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D. Harari

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WEAK APPROXIMATION AND NON-ABELIAN FUNDAMENTAL GROUPS

BY D. HARARI

ABSTRACT. – We introduce a new obstruction to weak approximation which is related to étale non-abelian coverings of a proper and smooth algebraic variety X defined over a number field k. This enables us to give some counterexamples to weak approximation which are not accounted for by the Brauer–Manin obstruction, for example bielliptic surfaces. © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous introduisons une nouvelle obstruction à l'approximation faible liée aux revêtements étales non-abéliens d'une variété algébrique propre et lisse définie sur un corps de nombres. Cela permet d'obtenir des contre-exemples à l'approximation faible qui ne viennent pas de l'obstruction de Brauer–Manin, par exemple les surfaces bielliptiques. © 2000 Éditions scientifiques et médicales Elsevier SAS

0. Introduction

Let X be a smooth and proper algebraic variety over a number field k and let Ω_k be the set of places of k. Recall (cf. [5,32]) that X is a *counterexample to the Hasse principle* if the set $X(\mathbf{A}_k) := \prod_{v\in\Omega_k} X(k_v)$ of adelic points of X is non-empty, but the set $X(k)$ of k-rational points of X is empty. A k-variety X (such that $X(k) \neq \emptyset$) satisfies *weak approximation* if $X(k)$ is dense in $X(\mathbf{A}_k)$ (equipped with the product of the v-adic topologies). It satisfies *weak approximation outside* S (where S is a finite set of places of k) if $X(k)$ is dense in $X(\mathbf{A}_k^S) := \prod_{v\notin S} X(k_v)$.

In his talk at the ICM in 1970 [25], Manin defined an obstruction to the Hasse principle, the so-called *Brauer–Manin obstruction*. A similar obstruction to weak approximation was later defined by Colliot-Thélène and Sansuc (cf. [5], Section 3). These obstructions are related to the Brauer group $\operatorname{Br} X$ of X (we shall recall their precise definitions in Section 1.3). For a long time, all known counterexamples to the Hasse principle and to weak approximation outside the archimedean places could be explained by means of the Brauer–Manin obstruction (it is well-known that the Brauer–Manin obstruction does not give much information at the archimedean places: see, for example, [33]).

Assuming a conjecture of Lang (the finiteness of $X(\mathbf{Q})$ if $X(\mathbf{C})$ is hyperbolic), Sarnak and Wang [30] found a smooth hypersurface of degree 1130 in $\mathbf{P}^4_{\mathbf{Q}}$ which is a counterexample to the Hasse principle not accounted for by the Brauer–Manin obstruction.

The first unconditional proof that the Brauer–Manin obstruction to the Hasse principle is not always the only one was very recently given by Skorobogatov [32]: he considers an elliptic surface over $\mathbf{Q}$ which is the quotient of the product of two curves of genus 1 by a fixed-point free involution. A natural question was to solve the similar problem for weak approximation (outside the archimedean places, or, more generally, outside a finite set of places) with the additional restriction that the variety X contains a k-rational point (or even that the set of k-rational points of X is Zariski-dense).

In this paper, we introduce a new obstruction to weak approximation related to **non-abelian** étale coverings of X (Proposition 2.2). We show (Theorem 5.1) that it is possible to use this obstruction to give sufficient conditions for X to be a counterexample to weak approximation (and actually to weak approximation outside S for any finite set $S \subset \Omega_k$) not accounted for by the Brauer–Manin obstruction. An interesting point is that these conditions (except the arithmetic assumption $X(k) \neq \emptyset$) are purely geometric. This reflects the fact that the arithmetic of a variety is expected to be closely related to its geometry. In particular $X \times_k K$ will be a counterexample of the same kind for any finite field extension K/k. We give several explicit examples, some of them with $X(k)$ Zariski-dense in X (Propositions 6.3 and 6.4).

The link to Skorobogatov's paper is the following: one of the ideas he uses is that the Brauer group may become bigger after passing to a finite unramified covering Y/X. In his example, this phenomenon is related to the existence of an abelian covering Z/Y such that the composite covering Z/X is **not** abelian. Using non-abelian torsors, it is possible to give a general formulation to explain Skorobogatov's counterexample to the Hasse principle and the counterexamples to weak approximation of the present paper. This consists of a generalization of the descent formalism developed in [32] (following the descent theory of Colliot-Thélène and Sansuc) in the abelian case. See [16] for more details.

