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INCOMPRESSIBLE FLOWS OF AN IDEAL FLUID WITH VORTICITY IN BORDERLINE SPACES OF BESOV TYPE

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ABSTRACT. – We prove a uniqueness theorem for the Euler equations for an ideal incompressible fluid under the condition that vorticity belongs to a space of Besov type. We also prove an existence theorem in dimension two. © Elsevier, Paris

RÉSUMÉ. – Nous démontrons un théorème d'unicité pour le système d'Euler pour un fluide incompressible idéal sous l'hypothèse que le tourbillon appartient à un espace de type Besov. Nous démontrons également un théorème d'existence en dimension deux. © Elsevier, Paris

0. Introduction

In this paper we study the nonstationary Euler equations of an ideal incompressible fluid

$$\begin{cases} \dot v_j(x,t) = -\sum_{i=1}^{n} v_i\partial_i v_j - \partial_j p \ , & 1 \le j \le n \ , \ x \in \mathbb{R}^n \ , \\ \operatorname{div} v = \sum_{j=1}^{n} \partial_j v_j = 0 \ , \\ v(x,0) = v_0(x) \ . \end{cases} \tag{0.1}$$

Here $v(x,t) = (v_1,\ldots,v_n)$ is the Eulerian velocity of a fluid flow. For incompressible fluids the key characteristic of the flow is vorticity

$$\omega_{ij}(x,t) = \partial_j v_i - \partial_i v_j \ , \qquad 1 \le i,j \le n \ .$$

The mathematical theory of Euler equations (0.1) is an old subject. In [C1] J.-Y. Chemin develops the theory in detail and gives an account of more recent results including his work [C2] on the regularity of vortex patches for the two-dimensional Euler equations.

Existence and uniqueness theorems are obtained for the problem (0.1) (locally in time for $n \ge 3$) for vorticity in various function spaces with *supercritical* smoothness. Here

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we refer to the condition $s > n/p$ where s is smoothness of vorticity in the sense of a particular scale of function spaces based on L^p.

The papers of L. Lichtenstein [L], N. Gunther [G], Wolibner [W] that started the subject deal with Hölder classes. Some of the more recent papers include those of D. Ebin and J. Marsden [EM], J. Bourguignon and H. Brezis [BB], R. Temam [T], T. Kato and G. Ponce [KP] (Sobolev spaces).

In [V] we proved the global existence for $n = 2$ where vorticity belongs to the Besov space $B^s_{2/s,1}$.

In the present paper we continue to investigate the *borderline* case $s = n/p$ which corresponds to the critical case of Sobolev (-Besov-Lizorkin-Triebel) embedding.

V. Yudovich [Y1] proved the basic uniqueness theorem for the weak solutions of (0.1) under the condition (bounded domain in $\mathbb{R}^n$) $\omega \in \mathrm{L}^\infty$. For $n = 2$ he was able to construct a weak solution in this class that exists globally in time. The existence of a weak solution in two dimensions (possibly without uniqueness) was proved for a wider class L^p, $1 < p < \infty$, by V. Yudovich [Y1] and by R. DiPerna and A. Majda [DM]. D. Chae [C] proved the existence theorem for vorticity in $L \log L(\mathbb{R}^2)$. J.-M. Delort [De] constructed a weak solution with initial vorticity arbitrary sign definite measure. A Bourbaki talk by P. Gérard [Ge] gives an account of this result as well as the results of [C2]. A different proof of J.-M. Delort's result was given by L.C. Evans and S. Műller [EvM] and by A. Majda [M].

V. Scheffer [S] and A. Shnirelman [Sh] constructed the first examples of *nonuniqueness* for weak solutions with $v(\cdot,\cdot) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R})$.

Recently V. Yudovich [Y2] further improved his uniqueness theorem [Y1] allowing (case of a bounded domain) $\omega \in \bigcap_{p_0 \le p < \infty} \mathrm{L}^p$ so that $\|\omega\|_{\mathrm{L}^p} \le C\theta(p)$ and $\theta(p)$ grows *moderately* in p. For example $\theta(p) = \log p$ guarantees uniqueness.[1] More precisely, V. Yudovich proved that for $Z(\alpha) = \inf_{\varepsilon \in (0, p_0^{-1}]} \{\varepsilon^{-1} \alpha^\varepsilon \theta(\frac{1}{\varepsilon})\}$ the condition $\int_1^\infty \alpha^{-1} Z(\alpha)\, \mathrm{d}\alpha = \infty$ implies uniqueness. This result holds for arbitrary n while the existence theorem was proved in [Y2] (see also [Y3]) only for $n = 2$. The proof of uniqueness in [Y2] is based on the energy method. In case the growth condition for $\theta(p)$ *fails* V. Yudovich constructed *counterexamples to this method* of proving uniqueness (if not to uniqueness per se).

We introduce here a different uniqueness class which is a variant of a *borderline* Besov space. To describe a particular case of the results of the paper we define

$$B_\Pi = \left\{ f \in S'(\mathbb{R}^n) \;\Big|\; \sum_{j=-1}^{N} \|\Delta_j f\|_{\mathrm{L}^\infty} = O(\Pi(N)) \right\} .$$

Here the increasing function Π satisfies certain conditions (see (i)-(iii) below), $\Delta_j f$ denotes the terms in Littlewood-Paley decomposition of f. One possible choice is $\Pi(N) = (N+2)\log_2^\kappa(N+3)$, $0 \le \kappa \le 1$. Notice $f \in B_\Pi$ means the norm $\|f\|_{B^0_{\infty,1}}$ is generally divergent but in a controlled way.

We prove uniqueness for the problem (0.1), n arbitrary, under the condition

$$\int_1^\infty \Pi(\alpha)^{-1}\, \mathrm{d}\alpha = \infty \,, \tag{0.2}$$

[1] "Not much stronger than linear" on p. 28 of [Y2] is a misprint. Logarithmic singularities that lead to linear in p growth of $\|\omega\|_{\mathrm{L}^p}$ are in fact not allowed but singularities of the kind $\log\log|x|$ that produce $\log p$ growth of $\|\omega\|_{\mathrm{L}^p}$ as $p \to \infty$ are covered by the results of [Y2].

$\omega \in \mathrm{L}^{p_0}$, $p_0 \in (1, n)$ and the norm $\|\omega\|_{B_\Pi}$ is bounded. By the choice $\Pi(N) = N + 2$ this implies uniqueness for flows with vorticity in Hölder space $\Lambda^0 = B^0_{\infty,\infty}$. As a corollary, a solution with vorticity bounded in bmo is unique. Indeed, bmo $\hookrightarrow \Lambda^0$.

For Π as above the (unique) solution exists globally in time $(n = 2)$ for $0 < \kappa \leq \frac{1}{2}$ and (possibly only) locally in time for $\frac{1}{2} < \kappa \leq 1$. The initial vorticity belongs to the space B_Γ with $\Gamma(N) = \log_2^\kappa(N + 3)$.

The existence theorem is proved in §9.

The proof of the uniqueness theorem given in §8 uses a "paradifferential analogue" of the Osgood uniqueness argument and seems *unrelated to the energy method*. The condition (0.2) in fact guarantees that the Lagrangian flow satisfies conditions of the Osgood uniqueness theorem. It also looks similar to the growth condition in [Y2]. On the other hand there are functions in B_Γ with a *compact support* that fail to belong to L^p for *all* sufficiently large p (see §2). Therefore the growth condition for $\theta(p)$ is violated.

The main tool we use to get an a priori estimate in B_Π are *wavelet decompositions* of vorticity. In fact we obtain a complete description of all spaces of this type in terms of wavelet coefficients. This allows to get the information about the action of a volume preserving homeomorphism (given by the fluid flow) on B_Γ.

We use the method of H. Bahouri and J.-Y. Chemin [BC] to obtain the necessary "paradifferential estimates" of the terms in Littlewood-Paley decomposition of the vorticity ω.

Some of the immediate problems left open include whether the time of existence is indeed finite for e.g., $\kappa = 1$ or this restriction is due to the weakness of the method. Also the relation with the transport theory developed by R. DiPerna and P.-L. Lions [DL], B. Desjardins [D1], H. Bahouri and J.-Y. Chemin [BC], J.-Y. Chemin and N. Lerner [CL], B. Desjardins [D2], especially for the flows that are not quasi Lipschitz, seems of interest.

1. Function spaces

Let $\Gamma : \mathbb{R} \to [1, \infty)$ be a locally Lipschitz continuous monotonically nondecreasing function satisfying the following conditions:

(i) $\Gamma(\alpha) = 1$ for $\alpha \in (-\infty, -1]$, $\lim_{\alpha \to \infty} \Gamma(\alpha) = \infty$.

(ii) There is a constant $C > 0$ such that $C^{-1}\Gamma(\beta) \leq \Gamma(\alpha) \leq C\Gamma(\beta)$ for $\alpha, \beta \in [-1, \infty)$, $|\alpha - \beta| \leq 1$.

(iii) There is a constant $C > 0$ such that

$$C\Gamma(\alpha) \geq \int_\alpha^\infty 2^{-(\xi - \alpha)}\Gamma(\xi)\, \mathrm{d}\xi \ , \alpha \in [-1, \infty) \ .$$

EXAMPLE 1.1. $\Gamma(\alpha) = (N + 2)^\varepsilon \log_2^\kappa(\alpha + 3)$ for $\alpha \geq -1$, where $\varepsilon, \kappa \in [0, \infty)$, except $\varepsilon = \kappa = 0$.

We choose radial $\Phi \in S(\mathbb{R}^n)$ such that $\mathrm{supp}\hat{\Phi} \subset \{\xi \in \mathbb{R}^n \mid \xi| \leq 1\}$ and $|\hat{\Phi}(\xi)| \geq C > 0$ on $\{\xi \in \mathbb{R}^n \mid |\xi| \leq \frac{5}{6}\}$. We also choose radial $\varphi \in S(\mathbb{R}^n)$ with $\mathrm{supp}\hat{\varphi} \subset \{\xi \in \mathbb{R}^n \mid \frac{1}{2} \leq |\xi| \leq 2\}$, $|\hat{\varphi}(\xi)| \geq C > 0$ on $\{\xi \in \mathbb{R}^n \mid \frac{3}{5} \leq |\xi| \leq \frac{5}{3}\}$. Let $\varphi_j(x) = 2^{jn}\varphi(2^j x)$ [i.e., $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j}\xi)$], $j \in \mathbb{Z}$.

DEFINITION 1.2. – Let for $f \in S'(\mathbb{R}^n)$

$$\Delta_{-1} f = \hat{\Phi}\left(\frac{1}{i}\frac{\partial}{\partial x}\right) f = (\hat{\Phi}\cdot\hat{f})^{\vee} = \Phi * f \ ;$$
$$\Delta_j f = \hat{\varphi}_j\left(\frac{1}{i}\frac{\partial}{\partial x}\right) f = (\hat{\varphi}_j\cdot\hat{f})^{\vee} = \varphi_j * f \ , \qquad j \geq 0 \ ;$$
$$\Delta_j f = 0 \ , \qquad j \leq -2 \ ;$$
$$S_k f = \sum_{j\leq k} \Delta_j f \ .$$

DEFINITION 1.3. – Let Γ be the function satisfying (i)–(iii) above. Let

$$B_\Gamma = \left\{ f \in S'(\mathbb{R}^n) \ \Big| \ \sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty} \leq C\Gamma(N) \ , \ N \geq -1 \right\} . \tag{1.1}$$

The norm on B_Γ is defined as the best constant in (1.1):

$$\|f\|_\Gamma = \sup_{N\geq -1}\left(\sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty}\right)\Gamma(N)^{-1} \ .$$

We give the characterization of B_Γ via the wavelet expansions. For standard Besov spaces see [FJ], [LM], [M].

Let radial $\psi \in S(\mathbb{R}^n)$ satisfy

$$\text{supp}\hat{\psi} \subset \left\{\xi \in \mathbb{R}^n \ \Big| \ \frac{1}{2} \leq |\xi| \leq 2\right\}, \quad |\hat{\psi}(\xi)| \geq C > 0 \text{ on } \left\{\xi \in \mathbb{R}^n \ \Big| \ \frac{3}{5} \leq |\xi| \leq \frac{5}{3}\right\}.$$

Choose radial $\Psi \in S(\mathbb{R}^n)$ so that $\text{supp}\hat{\Psi} \subset \{\xi \in \mathbb{R}^n \mid |\xi| \leq \pi\}$, $|\hat{\Psi}(\xi)| \geq C > 0$ on $\{\xi \in \mathbb{R}^n \mid |\xi| \leq 1\}$. We define the set of dyadic cubes as follows: for $m \in \mathbb{Z}$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$

$$Q_{m,k} = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid k_i 2^{-m} \leq x_i < (k_i + 1)2^{-m} \ ; \ i = 1, \ldots, n\} \ .$$

We set for any dyadic cube $Q = Q_{m,k}$

$$\psi_{m,k}(x) = \psi(2^m x - k) \ .$$

PROPOSITION 1.4. – *Let $f \in B_\Gamma$. There is the following representation*

$$f(x) = \sum_{k\in\mathbb{Z}^n} a_k \Psi(x-k) + \sum_{m=0}^{\infty}\sum_{k\in\mathbb{Z}^n} a_{m,k}\psi_{m,k}(x) \ . \tag{1.2}$$

The series in (1.2) *is convergent in $S'(\mathbb{R}^n)$. The coefficients a_k, $a_{m,k}$ satisfy the following estimate*

$$C^{-1}\|f\|_\Gamma \leq \sup_{N\geq -1}\left(\sum_{m=-1}^{N} \sup_{k\in\mathbb{Z}^n} |a_{m,k}|\right)\Gamma(N)^{-1} \leq C\|f\|_\Gamma \ , \tag{1.3}$$

where the constant $C > 0$ does not depend on $f \in B_\Gamma$. For $m = -1$ we set $a_{-1,k} = a_k$, $k \in \mathbb{Z}^n$.

Proof. – It is possible to make the above choices of $\varphi, \Phi, \psi, \Psi$ so that

$$\hat{\Phi}(\xi)\hat{\Psi}(\xi) + \sum_{m=0}^{\infty} \hat{\varphi}(2^{-m}\xi)\hat{\psi}(2^{-m}\xi) = 1 \ , \qquad \xi \in \mathbb{R}^n \ .$$

Then, following [FJ]

$$f(\cdot) = \sum_{k\in\mathbb{Z}^n} (\Phi * f)(k)\Psi(\cdot - k) + \sum_{m=0}^{\infty} 2^{-mn}(\varphi_m * f)(2^{-m}k)\psi_m(\cdot - 2^{-m}k) \tag{1.4}$$

where $\psi_m(x) = 2^{mn}\psi(2^m x)$, $m \geq 0$. The series in the right side of (1.4) is convergent in $S'(\mathbb{R}^n)$. Therefore, we define $a_k = \Phi * f(k)$; $a_{m,k} = \varphi_m * f(2^{-m}k)$. By construction

$$|a_k| \leq \|\Phi * f\|_{L^\infty} = \|\Delta_{-1} f\|_{L^\infty} \ ,$$
$$|a_{m,k}| \leq \|\varphi_m * f\|_{L^\infty} = \|\Delta_m f\|_{L^\infty}$$

for $m \geq 0$, $k \in \mathbb{Z}^n$. This proves the second inequality in 1.3. It remains to prove the first inequality. We need the following estimate from [FJ].

LEMMA 1.5 ([FJ]). – *The following two inequalities hold true for any* $M \in \mathbb{Z}_+$, $j, m \geq 0$

$$|(\Delta_j \psi_{m,k})(x)| \leq C 2^{-(m-j)(n+1)}(1 + 2^j|x - k2^{-m}|)^{-M} \text{ for } j \leq m \ , \text{ and} \tag{1.5}$$
$$|(\Delta_j \psi_{m,k})(x)| \leq C 2^{-(j-m)}(1 + 2^m|x - k2^{-m}|)^{-M} \text{ for } m \leq j \ . \tag{1.6}$$

The same inequalities are valid for $j = -1$, $m \geq 0$ *in case* (1.5) *and for* $m = -1$, $j \geq 0$ *in case* (1.6) *where* $\psi_{-1,k}(\cdot) = \Psi(\cdot - k)$.

