q} -subalgebra, stable under $*$. The induced involution $*|_{B_0}$ is again positive, and we can consider V as a hermitian left B_0 -module. Let $G_0 = G(B_0, *, V, \langle \cdot, \cdot \rangle)$ be the associated algebraic group over \mathbb{Q} . Then G is a closed subgroup of G_0 . By composing h with this inclusion we get a homomorphism $h_0 : \mathbf{S} \rightarrow (G_0)_{\mathbb{R}}$, and $\mathcal{D}_0 = (B_0, *|_{B_0}, V, \langle \cdot, \cdot \rangle, h_0)$ is a PEL-datum. The inclusion $G \hookrightarrow G_0$ induces a map (1.2.6)

$$\gamma_{G, G_0} : G X_* / G \Omega \longrightarrow G_0 X_* / G_0 \Omega,$$

and we have

$$\gamma_{G, G_0}(c_0(\mathcal{D})) = c_0(\mathcal{D}_0).$$

1.4. Primes of good reduction

(1.4.1) Let $\mathcal{D} = (B, *, V, \langle \cdot, \cdot \rangle, h)$ be a PEL-datum and let G be its associated group. We say that a prime number $p > 0$ is a *prime of good reduction with respect to \mathcal{D}* , if the following conditions are fulfilled:

- (a) $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a product of matrix algebras over unramified field extensions of \mathbb{Q}_p .
- (b) There exists a $\mathbb{Z}_{(p)}$ -order O_B of B , stable under $*$, such that $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a maximal order of $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$.
- (c) There exists a \mathbb{Z}_p -lattice $\Lambda \subset V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, which is an O_B -submodule, such that the restriction of $\langle \cdot, \cdot \rangle_{\mathbb{Q}_p}$ to $\Lambda \times \Lambda$ is a perfect pairing of \mathbb{Z}_p -modules.
- (d) We have $p > 2$ if the following equivalent conditions are fulfilled:
 - (i) $\text{End}_B(V) \otimes_{\mathbb{Q}} \mathbb{R}$ has a factor isomorphic to $M_n(\mathbb{H})$.
 - (ii) G is not connected.

In particular $G_{\mathbb{Q}_p}$ has a reductive model \mathcal{G} over \mathbb{Z}_p whose R -valued points in some commutative \mathbb{Z}_p -algebra are given by

$$\mathcal{G}(R) = \left\{ g \in \text{GL}_{R \otimes_{\mathbb{Z}_p} \mathbb{Q}_p}(\Lambda_R) \mid \langle gm, gn \rangle = \eta(g) \langle m, n \rangle, \right. \\ \left. \eta(g) \in R^\times, \text{ for all } m, n \in \Lambda_R \right\}$$

i.e. $G_{\mathbb{Q}_p}$ is unramified.

(1.4.2) From now on we fix a prime p of good reduction with respect to \mathcal{D} . Further we fix a $\mathbb{Z}_{(p)}$ -order O_B and a lattice Λ of $V_{\mathbb{Q}_p}$ as in (b) and (c) above. Finally we choose some algebraic closure $\bar{\mathbb{Q}}_p$ of \mathbb{Q}_p and an embedding $\nu : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Via this embedding we get an inclusion

$$\Gamma(p) := \text{Gal}(\bar{\mathbb{Q}}_p / \mathbb{Q}_p) \hookrightarrow \Gamma := \text{Gal}(\bar{\mathbb{Q}} / \mathbb{Q}).$$

If $(X^*, R^*, X_*, R_*, \Delta)$ is the associated based root datum to G and if Ω_0 is the root Weyl group of G , the action of Γ on X_* / Ω_0 gives an action of $\Gamma(p)$ on X_* / Ω_0 . If we denote

by $E_\nu = E_\nu(\mathcal{D})$ the field of definition of $c_0(\mathcal{D})$ with respect to $\Gamma(p)$, then $E_\nu(\mathcal{D})$ is the ν -adic completion of $E(\mathcal{D})$.

(1.4.3) Let $(X^*, R^*, X_*, R_*, \Delta)$ be the based root datum of G , and let Ω_0 be the Weyl group of the root datum (X^*, R^*, X_*, R_*) . Let $\bar{C} \subset (X_*)_{\mathbb{Q}}$ be the closed Weyl chamber associated to the root base Δ . This is a fundamental domain for the action of Ω_0 on $(X_*)_{\mathbb{Q}}$.

Now $\Gamma(p)$ acts on $(X_*)_{\mathbb{Q}}$ and \bar{C} is $\Gamma(p)$ -stable with respect to this action (1.2.4). The element $c_0(\mathcal{D}) \in X_*/\Omega_0$ defined in (1.3.3) will also be considered as an element of \bar{C} . Let $\Gamma(p)_c$ be the stabilizer of $c_0(\mathcal{D})$ in $\Gamma(p)$ and set

$$\bar{\mu}(\mathcal{D}) = [\Gamma(p) : \Gamma(p)_c]^{-1} \sum_{\gamma \in \Gamma(p)/\Gamma(p)_c} \gamma(c_0(\mathcal{D})) \in \bar{C}.$$

We consider $\bar{\mu}(\mathcal{D})$ as an element in $(X_*)_{\mathbb{Q}}/\Omega_0$.

1.5. Newton points associated to points of the moduli space

(1.5.1) Let $\mathcal{D} = (B, *, V, \langle \cdot, \cdot \rangle, h)$ be a PEL-datum, let G be its associated group, let $E = E(\mathcal{D})$ be the field of definition of $c_0(\mathcal{D})$, and let p be a prime number of good reduction with respect to \mathcal{D} .

Kottwitz ([Ko1] 5) has defined a moduli problem over $O_E \otimes \mathbb{Z}_{(p)}$ associated to \mathcal{D} which we recall now: Let $C^p \subset G(\mathbf{A}_f^p)$ be an open compact subgroup, where \mathbf{A}_f^p denotes the ring of finite adeles over \mathbb{Q} with trivial p -th component. Define a set-valued contravariant functor $\mathbf{A}_{\mathcal{D}, C^p}$ on the category of schemes S over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ which associates to S the set of isomorphism classes of quadruples $(A, \bar{\lambda}, \iota, \bar{\eta})$ where

- A is an abelian scheme over S up to prime-to- p -isogeny.
- $\bar{\lambda}$ is a \mathbb{Q} -homogeneous polarization of A containing a polarization $\lambda \in \bar{\lambda}$ of degree prime to p ,
- $\iota: O_B \rightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is an involution preserving $\mathbb{Z}_{(p)}$ -algebra homomorphism, where the involution is $*$ on O_B and the Rosati-Involution given by $\bar{\lambda}$ on $\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$,
- $\bar{\eta}$ is a level structure of type C^p .

We require that $(A, \bar{\lambda}, \iota, \bar{\eta})$ satisfies the determinant condition, i.e. we have an identity of polynomial functions on O_B

$$(1.5.1.1) \quad \det_{O_S}(b | \text{Lie}(A)) = \det_E(b | V^{(0, -1)})$$

(see [Ko1] 5 or [RZ] 3.23 a) for a precise formulation of the determinant condition).

We will assume that C^p is sufficiently small so that the functor $\mathbf{A}_{\mathcal{D}, C^p}$ is representable by a quasiprojective smooth scheme over $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$.

(1.5.2) Let k be an algebraically closed extension of the residue class field κ of E_ν , let $W(k)$ its ring of Witt vectors, and denote by L the quotient field of $W(k)$. Consider a geometric point

$$x = (A_0, \bar{\lambda}_0, \iota_0, \bar{\eta}_0) \in \mathbf{A}_{\mathcal{D}, C^p}(k)$$

of $\mathbf{A}_{\mathcal{D},C^p}$ and fix a polarization $\lambda_0 \in \bar{\lambda}_0$ of degree prime to p . Let $N(x)$ be the isocrystal associated to A_0 . It is equipped with a perfect alternating form given by λ_0 and with an \mathcal{O}_B -module structure given by ι_0 . If we fix an isomorphism

$$N(x) \cong V \otimes_{\mathbb{Q}_p} L$$

of $(B \otimes_{\mathbb{Q}} L)$ -modules preserving the alternating pairings on both sides, $N(x)$ can be considered as isocrystal with G^0 -structure and we denote by

$$\bar{\nu}(x) \in (X_*)_{\mathbb{Q}}/\Omega_0$$

its Newton point ([RR] 3.4 and 3.5). For every Zariski point s of $\mathbf{A}_{\mathcal{D},C^p} \otimes \kappa$, let k be some algebraically closed extension of $\kappa(s)$ and let x be the associated k -valued point of $\mathbf{A}_{\mathcal{D},C^p}$. We set

$$\bar{\nu}(s) := \bar{\nu}(x).$$

This is independent of the choice of k .

1.6. The Density Theorem

(1.6.1) Denote by κ the residue class field of E_ν . By [Ko1], 4.3, the $G'(\mathbb{R})$ -conjugacy class of h does not depend on the particular choice of h . Therefore $c_0(\mathcal{D})$ and $\bar{\mu}(\mathcal{D})$ depend only on $(B, *, V, \langle \cdot, \cdot \rangle)$ if $G' = G'^0$, i.e. if G is connected. In this case define $\overline{\mathcal{M}}(\mathcal{D}) = \{\bar{\mu}(\mathcal{D})\}$.

If G is not connected, every conjugate of one chosen h by an element of $G'(\mathbb{R})$ satisfies the PEL-conditions (1.3.1) for h as well. Therefore if we have $c_0(\mathcal{D})$ given by one choice of h every G -conjugate of $c_0(\mathcal{D})$ can occur as well. Denote by $c^{(i)}$ ($i = 1, \dots, m$) these conjugates. For every $c^{(i)}$ we get a rational conjugacy class $\bar{\mu}^{(i)}$ by the construction in (1.4.3), and in this case we define

$$\overline{\mathcal{M}}(\mathcal{D}) = \{\bar{\mu}^{(i)} \mid i = 1, \dots, m\}.$$

Now define:

$$\mathbf{A}_{\mathcal{D},C^p}^{\mu\text{-ord}} = \{s \in \mathbf{A}_{\mathcal{D},C^p} \otimes \kappa \mid \bar{\nu}(s) \in \overline{\mathcal{M}}(\mathcal{D})\}.$$

(1.6.2) **Density Theorem:** *The subset $\mathbf{A}_{\mathcal{D},C^p}^{\mu\text{-ord}}$ is open and dense in $\mathbf{A}_{\mathcal{D},C^p} \otimes \kappa$. In particular it is nonempty.*

We call $\mathbf{A}_{\mathcal{D},C^p}^{\mu\text{-ord}}$ the \mathcal{D} -ordinary locus (or the μ -ordinary locus) of $\mathbf{A}_{\mathcal{D},C^p} \otimes \kappa$.

The density theorem will be proved in chapters 2 and 4.

(1.6.3) Denote by $\mathbf{A}_{\mathcal{D},C^p}^{\text{ord}}$ the subset of points $s \in \mathbf{A}_{\mathcal{D},C^p} \otimes \kappa$ where the underlying abelian variety is ordinary. This is an open subset and we call it the *ordinary locus* of $\mathbf{A}_{\mathcal{D},C^p} \otimes \kappa$. We have the following result:

Theorem: *Assume that G is connected. Then the following assertions are equivalent:*

- (1) *The ordinary locus $\mathbf{A}_{\mathcal{D},C^p}^{\text{ord}}$ is nonempty.*
- (2) *The ordinary locus $\mathbf{A}_{\mathcal{D},C^p}^{\text{ord}}$ is dense in $\mathbf{A}_{\mathcal{D},C^p} \otimes \kappa$.*

(3) We have $E_\nu = \mathbb{Q}_p$.

Proof (first part): Consider $B_0 = \mathbb{Q}$ as subalgebra of B and let \mathcal{D}_0 be the associated PEL-datum (1.3.5). Then we have $G_0 = \mathrm{GSp}(V, \langle \cdot, \cdot \rangle)$ and

$$\mathbf{A}_{\mathcal{D}, C^p}^{\mathrm{ord}} = \{s \in \mathbf{A}_{C^p} \otimes \kappa \mid \gamma_{G, G_0}(\bar{\nu}(s)) = c_0(\mathcal{D}_0)\}.$$

By the density theorem it follows that (1) implies $\mathbf{A}_{\mathcal{D}, C^p}^{\mathrm{ord}} \cap \mathbf{A}_{\mathcal{D}, C^p}^{\mu\text{-ord}} \neq \emptyset$ and therefore we have:

$$(1.6.3.1) \quad \gamma_{G, G_0}(\bar{\mu}(\mathcal{D})) = c_0(\mathcal{D}_0).$$

Conversely (1.6.3.1) implies $\mathbf{A}_{\mathcal{D}, C^p}^{\mu\text{-ord}} \subset \mathbf{A}_{\mathcal{D}, C^p}^{\mathrm{ord}}$ and again by the density theorem assertion (2) follows. Therefore it remains to prove that (1.6.3.1) and assertion (3) are equivalent. For this we have to calculate $\bar{\mu}(\mathcal{D})$ and this will be done in (2.3).

2. Reduction to a deformation problem in four special cases

2.1. Reduction to a deformation problem

(2.1.1) To simplify notations we set

$$\begin{aligned} Y &:= \mathbf{A}_{\mathcal{D}, C^p} \otimes \kappa, \\ Y^0 &:= \mathbf{A}_{\mathcal{D}, C^p}^{\mu\text{-ord}}. \end{aligned}$$

From Grothendieck’s specialization theorem for crystals generalized in [RR] to crystals with G -structure we know that Y^0 is a locally closed subset of Y ([RR] 3.6). Therefore it suffices to show the following assertion for any point $s \in Y$:

(2.1.1.1) There exists a generization η of s such that $\eta \in Y^0$.

(2.1.2) Let y be a closed point of Y which is a specialization of s . As Y is of finite type over κ the residue class field of y is a finite field, in particular it is perfect. Let $\mathcal{O}_{\bar{y}}$ be the completion of the strict henselization of $\mathcal{O}_{Y, y}$ and let $\bar{\kappa}$ be its residue class field. If \bar{s} is some geometric point of Y lying above s the choice of an embedding $\bar{\kappa} \hookrightarrow \kappa(\bar{s})$ induces a homomorphism $\mathcal{O}_{\bar{y}} \rightarrow \kappa(\bar{s})$. Denote by $(A_0, \bar{\lambda}_0, \iota_0, \bar{\eta}_0)$ the geometric point of Y given by $\mathrm{Spec}(\bar{\kappa}) \rightarrow Y$. If $\hat{\mathbf{C}}$ denotes the category of local noetherian complete rings of characteristic p with residue class field $\bar{\kappa}$, the morphisms being local ring homomorphisms, $\mathcal{O}_{\bar{y}}$ represents the functor

$$\begin{aligned} \mathbf{D} = \mathbf{D}(\mathcal{D}): \hat{\mathbf{C}} &\longrightarrow \mathbf{Ens}, \\ R &\longmapsto \{(A, \bar{\lambda}, \iota, \bar{\eta}) \in \mathbf{A}_{\mathcal{D}, C^p}(R) \mid (A, \bar{\lambda}, \iota, \bar{\eta}) \otimes_R \bar{\kappa} = (A_0, \bar{\lambda}_0, \iota_0, \bar{\eta}_0)\}. \end{aligned}$$

Therefore it suffices to find some $R \in \hat{\mathbf{C}}$, some $(A, \bar{\lambda}, \iota, \bar{\eta}) \in \mathbf{D}(R)$, and some geometric point \bar{z} of $\mathrm{Spec}(R)$ such that the image of

$$\mathrm{Spec}(\kappa(\bar{z})) \longrightarrow \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(\mathcal{O}_{\bar{y}}) \longrightarrow Y$$

is a generization of s and lies in Y^0 (here the second arrow is given by $(A, \bar{\lambda}, \iota, \bar{\eta}) \in \mathbf{D}(R) = \mathrm{Hom}(\mathcal{O}_{\bar{y}}, R)$ and the last arrow is the canonical morphism).

(2.1.3) Let O be a \mathbb{Z}_p -algebra (unitary but not necessarily commutative), equipped with an involution $*$: $O \rightarrow O$ of \mathbb{Z}_p -algebras. Then a *principally quasi-polarized p -divisible O -module over R* is a triple (X, λ, ι) where X is a p -divisible group over R , λ is an isomorphism of X in its Serre dual X^\vee , such that $\lambda^\vee = -\lambda$ and $\iota: O \rightarrow \text{End}(X)$ is a \mathbb{Z}_p -algebra homomorphism commuting with $*$ on O and with the Rosati involution given by λ on $\text{End}(X)$.

(2.1.4) By the usual arguments (Serre-Tate, Grothendieck’s algebraization theorem, and rigidity of étale covers) the functor \mathbf{D} is isomorphic to the functor which associates to each $R \in \hat{\mathbf{C}}$ the set of isomorphism classes of pairs $((X, \lambda, \iota), \varphi_0)$ consisting of a principally quasi-polarized p -divisible $(O_B \otimes \mathbb{Z}_p)$ -module (X, λ, ι) over R and an isomorphism

$$\varphi_0: (X, \lambda, \iota) \otimes_R \bar{\kappa} \xrightarrow{\sim} (A_0[p^\infty], \lambda_0[p^\infty], \iota_0[p^\infty])$$

where we denote by $\lambda_0[p^\infty]$ the restriction of a polarization $\lambda_0 \in \bar{\lambda}_0$ of degree prime to p to the p -divisible group $A_0[p^\infty]$ of A_0 and by $\iota_0[p^\infty]: O_B \otimes \mathbb{Z}_p \rightarrow \text{End}(A_0[p^\infty])$ the operation induced by ι_0 . Finally the determinant condition (1.5.1.1) translates into requiring an equality of polynomial functions on $O_B \otimes \mathbb{Z}_p$

$$(2.1.4.1) \quad \det_{O_S}(b | \text{Lie}(X)) = \det_F(b | V_0)$$

where F/\mathbb{Q}_p is the field of definition of some cocharacter in the $G^0(\bar{\mathbb{Q}}_p)$ -conjugacy class $c_0(\mathcal{D})$ and where $V_0 \subset V_F$ is the weight 0 space of this cocharacter (note that the right hand side of (2.1.4.1) does not depend on the choice of the cocharacter in $c_0(\mathcal{D})$ because the action of G^0 on V commutes with the action of O_B).