The paper is organised as follows: in section 1 we recall some basic results. In Section 2, we introduce the obstruction to weak approximation related to the geometric fundamental group of a variety (Proposition 2.2 and Corollary 2.4). In Section 3, sufficient conditions for the vanishing of the Brauer–Manin obstruction associated to an adelic point are given (Corollary 3.3). Section 4 is purely geometric: for a variety X defined over an algebraically closed field, we show that a connected non-abelian covering of X does not become trivial after specialization to a fibre of the Albanese map f, provided that some restrictions are made on f (Proposition 4.2). Section 5 is devoted to the proof of our main Theorem 5.1 and in Section 6, we give miscellaneous applications of our results.

1. Preliminaries

1.1. Notation

Let k be a field of characteristic zero. Fix an algebraic closure $\bar{k}$ of k and set $\mathcal{G}_k = \mathrm{Gal}(\bar{k}/k)$. For any abelian group (or group scheme) C, the notation C_{tors} stands for the torsion part of C. If C is a locally compact group, we denote by $C^\vee$ its Pontryagin dual.

We shall frequently write $H^i(k, C)$ instead of $H^i(\mathcal{G}_k, C)$ for Galois cohomology groups with values in a $\mathcal{G}_k$-module C. For any scheme X, let $\mathrm{Pic}\, X = H^1_{\text{ét}}(X, \mathbf{G}_m)$ be the Picard group of X and let $\mathrm{Br}\, X = H^2_{\text{ét}}(X, \mathbf{G}_m)$ be the (cohomological) Brauer group of X. If X is a smooth and projective variety over an algebraically closed field, the condition $H^2(X, \mathcal{O}_X) = 0$ implies $\mathrm{Br}\, X$ finite ([14], Corollary 3.4 and [15], 8.12).

Assume that X is a k-variety (that is a separated k-scheme of finite type) and set $\overline{X} = X \times_k \bar{k}$. We let X_{red} denote the reduced k-variety associated to X and we let $\bar{k}[X]^* := H^0(\overline{X}, \mathbf{G}_m)$ be the set of invertible regular functions on $\overline{X}$. If K is a field and $N \in X(K)$, we shall still denote by N the corresponding K-point of X_{red} (cf. [18], II.2.3, p. 79). For any étale group scheme F on X, we define:

$$H^i_a(X, F) := \mathrm{Ker}\big[H^i_{\text{ét}}(X, F) \to H^i_{\text{ét}}\left(\overline{X}, F\right)\big],$$
$$\mathrm{Br}_a X := \mathrm{Ker}\big[\mathrm{Br}\, X \to \mathrm{Br}\, \overline{X}\big].$$

We shall always assume that $\bar{k}$ admits a (fixed) embedding in the field $\mathbf{C}$ of complex numbers, and we shall write $X_{\mathbf{C}} := \overline{X} \times_{\bar{k}} \mathbf{C}$.

We let $\pi_1(X,m)$ (or $\pi_1(X)$ if m is irrelevant) denote the étale fundamental group (for the geometric base point m) of a scheme X. Assume that X is a proper, smooth and geometrically connected k-variety; then the comparison theorem for the fundamental group ([12], X.1.8 and XII.5.2) says that $\pi_1(\overline{X})$ is the profinite completion of the topological fundamental group $\pi_1^{\mathrm{top}}(X_{\mathbf{C}})$. For example the geometric fundamental group $\pi_1(\overline{A})$ of an abelian k-variety A of dimension g is isomorphic to $\widehat{\mathbf{Z}}^{2g}$ ([12], XI.2.1). For any profinite group G, let $D(G)$ be the derived subgroup of G (in the category of profinite groups) and $G^{\mathrm{ab}} = G/D(G)$ be the abelianized group of G. The neutral element of G will frequently be denoted by e when G is not assumed to be abelian. The Néron–Severi group of a proper, smooth and connected variety X over an algebraically closed field is denoted by $\mathrm{NS}\,X$. It is a finitely generated abelian group [24] which is the quotient of $\mathrm{Pic}\,X$ by the (divisible) group $\mathrm{Pic}^0 X$ of those classes of divisors which are algebraically equivalent to zero.