To prove Proposition 1.4 we have to estimate $\|\Delta_j f\|_{L^\infty}$ where f is given by (1.2). We have from Lemma 1.5

$$\begin{aligned}
\|\Delta_{-1} f\|_{L^\infty} &\leq \Big\|\Delta_{-1} \sum_{k\in\mathbb{Z}^n} a_k \Psi(\cdot - k)\Big\|_{L^\infty} + \sum_{m=0}^{\infty} \Big\|\Delta_{-1} \sum_{k\in\mathbb{Z}^n} a_{m,k}\psi_{m,k}\Big\|_{L^\infty} \\
&\leq C\Big\| \sum_{k\in\mathbb{Z}^n} a_k \Psi(\cdot - k)\Big\|_{L^\infty} + \sum_{m=0}^{\infty} \Big\| \sum_{k\in\mathbb{Z}^n} a_{m,k}\Delta_{-1}\psi_{m,k}\Big\|_{L^\infty} \\
&\leq C \sup_{k\in\mathbb{Z}^n} |a_k| \cdot \sup_x \sum_{k\in\mathbb{Z}^n} (1 + |x - k|)^{-M} \\
&\qquad + C \sum_{m=0}^{\infty} \sup_{k\in\mathbb{Z}^n} |a_{m,k}| \cdot 2^{-m(n+1)} \sup_x \sum_k (1 + |x - k2^{-m}|)^{-M} \\
&\leq C \sup_{k\in\mathbb{Z}^n} |a_k| + C \sum_{m=0}^{\infty} 2^{-m-1} \sup_{k\in\mathbb{Z}^n} |a_{m,k}| \ .
\end{aligned}$$

Likewise, for $j \geq 0$, applying Lemma 1.5

$$\begin{aligned}
\|\Delta_j f\|_{L^\infty} &\leq \Big\| \sum_{k\in\mathbb{Z}^n} a_k \Delta_j \Psi(\cdot - k)\Big\|_{L^\infty} \\
&\quad + \sum_{m=0}^{j} \Big\| \sum_{k\in\mathbb{Z}^n} a_{m,k}\Delta_j \psi_{m,k}\Big\|_{L^\infty} \\
&\quad + \sum_{m=j+1}^{\infty} \Big\| \sum_{k\in\mathbb{Z}^n} a_{m,k}\Delta_j \psi_{m,k}\Big\|_{L^\infty} \\
&\leq C2^{-j} \sup_{k\in\mathbb{Z}^n} |a_k| \Big\| \sum_{k\in\mathbb{Z}^n} (1+|\cdot - k|)^{-M}\Big\|_{L^\infty} \\
&\quad + C\sum_{m=0}^{j} 2^{-(j-m)} \sup_{k\in\mathbb{Z}^n} |a_{m,k}| \Big\| \sum_{k\in\mathbb{Z}^n} (1+2^m|\cdot - k2^{-m}|)^{-M}\Big\|_{L^\infty} \\
&\quad + C\sum_{m=j+1}^{\infty} 2^{-(m-j)(n+1)} \sup_{k\in\mathbb{Z}^n} |a_{m,k}| \Big\| \sum_{k\in\mathbb{Z}^n} (1+2^j|\cdot - k2^{-m}|)^{-M}\Big\|_{L^\infty} \\
&\leq C2^{-j-1} \sup_{k\in\mathbb{Z}^n} |a_k| + C\sum_{m=0}^{j} 2^{-(j-m)} \sup_{k\in\mathbb{Z}^n} |a_{m,k}| \\
&\quad + C\sum_{m=j+1}^{\infty} 2^{-(m-j)} \sup_{k\in\mathbb{Z}^n} |a_{m,k}| \ .
\end{aligned}$$

Let

$$\sup_{k\in\mathbb{Z}^n} |a_{m,k}| = b_m \ , \qquad m \geq -1 \ ;$$

$$\sum_{j=-1}^{m} b_j = d_m \ , \qquad m \geq -1 \ ; \ d_{-2} = 0 \ .$$

Then, for every $j \geq -1$, $N \geq -1$

$$\|\Delta_j f\|_{L^\infty} \leq C \sum_{m=-1}^{j} 2^{-(j-m)} b_m + C\sum_{m=j+1}^{\infty} 2^{-(m-j)} b_m \ ;$$

$$\begin{aligned}
\sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty} &\leq C \sum_{m=-1}^{N} \left(\sum_{j=-1}^{m-1} 2^{-(m-j)} + \sum_{j=m}^{N} 2^{-(j-m)} \right) b_m \\
&\quad + C\sum_{m=N+1}^{\infty} \left(\sum_{j=-1}^{N} 2^{-(m-j)} \right) b_m \\
&\leq C\sum_{m=-1}^{N} b_m + C\sum_{m=N+1}^{\infty} 2^{-(m-N)}(d_m - d_{m-1}) \\
&\leq Cd_N + C2^N \sum_{m=N+1}^{\infty} (2^{-m} - 2^{-(m+1)}) d_m \\
&\leq Cd_N + C\sum_{m=N+1}^{\infty} 2^{-(m-N)} d_m \ .
\end{aligned}$$

Suppose for any $m \geq -1$, $d_m \leq C\Gamma(m)$. Then the right side of the last inequality is

$$\leq C\Gamma(N) + C\int_N^\infty 2^{-(\xi-N)}\Gamma(\xi)\,\mathrm{d}\xi \leq C\Gamma(N)$$

because of the conditions (ii) and (iii) describing Γ. This concludes the proof of the proposition.

We now construct the predual space to B_Γ. Let H_Γ be defined as follows:

$$H_\Gamma = \left\{ f \in S'(\mathbb{R}^n) \;\middle|\; \exists\ \{d_j\}_{j=-1}^\infty\ ,\ d_j \geq 0\ ,\ \sum_{j=-1}^\infty d_j < \infty \right.$$
$$\left. \text{such that } \|\Delta_m f\|_{\mathrm{L}^1} \leq \sum_{j\geq m} d_j \Gamma(j)^{-1}\ ,\ m \geq -1 \right\}.$$

We set

$$\|f\|_{\textcircled{\scriptsize\Gamma}} = \inf_{\{d_j\}} \sum_{j=-1}^\infty d_j.$$

PROPOSITION 1.6. – *The dual space to H_Γ is isomorphic to B_Γ:*

$$H'_\Gamma \approx B_\Gamma\ .$$

Proof. – We omit the proof which is fairly standard (cf. [P]).

PROPOSITION 1.7. – *Let $f \in H_\Gamma$. There is the following representation:*

$$f(x) = \sum_{k\in\mathbb{Z}^n} a_k \Psi(x-k) + \sum_{m=0}^\infty \sum_{k\in\mathbb{Z}^n} a_{m,k} 2^{mn} \psi_{m,k}(x)\ . \tag{1.7}$$

The series in (1.7) *is convergent in $S'(\mathbb{R}^n)$. Moreover there is a constant $C > 0$ which is independent of f such that*

$$C^{-1}\|f\|_{\textcircled{\scriptsize\Gamma}} \leq \inf_{\substack{\{d_j\}_{j=-1}^\infty, d_j\geq 0,\ \sum_{j=-1}^\infty d_j<\infty \\ \sum_{k\in\mathbb{Z}^n} |a_{m,k}| \leq \sum_{j=m}^\infty d_j\Gamma(j)^{-1},\ m\geq -1}} \sum_{j=-1}^\infty d_j \leq C\|f\|_{\textcircled{\scriptsize\Gamma}}\ . \tag{1.8}$$

We omit the proof which is similar to the proof of Proposition 1.4.

REMARK 1.8. – In particular predual H_Γ of the Banach space B_Γ is separable, and $S \hookrightarrow H_\Gamma$ is a dense embedding.

REMARK 1.9. – Since the function spaces B_Γ and H_Γ do not depend upon the choice of Φ, φ satisfying standard conditions (see e.g., [P], [FJ]) Propositions 1.4, 1.7 provide a complete description of these spaces in terms of wavelet decompositions with respect to band-limited wavelets constructed by P. Lemarié and Y. Meyer [LM], [M2].

REMARK 1.10. – If $\Gamma(N) = N+2$, $N \geq -1$ then obviously $\mathrm{L}^\infty \hookrightarrow B_\Gamma$. In case $\Gamma(N)$ grows slower than $(N+2)$, that is $\Gamma(N) = o(N+2)$ as $N \to \infty$, the above embedding is no longer valid. It means for such a $\Gamma(N)$ the space B_Γ is not comparable with the space L^∞. As we mentioned in the introduction V. Yudovich proved the global existence and uniqueness theorem [Y1] in dimension 2 for the Euler equations with vorticity in L^∞ (uniqueness for arbitrary $n \geq 2$).

2. A counterexample

PROPOSITION 2.1. – *For any $\rho \in (1,\infty)$ there exists a function f such that $f \in B_\Gamma$, $f \in \bigcap_{p\in(1,\rho)} \mathrm{L}^p(\mathbb{R}^n)$ and $f \notin \mathrm{L}^p(\mathbb{R}^n)$ for any $p \geq \rho$.*

Proof. – We use the following fundamental characterization of L^p-spaces [LM], [M2]. To simplify the notation we give the construction in dimension $n = 1$. Let ψ, Ψ be obtained from the MRA procedure [M2]. Let

$$f(x) = \sum_{k\in\mathbb{Z}} a_k \Psi(x-k) + \sum_{m=0}^{\infty} \sum_{k\in\mathbb{Z}} a_{m,k}\psi(2^m x - k) .$$

THEOREM 2.2 ([LM], [M2]). – *Let $p \in (1,\infty)$. Then $f \in \mathrm{L}^p$ if and only if the following two conditions are satisfied:*

$$\{a_k\}_{k\in\mathbb{Z}} \in \ell^p ; \tag{2.1}$$

$$g = \left(\sum_{m=0}^{\infty} \sum_{k\in\mathbb{Z}} |a_{m,k}|^2 \chi_{Q_{m,k}}\right)^{1/2} \in \mathrm{L}^p . \tag{2.2}$$

We note that the function g in (2.2) is a discrete analogue of the Littlewood-Paley quadratic function; $\chi_{Q_{m,k}}$ denotes the characteristic function of a dyadic cube $Q_{m,k}$.

To construct the example in Proposition 2.1 let $a_k = 0$ for $k \in \mathbb{Z}$ so that (2.1) is satisfied.

Since $\Gamma(\alpha) \to \infty$ as $\alpha \to \infty$ there is a sequence $\{c_m\}_{m=0}^{\infty}$ satisfying the following 3 conditions:

$$c_m \to 0 \quad \text{as} \quad m \to 0 ; \tag{2.3}$$

$$\sum_{m=0}^{N} |c_m| \leq C\Gamma(N) , \qquad N \geq 0 ; \tag{2.4}$$

$$\sum_{m=0}^{\infty} |c_m|^2 = \infty . \tag{2.5}$$

Indeed we proceed by induction.

We construct a strictly monotonic sequence of nonnegative integers $\{n_i\}$, $n_1 = 0$ and set

$$c_m = \begin{cases} i^{-1/2} , & m = n_i \\ 0 , & m \neq n_i \text{ for all } i \geq 1. \end{cases}$$

Thus (2.3) and (2.5) are satisfied. We set $C = \Gamma(0)^{-1}$ and assume $n_1, \ldots, n_q$ are already constructed so that (2.4) is satisfied for all $N \leq n_q$. Choose $n_{q+1} > n_q$ so that

$$\sum_{i=1}^{q+1} i^{-1/2} \leq C\Gamma(n_{q+1}) .$$

Such a choice could be made since $\Gamma(\alpha) \to \infty$. This takes care of (2.4). We fix a sequence $\{c_m\}_{m=0}^{\infty}$ satisfying (2.3)–(2.5) and let

$$f = \sum_{m=0}^{\infty} c_m \sum_{k=0}^{2^m-1} \varepsilon_{m,k}\psi(2^m x - k) ,$$

where $\varepsilon_{m,k}$ are taking on values 0 and 1.

We now assert that we can choose $\varepsilon_{m,k}$ in such a way that

$$g(x) = \begin{cases} 0 , & x \notin Q_{0,0} = [0,1) \\ x^{-\rho^{-1}} , & x \in Q_{0,0}. \end{cases} \tag{2.6}$$

Indeed we start with

$$\varepsilon_{0,0} = \begin{cases} 1 , & \text{if } |c_0|^2 \leq 1 \\ 0 , & \text{otherwise.} \end{cases}$$

Assume we constructed $\varepsilon_{m,k}$, $m \leq \ell$ so that

$$\sum_{m=0}^{\ell} |c_m|^2 \sum_{k=1}^{2^m-1} \varepsilon_{m,k}\chi_{Q_{m,k}}(x) \leq x^{-2\rho^{-1}} , \qquad x \in Q_{0,0} . \tag{2.7}$$

To do the induction step we simply set $\varepsilon_{\ell+1,\kappa} = 1$ in case

$$\sum_{m=0}^{\ell} |c_m|^2 \sum_{k=1}^{2^m-1} \varepsilon_{m,k}\chi_{Q_{m,k}}(x) + |c_{\ell+1,\kappa}|^2 \chi_{Q_{\ell+1,\kappa}} \leq x^{-2\rho^{-1}} \tag{2.8}$$

for $x \in Q_{\ell+1,\kappa}$, $0 \leq \kappa \leq 2^{\ell+1} - 1$ and $\varepsilon_{\ell+1,\kappa} = 0$ otherwise.

Then (2.7) is satisfied on the $(\ell + 1)$-st step because of (2.8) and (2.6) follows from (2.3) and (2.5). Obviously $g(x)$ in (2.6) satisfies the conditions $g \in \bigcap_{p\in(1,\rho)} \mathrm{L}^p$, $g \notin \mathrm{L}^p$ for any $p \geq \rho$.

We conclude from the Theorem 2.2 that f satisfies the same properties. Furthermore, from Proposition 1.4 and (2.4) $f \in B_\Gamma$. This concludes the proof.

REMARK 2.3. – The function f in Proposition 2.1 could be chosen to have *compact support*. Indeed, using sufficiently regular wavelets with compact support [D], [M2] and repeating the same construction word for word produces such a function.

3. Estimates for the Biot-Savart law

Let the vorticity $\omega \in \mathrm{L}^{p_0} \cap B_\Gamma$, $1 < p_0 < n$. We define the velocity as

$$u = \mathcal{K} * \omega ,$$

$\mathcal{K}$ being the kernel in the Biot-Savart law.

THEOREM 3.1. – *The function u is continuous and bounded on $\mathbb{R}^n$ with the following modulus of continuity:*

$$|u(x) - u(y)| \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_\Gamma \Gamma(-\log_2|x-y|))|x-y|\,, \qquad x \neq y\,; \tag{3.1}$$

$$\|u\|_{L^\infty} \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_\Gamma)\,, \qquad 1 < p_0 < n\,. \tag{3.2}$$

Proof. – For any $N \geq -1$, $x \neq y$

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{j=-1}^{N} \|\Delta_j \nabla u\|_{L^\infty}|x-y| + 2\sum_{j=N+1}^{\infty}\|\Delta_j u\|_{L^\infty} \\ &\leq C\sum_{j=-1}^{N}\|\Delta_j\nabla u\|_{L^\infty}|x-y| + C\sum_{j=N+1}^{\infty}2^{-j}\|\Delta_j\omega\|_{L^\infty}\,. \end{aligned} \tag{3.3}$$

We estimate the second term in the right side of (3.3) as follows. Let

$$d_m = \sum_{j=-1}^{m}\|\Delta_j\omega\|_{L^\infty}\,.$$

Then,

$$\begin{aligned} \sum_{j=N+1}^{\infty} 2^{-j}\|\Delta_j\omega\|_{L^\infty} &\leq \sum_{j=N+1}^{\infty}2^{-j}(d_j - d_{j-1}) \\ &= -2^{-(N+1)}d_N + \sum_{j=N+1}^{\infty} d_j(2^{-j} - 2^{-(j+1)}) \\ &\leq -2^{-(N+1)}d_N + \|\omega\|_\Gamma \sum_{j=N+1}^{\infty}\Gamma(j)2^{-j} \\ &\leq C2^{-N}\Gamma(N)\|\omega\|_\Gamma \end{aligned} \tag{3.4}$$

because of the conditions (ii) and (iii). We estimate now the first term in the right side of (3.3)

$$\begin{aligned} \sum_{j=-1}^{N}\|\Delta_j\nabla u\|_{L^\infty} &\leq \|\Delta_{-1}\nabla u\|_{L^\infty} + C\sum_{j=0}^{N}\|\Delta_j\omega\|_{L^\infty} \\ &\leq C\|\Delta_{-1}\nabla u\|_{L^{p_0}} + C\Gamma(N)\|\omega\|_\Gamma \\ &\leq C\|\Delta_{-1}\omega\|_{L^{p_0}} + C\Gamma(N)\|\omega\|_\Gamma \\ &\leq C\|\omega\|_{L^{p_0}} + C\Gamma(N)\|\omega\|_\Gamma\,. \end{aligned} \tag{3.5}$$

Here we used Bernstein's inequality and boundedness of the Calderón-Zygmund operator $\omega \mapsto \nabla u$ in L^{p_0}. The inequalities (3.3)–(3.5) yield

$$|u(x) - u(y)| \leq C(\|\omega\|_{L^{p_0}} + \|\omega\|_\Gamma\Gamma(N))|x-y| + C2^{-N}\Gamma(N)\|\omega\|_\Gamma\,.$$

We choose $N = [-\log_2|x-y|]$ (in case $|x-y| \leq 2$, otherwise choose $N=-1$). This proves (3.1). It remains to prove (3.2). We have

$$\begin{aligned}\|u\|_{L^\infty} &\leq \|\Delta_{-1}u\|_{L^\infty} + \sum_{j=0}^{\infty}\|\Delta_j u\|_{L^\infty}\\ &\leq \|\Delta_{-1}u\|_{L^\infty} + C\sum_{j=0}^{\infty}2^{-j}\|\Delta_j\omega\|_{L^\infty}\ .\end{aligned}$$

The second term is $\leq C\|\omega\|_\Gamma$ as follows from (3.4) for $N=-1$. To estimate the first term we write for $\chi \in \mathcal{D}(\mathbb{R}^n)$, $\chi \equiv 1$ near the origin,

$$\begin{aligned}\|\Delta_{-1}u\|_{L^\infty} &\leq \|(\chi\mathcal{K}) * \Delta_{-1}\omega\|_{L^\infty} + \|((1-\chi)\mathcal{K}) * \Delta_{-1}\omega\|_{L^\infty}\\ &\leq \|\chi\mathcal{K}\|_{L^1}\|\Delta_{-1}\omega\|_{L^\infty} + \|(1-\chi)\mathcal{K}\|_{L^{q_0}}\|\Delta_{-1}\omega\|_{L^{p_0}}\\ &\leq C\|\Delta_{-1}\omega\|_{L^{p_0}}\end{aligned}\tag{3.6}$$

since $\chi\mathcal{K} \in L^1$ and $(1-\chi)\mathcal{K} \in L^{q_0}$ $(\frac{1}{q_0}+\frac{1}{p_0}=1)$. This concludes the proof.