In particular we see that the functor \mathbf{D} depends only on the data \mathcal{D}_p consisting of $B \otimes \mathbb{Q}_p$ and $O_B \otimes \mathbb{Z}_p$ with the induced involution $*$, $V \otimes \mathbb{Q}_p$ and Λ with the induced perfect pairing $\langle \cdot, \cdot \rangle$, and of $c_0(\mathcal{D})$ as an element in the $\Gamma(p)$ -set $(X_*)_{\mathbb{Q}}/\Omega_0$. We write $\mathbf{D} = \mathbf{D}(\mathcal{D}_p)$. From this it follows that the assertion of (2.1.1.1) depends only on \mathcal{D}_p .

(2.1.5) In the same way (by considering $\mathbf{A}_{\mathcal{D}, C^p}^{\text{ord}}$ instead of $\mathbf{A}_{\mathcal{D}, C^p}^{\mu\text{-ord}}$) we see that assertions (1) and (2) of theorem (1.6.3) depend only on \mathcal{D}_p , and trivially this holds for assertion (3) of (1.6.3) as well.

(2.1.6) From now on we change notations and write B (resp. O_B , resp. V , resp. G , resp. G') for $B \otimes \mathbb{Q}_p$ (resp. $O_B \otimes \mathbb{Z}_p$, resp. $V \otimes \mathbb{Q}_p$, resp. $G \otimes \mathbb{Q}_p$, resp. $G' \otimes \mathbb{Q}_p$).

(2.1.7) Let $\bar{s}_1 \rightarrow Y$ be an arbitrary geometric point $(A_1, \bar{\lambda}_1, \iota_1, \bar{\eta}_1)$ of Y and let $(X_1, \lambda_1, \iota_1)$ be its principally quasi-polarized p -divisible O_B -module (after the choice of some prime-to- p isogeny $\lambda_1 \in \bar{\lambda}_1$). Set $\kappa_1 = \kappa(\bar{s}_1)$ and let $R = \kappa_1[[t]]$ be the ring of power series over κ_1 . We will lift $(X_1, \lambda_1, \iota_1)$ in the following way to R : We have the decomposition of $(X_1, \lambda_1, \iota_1)$ in its étale-multiplicative part $(X_1, \lambda_1, \iota_1)_{\text{ét mult}}$ and in its bi-infinitesimal part $(X_1, \lambda_1, \iota_1)_{\text{bi}}$. We construct some lifting $(X, \lambda, \iota)_{\text{bi}}$ of $(X_1, \lambda_1, \iota_1)_{\text{bi}}$ by the methods of chapter 3 and this gives a lifting (X, λ, ι) of $(X_1, \lambda_1, \iota_1)$ by setting

$$(X, \lambda, \iota) = ((X_1, \lambda_1, \iota_1)_{\text{ét mult}} \otimes_{\kappa_1} R) \times (X, \lambda, \iota)_{\text{bi}}$$

We get a lift $(A, \bar{\lambda}, \iota, \bar{\eta})$ of $(A_1, \lambda_1, \iota_1, \bar{\eta}_1)$ to $\kappa_1[[t]]$. Base changing $(A, \lambda, \iota, \bar{\eta})$ to an algebraic closure κ_2 of $\kappa_1((t))$ defines a geometric point $\bar{s}_2 \rightarrow Y$. We can repeat the process and get a sequence $(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_m)$ of geometric points of Y where \bar{s}_j is a generalization of \bar{s}_i if $j \geq i$. We are done with the proof of the density theorem if we get finally a geometric point $\bar{s}_n \rightarrow Y$ whose image lies in Y^0 .

2.2. Reduction to four special cases

(2.2.1) Let $(B, *)$ be a product $(B, *) = (B_1, *) \times (B_2, *)$ of two semi-simple finite-dimensional \mathbb{Q}_p -algebras with involutions and choose maximal orders O_{B_i} of B_i ($i = 1, 2$) such that $O_B = O_{B_1} \times O_{B_2}$. We get a corresponding decomposition $(V, \langle \cdot, \cdot \rangle) = (V_1, \langle \cdot, \cdot \rangle) \oplus (V_2, \langle \cdot, \cdot \rangle)$, and the O_B -lattice $\Lambda \subset V$ may be written in a unique way $\Lambda = \Lambda_1 \oplus \Lambda_2$, where $\Lambda_i \subset V_i$ is an O_{B_i} -lattice, and the restriction of $\langle \cdot, \cdot \rangle$ to $\Lambda_i \times \Lambda_i$ is a perfect \mathbb{Z}_p -bilinear form. Denote by \mathbf{D}_i ($i = 1, 2$) the corresponding functor $\hat{\mathbf{C}} \rightarrow \mathbf{Ens}$. We have then

$$\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2.$$

For the corresponding algebraic groups G, G_1 , and G_2 over \mathbb{Q}_p , we have $G \subset G_1 \times G_2$ where for any \mathbb{Q}_p -algebra R the R -valued points are given by

$$G(R) = \{(g_1, g_2) \in G_1(R) \times G_2(R) \mid \eta_1(g_1) = \eta_2(g_2)\},$$

$\eta_i: G_i \rightarrow \mathbb{G}_{m, \mathbb{Q}_p}$ denoting the multiplier homomorphism. The inclusion $G \hookrightarrow G_1 \times G_2$ induces an injective map

$$(X_*)_{\mathbb{Q}}/\Omega_0 \hookrightarrow (X_{1*})_{\mathbb{Q}}/\Omega_{1,0} \times (X_{2*})_{\mathbb{Q}}/\Omega_{2,0}.$$

For every $R \in \hat{\mathbf{C}}$ every point $X = (X_1, X_2) \in \mathbf{D}(R) = \mathbf{D}_1(R) \times \mathbf{D}_2(R)$ induces a map

$$\varphi_X: \text{Spec}(R) \longrightarrow (X_*)_{\mathbb{Q}}/\Omega_0, \quad s \mapsto \bar{\nu}(X_{\bar{s}}),$$

and we have a commutative diagram

$$\begin{array}{ccc} & \text{Spec}(R) & \\ \varphi_X \swarrow & & \searrow \varphi_{X_1} \times \varphi_{X_2} \\ (X_*)_{\mathbb{Q}}/\Omega_0 & \hookrightarrow & (X_{1*})_{\mathbb{Q}}/\Omega_{1,0} \times (X_{2*})_{\mathbb{Q}}/\Omega_{2,0}. \end{array}$$

(2.2.2) Due to the arguments of the previous subsection we can assume for the proof of the density theorem and of theorem (1.6.3) that $(B, *)$ as \mathbb{Q}_p -algebra with involution is simple. Using Morita theory for Hermitian modules (see e.g. [Kn] chap. I, § 9) we can further assume that we are in one of the following cases (where K denotes an unramified field extension of \mathbb{Q}_p and O_K its ring of integers):

- (AL) (linear case): $B = K \times K, O_B = O_K \times O_K, (a, b)^* = (b, a)$ for all $a, b \in K$.
- (AU) (unitary case): $B = K, O_B = O_K, * \in \text{Gal}(K/\mathbb{Q}_p)$ an automorphism of order 2.
- (C) (symplectic case): $B = K, O_B = O_K, * = \text{id}$.
- (D) (orthogonal case): $B = M_2(K), O_B = M_2(O_K), A^* = J^t A J^{-1}$ for $A \in M_2(K)$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; further we assume in this case that the residue characteristic of K is not equal 2 by the conditions we imposed upon p (1.4.1).

Note that G is disconnected if and only if case (D) occurs.

2.3. Calculation of $\bar{\mu}(\mathcal{D})$

(2.3.1) (Calculation of $\bar{\mu}(\mathcal{D})$ in case (AL)): Set $n = [K : \mathbb{Q}_p]$. We can identify

$$G' = \text{Res}_{K/\mathbb{Q}_p}(G'_0)$$

where G'_0 is the reductive K -group $GL(V'_K)$ and where $V_K = V'_K \times V''_K$ is the decomposition of V_K given by the action of $K \times K$. Therefore we have

$${}_G X_* = X_* = \left\{ (x_{ji})_{j \in \mathbb{Z}/n\mathbb{Z}, 1 \leq i \leq 2d} \in \mathbb{Z}^{2nd} \mid \begin{array}{l} x_{ji} + x_{j,2d+1-i} = \text{const} \\ \text{for all } j \in \mathbb{Z}/n\mathbb{Z}, i = 1, \dots, d \end{array} \right\}$$

if $d = \dim(V)/2n$.

The root Weyl group Ω_0 of G is given by $(S_d)^n$, and the action of an n -tuple of permutations (π_1, \dots, π_n) is given by acting on (x_{ji}) by simultaneously permuting (x_{j1}, \dots, x_{jd}) and $(x_{j,2d}, \dots, x_{j,d+1})$ with π_j for each $j \in \mathbb{Z}/n\mathbb{Z}$. A fundamental domain for the action of Ω_0 on X_* is given by

$$\{(x_{ji})_{j \in \mathbb{Z}/n\mathbb{Z}, 1 \leq i \leq 2d} \in X_* \mid x_{j1} \geq \dots \geq x_{jd} \text{ for all } j \in \mathbb{Z}/n\mathbb{Z}\}.$$

We will identify this set with X_*/Ω_0 .

The action of $\Gamma(p)$ factors through $\text{Gal}(K/\mathbb{Q}_p) = \langle \sigma^r \mid r \in \mathbb{Z}/n\mathbb{Z} \rangle$ where σ is the Frobenius and is given by

$$\sigma((x_{ji})) = (x_{j+1,i}).$$

Via these identifications $c_0(\mathcal{D}) \in X_*/\Omega_0$ is given by (c_{ji}) with

$$(c_{j1}, \dots, c_{jd}, c_{j,d+1}, \dots, c_{j,2d}) = (1^r, 0^s, 1^s, 0^r),$$

where $r, s \geq 0$ are integers satisfying $r + s = d$ (here n^q denotes the q -tuple where every entry is equal to n). In particular we see that $c_0(\mathcal{D})$ is fixed by $\Gamma(p)$ and it follows

$$\bar{\mu}(\mathcal{D}) = c_0(\mathcal{D}).$$

Further we see that $E_\nu = \mathbb{Q}_p$ and that (1.6.3.1) holds; therefore theorem (1.6.3) is proved in case (AL) under the assumption of the density theorem.

(2.3.2) (Calculation of $\bar{\mu}(\mathcal{D})$ in case (AU)): Let K_0 be the fixed field of $*$ in K and set $n = [K_0 : \mathbb{Q}_p]$. If σ is the Frobenius of K , then $*$ = σ^n . We have

$$G' = \text{Res}_{K_0/\mathbb{Q}_p}(G'_0)$$

where G'_0 is a quasi-split unitary group over K_0 determined by the quadratic extension K/K_0 . Therefore we can identify

$${}_G X_* = X_* = \{(x_{ji})_{j \in \mathbb{Z}/2n\mathbb{Z}, 1 \leq i \leq d} \in \mathbb{Z}^{2nd} \mid x_{ji} + x_{j+n,i} = \text{const for all } j, i\}$$

where $d = \dim(V)/2n$. The root Weyl group Ω_0 of G is given by $(S_d)^n$ acting on X_* via

$$((\pi_0, \dots, \pi_{n-1}), (x_{ji})) \mapsto (x_{j, \pi_j^{-\delta(j)n}(i)}).$$

where $\delta: \mathbb{Z}/2n\mathbb{Z} \rightarrow \{0, 1\}$ is defined as

$$\delta(l) = \begin{cases} 0, & \text{if } l = 0, \dots, n-1, \\ 1, & \text{if } l = n, \dots, 2n-1. \end{cases}$$

A fundamental domain for the action of Ω_0 on X_* is given by

$$\{(x_{ji}) \in X_* \mid x_{j1} \geq \dots \geq x_{jd} \text{ for } j = 0, \dots, n-1\}.$$

We will identify this set with X_*/Ω_0 .

The action of $\Gamma(p)$ factors through $\text{Gal}(K/\mathbb{Q}_p) = \langle \sigma^r \mid r \in \mathbb{Z}/2n\mathbb{Z} \rangle$ and is given by

$$\sigma((x_{ji})) = \begin{cases} (x_{j+1,i}), & \text{if } \delta(j+1) = \delta(j), \\ (x_{j+1,d+1-i}), & \text{if } \delta(j+1) \neq \delta(j). \end{cases}$$

Via these identifications $c_0(\mathcal{D}) \in X_*/\Omega_0$ is given by (c_{ji}) with

$$(c_{j1}, \dots, c_{jd}) = (1^{r(j)}, 0^{d-r(j)}),$$

where $r(j) \geq 0$ ($j \in \mathbb{Z}/2n\mathbb{Z}$) are integers satisfying $r(j) + r(j+n) = d$. In particular we see that $E_\nu = \mathbb{Q}_p$ if and only if $r(j) = d/2$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$. This case can only occur if d is even. Let

$$(m_{ji}) := \bar{\mu}(\mathcal{D}) = \frac{1}{2n} \sum_{r \in \mathbb{Z}/2n\mathbb{Z}} \sigma^r((c_{ji})) \in (X_*)_{\mathbb{Q}}/\Omega_0.$$

Then we have for all $i = 1, \dots, d$:

$$m_{0,i} = \dots = m_{n-1,i} = 1 - m_{n,i} = \dots = 1 - m_{2n-1,i},$$

and for $j = 0, \dots, n-1$, $i = 1, \dots, d$ we have

$$m_{ji} = 1 - \frac{1}{2n} \sum_{h=0}^{i-1} k(h)$$

with

$$k(h) = \#\{j \in \mathbb{Z}/2n\mathbb{Z} \mid r(j) = h\}.$$

Therefore we have $r(j) = d/2$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$ if and only if we have for $j = 0, \dots, n-1$:

$$m_{j,1} = \dots = m_{j,d/2} = 1, \quad m_{j,d/2+1} = \dots = m_{j,d} = 0,$$

i.e. if and only if (1.6.3.1) holds, and this proves theorem (1.6.3) in case (AU) (under the assumption of the density theorem).

(2.3.3) (Calculation of $\bar{\mu}(\mathcal{D})$ in case (C)): Set $n = [K : \mathbb{Q}_p]$. We can identify

$$G' = \text{Res}_{K/\mathbb{Q}_p}(G'_0)$$

where G'_0 is the symplectic group $\mathrm{Sp}(V_K, \langle \cdot, \cdot \rangle)$ over K . Therefore we can identify

$${}_G X_* = X_* = \left\{ (x_{ji})_{j \in \mathbb{Z}/n\mathbb{Z}, 1 \leq i \leq 2d} \in \mathbb{Z}^{2nd} \mid \begin{array}{l} x_{ji} + x_{j,2d+1-i} = \text{const} \\ \text{for all } j \in \mathbb{Z}/n\mathbb{Z}, i = 1, \dots, d \end{array} \right\}$$

where $d = \dim(V)/2n$.

The root Weyl group Ω_0 of G is given by $((S_d) \times \{\pm 1\}^d)^n$, and the action of an n -tuple $((\pi_1, (\varepsilon_{1i})_{1 \leq i \leq d}), \dots, (\pi_n, (\varepsilon_{ni})_{1 \leq i \leq d}))$ is given by acting on (x_{ji}) by simultaneously permuting (x_{j1}, \dots, x_{jd}) and $(x_{j,2d}, \dots, x_{j,d+1})$ with π_j and by exchanging x_{ji} and $x_{j,2d+1-i}$ if $\varepsilon_{ji} = -1$ (for $j \in \mathbb{Z}/n\mathbb{Z}$, $i = 1, \dots, d$).

A fundamental domain for the action of Ω_0 on X_* is given by

$$\left\{ (x_{ji})_{j \in \mathbb{Z}/n\mathbb{Z}, 1 \leq i \leq 2d} \in X_* \mid \begin{array}{l} x_{j1} \geq \dots \geq x_{jd} \geq x_{j,d+1} \geq \dots \geq x_{j,2d} \\ \text{for all } j \in \mathbb{Z}/n\mathbb{Z} \end{array} \right\}.$$

We will identify this set with X_*/Ω_0 .

The action of $\Gamma(p)$ factors through $\mathrm{Gal}(K/\mathbb{Q}_p) = \langle \sigma^r \mid r \in \mathbb{Z}/n\mathbb{Z} \rangle$ and is given by

$$\sigma((x_{ji})) = (x_{j+1,i}).$$

Via these identifications $c_0(\mathcal{D}) \in X_*/\Omega_0$ is given by (c_{ji}) with

$$(c_{j1}, \dots, c_{jd}, c_{j,d+1}, \dots, c_{j,2d}) = (1^d, 0^d).$$

In particular we see that $c_0(\mathcal{D})$ is fixed by $\Gamma(p)$ and it follows

$$\bar{\mu}(\mathcal{D}) = c_0(\mathcal{D}).$$

Further we see that $E_\nu = \mathbb{Q}_p$ and that (1.6.3.1) holds, therefore theorem (1.6.3) is proved in case (C) under the assumption of the density theorem.

(2.3.4) (Calculation of $\bar{\mu}(\mathcal{D})$ in case (D)): Set $n = [K : \mathbb{Q}_p]$. We have

$$G' = \mathrm{Res}_{K/\mathbb{Q}_p} G'_0$$

where G'_0 is a quasi-split orthogonal group over K . Therefore we can identify

$${}_G X_* = X_* = \left\{ (x_{ji})_{j \in \mathbb{Z}/n\mathbb{Z}, 1 \leq i \leq 2d} \in \mathbb{Z}^{2dn} \mid \begin{array}{l} x_{ji} + x_{j,2d+1-i} = \text{const}, \\ j \in \mathbb{Z}/n\mathbb{Z}, i = 1, \dots, d \end{array} \right\}$$

where $d = \dim(V)/2n$.

If we set $\Theta_0 = \left\{ (\pi, (\varepsilon_i)_{i=1, \dots, d}) \in S_d \times \{\pm 1\}^d \mid \prod_{1 \leq i \leq d} \varepsilon_i = 1 \right\}$, the root Weyl group Ω_0 of G is given by $(\Theta_0)^n$, and with the action on X_* as in case (C). A fundamental domain for the action of Ω_0 on X_* is given by

$$\left\{ (x_{ji})_{j \in \mathbb{Z}/n\mathbb{Z}, 1 \leq i \leq 2d} \in X_* \mid \begin{array}{l} x_{j1} \geq \dots \geq x_{j,d-1} \geq \max\{x_{j,d}, x_{j,d+1}\} \\ \geq \min\{x_{j,d}, x_{j,d+1}\} \geq x_{j,d+2} \geq \dots \geq x_{j,2d} \\ \text{for all } j \in \mathbb{Z}/n\mathbb{Z} \end{array} \right\}.$$

We will identify this set with X_*/Ω_0 .

