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DESSINS D'ENFANTS AND HUBBARD TREES

BY KEVIN M. PILGRIM *

ABSTRACT. – We show that the absolute Galois group acts faithfully on the set of Hubbard trees. Hubbard trees are finite planar trees, equipped with self-maps, which classify postcritically finite polynomials as holomorphic dynamical systems on the complex plane. We establish an explicit relationship between certain Hubbard trees and the trees known as "dessins d'enfants" introduced by Grothendieck. © 2000 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous montrons que le groupe de Galois absolu opère fidèlement sur l'ensemble des arbres de Hubbard. Ce sont des arbres finis, munis chacun d'une application préservant l'arbre. Ces arbres classifient les polynômes à ensemble postcritique fini en tant que systèmes dynamiques holomorphes du plan complexe. Nous établissons une relation explicite entre les arbres de Hubbard et des arbres combinatoires statiques introduits par Grothendieck sous le nom de « dessins d'enfants ». © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

Recently there has been an attempt to gain an understanding of the structure of the *absolute Galois group* $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by exploiting the remarkable fact that there is a faithful action of Γ on a certain infinite set of finite, planar trees, called *dessins*. These dessins are combinatorial objects which classify planar covering spaces $X \xrightarrow{f} \mathbb{C} - \{0,1\}$ given by polynomial maps f unramified above $\{0,1\}$. The action of Γ on the set of dessins is obtained by letting Γ act on the coefficients of f, which one may take to be algebraic.

The main result of this paper (Theorem 3.8) is that there is also a faithful action of Γ on the infinite set of *Hubbard trees*, which are finite planar trees equipped with self-maps, and which arise in the study of holomorphic dynamical systems. These Hubbard trees are combinatorial objects which classify *postcritically finite* polynomials $f:\mathbb{C} \to \mathbb{C}$ as dynamical systems (a polynomial f is postcritically finite if the *postcritical set* $\mathrm{P}_f = \bigcup_{n>0} f^{\circ n}(\mathrm{C}_f)$ is finite, where C_f is the set of critical points in $\mathbb{C}$). Again, one may take the coefficients of such a map to be algebraic, and the action of Γ is obtained by letting Γ act on the coefficients of f. In fact, we prove that Γ acts faithfully on a highly restricted subset DBP ("dynamical Belyi polynomials") consisting of postcritically finite polynomials f whose iterates are all unramified over $\{0,1\}$ and whose Hubbard tree is uniquely determined by the dessin associated to f as a covering space, plus a small amount of additional data (*see* Definition 3.3).

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There are several intriguing aspects to this dynamical point of view. First, it turns out that the natural class of objects with which to work consists of actual polynomials as opposed to equivalence classes of polynomials. Second, the dynamical theory is richer. In particular, we will introduce a special class of dynamical Belyi polynomials which we call *extra-clean* and which is closed under composition, hence under iteration. This will allow us to associate a *tower* of invariants to a single given polynomial f, namely the monodromy groups $\mathrm{Mon}(f^{\circ n})$ of its iterates. Finally, the dynamical theory here embeds into the non-dynamical one in the following sense: there is a Γ-equivariant injection of the set of extra-clean *dynamical* Belyi polynomials into the set of *non-dynamical* isomorphism classes of Belyi polynomials given by $f \mapsto f^{\circ 2}$ (Theorem 3.4). From the point of view of dynamics, this is remarkable: the dynamics of such an f, which involves an identification of domain and range, is completely determined by the isomorphism class of $f^{\circ 2}$ as a *covering space*, which does not require such an identification.

Organization of this paper

In Section 2 we recall the Grothendieck correspondence giving the combinatorial classification of algebraic curves defined over $\overline{\mathbb{Q}}$; throughout, we concentrate on the case of polynomials and planar tree dessins. In Section 3, we introduce dynamical Belyi polynomials, relate them with non-dynamical ones via the notion of a normalization, and prove a preliminary variant (Theorem 3.7) of our main result in terms of normalized dessins. In Section 4 we discuss the use of towers of monodromy groups to distinguish Galois orbits, give some examples, and derive recursive formulae for monodromy generators of iterates of maps (Theorem 4.2). In Section 5 we discuss various algebraic invariants, e.g., fields of moduli and definition, attached to dynamical Belyi polynomials. We prove (Theorem 5.2) that the field $K_{\mathrm{Coeff}}(f)$ generated by the coefficients of a dynamical Belyi polynomial f coincides with the *field of moduli* of the conjugacy class of f introduced by Silverman [10]. Section 6 is essentially independent, consisting of a translation of our preliminary main theorem into the language of Hubbard trees.

2. Dessins d'enfants

In 1979, Belyi proved a remarkable theorem: an algebraic curve X defined over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ only if there is a holomorphic function $f: X \to \mathbb{P}^1\mathbb{C}$, called a *Belyi morphism*, all of whose critical values lie in $\{0, 1, \infty\}$, i.e., X is a branched covering of $\mathbb{P}^1\mathbb{C}$ ramified only over $\{0, 1, \infty\}$ [1]. This in turn led Grothendieck to the observation that there is a faithful action of the absolute Galois group $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on a set of simple, concrete, combinatorial objects, called *dessins*, which in fact one may take to be certain finite planar trees. The structure of the orbits of Γ under this action remains quite mysterious, and the development of effective combinatorial invariants for distinguishing them has been the subject of recent work (*see*, e.g., [7]).

Combinatorial classification of algebraic curves

We begin by outlining the combinatorial classification of algebraic curves X defined over $\overline{\mathbb{Q}}$. For a good introduction to the subject, see, e.g., the article by Schneps in [9]. The statements are cleanest provided one first introduces a minor, commonly adopted technical notion. A Belyi morphism $f: X \to \mathbb{P}^1\mathbb{C}$ is called *clean* if the ramification at each point lying over 1 is exactly equal to two. Let $q(z) = 4z(1-z)$. If f is a Belyi morphism, then $q \circ f = 4f(1-f)$ is a clean Belyi morphism, and so X is defined over $\overline{\mathbb{Q}}$ if and only if there is a clean Belyi morphism from X to $\mathbb{P}^1\mathbb{C}$. If f is clean we call the pair (X, f) a *clean Belyi pair*. Two such pairs (X_1, f_1), (X_2, f_2) are called *isomorphic* if there is an isomorphism $\phi: X_1 \to X_2$ with $f_1 = f_2 \circ \phi$. We will be mainly interested in the case when $X = \mathbb{P}^1\mathbb{C}$ and f is a polynomial.

On the combinatorial side, a *Grothendieck dessin* is an abstract simplicial 2-complex with 0-cells, 1-cells, and 2-cells denoted respectively X_0, X_1, X_2 such that the underlying space is homeomorphic to a closed, connected, oriented surface, and such that the vertices X_0 are given a bipartite structure, i.e., are colored black and white such that each edge of X_1 has exactly one black and one white vertex. Two such dessins are called *isomorphic* if there is an orientation- and color-preserving isomorphism of complexes. A dessin is called *clean* if each white vertex is the endpoint of exactly two edges. The *genus* of a dessin is the genus of the underlying surface. We will be mainly interested in the case when the genus is zero and the union of edges and vertices forms a tree, i.e., there is a single two-cell in X_2. In this case we shall specify a dessin by specifying a planar tree with a bicoloring of vertices.

A clean Belyi pair (X, f) determines a clean dessin $\mathbf{D}_f$ whose white vertices are preimages of 1, whose black vertices are preimages of 0, whose edges are preimages of the segment $[0,1]$, and whose 2-cells are preimages of $\mathbb{P}^1\mathbb{C} - [0,1]$. The classification may now be formulated as follows ([9], Theorem I.5):

THEOREM 2.1 (Grothendieck correspondence). – *The map $(X, f) \mapsto \mathbf{D}_f$ descends to a bijection between isomorphism classes of clean Belyi pairs and clean dessins.*

Convention

We are mainly concerned with the case when $X = \mathbb{P}^1\mathbb{C} = \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and f is a polynomial. A clean Belyi pair (X, f) is determined by a clean Belyi polynomial $f \in \overline{\mathbb{Q}}[z]$. Two clean Belyi polynomials are then isomorphic if and only if there is an affine map $A \in \mathrm{Aut}(\mathbb{C})$ with $f = g \circ A$. Note that if $f = g \circ A$, with $f, g \in \overline{\mathbb{Q}}[z]$ of degree at least one, then necessarily $A \in \overline{\mathbb{Q}}[z]$ since A must send the set $f^{-1}(\{0,1\})$ onto the set $g^{-1}(\{0,1\})$ and both sets consist of a collection of at least two algebraic numbers.

Throughout the remainder of this work, *we will deal exclusively with clean Belyi polynomials and clean dessins of genus* 0, *i.e., planar tree dessins. We therefore now adopt the convention that the term "dessin" means clean planar tree dessin, and that "Belyi polynomial" means clean Belyi polynomial, unless otherwise specified.*

Notation

- $\mathrm{Aut}(\mathbb{C})$, the group of affine maps $az + b$, $a \neq 0$;
- A, B, elements of $\mathrm{Aut}(\mathbb{C})$;
- f, g, f_1, f_2, clean Belyi polynomials;
- BP, the set of clean Belyi polynomials;
- [BP], the set of isomorphism classes of clean Belyi polynomials;
- $[f]$, the isomorphism class of f as a Belyi polynomial;
- $\mathbf{D}_f$, the dessin of f;
- $[\mathbf{D}_f]$, the isomorphism class of $\mathbf{D}_f$;
- Γ, the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of dessins

The group Γ acts on clean Belyi polynomials by twisting coefficients, i.e., if $\sigma \in \Gamma$ and $f(z) = a_d z^d + \cdots + a_0$, then

$$f^\sigma = \sigma(a_d) z^d + \cdots + \sigma(a_0).$$

This action descends to an action on the set [BP] of isomorphism classes of Belyi polynomials, since if $f_1 = f_2 \circ A$, then $f_1^\sigma = (f_2 \circ A)^\sigma = f_2^\sigma \circ A^\sigma$. By the Grothendieck correspondence, we get an action of Γ on isomorphism classes of dessins. Lenstra and Schneps ([9], Theorem II.4) have shown:

THEOREM 2.2. – *The action of Γ on the set* [BP] *of isomorphism classes of Belyi polynomials, hence on the set of dessins, is faithful.*

In fact, their argument is constructive: given any $\sigma \in \Gamma$ and $\alpha, \beta \in \overline{\mathbb{Q}}$ with $\sigma(\alpha) = \beta \neq \alpha$, they produce, by using the arguments in the proof of Belyi's theorem and an elementary, technical, algebraic lemma (Lemma 3.1 below), a pair f_α, f_β of nonisomorphic Belyi polynomials with $f_\alpha^\sigma = f_\beta$.

