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MOST AUTOMORPHISMS OF A HYPERBOLIC GROUP HAVE VERY SIMPLE DYNAMICS

BY GILBERT LEVITT AND MARTIN LUSTIG

ABSTRACT. – Let G be a non-elementary hyperbolic group (e.g. a non-abelian free group of finite rank). We show that, for "most" automorphisms α of G (in a precise sense), there exist distinct elements X^+, X^- in the Gromov boundary ∂G of G such that $\lim_{n\to+\infty} \alpha^{\pm n}(g) = X^{\pm}$ for every $g \in G$ which is not periodic under α. This follows from the fact that the homeomorphism $\partial\alpha$ induced on ∂G has North–South (loxodromic) dynamics. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Soit G un groupe hyperbolique non élémentaire (par exemple un groupe libre non abélien de rang fini). Nous montrons que, pour "la plupart" des automorphismes α de G (en un sens bien précis), il existe deux éléments distincts X^+, X^- dans le bord de Gromov ∂G de G tels que $\lim_{n\to+\infty} \alpha^{\pm n}(g) = X^{\pm}$ pour tout $g \in G$ non périodique sous l'action de α. Ceci résulte du fait que l'homéomorphisme $\partial\alpha$ induit sur ∂G a une dynamique Nord–Sud (loxodromique). © 2000 Éditions scientifiques et médicales Elsevier SAS

0. Introduction and statement of results

Let α be an automorphism of a (word) hyperbolic group G. Fixing $g \in G$, we consider the sequence of iterates $\alpha^n(g)$, for $n \geqslant 1$. We assume that g is not α-periodic, so that $\alpha^n(g)$ goes off to infinity in G.

We will show that, for "most" automorphisms α of G (in a sense that will be made precise), there exists a point X^+ in the Gromov boundary ∂G such that $\alpha^n(g)$ converges to X^+ for every nonperiodic g. If G is free on a finite set A, this says that there exists a sequence of letters $a_k^{\pm 1} \in A \cup A^{-1}$ such that, for any non-periodic g, the kth letter of $\alpha^n(g)$ equals $a_k^{\pm 1}$ for n large.

This dynamical behavior is best expressed in terms of the homeomorphism $\partial\alpha$ induced by α on ∂G: for most $\alpha \in \operatorname{Aut} G$, the map $\partial\alpha$ has North–South dynamics in the following sense. We say that $\partial\alpha$, or α, has *North–South dynamics* (also called loxodromic dynamics) if $\partial\alpha$ has two distinct fixed points X^+, X^-, and $\lim_{n\to+\infty} \partial\alpha^{\pm n}(X) = X^{\pm}$ uniformly on compact subsets of $\partial G \setminus \{X^{\mp}\}$.

This implies (see Proposition 2.3) that *the set of α-periodic elements $g \in G$ is a virtually cyclic subgroup* (*possibly finite*), *and* $\lim_{n\to+\infty} \alpha^{\pm n}(g) = X^{\pm}$ *if $g \in G$ is not α-periodic*. For an arbitrary automorphism, it is proved in [15] that $\alpha^n(g)$ limits onto a finite subset of ∂G (that may depend on g).

If for instance α is conjugation i_m by $m \in G$, then $\partial\alpha$ is simply left-translation by m, and $\partial\alpha$ has North–South dynamics for all m outside of a finite set of conjugacy classes (those consisting of torsion elements), see [5,11,12].

In general, we consider an outer automorphism $\Phi \in \mathrm{Out}\, G$, viewed as a collection of ordinary automorphisms $\alpha \in \mathrm{Aut}\, G$. For a topological motivation, induce Φ by a continuous map $f: X \to X$ with $\pi_1 X \simeq G$. Automorphisms $\alpha \in \Phi$ correspond to lifts of f to the universal covering of X. Different lifts may have very different properties. On the other hand, conjugate maps have similar dynamical properties. This led Nielsen [18] to define lifts of f to be isogredient if they are conjugate by a covering transformation.

Going back to group automorphisms, we therefore define $\alpha, \beta \in \Phi$ to be *isogredient* if $\beta = i_h \circ \alpha \circ i_h^{-1}$ for some $h \in G$, with $i_h(g) = hgh^{-1}$ (the word "similar" was used in [9]).

We denote $\mathcal{S}(\Phi)$ the set of isogredience classes of automorphisms representing Φ. If $\Phi = 1$, then $\mathcal{S}(\Phi)$ may be identified to the set of conjugacy classes of G modulo its center. We say that $s \in \mathcal{S}(\Phi)$ has North–South dynamics if automorphisms $\alpha \in s$ have North–South dynamics on ∂G.