For the action of $\Gamma(p)$ on X_* we have to distinguish two cases:

G'_0 is split: The action of $\Gamma(p)$ factors through $\text{Gal}(K/\mathbb{Q}_p) = \langle \sigma^s \mid s \in \mathbb{Z}/n\mathbb{Z} \rangle$ and is given by

$$\sigma((x_{ji})) = (x_{j+1,i}).$$

G'_0 is non-split: Let K_1 be the unramified extension of degree 2 of K in $\overline{\mathbb{Q}_p}$ which splits G'_0 . The action of $\Gamma(p)$ factors through $\text{Gal}(K_1/\mathbb{Q}_p) = \langle \sigma^s \mid s \in \mathbb{Z}/2n\mathbb{Z} \rangle$ and is given by

$$\sigma((x_{ji})) = \begin{cases} (x_{j+1,1}, \dots, x_{j+1,2d}), & \text{if } j+1 \neq 0, \\ (x_{j+1,1}, \dots, x_{j+1,d+1}, x_{j+1,d}, \dots, x_{j+1,2d}), & \text{if } j+1 = 0. \end{cases}$$

Via these identifications $c_0(\mathcal{D}) \in X_*/\Omega_0$ is given by (c_{ji}) with two cases for every $j \in \mathbb{Z}/n\mathbb{Z}$ (1.6.1):

$$(+) \quad (c_{j1}, \dots, c_{j,2d}) = (1^{d-1}, 1, 0, 0^{d-1}),$$

or

$$(-) \quad (c_{j1}, \dots, c_{j,2d}) = (1^{d-1}, 0, 1, 0^{d-1}).$$

Set

$$s := \#\{j \in \mathbb{Z}/n\mathbb{Z} \mid \text{we are in case (+) for } j\}.$$

For the calculation of $\bar{\mu}(\mathcal{D}) := (m_{ji}) \in X_*/\Omega_0$ we have again to distinguish between the split and the non-split case:

In the split case $\bar{\mu}(\mathcal{D})$ is given by

$$(m_{ji})_{1 \leq i \leq 2d} = \left(1^{d-1}, \frac{s}{n}, 1 - \frac{s}{n}, 0^{d-1}\right)$$

for all $j \in \mathbb{Z}/n\mathbb{Z}$. Therefore we have

$$\overline{\mathcal{M}} = \left\{ \left(1^{d-1}, \frac{s}{n}, 1 - \frac{s}{n}, 0^{d-1}\right)^n \mid s = 0, \dots, n \right\}.$$

In the non-split case $\bar{\mu}(\mathcal{D})$ is given by

$$(m_{ji})_{1 \leq i \leq 2d} = \left(1^{d-1}, \frac{1}{2}, \frac{1}{2}, 0^{d-1}\right)$$

for all $j \in \mathbb{Z}/n\mathbb{Z}$, and we have

$$\overline{\mathcal{M}} = \left\{ \left(1^{d-1}, \frac{1}{2}, \frac{1}{2}, 0^{d-1}\right)^n \right\}.$$

3. Construction of deformations

3.1. Displays

(3.1.1) Notations: Throughout this chapter we fix the following notations: Let p be a prime number, and let R be a (commutative unitary) ring of characteristic p . Denote by

$W(R)$ the ring of Witt vectors of R , equipped with the Verschiebung τ and the Frobenius σ . We have a canonical surjective homomorphism

$$w_0: W(R) \longrightarrow R, (x_0, \dots, x_n, \dots) \mapsto x_0;$$

denote by I_R its kernel.

We briefly recall some definitions and results of Zink's theory of displays (see [Z1] for details).

(3.1.2) Let P_1 and P_2 be $W(R)$ -modules. For a σ -linear homomorphism $\Phi: P_1 \rightarrow P_2$ we denote by

$$\Phi^\#: W(R) \otimes_{\sigma, W(R)} P_1 \longrightarrow P_2$$

the linearization of Φ . We will call Φ a *monomorphism* (resp. an *epimorphism* resp. an *isomorphism*) if $\Phi^\#$ is a monomorphism (resp. an epimorphism resp. an isomorphism).

(3.1.3) Definition: A $3n$ -Display over R is a tuple (P, Q, F, V^{-1}) , where

- P is a finitely generated projective $W(R)$ -module.
- $Q \subset P$ is a submodule.
- $F: P \rightarrow P$ and $V^{-1}: Q \rightarrow P$ are σ -linear maps.

These data shall satisfy the following properties:

- (a) We have $I_R P \subset Q \subset P$ and the quotient P/Q is a direct summand of $P/I_R P$.
- (b) $V^{-1}: Q \rightarrow P$ is a σ -linear epimorphism.
- (c) For $x \in P$ and $w \in W(R)$ we have

$$V^{-1}(\tau(w)x) = wFx.$$

A morphism $\varphi: (P, Q, F, V^{-1}) \rightarrow (P', Q', F', V^{-1'})$ of $3n$ -displays is a $W(R)$ -linear mapping $\varphi: P \rightarrow P'$, such that $\varphi(Q) \subset Q'$, $F' \circ \varphi = \varphi \circ F$ and $V^{-1'} \circ \varphi|_Q = \varphi \circ V^{-1}$.

Thus we get the category of $3n$ -Displays over R . It is a \mathbb{Z}_p -linear category.

(3.1.4) Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$ -display over R . Then for every element $y \in Q$ we have

$$Fy = p \cdot V^{-1}y.$$

In particular we have $F(Q) \subset p \cdot P \subset I_R P \subset Q$ so that F induces a Frobenius-linear map $F: P/Q \rightarrow P/Q$. Note that this holds only because $pR = 0$.

(3.1.5) Lemma: Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$ -Display. There exists a unique $W(R)$ -linear map $V^\#: P \rightarrow W(R) \otimes_{\sigma, W(R)} P$, such that

$$\begin{aligned} V^\#(wFx) &= pw \otimes x & w \in W(R), x \in P, \\ V^\#(wV^{-1}y) &= w \otimes y & w \in W(R), y \in Q. \end{aligned}$$

The cokernel of $V^\#$ is a locally free $W(R)/pW(R)$ -module of the same rank as the R -module P/Q .

(3.1.6) We denote by $\sigma^i V^\#$ the $W(R)$ -linear map

$$\text{id} \otimes_{\sigma^i, W(R)} V^\#: W(R) \otimes_{\sigma^i, W(R)} P \longrightarrow W(R) \otimes_{\sigma^{i+1}, W(R)} P$$

and by $V^{n\#}$ the composite $\sigma^{n-1} V^\# \circ \dots \circ \sigma V^\# \circ V^\#$ for an integer $n \geq 1$.

Definition: A $3n$ -display (P, Q, F, V^{-1}) is called a display, if, locally on R , there exists an integer $N \geq 1$ such that the map

$$V^{N\#}: P \longrightarrow W(R) \otimes_{\sigma^N, W(R)} P$$

is zero modulo I_R .

The category of displays is defined as the full subcategory of the category of $3n$ -displays.

(3.1.7) Lemma: Let $\mathcal{P} = (P, Q, F, V^{-1})$ and $\mathcal{P}' = (P', Q', F', V^{-1'})$ be two $3n$ -displays and let $\varphi: \mathcal{P} \rightarrow \mathcal{P}'$ be a morphism of $3n$ -displays.

(1) We have a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{V^\#} & W(R) \otimes_{\sigma, W(R)} P \\ \varphi \downarrow & & \downarrow \text{id} \otimes \varphi \\ P' & \xrightarrow{V'^\#} & W(R) \otimes_{\sigma, W(R)} P'. \end{array}$$

(2) The morphism φ is an isomorphism of $3n$ -displays if and only if φ is bijective.

Proof: (1) follows immediately from the definition of $V^\#$.

(2) If φ is an isomorphism of $W(R)$ -modules, we have by (1) that

$$\text{rk}_{W(R)/pW(R)}(\text{coker}(V^\#)) = \text{rk}_{W(R)/pW(R)}(\text{coker}(V'^\#)).$$

By the last assertion in (3.1.5) it follows that $\text{rk}_R(P/Q) = \text{rk}_R(P'/Q')$. But φ induces a surjective homomorphism of locally free R -modules $P/Q \rightarrow P'/Q'$ which is therefore an isomorphism. It follows that $\varphi(Q) = Q'$. That φ^{-1} commutes with F and F' and with V^{-1} and $V^{-1'}$ is obvious.

(3.1.8) Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$ -display. Set

$$P^\vee = \text{Hom}_{W(R)}(P, W(R)), \quad Q^\vee = \{\Phi \in P^\vee \mid \Phi(Q) \subset I_R\}.$$

Let $(,) : P \times P^\vee \rightarrow W(R)$ be the canonical pairing. There are unique σ -linear maps $F: P^\vee \rightarrow P^\vee$ and $V^{-1}: Q^\vee \rightarrow P^\vee$, such that

$$(3.1.8.1) \quad \begin{array}{lll} (V^{-1}x, Fz) & = & (x, z)^\sigma \quad x \in Q, z \in P^\vee, \\ (Fx, Fz) & = & p(x, z)^\sigma \quad x \in P, z \in P^\vee, \\ (Fx, V^{-1}z) & = & (x, z)^\sigma \quad x \in P, z \in Q^\vee, \\ (V^{-1}x, V^{-1}z)^\tau & = & (x, z) \quad x \in Q, z \in Q^\vee. \end{array}$$

Then $\mathcal{P}^\vee = (P^\vee, Q^\vee, F, V^{-1})$ is a $3n$ -display, called the *dual $3n$ -display of \mathcal{P}* .

(3.1.9) Let $R = k$ be a perfect field. A *Dieudonné module* over k is a free $W(k)$ -module M of finite type with a σ -linear map $F: M \rightarrow M$ and a σ^{-1} -linear map $V: M \rightarrow M$, such that $FV = VF = p$. For every Dieudonné-module (M, F, V) over k we get a $3n$ -display (P, Q, F, V^{-1}) by setting $P = M$, $Q = VM$ with the obvious operators $F: M \rightarrow M$ and $V^{-1}: VM \rightarrow M$. This construction gives us a natural equivalence of the category of $3n$ -displays over k and the category of Dieudonné modules over k which commutes with duals. Combining this equivalence with covariant Dieudonné theory we get an equivalence BT' of the category of $3n$ -displays over k with the category of p -divisible groups over k .

In particular we get a decomposition of $3n$ -displays over perfect fields

$$\mathcal{P} = \mathcal{P}_{\text{ét}} \oplus \mathcal{P}_{\text{inf}} = \mathcal{P}_{\text{ét}} \oplus \mathcal{P}_{\text{mult}} \oplus \mathcal{P}_{\text{bi}}.$$

By definition \mathcal{P} is a display if and only if $\mathcal{P}_{\text{ét}} = 0$.

(3.1.10) Let $\varphi: R \rightarrow S$ be a ring homomorphism and let $\mathcal{P} = (P, Q, F, V^{-1})$ be a $3n$ -display over R . We will now define the $3n$ -display $\mathcal{P}_S = \mathcal{P} \otimes_R S = (P_S, Q_S, F_S, V_S^{-1})$ over S obtained by base change:

We set $P_S = W(S) \otimes_{W(R)} P$ and

$$F_S: P_S \longrightarrow P_S, \quad w \otimes x \mapsto \sigma(w) \otimes Fx.$$

Let Q_S be the kernel of the morphism $W(S) \otimes_{W(R)} P \rightarrow S \otimes_R P/Q$. Finally there exists a unique σ -linear map $V_S^{-1}: Q_S \rightarrow P_S$, such that

$$\begin{aligned} V_S^{-1}(w \otimes y) &= \sigma(w) \otimes V^{-1}y & w \in W(S), y \in Q, \\ V_S^{-1}(\tau(w) \otimes x) &= w \otimes Fx & w \in W(S), x \in P. \end{aligned}$$

It is easy to see that \mathcal{P}_S is in fact a $3n$ -display. Further \mathcal{P}_S is a display if \mathcal{P} is a display.

(3.1.11) Using Cartier theory Zink has constructed a functor \mathcal{BT} from the category of $3n$ -displays over R into the category of smooth formal groups of finite dimension over R (see [Z1]), such that if R is a perfect field we have for a $3n$ -display \mathcal{P}

$$\mathcal{BT}(\mathcal{P}) = \mathcal{BT}'(\mathcal{P})_{\text{inf}}.$$

Further Zink has shown:

(3.1.12) Theorem: *The restriction of \mathcal{BT} to the category of displays gives a functor of the category of displays $\mathcal{P} = (P, Q, F, V^{-1})$ over R into the category of p -divisible formal groups X over R , commuting with arbitrary base change. This is an equivalence of categories if R is a local complete noetherian ring with perfect residue class field. Further we have*

$$\text{Lie}(\mathcal{BT}(\mathcal{P})) = P/Q, \quad \text{ht}(\mathcal{BT}(\mathcal{P})) = \text{rk}_{W(R)}(P).$$

If \mathcal{P} is a bi-infinitesimal display (i.e. \mathcal{P} and \mathcal{P}^\vee are displays) $\mathcal{BT}(\mathcal{P}^\vee)$ can be identified with the Serre dual of $\mathcal{BT}(\mathcal{P})$.

3.2. Construction of liftings of displays with additional structures

(3.2.1) Let O be a \mathbb{Z}_p -algebra (unitary but not necessarily commutative), equipped with an involution $*$: $O \rightarrow O$ of \mathbb{Z}_p -algebras. Then a *principally quasi-polarized $3n$ -display with O -module structure over R* is a triple $(\mathcal{P}, \lambda, \iota)$ where \mathcal{P} is a $3n$ -display over R , λ is an isomorphism $\mathcal{P} \xrightarrow{\sim} \mathcal{P}^\vee$ satisfying $\lambda^\vee = -\lambda$ and where $\iota: O \rightarrow \text{End}(\mathcal{P})$ is a \mathbb{Z}_p -algebra homomorphism commuting with $*$ on O and with the Rosati involution given by λ on $\text{End}(\mathcal{P})$.

(3.2.2) From now on in this chapter k will be a perfect field of characteristic p , $R = k[[t]]$ the ring of power series over k , $k((t))^{\text{perf}}$ will be a perfect closure of $k((t))$, and $\mathcal{P}_0 = (P_0, Q_0, F_0, V_0^{-1})$ will denote a $3n$ -display over k .

(3.2.3) Let $\mathcal{P} = (P, Q, F, V^{-1})$ be the base change $(\mathcal{P}_0)_R$ to R . This is a lifting of \mathcal{P}_0 over R . By definition we have $P = P_0 \otimes_{W(k)} W(R)$. To any homomorphism $\alpha \in \text{Hom}_{W(R)}(P, W(tR)P)$ we associate another lifting $\mathcal{P}_\alpha = (P_\alpha, Q_\alpha, F_\alpha, V_\alpha^{-1})$ of $(P_0, Q_0, F_0, V_0^{-1})$ as follows: We set $P_\alpha := P$, $Q_\alpha := Q$ and

$$\begin{aligned} F_\alpha x &= Fx + \alpha(Fx), & x \in P, \\ V_\alpha^{-1}y &= V^{-1}y + \alpha(V^{-1}y), & y \in Q. \end{aligned}$$

By Nakayama's lemma V_α^{-1} is again a σ -linear epimorphism and we have obviously $V_\alpha^{-1}(\tau(w)x) = wF_\alpha(x)$ for $w \in W(R)$, $x \in P$. Therefore \mathcal{P}_α is again a $3n$ -display which lifts \mathcal{P} .

(3.2.4) For every $W(k)$ -linear endomorphism N of P_0 we get a $W(R)$ -linear map $\alpha_N \in \text{Hom}_{W(R)}(P, W(tR)P)$ by setting

$$\alpha_N(w \otimes m) = [t]w \otimes Nx$$

for $w \in W(R)$, $x \in M$. Here $[t] \in W(tR)$ denotes the Teichmüller representative of t . We set $\mathcal{P}_N = (P_N, Q_N, F_N, V_N^{-1}) := \mathcal{P}_{\alpha_N}$.

(1) For $x \in P_0 \subset P_N = W(R) \otimes_{W(k)} P_0$ we have

$$F_N(x) = (1 + [t]N)F_0(x).$$

For $y \in Q_0 \subset Q_N$ we have

$$V_N^{-1}(y) = (1 + [t]N)V_0^{-1}(y).$$

(2) We have in P_N :

$$P_0 \cap Q_N = Q_0.$$

(3.2.5) Let $\mathcal{P}'_0 = (P'_0, Q'_0, F'_0, V_0^{-1})$ be a second $3n$ -display over k , let N (resp. N') be a $W(k)$ -linear endomorphism of P_0 (resp. P'_0), and denote by $\varphi_0: \mathcal{P}_0 \rightarrow \mathcal{P}'_0$ be a morphism of $3n$ -displays. Let $\mathcal{P}_N = (P_N, Q_N, F_N, V_N^{-1})$ (resp. $\mathcal{P}'_{N'} = (P'_{N'}, Q'_{N'}, F'_{N'}, V_{N'}^{-1})$) the lifting of \mathcal{P}_0 (resp. \mathcal{P}'_0) to R defined by N (resp. N'). As $P_N = W(R) \otimes_{W(k)} P_0$ and $P'_{N'} = W(R) \otimes_{W(k)} P'_0$ we get a homomorphism $\varphi = \text{id}_{W(R)} \otimes \varphi_0: P_N \rightarrow P'_{N'}$ of $W(R)$ -modules.