PROPOSITION 3.2. – *Let $\omega \in L^{p_0} \cap L^{p_1}$ where $1 < p_0 < n$, $n < p_1 < \infty$. Then for all $x, y \in \mathbb{R}^n$, $|x-y| \leq 1$ we have*

$$|u(x)-u(y)| \leq C|x-y|^{1-\frac{n}{p_1}}\|\omega\|_{L^{p_1}}\ ,\tag{3.7}$$

where $C > 0$ depends on n and p_1,

$$\|u\|_{L^\infty} \leq C\|\omega\|_{L^{p_0}}^{\frac{p_0(p_1-n)}{n(p_1-p_0)}}\|\omega\|_{L^{p_1}}^{\frac{p_1(n-p_0)}{n(p_1-p_0)}}\ .\tag{3.8}$$

Proof. – For any $N \geq -1$

$$\begin{aligned}|u(x)-u(y)| &\leq \sum_{j=-1}^{N}|\Delta_j u(x) - \Delta_j u(y)| + 2\sum_{j=N+1}^{\infty}\|\Delta_j u\|_{L^\infty}\\ &\leq \sum_{j=-1}^{N}\|\Delta_j\nabla u\|_{L^\infty}|x-y| + C\sum_{j=N+1}^{\infty}2^{-j}\|\Delta_j\omega\|_{L^\infty}\\ &\leq \|\Delta_{-1}\nabla u\|_{L^\infty}|x-y| + C\sum_{j=0}^{N}\|\Delta_j\omega\|_{L^\infty}|x-y|\\ &\quad + C\sum_{j=N+1}^{\infty}2^{-j+\frac{nj}{p_1}}\|\Delta_j\omega\|_{L^{p_1}}\\ &\leq C\|\Delta_{-1}\omega\|_{L^{p_1}}|x-y| + \sum_{j=0}^{N}2^{nj/p_1}\|\omega\|_{L^{p_1}}|x-y|\\ &\quad + C2^{-N(1-\frac{n}{p_1})}\|\omega\|_{L^{p_1}}\\ &\leq C\|\omega\|_{L^{p_1}}\left(|x-y| + 2^{Nn/p_1}|x-y| + 2^{-N(1-\frac{n}{p_1})}\right)\ .\end{aligned}$$

We choose $N = [-\log_2|x-y|]$. This proves (3.7).

To estimate $\|u\|_{L^\infty}$ we proceed as in (3.6)

$$\|u\|_{L^\infty} \leq \|(1-\chi)\mathcal{K}\|_{L^{q_0}}\|\omega\|_{L^{p_0}} + \|\chi\mathcal{K}\|_{L^{q_1}}\|\omega\|_{L^{p_1}} \ .$$

Replacing $\chi(\cdot)$ by $\chi(\cdot/R)$ and choosing the appropriate R yields (3.8).

PROPOSITION 3.3. – *Let* $1 < p_0 < n$, $\omega \in L^{p_0}$. *Then* $u = \mathcal{K} * \omega \in L^{np_0/(n-p_0)}$ *and*

$$\|u\|_{L^{np_0/(n-p_0)}} \leq C\|\omega\|_{L^{p_0}} \ . \tag{3.9}$$

Proof. – Follows immediately from the Hardy-Littlewood-Sobolev inequality.

4. Action of volume preserving homeomorphisms on B_Γ

Let $g : \mathbb{R}^n \xrightarrow{\text{onto}} \mathbb{R}^n$ be a volume preserving homeomorphism. Assume there are decreasing functions $\sigma, \rho : \mathbb{R} \to (0,\infty)$, $\lim_{\xi\to\infty}\sigma(\xi) = 0$, $\lim_{\xi\to\infty}\rho(\xi) = 0$, satisfying the following

$$|g^{-1}(x) - g^{-1}(y)| \leq c_{g^{-1}}\sigma(-\log_2|x-y|) \ ; \tag{4.1}$$

$$|g(x) - g(y)| \leq c_g\rho(-\log_2|x-y|) \ ; \tag{4.2}$$

$$\sigma(\xi) = 2^{-\xi} \quad , \quad \rho(\xi) = 2^{-\xi} \ \text{ for } \ \xi \leq 0 \ ; \tag{4.3}$$

$$\log_2\sigma(\xi) \quad , \quad \log_2\rho(\xi) \ \text{ are convex.} \tag{4.4}$$

It follows from (4.3), (4.4) that

$$\sigma(\xi) \geq 2^{-\xi} \quad , \quad \rho(\xi) \geq 2^{-\xi} \ , \qquad \xi \in \mathbb{R} \ .$$

We need the following technical statement which is the main ingredient of the proof of the a priori estimate.

PROPOSITION 4.1. – *Assume* g, ρ, σ *satisfy the above properties and for* $m \geq -1$

$$f_m(x) = \sum_{k\in\mathbb{Z}^n} a_{m,k}\psi(2^m x - k) \ , \tag{4.5}$$

where

$$\sup_{k\in\mathbb{Z}^n}|a_{m,k}| \leq 1 \ , \tag{4.6}$$

$$\left[f_{-1}(x) = \sum_{k\in\mathbb{Z}^n} a_k\Psi(x-k)\right] .$$

Then there is a constant $\gamma = \gamma(n)$ *such that for* $j \leq m$

$$\|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} \leq C\,c_{g^{-1}}^n c_g^{n+1} 2^{\gamma j}\rho(m) \ . \tag{4.7}$$

Proof. – We have

$$\Delta_j(f_m \circ g^{-1})(x) = \sum_k a_{m,k} 2^{nj} \int \varphi(2^j y)\psi(2^m g^{-1}(x-y) - k)\,\mathrm{d}y \ . \tag{4.8}$$

Since the statement is evident for $j = m = -1$,

$$\left[\Big\| \sum_k a_k \Psi(x-k) \Big\|_{\mathrm{L}^\infty} \leq C \sup_k |a_k| \right]$$

we will assume that $m \geq 0$.

Let

$$\begin{aligned} h_{j,m,k}(x) &= 2^{nj} \int \varphi(2^j y)\psi(2^m g^{-1}(x-y) - k)\,\mathrm{d}y \\ &= \int \psi(2^m g^{-1}(x - 2^{-j}y) - k)\varphi(y)\,\mathrm{d}y \ , \end{aligned} \tag{4.9}$$

where we made a change of variables $y \to 2^{-j}y$. Then since $\int \psi(y)\,\mathrm{d}y = 0$,

$$\begin{aligned} & h_{j,m,k}(2^{-j}x + g(2^{-m}k)) \\ &= \int \psi(2^m g^{-1}(2^{-j}(x-y) + g(2^{-m}k)) - k)\varphi(y)\,\mathrm{d}y \\ &= \Big(\int_{|x-y| < \frac{|x|}{2}} + \int_{|x-y| \geq \frac{|x|}{2}} \Big) q(x-y) \int_0^1 \nabla\varphi(x + \tau(y-x)) \cdot (y-x)\,\mathrm{d}\tau\,\mathrm{d}y \\ &= I + J \ , \end{aligned} \tag{4.10}$$

where

$$q(z) = \psi(2^m g^{-1}(2^{-j}z + g(2^{-m}k)) - k) \ .$$

We estimate I first.

Since $\varphi \in S(\mathbb{R}^n)$ we have for any real M

$$\begin{aligned} |I| &\leq \int_{|x-y| < \frac{|x|}{2}} |q(x-y)|\,|x-y| \int_0^1 |\nabla\varphi(x + \tau(y-x))|\,\mathrm{d}\tau\,\mathrm{d}y \\ &\leq C \int_{|z| < \frac{|x|}{2}} |q(z)|\,|z|\,\mathrm{d}z\ (1+|x|)^{-M} \ . \end{aligned} \tag{4.11}$$

We substitute

$$z = 2^j(g(2^{-m}(w+k)) - g(2^{-m}k)) \ ; \tag{4.12}$$

$$g(2^{-m}(w+k)) = 2^{-j}z + g(2^{-m}k) \ ;$$

$$w = 2^m g^{-1}(2^{-j}z + g(2^{-m}k)) - k \ . \tag{4.13}$$

Since g is a volume preserving homeomorphism, for any real M_1, M_2 and since ρ is a decreasing function,

$$(4.14)\quad \int_{|z|<\frac{|x|}{2}} |q(z)|\,|z|\,\mathrm{d}z = 2^{(j-m)n+j} \int_{|z|<\frac{|x|}{2}} |\psi(w)|\,|g(2^{-m}(w+k)) - g(2^{-m}k)|\,\mathrm{d}w$$

$$\le 2^{(j-m)n+j} c_g \Big(\int_{|w|<2^m} + \int_{|w|\ge 2^m} \Big) |\psi(w)| \rho(m - \log_2 |w|)\,\mathrm{d}w$$

$$\le 2^{(j-m)n+j} c_g \int_{|w|\ge 2^m} |\psi(w)| 2^{-m} |w|\,\mathrm{d}w$$

$$+ 2^{(j-m)n+j} C c_g \sum_{\ell=-m}^{\infty} \rho(m+\ell) 2^{-n\ell} (1+2^{-\ell})^{-M_1}$$

$$\le C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \Big(\rho(0) 2^{-(M_1-n)m}$$

$$+ \rho(1) 2^{-(M_1-n)(m-1)} + \cdots + \rho(m) 2^0$$

$$+ \rho(m+1) 2^{-n} + \rho(m+2) 2^{-2n}$$

$$+ \cdots + \rho(m+\ell) 2^{-\ell n} + \cdots \Big)$$

$$\le C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \Big(\rho(0) 2^{-(M_1-n)m}$$

$$+ \rho(1) 2^{-(M_1-n)(m-1)} + \cdots + \rho(m) 2^0 \Big) .$$

LEMMA 4.2. – *Let ρ be any function that satisfies* $(4,3), (4.4)$. *Let* $M_3 > M_4 > 0$. *Then*

$$\rho(0)^{M_4} 2^{-M_3 m} + \rho(1)^{M_4} 2^{-M_3(m-1)} + \cdots + \rho(m)^{M_4} 2^0 \le C \rho(m)^{M_4} .$$

Proof. – Since $\log_2 \rho$ is convex,

$$\rho(0)^{M_4} 2^{-mM_3} + \rho(1)^{M_4} 2^{-(m-1)M_3} + \cdots + \rho(m)^{M_4} 2^0$$

$$\le \rho(0)^{M_4} 2^{-mM_3} + \rho(0)^{\frac{m-1}{m} M_4} \rho(m)^{\frac{1}{m} M_4} 2^{-(m-1)M_3} + \cdots + \rho(m)^{M_4} 2^0$$

$$\le \rho(m)^{M_4} \left\{ \left(\frac{\rho(0)}{\rho(m)}\right)^{\frac{m}{m} M_4} 2^{-M_3 m} + \left(\frac{\rho(0)}{\rho(m)}\right)^{\frac{m-1}{m} M_4} 2^{-M_3(m-1)} + \cdots + \left(\frac{\rho(0)}{\rho(m)}\right)^0 2^0 \right\}$$

$$\le \rho(m)^{M_4} \Big(1 - 2^{-M_3} \rho(m)^{-M_4/m} \Big)^{-1} .$$

But $\rho(m) \ge 2^{-m}$; therefore, $\rho(m)^{1/m} \ge \frac{1}{2}$. This implies the statement.

REMARK 4.3. – In particular,

$$\sigma(0)^{M_4} 2^{-M_3 m} + \sigma(1)^{M_4} 2^{-M_3(m-1)} + \cdots + \sigma(m)^{M_4} 2^0 \le C \sigma(m)^{M_4} .$$

To continue the proof of the proposition we note that the right side of (4.14) is

$$\le C c_g 2^{(j-m)(n+1)-mM_2} + C c_g 2^{(j-m)n+j} \rho(m) \le C c_g 2^{(j-m)n+j} \rho(m)$$

in case $M_2 \ge 0$.

Choosing $M_2 = 0$, $M_1 = n + 2$ we obtain from (4.11), (4.14)

$$|I| \le Cc_g 2^{(j-m)n+j} \rho(m)(1 + |x|)^{-M} .$$

We now turn to J in the right side of (4.10)

$$(4.15) \qquad |J| \le \int\limits_{|x-y| \ge \frac{|x|}{2}} |q(x-y)| \int_0^1 \mathrm{d}\tau |\nabla\varphi(x + \tau(y-x))| \, |y-x| \, \mathrm{d}y$$
$$\le C \int\limits_{|x-y| \ge \frac{|x|}{2}} |q(x-y)| \, |x-y| (1 + |x-y|)^{-1} \, \mathrm{d}y$$

since

$$\int_0^1 |\nabla\varphi(x + \tau(y-x))| \, \mathrm{d}\tau \le C(1 + |x-y|)^{-1} .$$

The right side of (4.15) after the substitution (4.12), (4.13) becomes

$$(4.16) \quad \le C \int\limits_{|z| \ge \frac{|x|}{2}} |q(z)| \, |z| (1 + |z|)^{M-1} \mathrm{d}z (1 + |x|)^{-M}$$
$$\le C \int\limits_{|z| \ge \frac{|x|}{2}} |q(z)| (|z| + |z|^M) \, \mathrm{d}z (1 + |x|)^{-M}$$
$$\le C 2^{(j-m)n} \int |\psi(w)| \{ 2^j |g(2^{-m}(w+k)) - g(2^{-m}k)|$$
$$+ 2^{Mj} |g(2^{-m}(w+k)) - g(2^{-m}k)|^M \} \, \mathrm{d}w (1 + |x|)^{-M}$$
$$\le C 2^{(j-m)n} \Big(\int\limits_{|w| \ge 2^m} + \int\limits_{|w| < 2^m} \Big) |\psi(w)| \Big((c_g 2^j \rho(m - \log_2 |w|)$$
$$+ c_g^M 2^{Mj} \rho(m - \log_2 |w|)^M \Big) \, \mathrm{d}w \, (1 + |x|)^{-M}$$
$$\le C 2^{(j-m)n} \int\limits_{|w| \ge 2^m} |\psi(w)| \Big(2^{j-m} c_g |w| + 2^{M(j-m)} c_g^M |w|^M \Big) \, \mathrm{d}w \, (1 + |x|)^{-M}$$
$$+ C 2^{(j-m)n} \Big(\sum_{\ell=-m}^{\infty} \Big\{ 2^{-n\ell} (1 + 2^{-\ell})^{-M_5} 2^j c_g \rho(m + \ell)$$
$$+ 2^{-n\ell} (1 + 2^{-\ell})^{-M_5} 2^{Mj} c_g^M \rho(m + \ell)^M \Big\} \Big) (1 + |x|)^{-M}$$
$$\le C c_g^M 2^{(j-m)(n+1) - M_6 m} (1 + |x|)^{-M}$$
$$+ \Big(C c_g 2^{(j-m)n+j} \rho(m) + C c_g^M 2^{(j-m)n + Mj} \rho(m)^M \Big) (1 + |x|)^{-M}$$
$$\le C 2^{(j-m)n} \Big(c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M \Big) (1 + |x|)^{-M}$$

for appropriate choice of M_5, M_6. Combining the estimates for I and J and using (4.10) we obtain $(m \geq j)$

$$(4.17)\quad \begin{aligned}\|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} &\leq \sup_x \sum_k |a_{m,k}|\,|h_{j,m,k}(x)| \\ &\leq \kappa_{j,m} \sup_k |a_{m,k}| \sup_x \sum_k (1 + 2^j|x - g(2^{-m}k)|)^{-M} \\ &\leq \kappa_{j,m} \sup_x \sum_k (1 + 2^j|x - g(2^{-m}k)|)^{-M} ,\end{aligned}$$

where

$$(4.18)\quad \kappa_{j,m} = C2^{(j-m)n}\Big(c_g 2^j \rho(m) + c_g^M 2^{Mj} \rho(m)^M\Big) .$$

Next we have to take care of the sum in the right side of (4.17). Let $x \in \mathbb{R}^n$ be fixed. Let

$$(4.19)\quad N(x, \lambda) = \#\ \{k \in \mathbb{Z}^n \mid |x - g(2^{-m}k)| \leq \lambda\} .$$

Then,

$$(4.20)\quad \begin{aligned}\sum_k \Big(1 + 2^j|x - g(2^{-m}k)|\Big)^{-M} &= \int_{-\infty}^{\infty} (1 + 2^j\lambda)^{-M} dN(x, \lambda) \\ &= M2^j \int_0^{\infty} (1 + 2^j\lambda)^{-M-1} N(x, \lambda)\, \mathrm{d}\lambda .\end{aligned}$$

We need the estimate for $N(x, \lambda)$.

$$(4.21)\quad \begin{aligned}N(x, \lambda) &\leq \#\Big\{k \in \mathbb{Z}^n \mid |g^{-1}(x) - 2^{-m}k| \leq c_{g^{-1}}\sigma(-\log_2 \lambda)\Big\} \\ &= \#\Big\{k \in \mathbb{Z}^n \mid |2^m g^{-1}(x) - k| \leq 2^m c_{g^{-1}}\sigma(-\log_2 \lambda)\Big\} \\ &\leq C(1 + 2^{mn} c_{g^{-1}}^n \sigma(-\log_2 \lambda)^n) .\end{aligned}$$

Combining (4.19)–(4.21) we obtain

$$(4.22)\quad \begin{aligned}&\sum_k (1 + 2^j|x - g(2^{-m}k)|)^{-M} \\ &\leq C2^j \int_0^{\infty} (1 + 2^j\lambda)^{-M-1}(1 + 2^{mn} c_{g^{-1}}^n \sigma(-\log_2 \lambda)^n)\, \mathrm{d}\lambda \\ &\leq C + Cc_{g^{-1}}^n 2^{mn} \int_0^{\infty} (1 + \lambda)^{-M-1} \sigma(j - \log_2 \lambda)^n\, \mathrm{d}\lambda .\end{aligned}$$

To estimate the integral we split

$$(4.23)\quad \begin{aligned}&\Big(\int_0^{2^j} + \int_{2^j}^{\infty}\Big)(1 + \lambda)^{-M-1}\sigma(j - \log_2 \lambda)^n\, \mathrm{d}\lambda \\ &\leq 2^{-jn} \int_{2^j}^{\infty} (1 + \lambda)^{-M-1}\lambda^n\, \mathrm{d}\lambda + C \sum_{\ell=-j}^{\infty} \sigma(j + \ell)^n 2^{-\ell}(1 + 2^{-\ell})^{-M-1} \\ &\leq C2^{-jM} + C\sigma(j)^n \leq C\sigma(j)^n ,\end{aligned}$$

provided M is sufficiently large ($M > n$). From (4.17)–(4.23) and since $\sigma(j) \geq 2^{-j}$

$$(4.24)\quad \|\Delta_j(f_m \circ g^{-1})\|_{L^\infty} \leq C2^{(j-m)n}(c_g 2^j \rho(m) + c_g^M 2^{Mj}\rho(m)^M)(1+2^{mn}c_{g^{-1}}^n \sigma(j)^n) \leq C c_{g^{-1}}^n 2^{jn}\sigma(j)^n (c_g 2^j \rho(m) + c_g^M 2^{Mj}\rho(m)^M)\ .$$

We now choose $M = (n+1)$. The right side of (4.24) is

$$\leq C c_{g^{-1}}^n c_g^{n+1} 2^{(2n+1)j}\rho(m)\ .$$

We set $\gamma = 2n+1$. This concludes the proof.