THEOREM 0.1. – *Let G be a hyperbolic group, and $\Phi \in \mathrm{Out}\, G$. Assume G is non-elementary (i.e. G is not virtually cyclic).*

(1) *All but finitely many $s \in \mathcal{S}(\Phi)$ have North–South dynamics.*

(2) *The set $\mathcal{S}(\Phi)$ of isogredience classes is infinite.*

Example 0.2. – When Φ is induced by a pseudo-Anosov homeomorphism φ of a closed surface Σ, the "exceptional" automorphisms $\alpha \in \Phi$ (those that do not have North–South dynamics) correspond to lifts of φ having a fixed point in the universal covering of Σ. The set of exceptional classes in $\mathcal{S}(\Phi)$ is in one-to-one correspondence with the set of fixed points of φ. It may be empty, see [8] for an explicit example. On the other hand, the number of fixed points of φ^k goes to infinity with k. Thus the number of exceptional isogredience classes cannot be bounded in terms of G only.

This example suggests the possibility of using exceptional isogredience classes to develop a fixed point theory for general outer automorphisms of free groups. Exceptional isogredience classes would be the algebraic analogue of Nielsen classes of fixed points, and there should be a (rational) zeta function obtained as a sum over exceptional classes of powers of Φ (compare [7]).

Example 0.3. – Suppose G is free. It follows from [3, Lemma 5.1] that some power of Φ contains an exceptional isogredience class. It may be shown using [4] and [14] that the isogredience class of α is the only exceptional class when α is the irreducible automorphism $a \mapsto abc$, $b \mapsto bab$, $c \mapsto cabc$ studied in [13].

The proof of the first assertion of Theorem 0.1 when G is not free requires the following fact, which is of independent interest:

PROPOSITION 0.4 (Quasiisometries of hyperbolic spaces have a quasi-fixed point or a quasi-axis). – *Let f be a (λ, C)-quasiisometry of a δ-hyperbolic proper geodesic metric space (E, d) to itself. There exists $M = M(\delta, \lambda, C)$, independent of E and f, with the following property: if $d(f(x), x) > M$ for all $x \in E$, then there exists a bi-infinite geodesic γ such that the Hausdorff distance between γ and $f(\gamma)$ is finite.*

Increasing M if necessary, we conclude (Corollary 1.4) that the action of f on ∂E has North–South dynamics, with fixed points the two endpoints of γ.

1. Quasiisometries of hyperbolic spaces

We start by proving Proposition 0.4. The proof may be seen as a generalization of the well-known argument constructing the axis of an isometry of an $\mathbf{R}$-tree having no fixed points (see [17]). We refer the reader to [5,11,12,23] for basic facts about hyperbolic spaces, quasiisometries, and hyperbolic groups.

Let (E,d) be a proper δ-hyperbolic geodesic metric space. Properness is assumed mostly for convenience, in particular E could be an $\mathbf{R}$-tree in what follows.

For $x,y\in E$, we denote $[x,y]$ any geodesic segment from x to y. Given a point z, any point $p\in[x,y]$ that is δ-close to both segments $[x,z]$ and $[y,z]$ will be called a projection of z onto $[x,y]$ (two projections are only a few δ's apart).

Recall that $f:E\to E$ is a (λ,C)-quasiisometry if

$$\frac{1}{\lambda}d(x,y)-C\leqslant d\big(f(x),f(y)\big)\leqslant \lambda d(x,y)+C$$

for all $x,y\in E$, and there exists g satisfying the same inequalities such that $f\circ g$ and $g\circ f$ are C-close to the identity. We let $\ell(f)=\inf_{x\in E}d(f(x),x)$ be the minimum displacement of f. Note that $\ell(g)\leqslant\lambda\ell(f)+2C$.

The following lemma is left as an exercise.

LEMMA 1.1. – *If f is a (λ,C)-quasiisometry of a compact interval to itself, then $\ell(f)\leqslant C$.*

From now on, we fix δ,λ,C. The quantities C_1,M_1,C_2 introduced below depend only on these three numbers, not on E, f, or the points under consideration. We also say that two points x,y are close, or have bounded distance, if their distance may be bounded a priori by some number depending only on δ,λ,C.

The quasiisometry f has the following basic property: there exists C_1 such that, for any geodesic segment $[x,y]$, the image of $[x,y]$ is contained in the C_1-neighborhood of $[f(x),f(y)]$.

Consider a geodesic triangle $a,f(a),f^2(a)$. Let u be a projection of a onto $[f(a),f^2(a)]$, and v a projection of $f(u)$ onto $[f(a),f^2(a)]$.

LEMMA 1.2. – *There exists M_1 such that, if $\ell(f)>M_1$, then $v\in[u,f^2(a)]$.*

Proof. – Suppose $v\in[f(a),u]$. Since $f(v)$ is close to $[f^2(a),f(u)]$ and $f(u)$ is close to v, the point $f^2(u)$ is close to $[f^2(a),f(u)]$. Thus, up to a bounded error, the points u and $f^2(u)$ both lie on the segment $[f^2(a),f(u)]$. It follows that f or g is close to a map sending $[u,f(u)]$ into itself. Lemma 1.1 implies that some point of $[u,f(u)]$ is close to its image by f. □

We assume from now on that $\ell(f)>M_1$.