Proposition: *The $W(R)$ -linear map φ is a morphism $\mathcal{P}_N \rightarrow \mathcal{P}'_{N'}$ of $3n$ -displays if and only if*

$$(*) \quad \varphi_0 \circ N = N' \circ \varphi_0.$$

Proof: We have always $\varphi(Q_N) \subset Q'_{N'}$, because $\varphi_0(Q_0) \subset Q'_0$. For $x \in P_0$ we have

$$\begin{aligned} & \varphi(F_N(1 \otimes x)) - F'_{N'}(\varphi(1 \otimes x)) \\ &= \varphi(1 \otimes F_0(x) + [t] \otimes NF_0(x)) - F'_{N'}(1 \otimes \varphi_0(x)) \\ &= 1 \otimes \varphi_0 F_0(x) + [t] \otimes \varphi_0 NF_0(x) - 1 \otimes F'_0 \varphi_0(x) - [t] \otimes N' F'_0 \varphi_0(x) \\ &= [t] \otimes \varphi_0 NF_0(x) - [t] \otimes N' \varphi_0 F_0(x). \end{aligned}$$

As $W(R)$ is flat over $W(k)$ and $[t]$ is not a zero divisor in $W(R)$, the last expression is zero for all $x \in P_0$ if and only if $\varphi_0 N F_0(x) = N' \varphi_0 F_0(x)$ for all $x \in P_0$. As the cokernel of F_0 is torsion, this is equivalent to the relation (*). It remains to show that (*) implies that φ commutes with V_N^{-1} and $V_{N'}^{-1}$ which can be seen by a direct calculation, analogous to the one above.

(3.2.6) Proposition: *Let N be a $W(k)$ -linear endomorphism of P_0 and let \mathcal{P}_N be the associated lifting of \mathcal{P}_0 to R . Let $(\mathcal{P}_0)^\vee = ((P_0)^\vee, (Q_0)^\vee, (F_0)^\vee, (V_0^{-1})^\vee)$ be the dual $3n$ -display. As $(P_0)^\vee$ is the $W(k)$ -dual P_0^* of P_0 , the dual N^* of N is a $W(k)$ -endomorphism of $(P_0)^\vee$. Denote by $\mathcal{P}_{-N^*} = (P_{-N^*}, Q_{-N^*}, F_{-N^*}, V_{-N^*}^{-1})$ the lifting of $(\mathcal{P}_0)^\vee$ associated to $-N^*$. If we have $N^2 = 0$, the canonical isomorphism of $W(R)$ -modules*

$$\begin{aligned} \varphi: P_{-N^*} &= W(R) \otimes_{W(k)} \text{Hom}_{W(k)}(P_0, W(k)) \\ &\longrightarrow (P_N)^\vee = \text{Hom}_{W(R)}(W(R) \otimes_{W(k)} P_0, W(R)) \end{aligned}$$

is an isomorphism

$$\mathcal{P}_{-N^*} \xrightarrow{\sim} (\mathcal{P}_N)^\vee$$

of $3n$ -displays.

Proof: By (3.1.7) (2) we only have to show that φ is a morphism of $3n$ -displays. As duality commutes with base change and as $(P_N, Q_N) = (P_0, Q_0) \otimes_k R$ and $(P_{-N^*}, Q_{-N^*}) = (P^\vee, Q^\vee) \otimes_k R$ in the sense of (3.1.10), we have $\varphi(Q_{-N^*}) = (Q_N)^\vee$. It remains to show that $\varphi \circ F_{-N^*} = (F_N)^\vee \circ \varphi$ and $\varphi \circ V_{-N^*}^{-1} = (V_N^{-1})^\vee \circ \varphi \mid Q_{-N^*}$. For $\lambda \in \text{Hom}_{W(k)}(P_0, W(k))$ we have:

$$\varphi F_{-N^*}(1 \otimes \lambda) = \varphi(1 \otimes (F_0)^\vee(\lambda)) - \varphi([t] \otimes N^*(F_0)^\vee(\lambda)).$$

For $x \in P$ we therefore have

$$\begin{aligned} &(\varphi F_{-N^*}(1 \otimes \lambda))(F_N(1 \otimes x)) \\ &= \varphi(1 \otimes (F_0)^\vee(\lambda))(1 \otimes F_0x + [t] \otimes N F_0x) \\ &\quad - \varphi([t] \otimes (N^*(F_0)^\vee(\lambda)))(1 \otimes F_0x + [t] \otimes N F_0x) \\ &= \varphi(1 \otimes (F_0)^\vee(\lambda))(1 \otimes F_0x) \\ &\quad + \varphi(1 \otimes (F_0)^\vee(\lambda))([t] \otimes N F_0x) \\ &\quad - \varphi([t] \otimes (N^*(F_0)^\vee(\lambda)))(1 \otimes F_0x) \\ &\quad - \varphi([t] \otimes (N^*(F_0)^\vee(\lambda)))([t] \otimes N F_0x) \\ &= (F_0x, (F_0)^\vee(\lambda)) \\ &\quad + [t](N F_0x, (F_0)^\vee(\lambda)) - [t](F_0x, N^*(F_0)^\vee(\lambda)) \\ &\quad - [t]^2(N F_0x, N^*(F_0)^\vee(\lambda)) \\ &= p\lambda(x)^\sigma, \end{aligned}$$

and the last equality holds, because $N^2 = 0$. On the other hand we have:

$$\begin{aligned} &(F_N)^\vee \varphi(1 \otimes \lambda)(F_N(1 \otimes x)) \\ &= (F_N(1 \otimes x), (F_N)^\vee \varphi(1 \otimes \lambda)) \\ &= p(1 \otimes x, \varphi(1 \otimes \lambda))^\sigma \\ &= p\lambda(x)^\sigma. \end{aligned}$$

The verification of the other three identities of (3.1.8.1) is entirely analogous, and we omit this.

(3.2.7) Let $(\mathcal{P}_0, \lambda_0, \iota_0)$ be a principally quasi-polarized bi-infinitesimal display with O -module structure over k , and let $(\mathcal{P}, \lambda, \iota)$ be a principally quasi-polarized $3n$ -display with O -module structure over R which lifts $(\mathcal{P}_0, \lambda_0, \iota_0)$. For all $n \geq 0$ we set

$$(\mathcal{P}_n, \lambda_n, \iota_n) = (\mathcal{P}, \lambda, \iota) \otimes_R R/(t)^{n+1},$$

and let $(X_n, \lambda_n, \iota_n)$ be the associated principally quasi-polarized bi-infinitesimal p -divisible O -module over $R_n = R/(t)^{n+1}$. The $(R_n)_{n \geq 0}$ -adic system $(X_n, \lambda_n, \iota_n)_{n \geq 0}$ now defines a principally quasi-polarized p -divisible O -module (X, λ, ι) over R . This is a lifting of $(X_0, \lambda_0, \iota_0)$. We call (X, λ, ι) *the principally quasi-polarized p -divisible O -module associated to $(\mathcal{P}, \lambda, \iota)$* .

Note that in general X and $\mathcal{BT}(\mathcal{P})$ are not isomorphic; more precisely they are isomorphic if and only if \mathcal{P} is a display.

(3.2.8) Let N be a $W(k)$ -linear endomorphism of P_0 such that $N^2 = 0$, let $\mathcal{P}_N = (P_N, Q_N, F_N, V_N^{-1})$ be the associated deformation of \mathcal{P}_0 over R . Set $\lambda = \lambda_0 \otimes \text{id}_{W(R)}$ and define $\iota: O \rightarrow \text{End}_{W(R)}(P_N)$ by $\iota(a) = \iota_0 \otimes \text{id}_{W(R)}$. Then:

Theorem: *With the notations above we have:*

- (1) *The triple $(\mathcal{P}_N, \lambda, \iota)$ is a principally quasipolarized $3n$ -display with an O -module structure over R if and only if the following two conditions hold:*
 - (i) *N is skew symmetric with respect to λ_0 , i.e. $N^* \circ \lambda_0 = -\lambda_0 \circ N$.*
 - (ii) *N is O -linear.*
- (2) *Let \mathcal{P}_0 be bi-infinitesimal and assume that the conditions (i) and (ii) of (1) hold. Let (X_N, λ, ι) be the principally quasi-polarized p -divisible O -module over R associated to $(\mathcal{P}_N, \lambda, \iota)$. Then we have an isomorphism*

$$\mathcal{BT}'((\mathcal{P}_N, \lambda, \iota) \otimes_R k((t))^{\text{perf}}) \cong (X_N, \lambda, \iota) \otimes_R k((t))^{\text{perf}}.$$

Proof: Part (1) follows from (3.2.5) and (3.2.6). Let us prove part (2): To shorten notations set $K := k((t))^{\text{perf}}$. Now $X_n = X_N \otimes_R R/(t)^{n+1}$ is a formal p -divisible group for all $n \geq 0$. In particular $(X_n)_{n \geq 0}$ is a (R_n) -adic system of formal groups. If we denote by Y the associated formal group over R , we have $Y = \mathcal{BT}(\mathcal{P}_N)$ by definition of the functor \mathcal{BT} ([Z1]). As X_n is an infinitesimal p -divisible group, we have $X_n = \overline{X}_n := \varinjlim \text{Inf}^k(X_n)$ where $\text{Inf}^k(Z)$ denotes the infinitesimal neighborhood of order k of the zero section of some *fppf*-sheaf Z in groups, and this implies $Y = \overline{X}_N = \varinjlim \text{Inf}^k(X_N)$ ([Me] chap. II, §4) (note that this holds only because we have $pR = 0$). As the functor $X \mapsto \overline{X}$ from the category of p -divisible groups into the category of formal groups commutes with base change we see that

$$\mathcal{BT}(\mathcal{P}_N) \otimes_R K \cong \overline{(X_N \otimes_R K)} = (X_N \otimes_R K)_{\text{inf}}.$$

As \mathcal{BT} is compatible with base change we have by (3.1.11):

$$\mathcal{BT}(\mathcal{P}_N) \otimes_R K \cong \mathcal{BT}(\mathcal{P}_N \otimes_R K) = \mathcal{BT}'(\mathcal{P}_N \otimes_R K)_{\text{inf}}.$$

Combining these functorial isomorphisms we get

$$(*) \quad \mathcal{BT}'(((\mathcal{P}_N, \iota) \otimes_R K)_{\text{inf}}) \cong ((X_N, \iota) \otimes_R K)_{\text{inf}}$$

and

$$(**) \quad \mathcal{BT}'(((\mathcal{P}_N, \lambda, \iota) \otimes_R K)_{\text{bi}}) \cong ((X_N, \lambda, \iota) \otimes_R K)_{\text{bi}}.$$

By making the same argument for the dual p -divisible group we get

$$\mathcal{BT}'(((\mathcal{P}_N, \iota)^\vee \otimes_R K)_{\text{inf}}) \cong ((X_N, \iota)^\vee \otimes_R K)_{\text{inf}}.$$

Here $(\mathcal{P}_N, \iota)^\vee$ (resp. $(X_N, \iota)^\vee$) is the $3n$ -display $(\mathcal{P}_N)^\vee$ (resp. the p -divisible group X_N^\vee) equipped with the O -module structure $a \mapsto \iota(a^*)^\vee$ for $a \in O$. But the principal quasi-polarisation λ on \mathcal{P}_N induces an isomorphism of $3n$ -displays from (\mathcal{P}_N, ι) onto $(\mathcal{P}_N, \iota)^\vee$ and this induces an isomorphism

$$((\mathcal{P}_N, \iota) \otimes_R K)_{\text{ét}} \cong ((\mathcal{P}_N, \iota)^\vee \otimes_R K)_{\text{mult}}.$$

Therefore we get

$$\begin{aligned} \mathcal{BT}'(((\mathcal{P}_N, \iota) \otimes_R K)_{\text{ét}}) &\cong \mathcal{BT}'(((\mathcal{P}_N, \iota)^\vee \otimes_R K)_{\text{mult}}) \\ (***) \quad &\cong (((X_N, \iota)^\vee \otimes_R K)_{\text{mult}}) \\ &\cong ((X_N, \iota) \otimes_R K)_{\text{ét}}. \end{aligned}$$

By combining (*) and (***) we obtain

$$\mathcal{BT}'((\mathcal{P}_N, \iota) \otimes_R K) \cong (X_N, \iota) \otimes_R K,$$

And this isomorphism induces

$$\mathcal{BT}'(((\mathcal{P}_N, \lambda, \iota) \otimes_R K)_{\text{ét mult}}) \cong ((X_N, \lambda, \iota) \otimes_R K)_{\text{ét mult}}$$

and we are done by (**).

(3.2.9) Let $(X_0, \lambda_0, \iota_0)$ be a principally quasi-polarized p -divisible O -module over k , such that X_0 is bi-infinitesimal. Let (M, F, V) be its covariant Dieudonné module, and let $\langle \ , \ \rangle$ be the perfect alternating form on M associated to λ_0 .

A deformation endomorphism of $(X_0, \lambda_0, \iota_0)$ is a $W(k)$ -linear endomorphism N of M satisfying

- (a) $N^2 = 0$,
- (b) N is skew-symmetric with regard to $\langle \ , \ \rangle$ (i.e. $\langle Nx, y \rangle = -\langle x, Ny \rangle$ for all $x, y \in M$),
- (c) N is O -linear

With this terminology theorem (3.2.8) and (3.2.4)(1),(2) together with the equivalence of $3n$ -displays and Dieudonné modules over perfect fields imply

Corollary: For every deformation endomorphism N of $(X_0, \lambda_0, \iota_0)$ we get a lifting $(X_N, \lambda_N, \iota_N)$ over R such that if (M_N, F_N, V_N) denotes the covariant Dieudonné module of $X_N \otimes_R k((t))^{\text{perf}}$ we have an isomorphism of $W(k((t))^{\text{perf}})$ -modules $W(k((t))^{\text{perf}}) \otimes_{W(k)} M \cong M_N$ such that via this isomorphism we have

- 1) $F_N(1 \otimes x) = 1 \otimes Fx + [t] \otimes NFx, \quad x \in M$
- 2) $V_N M_N \cap M = VM$,
- 3) $k((t))^{\text{perf}} \otimes_k M/VM = M_N/VM_N$.

4. Deformation in four special cases

4.1. Deformation sequences

(4.1.1) In this section we assume that we are in one of the cases (AL), (AU), (C) or (D) of (2.2.2), i.e. $B = K \times K$, $B = K$, or $B = M_2(K)$ where K is a finite unramified extension of \mathbb{Q}_p ; set $n = [K : \mathbb{Q}_p]$. Denote by O_K the ring of integers of K and by $\kappa(O_K)$ its residue class field. We identify the cyclic group $\text{Gal}(K/\mathbb{Q}_p)$ with $\mathbb{Z}/n\mathbb{Z}$ via $\sigma \mapsto 1$ where σ denotes the Frobenius.

Denote by κ' the composite of $\kappa(O_K)$ and κ . By replacing $\mathbf{A}_{\mathcal{D},C^p} \otimes \kappa$ by the étale cover $\mathbf{A}_{\mathcal{D},C^p} \otimes \kappa'$ and replacing $\mathbf{A}_{\mathcal{D},C^p}^{\mu\text{-ord}}$ by its inverse image in $\mathbf{A}_{\mathcal{D},C^p} \otimes \kappa'$ we can assume that the moduli space is defined over $\kappa(O_K)$.

(4.1.2) Let k be a perfect field extension of $\kappa(O_K)$ so that the embedding $\kappa(O_K) \hookrightarrow k$ induces an embedding $O_K \hookrightarrow W(k)$. Let X be a p -divisible group over k with an O_K -module structure $\iota: O_K \rightarrow \text{End}(X)$. If (M, F, V) is the covariant Dieudonné module of X , the induced action of O_K on M is given by a $\mathbb{Z}/n\mathbb{Z}$ -grading

$$M = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M(j)$$

of $W(k)$ -modules such that F (resp. V) is homogeneous of degree -1 (resp. $+1$) by setting

$$M(j) = \{m \in M \mid \iota(a)m = \sigma^{-j}(a)m\}.$$

A $W(k)$ -linear endomorphism N of M is O_K -linear if and only if it is homogeneous of degree 0.

(4.1.3) Let (X, ι) be a p -divisible O_K -module over a perfect extension k of $\kappa(O_K)$ and let $M = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M(j)$ be the Dieudonné module with O_K -module structure associated to (X, ι) . For $j \in \mathbb{Z}/n\mathbb{Z}$ we define k -vector spaces

$$\begin{aligned} \bar{M}(j) &= M(j)/pM(j), \\ L(j) &= M(j)/VM(j-1). \end{aligned}$$

The σ -linear endomorphism F of M induces for all $j \in \mathbb{Z}/n\mathbb{Z}$ Frobenius-linear maps

$$\begin{aligned} F: \bar{M}(j) &\longrightarrow \bar{M}(j-1), \\ F: L(j) &\longrightarrow L(j-1). \end{aligned}$$

For $c \in L = \bigoplus L(j)$ we define

$$\varpi(c) := \inf\{n \geq 0 \mid F^n(c) = 0\}.$$

This is a finite number for all $c \in L$ if and only if F is nilpotent on L (or equivalently on $\bar{M} = M/pM$). For $x \in \bar{M}$ we define $\varpi(x) := \varpi(c_x)$ where c_x is the image of x in L .

Definition: A family $(x(j))_{j \in \mathbb{Z}/n\mathbb{Z}}$ of elements $x(j) \in \bar{M}(j)$ is called deformation sequence (of (X, ι)) if it satisfies the following conditions:

- (a) $x(j) \notin VM$ for all $j \in \mathbb{Z}/n\mathbb{Z}$.
- (b) $x(j) = Fx(j+1)$ if $\varpi(x(j+1)) > 1$.

(4.1.4) Proposition: *If X is bi-infinitesimal and if $\dim_k L(j) > 0$ for all $j \in \mathbb{Z}/n\mathbb{Z}$ there always exists a deformation sequence.*

Proof: For $j \in \mathbb{Z}/n\mathbb{Z}$ we define

$$\begin{aligned}\Omega(j) &= \{\varpi(x(j)) \mid x(j) \in \bar{M}(j)\} \subset \mathbb{N}_0, \\ \gamma(j) &= \min\{\omega \in \Omega(j) \mid \omega \neq 0\} \in \mathbb{N}, \\ \Gamma &= \max\{\gamma(j) \mid j \in \mathbb{Z}/n\mathbb{Z}\} \in \mathbb{N}.\end{aligned}$$

Let $j_0 \in \mathbb{Z}/n\mathbb{Z}$ be such that $\gamma(j_0) = \Gamma$. Then

$$(*) \quad \Omega(j_0 + 1) \cap \{1, \dots, \Gamma\} = \{1\}.$$

Choose $x(j_0) \in \bar{M}(j_0)$ such that $\varpi(x(j_0)) = \Gamma$ and define successively for $j = j_0 - 1, \dots, j_0 - (n - 1)$ elements $x(j) \in \bar{M}(j)$ as follows; we distinguish two cases:

Case 1 ($\varpi(x(j + 1)) > 1$): $x(j) := Fx(j + 1)$.