5. A priori estimate

Let Γ be a function $\mathbb{R} \to [1,\infty)$ satisfying the conditions (i)–(iii) above. Let $\Gamma_1(\alpha) = (\alpha+2)\Gamma(\alpha)$ for $\alpha \in [-1,\infty)$, $\Gamma_1(\alpha) = 1$ for $\alpha \in (-\infty,-1)$. We assume

(iv) Γ_1 satisfies the condition (iii);

(v) Γ_1 is convex;

(vi) $\int_1^\infty \Gamma_1(\alpha)^{-1}\,d\alpha = \infty$.

The conditions (i), (ii) are automatically satisfied for Γ_1. It is clear also that (ii)–(iv) follow from one condition

$$\lim \mathrm{ess}_{\alpha\to\infty}\Gamma'(\alpha)\Gamma(\alpha)^{-1} = 0\ .$$

Indeed, assuming this last condition, (ii) follows immediately while (iii) and (iv) are verified using integration by parts in the right side of (iii) (also Γ being replaced by Γ_1). For example, $\Gamma(\alpha) = \log_2^\kappa(\alpha+3)$, $\alpha \geq -1$, where $\kappa \in (0,1]$ satisfies all the conditions (i)–(vi). So does $\Gamma(\alpha) = \log_2(\alpha+3)\log_2\log_2(\alpha+5)$, etc.

We fix $f \in B_\Gamma \cap L^{p_0} \cap L^{p_1}$ where $1 < p_0 < 2 < p_1 < \infty$ and consider the Euler equation for the vorticity of an incompressible fluid:

$$(5.1)\qquad \dot\omega(x,t) = -(v,\nabla)\omega \equiv -\sum_{i=1}^{2} v_i\partial_i\omega\ ;\ (x,t) \in \mathbb{R}^2 \times [0,T_1]\ ,$$

$$(5.2)\qquad v = \mathcal{K} * \omega\ ,$$

$$(5.3)\qquad \omega(\cdot,0) = f(\cdot) \in B_\Gamma \cap L^{p_0} \cap L^{p_1}\ ,$$

where $\mathcal{K}$ is the kernel in Biot-Savart law. We fix $T_1 > 0$ and assume we are given a flow

$$(5.4)\qquad v(\cdot) \in \mathcal{K} * C([0,T_1]; B_{\Gamma_1} \cap L^{p_0} \cap L^{p_1})\ .$$

According to Theorem 3.1 the classical trajectories are well defined. Indeed, the conditions of the Osgood uniqueness theorem are satisfied as follows from (vi). We refer to [CL] and

[C1] for an appropriate version of the Osgood uniqueness theorem. Since $\operatorname{div} v = 0$, the corresponding flow is volume preserving.

We denote the flow by $g_v(t) = g(t)$, i.e.

$$\dot{g}_v(x,t) = v(g_v(x,t),t) \; ; \; g_v(x,0) = x \; , \qquad x \in \mathbb{R}^2 \; . \tag{5.5}$$

For any $t \in [0, T_1]$, $g_v(\cdot, t)$ is a volume preserving homeomorphism $\mathbb{R}^2 \overset{\text{onto}}{\longrightarrow} \mathbb{R}^2$.

Let $\lambda(t)$, $t \in [0,T]$, $0 < T \le T_1$ be a continuous positive nondecreasing function defined as follows:

$$\dot{\lambda} = C\lambda^2 \; , \tag{5.6}$$

$$\lambda(0) = \max(\|\omega(0)\|_{B_{\Gamma_1}}, 1) \tag{5.7}$$

The constant $C > 0$ will be chosen later.

THEOREM 5.1. – *Let $f \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$. Let $v(\cdot)$ be a regular solution to* (5.1)–(5.3) *satisfying* (5.4). *Let*

$$(\alpha + 2)\Gamma'(\alpha) \le C \textit{ for a.e. } \alpha \in [-1, \infty) \; ; \tag{5.8}$$

$$\|f\|_{B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}} \le C \; . \tag{5.9}$$

Then there is a $T > 0$, $T \le T_1$ such that

$$\|\omega(t)\|_{B_{\Gamma_1}} \le \lambda(t) \; , \qquad t \in [0,T] \; . \tag{5.9'}$$

Both $T > 0$ and the constant C in (5.6) *that defines λ depend only on the constant in* (5.9) *and on Γ. In addition, $\omega(t)$ is equimeasurable with f.*

REMARK 5.2. – Obviously (5.8) implies (vi).

Proof. – We define first $\lambda(\cdot)$ on the whole interval $[0, T_1]$ as follows:

$$\lambda(t) = \max(\sup_{0 \le \tau \le t} \|\omega(\tau)\|_{B_{\Gamma_1}}, 1) \; , \qquad t \in [0, T_1] \; .$$

Let C_0 be a sufficiently large constant such that (see (3.7))

$$|w(x) - w(y)| \le C_0 |x - y| \; , \qquad |x - y| \ge \frac{1}{2}$$

for any $w = \mathcal{K} * f_1$ where $f_1 \in \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$ is equimeasurable with f. In particular

$$|v(x,t) - v(y,t)| \le C_0 |x - y| \; , \qquad |x - y| \ge \frac{1}{2} \; , \; t \in [0, T_1] \; . \tag{5.10}$$

Also from (3.1) for C_0 sufficiently large

$$|v(x,t) - v(y,t)| \le C_0 \lambda(t) \Gamma_1(-\log_2 |x - y|) |x - y| \; , \qquad |x - y| \le \frac{1}{2} \; . \tag{5.11}$$

We define

$$\tilde\Gamma_1(m,t) = \begin{cases} \lambda(t)^{-1} , & -\infty < m \le m_1 \\ 1 + (m+1)\Gamma_1'(-1+) , & m_1 \le m \le -1 \\ \Gamma_1(m) , & -1 < m < \infty \end{cases}$$

with $m_1 = -1 - (1 - \lambda(t)^{-1})\Gamma_1'(-1+)^{-1}$.

Since $\Gamma_1'(-1+) \ge 1$, we have $-2 \le m_1 \le 1$. It is clear that $\tilde\Gamma_1$ is *convex* on $\mathbb{R}$. It follows from (5.10), (5.11) that

$$|v(x,t) - v(y,t)| \le C_0\lambda(t)\tilde\Gamma_1(-\log_2|x-y|, t)|x-y| , \qquad x \ne y . \tag{5.12}$$

We estimate stretching in the flow given by the vector-field $v(x,t)$, $t \in [0,T_1]$.

We denote by $\mu(t) = \mu(m,t)$ solution to the Cauchy problem

$$\dot\mu(m,t) = -C_0(\log_2 e)\lambda(t)\tilde\Gamma_1(\mu(m,t),t) , \qquad \mu(m,0) = m \in \mathbb{R} . \tag{5.13}$$

The solution exists and is unique for all $t \in [0,T_1]$ as follows easily from our construction of $\tilde\Gamma_1$.

PROPOSITION 5.2. – *Let $x, y \in \mathbb{R}^n$, $x \ne y$. Then*

$$|g_v(t)x - g_v(t)y| \le 2^{-\mu(m,t)} , \qquad t \in [0,T_1]$$

where $m = -\log_2|x-y|$.

Proof. – Let $\xi(t) = |g_v(t)x - g_v(t)y|$. Then (5.12) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\xi(t) &\le |v(g_v(t)x,t) - v(g_v(t)y,t)| \\ &\le C_0\lambda(t)\tilde\Gamma_1(-\log_2\xi(t),t)\xi(t) .\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(-\log_2\xi(t)) \ge -C_0\lambda(t)(\log_2 e)\tilde\Gamma_1(-\log_2\xi(t),t) .$$

Since the right side is *nonincreasing* on $\mathbb{R}$ as a function of $(-\log_2\xi(t))$ it follows from the Gronwall type of argument that

$$-\log_2\xi(t) \ge \mu(m,t) , \qquad t \in [0,T_1] .$$

This proves the proposition.

PROPOSITION 5.3. – *For any fixed $t \in [0,T_1]$ the function*

$$m \longmapsto \mu(m,t)$$

is concave.

Proof. – To simplify the notation we change time and reduce the equation (5.13) to

$$\dot\mu(m,t) = -\tilde\Gamma_1(\mu(m,t),t) , \qquad \mu(m,0) \equiv m .$$

Let $m_1, m_2 \in \mathbb{R}$, $m_1 \neq m_2$. For any $\theta \in (0,1)$

$$(\theta\mu(m_1,t) + (1-\theta)\mu(m_2,t))^{\cdot} = -\theta\tilde{\Gamma}_1(\mu(m_1,t),t)$$
$$-(1-\theta)\tilde{\Gamma}_1(\mu(m_2,t),t) \leq -\tilde{\Gamma}_1(\theta\mu(m_1,t) + (1-\theta)\mu(m_2,t),t) \ .$$

Hence,

$$-(\theta\mu(m_1,t) + (1-\theta)\mu(m_2,t))^{\cdot} \geq \tilde{\Gamma}_1(\theta\mu(m_1,t) + (1-\theta)\mu(m_2,t),t) \ .$$

Since the right side is a *nonincreasing* function of $-(\theta\mu(m_1,t)+(1-\theta)\mu(m_2,t))$ employing the same Gronwall's argument as before yields

$$-(\theta\mu(m_1,t) + (1-\theta)\mu(m_2,t)) \geq -\mu(\theta m_1 + (1-\theta)m_2,t) \ .$$

This completes the proof.

We now set

$$\tilde{\rho}(m,t) = 2^{-\mu(m,t)} \ .$$

It follows from Proposition 5.3 that $\log_2 \tilde{\rho}(m,t)$ is convex. For $m \leq -2$ $\mu(m,t) = m - C_0(\log_2 e)t$ thus $\tilde{\rho}(m,t) = e^{C_0 t}2^{-m}$.

We now set

$$\rho(m,t) = \frac{1}{4}e^{-C_0 t}\tilde{\rho}(m-2,t) \ ;$$
$$c_{g_v(t)} = 4e^{C_0 t} \ .$$

Then all the conditions for ρ at the beginning of §4 are satisfied for $g = g_v(t)$, $\rho(m) = \rho(m,t)$. In particular, (4.2) follows from Proposition 5.2.

There are similar statements concerning σ. We just replace $v(t)$ by $-v(T_1-t)$, $\tilde{\Gamma}_1(m,t)$ by $\tilde{\Gamma}_1(m,T_1-t)$, etc. Let $\eta(m,t)$ solve the Cauchy problem

$$\dot{\eta}(m,t) = -C_0(\log_2 e)\lambda(T_1-t)\tilde{\Gamma}_1(\eta(m,t),T_1-t) \ , \qquad \eta(m,0) = m \ .$$

PROPOSITION 5.2′. – *Let $x, y \in \mathbb{R}^n$, $x \neq y$. Then*

$$|g_{-v(T_1-\cdot)}(t)x - g_{-v(T_1-\cdot)}(t)y| \leq 2^{-\eta(m,t)} \ , \qquad t \in [0,T_1]$$

where $m = -\log_2|x-y|$.

PROPOSITION 5.3′. – *For any fixed $t \in [0,T_1]$ the function*

$$m \longmapsto \eta(m,t)$$

is concave.

We set

$$\sigma(m,T_1) = \frac{1}{4}e^{-C_0 T_1}2^{-\eta(m-2,T_1)} \ .$$

Then (4.1), (4.3) and (4.4) are satisfied for $g = g_v(T_1)$, $\sigma(m) = \sigma(m,T_1)$, $C_{g^{-1}} = 4e^{C_0 T_1}$. Replacing T_1 by arbitrary $t \in [0,T_1]$ we recover the desirable estimates for the homeomorphism $(g_v(t))^{-1}$.

To continue the proof of Theorem 5.1 we have to estimate $\|f\circ g_v(t)^{-1}\|_{B_{\Gamma_1}}$ for $t\in[0,T_1]$.

PROPOSITION 5.4. – *Let f, $g_v(t)$ $t\in[0,T_1]$ be as above. Then*

$$\|f\circ g_v(t)^{-1}\|_{\Gamma_1}\le C\|f\|_\Gamma 2^{C\int_0^t\lambda(\tau)\,\mathrm{d}\tau}\,,\qquad t\in[0,T_1]\,.$$

Proof. – Let $N\ge -1$. We use the "wavelet" decomposition (1.2) and Proposition 4.1. We have for arbitrary $m\ge N$ (see (4.5)):

$$\begin{aligned}(5.14)\qquad &\sum_{j=-1}^{N}\|\Delta_j(f\circ g_v(t)^{-1})\|_{L^\infty}\\ &\le\sum_{j=-1}^{N}\sum_{\ell=-1}^{\infty}\|\Delta_j(f_\ell\circ g_v(t)^{-1})\|_{L^\infty}\\ &=\sum_{j=-1}^{N}\Big(\sum_{\ell=-1}^{m}+\sum_{\ell=m+1}^{\infty}\Big)\|\Delta_j(f_\ell\circ g_v(t)^{-1})\|_{L^\infty}\\ &\le C(N+2)\sum_{\ell=-1}^{m}\|f_\ell\|_{L^\infty}+\sum_{j=-1}^{N}C2^{\gamma j}\sum_{\ell=m+1}^{\infty}\rho(\ell,t)\sup_{k\in\mathbb{Z}^n}|a_{\ell,k}|\\ &\le C(N+2)\Gamma(m)\|f\|_\Gamma+C2^{\gamma N}\sum_{\ell=m+1}^{\infty}\rho(\ell,t)\sup_{k\in\mathbb{Z}^n}|a_{\ell,k}|\,.\end{aligned}$$

We used in (5.14) that the constants $c_{g_v(t)^{\pm1}}$ are uniformly bounded with respect to $t\in[0,T_1]$.