LEMMA 1.3. – *There exists C_2 with the following property: for any $a\in E$, there exist three points p,q,r, lying in this order on $[a,f^2(a)]$, such that*

(1) *q is C_2-close to a projection of $f(a)$ onto $[a,f^2(a)]$;*

(2) *p is C_2-close to $g(q)$;*

(3) *r is C_2-close to $f(q)$.*

Proof. – With the same notations as above, it follows from Lemma 1.2 that u is close to $[f(a),f(u)]$. Therefore $g(u)$ is close to $[a,u]$. We also know that $f(u)$ is close to $[u,f^2(a)]$. Let p,q,r be projections onto $[a,f^2(a)]$ of $g(u),u,f(u)$ respectively. Either they are in the correct order $a,p,q,r,f^2(a)$, or this may be achieved by moving them by a bounded amount. □

Note that $f(q)$ is close to $[f(p),f(r)]$, hence to $[q,f^2(q)]$.

We now complete the proof of Proposition 0.4. View Lemma 1.3 as a way of assigning a point q to any point a. We construct a sequence q_n by iterating this process, with q_0 the point assigned by Lemma 1.3 to an arbitrary starting point $a\in E$. Since $f(q_n)$ is close to $[q_n,f^2(q_n)]$, the point q_{n+1} is close to $f(q_n)$.

Note that by construction $q_{n+1} \in [q_n, f^2(q_n)]$, while q_{n+2} is close to $f(q_{n+1})$, hence to $[q_n, f^2(q_n)]$ by assertion (3) of Lemma 1.3. Thus the broken geodesics $\gamma_n = [q_n, q_{n+1}] \cup [q_{n+1}, q_{n+2}]$ are uniformly quasigeodesic. Also note that by assertion (2) of Lemma 1.3 we have

$$d(q_n, q_{n+1}) \geqslant d\big(g(q_{n+1}), q_{n+1}\big) - C_2 \geqslant \ell(g) - C_2,$$

showing that the overlap between γ_n and γ_{n+1} is bounded below by a linear function of $\ell(f)$.

It follows from [5, Théorème 3.1.4] or [11, Théorème 5.25] that the sequence q_n is an infinite quasigeodesic γ^+ if $\ell(f)$ is large enough. Since $d(f(q_n), q_{n+1})$ is bounded, the point at infinity of γ^+ is fixed by ∂f (the homeomorphism induced by f on ∂E). The quasigeodesic may be extended in the other direction by applying the same construction to g, yielding a bi-infinite quasigeodesic, hence a second fixed point for ∂f. This proves Proposition 0.4.

COROLLARY 1.4. – *Let f be a (λ, C)-quasiisometry of a δ-hyperbolic proper geodesic metric space (E, d) to itself. There exists $N = N(\delta, \lambda, C)$, independent of E and f, with the following property: if $d(f(x), x) > N$ for all $x \in E$, then ∂f has North–South dynamics.*

Proof. – Suppose $\ell(f) > M$. Let γ be a bi-infinite geodesic joining two fixed points X_0, X_1 of ∂f. Consider $X \neq X_0, X_1$ in ∂E. Let θ be a projection of X onto γ. A projection θ' of $\partial f(X)$ is close to $f(\theta)$. If $\ell(f)$ is large enough, the distance from θ to θ' is bounded below and the oriented segment $\theta\theta'$ always points towards the same endpoint X_i of γ, independently of the choice of X. Applying this argument to both f and g, we deduce that ∂f has North–South dynamics. □

2. North–South dynamics

We first prove:

THEOREM 2.1. – *Let $\Phi \in \operatorname{Out} G$, with G hyperbolic. All but finitely many isogredience classes $s \in \mathcal{S}(\Phi)$ have North–South dynamics on ∂G.*

Proof. – Let E be the Cayley graph of G with respect to some finite generating set A, with the natural left-action of G. We identify the set of vertices of E with G, and ∂E with ∂G. We fix a "basepoint" $\alpha \in \Phi$, and we represent it by a quasiisometry $J: E \to E$ sending a vertex g to the vertex $\alpha(g)$, equivariant in the sense that $\alpha(h)J = Jh$ for every $h \in G$.

Given $\beta \in \Phi$, we write $\beta = i_m \circ \alpha$ and we consider the map $J_\beta = mJ$ (this involves a choice for m if the center of G is not trivial). Note that it maps a vertex g onto $m\alpha(g)$ (not onto $\beta(g) = m\alpha(g)m^{-1}$).

The map J_β satisfies $\beta(g)J_\beta = J_\beta g$, it induces $\partial\beta$ on ∂E (because a right-translation of G induces the identity on the boundary), and the maps J_β are uniformly quasiisometric (because they differ by left-translations).

If two maps J_β, J_γ, with $\beta, \gamma \in \Phi$, coincide at some point of E, then clearly $\beta = \gamma$. More generally:

LEMMA 2.2. – *Let $\beta, \gamma \in \Phi$. If there exist $g, h \in G$ with*

$$g^{-1}J_\beta(g) = h^{-1}J_\gamma(h),$$

then β and γ are isogredient.