Since the notion of isomorphism between Belyi polynomials involves a coordinate change in the domain, but not in the range, they cannot be considered as dynamical objects. In the next section, we replace the notion of isomorphism with that of affine conjugacy, and show that the action of Γ on a suitable set of affine conjugacy classes is faithful.

3. Dynamical Belyi polynomials

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geqslant 2$. Complex dynamics is concerned with the behavior of points under *iteration* of such a function, i.e., with the behavior of *orbits*

$$\{z, f(z), f^{\circ 2}(z), f^{\circ 3}(z), \dots\},$$

where $f^{\circ n}$ denotes the n-fold composition of f with itself. It turns out that understanding the orbits of the critical points (i.e., those $c \in \mathbb{C}$ for which $f'(c) = 0$) is crucial to understanding the global dynamics of f. Let C_f denote the set of critical points of f and $\mathrm{V}_f = f(\mathrm{C}_f)$ the set of critical values. We define the *postcritical set* of f by

$$\mathrm{P}_f = \overline{\bigcup_{n>0} f^{\circ n}(\mathrm{C}_f)}.$$

Then $\mathrm{V}_f \subset \mathrm{P}_f$, $f(\mathrm{P}_f) \subset \mathrm{P}_f$, and $\mathrm{P}_{f^{\circ n}} = \mathrm{P}_f$ for all $n > 0$.

Definition 3.1. – A *dynamical Belyi polynomial* is a Belyi polynomial f for which $\mathrm{V}_f = \mathrm{P}_f = \{0, 1\}$. We denote by DBP the set of all dynamical Belyi polynomials.

Recall that, by convention, f is assumed clean.

PROPOSITION 3.1. – *Let f be a dynamical Belyi polynomial. Then $f^{-1}(\{0,1\}) \supset \{0,1\}$ and hence* 0 *and* 1 *are vertices of the dessin* $\mathbf{D}_f$.

Proof. – The statement follows since $f(\{0,1\}) \subset \{0,1\}$. □

PROPOSITION 3.2. – *Let $f, g \in$ DBP, and suppose $g = B \circ f \circ A$, where $A, B \in \mathrm{Aut}(\mathbb{C})$. Then $B = \mathrm{id}$.*

Proof. – We have

$$\{0,1\} = \mathrm{V}_g = \mathrm{V}_{B \circ f \circ A} = B(\mathrm{V}_{f \circ A}) = B(\mathrm{V}_f) = B(\{0,1\}),$$

where the first equality follows from the preceding definition. So $B = \mathrm{id}$ or $B(z) = 1 - z$. To rule out the latter case, choose $z \in f^{-1}(0)$ mapping to 0 with local degree one. Then

$$g\big(A^{-1}(z)\big) = (B \circ f \circ A)\big(A^{-1}(z)\big) = B(0) = 1$$

mapping by local degree one, which violates the cleanness of g. □

Two polynomials $f, g: \mathbb{C} \to \mathbb{C}$ for which there is an affine map $A: \mathbb{C} \to \mathbb{C}$ satisfying $g = A^{-1} f A$ are called *conjugate*. As dynamical systems, they are the same, just viewed in different coordinates. A conjugacy from f to itself is called an *automorphism* of f. As a corollary to the previous proposition, upon setting $B = A^{-1}$ we obtain:

THEOREM 3.1. – *No two distinct elements f, g of* DBP *are conjugate, and no element of* DBP *has a nontrivial automorphism.*

THEOREM 3.2. – *Let $f \in$ DBP, and let*

$$Z_f = \{z \mid f(z) = 0\}, \qquad O_f = \{z \mid f(z) = 1\}, \qquad \mathrm{Fix}(f) = \{z \mid f(z) = z\}.$$

Then f is uniquely determined by any one of the three sets Z_f, O_f, $\mathrm{Fix}(f)$*, counted with multiplicity.*

Proof. – If $f, g \in$ DBP and either $Z_f = Z_g$ or $O_f = O_g$, then $f = B \circ g$ where $B \in \mathrm{Aut}(\mathbb{C})$, and so by Theorem 3.1 $f = g$. If $\mathrm{Fix}(f) = \mathrm{Fix}(g)$ then since $f(\{0,1\}), g(\{0,1\}) \subset \{0,1\}$ we must have $f|\{0,1\} = g|\{0,1\}$. Since $\mathrm{Fix}(f) = \mathrm{Fix}(g)$,

$$f(z) - z = \lambda(g(z) - z), \quad \text{for some } \lambda \in \mathbb{C}. \tag{1}$$

If either $f(0) = g(0) = 1$ or $f(1) = g(1) = 0$ then substituting into (1) implies $\lambda = 1$ and $f = g$. Otherwise, $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$. Differentiating Eq. (1) we obtain

$$f'(z) - 1 = \lambda\big(g'(z) - 1\big). \tag{2}$$

The cleanness criterion implies that $f'(1) = g'(1) = 0$, and substituting this into Eq. (2) implies $\lambda = 1$ and $f = g$. □

DBPs and normalized Belyi polynomials

We next relate dynamical Belyi polynomials and non-dynamical ones via the notion of a normalization.

Definition 3.2. – A *normalized Belyi polynomial* is a pair $(f, (z, w))$ where f is a Belyi polynomial and (z, w) is an ordered pair of distinct (algebraic) numbers with $\{z, w\} \subset f^{-1}(\{0,1\})$. Two normalized Belyi polynomials $(f_1, (z_1, w_1))$ and $(f_2, (z_2, w_2))$ are called *isomorphic* if there is an $A \in \mathrm{Aut}(\mathbb{C})$ for which $f_1 = f_2 \circ A$, $z_2 = A(z_1)$, and $w_2 = A(w_1)$. We denote the set of isomorphism classes of normalized Belyi polynomials by $[\mathrm{BP}^*]$.

Note that we do *not* require that $z \in f^{-1}(0)$ and $w \in f^{-1}(1)$.

Definition 3.3. – A *normalized dessin* $\mathbf{D}^*$ is a dessin $\mathbf{D}$ together with an ordered pair (z, w) of vertices of $\mathbf{D}$. Two normalized dessins $\mathbf{D}_1^*$, $\mathbf{D}_2^*$ are called *isomorphic* if there is an isomorphism $\mathbf{D}_1 \to \mathbf{D}_2$ of dessins carrying one ordered pair of vertices to the other.

An immediate consequence of the definitions and the Grothendieck correspondence is that the natural map sending a normalized Belyi polynomial $(f, (z, w))$ to the normalized dessins $(\mathbf{D}_f, (z, w))$ descends to a bijection between isomorphism classes of geometric objects (normalized Belyi polynomials) and combinatorial ones (normalized abstract dessins).

Recall that if $f \in \mathrm{DBP}$ then 0 and 1 are vertices of $\mathbf{D}_f$. There are natural maps

$$\begin{aligned} \mathrm{DBP} &\longrightarrow [\mathrm{BP}^*] \quad \text{given by} \quad & f &\longmapsto [(f,(0,1))], \\ \mathrm{DBP} &\longrightarrow [\mathrm{BP}] \quad \text{given by} \quad & f &\longmapsto [f], \\ [\mathrm{BP}^*] &\longrightarrow [\mathrm{BP}] \quad \text{induced by} \quad & (f,(z,w)) &\longmapsto [f], \end{aligned}$$

where $[f]$ is the isomorphism class of f as a Belyi polynomial. The following diagram then commutes:

$$\begin{tikzcd} \mathrm{DBP} \arrow[rr] \arrow[dr] & & {[\mathrm{BP}^*]} \arrow[dl] \\ & {[\mathrm{BP}]} & \end{tikzcd}$$

THEOREM 3.3. – *The map* $\mathrm{DBP} \to [\mathrm{BP}^*]$ *given by*

$$f \longmapsto [(f,(0,1))]$$

is a bijection.

Proof. – The map is clearly injective, since if $(f_1,(0,1))$ is isomorphic to $(f_2,(0,1))$, then the affine map A giving the isomorphism must send zero to zero and one to one. Hence A is the identity and $f_1 = f_2$. The map is surjective as well. First, any isomorphism class of normalized clean Belyi polynomial contains a representative where $z = 0$ and $w = 1$, which can be constructed as follows. Choose any representative $(f,(z,w))$, and let A be the unique affine map which sends 0 to z and 1 to w. Then $(f \circ A,(0,1))$ is equivalent to $(f,(z,w))$. We now claim that $f \circ A \in \mathrm{DBP}$. First, $\mathrm{V}_{f\circ A} = \mathrm{V}_f = \{0,1\}$ since f is assumed clean and precomposing f by an affine map does not change the set of critical values. On the other hand, by construction, $f \circ A(\{0,1\}) \subset \{0,1\}$. Hence $\mathrm{P}_{f\circ A} \subset \{0,1\}$ and so $f \circ A \in \mathrm{DBP}$. By the definition of isomorphism in BP^*, $[f \circ A,(0,1)] = [f,(z,w)]$ and so the map is surjective. □

Thus, a normalized Belyi polynomial $(f,(z,w))$ determines a holomorphic dynamical system $g = f \circ A \in \mathrm{DBP}$ by identifying range and domain via an affine map A sending 0 to z and 1 to w. The above theorem implies that DBP is in bijective correspondence with $[\mathrm{BP}^*]$, which in turn is in bijective correspondence with the set of normalized clean dessins.