The last sum in (5.14) requires some attention. We proceed as follows. Let

$$d_\ell=\sum_{j=-1}^{\ell}\sup_{k\in\mathbb{Z}^n}|a_{j,k}|\,.$$

Then,

$$\begin{aligned}(5.15)\qquad &\sum_{\ell=m+1}^{\infty}\rho(\ell,t)\sup_{k\in\mathbb{Z}^n}|a_{\ell,k}|\\ &=\sum_{\ell=m+1}^{\infty}\rho(\ell,t)(d_\ell-d_{\ell-1})\\ &=-d_m\rho(m+1,t)+\sum_{\ell=m+1}^{\infty}(-\rho(\ell+1,t)+\rho(\ell,t))\,d_\ell\\ &\le -d_m\rho(m+1,t)+\|f\|_\Gamma\sum_{\ell=m+1}^{\infty}(-\rho(\ell+1,t)+\rho(\ell,t))\Gamma(\ell)\\ &\le -d_m\rho(m+1,t)+\|f\|_\Gamma\sum_{\ell=m+1}^{\infty}\int_\ell^{\ell+1}-\partial_\xi\rho(\xi,t)\,\mathrm{d}\xi\;\Gamma(\ell)\\ &\le -d_m\rho(m+1,t)+C\|f\|_\Gamma\int_{m+1}^{\infty}-\partial_\xi\rho(\xi,t)\Gamma(\xi)\,\mathrm{d}\xi\end{aligned}$$

since Γ satisfies (ii). Since (5.8) is satisfied we can estimate the integral in the right side of (5.15) as follows:

$$\begin{aligned}\int_{m+1}^{\infty} & -\partial_\xi \rho(\xi,t)\Gamma(\xi)\,\mathrm{d}\xi \\ & \leq C\int_{m+1}^{\infty} \partial_\xi \mu(\xi-2,t)2^{-\mu(\xi-2,t)}\Gamma(\xi)\,\mathrm{d}\xi \\ & = C\int_{m-1}^{\infty} \partial_\xi \mu(\xi,t)2^{-\mu(\xi,t)}\Gamma(\xi+2)\,\mathrm{d}\xi \\ & \leq C\int_{m-1}^{\infty} \partial_\xi \mu(\xi,t)2^{-\mu(\xi,t)}\Gamma(\xi)\,\mathrm{d}\xi \\ & = C\int_{\mu(m-1,t)}^{\infty} 2^{-\mu}\Gamma(\xi(\mu,t))\,\mathrm{d}\mu\ .\end{aligned}$$

We have from (5.13)

$$\Gamma(\xi)-\Gamma(\mu) = C_0(\log_2 e)\int_0^t \lambda(\tau)\Gamma'(\mu(\tau))\tilde{\Gamma}_1(\mu(\tau),\tau)\,\mathrm{d}\tau\ . \tag{5.16}$$

This combined with (5.8) yields

$$\Gamma(\xi) \leq \Gamma(\mu)2^{C\int_0^t \lambda(\tau)\,\mathrm{d}\tau}\ .$$

Hence,

$$\begin{aligned}\sum_{\ell=m+1}^{\infty} & \rho(\ell,t)\sup_{k\in\mathbb{Z}^n}|a_{\ell,k}| \\ & \leq C\|f\|_\Gamma 2^{C\int_0^t \lambda(\tau)\,\mathrm{d}\tau}\Gamma(m)2^{-\mu(m-1,t)} \\ & \leq C\|f\|_\Gamma 2^{C\int_0^t \lambda(\tau)\,\mathrm{d}\tau}\Gamma(m)\rho(m+1,t)\ .\end{aligned} \tag{5.17}$$

The choice of m in (5.14) so far was arbitrary. We now choose m so that (assuming $N \geq 1$)

$$\mu(m-1,t) \geq \gamma N > \mu(m-2,t)\ . \tag{5.18}$$

As follows from (5.17) the second term in the right side of (5.14) is

$$\leq C\|f\|_\Gamma 2^{C\int_0^t \lambda(\tau)\,\mathrm{d}\tau}\Gamma(m)\ . \tag{5.19}$$

We have to estimate $\Gamma(m)$ as above

$$\Gamma(m) \leq \Gamma(\mu(m,t))2^{C\int_0^t \lambda(\tau)\,\mathrm{d}\tau}\ .$$

Furthermore, in general, for all m

$$\partial_m \mu(m,t) \leq 1\ .$$

Therefore, using (5.8)

$$\Gamma(m) \leq C\Gamma(\gamma N)2^{C\int_0^t \lambda(\tau)\,\mathrm{d}\tau} \leq C\Gamma(N)2^{C\int_0^t \lambda(\tau)\,\mathrm{d}\tau} . \tag{5.20}$$

It follows from (5.14), (5.19), (5.20) that for $N \geq 1$

$$\sum_{j=-1}^{N} \|\Delta_j(f \circ g_v^{-1}(t))\|_{\mathrm{L}^\infty} \leq C\Gamma_1(N)\|f\|_\Gamma 2^{C\int_0^t \lambda(\tau)\,\mathrm{d}\tau} . \tag{5.21}$$

This concludes the proof of the Proposition.

We can now finish the proof of the Theorem. Let $\tilde{\lambda}(\cdot)$ satisfy the following ordinary differential equation:

$$\dot{\tilde{\lambda}} = C\tilde{\lambda}^2 , \qquad \tilde{\lambda}(0) = \lambda(0) , \ t \in [0, T] , \tag{5.22}$$

where the constant C is sufficiently large. The time $T > 0$ is chosen to be less than the blow-up time for (5.22). Then (5.9′) is satisfied because (5.21) implies $\lambda(t) \leq \tilde{\lambda}(t)$, $t \in [0, T]$. In addition to this $f \circ g_v(t)^{-1}$ is equimeasurable with f.

This concludes the proof.

We now give an a priori estimate on an infinite interval.

THEOREM 5.5. – *Let $f \in B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$ where Γ satisfies (i)–(v). Let $v(t)$ be a regular solution to* (5.1)–(5.3) *such that*

$$v(\cdot) \in \mathcal{K} * C_{\mathrm{loc}}([0,\infty); B_{\Gamma_1} \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}) .$$

Assume

$$\Gamma'(\alpha)\Gamma_1(\alpha) \leq C \ \ \textit{a.e.} \ \ \alpha \geq -1 ; \tag{5.23}$$

$$\|f\|_{B_\Gamma \cap \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}} \leq C . \tag{5.24}$$

Then there exists a positive continuous nondecreasing function $\lambda(\cdot)$ on $[0,\infty)$ such that

$$\|\omega(t)\|_{B_{\Gamma_1}} \leq \lambda(t) , \qquad t \in [0, \infty) \tag{5.25}$$

This function depends only on the constant in (5.24) *and on* Γ.

Proof. – We have from (5.16), (5.23)

$$\Gamma(\xi) \leq \Gamma(\mu) + C\int_0^t \lambda(\tau)\,\mathrm{d}\tau . \tag{5.26}$$

Therefore,

$$\int_{m+1}^{\infty} -\partial_\xi \rho(\xi,t)\Gamma(\xi)\,\mathrm{d}\xi \leq C\rho(m+1,t)\left\{\Gamma(\mu(m,t)) + \int_0^t \lambda(\tau)\,\mathrm{d}\tau\right\} . \tag{5.27}$$

This yields

$$\sum_{\ell=m+1}^{\infty} \rho(\ell,t) \sup_{k\in\mathbb{Z}^n} |a_{\ell,k}| \leq C\|f\|_\Gamma \rho(m+1,t)\left\{\Gamma(\mu(m,t)) + \int_0^t \lambda(\tau)\,\mathrm{d}\tau\right\} .$$

We choose m so that (5.18) is satisfied. Then the second term in the right side of (5.14) is

$$(5.28)\qquad \begin{aligned} &\leq C\|f\|_\Gamma\left\{\Gamma(\mu(m,t)) + \int_0^t \lambda(\tau)\,\mathrm{d}\tau\right\} \\ &\leq C\|f\|_\Gamma\left\{\Gamma(N) + \int_0^t \lambda(\tau)\,\mathrm{d}\tau\right\} . \end{aligned}$$

Using (5.26), (5.28), (5.14) we get for $N \geq 1$

$$\begin{aligned} \sum_{j=-1}^{N} \|\Delta_j(f\circ g_v(t)^{-1})\|_{\mathrm{L}^\infty} &\leq C\|f\|_\Gamma N\left(\Gamma(N) + \int_0^t \lambda(\tau)\,\mathrm{d}\tau\right) \\ &\leq C\left(1 + \int_0^t \lambda(\tau)\,\mathrm{d}\tau\right)\|f\|_\Gamma \Gamma_1(N) . \end{aligned}$$

We fix an arbitrary $T > 0$ and choose $\tilde{\lambda}$ to satisfy the initial value problem

$$\dot{\tilde{\lambda}} = C\tilde{\lambda} , \quad \tilde{\lambda}(0) = C , \qquad t \in [0,T] ,$$

where C is a sufficiently large constant. Then

$$\|f\circ g_v(t)^{-1}\|_{B_{\Gamma_1}} \leq \lambda(t) \leq \tilde{\lambda}(t) , \qquad t\in[0,T] .$$

This concludes the proof.

EXAMPLE 5.6. – The function

$$\Gamma(\alpha) = \log_2^\kappa(\alpha+3) , \qquad \alpha \geq -1$$

satisfies (5.23) for $0 < \kappa \leq \frac{1}{2}$.

6. Commutator estimates

Let u be a vector field, $\mathrm{div} u = 0$, $u \in \mathrm{L}^\infty$, $\nabla u \in B_\Pi$. Let the function $\Pi : \mathbb{R} \to [1,\infty)$ satisfies the conditions (i)–(iii) above. Let (e.g.) $w \in \mathrm{L}^\infty$. We define as in [BC] for $j \geq 1$

$$R_j(u,w) = \Delta_j(u,\nabla)w - (S_{j-2}u,\nabla)\Delta_j w .$$

We use the normalization $\hat{\Phi}(\xi) + \sum_{j=0}^{\infty} \hat{\varphi}(2^{-j}\xi) = 1$ with the same conditions on the support as in §1. Then $-\hat{\Phi}(\xi/2) + \hat{\Phi}(\xi) = \hat{\varphi}(\xi)$.

For $|j-j'|\geq 2$ we have $\Delta_j\Delta_{j'}=0$. We also use the standard notation (when a and b are functions or distributions)

$$T_a b = \sum_{j=-1}^{\infty} S_{j-2}a\Delta_j b\ ;$$

$$R(a,b) = \sum_{|j-j'|\leq 1} \Delta_j a\Delta_{j'} b\ .$$

Then Bony's [B] formula (see also [M1]) reads

$$a\cdot b = T_a b + T_b a + R(a,b)\ .$$

THEOREM 6.1. – *There is an absolute constant M_0 so that the following inequality holds true:*

$$\text{(6.1)}\quad \|R_j(u,w)\|_{L^\infty} \leq C \sum_{|j'-j|\leq M_0} \Big\{\|S_{j'-2}\nabla w\|_{L^\infty}\|\Delta_{j'}u\|_{L^\infty} + \|S_{j'-2}\nabla u\|_{L^\infty}\|\Delta_{j'}w\|_{L^\infty}\Big\}$$
$$+ C2^j \sum\nolimits'_{\substack{j'\geq j-M_0\\ |j'-j''|\leq 1}} 2^{-j'}\|\Delta_{j'}\nabla u\|_{L^\infty}\|\Delta_{j''}w\|_{L^\infty}\ .$$

When $j'=-1$ the factor $\|\Delta_{j'}\nabla u\|_{L^\infty}$ in the last sum ought to be replaced by $\|\Delta_{-1}u\|_{L^\infty}$.

Here and below the notation $\sum'$ is to indicate this convention: $\|\Delta_{-1}u\|_{L^\infty}$ instead of $\|\Delta_{-1}\nabla u\|$ (also for u_1, w in the next section).

REMARK 6.2. – The first term in the first sum could be estimated as follows:

$$\|S_{j'-2}\nabla w\|_{L^\infty}\|\Delta_{j'}u\|_{L^\infty} \leq C\|S_{j'-2}w\|_{L^\infty}\|\Delta_{j'}\nabla u\|_{L^\infty}\ ,\qquad j'\geq -1\ .$$

Notice that both the left side and the right side vanish for $j'\leq 0$.

Proof. – The proof will follow very closely the arguments of H. Bahouri and J.-Y. Chemin [BC]. We have

$$R_j(u,w) = \sum_{\ell=1}^{4} R_j^\ell(u,w)\ , \tag{6.2}$$

where

$$\text{(6.3)}\quad \begin{cases} R_j^1(u,w) = \sum_{k=1}^{n} \Delta_j T_{\partial_k w} u_k \\ R_j^2(u,w) = -\sum_{k=1}^{n} [T_{u_k}\partial_k, \Delta_j] w \\ R_j^3(u,w) = \sum_{k=1}^{n} T_{u_k - S_{j-2}u_k}\partial_k\Delta_j w \\ R_j^4(u,w) = \sum_{k=1}^{n} \{\Delta_j R(u_k, \partial_k w) - R(S_{j-2}u_k, \Delta_j\partial_k w)\}\ . \end{cases}$$

Indeed,

$$\begin{aligned}
\Delta_j(u,\nabla)w &= \Delta_j \sum_{k=1}^{n} u_k \partial_k w \\
&= \Delta_j \sum_{k=1}^{n} \{T_{\partial_k w} u_k + T_{u_k} \partial_k w + R(u_k, \partial_k w)\} \\
&= \sum_{\ell=1}^{2} R_j^\ell(u,w) + \sum_{k=1}^{n} \{T_{u_k} \partial_k \Delta_j w + \Delta_j R(u_k, \partial_k w)\} \\
&= \sum_{\ell=1}^{3} R_j^\ell(u,w) + \sum_{k=1}^{n} \{T_{S_{j-2}u_k} \partial_k \Delta_j w + \Delta_j R(u_k, \partial_k w)\} \\
&= \sum_{\ell=1}^{4} R_j^\ell(u,w) + \sum_{k=1}^{n} \Big\{T_{S_{j-2}u_k} \partial_k \Delta_j w + T_{\partial_k \Delta_j w} S_{j-2} u_k \\
&\qquad + R(S_{j-2}u_k, \Delta_j \partial_k w) - T_{\partial_k \Delta_j w} S_{j-2} u_k \Big\} \\
&= \sum_{\ell=1}^{4} R_j^\ell + (S_{j-2}u, \nabla)\Delta_j w - \sum_{k=1}^{n} T_{\partial_k \Delta_j w} S_{j-2} u_k \ .
\end{aligned}$$

We claim the last term is vanishing.

Indeed,

$$\sum_{k=1}^{n} T_{\partial_k \Delta_j w} S_{j-2} u_k = \sum_{k=1}^{n} \sum_{j'} S_{j'-2} \partial_k \Delta_j w \cdot \Delta_{j'} S_{j-2} u_k \ .$$

The second factor vanishes for $j' \geq j$. The first factor vanishes for $j' \leq j$. This proves (6.2). We estimate the terms in the right side of (6.2) in L^∞. We have

$$R_j^1(u,w) = \sum_{k=1}^{n} \Delta_j \sum_{j'} S_{j'-2} \partial_k w \Delta_{j'} u_k \ .$$

All the terms in this sum with $|j - j'| > M_0$ vanish. Hence,

$$\|R_j^1(u,w)\|_{\mathrm{L}^\infty} \leq \sum_{k=1}^{n} \sum_{|j'-j| \leq M_0} \|S_{j'-2} \partial_k w\|_{\mathrm{L}^\infty} \|\Delta_{j'} u_k\|_{\mathrm{L}^\infty} \ . \tag{6.4}$$

We may assume $j' \geq 0$ in the right side of (6.4) since otherwise $S_{j'-2}\partial_k w = 0$. Hence,

$$\|\Delta_{j'} u_k\|_{\mathrm{L}^\infty} \leq C 2^{-j} \|\Delta_{j'} \nabla u\|_{\mathrm{L}^\infty} \ . \tag{6.5}$$

Also

$$\|S_{j'-2} \partial_k w\|_{\mathrm{L}^\infty} \leq C 2^{j} \|S_{j-2} w\|_{\mathrm{L}^\infty} \ . \tag{6.6}$$

The combination of (6.4)–(6.6) yields

$$\begin{aligned}
\|R_j^1(u,w)\|_\infty &\leq C \sum_{|j'-j| \leq M_0} \|S_{j'-2} \nabla w\|_{\mathrm{L}^\infty} \|\Delta_{j'} u\|_{\mathrm{L}^\infty} \\
&\leq C \sum_{|j'-j| \leq M_0} \|S_{j'-2} w\|_{\mathrm{L}^\infty} \|\Delta_{j'} \nabla u\|_{\mathrm{L}^\infty} \ , \qquad j \geq -1 \ .
\end{aligned} \tag{6.7}$$

We now study $R_j^2(u,w)$

$$\begin{aligned} R_j^2(u,w) &= -\sum_{j'}\sum_{k=1}^{n}[S_{j'-2}u_k\partial_k\Delta_{j'},\Delta_j]w \\ &= -\sum_{k=1}^{n}\sum_{j'}[S_{j'-2}u_k,\Delta_j]\partial_k\Delta_{j'}w \\ &= -\sum_{k=1}^{n}\sum_{|j'-j|\le M_0}[S_{j'-2}u_k,\Delta_j]\partial_k\Delta_{j'}w\ . \end{aligned}$$

We use as in [BC] the following explicit representation of the operator in the right side

$$\begin{aligned} &R_j^2(u,w)(x) \\ &\quad = -\sum_{k=1}^{n}\sum_{|j'-j|\le M_0} 2^{jn}\int \varphi(2^j(x-y))(S_{j'-2}u_k(x)-S_{j'-2}u_k(y))\partial_k\Delta_{j'}w(y)\,\mathrm{d}y \\ &\quad = -\sum_{|j'-j|\le M_0}\sum_{k=1}^{n} 2^{j(n+1)}\int \partial_k\varphi(2^j(x-y))(S_{j'-2}u_k(x)-S_{j'-2}u_k(y))\Delta_{j'}w(y)\,\mathrm{d}y \end{aligned}$$

since $\mathrm{div}S_{j'-2}u(y) = S_{j'-2}\mathrm{div}u(y) = 0$.