Proof. – Writing $\beta = i_m \circ \alpha$ and $\gamma = i_n \circ \alpha$ we get

$$g^{-1}m\alpha(g) = h^{-1}n\alpha(h)$$

which we rewrite as

$$nm^{-1} = hg^{-1}m\alpha(gh^{-1})m^{-1} = hg^{-1}\beta(gh^{-1}),$$

showing that $\gamma = i_{nm^{-1}} \circ \beta = i_{hg^{-1}} \circ \beta \circ (i_{hg^{-1}})^{-1}$ is isogredient to β. □

By Corollary 1.4, there exists a number N (independent of β) such that, if J_β moves every point of E more than N, then $\partial\beta$ has North–South dynamics. Since E is a locally finite graph, Lemma 2.2 implies that this condition is fulfilled for all $\beta \in \Phi$ outside of a finite set of isogredience classes. This completes the proof of Theorem 2.1. □

Remark. – When G is a free group F_n, there is (using the notations of [4]) a one-to-one correspondence between $\mathcal{S}(\Phi)$ and the set of connected components of the graph $D(\varphi)$, for $\varphi \in \Phi$. In this case one may use Lemma 5.1 of [4] instead of Proposition 0.4 in the above proof. Also note that, as a corollary of Theorem 4 of [9], the map $\partial\beta$ has at most 4 fixed points for $\beta \in \Phi$ outside of at most $4n-4$ isogredience classes. Another remark: $\mathcal{S}(\Phi)$ is infinite when $\Phi \in \operatorname{Out} F_n$ fixes a nontrivial conjugacy class, by Proposition 5.4 of [4].

PROPOSITION 2.3. – *Suppose $\partial\alpha$ has North–South dynamics, with attracting fixed point X^+ and repelling fixed point X^-. Then*:

(1) *The subgroup $P(\alpha) \subset G$ consisting of all α-periodic elements is either finite or virtually* $\mathbf{Z}$ *with limit set $\{X^+, X^-\}$.*
(2) *If $g \in G$ is not α-periodic, then* $\lim_{n\to+\infty} \alpha^{\pm n}(g) = X^\pm$.

Proof. – Given $g \in G$ of infinite order, we denote $g^{\pm\infty} = \lim_{n\to+\infty} g^{\pm n}$. These are distinct points of ∂G. Note that $\partial\alpha(g^{\pm\infty}) = \alpha(g)^{\pm\infty}$. The subgroup of G consisting of elements whose action on ∂G leaves $\{g^\infty, g^{-\infty}\}$ invariant is the maximal virtually cyclic subgroup N_g containing g. If $h \notin N_g$, then $\{g^\infty, g^{-\infty}\}$ is disjoint from its image by h. If $\partial\alpha(g^\infty) = g^\infty$, then N_g is α-invariant (i.e. $\alpha(N_g) = N_g$).

Suppose (1) is false. Then there exist two α-periodic elements g, h of infinite order generating a non-elementary group. The points $g^{\pm\infty}$ and $h^{\pm\infty}$ are four distinct periodic points of $\partial\alpha$, a contradiction.

To prove (2), first suppose G is virtually cyclic. Then G maps onto $\mathbf{Z}$ or $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ with finite kernel (see [21]). From this one deduces that the periodic subgroup $P(\alpha)$ has index at most 2 and contains all elements of infinite order (an instructive example is conjugation by ab in $\langle a, b \mid a^2 = b^2 = 1\rangle$). Both ends of G are fixed by $\partial\alpha$; all non-periodic torsion elements (if any) converge towards one end under iteration of α, towards the other end under iteration of α^{-1}.

Now consider the general case. It suffices to show $\lim_{n\to+\infty} \alpha^n(g) = X^+$. Since X^+ is a fixed point of $\partial\alpha$, we are free to replace α by a power if needed. We first note that there exists a number C such that

$$(g, g^\infty) \geqslant \frac{1}{2}|g| - C$$

for every g of infinite order (where $(\ ,\ )$ denotes Gromov's scalar product based at the identity in the Cayley graph, and $|\ |$ is word length). This follows easily from Lemma 3.5 of [20] (if G is free and A is a basis, $C = -1/2$ clearly works).

Suppose g is not α-periodic. Then $\lim_{n\to\infty} |\alpha^n(g)| = \infty$. If furthermore g has infinite order, applying the previous inequality to $\alpha^n(g)$ yields

$$\lim_{n\to+\infty} \alpha^n(g) = \lim_{n\to+\infty} \bigl(\alpha^n(g)\bigr)^\infty = \lim_{n\to+\infty} \partial\alpha^n(g^\infty) = X^+$$

(note that $g^\infty \neq X^-$, since otherwise $N(g)$ would be α-invariant and g would be periodic).