Let X be a smooth and projective k-variety. A line bundle L on X is said to be *nef* if $L.C \geqslant 0$ for all complete curve $C \subset X$. Let E be a vector bundle on X with projectivized bundle of hyperplanes $\mathbf{P}(E)$ and associated canonical line bundle $\mathcal{O}_E(1)$. Then E is said to be *nef* if $\mathcal{O}_E(1)$ is nef over $\mathbf{P}(E)$ [7].

1.2. Torsors and coverings

In this paragraph we recall some facts about torsors, which were used by Colliot-Thélène and Sansuc in their theory of descent [5] and by Skorobogatov in [32].

Let X be a k-variety. Let F be a $\mathcal{G}_k$-module which is of finite type as an abelian group, and S be the k-group of multiplicative type which is dual of F. Recall [5,32] that an *X-torsor* Y under S is a principal homogeneous space (over X) $f: Y \to X$ under S. The X-torsors under S (up to isomorphisms) correspond to the elements of $H^1_{\text{ét}}(X,S)$ ([27], III.4.6).

Let $K \supset k$ be a field and let $M \in X(K) = \mathrm{Hom}_k(\mathrm{Spec}\,K, X)$. We shall say that the torsor Y *splits* at M if there exists a K-point M' of Y such that $f(M') = M$. This is equivalent to saying that the evaluation $[Y](M) \in H^1(K,S)$ is trivial, where $[Y]$ is the class of Y in $H^1_{\text{ét}}(X,S)$.

The following fundamental exact sequence was introduced by Colliot-Thélène and Sansuc ([5], Theorem 1.5.1):

$$0 \to \mathrm{Ext}^1_{\mathcal{G}_k}\left(F, \bar{k}[X]^*\right) \to H^1_{\text{ét}}(X,S) \xrightarrow{\chi} \mathrm{Hom}_{\mathcal{G}_k}\left(F, \mathrm{Pic}\,\overline{X}\right) \tag{1}$$

with:

$$\mathrm{Coker}\,\chi = \mathrm{Ker}\left[\mathrm{Ext}^2_{\mathcal{G}_k}\left(F, \bar{k}[X]^*\right) \to H^2_{\text{ét}}(X,S)\right]$$

and the *type* of Y is the image of $[Y]$ by χ. If X is proper, reduced, and geometrically connected, then the kernel of the map χ is just $H^1(k,F)$: indeed $\bar{k}[X]^* = \bar{k}^*$ in this case ($\mathrm{char}\,k = 0$). This also implies ([5], 1.5) that the cokernel of χ is just the kernel of the pull-back (with respect of the structural morphism $X \to \mathrm{Spec}\,k$) $H^2(k,S) \to H^2_{\text{ét}}(X,S)$. Thus there exists a torsor of type λ (for any $\lambda \in \mathrm{Hom}_{\mathcal{G}_k}(F, \mathrm{Pic}\,\overline{X})$) as soon as $X(k) \neq \emptyset$ (because of the existence of a section for the structural map $X \to \mathrm{Spec}\,k$).

A *covering* of X is a finite, flat and surjective k-morphism $f: Z \to X$. Let n_Z (respectively n_X) be the number of connected components of $\overline{Z}$ (respectively $\overline{X}$) and d be the degree of f. The covering f is said to be *geometric* (respectively geometrically non-trivial) if $n_Z = n_X$ (respectively $n_Z < d n_X$). We shall use "non-trivial" instead of "geometrically non-trivial" when k is algebraically closed. If X is smooth and geometrically integral over k, then an étale covering $Z \to X$ is geometric if and only if the k-variety Z is geometrically integral.

Let $f:Z\to X$ be a covering and $M\in X(K)$, where K is some field containing k. We shall say that f *splits* at M if f is étale at M and there exists a K-point N of Z such that $f(N)=M$. If f is Galois ([27], I.5.4), this is equivalent to saying that f splits completely at M, that is f is étale at M and the fibre of f at M consists of K-points.