The fibers of the map $\mathrm{DBP} \to [\mathrm{BP}]$

The fiber of the forgetful map $\mathrm{DBP} \to [\mathrm{BP}]$ over a given element $[f] \in [\mathrm{BP}]$ is a disjoint union of four nonempty subsets, which we describe in terms of the identification of DBP with $[\mathrm{BP}^*]$ given above. In our normalization of f, we may freely and independently choose z or w to be a black vertex (a preimage of 0) or a white vertex (a preimage of 1) of $\mathbf{D}_f$, giving us four possibilities, all of which can occur. In terms of dynamics, suppose g is the element of DBP corresponding to $[(f,(z,w))] \in [\mathrm{BP}^*]$ in Theorem 3.3. Then

$$\begin{aligned} z \text{ is black} &\iff g(0) = 0, \\ z \text{ is white} &\iff g(0) = 1, \\ w \text{ is black} &\iff g(1) = 0, \\ w \text{ is white} &\iff g(1) = 1. \end{aligned}$$

Within each of these classes, one can further classify points g in the fiber over $[f]$ by recording the local degrees of g near 0 and 1.

Extra-clean dynamical Belyi polynomials

Dynamics is concerned with iteration, and although an iterate of a dynamical Belyi polynomial is again a Belyi polynomial, the property of cleanness may be lost. For example, if we choose w to be a white vertex, then $g(1) = 1$ by local degree two, and so $g^{\circ 2}(1) = 1$ but now mapping by local degree four, violating cleanness. To remedy this, we formulate

Definition 3.4. – An element $g \in \mathrm{DBP}$ is called *extra-clean* if $g(1) = g(0) = 0$, and the local degrees of g near 0 and 1 are both equal to one. The set of all extra-clean dynamical Belyi polynomials we denote by XDBP.

In terms of normalizations, suppose $g \in \mathrm{DBP}$ corresponds to the class of normalized dessins $[(\mathbf{D}, (z, w))]$. Then $g \in \mathrm{XDBP}$ if and only if z, w are both ends of the dessin $\mathbf{D}$, which are necessarily black since, by cleanness, white vertices are always incident to two edges. We denote by $[\mathrm{XBP}^*]$ the subset of $[\mathrm{BP}^*]$ corresponding to XDBP, and refer to the associated normalized dessins as *extra-clean normalized dessins*. We obtain the following commutative diagram:

$$\begin{tikzcd} \mathrm{XDBP} \arrow[rr] \arrow[dr] & & {[\mathrm{XBP}^*]} \arrow[dl] \\ & {[\mathrm{BP}]} & \end{tikzcd}$$

where the map $\mathrm{XDBP} \to [\mathrm{BP}]$ is surjective.

Remark. – We emphasize here that DBP is a set of maps, not a set of maps modulo an equivalence relation. Composition does not descend from BP to a well-defined operation on isomorphism classes of non-dynamical Belyi polynomials; see the example in Section 4 for two polynomials f, g with $\mathbf{D}_f$ isomorphic to $\mathbf{D}_g$ but with $\mathbf{D}_{g^{\circ 2}}$ and $\mathbf{D}_{f^{\circ 2}}$ non-isomorphic.

Indeed, if $f, g \in \mathrm{XDBP}$, this is always the case:

THEOREM 3.4. – *If $f, g \in \mathrm{XDBP}$, then $f \circ g \in \mathrm{XDBP}$. Moreover,*

$$\mathbf{D}_{f^{\circ 2}} \simeq \mathbf{D}_{g^{\circ 2}} \iff f = g.$$

Thus the map

$$\mathrm{XDBP} \longrightarrow [\mathrm{BP}],$$

given by

$$f \longmapsto [f^{\circ 2}],$$

is injective and Γ-equivariant.

This is perhaps remarkable, since it implies that the dynamical system generated by f is completely determined by the topology of $f^{\circ 2}$ as a covering space. This property fails even for the highly restricted set of postcritically finite quadratic polynomials p. Apart from $p(z) = z^2$, $p^{\circ 2}$ will have two finite critical values, and as covering spaces of the twice-punctured plane the second iterates of any two such p are isomorphic.

The proof relies on the following lemma of Lenstra and Schneps used in their proof of Theorem 2.2, and a fact from holomorphic dynamics:

LEMMA 3.1 ([9], Lemma II.3). – *Suppose* G, H, $\widetilde{G}$, $\widetilde{H}$ *are polynomials with* $G \circ H = \widetilde{G} \circ \widetilde{H}$ *and* $\deg(H) = \deg(\widetilde{H})$. *Then there exist constants* c, d *for which* $\widetilde{H} = cH + d$, *i.e.,* $\widetilde{H} = B \circ H$ *for some* $B \in \mathrm{Aut}(\mathbb{C})$.

Proof of Theorem 3.4. – For a polynomial f, set $\mathrm{Fix}(f) = \{p \mid f(p) - p = 0\}$. From dynamics, one knows that if there is a $p \in \mathrm{Fix}(f)$ which is a multiple root of $f(z) - z$ (i.e., the *multiplier* of f at p is equal to one), then there is a critical point of f whose forward orbit is infinite and accumulates at p ([2], Theorem 9.3.2). For dynamical Belyi polynomials this cannot occur, since $\mathrm{P}_f = \{0, 1\}$. Hence, for any dynamical Belyi polynomial, the fixed points are all simple. Thus

$$f(z) - z = c \cdot \prod_{p \in \mathrm{Fix}(f)} (z - p)$$

for a nonzero constant c. By Theorem 3.2 it suffices to show $\mathrm{Fix}(f) = \mathrm{Fix}(g)$.

The hypothesis and the Grothendieck correspondence imply that there is an $A \in \mathrm{Aut}(\mathbb{C})$ with $f^{\circ 2} = g^{\circ 2} \circ A$. We write this as $f \circ f = g \circ (g \circ A)$. The polynomials f and g have the same degree, since their second iterates have the same degree. Applying Lemma 3.1 with $G = g$, $H = g \circ A$, $\widetilde{G} = \widetilde{H} = f$ we obtain an affine map B for which $f = B \circ (g \circ A)$. By Proposition 3.2, $B = \mathrm{id}$ and so $f = g \circ A$.

Now let $p \in \mathrm{Fix}(f)$. Then by the previous paragraph

$$f(p) = g \circ A(p) = p$$

and by hypothesis

$$f^{\circ 2}(p) = g^{\circ 2} \circ A(p) = p.$$

Applying g to both sides of the last equality in the first equation, and comparing with the second we get

$$g(p) = g^{\circ 2} \circ A(p) = p.$$

Hence $\mathrm{Fix}(f) \subset \mathrm{Fix}(g)$. Equality follows, either by appealing to the symmetry of the roles of f and g, or the fact that $\mathrm{Fix}(f)$, $\mathrm{Fix}(g)$ have the same size. □

Galois action on DBP

The group Γ acts on polynomials $f \in \overline{\mathbb{Q}}[z]$ by acting on its coefficients. Note that this is a left action, i.e., $f^{\sigma\tau} = (f^{\tau})^{\sigma}$ since, e.g., if z is algebraic,

$$f^{\sigma}(z) = \sigma \circ f \circ \sigma^{-1}(z).$$

Hence the action of Γ on polynomials in $\overline{\mathbb{Q}}[z]$ by twisting coefficients is natural with respect to the dynamics in the following sense: if f is defined over $\overline{\mathbb{Q}}$ and $z \in \overline{\mathbb{Q}}$, then $\sigma(f(z)) = f^{\sigma}(\sigma(z))$.

Using this, and the fact that the property of being a critical point is algebraic, it is easy to show that the group action of Γ on $\overline{\mathbb{Q}}[z]$ preserves the set DBP. Similarly, the action of Γ must preserve local degrees, i.e., if f maps x to y by local degree k, then f^{σ} maps $\sigma(x)$ to $\sigma(y)$ by local degree k. Hence Γ acts on the set XDBP as well.

Recall that $[\mathrm{XBP}^*]$ corresponds to the set XDBP under the bijection given in Fig. 2 and Theorem 3.3. The group Γ also acts on $[\mathrm{XBP}^*]$ in the obvious way:

$$\sigma \cdot \big[\big(f, (z, w)\big)\big] = \big[f^{\sigma}, \big(\sigma(z), \sigma(w)\big)\big].$$

Together with the usual action of Γ on [BP] we find as a consequence of the definitions.

THEOREM 3.5. – *The diagrams in Figs.* 1 *and* 2 *are equivariant with respect to the action of* Γ.

There is a unique isomorphism class of clean Belyi polynomial with exactly one critical value, namely that of $q(z) = 4z(1-z)$. The isomorphism class of q is therefore fixed under the action of Γ. The theorem of Lenstra and Schneps (Theorem 2.2) asserts that the action of Γ on the entire set [BP], which includes $[q]$, is faithful. Thus Γ acts faithfully on $[\mathrm{BP}] - \{[q]\}$. By the preceding theorem, Galois orbits in XDBP and in DBP lie over orbits in $[\mathrm{BP}] - \{[q]\}$. However, the images of the forgetful maps XDBP $\to$ [BP] and DBP to [BP] are in both cases $[\mathrm{BP}] - \{[q]\}$, so we obtain:

THEOREM 3.6. – *The actions of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on* XDBP *and on* DBP *by twisting coefficients are faithful.*

Since the correspondence between XDBP and [XBP*] is a bijection, and the latter set is in bijective correspondence with isomorphism classes of normalized dessins where the chosen points are both ends of the dessins, we obtain a combinatorial version of the preceding theorem:

THEOREM 3.7. – *The group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on both the set of isomorphism classes of extra-clean normalized dessins and on the set of normalized* (*clean*) *dessins.*

In Section 6, we will show that the sets DBP and XDBP are *naturally* isomorphic to respectively the sets BHT of (isomorphism classes of) clean Belyi-type Hubbard trees and XBHT of extra-clean Belyi type Hubbard trees. Our main theorem then follows:

THEOREM 3.8. – *The group* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts faithfully on both* BHT *and* XBHT.