Applying the Taylor's formula we obtain

$$\begin{aligned} &R_j^2(u,w)(x) \\ &\quad = -\sum_{|j'-j|\le M_0}\sum_{k=1}^{n} 2^{j(n+1)}\int \partial_k\varphi(2^j(x-y)) \\ &\quad \sum_{m=1}^{n}\int_0^1 S_{j'-2}\partial_m u_k(x+\tau(y-x))\cdot(x_m-y_m)\,\mathrm{d}\tau\Delta_{j'}w(y)\,\mathrm{d}y \\ &\quad = -\sum_{|j'-j|\le M_0}\sum_{k=1}^{n}\sum_{m=1}^{n}\int \partial_k\varphi(z)z_m\int_0^1 S_{j'-2}\partial_m u_k(x-\tau 2^{-j}z)\Delta_{j'}w(x-2^{-j}z)\,\mathrm{d}\tau\,\mathrm{d}z\ . \end{aligned}$$

Minkowski integral inequality yields:

$$\begin{aligned} (6.8)\qquad \|R_j^2(u,w)\|_{\mathrm{L}^\infty} &\le C\sum_{m=1}^{n}\sum_{k=1}^{n}\sum_{|j'-j|\le M_0}\|S_{j'-2}\partial_m u_k\|_{\mathrm{L}^\infty}\|\Delta_{j'}w\|_{\mathrm{L}^\infty} \\ &\le C\sum_{|j'-j|\le M_0}\|S_{j'-2}\nabla u\|_{\mathrm{L}^\infty}\|\Delta_{j'}w\|_{\mathrm{L}^\infty}\ . \end{aligned}$$

Next we estimate the term $R_j^3(u,w)$. The argument similar to the one used for R_j^1 leads to the estimate

$$(6.9)\qquad \|R_j^3(u,w)\| \le C\sum_{m=j-1}^{j}\|\Delta_m\nabla u\|_{\mathrm{L}^\infty}\|\Delta_j w\|_{\mathrm{L}^\infty}\ ,\qquad j\ge -1\ ,$$

except when $m = -1$ the factor $\|\Delta_{-1}\nabla u\|_{L^\infty}$ ought to be replaced by $\|\Delta_{-1}u\|_{L^\infty}$. We now turn to $R_j^4(u,w)$:

$$R_j^4(u,w) = R_j^{4,1}(u,w) + R_j^{4,2}(u,w) \ , \tag{6.10}$$

$$R_j^{4,1}(u,w) = \sum_{k=1}^{n} \Delta_j \partial_k R(u_k - S_{j-2}u_k, w) \ , \tag{6.11}$$

$$R_j^{4,2}(u,w) = \sum_{k=1}^{n} \Delta_j R(S_{j-2}u_k, \partial_k w) - R(S_{j-2}u_k, \Delta_j \partial_k w) \ . \tag{6.12}$$

We have:

$$R_j^{4,1}(u,w) = \sum_{k=1}^{n} \partial_k \Delta_j \sum_{|j'-j''|\le 1} \Delta_{j'}(u_k - S_{j-2}u_k)\Delta_{j''} w \ .$$

Therefore,

$$\begin{aligned} \|R_j^{4,1}(u,w)\|_{L^\infty} &\le C2^j \sum_{j'\ge j-M_0} \sum_{|j'-j''|\le 1} \|\Delta_{j'}u\|_{L^\infty}\|\Delta_{j''}w\| \\ &\le C2^j \sum_{j'\ge j-M_0} \sum_{|j'-j''|\le 1} 2^{-j'}\|\Delta_{j'}\nabla u\|_{L^\infty}\|\Delta_{j''}w\|_{L^\infty} \ , \end{aligned} \tag{6.13}$$

except when $j' = -1$ the factor $\|\Delta_{j'}\nabla u\|_{L^\infty}$ ought to be replaced by $\|\Delta_{-1}u\|_{L^\infty}$. We now estimate the term $R_j^{4,2}(u,w)$:

$$\begin{aligned} R_j^{4,2}(u,w) &= \sum_{k=1}^{n} \sum_{|j'-j''|\le 1} \Big\{\Delta_j((\Delta_{j'}S_{j-2}u_k)\cdot \Delta_{j''}\partial_k w) \\ &\qquad - (\Delta_{j'}S_{j-2}u_k)(\Delta_{j''}\Delta_j\partial_k w)\Big\} \\ &= \sum_{k=1}^{n} \sum_{j-1\ge j'\ge j-M_0} \sum_{|j'-j''|\le 1} [\Delta_j, \Delta_{j'}S_{j-2}u_k]\Delta_{j''}\partial_k w \ . \end{aligned} \tag{6.14}$$

Using the explicit integral representation as above and proceeding as with the term R_j^2 we arrive at

$$\begin{aligned} \|R_j^{4,2}(u,w)\|_{L^\infty} &\le C \sum_{\substack{|j'-j|\le M_0 \\ |j'-j''|\le 1}} \|\Delta_{j'}S_{j-2}\nabla u\|_{L^\infty}\|\Delta_{j''}w\|_{L^\infty} \\ &\le C \sideset{}{'}\sum_{\substack{|j'-j|\le M_0 \\ |j'-j''|\le 1}} \|\Delta_{j'}\nabla u\|_{L^\infty}\|\Delta_{j''}w\|_{L^\infty} \end{aligned} \tag{6.15}$$

Adding up the estimates (6.7), (6.8), (6.9), (6.13) and (6.15) and using (6.2), (6.3), (6.10)–(6.12) yield the statement of the Theorem.

7. Uniqueness theorem

Let Π be a function $\mathbb{R} \to [1, \infty)$ that satisfies the conditions (i)–(iii) above. In this section the dimension $n \geq 2$ is *arbitrary*.

THEOREM 7.1 (Uniqueness theorem). – *Let the following conditions be satisfied:*

$$\int_1^\infty \Pi(\alpha)^{-1}\, \mathrm{d}\alpha = \infty \; ; \tag{7.1}$$

$$\Pi(\alpha)2^{-\alpha} \textit{ is nonincreasing for } \alpha \geq C \;, \; \lim_{\alpha\to\infty} \Pi(\alpha)2^{-\alpha} = 0 \;. \tag{7.1'}$$

Let for $t \in [0, T]$, $u_1(x,t)$, $u_2(x,t)$ *be two vector fields in* $\mathbb{R}^n$ *and* $\omega_{1,2} = \operatorname{curl} u_{1,2}$. *Assume the following:*

$$\omega_{1,2} \in \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0}) \;, \; \|\omega_{1,2}(\cdot)\|_\Pi \in \mathrm{L}^\infty([0,T]) \; ; \tag{7.2}$$

$$u_{1,2} = \mathcal{K} * \omega_{1,2} \; ; \tag{7.3}$$

$$\operatorname{div} u_{1,2} = 0 \; ; \tag{7.4}$$

$$\dot{u}_{1,2} = -(u_{1,2}, \nabla)u_{1,2} - \nabla p_{1,2} \; ; \tag{7.5}$$

$$\omega_{1,2}(\cdot, 0) = f(\cdot) \in B_\Pi \cap \mathrm{L}^{p_0} \;, \tag{7.6}$$

where $p_0 \in (1, n)$.
Then, $u_1 = u_2$ *on* $[0, T]$.

Proof. – Let $w = u_1 - u_2$, $\omega = \operatorname{curl} w$, $p = p_1 - p_2$. Then,

$$\dot{w} = -(u_1, \nabla)w - (w, \nabla)u_2 - \nabla p \;, \qquad t \in [0, T] \; ; \tag{7.7}$$

$$\operatorname{div} w = 0 \; ; \tag{7.8}$$

$$w\big|_{t=0} = 0 \;. \tag{7.9}$$

Note that $w \in \mathrm{L}^\infty(\mathbb{R}^n \times [0,T])$ as follows from (7.2), (7.3) and Theorem 3.1.

It follows from (7.7) that for any $j \geq -1$

$$\begin{aligned} \Delta_j \dot{w} = &- (S_{j-2}u_1, \nabla)\Delta_j w - R_j(u_1, w) \\ &- (S_{j-2}w, \nabla)\Delta_j u_2 - R_j(w, u_2) \\ &- \Delta_j \nabla p \;. \end{aligned} \tag{7.10}$$

We have according to Theorem 6.1

$$\|R_j(u_1,w)\|_{L^\infty} \le C \sum_{|j'-j|\le M_0} \Big\{ \|S_{j'-2}\nabla u_1\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} + \|S_{j'-2}\nabla w\|_{L^\infty} \|\Delta_{j'} u_1\|_{L^\infty} \Big\} + C2^j \sum\nolimits'_{\substack{j'\ge j-M_0\\ |j'-j''|\le 1}} 2^{-j'} \|\Delta_{j'}\nabla u_1\|_{L^\infty} \|\Delta_{j''} w\|_{L^\infty} , \tag{7.11}$$

where $\sum'$ indicates the same agreement as in Theorem 6.1.

Likewise,

$$\|R_j(w,u_2)\|_{L^\infty} \le C \sum_{|j'-j|\le M_0} \{ \|S_{j'-2}\nabla w\|_{L^\infty} \|\Delta_{j'} u_2\|_{L^\infty} + \|S_{j'-2}\nabla u_2\|_{L^\infty} \|\Delta_{j'} w\|_{L^\infty} \} + C2^j \sum\nolimits'_{\substack{j'\ge j-M_0\\ |j'-j''|\le 1}} 2^{-j'} \|\Delta_{j'}\nabla w\|_{L^\infty} \|\Delta_{j''} u_2\|_{L^\infty} . \tag{7.12}$$

We have to estimate $\Delta_j \nabla p$.

Taking div of both sides of (7.10) we get

$$\Delta_j \Delta p = -\mathrm{div} R_j(u_1,w) - \mathrm{div} R_j(w,u_2) - \mathrm{tr}(\nabla\Delta_j w \cdot \nabla S_{j-2} u_1) - \mathrm{tr}(\nabla \Delta_j u_2 \cdot \nabla S_{j-2} w) . \tag{7.12'}$$

Here Δ in the left side stands for Laplacian and we have used the solenoidality condition for w and u_2.

We consider two cases.

Case I. $j \ge 0$. Then from (7.10), (7.8), (7.12′) since

$$-\nabla\Delta_j p(x) = \mathcal{F}_{\xi\to x}(i\xi|\xi|^{-2}(\Delta_j \Delta p)^\wedge(\xi)) ,$$

we can use a standard Littlewood-Paley argument in conjunction with the Bernstein's inequality and arrive at

$$\begin{aligned} \|\Delta_j \nabla p\|_{L^\infty} &\le C\|R_j(u_1,w)\|_{L^\infty} + C\|R_j(w,u_2)\|_{L^\infty} \\ &\quad + C2^{-j}\|S_{j-2}\nabla u_1\|_{L^\infty}\|\Delta_j \nabla w\|_{L^\infty} \\ &\quad + C2^{-j}\|S_{j-2}\nabla w\|_{L^\infty}\|\Delta_j \nabla u_2\|_{L^\infty} \\ &\le C\|R_j(u_1,w)\|_{L^\infty} + C\|R_j(w,u_2)\|_{L^\infty} \\ &\quad + C\|S_{j-2}\nabla u_1\|_{L^\infty}\|\Delta_j w\|_{L^\infty} \\ &\quad + C\|S_{j-2}\nabla w\|_{L^\infty}\|\Delta_j u_2\|_{L^\infty} . \end{aligned} \tag{7.13}$$

Case II. $j=-1$. Then from (7.7) applying the Bernstein's inequality, choosing arbitrary $p_2 \in [\frac{np_0}{n-p_0}, \infty)$ and using boundedness of H. Weyl projection onto gradients we get

$$
\begin{aligned}
(7.14)\qquad \|\Delta_{-1}\nabla p\|_{L^\infty} &\le C\|\Delta_{-1}\nabla p\|_{L^{p_2}} \\
&\le C\Big\| -\sum_{k=1}^{n} \partial_k \Delta_{-1}\{u_{1k}w + w_k u_2\}\Big\|_{L^{p_2}} \\
&\le C\|\Delta_{-1}(u_1 \otimes w)\|_{L^{p_2}} + C\|\Delta_{-1}(w\otimes u_2)\|_{L^{p_2}} \\
&\le C \sum_{|j'-j''|\le M_0} (\|\Delta_{j'}u_1\|_{L^{p_2}} + \|\Delta_{j'}u_2\|_{L^{p_2}})\|\Delta_{j''}w\|_{L^\infty}\ .
\end{aligned}
$$

We now fix an $N \ge -1$ and estimate the sum

$$
\begin{aligned}
(7.15)\qquad &\sum_{j=-1}^{N} \|R_j(u_1,w)\|_{L^\infty} + \sum_{j=-1}^{N} \|R_j(w,u_2)\|_{L^\infty} \\
&\quad + \sum_{j=-1}^{N} \|\Delta_j \nabla p\|_{L^\infty} + \sum_{j=-1}^{N} \|(S_{j-2}w,\nabla)\Delta_j u_2\|_{L^\infty} \equiv Q_1+Q_2+Q_3+Q_4\ .
\end{aligned}
$$

We will prove below that

$$
Q_j \le C\Pi(N)\|w\|_{B^0_{\infty,1}}\ , \qquad j=1,\dots,4\ .
$$

Here the norm in Besov space $B^0_{\infty,1}$ is defined as follows:

$$
B^0_{\infty,1} = \left\{ w\in S' \ \Big|\ \|w\|_{B^0_{\infty,1}} = \sum_{j=-1}^{\infty} \|\Delta_j w\|_{L^\infty} < \infty \right\}.
$$

The inequality (7.11) yields (see Remark 6.2)

$$
\begin{aligned}
(7.16)\qquad Q_1 &\le C \sum_{j=-1}^{N} \sum_{|j'-j|\le M_0} \Big\{ \|S_{j'-2}\nabla u_1\|_{L^\infty}\|\Delta_{j'}w\|_{L^\infty} + \|S_{j'-2}w\|_{L^\infty}\|\Delta_{j'}\nabla u_1\|_{L^\infty}\Big\} \\
&\quad + C\sum_{j=-1}^{N} 2^j \sum_{\substack{j'\ge j-M_0 \\ |j'-j''|\le 1}}{}' 2^{-j'} \|\Delta_{j'}\nabla u_1\|_{L^\infty}\|\Delta_{j''}w\|_{L^\infty}\ .
\end{aligned}
$$

The first sum in (7.16) can be taken care of as follows. Using (7.2), (7.3) and Proposition 3.3 we obtain:

$$
\begin{aligned}
(7.17)\qquad &\sum_{j=-1}^{N} \sum_{|j'-j|\le M_0} \{\|S_{j'-2}\nabla u_1\|_{L^\infty}\|\Delta_{j'}w\|_{L^\infty} + \|S_{j'-2}w\|_{L^\infty}\|\Delta_{j'}\nabla u_1\|_{L^\infty}\} \\
&\quad \le C\Big(\sup_{-1\le j'\le N+M_0} \|S_{j'-2}\nabla u_1\|_{L^\infty}\Big) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'}w\|_{L^\infty}
\end{aligned}
$$

$$
\begin{aligned}
&+C\Big(\sum_{j'=-1}^{N+M_0}\|\Delta_{j'}\nabla u_1\|_{\mathrm{L}^\infty}\Big)\sup_{-1\le j'\le N+M_0}\|S_{j'-2}w\|_{\mathrm{L}^\infty}\\
&\le C\|\omega_1\|_{B_\Pi\cap \mathrm{L}^{p_0}}\Pi(N+M_0)\sum_{j'=-1}^{N+M_0}\|\Delta_{j'}w\|_{\mathrm{L}^\infty}\\
&\le C\Pi(N)\sum_{j'=-1}^{N+M_0}\|\Delta_{j'}w\|_{\mathrm{L}^\infty}\ .
\end{aligned}
$$

We now turn to the second sum in the right side of (7.16)

$$
\begin{aligned}
(7.18)\quad &\sum_{j=-1}^{N}2^j\sideset{}{'}\sum_{\substack{j'\ge j-M_0\\ |j'-j''|\le 1}}2^{-j'}\|\Delta_{j'}\nabla u_1\|_{\mathrm{L}^\infty}\|\Delta_{j''}w\|_{\mathrm{L}^\infty}\\
&\le C\sum_{j''=-1}^{\infty}\|\Delta_{j''}w\|_{\mathrm{L}^\infty}\Big(\sum_{j=-1}^{\min(N,j''+M_0+1)}2^{j-j''}\Big)\sideset{}{'}\sum_{|j'-j''|\le 1}\|\Delta_{j'}\nabla u_1\|_{\mathrm{L}^\infty}\\
&\le C2^{\min(N,j'')-j''}\sideset{}{'}\sum_{|j'-j''|\le 1}\|\Delta_{j'}\nabla u_1\|_{\mathrm{L}^\infty}\Big(\sum_{j''=-1}^{\infty}\|\Delta_{j''}w\|_{\mathrm{L}^\infty}\Big)\ .
\end{aligned}
$$

We estimate the coefficient in front of $\sum_{j''=1}^{\infty}\|\Delta_{j''}w\|_{\mathrm{L}^\infty}$. We consider two cases. In case $j''\le N$

$$
(7.19)\qquad 2^{\min(N,j'')-j''}\sideset{}{'}\sum_{|j'-j''|\le 1}\|\Delta_{j'}\nabla u_1\|_{\mathrm{L}^\infty}\le C\Pi(N)\|\omega_1\|_{B_\Pi\cap \mathrm{L}^{p_0}}\le C\Pi(N)\ .
$$

In case $j''>N$

$$
\begin{aligned}
(7.20)\qquad &2^{N-j''}\sideset{}{'}\sum_{|j'-j''|\le 1}\|\Delta_{j'}\nabla u_1\|_{\mathrm{L}^\infty}\\
&\le C2^{N-j''}\Pi(j'')\|\omega_1\|_{B_\Pi\cap \mathrm{L}^{p_0}}\\
&\le C2^{N-j''}\Pi(j'')\le C\int_N^\infty 2^{N-\xi}\Pi(\xi)\,\mathrm{d}\xi\\
&\le C\Pi(N)\ .
\end{aligned}
$$