Now we consider a non α-periodic element g of finite order. We distinguish two cases. Suppose first that $\{X^+, X^-\}$ is the limit set of an infinite α-invariant virtually cyclic subgroup H. We may assume that H is maximal (it then contains all periodic elements). If $g \notin H$, choose $h \in H$ of infinite order, with $h^{\pm\infty} = X^{\pm}$. Replacing α by a power, we may assume $\alpha(h) = h$. We have $gh^{\infty} \neq h^{-\infty}$, and therefore $g_k = h^k g h^k$ has infinite order for k large enough. Since $\alpha^n(g_k)$ converges to X^+ as $n \to +\infty$, we find that $\alpha^n(g)$ converges to $h^{-k}X^+ = X^+$, as desired. There remains to rule out the possibility that non-periodic torsion elements $g \in H$ converge towards X^- under iteration of α. If this happens, choose $j \notin H$. Since j and gj are not α-periodic (they don't belong to H), we know that $\alpha^n(j)$ and $\alpha^n(gj)$ are close to X^+ for n large. But $\alpha^n(g)$ and $\alpha^n(g^{-1})$ are close to X^-. This is impossible.

If $\{X^+, X^-\}$ is not as above, then X^+ (respectively X^-) is an attracting (respectively repelling) fixed point for the action of $\alpha \cup \partial\alpha$ on the compact space $G \cup \partial G$ (see [14]). The desired result $\lim_{n\to+\infty} \alpha^n(g) = X^+$ follows from an elementary dynamical argument. Indeed, the sequence $\alpha^n(g)$, with $n > 0$, has some limit point $X \in \partial G$. We have $X \neq X^-$ because X^- is repelling on $G \cup \partial G$, and therefore $\partial\alpha^n(X)$ converges to X^+. We then deduce that X^+ is a limit point of $\alpha^n(g)$, and finally that $\alpha^n(g)$ converges to X^+ because X^+ is attracting on $G \cup \partial G$. □

3. Isogredience classes

The main result of this section is the infiniteness of $\mathcal{S}(\Phi)$ (but see also Proposition 3.7). We first study four different situations where we can reach this conclusion. For now, we only assume that G is any finitely generated group. We fix $\Phi \in \operatorname{Out} G$ and $\alpha \in \Phi$.

• By definition, the automorphisms $\beta = i_m \circ \alpha$ and $\gamma = i_n \circ \alpha$ are isogredient if and only if there exists $g \in G$ with $\gamma = i_g \circ \beta \circ i_g^{-1}$, or equivalently $n = gm\alpha(g^{-1})c$ with c in the center of G. Though we will not use it, we note that *$\mathcal{S}(\Phi)$ is infinite if the center of G is finite and the action of Φ on $H_1(G;\mathbf{R})$ has 1 as an eigenvalue.*

Now assume that Φ preserves some $\mathbf{R}$-tree (see [6], [17], [22] for basics about $\mathbf{R}$-trees). This means that there is an $\mathbf{R}$-tree T equipped with an isometric action of G whose length function satisfies $\ell \circ \Phi = \lambda\ell$ for some $\lambda \geqslant 1$. We always assume that the action is minimal and irreducible (no global fixed point, no invariant line, no invariant end). We say $g \in G$ is hyperbolic if it is hyperbolic as an isometry of T. We shall use the following fact due to Paulin [19]: any segment $[a,b] \subset T$ is contained in the axis of some hyperbolic $g \in G$.

Because $\ell \circ \Phi = \lambda\ell$, it follows from [6] (see also [9], [16]) that, given $\alpha \in \Phi$, there is a (unique) map $H = H_\alpha : T \to T$ with the following properties: H is a homothety with stretching factor λ (i.e. $d(Hx, Hy) = \lambda d(x,y)$), and it satisfies $\alpha(g)H = Hg$ for every $g \in G$. If $\beta = i_m \circ \alpha$, then $H_\beta = mH_\alpha$. If $\beta = i_g \circ \alpha \circ i_g^{-1}$ is isogredient to α, then $H_\beta = gH_\alpha g^{-1}$ is conjugate to H_α.

• First consider the case when $\lambda = 1$. In this case the translation length of the isometry H_β is an isogredience invariant of β and we easily get:

PROPOSITION 3.1. – *Suppose $\ell \circ \Phi = \ell$, where ℓ is the length function of an irreducible action of G on an $\mathbf{R}$-tree. Then $\mathcal{S}(\Phi)$ is infinite.*

Proof. – Fix $\alpha \in \Phi$. Using Paulin's lemma, it is easy to construct $m \in G$ with the translation length of mH_α arbitrarily large. The corresponding automorphisms $i_m \circ \alpha$ are in distinct isogredience classes. □

• The case $\lambda > 1$ is harder.

PROPOSITION 3.2. – *Suppose $\ell \circ \Phi = \lambda\ell$, where $\lambda > 1$ and ℓ is the length function of an irreducible action of G on an $\mathbf{R}$-tree T. Assume that arc stabilizers are finite, and there exists*

$N_0 \in \mathbf{N}$ *such that, for every* $Q \in T$, *the action of* $\operatorname{Stab} Q$ *on* $\pi_0(T \setminus \{Q\})$ *has at most* N_0 *orbits. Then* $\mathcal{S}(\Phi)$ *is infinite.*

An arc stabilizer is the pointwise stabilizer of a nondegenerate segment $[a,b]$, and $\operatorname{Stab} Q$ denotes the stabilizer of Q.