4. Distinguishing Galois orbits

It is known (*see*, e.g., [7]) that if f, g are two Belyi polynomials (not necessarily clean) which are Galois conjugate, and if $\mathbf{D}_f, \mathbf{D}_g$ are their corresponding dessins, then

(1) $\mathbf{D}_f, \mathbf{D}_g$ have the same number of edges (i.e., f and g have the same degree);

(2) $\mathbf{D}_f, \mathbf{D}_g$ have the same set of valencies (i.e., the unordered sets of local degrees of f and g at points lying over 0 and 1 are the same);

(3) the *monodromy groups* $\mathrm{Mon}(f)$, $\mathrm{Mon}(g)$ are permutation-isomorphic.

The monodromy group[1] $\mathrm{Mon}(f)$ of a Belyi polynomial is the monodromy group of the covering $f : \mathbb{C} - f^{-1}(\{0,1\}) \to \mathbb{C} - \{0,1\}$ and can be described in many equivalent ways (*see*, e.g., [7]). Here, we think of it as a finite permutation group acting transitively (but not freely) on the set of edges E_f of $\mathbf{D}_f$ (equivalently, on the fiber $f^{-n}(b)$ of an arbitrary basepoint $b \neq 0$, 1, usually $b = 1/2$), and define it as the group generated by the two permutations $\sigma_0(f)$ which rotates edges counterclockwise about black vertices, and $\sigma_1(f)$ which rotates edges counterclockwise about white vertices (*see* Fig. 7). The polynomial f is clean exactly when $\sigma_1(f)$ is a fixed-point free involution. The condition that $\mathrm{Mon}(f)$ and $\mathrm{Mon}(g)$ are permutation-isomorphic means that there is a bijection $\tau : E_f \to E_g$ such that $\mathrm{Mon}(g) = \tau\,\mathrm{Mon}(f)\tau^{-1}$. The dessins $\mathbf{D}_f$ and $\mathbf{D}_g$ are isomorphic exactly when one can choose a single bijection τ so that $\sigma_0(g) = \tau\sigma_0(f)\tau^{-1}$ and $\sigma_1(g) = \tau\sigma_1(f)\tau^{-1}$ simultaneously; indeed, this is a special case of the well-known *Hurwitz classification* of coverings.

[1] If F is a Belyi polynomial which is not necessarily clean, the monodromy group of the clean polynomial $q \circ F$ is usually called the *cartographic group* of F.

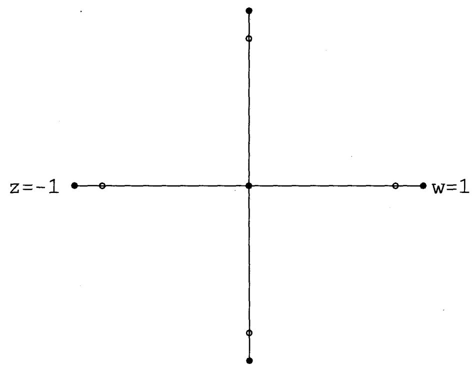

Fig. 1. The normalized dessin of f.

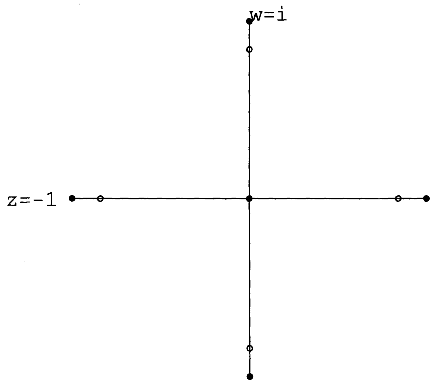

Fig. 2. The normalized dessin of g.

Thus the monodromy group is an invariant of the Galois orbit. However, even for clean polynomials, it is known that this invariant can fail to distinguish Galois orbits (*see* [7], Example 6). Below, we take up similar considerations for dynamical Belyi polynomials.

Recall that Γ acts faithfully on the set XDBP of extra-clean dynamical Belyi polynomials, and that XDBP is closed under composition and iteration. For any two polynomials $f, g \in \overline{\mathbb{Q}}[z]$, we have $(f \circ g)^\sigma = f^\sigma \circ g^\sigma$. So in particular if $f, g \in$ XDBP and $g = f^\sigma$, then for all $n > 0$, $g^{\circ n} = (f^{\circ n})^\sigma$. Hence by (3) above we have:

THEOREM 4.1. – *If $f, g \in$ XDBP and $g = f^\sigma$, then* $\mathrm{Mon}(f^{\circ n})$ *and* $\mathrm{Mon}(g^{\circ n})$ *are permutation-isomorphic for all $n > 0$.*

QUESTION. – *To what extent is the converse to Theorem* 4.1 *true?*

Recall that by Theorem 3.4, the dessins of $f^{\circ 2}$ and $g^{\circ 2}$ are distinct if $f \neq g$. So a tower of permutation-isomorphic monodromy groups cannot arise from the trivial situation where the dessins of $f^{\circ n}$ and $g^{\circ n}$ are isomorphic for all n. Below, we give a simple example where we use this criterion to distinguish orbits lying over a single element of [BP].

An example

Figs. 1 and 2 show two normalized dessins of degree eight; the underlying dessin, without normalization, is that arising from the Belyi polynomial $h(z) = 4z^4(1 - z^4) = q \circ (z \mapsto z^4)$, where $q(z) = 4z(1 - z)$. Denoting by f and g the corresponding dynamical Belyi polynomials, we see that $f = h \circ A_f$, $g = h \circ A_g$, where A_f, A_g are affine maps sending the pair $(0, 1)$ to the indicated vertices, which are $(-1, 1)$ for f and $(-1, i)$ for g. Thus $A_f(z) = 2z - 1$ and $A_g(z) = (1 + i)z - 1$. Since q and h are both defined over $\mathbb{Q}$, it follows that since A_f and A_g are not Galois conjugate, the maps f and g are not Galois conjugate either.

We now prove this combinatorially by appealing to Theorem 4.1 with $n = 2$, i.e., we shall show that $f^{\circ 2}$ and $g^{\circ 2}$ have nonisomorphic monodromy groups. To ease the computation we exploit the fact that $f^{\circ 2} = q \circ F$ and $g^{\circ 2} = q \circ G$. Since q is defined over $\mathbb{Q}$, it suffices to show that F and G have nonisomorphic monodromy groups. The (non-clean) dessins of the degree 32 polynomials F and G are shown in Figs. 3 and 4. An obvious difference is the presence of dihedral symmetry in the dessin of F which is absent in that of G. Note that as abstract one-complexes, the dessins of F and G are homeomorphic; they differ only in the way in which they are embedded in the plane. The dessins of f and g are obtained from those of F and G by replacing each white vertex with a black vertex, and then bisecting each edge with a white

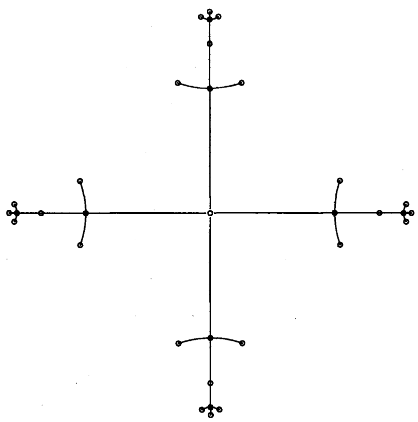

Fig. 3. The dessin of F.

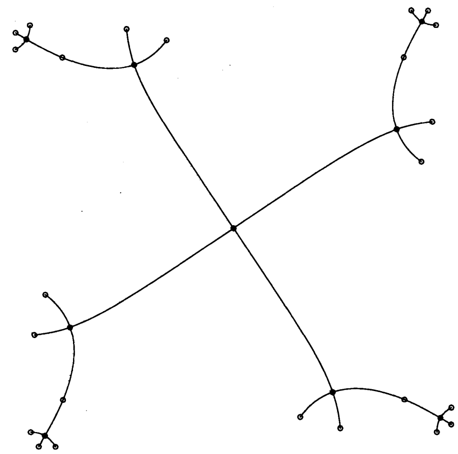

Fig. 4. The dessin of G.

vertex. Labelling the edges of $\mathbf{D}_F, \mathbf{D}_G$ more or less arbitrarily and writing down the elements of $\mathfrak{S}_{32}$ corresponding to the generators σ_0, σ_1 for each map, a brief (1-second) computation in Maple shows that the order of $\mathrm{Mon}(F)$ is 2^{16} while the order of $\mathrm{Mon}(G)$ is 2^{18}, proving our claim. Alternatively, one can work directly with $\mathbf{D}_{f^{\circ 2}}$, $\mathbf{D}_{g^{\circ 2}}$ and determine that the orders of their monodromy groups are respectively 2^{27} and 2^{29}.

Remark. – As dynamical systems, elements of XDBP are highly expanding with respect to a suitable (orbifold) metric on a neighborhood of their Julia sets. This, and standard arguments from complex dynamics, can be used to prove that in fact the dessins of $f^{\circ n}$ converge exponentially fast to the Julia set of f. The Julia sets of f and g are depicted in Figs. 5 and 6.

Recursive formulae for monodromy generators

Let $f \in$ XDBP and let $\sigma_0^1 := \sigma_0$, $\sigma_1^1 := \sigma_1$ be the generators for $\mathrm{Mon}(f)$ defined by the action of simple counterclockwise-oriented loops α_0, $\alpha_1 \in \pi_1(\mathbb{C} - \{0,1\}, b)$ on the fiber $f^{-1}(b)$ (equivalently, on the set of edges of $\mathbf{D}_f$). Here, we derive recursive formulae for the generators σ_0^n, σ_1^n of $\mathrm{Mon}(f^{\circ n})$.