The combination of (7.18)–(7.20) yields

$$
(7.21)\qquad \sum_{j=-1}^{N}2^j\sideset{}{'}\sum_{\substack{j'\ge j-M_0\\ |j'-j''|\le 1}}2^{-j'}\|\Delta_{j'}\nabla u_1\|_{\mathrm{L}^\infty}\|\Delta_{j''}w\|_{\mathrm{L}^\infty}\le C\Pi(N)\sum_{j=-1}^{\infty}\|\Delta_j w\|_{\mathrm{L}^\infty}\ .
$$

It follows from (7.16), (7.17), (7.21) that

$$
(7.22)\qquad Q_1\le C\Pi(N)\sum_{j=-1}^{\infty}\|\Delta_j w\|_{\mathrm{L}^\infty}\ .
$$

We now estimate Q_2. From (7.12)

$$Q_2 \le C \sum_{j=-1}^{N} \sum_{|j'-j|\le M_0} \Big\{ \|S_{j'-2}w\|_{L^\infty} \|\Delta_{j'}\nabla u_2\|_{L^\infty} + \|S_{j'-2}\nabla u_2\|_{L^\infty} \|\Delta_{j'}w\|_{L^\infty} \Big\} + C \sum_{j=-1}^{N} 2^j \sideset{}{'}\sum_{\substack{j'\ge j-M_0\\ |j'-j''|\le 1}} 2^{-j'} \|\Delta_{j'}\nabla w\|_{L^\infty} \|\Delta_{j''}u_2\|_{L^\infty} . \tag{7.23}$$

We take care of the first sum in the right side of (7.23) as follows:

$$\begin{aligned}\sum_{j=-1}^{N} \sum_{|j'-j|\le M_0} &\Big\{ \|S_{j'-2}w\|_{L^\infty} \|\Delta_{j'}\nabla u_2\|_{L^\infty} + \|S_{j'-2}\nabla u_2\|_{L^\infty} \|\Delta_{j'}w\|_{L^\infty} \Big\} \\ &\le C\Big(\sup_{-1\le j'\le N+M_0} \|S_{j'-2}w\|_{L^\infty}\Big) \sum_{j'=-1}^{N+M_0} \|\Delta_{j'}\nabla u_2\|_{L^\infty} \\ &\quad + C\Big(\sup_{-1\le j'\le N+M_0} \|S_{j'-2}\nabla u_2\|_{L^\infty}\Big) \sum_{j'=-1}^{N+M_0} \|\Delta_j w\|_{L^\infty} \\ &\le C\Pi(N) \sum_{j'=-1}^{N+M_0} \|\Delta_j w\|_{L^\infty}\end{aligned} \tag{7.24}$$

using the same argument as in (7.17). To estimate the second sum in (7.23) we notice that

$$\begin{aligned}\sum_{j=-1}^{N} 2^j \sideset{}{'}\sum_{\substack{j'\ge j-M_0\\ |j'-j''|\le 1}} &2^{-j'} \|\Delta_{j'}\nabla w\|_{L^\infty} \|\Delta_{j''}u_2\|_{L^\infty} \\ &\le C \sum_{j=-1}^{N} 2^j \sum_{\substack{j'\ge j-M_0\\ |j'-j''|\le 1}} \|\Delta_{j'}w\|_{L^\infty} \|\Delta_{j''}u_2\|_{L^\infty} \\ &\le C \sum_{j'=-1}^{\infty} \|\Delta_{j'}w\|_{L^\infty} \Big(\sum_{j=-1}^{\min(N,j'+M_0)} 2^j \Big) \sum_{|j''-j'|\le 1} \|\Delta_{j''}u_2\|_{L^\infty} \\ &\le C \sum_{j=-1}^{\infty} \|\Delta_{j'}w\|_{L^\infty} 2^{\min(N,j')} \sum_{|j''-j'|\le 1} \|\Delta_{j''}u_2\|_{L^\infty} .\end{aligned} \tag{7.25}$$

In case $j' \le N$,

$$\begin{aligned}2^{\min(N,j')} \sum_{|j''-j'|\le 1} \|\Delta_{j''}u_2\|_{L^\infty} &\le C 2^{j'} 2^{-j'} \Pi(j') \|\omega_2\|_{B_\Pi \cap L^{p_0}} \\ &\le C\Pi(N) .\end{aligned} \tag{7.26}$$

In the opposite case $j' > N$,

$$2^{\min(N,j')} \sum_{|j''-j'|\le 1} \|\Delta_{j''} u_2\|_{\mathrm{L}^\infty} \le C 2^{N-j'} \Pi(j') \|\omega_2\|_{B_\Pi \cap \mathrm{L}^{p_0}}$$

$$(7.27) \qquad \le C 2^{N-j'} \Pi(j') \le C \int_N^\infty 2^{N-\xi} \Pi(\xi)\, d\xi$$

$$\le C\Pi(N) \; .$$

We conclude from (7.23)–(7.27) that

$$(7.28) \qquad Q_2 \le C\Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{\mathrm{L}^\infty} \; .$$

We turn now to Q_3 in the right side of (7.15). From (7.13), (7.14), (7.15), (7.22), (7.28) [notice that the last two terms in the right side of (7.13) are already present in the estimates (7.11), (7.12) for $\|R_j(u_1, w)\|_{\mathrm{L}^\infty}$, $\|R_j(w, u_2)\|_{\mathrm{L}^\infty}$] we obtain:

$$(7.29) \qquad Q_3 \le C\Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{\mathrm{L}^\infty} + C\|\Delta_{-1} \nabla p\|_{\mathrm{L}^\infty}$$

$$\le C\Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{\mathrm{L}^\infty} \; .$$

Indeed, the factor in the right side on (7.14) is taken care of as follows:

$$\|\Delta_{j'} u_1\|_{\mathrm{L}^{p_2}} + \|\Delta_{j'} u_2\|_{\mathrm{L}^{p_2}} \le C\|u_1\|_{\mathrm{L}^{p_2}} + C\|u_2\|_{\mathrm{L}^{p_2}} \; .$$

We choose $p_2 = \frac{np_0}{n-p_0}$. Then from (3.9)

$$\|u_{1,2}\|_{\mathrm{L}^{p_2}} \le C\|\omega_{1,2}\|_{\mathrm{L}^{p_0}} \le C \; .$$

$$(7.29') \qquad Q_4 \le \sum_{j=-1}^{N} \|S_{j-2} w\|_{\mathrm{L}^\infty} \|\Delta_j \nabla u_2\|_{\mathrm{L}^\infty}$$

$$\le \sup_{-1\le j\le N} \|S_{j-2} w\|_{\mathrm{L}^\infty} \sum_{j=-1}^{N} \|\Delta_j \nabla u_2\|_{\mathrm{L}^\infty}$$

$$\le C\Pi(N) \sum_{j=-1}^{N-2} \|\Delta_j w\|_{\mathrm{L}^\infty}$$

$$\le C\Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{\mathrm{L}^\infty} \; .$$

It follows from (7.15), (7.22), (7.28), (7.29), (7.29′) that

$$(7.30) \qquad \sum_{j=-1}^{N} \|R_j(u_1, w)\|_{\mathrm{L}^\infty} + \sum_{j=-1}^{N} \|R_j(w, u_2)\|_{\mathrm{L}^\infty}$$

$$+ \sum_{j=-1}^{N} \|\Delta_j \nabla p\|_{\mathrm{L}^\infty} + \sum_{j=-1}^{N} \|(S_{j-2} w, \nabla)\Delta_j u_2\|_{\mathrm{L}^\infty}$$

$$\le C\Pi(N) \sum_{j=-1}^{\infty} \|\Delta_j w\|_{\mathrm{L}^\infty} \; .$$

Solving the equation (7.10) along characteristics and using (7.9) $[\Delta_j w|_{t=0} = 0]$ we obtain, for any $t \in [0,T]$ and any $j \geq -1$,

$$(7.31)\quad \|\Delta_j w(t)\|_{L^\infty} \leq C \int_0^t \Big\{ \|R_j(u_1(\tau), w(\tau))\|_{L^\infty} + \|R_j(w(\tau), u_2(\tau))\|_{L^\infty} + \|\Delta_j \nabla p(\tau)\|_{L^\infty} + \|(S_{j-2}w(\tau), \nabla)\Delta_j u_2(\tau)\|_{L^\infty} \Big\}\, d\tau \ .$$

We have for arbitrary $t \in [0,T]$, $N \geq -1$

$$(7.32)\quad \sum_{j=-1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} = \sum_{j=-1}^{N} \|\Delta_j w(t)\|_{L^\infty} + \sum_{j=N+1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} \leq \sum_{j=-1}^{N} \|\Delta_j w(t)\|_{L^\infty} + C \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \nabla w(t)\|_{L^\infty} \ .$$

We estimate the second term in the right side of (7.32). Let

$$(7.33)\quad d_m = \sum_{k=-1}^{m} \|\Delta_j \nabla w(t)\|_{L^\infty} \ .$$

Then, from (7.2)

$$(7.34)\quad \sum_{j=N+1}^{\infty} 2^{-j} \|\Delta_j \nabla w(t)\|_{L^\infty} = \sum_{j=N+1}^{\infty} 2^{-j}(d_j - d_{j-1}) = -2^{-(N+1)} d_N + \sum_{j=N+1}^{\infty} (2^{-j} - 2^{-(j+1)})\, d_j \leq C \sum_{j=N+1}^{\infty} 2^{-j} \Pi(j) \|\omega(t)\|_{B_\Pi \cap L^{p_0}} \leq C 2^{-N} \Pi(N) \ .$$

Using (7.30)–(7.34) we get:

$$(7.35)\quad \sum_{j=-1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} \leq C\Pi(N) \int_0^t \sum_{j=-1}^{\infty} \|\Delta_j w(\tau)\|_{L^\infty}\, d\tau + C 2^{-N} \Pi(N) \ .$$

We denote by $\zeta(t)$ the following function on $[0,T]$

$$(7.36)\quad \zeta(t) = \int_0^t \sum_{j=-1}^{\infty} \|\Delta_j w(\tau)\|_{L^\infty}\, d\tau \ .$$

Since

$$\Big\| \sum_{j=-1}^{\infty} \|\Delta_j w(\cdot)\|_{L^\infty} \Big\|_{L^\infty([0,T])} \leq C \Big\| \|\omega(\cdot)\|_\Pi \Big\|_{L^\infty([0,T])} + C\|\omega\|_{L^\infty([0,T];L^{p_0})} \leq C \ ,$$

$\zeta(t)$ is monotonically nondecreasing absolutely continuous function. Since $\zeta(0) = 0$ there exists a $t_0 \in [0,T]$ such that

$$\zeta(t) \equiv 0 \text{ on } [0,t_0] \quad ; \quad \zeta(t) > 0 \text{ on } (t_0,T] \ .$$

If $t_0 = T$ then the uniqueness follows. Therefore, we assume $t_0 < T$. Let ε be small enough so that $t_0 + \varepsilon < T$ and $\zeta(t) < 2^{-M_1-1}$ on $(t_0, t_0+\varepsilon)$. We will choose M_1 later. For any $t \in (t_0, t_0 + \varepsilon)$ we choose N in (7.35) as follows:

$$N = [-\log_2 \zeta(t)],$$

where the square bracket means the entire part of a real number. Then (7.35) implies:

$$\dot{\zeta}(t) \le C\Pi(-\log_2 \zeta(t)) \cdot \zeta(t) \ , \qquad \zeta(0) = 0 \ . \tag{7.37}$$

Notice that from (7.1)

$$\int_0^{1/2} \zeta^{-1}\Pi(-\log_2 \zeta)^{-1} \mathrm{d}\zeta = (\log_2 e)^{-1} \int_1^{\infty} \Pi(\alpha)^{-1}\, \mathrm{d}\alpha = \infty \ .$$

Therefore, the Osgood uniqueness theorem applies to the initial value problem

$$\dot{\eta}(t,\delta) = C\Pi(-\log_2 \eta)\eta \ , \qquad \eta(t_0,\delta) = \delta \ , \tag{7.38}$$

where $\delta > 0$ is small. The solution η exists on $(t_0, t_0 + \varepsilon)$ and depends continuously on the parameter δ. The simple Gronwall's argument now works for comparison of ζ and η. Then $\zeta(t) < \eta(t,\delta)$ for sufficiently small δ and for all $t \in [t_0, t_0+\varepsilon)$. For otherwise let $t_1 = \min_{t\in[t_0,t_0+\varepsilon)}\{t \mid \zeta(t) = \eta(t,\delta)\}$. From (7.37), (7.38)

$$\begin{aligned} \zeta(t_1) &\le C \int_{t_0}^{t_1} \Pi(-\log_2 \zeta(\tau))\zeta(\tau)\, \mathrm{d}\tau \\ &< \delta + C \int_{t_0}^{t_1} \Pi(-\log_2 \eta(\tau,\delta))\eta(\tau,\delta)\, \mathrm{d}\tau = \eta(t_1,\delta) \ . \end{aligned} \tag{7.39}$$

We have to use in (7.39) that the function $\zeta \mapsto \Pi(-\log_2 \zeta)\zeta$ is *monotonically* decreasing to 0 as $\zeta \to 0+$. This is just a reformulation of (7.1′).

We choose M_1 in such a way that $\Pi(\alpha)2^{-\alpha}$ is nonincreasing for $\alpha \ge M_1$. Now (7.39) contradicts the definition of t_1. Passing to the limit as $\delta \to 0+$ we have to conclude that $\zeta \equiv 0$ on $[t_0, t_0+\varepsilon)$. But this contradicts the definition of t_0. The proof of uniqueness is now completed.

COROLLARY 7.2. – *Assume* (7.3), (7.4), (7.5). *Let, in addition to this,*

$$\omega_{1,2} \in \mathrm{L}^\infty([0,T];\mathrm{L}^{p_0}) \ ;$$
$$\|\omega_{1,2}(\cdot)\|_{\mathrm{bmo}} \in \mathrm{L}^\infty([0,T]) \ ;$$
$$\omega_{1,2}(\cdot,0) = f(\cdot) \in \mathrm{bmo} \cap \mathrm{L}^{p_0} \ ,$$

where $p_0 \in (1,n)$. *Then* $u_1 = u_2$ *on* $[0,T]$.

Proof. – Let $\Pi(N) = (N+2)$, $N \ge -1$. Then $\mathrm{bmo} \hookrightarrow B_\Pi$. But

$$\int_1^\infty (N+2)^{-1}\mathrm{d}N = \infty$$

so (7.1) is satisfied. Condition (7.1′) is satisfied as well. Therefore uniqueness follows from the Theorem 7.1.

REMARK 7.3. – The same proof works for the space $\Lambda^0 = B^0_{\infty,\infty}$ replacing the space bmo.

8. Construction of the flow

In this section $n = 2$. Let Γ satisfy the conditions (i)–(v).

THEOREM 8.1. – *Assume* (5.8). *Let*

$$f \in B_\Gamma \cap L^{p_0} \cap L^{p_1} , \qquad 1 < p_0 < 2 < p_1 < \infty . \tag{8.1}$$

Then there exists a $T > 0$ *and a solution* $\omega(\cdot)$ *to the Euler equations*

$$\dot{u} = -(u, \nabla)u - \nabla p , \qquad \omega(0) = f , \tag{8.2}$$

$$u = \mathcal{K} * \omega , \qquad \mathrm{div} u = 0 , \tag{8.3}$$

satisfying the following condition

$$\omega(\cdot) \in L^\infty([0,T]; L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0,T]; B_{\Gamma_1}) , \tag{8.4}$$

where $C_{w^*}([0,T]; B_{\Gamma_1})$ *stands for the space of weak* continuous functions with values in* B_{Γ_1} *in the sense of duality* $H'_{\Gamma_1} = B_{\Gamma_1}$.

THEOREM 8.2. – *Assume* (5.23), (8.1). *Then there exists a solution*

$$\omega(\cdot) \in L^\infty_{\mathrm{loc}}([0,\infty); L^{p_0} \cap L^{p_1}) \cap C_{w^*}([0,\infty); B_{\Gamma_1})$$

to the problem (8.2), (8.3).

REMARK 8.3. – The solution $\omega(\cdot)$ in both theorems is unique provided (7.1′) is satisfied. Indeed, (5.8) implies (7.1) and uniqueness follows from the Theorem 7.1 with $\Pi = \Gamma_1$. Likewise (5.23) implies (7.1) with the same Π.

REMARK 8.4. – See the definition of H_{Γ_1} in Section 1 (also the Proposition 1.6).

To construct the solution in Theorems 8.1, 8.2 we use approximation by regular solutions. We need therefore the following result.

THEOREM 8.5 ([C1]). – *Let* $n = 2$, $p \in (1,\infty)$, $f \in \cap_{r>0}\Lambda^r \cap L^p$. *Then there exists a (unique) solution*

$$\omega(\cdot) \in \cap_{r>0} L^\infty_{\mathrm{loc}}([0,\infty); \Lambda^r \cap L^p)$$

to the problem (8.2), (8.3).

Here Λ^r, $r \in \mathbb{R}$ denotes the Hölder space. For convenience we recall the definition.

DEFINITION 8.6. – Let $r \in \mathbb{R}$.

$$\Lambda^r = B^r_{\infty,\infty} = \left\{ f \in S' \mid \|f\|_{\Lambda^r} = \sup_{j \ge -1} 2^{rj} \|\Delta_j f\|_{L^\infty} < \infty \right\} .$$

In particular for $r > 0$, $\Lambda^r \subset L^\infty$.

PROPOSITION 8.7. – *For* $r > 0$, $\Lambda^r \subset B_\Gamma$.