Case 2 ($\varpi(x(j + 1)) = 1$): Let $x(j)$ be an arbitrary element of $\bar{M}(j)$ such that $\varpi(x(j)) = \gamma(j)$.

By definition we have $1 \leq \varpi(x(j)) \leq \Gamma$ for all $j \in \mathbb{Z}/n\mathbb{Z}$. Therefore (*) implies $\varpi(x(j_0 + 1)) = 1$, and $(x(j))_{j \in \mathbb{Z}/n\mathbb{Z}}$ is a deformation sequence.

(4.1.5) Let (X, λ, ι) be a principally polarized p -divisible O_B -module over a perfect field k . Let (M, λ, ι) be its Dieudonné module and let $M = \bigoplus M(j)$ be the decomposition of M induced by the O_K -module structure. Then we have:

Proposition: *Suppose that X is bi-infinitesimal. Let $(x(j))_{j \in \mathbb{Z}/n\mathbb{Z}}$ be a deformation sequence, such that the following condition holds (which depends in general on the deformation sequence):*

(E) *There exists a deformation endomorphism $N \in \text{End}_{W(k)}(M)$ of (X, λ, ι) such that, if we denote by \bar{N} the endomorphism of \bar{M} induced by N , we have for all $j \in \mathbb{Z}/n\mathbb{Z}$*

$$\begin{aligned}\bar{N}x(j) &= 0, \\ \bar{N}(Fx(j + 1)) &= \begin{cases} 0, & \text{if } \varpi(x(j + 1)) > 1; \\ x(j), & \text{if } \varpi(x(j + 1)) = 1. \end{cases}\end{aligned}$$

Let $(X_N, \lambda_N, \iota_N)$ be the lifting of (X, λ, ι) to $k[[t]]$ associated to N (3.2.9). Then the generic fibre of X_N is not bi-infinitesimal.

Proof: Let (M_N, F_N, V_N) be the Dieudonné module of the base change of X_N to some perfect closure of $k((t))$. For all $j \in \mathbb{Z}/n\mathbb{Z}$ we have modulo $V_N M_N$:

$$\begin{aligned}F_N(1 \otimes x(j + 1)) &= 1 \otimes Fx(j + 1) + [t] \otimes NFx(j + 1) \\ &\equiv \begin{cases} 1 \otimes x(j), & \text{if } \varpi(x(j + 1)) > 1; \\ [t] \otimes x(j), & \text{if } \varpi(x(j + 1)) = 1. \end{cases}\end{aligned}$$

Therefore we have $F_N^n x(j) \equiv [t]^q x(j) \pmod{V_N M_N}$ for some $q > 0$ ($q = 0$ cannot occur as X is bi-infinitesimal). As $x(j) \notin VM = V_N M_N \cap M$ we have $x(j) \not\equiv 0 \pmod{V_N M_N}$ and in particular F_N is not nilpotent on $M_N/V_N M_N$.

4.2. The linear case

(4.2.1) We are now in case (AL) of (2.2.2) and fix the following notations: K will denote an unramified extension of \mathbb{Q}_p , O_K its ring of integers, $\kappa(O_K)$ its residue class field. We set $n = [K : \mathbb{Q}_p]$, and $\sigma \in \text{Gal}(K/\mathbb{Q}_p)$ will denote the Frobenius on K . Let $B = K \times K$, equipped with the involution $*$: $(a, b) \mapsto (b, a)$, $O_B = O_K \times O_K$.

For a principally quasi-polarized p -divisible O_B -module (X, λ, ι) over some $\kappa(O_K)$ -algebra R the action of O_B on $\text{Lie}(X)$ defines a decomposition of locally free R -modules

$$\text{Lie}(X) = L' \oplus L'' = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} L'(j) \oplus \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} L''(j)$$

where e.g. $L'(j) = \{d \in \text{Lie}(X) \mid \iota(a, b)d = \sigma^{-j}(a)d \text{ for all } (a, b) \in O_K \times O_K\}$. If (X, λ, ι) comes from a point in the moduli space $\mathbf{A}_{\mathcal{D}, C^p}$ the determinant condition (2.1.4.1) is equivalent to requiring an identity of polynomial functions on O_K

$$\det(\iota(a)|L') = \prod_{j \in \mathbb{Z}/n\mathbb{Z}} \sigma^{-j}(a)^r$$

for the r defined in (2.3.1), i.e. $\text{rk}_R(L'(j)) = r$ and therefore $\text{rk}_R(L''(j)) = s$ for all $j \in \mathbb{Z}/n\mathbb{Z}$.

(4.2.2) Let k be a perfect field extension of κ . Let (X, λ, ι) be a principally quasi-polarized p -divisible O_B -module over k and let $(M, \langle \cdot, \cdot \rangle, \iota)$ be its associated covariant Dieudonné module. To give an O_B -action on M is the same as to give a decomposition $M = M' \oplus M''$ of Dieudonné modules with O_K -action. As ι commutes with the involutions, M' and M'' are totally isotropic and in perfect duality with respect to $\langle \cdot, \cdot \rangle$. Therefore we obtain an equivalence of the category of principally quasi-polarized p -divisible O_B -modules (X, λ, ι) over k and the category of Dieudonné modules M' over k whose underlying $W(k)$ -module is $\mathbb{Z}/n\mathbb{Z}$ -graded:

$$M' = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M'(j),$$

such that F (resp. V) is homogeneous of degree -1 (resp. $+1$).

Via this equivalence giving a deformation endomorphism for (X, λ, ι) , (3.2.9) is equivalent to giving a $W(k)$ -linear endomorphism N' of M' with $N'^2 = 0$ which is homogeneous of degree 0.

(4.2.3) Proposition: *Let k be a perfect field extension of κ . Let (X, λ, ι) be a principally quasi-polarized p -divisible O_B -module over k , and let $M' = \bigoplus M'(j)$ be the associated Dieudonné module with O_K -module structure. Let X be bi-infinitesimal and assume that $\dim_k(L'(j)) > 0$ for all $j \in \mathbb{Z}/n\mathbb{Z}$. Then there exists a lifting $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ whose generic fibre is not bi-infinitesimal.*

Proof: Let $(x'(j))_{j \in \mathbb{Z}/n\mathbb{Z}}$ be an arbitrary deformation sequence of elements $x'(j) \in M'(j)/pM'(j)$ (this exists by (4.1.4)). Choose some homogeneous endomorphism

$N' \in \text{End}_{W(k)}(M')$ of degree 0 with $N'^2 = 0$ such that the induced endomorphism \bar{N}' on M'/pM' satisfies

$$\begin{aligned} \bar{N}'x'(j) &= 0, \\ \bar{N}'(Fx'(j+1)) &= \begin{cases} 0, & \text{if } \varpi(x'(j+1)) > 1 \\ x'(j), & \text{if } \varpi(x'(j+1)) = 1 \end{cases} \end{aligned}$$

for all $j \in \mathbb{Z}/n\mathbb{Z}$. By (4.2.2) this defines a deformation endomorphism N for (X, λ, ι) , and by (4.1.5) the generic fibre of the deformation of (X, λ, ι) associated to N is not bi-infinitesimal.

(4.2.4) From (4.2.3) we deduce that we can find for every point $s \in \mathbf{A}_{\mathcal{D}, C^p} \otimes \kappa$ a generalization $y \in \mathbf{A}_{\mathcal{D}, C^p} \otimes \kappa$ of s such that if $\bar{y} = (A, \bar{\lambda}, \iota, \bar{\eta})$ is a geometric point over y the following condition (depending only on y) holds: Let (X, λ, ι) be the principally quasi-polarized p -divisible O_B -module associated to $(A, \bar{\lambda}, \iota)$ (after choosing some prime-to- p isogeny $\lambda \in \bar{\lambda}$), and let

$$\begin{aligned} M &= M' \oplus M'', \\ M' &= M'_{\text{bi}} \oplus M'_{\text{mult}} \oplus M'_{\text{ét}} \\ &= \bigoplus M'_{\text{bi}}(j) \oplus \bigoplus M'_{\text{mult}}(j) \oplus \bigoplus M'_{\text{ét}}(j) \end{aligned}$$

be its covariant Dieudonné module. Then there exists a $j_0 \in \mathbb{Z}/n\mathbb{Z}$ such that $M'_{\text{bi}}(j_0)/VM'_{\text{bi}}(j_0-1) = 0$. By the determinant condition we know that $M'_{\text{bi}}(j)/VM'_{\text{bi}}(j-1)$ has the same rank for every $j \in \mathbb{Z}/n\mathbb{Z}$ (4.2.1), and as M'_{bi} and M''_{bi} are dual to each other we see that the condition above is equivalent to the condition that X is an ordinary p -divisible group, i.e. the only slopes of X are 0 and 1. Further we know that $M'(j)/VM'(j-1)$ has rank r , which means that we have $\text{rk}(M'_{\text{mult}}(j)) = r$ for all $j \in \mathbb{Z}/n\mathbb{Z}$. Therefore the Newton polygon of M' is given by $(n_{ji})_{j \in \mathbb{Z}/n\mathbb{Z}, 1 \leq i \leq d}$ with

$$(n_{j1}, n_{jd}) = (1^r, 0^{d-r}),$$

and we are done by the calculation of $\bar{\mu}(\mathcal{D})$ in (2.3.1).

4.3. The unitary case

(4.3.1) This is the case (AU) of (2.2.2); therefore throughout this section we fix the following notations: K will denote an unramified extension of \mathbb{Q}_p , equipped with a \mathbb{Q}_p -automorphism $*$ of order 2. Let K_0 be the fixed field of $*$ and set $n = [K_0 : \mathbb{Q}_p]$. Note that we changed the meaning of n in comparison to the section on deformation sequences. If $\sigma \in \text{Gal}(K/\mathbb{Q}_p)$ denotes the Frobenius automorphism of K we have $*$ = σ^n . Let O_K be the ring of integers of K and let $\kappa(O_K)$ be its residue class field.

For a principally quasi-polarized p -divisible O_K -module (X, λ, ι) over some $\kappa(O_K)$ -algebra R the action of O_K on $\text{Lie}(X)$ defines a decomposition of locally free R -modules

$$\text{Lie}(X) = \bigoplus_{j \in \mathbb{Z}/2n\mathbb{Z}} L(j)$$

where $L(j) = \{d \in \text{Lie}(X) \mid \iota(a)d = \sigma^{-j}(a)d \text{ for all } a \in O_B\}$. If (X, λ, ι) comes from a point in the moduli space $\mathbf{A}_{\mathcal{D}, C^p}$ the determinant condition (2.1.4.1) is equivalent to requiring an identity of polynomial functions on O_K

$$\det(\iota(a) \mid \text{Lie}(X)) = \prod_{j \in \mathbb{Z}/2n\mathbb{Z}} \sigma^{-j}(a)^{r(j)}$$

where $r(j) = rk_R(L(j)) \geq 0$ are fixed integers satisfying

$$r(j) + r(j + n) = \dim(V)/2n$$

for every $j \in \mathbb{Z}/2n\mathbb{Z}$.

(4.3.2) Let k be a perfect field extension of $\kappa(O_K)$. Let (X, λ, ι) be a principally quasi-polarized p -divisible O_K -module over k and let (M, λ, ι) be its associated covariant Dieudonné module; we denote by $\langle \cdot, \cdot \rangle$ the perfect alternating form associated to λ . To give an O_K -action on M is the same as to give a decomposition

$$(4.3.2.1) \quad M = \bigoplus_{j \in \mathbb{Z}/2n\mathbb{Z}} M(j)$$

of $W(k)$ -modules such that F (resp. V) is homogeneous of degree -1 (resp. $+1$) (4.1.2). That ι commutes with the involutions is equivalent to:

$$(4.3.2.2) \quad \begin{aligned} \langle M(j), M(j') \rangle &= \{0\} \quad \text{for all } j, j' \in \mathbb{Z}/2n\mathbb{Z} \text{ with } j \neq j' + n; \\ \langle \cdot, \cdot \rangle \mid M(j) \times M(j + n) &\text{ is perfect on both sides for all } j \in \mathbb{Z}/2n\mathbb{Z}. \end{aligned}$$

We obtain an equivalence of the category of principally quasi-polarized p -divisible O_B -modules (X, λ, ι) over k and the category of principally quasi-polarized Dieudonné modules $(M, \langle \cdot, \cdot \rangle)$ over k whose underlying $W(k)$ -module is $\mathbb{Z}/2n\mathbb{Z}$ -graded, such that F (resp. V) is homogeneous of degree -1 (resp. $+1$) and such that (4.3.2.2) holds. Via this equivalence giving a deformation endomorphism of (X, λ, ι) is equivalent to giving a $W(k)$ -linear endomorphism N of M with $N^2 = 0$ which is homogeneous of degree 0, such that the restriction to $M(j) \oplus M(j + n)$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$. This is further equivalent to giving a $W(k)$ -linear endomorphism $N(j)$ of $M(j)$ with $N(j)^2 = 0$ for $j = 0, \dots, n - 1$ (then $N \mid M(j + n)$ is given by $-\lambda^{-1} \circ N(j)^* \circ \lambda \mid M(j + n)$).

(4.3.3) We are now going to deform principally quasi-polarized p -divisible O_K -modules (X, λ, ι) in the way described in (2.1.7). This will happen in three steps. First we will make a deformation into the locus where $F \mid M(j)/VM(j - 1)$ or $F \mid M(j + n)/VM(j + n - 1)$ is injective for all $j \in \mathbb{Z}/2n\mathbb{Z}$ (4.3.4) (here $M = \bigoplus M(j)$ is the bi-infinitesimal part of the Dieudonné module of a point). This will simplify the second and the third step. In the second step (4.3.5) we will use the theory of deformation sequences developed in 4.1 to raise the p -rank as far as possible. This brings us into the locus where $M(j_0)/VM(j_0 - 1) = \{0\}$ for some $j_0 \in \mathbb{Z}/2n\mathbb{Z}$. Finally we deform in (4.3.7) into a locus where (X, λ, ι) satisfies a certain additional condition (see (4.3.8) for the precise formulation) which will ensure (4.3.9) that we really deformed into the μ -ordinary locus.

(4.3.4) We use the notations of (4.1.3) and (4.3.1). Further assume that X is bi-infinitesimal and that there exists a $j_0 \in \mathbb{Z}/2n\mathbb{Z}$, such that $F \mid L(j_0 + 1)$ and $F \mid L(j_0 + n + 1)$

are not injective. For every deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ we denote by $\tilde{M} = \bigoplus \tilde{M}(j)$ the covariant Dieudonné module associated to the base change of $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ to some perfect closure $k((t))^{\text{perf}}$ of $k((t))$; set $\tilde{L}(j) = \tilde{M}(j)/V\tilde{M}(j-1)$. We have:

Proposition (first step): *There exists a deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ such that*

$$\dim(\text{Ker}(\tilde{F}|\tilde{L}(j_0+1))) < \dim(\text{Ker}(F|L(j_0+1))),$$

and

$$\dim(\text{Ker}(\tilde{F}|\tilde{L}(j_0+n+1))) < \dim(\text{Ker}(F|L(j_0+n+1))).$$

Proof: For all $j \in \mathbb{Z}/2n\mathbb{Z}$ and for $c \in L = \bar{M}/V\bar{M}$ choose a lift x of c to \bar{M} and define

$$c^F := Fx.$$

As $\text{Ker}(F|\bar{M}) = V\bar{M}$ the element c^F depends only on c and we get an injective Frobenius-linear map $L \rightarrow \bar{M}$, $c \mapsto c^F$. This induces for all $j \in \mathbb{Z}/2n\mathbb{Z}$ an injective map $\text{Ker}(F|L(j+1)) \rightarrow V\bar{M}(j-1)$ whose image is $V\bar{M}(j-1) \cap F\bar{M}(j+1)$. In particular we have for all $j \in \mathbb{Z}/2n\mathbb{Z}$

$$(4.3.4.1) \quad \dim_k(\text{Ker}(F|L(j+1))) = \dim_k(V\bar{M}(j-1) \cap F\bar{M}(j+1)).$$

Therefore the hypothesis implies that we can find non zero elements $y(j_0) \in V\bar{M}(j_0-1) \cap F\bar{M}(j_0+1)$ and $y(j_0+n) \in V\bar{M}(j_0+n-1) \cap F\bar{M}(j_0+n+1)$. As $\langle \cdot, \cdot \rangle$ is a perfect pairing of $\bar{M}(j_0)$ with $\bar{M}(j_0+n)$ there exist $z(j_0) \in \bar{M}(j_0)$ and $z(j_0+n) \in \bar{M}(j_0+n)$ such that

$$(*) \quad \langle y(j_0), z(j_0+n) \rangle = -\langle y(j_0+n), z(j_0) \rangle \neq 0.$$

This in particular implies $z(j_0), z(j_0+n) \notin V\bar{M} + F\bar{M}$. By modifying $z(j_0)$ and $z(j_0+n)$ respectively by an element from $V\bar{M}(j_0-1)$ and $V\bar{M}(j_0+n-1)$, we can further assume that

$$(**) \quad \langle z(j_0), z(j_0+n) \rangle = 0.$$

This does not change (*).

Let \bar{M}' be the k -span of the set $\{y(j_0), y(j_0+n), z(j_0), z(j_0+n)\}$ which is linearly independent. Because of (*) the restriction of $\langle \cdot, \cdot \rangle$ to $\bar{M}' \times \bar{M}'$ is perfect. Define a k -linear endomorphism \bar{N} of \bar{M} by setting

$$\begin{aligned} \bar{N}y(j_0) &= z(j_0), & \bar{N}z(j_0) &= 0, \\ \bar{N}y(j_0+n) &= z(j_0+n), & \bar{N}z(j_0+n) &= 0, \\ \bar{N}|_{\bar{M}'^\perp} &= 0. \end{aligned}$$

Then we obviously have $\bar{N}^2 = 0$, \bar{N} is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$ because of (*) and (**), and \bar{N} is homogeneous of degree 0. It follows immediately from (4.3.2) that \bar{N} can be lifted to a deformation endomorphism $N \in \text{End}_{W(k)}(M)$ for (X, λ, ι) . Let $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ be the deformation to $k[[t]]$ associated to N and let $\tilde{M} = \bigoplus \tilde{M}(j)$ be the covariant Dieudonné module associated to $(\tilde{X}, \tilde{\lambda}, \tilde{\iota}) \otimes_{k[[t]]} k((t))^{\text{perf}}$.