Since f is extra-clean, each of the vertices 0 and 1 is incident to exactly one edge of $\mathbf{D}_f$. Let $E = \{\varepsilon_0, \varepsilon_1, \ldots\}$ denote the set of edges of $\mathbf{D}_f$, where ε_0 is the unique edge incident to 0 and ε_1 is the unique edge incident to 1 (*see* Fig. 7). We will show that there is a canonical identification of the set E^n of edges of $\mathbf{D}_{f^{\circ n}}$ with the n-fold Cartesian product $E \times E \times \cdots \times E$ such that the following theorem holds:

THEOREM 4.2. – *For all $n \geqslant 2$,*

$$\sigma_1^n(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_n}) = \big(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}, \sigma_1(\varepsilon_{i_n})\big),$$

and

$$\sigma_0^n(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_n}) = \begin{cases} (\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}, \sigma_0(\varepsilon_{i_n})) & \text{if } \varepsilon_{i_n} \neq \varepsilon_0, \varepsilon_1, \\ (\sigma_1^{n-1}(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}), \varepsilon_{i_n}) & \text{if } \varepsilon_{i_n} = \varepsilon_1, \\ (\sigma_0^{n-1}(\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}), \varepsilon_{i_n}) & \text{if } \varepsilon_{i_n} = \varepsilon_0. \end{cases}$$

We will make use of a convenient Markov partition for the dynamics of f.

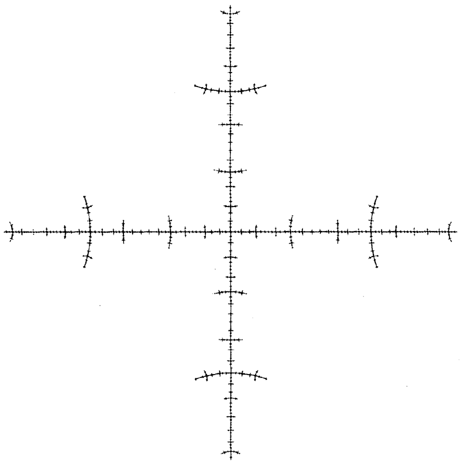

Fig. 5. The Julia set of f.

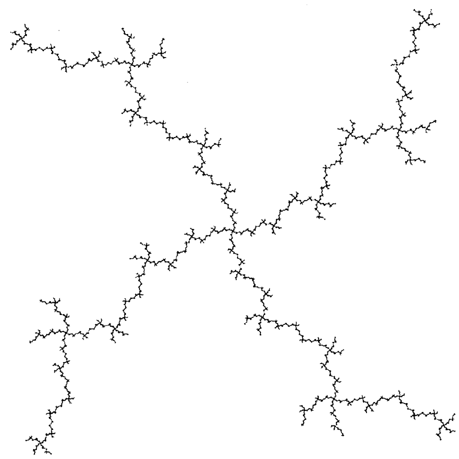

Fig. 6. The Julia set of g.

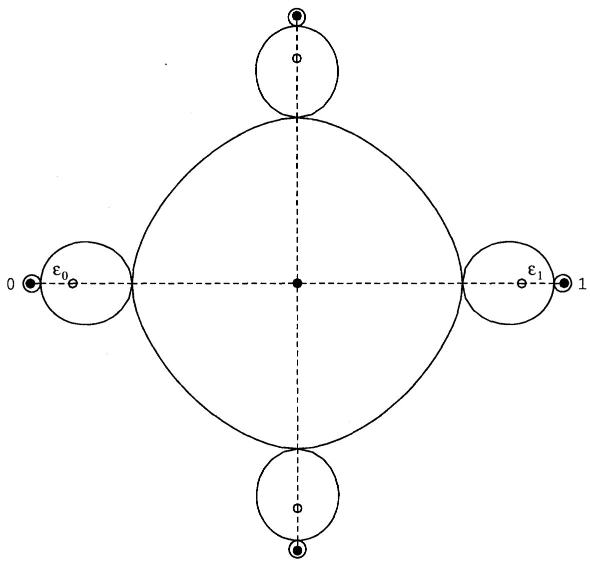

Fig. 7. Here, the lifts of α_0 and α_1 are the loops going once around the black and white vertices, respectively. The edges ε_0, ε_1 are indicated. Note that the lift $\alpha_0^1(\varepsilon_1)$ of α_0 based at ε_1 goes once around 1, and the lift $\alpha_0^1(\varepsilon_0)$ of α_0 based at ε_0 goes once around 0. The dashed tree is the dessin of the map f of Fig. 5. We have $U^0 = \mathbb{C} - (-\infty, 0] \cup [1, +\infty)$, and all lifts of α_0, α_1 except $\alpha_0^1(\varepsilon_0)$, $\alpha_0^1(\varepsilon_1)$ are contained in U^0.

The partitions U^n

Results from holomorphic dynamics imply the existence of two canonical disjoint arcs γ_0, γ_1 joining 0 and 1 respectively to infinity, called *external rays*, such that $f(\gamma_0) = \gamma_0$ and $f(\gamma_1) = \gamma_0$. Set $\Gamma^0 = \gamma_0 \cup \gamma_1$ and let $U^0 = \mathbb{C} - \Gamma^0$; note that U^0 is simply-connected and contains no critical values of $f^{\circ n}$. Since, conceivably, Γ^0 may intersect $(0, 1)$, we choose any arc λ joining 0 and 1, avoiding Γ^0, and passing through a rational basepoint $b \in (0, 1)$ for the construction of the dessins $\mathbf{D}_{f^{\circ n}}$. For $n \geqslant 2$, define inductively $U^n := f^{-1}(U^{n-1})$. Then each connected component U of U^n is an open disc mapping biholomorphically onto its image (which is a connected component of U^{n-1}), and is also contained in a unique connected component of U^{n-1}. Clearly, the set of connected components of U^n can be canonically identified with either the set E^n, or with the full preimage $f^{-n}(b)$. Moreover, since $U^1 \subset U^0$, and each of the d connected components of U^1 maps biholomorphically to U^0, each of the sets $U^n, E^n, f^{-n}(b)$ have d^n elements, where $d = \deg(f)$.

Itineraries

We now show that E^n may be canonically identified with the n-fold Cartesian product $E \times E \times \cdots \times E$. Given a connected component U of U^n and a subset $X \subset U$, we may form its *itinerary* $\iota(X)$ with respect to the partition U^1 by setting

$$\iota(X) = (\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_n}),$$

where

$$\varepsilon_{i_k} = \text{the unique component of } U^1 \text{ containing } f^{\circ k}(X), \quad k = 0, 1, \ldots, n-1.$$

Applying this construction with X equal to an edge of $\mathbf{D}_{f^{\circ n}}$ (equivalently, with an element of $f^{-n}(b)$) we have that ι determines a bijection

$$\iota : E^n \longrightarrow \underbrace{E \times E \times \cdots \times E}_{n}.$$

Since this identification is canonical, we will use the notation E^n to denote the n-fold Cartesian product of E with itself.

Canonical associated maps

The map $f : U^n \to U^{n-1}$ and the inclusion map $j^n : U^n \hookrightarrow U^{n-1}$ induce maps $f : E^n \to E^{n-1}$ and $j^n : E^n \to E^{n-1}$ given by

$$f\big((\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}, \varepsilon_{i_n})\big) = (\varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}, \varepsilon_{i_n})$$

and

$$j^n\big((\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}, \varepsilon_{i_n})\big) = (\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}),$$

i.e., are the left- and right-shift maps, respectively.

Proof of Theorem 4.2. – To ease notation, set

$$(\varepsilon) := (\varepsilon_{i_1}, \varepsilon_{i_2}, \ldots, \varepsilon_{i_{n-1}}, \varepsilon_{i_n}).$$

Given (ε), thought of as an element of $f^{-n}(b)$, we denote by $\alpha_0^n((\varepsilon))$, $\alpha_1^n((\varepsilon))$ the lifts of α_0, α_1 under $f^{\circ n}$ based at $((\varepsilon))$. Note that for $1 \leqslant k \leqslant n-1$,

$$f^{\circ k}\big(\alpha_0^n((\varepsilon))\big) = \alpha_0^{n-k}(\varepsilon_{i_{k+1}}, \ldots, \varepsilon_{i_n})$$

and

$$f^{\circ k}\big(\alpha_1^n((\varepsilon))\big) = \alpha_1^{n-k}(\varepsilon_{i_{k+1}}, \ldots, \varepsilon_{i_n}).$$

Case of σ_1^n. We have

$$f^{\circ(n-1)}\big(\alpha_1^n(\varepsilon)\big) = \alpha_1^1(\varepsilon_{i_n}) \subset U^0, \tag{3}$$

where the inclusion at right follows since f is clean (*cf.* Fig. 7). This implies that

$$\alpha_1^n\big((\varepsilon)\big) \text{ is contained in a unique component of } U^{n-1}. \tag{4}$$

By definition, $\sigma_1^n((\varepsilon))$ is the endpoint of the curve $\alpha_1^n((\varepsilon))$ and $((\varepsilon))$ is its starting point. Thus by (4), we have

$$j^n\big(\sigma_1^n(\varepsilon)\big) = j^n\big((\varepsilon)\big). \tag{5}$$

But (3) also implies that the image of the endpoint of $\alpha_1^n((\varepsilon))$ under $f^{\circ(n-1)}$ is the endpoint of $\alpha_1^1(\varepsilon_{i_n})$, which is $\sigma_1^1(\varepsilon_{i_n}) = \sigma_1(\varepsilon_{i_n})$ by the definition of σ_1. This, and (5), prove the formula.

Case of σ_0^n. The formula for the case when $\varepsilon_{i_n} \neq \varepsilon_0, \varepsilon_1$ follows from the same argument as above, using the fact that since f is extra-clean, $\alpha_0^1(\varepsilon_{i_n}) \subset U^0$ if $\varepsilon_{i_n} \neq \varepsilon_0, \varepsilon_1$.