Proof. – Fix $\alpha \in \Phi$ and consider $H = H_\alpha$. We choose a point $P \in T$ as follows. It is the unique fixed point of H if H has a fixed point in T. Otherwise H has a unique fixed point Q in the metric completion $\overline{T}$ of T, and a unique eigenray ρ (by definition, ρ is the image of an isometric embedding $\rho\colon (0,\infty) \to T$ such that $H\rho(t) = \rho(\lambda t)$ for all $t > 0$, see [9]). We let P be any point on ρ. In both cases $P \in [H^{-1}P, HP]$.

For further reference, we note that the stabilizer of any initial segment $\rho(0,t)$ of an eigenray is the same as the stabilizer of the whole eigenray, because $\operatorname{Stab}\rho(0,t)$ and $\operatorname{Stab}\rho(0,\lambda t) = \alpha(\operatorname{Stab}\rho(0,t))$ are finite groups with the same order. Suppose furthermore that H has two eigenrays ρ, ρ', and $g \in G$ maps an initial segment of ρ onto an initial segment of ρ'. From the basic equation $\alpha(g)H = Hg$ it follows that $g^{-1}\alpha(g)$ fixes an initial segment of ρ, hence all of ρ, and we deduce that g maps the *whole* of ρ onto ρ'.

Returning to the main line of proof, we want to find $v, w \in G$ generating a free subgroup of rank 2, such that:

(i) vP and wP belong to a component T^+ of $T \setminus \{P\}$.
(ii) $v^{-1}P$ and $w^{-1}P$ belong to another component T^-.
(iii) If $HP \neq P$, then $H^{\pm 1}P \in T^{\pm}$.
(iv) If $HP = P$, then $H(T^+) \neq T^-$.

Note that these conditions force v and w to be hyperbolic, with axes intersecting in a nondegenerate segment containing P in its interior. Furthermore, the two axes induce the same orientation on their intersection.

It is easy to construct v, w using Lemma 2.6 of [6] and Paulin's lemma, except in one "bad" situation where (iv) cannot be achieved: $HP = P$, and $T \setminus \{P\}$ has exactly two components, which are permuted by H.

If H is bad, we have to change our initial choice of $\alpha \in \Phi$. We use the following observation. Suppose H_1, H_2 are homotheties with the same dilation factor $\lambda > 1$ and distinct fixed points P_1, P_2; if H_1 (respectively H_2) does not send the component of $T \setminus \{P_1\}$ (respectively $T \setminus \{P_2\}$) containing P_2 (respectively P_1) into itself, then $H_2 H_1^{-1}$ is a hyperbolic isometry whose axis contains $[P_1, P_2]$.

We choose $m \in G$ acting on T as a hyperbolic isometry with axis not containing P, and we replace α by $\alpha' = i_m \circ \alpha$. Let $H' = mH = H_{\alpha'}$. We claim that H' cannot be bad (with respect to its fixed point P'). Indeed, this follows from the above observation because the axis of $H'H^{-1}$ does not contain P. Thus, when H is bad, we can find v, w satisfying the above conditions with respect to H'. For simplicity, we keep writing H, α rather than H', α'.

Now assume by way of contradiction that there are only K isogredience classes in $\mathcal{S}(\Phi)$. Given an integer p, consider the set W consisting of words in the letters v, w containing each letter exactly p times (we do not use v^{-1} or w^{-1}). We fix p such that W has more than Ks^2N_0 elements, where s is the order of the stabilizer of the arc $I = [P, vP] \cap [P, wP]$ and N_0 is defined in the statement of Proposition 3.2. We will consider the automorphisms $i_\sigma \circ \alpha$, for $\sigma \in W$, and the corresponding homotheties σH.

Consider $\sigma = u_1 \dots u_{2p} \in W$, with each u_i equal to v or w. The elements v, w were chosen in such a way that the points

$$P, u_1P, u_1u_2P, \dots, u_1 \dots u_{2p}P, u_1 \dots u_{2p}HP = \sigma HP$$

all lie in this order on the segment $[P, \sigma HP]$ (with the last two points possibly equal). Since P belongs to the axis of both v and w, we find that, for any $\sigma \in W$, the length of $[P, \sigma HP]$ equals

$$L = p\ell(v) + p\ell(w) + d(P, HP)$$

independently of σ. We also observe that, if $\sigma, \tau \in W$, then $[P, \sigma HP] \cap [P, \tau HP]$ contains the segment $I = [P, vP] \cap [P, wP]$.

Furthermore the intersection $[P, \sigma HP] \cap [\sigma HP, (\sigma H)^2 P]$ consists only of σHP: this follows from $P \in [u_{2p}^{-1} P, Hu_1 P]$ if $HP = P$, from $P \in [H^{-1}P, u_1 P]$ if $HP \neq P$. This implies that $[P, \sigma HP]$ is contained in an eigenray ρ_σ of the homothety σH. Let Q_σ denote the fixed point of σH in the completion $\overline{T}$ (the origin of ρ_σ).

Now we remark that $[P, \sigma HP]$ is *the only fundamental domain of length* L for the action of σH on its eigenray ρ_σ. In particular, $d(Q_\sigma, P) = \frac{L}{\lambda - 1}$ is independent of $\sigma \in W$.