Now the proposition follows from the following claim:

Claim: *We have*

$$(a) \dim(\operatorname{Im}(\tilde{F}|\tilde{L}(j_0 + 1))) > \dim(\operatorname{Im}(F|L(j_0 + 1))).$$

$$(a') \dim(\operatorname{Im}(\tilde{F}|\tilde{L}(j_0 + n + 1))) > \dim(\operatorname{Im}(F|L(j_0 + n + 1))).$$

Proof of the claim: We show (a) (the proof for (a') is entirely analogous). By (3.2.9)(3) we can identify \tilde{L} with $k((t))^{\text{perf}} \otimes_k L$. For any $c(j_0) \in \operatorname{Im}(F|L(j_0 + 1))$ we have $1 \otimes c(j_0) \in \operatorname{Im}(\tilde{F}|\tilde{L}(j_0 + 1))$. Indeed let $m(j_0) \in \bar{M}(j_0)$ be a lift of $c(j_0)$, such that $m(j_0) \in F\bar{M}(j_0 + 1)$. As $y(j_0 + n) \in F\bar{M}$ we have, for every such lift, $\langle m(j_0), y(j_0 + n) \rangle = 0$. As $\langle y(j_0), z(j_0 + n) \rangle \neq 0$ we can modify $m(j_0)$ by adding a multiple of $y(j_0)$ such that $\langle m(j_0), z(j_0 + n) \rangle = 0$. Thus we can assume that $m(j_0) \in \bar{M}'^\perp \cap F\bar{M}(j_0 + 1)$. Let $n(j_0 + 1) \in \bar{M}(j_0 + 1)$ be an element with $Fn(j_0 + 1) = m(j_0)$; then we have (mod $\tilde{V}\tilde{M}$):

$$\begin{aligned} \tilde{F}(1 \otimes n(j_0 + 1)) &= 1 \otimes Fn(j_0 + 1) + t \otimes \bar{N}Fn(j_0 + 1) \\ &= 1 \otimes m(j_0) + t \otimes \bar{N}m(j_0) \\ &= 1 \otimes m(j_0) \end{aligned}$$

because $\bar{N}|\bar{M}'^\perp = 0$.

As $z(j_0) \notin \operatorname{Im}(F|L(j_0 + 1))$ it remains to show that $1 \otimes z(j_0) \in \operatorname{Im}(\tilde{F}|\tilde{L}(j_0 + 1))$. For this let $x(j_0 + 1) \in \bar{M}(j_0 + 1)$ be an element with $Fx(j_0 + 1) = y(j_0)$. Then we have

$$\begin{aligned} \tilde{F}(1 \otimes x(j_0 + 1)) &= 1 \otimes Fx(j_0 + 1) + t \otimes \bar{N}Fx(j_0 + 1) \\ &= 1 \otimes y(j_0) + t \otimes \bar{N}y(j_0) \\ &\equiv t \otimes z(j_0) \pmod{\tilde{V}\tilde{M}}, \end{aligned}$$

and we are done.

(4.3.5) Proposition (second step): *Again let X be bi-infinitesimal. We further assume*

$$(4.3.5.1) \quad \operatorname{Ker}(F|L(j + 1)) = 0 \quad \text{or} \quad \operatorname{Ker}(F|L(j + n + 1)) = 0$$

for all $j \in \mathbb{Z}/2n\mathbb{Z}$. Then if $r(j) > 0$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$, there exists a deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ such that $\tilde{X} \otimes_{k[[t]]} k((t))$ is not bi-infinitesimal.

Proof: We will make use of (4.1.5); therefore we have to find a deformation sequence such that condition (E) holds. By (4.1.4) there exists a deformation sequence $(x(j))_{j \in \mathbb{Z}/2n\mathbb{Z}}$ of elements $x(j) \in \bar{M}(j)$. For every $j = 0, \dots, n - 1$ we will now modify $x(j)$; the map

$$\begin{aligned} \bar{M}(j - 1) &\longrightarrow k, \\ m(j - 1) &\mapsto \langle x(j + n), Vm(j - 1) \rangle = \langle Fx(j + n), m(j - 1) \rangle^{1/p} \end{aligned}$$

is surjective because $Fx(j + n) \neq 0$. In particular there exists an element $m(j - 1) \in \bar{M}(j - 1)$ such that

$$\langle x(j + n), x(j) + Vm(j - 1) \rangle = 0.$$

Replacing $x(j)$ by $x(j) + Vm(j - 1)$, $(x(j))_{j \in \mathbb{Z}/2n\mathbb{Z}}$ is still a deformation sequence and we have

$$(*) \quad \langle x(j), x(j + n) \rangle = 0$$

for all $j \in \mathbb{Z}/2n\mathbb{Z}$.

We claim that for every deformation sequence $(x(j))$ satisfying $(*)$ the condition (E) holds (under the assumption of (4.3.5.1)): For all $j \in \mathbb{Z}/2n\mathbb{Z}$ let $\bar{M}'(j)$ be the k -span of $\{x(j), Fx(j + 1)\}$. Note that $x(j)$ and $Fx(j + 1)$ are linearly independent if and only if $Fx(j + 1) \in VM(j - 1)$, otherwise we have $x(j) = Fx(j + 1)$. Define an endomorphism $\bar{N}'(j)$ of $\bar{M}'(j)$:

$$\begin{aligned} \bar{N}'(j)x(j) &= 0, \\ \bar{N}'(j)(Fx(j + 1)) &= \begin{cases} 0, & \text{if } Fx(j + 1) = x(j) \\ x(j), & \text{if } Fx(j + 1) \in VM. \end{cases} \end{aligned}$$

Set $\bar{N}' = \bigoplus \bar{N}'(j)$; this is an endomorphism of $\bar{M}' = \bigoplus \bar{M}'(j)$. Obviously $\bar{N}'^2 = 0$, and \bar{N}' is homogeneous of degree 0. Further \bar{N}' is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$: Indeed it suffices to show that $\bar{N}'(j) \oplus \bar{N}'(j + n)$ is skew-symmetric for all $j \in \mathbb{Z}/2n\mathbb{Z}$. By the assumption (4.3.5.1) we know that $\bar{N}'(j) = 0$ or $\bar{N}'(j + n) = 0$. If both are zero we are done; therefore we can assume that one of them is not zero, say $\bar{N}'(j) \neq 0$; this implies

$$(**) \quad \begin{aligned} \bar{M}'(j) &= k \cdot x(j) \oplus k \cdot Fx(j + 1), \\ \bar{M}'(j + n) &= k \cdot x(j + n). \end{aligned}$$

It is now easy to check explicitly that \bar{N}' is skew-symmetric.

Now if we can extend \bar{N}' to an endomorphism \bar{N} of \bar{M} such that $\bar{N}^2 = 0$, \bar{N} homogeneous of degree 0, and \bar{N} skew-symmetric with respect to $\langle \cdot, \cdot \rangle$, we can lift it to a deformation endomorphism N for (X, λ, ι) because this is equivalent to lifting endomorphisms of square zero by (4.3.2) and we are done. It suffices to extend $\bar{N}'_j := \bar{N}'(j) \oplus \bar{N}'(j + n)$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$ to an endomorphism \bar{N}_j of $\bar{M}_j := \bar{M}(j) \oplus \bar{M}(j + n)$. If $\bar{N}'_j = 0$ we simply set $\bar{N}_j = 0$ and we are done. Therefore we can assume that $\bar{N}'(j) \neq 0$, i.e. $(**)$ holds. If $\langle x(j + n), Fx(j + 1) \rangle \neq 0$, we choose some $m(j + n) \in \bar{M}(j + n)$ such that $\langle m(j + n), x(j) \rangle = -\langle x(j + n), Fx(j + 1) \rangle$ and $\langle m(j + n), Fx(j + 1) \rangle = 0$; define \bar{N}_j by setting $\bar{N}_j m(j + n) = x(j + n)$ and equal to zero on the orthogonal complement of the subspace generated by $\{x(j), Fx(j + 1), x(j + n), m(j + n)\}$. If $\langle x(j + n), Fx(j + 1) \rangle = 0$, $\bar{M}'_j := \bar{M}'(j) \oplus \bar{M}'(j + n)$ is totally isotropic and we can find some totally isotropic complement \bar{U}'_j in \bar{M}_j which is homogeneous with respect to the decomposition $\bar{M}_j = \bar{M}(j) \oplus \bar{M}(j + n)$ and such that $\langle \cdot, \cdot \rangle$ induces a perfect pairing of \bar{M}'_j and \bar{U}'_j . There is a unique way to extend \bar{N}'_j to $\bar{M}'_j \oplus \bar{U}'_j$ such that the extension is still skew-symmetric, and this extension is automatically homogeneous and of square zero. This can be further extended by zero on the orthogonal complement of $\bar{M}'_j \oplus \bar{U}'_j$ to \bar{M}_j .

(4.3.6) By (4.3.4) and (4.3.5) we can find for every point $s \in \mathbf{A}_{\mathcal{D}, C^p} \otimes \kappa(O_K)$ a generalization y such that the following condition holds: Let $\bar{y} = (A, \bar{\lambda}, \iota, \bar{\eta})$ be a geometric point over y and let $(M = \bigoplus M(j), \langle \cdot, \cdot \rangle)$ be the bi-infinitesimal part of the principally quasi-polarized Dieudonné module with O_K -module structure over $k = \kappa(\bar{y})$ associated to \bar{y} (after choosing some prime-to- p isogeny $\lambda \in \bar{\lambda}$). Set $L(j) = M(j)/VM(j - 1)$ for $j \in \mathbb{Z}/2n\mathbb{Z}$. Then we have:

(C₀) There exists a $j_0 \in \mathbb{Z}/2n\mathbb{Z}$ such that $L(j_0) = \{0\}$.

(C₁) One of the maps $F: L(j+1) \rightarrow L(j)$ or $F: L(j+n+1) \rightarrow L(j+n)$ is injective.

(4.3.6.1) Lemma: Set $r(j) = \dim(L(j))$, $\bar{M}(j) = M(j)/pM(j)$, $\bar{T}(j) = V\bar{M}(j-1)$, and $d = \dim(\bar{M}(j))$. By condition (C₁) the following conditions are equivalent for some $j \in \mathbb{Z}/2n\mathbb{Z}$:

- (1) We have $r(j+1) \leq r(j)$.
- (2) The map $F: L(j+1) \rightarrow L(j)$ is injective.
- (3) We have $\bar{T}(j) \cap F\bar{M}(j+1) = \{0\}$.
- (4) The map $V: \bar{T}(j-1) \rightarrow \bar{T}(j)$ is injective.
- (5) We have $r(j+n+1) \geq r(j+n)$.
- (6) The map $F: L(j+n+1) \rightarrow L(j+n)$ is surjective.
- (7) We have $FM(j+n+1) + VM(j+n-1) = M(j+n)$.
- (8) The map $V: \bar{T}(j+n-1) \rightarrow \bar{T}(j+n)$ is surjective.

If these equivalent conditions hold we have:

$$\begin{aligned} r(j+n) - r(j+n+1) &= d - \dim(F\bar{M}(j+n+1)) - \dim(\bar{T}(j+n)), \\ r(j+1) - r(j) &= \dim(F\bar{M}(j+1) \cap \bar{T}(j)). \end{aligned}$$

Proof: The equivalence of (2) and (3) resp. of (6) and (7) is clear.

The map $F: \bar{M}(j+1) \rightarrow \bar{M}(j)$ induces a Frobenius-linear isomorphism from $\text{Ker}(F|L(j+1))$ onto $\text{Ker}(V|\bar{T}(j))$. Further the perfect duality of $\bar{M}(j)$ and $\bar{M}(j+n)$ via $\langle \cdot, \cdot \rangle$ induces a perfect duality of $\bar{T}(j)$ with $L(j+n)$ and it is easily seen that we have, via this duality, $\text{Ker}(V|\bar{T}(j)) = (FL(j+n+1))^\perp$. Therefore we have

$$\begin{aligned} \dim(\text{Ker}(F|L(j+1))) + \dim(FL(j+n+1)) &= \dim(\text{Ker}(V|\bar{T}(j))) + \dim(FL(j+n+1)) \\ &= \dim(L(j+n)) \\ &= \dim(\bar{T}(j)). \end{aligned}$$

This implies the equivalence of (2), (4), (6), and (8).

Finally we know by (C₁) that $F|L(j+1)$ or $F|L(j+n+1)$ is injective. Therefore if $r(j+1) \leq r(j)$ and if $F|L(j+1)$ were not injective we would have by the equivalence of (6) and (2) applied for $j+n$ instead of j that $F|L(j+1)$ would be surjective and we would have therefore $r(j+1) = r(j)$, and $F|L(j+1)$ would be an isomorphism, which is absurd proving the equivalence of (1) and (2). By the same argument we see the equivalence of (5) and (6).

(4.3.7) Now we come to the third step: Let (X, λ, ι) be a bi-infinitesimal principally quasi-polarized p -divisible O_K -module over an algebraically closed extension k of $\kappa(O_K)$ and let $(M = \bigoplus M(j), \langle \cdot, \cdot \rangle)$ be its Dieudonné module. We assume that the conditions (C₀) and (C₁) are satisfied. Further fix a decomposition $M(j) = \Lambda(j) \oplus T(j)$ of the $W(k)$ -module $M(j)$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$ such that $VM(j-1) = p\Lambda(j) \oplus T(j)$, which is compatible with F , i.e. $F\Lambda(j+1)$ and $FT(j+1)$ are homogeneous with respect to this decomposition. Then $\text{rk}_{W(k)}(\Lambda(j)) = r(j)$. This will be called a *homogeneous normal decomposition*. Such a decomposition can be constructed by (C₀) and (C₁). Note that

by reducing modulo p it is immediat that $\langle \cdot, \cdot \rangle$ induces a perfect pairing of $\Lambda(j)$ and $T(j+n)$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$.

Let s be an integer with $1 \leq s \leq n$. Let us assume that there exists a $j_0 \in \mathbb{Z}/2n\mathbb{Z}$ such that

$$(4.3.7.1) \quad \text{rk}_{W(k)}(F^s \Lambda(j_0 + s) \cap T(j_0)) > \max\{r(j_0 + s) - r(j_0), 0\}.$$

Note that the left hand side of (4.3.7.1) is always greater or equal than the right hand side.

For every deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ we denote by $\tilde{M} = \bigoplus \tilde{M}(j)$ the covariant Dieudonné module associated to the base change of $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ to some perfect closure $k((t))^{\text{perf}}$ of $k((t))$; set

$$\begin{aligned} \tilde{L}(j) &= \tilde{M}(j)/\tilde{V}\tilde{M}(j-1), \\ \tilde{\Lambda}(j) &= \Lambda \otimes_{W(k)} W(k((t))^{\text{perf}}), \\ \tilde{T}(j) &= T \otimes_{W(k)} W(k((t))^{\text{perf}}). \end{aligned}$$

We have:

Proposition (third step): *There exists a deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ such that*

$$\text{rk}_{W(k((t))^{\text{perf}})}(\tilde{F}^s \tilde{\Lambda}(j_0 + s) \cap \tilde{T}(j_0)) < \text{rk}_{W(k)}(F^s \Lambda(j_0 + s) \cap T(j_0)).$$

Proof: The proof is analogous to the one of proposition (4.3.4) and we omit it (take an element $0 \neq m(j_0 + 1) \in F^{s-1}\Lambda(j_0 + s)$ such that $Fm(j_0 + 1) \in T(j_0)$ and construct a deformation endomorphism which is zero on $M(j)$ for $j \notin \{j_0, j_0 + n\}$ such that after deformation $\tilde{F}m(j_0 + 1) \notin \tilde{T}(j_0)$).

(4.3.8) By (4.3.7) we can now assume for the proof of the density theorem that we have deformed every point into the locus where in addition to (C_0) and (C_1) the following condition holds:

(C₂) For all $j \in \mathbb{Z}/2n\mathbb{Z}$ and for all $s \in \{1, \dots, n\}$ we have

$$\text{rk}_{W(k)}(F^s \Lambda(j + s) \cap T(j)) = \max\{r(j + s) - r(j), 0\}.$$

(4.3.9) Let $(M = \bigoplus M(j), \langle \cdot, \cdot \rangle)$ be a principally quasi-polarized Dieudonné module with O_K -module structure over an algebraically closed extension k of $\kappa(O_K)$ with $\dim_k(M(j)/VM(j-1)) = r(j)$ for some integers $r(j) \geq 0$. Note that this implies $r(j) + r(j+n) = \text{rk}_{W(k)}(M)/2n =: d$ for all j . Assume that the conditions (C_0) and (C_1) are satisfied, choose a homogeneous normal decomposition $M(j) = \Lambda(j) \oplus T(j)$, and assume that condition (C_2) holds.

For one (or equivalently for all) $j_0 \in \mathbb{Z}/2n\mathbb{Z}$ let

$$(\lambda_1, \dots, \lambda_d, 2n - \lambda_1, \dots, 2n - \lambda_d)$$

be the slope sequence of the σ^{2n} -crystal $(M(j_0) \oplus M(j_0 + n), F^{2n})$ (independent of the choice of j_0). We claim that we have, up to order,

$$\lambda_i = \sum_{h=0}^{i-1} k(h), \quad i = 1, \dots, d$$

where $k(h) = \#\{j \in \mathbb{Z}/2n\mathbb{Z} \mid r(j) = h\}$. By (2.3.2) this proves the density theorem in the unitary case.

First consider the map $k: \{0, \dots, d\} \rightarrow \{0, \dots, 2n\}$. By (C_0) we know that

$$(*) \quad k(0) > 0.$$

If we set $Q = \{h \mid k(h) > 0\}$ this implies $q := \#Q \geq 1$. Further we have

$$(**) \quad k(h) = k(d - h)$$

because $r(j) + r(j + n) = d$. Finally by definition we have

$$(***) \quad \sum_{h=0}^d k(h) = 2n.$$

Note that these conditions imply that $k(h) \leq n$ for all h if $d \neq 0$. The claim will now be proved by induction on $q \geq 1$:

If $q = 1$, $(*)$ and $(**)$ imply $d = 0$, and this is trivial. If $q = 2$, we have $k(0) = k(d) = n$, and this implies $F^{2n}M(j) = p^n M(j)$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$. Therefore $\lambda_i = n$ for all $i = 1, \dots, d$, and this proves the case $q = 2$.