Similarly an étale algebra L of degree d over a number field k is said to be *split* at a place v of k if L/k is unramified at v and k_v is a direct summand of $L\otimes_k k_v$; it is *completely split* (or totally split) at v if $L\otimes_k k_v\simeq k_v^d$; the two conditions are equivalent if L/k is Galois.

1.3. The Brauer–Manin obstruction to the Hasse principle and weak approximation

Let k be a number field. We let Ω_k denote the set of places of k and we let k_v be the completion of k at the place v. For any finite place v, let $\mathcal{O}_v$ be the ring of integers of k_v and $\mathbf{F}_v$ be its residue field. Local class field theory gives an injective map $j_v:\operatorname{Br}k_v\to\mathbf{Q}/\mathbf{Z}$ which is an isomorphism for v finite. Let S be a finite set of places of k containing the archimedean places, and let $\mathcal{O}_k$ be the ring of integers of k. The *ring of S-integers* $\mathcal{O}_{k,S}$ is the set of elements $x\in k$ such that $v(x)\geqslant 0$ for any $v\notin S$.

Let X be a smooth, proper and geometrically integral k-variety and assume $X(k)\neq\emptyset$. Recall that X satisfies *weak approximation* if $X(k)$ is dense in the set $X(\mathbf{A}_k)=\prod_{v\in\Omega_k}X(k_v)$ of adelic points of X (equipped with the product of the v-adic topologies). Set:

$$X(\mathbf{A}_k)^{\mathrm{Br}}=\left\{(M_v)\in X(\mathbf{A}_k),\ \forall A\in\operatorname{Br}X,\ \sum_{v\in\Omega_k}j_v\bigl(A(P_v)\bigr)=0\right\}.$$

(The sum is well-defined: indeed X is proper, hence the specialization $A(P_v)$ comes from $\operatorname{Br}\mathcal{O}_v$ outside the finite set S of places of bad reduction of X and A; so, by [27], IV.2.13, $A(P_v)=0$ for $v\notin S$.)

Let $\overline{X(k)}$ be the closure of $X(k)$ in $X(\mathbf{A}_k)$. The reciprocity law of global class field theory implies:

$$\overline{X(k)}\subset X(\mathbf{A}_k)^{\mathrm{Br}}.$$

In particular, the condition $(M_v)\notin X(\mathbf{A}_k)^{\mathrm{Br}}$ for some adelic point (M_v) is an obstruction to weak approximation, the so-called *Brauer–Manin obstruction* (see [5], 3.1 for more details). The condition $X(\mathbf{A}_k)^{\mathrm{Br}}=\emptyset$ is the Brauer–Manin obstruction to the Hasse principle.

Let S be a finite set of places of k; set $X(\mathbf{A}_k^S)=\prod_{v\notin S}X(k_v)$. We let $\overline{X(k)}^S$ denote the set of elements of $X(\mathbf{A}_k)$ whose projection to $X(\mathbf{A}_k^S)$ belongs to the closure of $X(k)$ in $X(\mathbf{A}_k^S)$. Namely $\overline{X(k)}^S$ is the product of $\prod_{v\in S}X(k_v)$ by the closure of $X(k)$ in $\prod_{v\notin S}X(k_v)$.

Note that if k is totally imaginary, then we have $\overline{X(k)}^{\Omega_\infty}\subset X(\mathbf{A}_k)^{\mathrm{Br}}$, where Ω_∞ is the set of archimedean places of k.

The goal of this paper is to construct some classes of varieties X such that the relation $X(\mathbf{A}_k)^{\mathrm{Br}}\not\subset\overline{X(k)}^S$ holds for any finite set $S\subset\Omega_k$. In other terms, there exist adelic points on X for which there is no Brauer–Manin obstruction, but which cannot be approximated by a rational point. More precisely, for any finite set $S\subset\Omega_k$, there exists a finite set T of places of k, with $T\cap S=\emptyset$, and an element $(M_v)_{v\in T}$ in $\prod_{v\in T}X(k_v)$ with the following properties: $(M_v)_{v\in T}$ does not belong to the closure of $X(k)$ in $\prod_{v\in T}X(k_v)$, but $(M_v)_{v\in T}$ is the projection of an element $(M_v)_{v\in\Omega_k}$ of $X(\mathbf{A}_k)^{\mathrm{Br}}$.