Proof.

$$\sum_{j=-1}^{N} \|\Delta_j f\|_{L^\infty} \le \sum_{j=-1}^{N} 2^{-rj}\|f\|_{\Lambda^r} \le (1-2^{-r})^{-1}\|f\|_{\Lambda^r} \le C\Gamma(N)\ , \qquad N \ge -1\ .$$

Proof of Theorem 8.1. – We construct for any $m \ge 1$ the solution

$$\omega_m(\cdot) \in \cap_{r>0} L^\infty_{\mathrm{loc}}([0,\infty); \Lambda^r \cap L^{p_0}) \tag{8.5}$$

provided by Theorem 8.5 such that

$$\omega_m(0) = S_m f \in \cap_{r>0}\Lambda^r\ . \tag{8.6}$$

It follows from Littlewood-Paley theory and (8.1) that

$$\|\omega_m(0)\|_{L^{p_0}} \le C \quad , \quad \|\omega_m(0)\|_{L^{p_1}} \le C\ . \tag{8.7}$$

We also have

$$\|\omega_m(0)\|_\Gamma = \sup_{N\ge -1} \sum_{j=-1}^{N} \|\Delta_j \omega_m(0)\|_{L^\infty}\Gamma(N)^{-1}\ . \tag{8.8}$$

But $\Delta_j S_m = 0$ for $j \ge m+2$; $\Delta_j S_m = \Delta_j$ for $j \le m-2$ and in any case

$$\|\Delta_j S_m f\|_{L^\infty} = \|S_m \Delta_j f\|_{L^\infty} \le C\|\Delta_j f\|_{L^\infty}\ . \tag{8.9}$$

Using properties of Γ we conclude from (8.8) that

$$\|\omega_m(0)\|_\Gamma \le C\|f\|_\Gamma\ . \tag{8.10}$$

In addition (8.5) together with Proposition 8.7 imply

$$\omega_m(\cdot) \in L^\infty_{\mathrm{loc}}([0,\infty); B_{\Gamma_1})\ . \tag{8.11}$$

From Theorem 5.1 using (8.7), (8.10), (8.11) we find a $T > 0$ and a constant $C > 0$ independent of m such that

$$\|\omega_m\|_{L^\infty([0,T];L^{p_0}\cap L^{p_1})} \le C\ ,\ \|\omega_m\|_{L^\infty([0,T];B_{\Gamma_1})} \le C\ . \tag{8.12}$$

Let $m > \ell \ge 1$ are two indices. We use the estimates of §7 where by abusing notation we set $\Pi = \Gamma_1$,

$$u_m = \mathcal{K} * \omega_m \quad , \quad u_\ell = \mathcal{K} * \omega_\ell\ , \tag{8.13}$$

$$w = u_m - u_\ell = \mathcal{K} * (\omega_m - \omega_\ell)\ . \tag{8.14}$$

The estimate (7.31) ought to be replaced by

$$\begin{aligned} \|\Delta_j w(t)\|_{L^\infty} \le \|\Delta_j w(0)\|_{L^\infty} + C\int_0^t \Big\{ &\|R_j(u_m(\tau), w(\tau))\|_{L^\infty} \\ &+ \|R_j(w(\tau), u_\ell(\tau))\|_{L^\infty} + \|\Delta_j \nabla p(\tau)\|_{L^\infty} \\ &+ \|(S_{j-2}w(\tau), \nabla)\Delta_j u_\ell(\tau)\|_{L^\infty} \Big\}\, d\tau \end{aligned} \tag{8.15}$$

since $\Delta_j w(0)$ does not have to vanish. Combining (7.30), (8.15), (7.32)–(7.34) yields:

$$\sum_{j=-1}^{\infty} \|\Delta_j w(t)\|_{L^\infty} \le \sum_{j=-1}^{N} \|\Delta_j w(0)\|_{L^\infty} + C\Pi(N)\int_0^t \sum_{j=-1}^{\infty} \|\Delta_j w(\tau)\|_{L^\infty}\,d\tau + C2^{-N}\Pi(N) \tag{8.16}$$

for any $N \ge -1$.

We use the same function $\zeta(t)$ as in (7.36). The estimate (8.16) implies

$$\dot\zeta(t) \le \sum_{j=-1}^{\infty} \|\Delta_j w(0)\|_{L^\infty} + C\Gamma_1(-\log_2 \zeta(t))\zeta(t) \tag{8.17}$$

for $t \in [0,T]$ by making the same choice of N as in §7.

We now observe that

$$\begin{aligned} \kappa_{m,\ell} &= \sum_{j=-1}^{\infty} \|\Delta_j w(0)\|_{L^\infty} = \sum_{j=-1}^{\infty} \|\Delta_j \mathcal{K} * (S_m - S_\ell) f\|_{L^\infty} \\ &= \sum_{j=-1}^{\infty} \|\Delta_j \mathcal{K} * \sum_{q=\ell+1}^{m} \Delta_q f\|_{L^\infty} \\ &\le \sum_{q=\ell+1}^{m} \sum_{|j-q|\le 1} \|\mathcal{K} * \Delta_j \Delta_q f\|_{L^\infty} \\ &\le C \sum_{q=\ell+1}^{\infty} \|\mathcal{K}\Delta_q f\|_{L^\infty} \\ &\le C \sum_{q=\ell+1}^{\infty} 2^{-q} \|\Delta_q f\|_{L^\infty}\ . \end{aligned} \tag{8.18}$$

Let $d_m = \sum_{j=-1}^{m} \|\Delta_j f\|_{L^\infty}$. Then (8.18) implies

$$\begin{aligned} \kappa_{m,\ell} &\le C \sum_{q=\ell+1}^{\infty} 2^{-q}(d_q - d_{q-1}) \\ &= -C2^{-(\ell+1)} d_\ell + C \sum_{q=\ell+1}^{\infty} 2^{-q-1}\, d_q \\ &\le C \sum_{q=\ell+1}^{\infty} 2^{-q}\Gamma(q) \le C \int_{\ell+1}^{\infty} 2^{-\xi}\Gamma(\xi)\,d\xi \\ &\le C2^{-\ell}\Gamma(\ell)\ . \end{aligned} \tag{8.19}$$

In (8.19) we used properties of Γ. Integrating both sides of (8.17) and using (8.18), (8.19) we obtain

$$\zeta(t) \le C2^{-\ell}\Gamma(\ell) + C\int_0^t \Gamma_1(-\log_2 \zeta(\tau))\zeta(\tau)\,d\tau\ .$$

Using monotonicity of $\Gamma_1(-\log\zeta)\cdot\zeta$ for small $\zeta\geq 0$ as in §8 we apply a similar Gronwall's argument to conclude

$$\zeta(t)\leq\eta(t,C2^{-\ell}\Gamma(\ell))\ ,\qquad t\in[0,T]\ . \tag{8.20}$$

Going back to (8.17) and using again (8.18)–(8.19) and monotonicity of $\Gamma_1(-\log\zeta)\cdot\zeta$ for small $\zeta\geq 0$ we obtain

$$\begin{aligned}\dot{\zeta}(t)\leq C2^{-\ell}\Gamma(\ell)+C\int_0^t &\Gamma_1\big(-\log_2\eta(\tau,C2^{-\ell}\Gamma(\ell))\big)\\ &\cdot\eta(\tau,C2^{-\ell}\Gamma(\ell))\,\mathrm{d}\tau\ ,\qquad t\in[0,T]\ .\end{aligned}$$

Therefore, $\{u_m(\cdot)\}_{m=1}^{\infty}$ is a Cauchy sequence in the Banach space $\mathrm{L}^\infty([0,T];B^0_{\infty,1})$. We remind that the norm in the Besov space $B^0_{\infty,1}$ is defined as follows

$$B^0_{\infty,1}=\left\{w\in S'\ \Big|\ \|w\|_{B^0_{\infty,1}}=\sum_{j=-1}^{\infty}\|\Delta_j w\|_{\mathrm{L}^\infty}<\infty\right\}\ .$$

Therefore, there exists a strong limit in $\mathrm{L}^\infty([0,T];B^0_{\infty,1})$

$$u_m\to u\in\mathrm{L}^\infty([0,T];B^0_{\infty,1})\ \text{ as }\ m\to\infty\ . \tag{8.21}$$

We claim now that in fact $\omega=\mathrm{curl}\,u$ satisfies

$$\|\omega(\cdot)\|_{\Gamma_1}\in\mathrm{L}^\infty([0,T])\ . \tag{8.22}$$

Let $N\geq -1$ be an arbitrary integer.

Consider the seminorm on $\mathrm{L}^\infty([0,T];B^0_{\infty,1})$

$$\rho_N(v)=\left\|\sum_{j=-1}^{N}\|\Delta_j v(\cdot)\|_{L^\infty}\right\|_{\mathrm{L}^\infty([0,T])}\ .$$

Obviously $\rho_N\ (u_m-u)\to 0$ as $m\to\infty$. On the other hand, from Bernstein's inequality

$$\left\|\sum_{j=-1}^{N}\|\Delta_j(\omega_m-\omega)(\cdot)\|_{\mathrm{L}^\infty}\right\|_{\mathrm{L}^\infty([0,T])}\leq C2^N\rho_N(u_m-u)\ . \tag{8.23}$$

Hence,

$$\begin{aligned}&\left\|\sum_{j=-1}^{N}\|\Delta_j\omega_m(\cdot)\|_{\mathrm{L}^\infty}-\sum_{j=-1}^{N}\|\Delta_j\omega(\cdot)\|_{\mathrm{L}^\infty}\right\|_{\mathrm{L}^\infty([0,T])}\\ &\quad\leq\left\|\sum_{j=-1}^{N}\Big|\ \|\Delta_j\omega_m(\cdot)\|_{\mathrm{L}^\infty}-\|\Delta_j\omega(\cdot)\|_{\mathrm{L}^\infty}\Big|\ \right\|_{\mathrm{L}^\infty([0,T])}\\ &\quad\leq\left\|\sum_{j=-1}^{N}\|\Delta_j(\omega_m-\omega)(\cdot)\|_{\mathrm{L}^\infty}\right\|_{\mathrm{L}^\infty([0,T])}\\ &\quad\leq C2^N\rho_N(u_m-u)\to 0\ \text{ as }\ m\to\infty\ ,\end{aligned} \tag{8.24}$$

where on the last step we used (8.23). From (8.12) we have

$$\sum_{j=-1}^{N} \|\Delta_j \omega_m(t)\|_{\mathrm{L}^\infty} \le C\Gamma_1(N) \ , \tag{8.25}$$

where the constant C does not depend on m. Passing to the limit as $m \to \infty$ in (8.24) and using (8.25) yields

$$\Big\| \sum_{j=-1}^{N} \|\Delta_j \omega(\cdot)\|_{\mathrm{L}^\infty} \Big\|_{\mathrm{L}^\infty([0,T])} \le C\Gamma_1(N) \ .$$

This implies (8.22).

In fact since $\{\omega_m\}$ is a Cauchy sequence in $C([0,T]; B^0_{\infty,1})$ this argument shows that $\omega(t) \in B_{\Gamma_1}$ *for all* $t \in [0,T]$. We note that (8.21) implies

$$u_m \to u \ \text{ strongly in } \ \mathrm{L}^\infty(\mathbb{R}^2 \times [0,T]) \cap C(\mathbb{R}^2 \times [0,T]) \ . \tag{8.26}$$

We have in addition the following facts after possibly choosing a subsequence

$$\omega_m \to \omega \ \text{ weak* in } \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0}) \text{ and in } \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_1}) \ ; \tag{8.27}$$

$$\dot{u}_m \to \dot{u} \ \text{ weak* in } \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_0}) \text{ and in } \mathrm{L}^\infty([0,T]; \mathrm{L}^{p_1}) \ . \tag{8.28}$$

Let $\rho \in S$, $\mathrm{div}\rho = 0$ is a test function. Also let $\theta \in \mathcal{D}([0,T))$. Then, by construction

$$\langle u_m(0), \rho\rangle \theta(0) + \int_0^T \langle u_m(\tau), \rho\rangle \dot{\theta}(\tau)\, \mathrm{d}\tau + \int_0^T \langle u_m(\tau), (u_m(\tau), \nabla)\rho\rangle \theta(\tau)\, \mathrm{d}\tau = 0 \ .$$

We have $\langle u_m(0), \rho\rangle \to \langle \mathcal{K} * f, \rho\rangle$ as $m \to \infty$ because of (8.1), (8.6). Also from (8.26)

$$\int_0^T \langle u_m(\tau), \rho\rangle \dot{\theta}(\tau)\, \mathrm{d}\tau \to \int_0^T \langle u(\tau), \rho\rangle \dot{\theta}(\tau)\, \mathrm{d}\tau \ ;$$
$$\int_0^T \langle u_m(\tau), (u_m(\tau), \nabla)\rho\rangle \theta(\tau)\, \mathrm{d}\tau \to \int_0^T \langle u(\tau), (u(\tau), \nabla)\rho\rangle \theta(\tau)\, \mathrm{d}\tau \text{ as } m \to \infty \ .$$

Therefore, the limit u satisfies the Euler equation and the initial condition

$$\begin{cases} \dot{u} = -(u, \nabla)u - \nabla p \ , \\ \mathrm{div} u = 0 \ , \qquad u(0) = \mathcal{K} * f \ . \end{cases}$$

The remaining part of the proof is contained in the following Lemma.

LEMMA 8.4. – *The vorticity constructed above* $\omega(\cdot)$ *is weak* continuous with values in* $B_{\Gamma_1} = H'_{\Gamma_1}$.

Proof. – Since $\|u - u_m\|_{C([0,T]; B^0_{\infty,1})} \to 0$ as $m \to \infty$ we have

$$\|\omega - \omega_m\|_{C([0,T]; B^{-1}_{\infty,1})} \to 0 \ \text{ as } \ m \to \infty \ . \tag{8.29}$$

Let $h \in H_{\Gamma_1}$. Consider the function $t \mapsto \langle \omega(t), h \rangle = \varphi(t)$ on $[0,T]$. Let $\varphi_m(t) = \langle \omega_m(t), h \rangle$. Then, for any $t_0 \in [0,T]$

$$\varphi(t) - \varphi(t_0) = (\varphi - \varphi_m)(t) - (\varphi - \varphi_m)(t_0) + (\varphi_m(t) - \varphi_m(t_0)) \ . \tag{8.30}$$

For a fixed m $\varphi_m(t) - \varphi_m(t_0) \to 0$ as $t \to t_0$ since $t \mapsto \omega_m(t)$ is (strongly) continuous with values in B_{Γ_1}. To obtain the statement of the lemma we estimate the first and the second term in (8.30). We have

$$|(\varphi - \varphi_m)(t)| \leq |\langle (\omega - \omega_m)(t), h - \tilde{h} \rangle| + |\langle (\omega - \omega_m)(t), \tilde{h} \rangle| \ ,$$

where $\tilde{h} \in H_{\Gamma_1}$.
We choose $\tilde{h} \in B^1_{1,1+\delta^{-1}}$ so that

$$\|\tilde{h} - h\|_{\textcircled{\scriptsize\Gamma_1}} < \varepsilon \ . \tag{8.31}$$

Obviously $B^1_{1,1+\delta^{-1}} \subset B^1_{1,\infty} \subset H_{\Gamma_1}$. Indeed, if

$$\|\Delta_j f\|_{\mathrm{L}^1} \leq C 2^{-j} = C 2^{-j} \Gamma_1(j) \Gamma_1(j)^{-1} \ ,$$

this implies

$$\|f\|_{\textcircled{\scriptsize\Gamma_1}} \leq C \sum_{j=-1}^{\infty} 2^{-j} \Gamma_1(j) \leq C$$

according to (ii)-(iv). It is clear that $B^1_{1,1+\delta^{-1}}$ is dense in H_{Γ_1} since it contains all functions from L^1 with a bounded Fourier spectrum.

Since $\|\omega(t)\|_{\Gamma_1}$, $\|\omega_m(t)\|_{\Gamma_1}$ are uniformly bounded on $[0,T]$,

$$|\langle (\omega - \omega_m)(t), h - \tilde{h} \rangle| < C\varepsilon \ . \tag{8.32}$$

Also, $B^{-1}_{\infty,1} \hookrightarrow B^{-1}_{\infty,1+\delta}$ and thus

$$|\langle (\omega - \omega_m)(t), \tilde{h} \rangle| \leq C \|\omega - \omega_m\|_{C([0,T]; B^{-1}_{\infty,1})} \|\tilde{h}\|_{B^1_{1,1+\delta^{-1}}} \ . \tag{8.33}$$

We need here the duality $(B^1_{1,1+\delta^{-1}})' = B^{-1}_{\infty,1+\delta}$ (see e.g., [P], [Tr]). We now fix any $\delta > 0$ (say $\delta = 1$). For any $\varepsilon > 0$ we choose $\tilde{h}$ to satisfy (8.31) and using (8.29) choose m so that the right side of (8.33) becomes $< \varepsilon$.

Then from (8.31)–(8.33) $\overline{\lim}_{t \to t_0} |\varphi(t) - \varphi(t_0)| \leq C\varepsilon$. This concludes the proof of the lemma.

Theorem 8.1 is proved.

Proof of Theorem 8.2. – We repeat the proof of Theorem 8.1 word for word except now the choice of $T > 0$ is arbitrary. Here we have to refer to Theorem 5.5 instead of Theorem 5.1. Thus condition (5.23) has to be satisfied for this argument to work. This completes the proof.

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Last minute addition: A result very similar to [Y2] was obtained independently in Philippe Serfati, Pertes de régularité pour le laplacien et l'équation d'Euler sur $\mathbb{R}^n$, preprint, 15 pp., 1994.