Now assume $q > 2$. Then

$$a := \min\{\alpha \in Q \mid \alpha > 0\} < d.$$

Let $j_0 \in \mathbb{Z}/2n\mathbb{Z}$ be an index such that $r(j_0) = a$. Note that by definition of a we have $r(j_0 + n) \geq a$. Now define direct summands $\Lambda'(j)$ of $M(j)$ which are direct summands of $\Lambda(j)$ if $r(j) > 0$. Do this successively for $j = j_0, j_0 - 1, \dots, j_0 + 1$ as follows: For $j = j_0$ set $\Lambda'(j) = \Lambda(j_0)$. Let $\Lambda'(j + 1)$ already be constructed. If $r(j + 1) > 0$ (and therefore $r(j + 1) \geq a$) the condition (C_2) implies that $F\Lambda'(j + 1)$ is a direct summand of $M(j)$ which lies in $\Lambda(j)$ if $\Lambda(j) \neq \{0\}$; in this case set $\Lambda'(j) := F\Lambda'(j + 1)$. If $r(j + 1) = 0$ we have $FM(j + 1) = pM(j)$ and in this case we set $\Lambda'(j) = p^{-1}F\Lambda'(j + 1)$; this is a direct summand of $M(j)$ and lies in $\Lambda(j)$ if $r(j) > 0$. Again, by condition (C_2) , we see that $F\Lambda'(j_0 + 1) \cap T(j_0) = \{0\}$, and by the homogeneity with respect to F , this implies $F\Lambda'(j_0 + 1) \subset \Lambda'(j_0)$ with equality if and only if $r(j_0 + 1) > 0$ and with $F\Lambda'(j_0 + 1) = p\Lambda'(j_0)$ if $r(j_0 + 1) = 0$. It follows that $F^{2n}\Lambda'(j_0) = p^{k(0)}\Lambda'(j_0)$; in particular $(\Lambda'(j_0), F^{2n})$ is an isoclinic subcrystal of $(M(j_0), F^{2n})$ of height a with slope $k(0)$. Now consider $T(j_0 + n)$ which has also rank a . By making an analogous construction for $T(j_0 + n)$ instead of $\Lambda(j_0)$ (or by considering the dual crystal, identified with the original one by $\langle \cdot, \cdot \rangle$) we get a family $T'(j)$ of direct summands of $M(j)$ with $T'(j) \subset T(j)$ if $T(j) \neq \{0\}$ and with $F^{2n}T'(j_0 + n) = p^{2n-k(0)}T'(j_0)$. It follows that

$$\Lambda'(j_0) \oplus T'(j_0) \oplus \Lambda'(j_0 + n) \oplus T'(j_0 + n)$$

is a subcrystal of $M(j_0) \oplus M(j_0 + n)$ such that the pairing induced by $\langle \cdot, \cdot \rangle$ is perfect and which has slopes $(k(0)^a, (2n - k(0))^a, k(d)^a, (2n - k(d))^a)$.

If we set

$$M' = \bigoplus_{j \in \mathbb{Z}/2n\mathbb{Z}} \Lambda'(j) \oplus \bigoplus_{j \in \mathbb{Z}/2n\mathbb{Z}} T'(j),$$

$M'' = (M')^\perp$ inherits the structure of a principally quasi-polarized Dieudonné module with O_K -module structure from M . We have $d'' := \text{rk}_{W(k)}(M''(j)) = \text{rk}_{W(k)}(M(j)) - 2a$ for all $j \in \mathbb{Z}/2n\mathbb{Z}$. Further if we set $\Lambda''(j) = \Lambda(j) \cap M''(j)$ and $T''(j) = T(j) \cap M''(j)$ for $j \in \mathbb{Z}/2n\mathbb{Z}$, this defines a homogeneous normal decomposition of M'' . Set $r''(j) = \text{rk}_{W(k)}(\Lambda''(j)) = \dim_k(M''(j)/VM''(j-1))$. Then we have

$$\text{rk}_{W(k)}(\Lambda''(j)) = \begin{cases} \text{rk}_{W(k)}(\Lambda(j)) - a, & \text{if } 0 < r(j) < d \\ 0, & \text{if } r(j) = 0 \\ \text{rk}_{W(k)}(\Lambda(j)) - 2a, & \text{if } r(j) = d \end{cases}$$

and

$$\text{rk}_{W(k)}(T''(j)) = \begin{cases} \text{rk}_{W(k)}(\Lambda(j)) - a, & \text{if } 0 < r(j) < d \\ 0, & \text{if } r(j) = d \\ \text{rk}_{W(k)}(T(j)) - 2a, & \text{if } r(j) = 0 \end{cases} = d'' - r''(j)$$

Therefore it is obvious that M'' satisfies condition (C_0) , and making a case by case verification (distinguishing the cases where $r(j) = 0$, $r(j) = d$ or $0 < r(j) < d$) it is easy to see that (C_2) holds as well for M'' . Further if we define Q'' for M'' like we defined Q for M , we have

$$Q'' = \{0\} \cup \{h - a \mid h \in Q \cap \{a, \dots, d - a\}\} \cup \{d - 2a\}.$$

Therefore we have $\#Q'' = q - 2$ and by applying the induction hypothesis for M'' the claim follows.

4.4. The symplectic case

(4.4.1) We are now in case (C) of (2.2.2) and fix the following notations: K will denote an unramified extension of \mathbb{Q}_p . Let O_K be the ring of integers of K and let $\kappa(O_K)$ be its residue class field. Let n be the degree of the field extension of K over \mathbb{Q}_p , and let us denote by $\sigma \in \text{Gal}(K/\mathbb{Q}_p)$ the Frobenius on K . Finally we take $B = K$, equipped with the trivial involution $*$ = id_K .

For a principally quasi-polarized p -divisible O_K -module (X, λ, ι) over some $\kappa(O_K)$ -algebra R the action of O_K on $\text{Lie}(X)$ defines a decomposition of locally free R -modules

$$\text{Lie}(X) = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} L(j)$$

where $L(j) = \{d \in \text{Lie}(X) \mid \iota(a)d = \sigma^{-j}(a)d \text{ for all } a \in O_K\}$. If (X, λ, ι) comes from a point in the moduli space $\mathbf{A}_{\mathcal{D}, C^p}$ the determinant condition (2.1.4.1) is equivalent to requiring an identity of polynomial functions on O_K

$$\det(\iota(a) \mid \text{Lie}(X)) = \prod_{j \in \mathbb{Z}/n\mathbb{Z}} \sigma^{-j}(a)^r$$

with $r = \dim(V)/2n$, i.e. $\text{rk}_R(L(j)) = r$ for all $j \in \mathbb{Z}/n\mathbb{Z}$.

(4.4.2) Let k be a perfect field extension of $\kappa(O_K)$. Let (X, λ, ι) be a principally quasi-polarized p -divisible O_K -module over k and let (M, λ, ι) be its associated covariant Dieudonné module; we denote by $\langle \cdot, \cdot \rangle$ the perfect alternating form associated to λ . The

O_K -action is defined by a $\mathbb{Z}/n\mathbb{Z}$ -grading of the underlying $W(k)$ -module of M with alternating form, such that F (resp. V) is homogeneous of degree -1 (resp. $+1$). Via this equivalence giving a deformation endomorphism of (X, λ, ι) is equivalent to giving a $W(k)$ -linear endomorphism N of M with $N^2 = 0$ which is homogeneous of degree 0, such that the restriction to $M(j)$ is skew-symmetric with respect to $\langle \cdot, \cdot \rangle$ for all $j \in \mathbb{Z}/n\mathbb{Z}$.

(4.4.3) Proposition: *Let X be bi-infinitesimal and assume $\dim_k L(j) > 0$ for all $j \in \mathbb{Z}/n\mathbb{Z}$. Then there exists a deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ such that $\tilde{X} \otimes_{k[[t]]} k((t))$ is not bi-infinitesimal.*

Proof: We use (4.1.5). More precisely we will show that for every deformation sequence the condition (E) holds. As deformation sequences always exist (4.1.4) this proves the proposition. Let $(x(j))_{j \in \mathbb{Z}/n\mathbb{Z}}$ be a deformation sequence of elements $x(j) \in \bar{M}(j) = M(j)/pM(j)$. It suffices to construct a skew-symmetric endomorphism $\bar{N}(j) \in \text{End}_{W(k)}(M(j))$ of square zero for all $j \in \mathbb{Z}/n\mathbb{Z}$ such that the induced endomorphism $\bar{N}(j)$ of $\bar{M}(j)$ satisfies

$$(*) \quad \begin{aligned} \bar{N}(j)x(j) &= 0, \\ \bar{N}(j)(Fx(j+1)) &= \begin{cases} 0, & \text{if } Fx(j+1) = x(j), \\ x(j), & \text{if } Fx(j+1) \in VM. \end{cases} \end{aligned}$$

First define $\bar{N}(j)$ on the k -span $\bar{M}'(j)$ of $\{x(j), Fx(j+1)\}$ by (*). Then $\bar{N}(j)$ is automatically skew-symmetric and of square zero. Extend $\bar{N}(j)$ to a skew-symmetric endomorphism of $\bar{M}(j)$ of square zero:

If $Fx(j+1) = x(j)$, simply set $\bar{N}(j) = 0 \in \text{End}_k(\bar{M}(j))$.

If $\langle x(j), Fx(j+1) \rangle \neq 0$, extend it by zero on $\bar{M}'(j)^\perp$.

Finally if $x(j) \neq Fx(j+1)$ and $\langle x(j), Fx(j+1) \rangle = 0$ choose a totally isotropic sub-vector space $\bar{U}'(j)$ of $\bar{M}(j)$ such that $\langle \cdot, \cdot \rangle$ induces a perfect duality of $\bar{U}'(j)$ and $\bar{M}'(j)$. Then there is a unique skew-symmetric extension of $\bar{N}(j)$ to $\bar{M}'(j) \oplus \bar{U}'(j)$ and this extension is automatically of square zero. Now extend $\bar{N}(j)$ by zero on $(\bar{M}'(j) \oplus \bar{U}'(j))^\perp$.

But for every skew-symmetric endomorphism f of square zero of some finite-dimensional symplectic k -vector space $(U, \langle \cdot, \cdot \rangle)$ there exists a base of U , such that $\langle \cdot, \cdot \rangle$ is given by $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and f is given by $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ with respect to this base. Applying this to $f = \bar{N}(j)$ for all $j \in \mathbb{Z}/n\mathbb{Z}$ we see that there exists a lift $N(j)$ of $\bar{N}(j)$ which is skew-symmetric and of square zero.

(4.4.4) Entirely analogously to (4.2.4) we deduce from (4.4.3) that we can find for every point $s \in \mathbf{A}_{\mathcal{D}, C^p} \otimes \kappa$ a generization $y \in \mathbf{A}_{\mathcal{D}, C^p} \otimes \kappa$ of s such that if $\bar{y} = (A, \bar{\lambda}, \iota, \bar{\eta})$ is a geometric point over y the abelian variety A is ordinary. Therefore in the case (C) we are done if we can show that the μ -ordinary locus is equal to the ordinary locus. But this follows from the calculation of $\bar{\mu}(\mathcal{D})$ in (2.3.3).

4.5. The orthogonal case

(4.5.1) We are now in case (D) of (2.2.2) and fix the following notations: K will denote an unramified extension of \mathbb{Q}_p . Let O_K be the ring of integers of K and let $\kappa(O_K)$ be its residue class field. Assume $\text{char}(\kappa(O_K)) \neq 2$. Let n be the degree of the field extension of K over \mathbb{Q}_p , and denote by $\sigma \in \text{Gal}(K/\mathbb{Q}_p)$ the Frobenius on K . Finally we take

$B = M_2(K)$, equipped with the involution $A^* = J^t A J^{-1}$ for $A \in B$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $O_B = M_2(O_K)$.

For a principally quasi-polarized p -divisible O_B -module (X, λ, ι) over some $\kappa(O_K)$ -algebra R the action of O_B on $\text{Lie}(X)$ defines a decomposition $\text{Lie}(X) = L'^2$ where L' is an O_K -module. Therefore we get a decomposition of locally free R -modules

$$\text{Lie}(X) = (L')^2 = \left(\bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} L(j) \right)^2$$

where $L(j) = \{d \in L' \mid \iota(a)d = \sigma^{-j}(a)d \text{ for all } a \in O_K\}$. If (X, λ, ι) comes from a point in the moduli space $\mathbf{A}_{\mathcal{D}, C^p}$ the determinant condition (2.1.4.1) is equivalent to requiring an identity of polynomial functions on O_K

$$\det(\iota(a)|L') = \prod_{j \in \mathbb{Z}/n\mathbb{Z}} \sigma^{-j}(a)^r$$

with $r = \dim(V)/4n$, i.e. $rk_R(L(j)) = r$ for all $j \in \mathbb{Z}/n\mathbb{Z}$.

(4.5.2) Let k be a perfect field extension of $\kappa(O_K)$. Let (X, λ, ι) be a principally quasi-polarized p -divisible O_B -module over k and let (M, λ, ι) be its associated covariant Dieudonné module; we denote by $\langle \cdot, \cdot \rangle$ the perfect alternating form associated to λ . To give an O_B -action on M is the same as to give a decomposition $M = M'^2$ of O_K -modules. The O_B -action ι commutes with the involutions if and only if $M' \oplus \{0\}$ and $\{0\} \oplus M'$ are totally isotropic with respect to $\langle \cdot, \cdot \rangle$ and if

$$\langle (0, m'_1), (m'_2, 0) \rangle = -\langle (m'_1, 0), (0, m'_2) \rangle$$

for all $m'_1, m'_2 \in M'$. As $\langle \cdot, \cdot \rangle$ is alternating and perfect, the bilinear form (\cdot, \cdot) on M' given by

$$(4.5.2.1) \quad (m'_1, m'_2) = \langle (0, m'_1), (m'_2, 0) \rangle$$

is symmetric and perfect. As λ commutes with F and V we have

$$(4.5.2.2) \quad (Fm'_1, m'_2) = (m'_1, Vm'_2)^\sigma.$$

Finally the O_K -module structure of M' gives a decomposition

$$M' = \bigoplus_{j \in \mathbb{Z}/n\mathbb{Z}} M'(j)$$

such that F (resp. V) is homogeneous of degree -1 (resp. $+1$), and this is an orthogonal decomposition with respect to (\cdot, \cdot) . Altogether we obtain an equivalence of the category of principally quasi-polarized p -divisible O_B -modules (X, λ, ι) over k of height $2h$ and the category of Dieudonné modules M' over k of height h , equipped with a perfect symmetric bilinear form (\cdot, \cdot) satisfying (4.5.2.2), whose underlying $W(k)$ -module is $\mathbb{Z}/n\mathbb{Z}$ -graded, such that F (resp. V) is homogeneous of degree -1 (resp. $+1$).

Via this equivalence giving a deformation endomorphism of (X, λ, ι) is equivalent to giving for all $j \in \mathbb{Z}/n\mathbb{Z}$ a $W(k)$ -linear endomorphism $N'(j)$ of $M'(j)$ with $N'(j)^2 = 0$ being skew-symmetric with respect to $(\ , \)$.

(4.5.3) Let k be a perfect field extension of $\kappa(O_K)$. Let (X, λ, ι) be a principally quasi-polarized p -divisible O_K -module over k and let $M' = \bigoplus M'(j)$ be its associated Dieudonné module, equipped with the perfect symmetric form $(\ , \)$ induced by λ (4.5.2). For $j \in \mathbb{Z}/n\mathbb{Z}$ we define k -vector spaces

$$\begin{aligned}\bar{M}(j) &= M'(j)/pM'(j), \\ L(j) &= M'(j)/VM'(j-1) = \bar{M}(j)/V\bar{M}(j-1);\end{aligned}$$

we further set

$$r(j) = \dim_k L(j).$$

We assume

- (1) $r(j) = \dim_k(\bar{M}(j))/2$ for all $j \in \mathbb{Z}/n\mathbb{Z}$, in particular the $r(j)$ are all equal to some $r \in \mathbb{N}_0$.
- (2) $\dim_k F\bar{M} = \dim_k V\bar{M}$ (this implies together with (1) that

$$\dim_k V\bar{M}(j) = \dim_k F\bar{M}(j) = r$$

for all $j \in \mathbb{Z}/n\mathbb{Z}$).

Note that for the proof of the density theorem we only deal with p -divisible groups where these conditions hold (condition (2) always holds if X is the p -divisible group of some abelian variety, and condition (1) is implied by the determinant condition (4.5.1)).

(4.5.4) We are now going to deform principally quasi-polarized p -divisible O_B -modules (X, λ, ι) in the way described in (2.1.7). This will be done in two steps. First we will make a deformation into the locus where $F|M(j)/VM(j-1) \neq 0$ for all $j \in \mathbb{Z}/n\mathbb{Z}$ (4.5.5) (here $M = \bigoplus M(j)$ is the bi-infinitesimal part of the Dieudonné module of a point). This will simplify the second step (4.5.6) where we will use the theory of deformation sequences developed in 4.1 to raise the p -rank as far as possible. This brings us into the locus where $\text{rk}(M(j)/VM(j-1)) \leq 1$ for all $j \in \mathbb{Z}/n\mathbb{Z}$. In (4.5.7)–(4.5.10) we will see that this condition implies already that we deformed into the μ -ordinary locus.