It has been conjectured by Colliot-Thélène and Sansuc (and proven in several special cases, cf. [5]) that $X(\mathbf{A}_k)^{\mathrm{Br}}=\overline{X(k)}$ for smooth and proper (geometrically) rational surfaces. The inclusion $X(\mathbf{A}_k)^{\mathrm{Br}}\subset\overline{X(k)}^{\Omega_\infty}$ is known to hold for an abelian variety X if one assumes the finiteness of the Tate–Shafarevich group of X [25,33]. Over an algebraically closed field, smooth and proper rational varieties are simply connected ([12], XI.1.2) and abelian varieties have an abelian fundamental group, so it is perhaps not surprising that varieties with non-abelian geometric fundamental groups arise in our work.

2. Obstruction to weak approximation associated to an étale covering

In this section, we introduce an obstruction to weak approximation (and in fact to weak approximation outside any finite set of places) related to geometric étale coverings of a variety. We begin with an obvious formal lemma:

LEMMA 2.1. – *Let $E = \{L_1, \dots, L_s\}$ be a finite set of étale k-algebras and let Σ be a finite set of places of k. Then there exists a finite set $S \subset \Omega_k$, with $S \cap \Sigma = \emptyset$, such that the following property holds*:

For each $i \in \{1, \dots, s\}$, the condition that L_i splits at any place v of S implies that L_i splits at any place $v \notin \Sigma$.

Proof. – For each $i \in \{1, \dots, s\}$, set:
- $S_i = \emptyset$ if L_i splits at any place $v \notin \Sigma$.
- $S_i = \{v_i\}$, where v_i is a place of $\Omega_k - \Sigma$ such that L_i does not split at v_i, if such a place exists.

Now $S := \bigcup_{i=1}^{s} S_i$ has the required property. □

We are now ready to prove:

PROPOSITION 2.2. – *Let X be a smooth, proper, and geometrically integral variety over a number field k. Fix a finite set of places Σ of k. Let $\rho: Z \to X$ be an étale covering of X, with Z geometrically integral.*

Then there exists a finite subset $S = S_{\rho,\Sigma}$ of $(\Omega_k - \Sigma)$ such that for any place $w \notin (S \cup \Sigma)$ and any point $(N_v)_{v \in S \cup \{w\}}$ of $\prod_{v \in S \cup \{w\}} X(k_v)$, the condition:

$N_v \in \rho(Z(k_v))$ for $v \in S$ but $N_w \notin \rho(Z(k_w))$ implies that (N_v) does not belong to the closure of $X(k)$ in $\prod_{v \in S \cup \{w\}} X(k_v)$.

Proof of Proposition 2.2. – Take a finite set of places S_0 (containing the archimedean places of k) such that the schemes X and Z extend to smooth and proper schemes $\mathcal{X}$ and $\mathcal{Z}$ over $U = \operatorname{Spec} \mathcal{O}_{k,S_0}$. We can also assume that the morphism ρ extends to a finite and étale morphism over U. Let d be the degree of ρ. There exist only finitely many number fields (hence only finitely many étale k-algebras $L_1, \dots, L_s$) which are unramified outside S_0 and of degree $\leqslant d$ over k ([23], V.4, Theorem 5). By Lemma 2.1, one can find a finite set of places $S = S_{\rho,\Sigma}$ (with $S \cap \Sigma = \emptyset$), such that for each $i \in \{1, \dots, s\}$, the condition that L_i splits at any place v of S implies that L_i splits at any place $v \notin \Sigma$.

Let w be a place of k outside $S \cup \Sigma$. Fix a point $(N_v)_{v \in S \cup \{w\}}$ which belongs to the closure of $X(k)$ in $\prod_{v \in S \cup \{w\}} X(k_v)$, and such that ρ splits at N_v for $v \in S$. Take a k-rational point Q which is close to N_v for $v \in S \cup \{w\}$ and denote by Q_v the v-adic component of Q. The fiber Z_Q of ρ at Q is the spectrum of an étale k-algebra which is of degree d and unramified outside S_0 (because the covering ρ extends to an étale covering of proper and smooth schemes over $\operatorname{Spec} \mathcal{O}_{k,S_0}$), hence $Z_Q \simeq \operatorname{Spec} L_i$ for some $i \in \{1, \dots, s\}$.