Now suppose that $\varepsilon_{i_n} = \varepsilon_1$. Since f is extra-clean, 1 maps to 0 under f by local degree one. Let us identify ε_1 with the unique preimage of b which lies on the edge incident to 1. Then the loop α_0 lifts (in the sense of covering spaces) under f to a simple loop $\alpha_1^1(\varepsilon_1)$ which goes exactly once counterclockwise around the point 1 (*see* Fig. 7). It then follows that σ_0^n maps $f^{-(n-1)}(\varepsilon_1)$ to itself bijectively. Let ω be an oriented embedded arc in U^0 going from ε_1 to b. Lifting ω determines a bijection

$$\tau^{n-1} : f^{-(n-1)}(\varepsilon_1) \longrightarrow f^{-(n-1)}(b)$$

and any two such arcs ω, ω' determine the same bijection since U^0 is simply-connected. The loops $\overline{\omega} * \alpha_1 * \omega$ (i.e., do ω, then α_1, then the reverse of ω) and $\alpha_0^1(\varepsilon_1)$ are both based at ε_1 and are homotopic, hence

$$(\tau^{n-1})^{-1} \circ \sigma_1^{n-1} \circ \tau^{n-1} = \sigma_0^n | f^{-(n-1)}(\varepsilon_1).$$

Note that since $\omega \subset U^0$, any lift of ω under $f^{\circ(n-1)}$ is contained in a unique connected component of U^{n-1}. It follows that ε and $\tau^{n-1}(\varepsilon)$ have itineraries which agree to the first $n-1$ places. Since

$$f^{-(k-1)}(\varepsilon_1) = E^{n-1} \times \{\varepsilon_1\}$$

and

$$f^{-(n-1)}(b) = E^{n-1},$$

the map τ^{n-1} is thus that given by

$$(\varepsilon_{i_1}, \ldots, \varepsilon_{i_{n-1}}, \varepsilon_1) \longmapsto (\varepsilon_{i_1}, \ldots, \varepsilon_{i_{n-1}});$$

that is,

$$\tau^{n-1} = j^{n-1} | E^{n-1} \times \{\varepsilon_1\}.$$

From this, the result follows immediately.

Finally, the case when $\varepsilon_{i_n} = \varepsilon_0$ is exactly analogous—since f is extra-clean, 0 maps to 0 by local degree one, hence the loop α_0 lifts to a loop $\alpha_0^1(\varepsilon_0)$ going exactly once counterclockwise around 0 (*see* Fig. 7). $\square$

Remarks. – The dessins $\mathbf{D}_{f^{\circ n}}$ may be inductively constructed as follows. Recall that by cleanness, each edge $(\varepsilon)^{n-1}$ of $\mathbf{D}_{f^{\circ(n-1)}}$ is incident to exactly one black and one white vertex. To obtain $\mathbf{D}_{f^{\circ n}}$, replace each closed edge $(\varepsilon)^{n-1}$ of $\mathbf{D}_{f^{\circ(n-1)}}$ with a copy of $\mathbf{D}_f$ by gluing 0 to the black vertex of $(\varepsilon)^{n-1}$ and 1 to the white vertex of $(\varepsilon)^{n-1}$. This makes synthetic drawing of the dessins of $f^{\circ n}$ easy. In contrast, the expanding nature of maps in XDBP implies that fine detail is rapidly lost in exact drawings, as Figs. 3 and 4 indicate.

5. Algebraic invariants

In this section, we formulate some algebraic invariants attached to a dynamical Belyi polynomial f. We wish these invariants to be dynamically meaningful, i.e., they should be invariants of the affine conjugacy class of f. Our discussion of fields of moduli is drawn from Silverman [10]. We begin with some background.

The group Γ

The group $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ inherits a natural topology known as the *Krull topology* in which it is compact, and one has the following as a special case of *Krull's theorem* [6]. Given a subgroup $G < \Gamma$ we denote by $\mathrm{Inv}(G)$ the subfield of $\overline{\mathbb{Q}}$ fixed by G.

THEOREM 5.1 (Krull). – *Let $\mathcal{G}$ be the set of closed subgroups G of Γ and $\mathcal{F}$ the set of subfields of $\overline{\mathbb{Q}}$. Then the maps*

$$\mathcal{G} \ni G \longrightarrow \mathrm{Inv}(G) \in \mathcal{F} \tag{6}$$

and

$$\mathcal{F} \ni F \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}/F) \in \mathcal{G} \tag{7}$$

are inverses and order-reversing. A subgroup $G \in \mathcal{G}$ is normal in Γ if and only if $F = \mathrm{Inv}(G)$ is Galois over $\overline{\mathbb{Q}}$, and if this is the case, then $\mathrm{Gal}(F/\mathbb{Q}) \cong \Gamma / \mathrm{Gal}(\overline{\mathbb{Q}}/F)$.

In what follows, we let Rat_d denote the set of all rational maps of a given degree $d \geqslant 2$ with coefficients in $\overline{\mathbb{Q}}$. We remark that Γ acts on Rat_d by twisting coefficients, as with the case of polynomials.

Fields of definition

Given $f \in \mathrm{Rat}_d$ and a field extension K of $\mathbb{Q}$, we say that f is *defined over* K if its coefficients are in K, and that K is a *field of definition* for f. It can easily be shown that f is defined over K if and only if $f^\sigma = f$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

Definition 5.1. – Given $f \in \mathrm{Rat}_d$, the *coefficient field* $K_{\mathrm{Coeff}}(f)$ of f is the field generated by its coefficients.

Fields of moduli

If $f, g \in \mathrm{Rat}_d$ are affine conjugate, i.e., $f = AgA^{-1}$, $A \in \mathrm{PSL}_2(\mathbb{C})$, then necessarily $A \in \mathrm{PSL}_2(\overline{\mathbb{Q}})$ since, e.g., A must send periodic points to periodic points: there are always at least three such points, which are necessarily algebraic, and A is determined by where it sends three points. Hence

$$f^\sigma = A^\sigma g^\sigma (A^\sigma)^{-1}$$

and so the action of Γ on Rat_d descends to an action on the *moduli space*

$$\mathrm{M}_d = \mathrm{Rat}_d/\text{conjugation}.$$

Denoting elements of M_d by φ and the action by $\varphi \mapsto \varphi^\sigma$, we note that by definition, for any $f \in \varphi$,

$$\varphi^\sigma = \text{the conjugacy class of } f^\sigma.$$

Given $\varphi \in \mathrm{M}_d$, we denote the stabilizer of ϕ under the action of Γ by

$$\mathrm{Stab}_\Gamma(\varphi) = \{\sigma \in \Gamma \mid \varphi^\sigma = \varphi\}.$$

Definition 5.2. – Let $\varphi \in \mathrm{M}_d$. The *field of moduli $K_{\mathrm{Moduli}}(\varphi)$ of φ* is defined by

$$K_{\mathrm{Moduli}}(\varphi) = \mathrm{Inv}\big(\mathrm{Stab}_\Gamma(\varphi)\big).$$

We say that a field $K \subset \overline{\mathbb{Q}}$ is a *field of definition for φ* if φ contains an element f defined over K.

The field of moduli of $\varphi \in \mathrm{M}_d$ is contained in any field of definition of ϕ. To see this, choose any $f \in \varphi$ defined over $K \subset \overline{\mathbb{Q}}$. Then $f^\sigma = f$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K) < \Gamma$. Hence $\varphi^\sigma = \varphi$ for all $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and so $\mathrm{Gal}(\overline{\mathbb{Q}}/K) < \mathrm{Stab}_\Gamma(\varphi)$. By Krull's theorem, $K = \mathrm{Inv}(\mathrm{Gal}(\overline{\mathbb{Q}}/K)) \supset \mathrm{Inv}(\mathrm{Stab}_\Gamma(\varphi)) = K_{\mathrm{Moduli}}(\varphi)$.

The question of whether the field of moduli of a dynamical system $\varphi \in \mathrm{M}_d$ is also a field of definition is quite subtle. Silverman [10] has shown that if d is even, or if φ contains a polynomial, then the field of moduli is always a field of definition. On the other hand, he shows that, e.g., the conjugacy class ϕ of the map

$$f(z) = i\left(\frac{z-1}{z+1}\right)^3$$

has field of moduli equal to $\mathbb{Q}$ (since f is Möbius conjugate to $\overline{f}$), but nonetheless is not definable over $\mathbb{Q}$. Couveignes [3] has studied the analogous question for non-dynamical Belyi polynomials (i.e., where the equivalence relation is $f \sim g$ if $f = g \circ A$, $A \in \mathrm{Aut}(\mathbb{C})$), and gives an example of a polynomial whose field of moduli is not a field of definition.

Recall that DBP is a set of polynomials, not "modded out" by an equivalence relation. In our case, it turns out that the coefficient field of an element $f \in \mathrm{DBP}$ is an intrinsic quantity of the conjugacy class of f:

THEOREM 5.2. – *Let $\varphi \in \mathrm{M}_d$ be the conjugacy class of the degree d map $f \in \mathrm{DBP}$. Then $K_{\mathrm{Coeff}}(f) = K_{\mathrm{Moduli}}(\varphi)$.*

Proof. – The stabilizer in Γ of $f \in \mathrm{DBP}$ is a closed subgroup of Γ. Since no two elements of DBP are affine conjugate, the stabilizers of f under the action of Γ on DBP and of φ under the action of Γ on M_d coincide. Hence their invariant subfields of $\overline{\mathbb{Q}}$ are the same, and so

$$K_{\mathrm{Moduli}}(\varphi) = \mathrm{Inv}\big(\mathrm{Stab}_\Gamma(\varphi)\big) = \mathrm{Inv}\big(\mathrm{Stab}_\Gamma(f)\big).$$

Since the field of moduli is always contained in any field of definition,

$$K_{\mathrm{Moduli}}(\varphi) \subset K_{\mathrm{Coeff}}(f).$$

$$
\begin{tikzcd}
& \mathbb{Q}(Z_f \cup O_f) \arrow[dl, dash] \arrow[ddr, dash] & \\
\mathbb{Q}(\mathrm{C}_f) \arrow[d, dash] & & \\
\mathbb{Q}(O_f) \arrow[dr, dash] & & \mathbb{Q}(Z_f) \arrow[dl, dash] \\
& K &
\end{tikzcd}
$$

Fig. 8.