(4.5.5) We now further assume that X is bi-infinitesimal and that there exists a $j_0 \in \mathbb{Z}/n\mathbb{Z}$, such that

$$F|L(j_0 + 1) = 0;$$

this is equivalent to the condition

$$F\bar{M}(j_0 + 1) = V\bar{M}(j_0 - 1)$$

by our general assumptions ((4.5.3)(1),(2)). For every deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ we denote by $\tilde{M} = \bigoplus \tilde{M}(j)$ the covariant Dieudonné module associated to the base change of $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ to some perfect closure $k((t))^{\text{perf}}$ of $k((t))$; set $\tilde{L}(j) = \tilde{M}(j)/V\tilde{M}(j-1)$. Now we have:

Proposition (first step): Assume that $r \geq 2$. Then there exists a deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ such that

$$F|\tilde{L}(j_0 + 1)) \neq 0.$$

Proof: Take a non zero element $y(j_0) \in F\bar{M}(j_0 + 1) = V\bar{M}(j_0 - 1)$. As $V\bar{M}(j_0 - 1)$ is a maximal totally isotropic subspace of $\bar{M}(j_0)$ and as $\text{char}(k) \neq 2$ there exists a totally isotropic complement $\bar{M}'(j_0)$ of $V\bar{M}(j_0 - 1)$ in $\bar{M}(j_0)$. As $r = \dim(V\bar{M}(j_0 - 1)) = \dim(\bar{M}'(j_0)) \geq 2$ we can find an element $z(j_0) \in \bar{M}'(j_0)$ such that $(y(j_0), z(j_0)) = 0$. Define $\bar{N}(j_0)$ on the subspace $\bar{U}(j_0)$ generated by $\{y(j_0), z(j_0)\}$ by

$$\bar{N}(j_0)y(j_0) = z(j_0), \quad \bar{N}(j_0)z(j_0) = 0.$$

As $\bar{U}(j_0)$ is totally isotropic, $\bar{N}(j_0)$ is trivially a skew-symmetric endomorphism of square zero. By choosing a totally isotropic subspace $\bar{U}'(j_0)$ of $\bar{M}(j_0)$ which is in perfect duality with $\bar{U}(j_0)$ via $(,)$ and by defining $\bar{N}(j_0)|(\bar{U}(j_0) \oplus \bar{U}'(j_0))^\perp = 0$ we can extend $\bar{N}(j_0)$ to an endomorphism of $\bar{M}(j_0)$ which is skew-symmetric with respect to $(,)$ and of square zero. We can now lift $\bar{N}(j_0)$ to $N'(j_0) \in \text{End}_{W(k)}(M'(j_0))$ such that if we define $N'(j) = 0$ for all $j \neq j_0$ this gives a deformation endomorphism N by (4.5.2):

By standard arguments from (bi-)linear algebra we can find a base of $\bar{M}(j_0)$ such that $(,)$ has the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and such that $\bar{N}(j_0)$ has a matrix of the form $\begin{pmatrix} 0 & \bar{A} \\ 0 & 0 \end{pmatrix}$ with respect to this base, and the skew-symmetry of \bar{N} is equivalent to ${}^t\bar{A} = -\bar{A}$ (use that $(,)$ restricted to $\bar{M}(j_0)$ is hyperbolic as $V\bar{M}(j_0 - 1)$ is totally isotropic and that $\text{char}(k) \neq 2$). Now we can lift the constructed k -base of \bar{M} to a $W(k)$ -base of M' , such that $(,)$ has still the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ with respect to this base ($W(k)$ is 2-henselian), and we define N with respect to this base by a matrix $\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ where A is some arbitrary skew-symmetric lift of \bar{A} .

Having constructed the deformation endomorphism N , let $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ be the associated deformation of (X, λ, ι) to $k[[t]]$ and denote by $\tilde{M}' = \bigoplus \tilde{M}(j)$ the covariant Dieudonné module associated to $(\tilde{X}, \tilde{\lambda}, \tilde{\iota}) \otimes_{k[[t]]} k((t))^{\text{perf}}$.

We have to show that $F|\tilde{L}(j_0 + 1) \neq 0$. Let $x(j_0 + 1) \in \bar{M}(j_0 + 1)$ be an element such that $Fx(j_0 + 1) = y(j_0)$. Identify $\tilde{L}(j_0) = k((t))^{\text{perf}} \otimes_k L(j_0)$. Then we have modulo $\tilde{V}\tilde{M}$:

$$\tilde{F}(1 \otimes x(j_0 + 1)) = 1 \otimes Fx(j_0 + 1) + [t] \otimes NFx(j_0 + 1) \equiv [t] \otimes z(j_0),$$

and this is an element not equal to zero in $\tilde{M}/\tilde{V}\tilde{M}$ since $z(j_0) \notin V\bar{M}$.

(4.5.6) Proposition (second step): Let X again be bi-infinitesimal and assume that

$$(4.5.6.1) \quad F\bar{M}(j + 1) \neq V\bar{M}(j - 1) \quad \text{for all } j \in \mathbb{Z}/n\mathbb{Z}.$$

Then there exists a deformation $(\tilde{X}, \tilde{\lambda}, \tilde{\iota})$ of (X, λ, ι) to $k[[t]]$ such that $\tilde{X} \otimes_{k[[t]]} k((t))$ is not bi-infinitesimal.

Proof: We will again use (4.1.5); therefore by (4.5.2) we have to construct a deformation sequence $(x(j))_{j \in \mathbb{Z}/n\mathbb{Z}}$ of elements $x(j) \in \bar{M}(j)$ such that the following condition holds

(E') For all $j \in \mathbb{Z}/n\mathbb{Z}$ there exists a $W(k)$ -linear endomorphism $N'(j)$ of $(M'(j))$ with $N'(j)^2 = 0$ which is skew-symmetric with respect to $(\ , \)$ such that, if we denote by $\bar{N}(j)$ the endomorphism of $\bar{M}(j)$ induced by $N'(j)$, we have

$$\begin{aligned} \bar{N}(j)x(j) &= 0, \\ \bar{N}(j)(Fx(j+1)) &= \begin{cases} 0, & \text{if } Fx(j+1) = x(j); \\ x(j), & \text{if } Fx(j+1) \in VM. \end{cases} \end{aligned}$$

We first show the following lemma (which holds even if we do not assume (4.5.6.1)):

(4.5.6.2) Lemma: *There exists a deformation sequence $(x'(j))$ such that (E') holds if and only if there exists a deformation sequence $(x(j))$ such that the following condition holds:*

(E'') For all $j \in \mathbb{Z}/n\mathbb{Z}$ with $x(j) \neq Fx(j+1)$ we have

$$(x(j), Fx(j+1)) = 0$$

Proof of the lemma: The condition is necessary: If $(x(j))$ is a deformation sequence such that there exists for $j \in \mathbb{Z}/n\mathbb{Z}$ an endomorphism $N'(j)$ with the property required in (E'), we have for $j \in \mathbb{Z}/n\mathbb{Z}$ with $Fx(j+1) \neq x(j)$:

$$\begin{aligned} (x(j), Fx(j+1)) &= (\bar{N}(j)Fx(j+1), Fx(j+1)) \\ &= -(Fx(j+1), \bar{N}(j)Fx(j+1)) \\ &= -(x(j), Fx(j+1)) \end{aligned}$$

and this implies $(x(j), Fx(j+1)) = 0$ because $\text{char}(k) \neq 2$.

The condition is sufficient: Let $(x(j))$ be a deformation sequence such that (E'') holds. Let Σ be the set of elements $j \in \mathbb{Z}/n\mathbb{Z}$ with $x(j) \neq Fx(j+1)$. For $j \in \Sigma$ let $m(j-1) \in \bar{M}(j-1)$ be an element such that

$$2(Fx(j), m(j-1))^{1/p} = -(x(j), x(j))$$

and set $x'(j) := x(j) + Vm(j-1)$. Then we have

$$\begin{aligned} (x'(j), x'(j)) &= (x(j) + Vm(j-1), x(j) + Vm(j-1)) \\ &= (x(j), x(j)) + 2(x(j), Vm(j-1)) \\ &= (x(j), x(j)) + 2(Fx(j), m(j-1))^{1/p} \\ &= 0. \end{aligned}$$

If we set $x'(j) := x(j)$ for $j \notin \Sigma$, $(x'(j))_{j \in \mathbb{Z}/n\mathbb{Z}}$ is still a deformation sequence which satisfies (E''). We claim that for this sequence the condition (E') holds.

For $j \notin \Sigma$, i.e. $x'(j) = Fx'(j+1)$, we can simply define $N'(j)$ to be the zero endomorphism of $M'(j)$. Therefore assume $j \in \Sigma$, i.e. $Fx'(j+1) \in VM$. The k -span $\bar{U}_0(j)$ of $\{x'(j), Fx'(j+1)\}$ is a two-dimensional subspace which is totally isotropic by (E''). Define an endomorphism $\bar{N}_0(j)$ of $\bar{U}_0(j)$ by

$$\bar{N}_0(j)Fx'(j+1) = x(j), \quad \bar{N}_0(j)x(j) = 0.$$

Trivially this is a skew-symmetric endomorphism of square zero. As $\text{char}(k) \neq 2$ we can find a totally isotropic subspace $\bar{U}_1(j)$ of $\bar{M}(j)$ such that $(\ , \)$ induces a perfect

duality of $\bar{U}_0(j)$ with $\bar{U}_1(j)$. There is a unique skew-symmetric extension $\bar{N}(j)$ of $\bar{N}_0(j)$ to $\bar{U}_0(j) \oplus \bar{U}_1(j)$ and this extension is automatically of square zero. Finally extend $\bar{N}(j)$ to an endomorphism $\bar{N}(j)$ of $\bar{M}(j)$ by $\bar{N}(j)|(\bar{U}_0(j) \oplus \bar{U}_1(j))^\perp = 0$. Now it remains to lift $\bar{N}(j)$ to an endomorphism $N'(j)$ of $M'(j)$ which is skew-symmetric and of square zero. This can be done as in the proof of (4.5.5).

Proof of the proposition (continued): By (4.5.6.2) it remains to show that there exists a deformation sequence $(x(j))$ such that (E'') holds. For this it suffices to construct a deformation sequence $(x(j))$ such that $x(j) \in F\bar{M}(j+1)$ for all $j \in \mathbb{Z}/n\mathbb{Z}$. For simplicity call an element $x(j) \in \bar{M}(j)$ “good” if we have $x(j) \in F\bar{M}(j+1) \setminus V\bar{M}(j-1)$ and $F^n x(j) = 0$. The assumption (4.5.6.1) is equivalent to $F\bar{M}(j+1) \setminus V\bar{M}(j-1) \neq \emptyset$ by our general hypothesis ((4.5.3)(1),(2)), and therefore for all $j \in \mathbb{Z}/n\mathbb{Z}$ there exist “good” elements $x(j) \in \bar{M}(j)$ because F is nilpotent on \bar{M} .

We will now construct a deformation sequence consisting of “good” elements. Choose some $j_0 \in \mathbb{Z}/n\mathbb{Z}$ and some “good” element $x(j_0) \in \bar{M}(j_0)$. Now define elements $x(j) \in \bar{M}(j)$ for $j = j_0 - 1, \dots, j_0 - (n-1)$. Distinguish two cases: If $\varpi(x(j+1)) > 1$ define $x(j) := Fx(j+1)$. If $\varpi(x(j+1)) = 1$ let $x(j)$ be an arbitrary “good” element of $\bar{M}(j)$. Thus we get a family $(x(j))$. If $\nu := \varpi(x(j_0+1)) = 1$ the sequence $(x(j))$ is a deformation sequence and we are done; but in general this will not be the case and then we replace $x(j_0), x(j_0-1), \dots, x(j_0-\nu+2)$ by $Fx(j_0+1), F^2x(j_0+1), \dots, F^{\nu-1}x(j_0+1)$ and this gives us a deformation sequence of “good” elements, and the proposition is proved.

(4.5.7) From (4.5.5) and (4.5.6) we deduce that we can find for every point $s \in \mathbf{A}_{\mathcal{D}, C^p} \otimes \kappa(O_K)$ a generization y of s such that if $\bar{y} = (A, \bar{\lambda}, \iota, \bar{\eta})$ is a geometric point over y the following condition (depending only on y) holds: Let (X, λ, ι) be the principally quasi-polarized p -divisible O_B -module over $k = \kappa(\bar{y})$ associated to $(A, \bar{\lambda}, \iota)$ (after choosing some prime-to- p isogeny $\lambda \in \bar{\lambda}$), and let $(M, \langle \cdot, \cdot \rangle, \iota)$ be its covariant Dieudonné module. Then:

$$\rho := rk_{W(k)}(M_{\text{bi}})/4n \leq 1.$$

Let $(M' = \bigoplus M'(j), \langle \cdot, \cdot \rangle)$ the associated Dieudonné module (4.5.2) and set $L_{\text{bi}}(j) = M'_{\text{bi}}(j)/VM'_{\text{bi}}(j-1)$. Then we have $\rho = \dim_k(L_{\text{bi}}(j))$ for all $j \in \mathbb{Z}/n\mathbb{Z}$.

On the other hand the hermitian $M_2(O_K)$ -module structure of $(\Lambda, \langle \cdot, \cdot \rangle)$ corresponds via Morita equivalence to a decomposition $\Lambda = \Lambda'^2$, where Λ' is a O_K -module and where $\langle \cdot, \cdot \rangle$ corresponds to a symmetric pairing (\cdot, \cdot) on Λ' . If we denote by \mathcal{G}'_0 the O_K -group $O(\Lambda', (\cdot, \cdot)_{O_K})$ and by G'_0 its generic fibre we have

$$G' = \text{Res}_{K/\mathbb{Q}_p}(G'_0).$$

If we set $\Phi = F^n|M'(0)$, $(M'(0), \Phi)$ is a σ^n -crystal equipped with a symmetric form (\cdot, \cdot) . Its associated isocrystal is an isocrystal with $(G_0)^0$ -structure in the sense of [Ko3] where G_0 denotes the group of orthogonal similitudes of the quadratic K -space $(V', (\cdot, \cdot)_K)$. Its Newton point is given by $(1^{d-1}, \alpha, 1-\alpha, 0^{d-1})$ with the identifications of (2.3.4), where $0 \leq \alpha \leq 1$ (note that the slopes of $(M'(0), \Phi)$ in the sense of [Z2] Kap. 6 are $(n^{d-1}, n\alpha, n(1-\alpha), 0^{d-1})$). We have $\rho = 0$ if and only if $\alpha \in \{0, 1\}$.

(4.5.8) To shorten notations let us write $H = (G_0)^0$, denote by (N, Φ) the σ^n -isocrystal with H -structure $(M'(0)_{\mathbb{Q}}, F^n)$, and define $L = \text{Quot}(W(k))$. Fix an isomorphism of

L -vector spaces with pairings

$$(4.5.8.1) \quad (N, (\ , \)) \xrightarrow{\sim} (V', (\ , \)) \otimes_K L.$$

Let $\nu \in \text{Hom}_L(\mathbb{D}_L, H_L)$ be the slope homomorphism associated to (N, Φ) via (4.5.8.1). As H is quasi-split we can choose the isomorphism (4.5.8.1) such that ν is defined over K (cf. [Ko4] 1.1.3 (a)). Let us identify N and $V' \otimes_K L$ via such an isomorphism. As ν is defined over K , the slope decomposition of N is K -rational.

In particular we have a K -rational decomposition

$$N = N_{\text{ét mult}} \oplus N_{\text{bi}} = N_{\text{ét}} \oplus N_{\text{mult}} \oplus N_{\text{bi}}.$$

Note that $N_{\text{bi}}^\perp = N_{\text{ét mult}}$ and that $N_{\text{ét mult}}$ is hyperbolic with $N_{\text{ét}}$ and N_{mult} as maximal isotropic subspaces. Set $V_{\text{bi}} = N_{\text{bi}} \cap V'$. We have $\dim_K(V_{\text{bi}}) = 2\rho$.

(4.5.9) Now suppose that H is a split reductive group over K . By (2.3.4) the density theorem is proved in this case if we can show that $n\alpha \in \{0, \dots, n\}$. If $\alpha \in \{0, 1\}$, we are done; therefore assume $0 < \alpha < 1$, i.e. $\rho = 1$. Then $(V_{\text{bi}}, (\ , \))$ is a hyperbolic plane over K , and if we choose a hyperbolic pair $v, w \in V_{\text{bi}}$, this base gives an identification

$$GO^0(V_{\text{bi}}, (\ , \)) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in K^\times \right\}.$$

This implies that the subspace of N_{bi} generated by v (resp. w) over L is a subisocrystal of N_{bi} and obviously the slope of an isocrystal of height one (in the sense of [Z2] Kap. 6) must be an integer, and we are done.

(4.5.10) Now suppose that H is non-split over K . Assume we have $\rho = 0$, i.e. $N = N_{\text{ét}} \oplus N_{\text{mult}}$. As this decomposition is K -rational (4.5.8) and as $N_{\text{ét}}$ and N_{mult} are totally isotropic, this would imply that $(V', (\ , \))$ is hyperbolic, what is absurd. Therefore we have $N_{\text{bi}} \neq 0$ if H is non-split. Further we see that the two-dimensional quadratic space $(V_{\text{bi}}, (\ , \))$ over K must be anisotropic (otherwise it would be hyperbolic, and this would again imply that $(V', (\ , \))$ is hyperbolic). Now we have:

(4.5.10.1) Lemma: *If the quadratic space $(V_{\text{bi}}, (\ , \))$ is anisotropic, the σ^n -isocrystal (N_{bi}, Φ) is isoclinic.*

Proof: If (N_{bi}, Φ) is not isoclinic we have a non-trivial decomposition $N_{\text{bi}} = U_1 \oplus U_2$ of isocrystals where the slopes of U_1 and U_2 are different. As the slope decomposition is K -rational (4.5.8), $V_i = U_i \cap V_{\text{bi}}$ ($i = 1, 2$) are one-dimensional K -vector spaces. The lemma is proved if we can show that they are totally isotropic. For any $0 \neq v_i \in V_i$ we have $\Phi(v_i) = p^{r_i} \gamma_i v_i$ for integers $r_i > 0$ and units γ_i of $W(k)$. We have $r_1 + r_2 = n$ and $r_1 \neq r_2$, and in particular $r_1 \neq \frac{n}{2} \neq r_2$. Therefore the relations

$$(v_i, v_i) = \gamma_i^{-1} p^{-r_i} (\Phi v_i, v_i) = p^{-r_i} \gamma_i^{-1} (v_i, p^n \Phi^{-1} v_i)^{\sigma^n} = p^{n-2r_i} \gamma_i^{-2} (v_i, v_i)^{\sigma^n}$$

imply, by looking at p -adic valuations, that V_i is totally isotropic.

By the lemma, (N_{bi}, Φ) is isoclinic, and therefore we have $\alpha = 1/2$. In view of the calculation of $\bar{\mu}(\mathcal{D})$ in (2.3.4) the density theorem is proved in the non-split case (D).

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