As ρ is étale, the map $Z(k_v) \to X(k_v)$ induced by ρ is a local isomorphism for each $v \in \Omega_k$ by the implicit function theorem. In particular, for $v \in S$, the condition that ρ splits at N_v implies that ρ splits at Q_v for Q_v close to N_v. Thus the k-algebra L_i splits at the places of S, hence at any place $v \notin \Sigma$ thanks to the choice of S. So the covering ρ splits at Q_v for $v \notin \Sigma$ (note that it does not necessarily split at Q).

Now the condition that Q_w is close to N_w implies that ρ splits at N_w as well and we are done. □

Remark. – Note that the covering ρ is not assumed to be Galois. The condition X proper is essential to ensure that for $v \notin S_0$, a k_v-point M_v of X extends to an $\mathcal{O}_v$-section of $\mathcal{X}$ ([18], II.4.7), which implies that the residue field of any point of Z which lies over M_v is unramified over k_v. This classical argument is used in the proof of the weak Mordell–Weil theorem (cf. [27], III.4.22). See [6], Proposition 1.2 for another example.

A similar technique can be used to deal with strong approximation on affine varieties ([20], 1.5 and 2.1; [28], Theorem 1).

The following lemma ensures the existence of points (N_v) satisfying the assumptions of Proposition 2.2. Later on (Theorem 5.1) we want to get adelic points $(N_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$, so we need a precise statement:

LEMMA 2.3. – *Let k be a number field and let $Z \xrightarrow{h} Y \xrightarrow{g} X$ be coverings of k-varieties, with X geometrically connected of dimension > 0. Assume that $\rho := (g \circ h) : Z \to X$ is Galois and that h is geometrically non-trivial (e.g., geometric of degree at least two). Then there exists a finite field extension K/k such that, for almost all places v which are totally split for K/k, there exists a k_v-point N_v of X with the property: g splits at N_v but ρ does not split at N_v.*

Remark. – By Tchebotarev's density theorem ([23], VIII.4, Theorem 10), infinitely many places v of k are totally split for K/k. "Almost all" means all but finitely many.

Proof. – The proof breaks into two steps:

a) *Reduction to the case when Z and X are smooth and geometrically integral.*

Take a finite Galois field extension K/k such that:

- Y_K (respectively Z_K) and $\overline{Y}$ (respectively $\overline{Z}$) have the same number of connected components.
- X_K and $\overline{X}$ have the same number of irreducible components

(where $X_K := X \times_k K$, $Y_K := Y \times_k K$, $Z_K := Z \times_k K$). Now we can choose a non-empty open subset $U_K \subset X_K$ which is included in an irreducible component of X_K of dimension $\geqslant 1$. Set $U = U_K^{\mathrm{red}}$; we may assume (shrinking U_K if necessary) that U is smooth over K and that the covering:

$$\rho_U : Z_U := Z_K \times_{X_K} U \to U$$

induced by ρ is étale. Note that U is geometrically integral over K (because $\overline{U}$ is an open subset of an irreducible component of $\overline{X}$) and $\dim U \geqslant 1$.

Set $Y_U = Y_K \times_{X_K} U$. The connected components of Y_U (respectively Z_U) are geometrically connected because of the choice of K, and the covering $h_U : Z_U \to Y_U$ induced by h is geometrically non-trivial. Therefore there exists a connected component Y_U^1 of Y_U such that $h_U^{-1}(Y_U^1)$ contains a connected component Z_U^1 of Z_U satisfying: the covering $h_U : Z_U^1 \to Y_U^1$ is geometric of degree at least 2. Now the K-variety Z_U^1 is geometrically connected and we get étale coverings:

$$Z_U^1 \xrightarrow{h_U^1} Y_U^1 \xrightarrow{g_U^1} U$$

with Z_U^1 geometrically integral over K (it is smooth and geometrically connected). The covering $\rho_U^1 := g_U^1 \circ h_U^1$ is Galois (because ρ is Galois), with h_U^1 geometric of degree $\geqslant 2$. Thus, considering the coverings $Z_U^1 \xrightarrow{h_U^1} Y_U^1 \xrightarrow{g_U^1} U$ of geometrically integral K-varieties, it is sufficient to prove Lemma 2.3 in the special case when Z and X are smooth and geometrically integral k-varieties.

b) *Proof of Lemma* 2.3 *when Z and X are smooth and geometrically integral.*

In this case the result is a corollary of a "geometric Tchebotarev's density theorem" proven by Ekedahl [9] and the statement holds with $K = k$.