To prove the other direction, it suffices to show that $\mathrm{Inv}(\mathrm{Stab}_\Gamma(f)) \supset K_{\mathrm{Coeff}}(f)$, and since $\mathrm{Stab}_\Gamma(f)$ is closed this is equivalent by Krull's theorem to showing that

$$\mathrm{Stab}_\Gamma(f) < \mathrm{Gal}\big(\overline{\mathbb{Q}}/K_{\mathrm{Coeff}}(f)\big).$$

But this is clear, since $\sigma \in \Gamma$ is the identity on $K_{\mathrm{Coeff}}(f)$ if and only if it is the identity on the generating set of $K_{\mathrm{Coeff}}(f)$, i.e., on the coefficients of f. $\square$

COROLLARY 5.1. – *If $f \in$ DBP and φ is the conjugacy class of f, then φ is definable over $\mathbb{Q}$ if and only if $f \in \mathbb{Q}[z]$.*

Since elements of DBP admit no nontrivial automorphisms, the set Z_f of zeros, the set O_f of ones, and the set $\mathrm{C}_f \supset O_f$[2] of critical points are all dynamically distinguished subsets of $\overline{\mathbb{Q}} \subset \mathbb{C}$. Hence they form invariants of the conjugacy class of f. For a subset S of $\overline{\mathbb{Q}}$ and a field K, we denote by $K(S)$ the extension of K generated by K and S. The above fields are related as follows:

THEOREM 5.3. – *Let $f \in$ DBP and set $K = K_{\mathrm{Coeff}}(f)$. Then we have the following collection of field extensions (see Fig. 8), each of which is Galois.*

We first establish:

LEMMA 5.1. – *Let $f \in$ DBP and let $K = K_{\mathrm{Coeff}}(f)$. Then $K \subset \mathbb{Q}(Z_f) \cap \mathbb{Q}(O_f)$.*

Proof of Lemma 5.1. – We have

$$f(z) = c \cdot \prod_{z_i \in Z_f} (z - z_i)^{d_i} = c \cdot \prod_{w_j \in O_f} (z - w_j)^2 + 1$$

for some $c \in \overline{\mathbb{Q}}$. It suffices to show $c \in \mathbb{Q}(Z_f)$ and $c \in \mathbb{Q}(O_f)$. Suppose $c \notin \mathbb{Q}(Z_f)$. Let E be a Galois extension of $\mathbb{Q}(Z_f)$ containing c and let $\sigma \in \mathrm{Gal}(E/\mathbb{Q}(Z_f))$ satisfy $\sigma(c) \neq c$. Then $f \neq f^\sigma$ but $Z_f = Z_{f^\sigma}$, violating Theorem 3.2. The case of O_f is similar. $\square$

Proof of Theorem 5.3. – By Lemma 5.1, the diagram in Fig. 8 is equivalent to the diagram in Fig. 9.

[2] This condition is guaranteed by the assumption of cleanness.

$$\begin{tikzcd}
& K(Z_f \cup O_f) \arrow[dl, dash] \arrow[ddr, dash] & \\
K(\mathrm{C}_f) \arrow[d, dash] & & \\
K(O_f) \arrow[dr, dash] & & K(Z_f) \arrow[dl, dash] \\
& K &
\end{tikzcd}$$

Fig. 9.

As extensions of K, the fields $K(O_f)$, $K(Z_f)$, $K(\mathrm{C}_f)$, and $K(Z_f \cup O_f)$ are the splitting fields of the separable polynomials in $K[z]$ given by $f-1$, f, f', and $f(f-1)$, respectively. Hence all indicated and implied extensions are Galois. □

Remark and example. – In general, it is possible that $\mathbb{Q}(\mathrm{C}_f)$ is a proper subfield of $Q(Z_f \cup O_f)$, as the following example shows. Consider the polynomial $f = q \circ g$, where $q(z) = 4z(1-z)$ and $g(z) = -3z^3(z - 4/3)$. It is easily verified that $f \in \mathrm{DBP}$, and that

$$\mathbb{Q}(O_f) = \mathbb{Q}(\mathrm{C}_f) = \mathbb{Q}\big(\text{Roots of } 6z^4 - 8z^3 + 1\big)$$

and

$$\mathbb{Q}(Z_f) = \mathbb{Q}\big(\sqrt{2}i\big).$$

The Galois group of the above quartic is $\mathfrak{S}_4$ and its discriminant is $\Delta = -2^{11} \cdot 3^3$. Since $\mathfrak{S}_4$ contains a unique normal subgroup of index 2, $\mathbb{Q}(\mathrm{C}_f)$ contains a unique Galois subfield of dimension two over $\mathbb{Q}$, namely $Q(\sqrt{\Delta}) = \mathbb{Q}(\sqrt{6}i) \neq \mathbb{Q}(\sqrt{2}i)$. Hence $\mathbb{Q}(\sqrt{2}i)$ is not a subfield of $\mathbb{Q}(\mathrm{C}_f)$.

6. Hubbard trees

In this section, we introduce Hubbard trees and conclude the proof of our main result: that Γ acts faithfully on Belyi-type Hubbard trees (Theorem 3.8). We also explicitly relate Hubbard trees and normalized dessins. Our proof relies on Thurston's combinatorial characterization of a certain class of polynomials regarded as holomorphic dynamical systems, and on work of Poirier connecting Thurston's characterization with Hubbard trees.

Thurston equivalence of branched coverings

We relax the structure on elements of DBP and consider instead the more flexible setting of *continuous* orientation-preserving, finite degree, postcritically finite branched coverings $F:\mathbb{C} \to \mathbb{C}$. Following Thurston, two such maps F, G are called *combinatorially equivalent* if there are orientation-preserving homeomorphisms $\psi_0,\ \psi_1 : \mathbb{C} \to \mathbb{C}$ for which $\psi_0 \circ F = G \circ \psi_1$ and for which ψ_0 is isotopic to ψ_1 through homeomorphisms fixing P_F, the postcritical set of F.

THEOREM 6.1 (Thurston rigidity). – *If $F:\mathbb{C}\to\mathbb{C}$ is a postcritically finite branched covering, then F is combinatorially equivalent to at most one (up to affine conjugacy) polynomial f.* [3]

As a very special case of Thurston's characterization of postcritically finite rational functions as branched coverings of the sphere to itself [4] we have:

THEOREM 6.2. – *If $F:\mathbb{C}\to\mathbb{C}$ is a postcritically finite branched covering for which $\#\mathrm{P}_F = 2$, then F is combinatorially equivalent to exactly one (up to affine conjugacy) polynomial f.*

Abstract minimal Hubbard trees

A Hubbard tree can be thought of as giving an almost normal form for a postcritically finite polynomial map from the plane to itself. Roughly, it is a finite planar tree T together with a map τ from T to itself sending vertices to vertices (but not necessarily edges to edges) which is extendable to a branched covering of the plane to itself, satisfies some reasonable minimality conditions, and is sufficiently expanding to be combinatorially equivalent to a polynomial. Our definitions are slightly condensed versions of those of Poirier [8].

Definition 6.1. – A *Hubbard tree* [4] is a triple $\mathbf{T}=(T,\tau,\delta)$ where:

(1) *Planar tree.* T is a planar tree with vertex set V;

(2) *Dynamics.* $\tau:T\to T$ is a continuous map sending V to V and which is injective on the closures of edges;

(3) *Local degree.* $\delta:T\to\{1,2,3,\ldots\}$ is the *local degree function*;

(4) *Uniquely extendable.* For each $x\in T$, τ extends to an orientation-preserving branched covering from a neighborhood of v to a neighborhood of $\tau(v)$, and which in local coordinates with $v=\tau(v)=0$ is given by $\tau(z)=z^{\delta(v)}$. If $\delta(v)>1$ we call v a *critical point* and denote the set of critical points by C. We define the *postcritical set* P by

$$\mathrm{P}=\bigcup_{n>0}\tau^{\circ n}(C)\subset \mathrm{V}.$$

(5) *Nontrivial.* The *degree* d of $\mathbf{T}$ defined by

$$d=1+\sum_{v\in V}\delta(v)-1,$$

is at least two;

(6) *Homogeneity.* $C\subset\tau(T)$ (i.e., critical points must have preimages), and for each $x\in\tau(T)$,

$$\sum_{\tau(y)=x}\delta(y)=d$$

(i.e., all points which have preimages have exactly d preimages, counting multiplicities);

(7) *Expansion.* A vertex v is called a *Fatou vertex* if there are integers $n\geqslant 0$, $p>0$ for which $\tau^n(v)=\tau^{n+p}(v)\in C$ (i.e., it lands on a period p critical point); otherwise, v is called a *Julia vertex*, and we require that if v, v' is any pair of distinct, adjacent Julia vertices, then there is an integer $n>0$ for which $\tau(v)$, $\tau(v')$ are nonadjacent.

[3] More generally, this holds for postcritically finite branched coverings $F:\mathrm{S}^2\to\mathrm{S}^2$, apart from the non-polynomial Lattès examples. (*See* [4,5].)

[4] Poirier would call these *dynamical, homogeneous, expanding, minimal Hubbard trees.*

(8) *Minimality.* Given two such triples $\mathbf{T}$, $\mathbf{T}'$, we say $T \preceq T'$ if there is a dynamically compatible orientation-preserving embedding of pairs, i.e., an embedding $\phi : (T, \mathrm{V}) \to (T', \mathrm{V}')$ such that $\tau' \circ \phi = \phi \circ \tau$ and $\delta = \delta' \circ \phi$.

Finally, we say two Hubbard trees $\mathbf{T}$, $\mathbf{T}'$ are *isomorphic* if $T \preceq T'$ and $T' \preceq T$. Equivalently, two Hubbard trees are isomorphic if there is an orientation-preserving homeomorphism of the plane to itself carrying (T, V) to (T', V'), conjugating the dynamics on the vertices, and preserving the local degree functions.

The minimality criterion is obviously necessary if we think of a Hubbard tree as a reasonable normal form for encoding a branched covering from the plane to itself. For example, consider a Hubbard tree with an edge which is fixed pointwise. Add a vertex at the midpoint, and transport this using the dynamics to obtain another new homogeneous tree. Finally, add any number of edges emanating from the new vertex and transport these new edges using the dynamics to obtain a new homogeneous tree. Dynamically, these added edges are extraneous.