Suppose for a moment that for every $\sigma \in W$ the map σH has only one eigenray (this happens in particular if $Q_\sigma \in \overline{T} \setminus T$). If i_c conjugates $i_\sigma \circ \alpha$ and $i_\tau \circ \alpha$ (with $\sigma, \tau \in W$ and $c \in G$), then c conjugates σH and τH. Therefore c sends ρ_σ onto ρ_τ, and the fundamental domain $[P, \sigma HP]$ onto $[P, \tau HP]$. Since these segments both contain I, we find $c \in \operatorname{Stab} I$. This contradicts the choice of p in this special case, since we obtain $|W|/s$ distinct isogredience classes in $\mathcal{S}(\Phi)$.

In general, if i_c conjugates $i_\sigma \circ \alpha$ and $i_\tau \circ \alpha$, we can only say that c sends Q_σ to Q_τ. Since $|W| > Ks^2 N_0$, we can find distinct elements $\sigma, \tau(1), \ldots, \tau(s^2+1)$ in W such that some $i_{c(j)}$ conjugates $i_\sigma \circ \alpha$ and $i_{\tau(j)} \circ \alpha$, and some element $h(j) \in \operatorname{Stab} Q_{\tau(j)}$ sends an initial segment of the $[\tau(j)H]$-eigenray $c(j)\rho_\sigma$ onto an initial segment of $\rho_{\tau(j)}$.

We have pointed out earlier that $h(j)$ sends the whole eigenray $c(j)\rho_\sigma$ onto $\rho_{\tau(j)}$. Therefore $h(j)c(j) \in \operatorname{Stab} I$. Thus there are at least $s+1$ values of j for which the maps $\tau(j)H$ have a common eigenray containing I. This is a contradiction because at most s elements of G can have the same action on I. This completes the proof of Proposition 3.2. □

• We also need:

PROPOSITION 3.3. – *$\mathcal{S}(\Phi)$ is infinite if G is hyperbolic, non-elementary, and Φ has finite order in* $\operatorname{Out} G$.

Proof. – Let J be the subgroup of $\operatorname{Aut} G$ consisting of all automorphisms whose image in $\operatorname{Out} G$ is a power of Φ. The exact sequence $\{1\} \to K \to J \to \langle \Phi \rangle \to \{1\}$, with $K = G/\text{Center}$ and $\langle \Phi \rangle$ finite, shows that J is hyperbolic, non-elementary. The set of automorphisms $\alpha \in \Phi$ is a coset of J mod K. If $\alpha, \beta \in \Phi$ are isogredient, they are conjugate in J. The proof of Proposition 3.3 is therefore concluded by applying the following fact, due to T. Delzant. □

LEMMA 3.4. – *Let J be a non-elementary hyperbolic group. Let K be a normal subgroup with abelian quotient. Every coset of J mod K contains infinitely many conjugacy classes.*

Proof. – Fix u in the coset C under consideration. Suppose for a moment that we can find $c, d \in K$, generating a free group of rank 2, such that $uc^\infty \neq c^{-\infty}$ and $ud^\infty \neq d^{-\infty}$ (recall that we denote $g^{\pm\infty} = \lim_{n \to +\infty} g^{\pm n}$ for g of infinite order). Consider $x_k = c^k u c^k$ and $y_k = d^k u d^k$. For k large, the above inequalities imply that these two elements have infinite order, and do not generate a virtually cyclic group because $x_k^{\pm\infty}$ (respectively $y_k^{\pm\infty}$) is close to $c^{\pm\infty}$ (respectively $d^{\pm\infty}$). Fix k, and consider the elements $z_n = x_k^{n+1} y_k^{-n}$. They belong to the coset C, because J/K is abelian, and their stable norm goes to infinity with n. Therefore C contains infinitely many conjugacy classes.

Let us now construct c, d as above. Choose $a, b \in K$ generating a free group of rank 2. We first explain how to get c. There is a problem only if $ua^\infty = a^{-\infty}$ and $ub^\infty = b^{-\infty}$. In that case there exist integers p, q with $ua^pu^{-1} = a^{-p}$ and $ub^qu^{-1} = b^{-q}$. We take $c = a^pb^q$, noting that $ucu^{-1} = a^{-p}b^{-q}$ is different from $c^{-1} = b^{-q}a^{-p}$.

Once we have c, we choose $c' \in K$ with $\langle c, c'\rangle$ free of rank 2, and we obtain d by applying the preceding argument using c' and cc' instead of a and b. The group $\langle c, d\rangle$ is free of rank 2 because d is a positive word in c' and cc'. □

Remark. – As pointed out by Delzant, similar arguments show that $\mathcal{S}(\Phi)$ is infinite when Φ has infinite order but is hyperbolic in the sense of [1] (because J is hyperbolic, see [1]).

We can now prove:

THEOREM 3.5. – *For every $\Phi \in \operatorname{Out} G$, with G a non-elementary hyperbolic group, the set $\mathcal{S}(\Phi)$ is infinite.*

Proof. – By Proposition 3.3, we may assume that Φ has infinite order. By Paulin's theorem [20], it preserves some $\mathbf{R}$-tree T with a nontrivial minimal small action of G (recall that an action of G is small if all arc stabilizers are virtually cyclic; the action of G on T is always irreducible).