Let G be the Galois group of ρ and $H \subset G$ be the Galois group of h.

Take a smooth model $\mathcal{X}$ (respectively $\mathcal{Y}$, $\mathcal{Z}$) of the k-variety X (respectively Y, Z) over a number ring $\mathcal{O}_{k,S}$ (where S is a finite set of places of k). The k-variety X is of dimension > 0. Thus, by Lang–Weil's theorem, the cardinal of $\mathcal{X}(\mathbf{F}_v)$ goes to infinity when the cardinal of the finite field $\mathbf{F}_v$ goes to infinity. Using Hensel's lemma and applying the main lemma of [9] to the covering $\rho : \mathcal{Z} \to \mathcal{X}$, we get, for almost all places v of k, a k_v-point N_v of X such that g splits at N_v and ρ does not split at N_v: indeed this condition means that the conjugacy class of the Frobenius F_{N_v} of the covering ρ at N_v meets H and is not trivial. Such a class exists because H is of cardinal $\geqslant 2$. $\square$

COROLLARY 2.4. – *Let X be a smooth, proper, and geometrically integral variety over a number field k and suppose that there exists an étale Galois covering $\rho : Z \to X$ of degree $\geqslant 2$, with Z geometrically integral. Assume $X(\mathbf{A}_k) \neq \emptyset$. Then for any finite field extension L/k and any finite set $\Sigma \subset \Omega_L$, the L-variety X does not satisfy weak approximation outside Σ.*

Proof. – Replacing X with $X \times_k L$ if necessary, we may assume $L = k$. The set $Z(k_v)$ is non-empty for almost all places v of k because of Lang–Weil's theorem and Hensel's lemma, so we may assume $Z(k_v) \neq \emptyset$ for $v \notin \Sigma$. Let S be a finite set of places as in Proposition 2.2. By Lemma 2.3, it is possible to find a place $w \notin \Sigma \cup S$ and a point $N_w \in X(k_w)$, such that ρ does not split at N_w. For $v \in S$, choose $N_v \in \rho(Z(k_v))$. Then the point $(N_v)_{v \in S \cup \{w\}}$ does not belong to the closure of $X(k)$ in $\prod_{v \in S \cup \{w\}} X(k_v)$. As $S \cup \{w\}$ does not meet Σ, the corollary is proven. $\square$

Remark. – In fact it is possible to prove Corollary 2.4 with the weaker assumption $\pi_1(\overline{X})$ non-trivial (instead of assuming that there exists a non-trivial Galois geometric covering of X defined over k); the proof is a bit more complicated (the ingredients appear in the proof of Lemma 5.2 below).

If the covering ρ is abelian with group G, then one can show (using Lemma 3.1 below applied to the constant k-group $S = G$) that the obstruction defined in Proposition 2.2 is coarser than the Brauer–Manin obstruction (see [32], Theorem 3 and [16], 4.3). But when ρ is not abelian, it may happen that $X(\mathbf{A}_k)^{\mathrm{Br}} \not\subset \overline{X(k)}^{\Sigma}$ (Theorem 5.1).

3. The set $X(\mathbf{A}_k)^{\mathrm{Br}}$

Let X be a smooth and proper variety over a number field k with $X(k) \neq \emptyset$. Our goal in this section is to find points in $X(\mathbf{A}_k)^{\mathrm{Br}}$ when the dimension of the image of the Albanese map is strictly lower than $\dim X$.

The first two results of this section are purely algebraic; we fix a field k of characteristic zero.

The following lemma is a slight generalization of [5], 1.5.1 and [32], Lemma 3. It will be used below to deal with a possibly non-reduced or non-irreducible variety.