THEOREM 6.3 ([8], Theorems I.4.7 and II.4.8, p. 28). – *The set of affine conjugacy classes of postcritically finite polynomials of degree at least two is in bijective correspondence with the set of isomorphism classes of Hubbard trees.*

We now outline the construction of this bijection. A Hubbard tree $\mathbf{T}$ determines a postcritically finite branched covering $F : \mathbb{C} \to \mathbb{C}$. Expansion and Thurston's characterization of postcritically finite rational maps as branched coverings guarantees that F is combinatorially equivalent to a postcritically finite polynomial, which is unique up to affine conjugacy by Thurston rigidity. So one has a well-defined map from Hubbard trees $\mathbf{T}$ to conjugacy classes of polynomials $f_{\mathbf{T}}$. The injectivity of this map follows from the minimality criterion — without it, one could, e.g., simply add additional orbits of periodic vertices to obtain a new tree which yields the same polynomial. As a consequence, two Hubbard trees, when extended to branched coverings of the plane, yield combinatorially equivalent coverings if and only if they are isomorphic.

A description of the inverse of this map requires some notions from complex dynamics.

Julia sets

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $\geqslant 2$. We define:
- the *filled Julia set* $\mathrm{K}_f = \{z \mid f^{\circ n}(z) \not\to \infty\}$;
- the *Julia set* $\mathrm{J}_f = \partial \mathrm{K}_f$;
- the *Fatou set* $\mathrm{F}_f = \mathbb{C} - \mathrm{J}_f$.

If f is postcritically finite, K_f and J_f are connected and locally connected. The set K_f has interior if and only if there are periodic critical points. Otherwise, $\mathrm{K}_f = \mathrm{J}_f$ is a *dendrite*, i.e., compact, connected, with no interior, and whose complement is connected. In particular, the Julia sets of maps in XDBP are always dendrites, and given any two points $x, y \in \mathrm{K}_f$, there is a unique (up to reparameterization) continuous embedding $\gamma : [0,1] \to \mathrm{K}_f$ joining x to y. In the case when K_f has interior, the connected components of the interior of K_f are bounded components Ω of F_f. These are Jordan domains whose closures meet in at most one point. Each such bounded component Ω contains a unique "center point" p which eventually maps onto a critical point. The pair (Ω, p) is conformally isomorphic to the unit disc $(\Delta, 0)$ and thus carries a canonical foliation by radial arcs $\{r\,\mathrm{e}^{2\pi i t} \mid r < 1\}$ called *internal rays.* Any two points $x, y \in \overline{\Omega}$ are joined by a unique arc, called a *regulated arc*, which is the union of two closed internal rays.

The tree $\mathbf{T}_f$ associated to a postcritically finite polynomial f may now be described as follows. The underlying tree T is the smallest subcontinuum of K_f containing $f^{-1}(\mathrm{P}_f)$ and whose intersection with the closure of any bounded Fatou component is either a regulated arc or a finite collection of points on the boundary. This is always a topological tree. The vertex set V one takes to be $f^{-1}(\mathrm{P}_f)$ together with the (necessarily finite) number of points v for which $T - \{v\}$ has three or more components. As τ one takes the function $f|T$, and it is known that $f(\mathrm{V}) \subset \mathrm{V}$ and $f(T) \subset T$. Finally, as δ one takes the local degree function associated to f. Poirier then shows that if $\mathbf{T}$ is a Hubbard tree and $f_{\mathbf{T}}$ the associated polynomial, then the tree obtained by applying the above construction to $f_{\mathbf{T}}$ is a Hubbard tree equivalent to $\mathbf{T}$.

We now make an explicit connection between Hubbard trees and tree dessins.

Definitions 6.2. – We denote by:

- [BH] the set of isomorphism classes of Hubbard trees $\mathbf{T}$ such that $P = V$ has exactly two points, and for which there exists a (necessarily unique) $v \in P$ such that $\delta(y) = 2$ for all $y \in V$ with $\tau(y) = v$, i.e., the local degree of τ near y is exactly equal to two. Here, P, V refer to the postcritical set and set of critical values, defined in the obvious way. We refer to [BH] as the set of isomorphism classes of *clean Belyi-type Hubbard trees*.
- [BBC] the set of combinatorial equivalence classes of branched coverings $F:\mathbb{C} \to \mathbb{C}$ such that $\mathrm{P}_F = \mathrm{V}_F$ has exactly two points, and for which there exists a (necessarily unique) $v \in \mathrm{P}_F$ such that the local degree of F near y is exactly equal to two for all y with $F(y) = v$. We refer to [BBC] as the set of combinatorial equivalence classes of *clean Belyi-type branched coverings* from the plane to itself.

We define extra-clean Belyi type Hubbard trees and extra-clean Belyi-type branched coverings analogously. Note that the homogeneity and extendability conditions guarantee that if $\mathbf{T} \in \mathrm{BH}$ and F is any extension of τ to a branched covering of the plane to itself, then F is a clean Belyi-type branched covering.

Proof of Theorem 3.8. – There are natural maps from DBP to [BBC] and [BHT] which send a dynamical Belyi polynomial f to its combinatorial class as a branched covering and the isomorphism class of its Hubbard tree, respectively. We then have the following commutative diagram, where all lines indicate bijections (*see* Fig. 10).

Here, the diagonal map at upper right is by definition the composition of the sides, so triangle (I) is commutative by definition. Triangle (II) commutes by extension of the Grothendieck correspondence to normalized clean tree dessins and normalized clean Belyi polynomials. Triangle (IV) commutes by restricting Poirier's theorems to the case of clean dynamical Belyi polynomials, Hubbard trees, and branched coverings.

By Poirier's theorem, DBP is in bijective correspondence with [BHT]. So we can use this correspondence to define a faithful action of Γ on [BHT]. This applies as well in the extra-clean case, and Theorem 3.8 is proved. $\square$

$$\begin{tikzcd}
 & {[\mathrm{BP}^*]} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
{[\mathrm{TD}^*]} \arrow[r, dash] \arrow[dr] & \mathrm{DBP} \arrow[r, dash] \arrow[d, dash] \arrow[ul, phantom, "\mathrm{II}"] \arrow[ur, phantom, "\mathrm{I}"] \arrow[dl, phantom, "\mathrm{III}"] \arrow[dr, phantom, "\mathrm{IV}"] & {[\mathrm{BBC}]} \arrow[dl, dash] \\
 & {[\mathrm{BHT}]} &
\end{tikzcd}$$

Fig. 10.

Hubbard trees and normalized dessins

The preceding proof is somewhat unsatisfactory, since the map sending $f \in \mathrm{DBP}$ to its Hubbard tree (which is a combinatorial object) is defined via the topology and dynamics of the Julia set, which is geometric. Below, we give an alternative description of the arrowed map in Triangle (III) of Fig. 10. We will construct, given $f \in \mathrm{DBP}$ and its normalized dessin, an associated Hubbard tree $\mathbf{T}$ and an extension of the dynamics to a branched covering which is combinatorially equivalent to f. By uniqueness of the Hubbard tree, this tree will be *the* Hubbard tree of f.

Given any isomorphism class $\mathbf{D}^*$ of normalized dessin, let f be the corresponding element of DBP, let $\mathbf{D}_f$ be the dessin of f with vertices X_0 (i.e., $\mathbf{D}_f$ is the preimage of $[0,1]$ with its tree structure and bicoloring of vertices). Then since $f \in \mathrm{DBP}$, $f(\{0,1\}) \subset \{0,1\}$, so $f^{-1}(\{0,1\}) \supset \{0,1\}$. Let α denote the unique subarc of D_f joining 0 and 1 in $\mathbf{D}_f$. Since any two arcs joining zero and one in the plane are ambient isotopic fixing endpoints, there is a homeomorphism $\psi_1 : \mathbb{C} \to \mathbb{C}$ be a homeomorphism such that $\psi_1(0) = 0$, $\psi_1(1) = 1$, $\psi_1([0,1]) = \alpha$, and ψ_1 is isotopic to the identity through maps fixing 0 and 1. Let $F : \mathbb{C} \to \mathbb{C}$ be the branched covering $f \circ \psi_1$. Then F is combinatorially equivalent to f (take ψ_0 to be the identity in the definition) and indeed this is true for any choice of ψ_1.

We now set $\mathbf{T}$ to be the Hubbard tree whose underlying tree T is $\psi_1^{-1}(D_f)$, whose vertex set V is $\psi_1^{-1}(X_0(f))$, whose local degree function is the restriction of the local degree of F, and whose dynamics τ is the restriction of F to T.

We next claim that $\mathbf{T}$ is indeed a Hubbard tree. By construction, edges map injectively on their interiors and the dynamics is extendable. Homogeneity is also clear, since the underlying tree is the full preimage of a set containing all the finite postcritical set. Minimality holds since each vertex is the preimage of a critical value, these are required by the homogeneity condition, and hence the tree cannot be modified by removing vertices to obtain a smaller tree with respect to the partial order $\preceq$ on trees. We now verify the expansion condition. Let v, v' be any pair of adjacent Julia vertices, and suppose that for all i, $\tau^{\circ i}(v)$ and $\tau^{\circ i}(v')$ are adjacent. Note that $\tau^{\circ i}(v) \neq \tau^{\circ i}(v')$ by requirement (2) in the definition of Hubbard tree. Since P_T is finite (in fact, has two elements), by replacing v and v' with $\tau^{\circ i}(v)$, $\tau^{\circ i}(v')$ we may assume v, v' are periodic. By the construction of $\mathbf{T}$, we may assume $v = 0$ and $v' = 1$. Then 0 and 1 are periodic Julia vertices, hence neither is a critical point. That is, both 0 and 1 are ends of the tree $\mathbf{D}_f$. Since the degree of f is at least two, any two ends of the tree $\mathbf{D}_f$ are nonadjacent, a contradiction.

The above function $[\mathrm{TD}^*] \to [\mathrm{BHT}]$ makes the triangle (III) commute, and so the map we have constructed is a bijection.

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