If $\lambda = 1$, we use Proposition 3.1. If $\lambda > 1$, we apply Proposition 3.2. The existence of N_0 follows from work of Bestvina and Feighn [2] (alternatively, one could for G torsion-free use ad hoc trees as in [15]). Finiteness of arc stabilizers is stated as the next lemma. □

LEMMA 3.6. – *Suppose $\ell \circ \Phi = \lambda\ell$, where ℓ is the length function of a nontrivial small action of a hyperbolic group G on an $\mathbf{R}$-tree T. If $\lambda > 1$, then T has finite arc stabilizers.*

Proof. – This is proved in [9, Lemma 2.8] when G is free. We sketch the proof of the general case. We may assume that the action is minimal. Let $c \subset T$ be an arc with infinite stabilizer S. Let p be the index of S in the maximal virtually cyclic subgroup $\overline{S}$ that contains it. Fix $\alpha \in \Phi$, and denote by H the associated homothety of T.

Since there is a finite union of arcs whose union meets every orbit, we can find, for k large, disjoint subarcs $c_0, \dots, c_p$ of $H^k(c)$ such that $c_i = v_ic_0$ for some $v_i \in G$. For each i, the stabilizer of c_i lies between $\alpha^k(S) = \operatorname{Stab} H^k(c)$ and $\alpha^k(\overline{S})$. From $\operatorname{Stab} c_i = v_i \operatorname{Stab} c_0 v_i^{-1}$ we get $\alpha^k(\overline{S}) = v_i\alpha^k(\overline{S})v_i^{-1}$, hence $v_i \in \alpha^k(\overline{S})$. This is a contradiction since $1, v_1, \dots, v_p$ all lie in different cosets of $\alpha^k(\overline{S})$ modulo $\alpha^k(S)$. □

If G is a free group F_n, we also prove:

PROPOSITION 3.7. – *There exists a number C_n such that, for any $\Phi \in \operatorname{Out} F_n$ and any integer $k \geqslant 2$, the natural map $\mathcal{S}(\Phi) \to \mathcal{S}(\Phi^k)$ is at most C_n-to-one.*

Proof. – Let α_i $(1 \leqslant i \leqslant N)$ be pairwise non-isogredient automorphisms in Φ having isogredient kth powers. We want to bound N in terms of n only. We may assume that α_i^k is a fixed automorphism β.

Let T be an $\mathbf{R}$-tree with trivial arc stabilizers preserved by Φ (see [9, Theorem 2.1]), and H_i the homothety associated to α_i. The H_i's all have the same kth power H_β. For $i \neq j$, we have $H_i = g_{ij}H_j$ for some nontrivial $g_{ij} \in F_n$. Note that H_i and H_j cannot coincide on more than one point since F_n acts on T with trivial arc stabilizers.

First suppose $\lambda > 1$. Then H_β and all maps H_i fix the same point $Q \in \overline{T}$. The stabilizer $\operatorname{Stab} Q \subset F_n$ is α_i-invariant and has rank $< n$ by [10] (see [9]).

If $\operatorname{Stab} Q$ is trivial (in particular if $Q \in \overline{T} \setminus T$), then $g_{ij} = 1$ and $\alpha_i = \alpha_j$.

If $\operatorname{Stab} Q$ has rank $\geqslant 2$, we use induction on n since the restrictions of the α_i's to $\operatorname{Stab} Q$ are non-isogredient automorphisms representing the same outer automorphism [9, Lemma 5.1].

If $\operatorname{Stab} Q$ is cyclic, generated by some u, we note that g_{ij} is a power of u and $\alpha_i(u)$ is independent of i. If $\alpha_i(u) = u$, then H_i commutes with u and $H_i^k = H_j^k$ implies $g_{ij} = 1$. If

$\alpha_i(u) = u^{-1}$, we write $u^{2p} = u^p \alpha_i(u^p)^{-1}$, showing that α_i is isogredient to α_j whenever g_{ij} is an even power of u.

Now suppose $\lambda = 1$. If H_β has no fixed point, then $N = 1$ since all H_i's coincide on the axis of H_β. Assume therefore that H_β has fixed points. If all maps H_i have a common fixed point Q, we can argue as above. We complete the proof by showing how to reduce to this situation.

Let Q_i be a fixed point of H_i, and e_i some edge containing Q_i and fixed by H_β. The action of F_n on pairs (Q_i, e_i) has at most $6n - 6$ orbits (twice the number of edges of the quotient graph T/F_n). After possibly dividing N by $6n - 6$ we may assume there is only one orbit. Note that the action on T of the element $c_{ij} \in F_n$ sending (Q_i, e_i) to (Q_j, e_j) commutes with H_β since e_i and e_j are both fixed by H_β. This implies that β fixes c_{ij}, and we can change α_i within its isogredience class so as to make all points Q_i the same, while retaining the property $\alpha_i^k = \beta$. □

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