LEMMA 3.1. – *Let X be a k-variety and $M \in X(k)$. Let F be a $\mathcal{G}_k$-module which is of finite type as an abelian group, equipped with a $\mathcal{G}_k$-homomorphism $\lambda : F \to \operatorname{Pic} \overline{X}$. Denote by S the k-group of multiplicative type which is dual to F.*

(1) *There is an exact sequence*:

$$\operatorname{Pic} X \to \left(\operatorname{Pic} \overline{X}\right)^{\mathcal{G}_k} \to H^2\left(k, \bar{k}[X]^*\right) \to \operatorname{Br}_a X \xrightarrow{r} H^1\left(k, \operatorname{Pic} \overline{X}\right). \tag{2}$$

If X is proper, reduced and geometrically connected, then

$$\operatorname{Ker} r = \operatorname{Im}[\operatorname{Br} k \to \operatorname{Br}_a X].$$

(2) *Assume that there exists an X-torsor Y under S of type λ (e.g.: X is proper, reduced and geometrically connected). Let Γ be the subgroup of* $\operatorname{Br}_a X$ *which consists of those elements γ such that $r(\gamma)$ comes from $H^1(k,F)$ (via λ_*). Then Γ is generated modulo* $\operatorname{Ker} r$ *by the cup-products $\alpha \cup [Y]$, $\alpha \in H^1(k,F)$.*

(3) *Let $K \supset k$ be a field and let $N \in X(K)$. Assume that X is proper and geometrically connected, and let λ' be the homomorphism $F \to \operatorname{Pic} \overline{X}_{\mathrm{red}}$ obtained by composing λ and the canonical map $\operatorname{Pic} \overline{X} \to \operatorname{Pic} \overline{X}_{\mathrm{red}}$. Then there exists an X_{red}-torsor Y under S of type λ' which splits at M. The condition that Y splits at N implies: $\gamma(M) = \gamma(N)$ for any $\gamma \in \Gamma$.*

Proof. – (1) This exact sequence follows from Hochschild–Serre's spectral sequence for the Galois covering $\overline{X} \to X$ and the étale sheaf $\mathbf{G}_m$ (cf. [5], 1.5.0).

If X is proper, reduced and geometrically connected, then $\bar{k}[X]^* = \bar{k}^*$ and $H^2(k, \bar{k}^*) = \operatorname{Br} k$, hence $\operatorname{Ker} r = \operatorname{Im}[\operatorname{Br} k \to \operatorname{Br}_a X]$.

(2) Let $\gamma \in \Gamma$. By definition, there exists $\alpha \in H^1(k,F)$ such that $\lambda_*(\alpha) = r(\gamma)$. Now, by [32] (Lemma 3), we have $r(\gamma) = r(\alpha \cup [Y])$ and the result follows from (1).

(3) We have $X_{\mathrm{red}}(k) \neq \emptyset$ (recall that we still denote by M and N the elements of $X_{\mathrm{red}}(k)$ and $X_{\mathrm{red}}(K)$ respectively associated to M and N). As X_{red} is proper, reduced, and geometrically connected, the existence of Y follows from $X_{\mathrm{red}}(k) \neq \emptyset$ (see Section 1.2; the condition that Y splits at M is obtained by replacing $[Y]$ with $[Y] - [Y](M)$ if necessary). The following diagram is commutative:

$$\begin{tikzcd}
\operatorname{Br}_a X \arrow[r, "r"] \arrow[d] & H^1(k, \operatorname{Pic} \overline{X}) \arrow[d] \\
\operatorname{Br}_a X_{\mathrm{red}} \arrow[r, "r'"] & H^1(k, \operatorname{Pic} \overline{X}_{\mathrm{red}})
\end{tikzcd}$$

In particular the image γ' of γ in $\operatorname{Br}_a X_{\mathrm{red}}$ satisfies: $r'(\gamma')$ comes from $H^1(k,F)$ via λ'_*. By (1), the kernel of r' is just $\operatorname{Im}[\operatorname{Br} k \to \operatorname{Br}_a X_{\mathrm{red}}]$. If $[Y](N) = 0$, we have $(\alpha \cup [Y])(N) = (\alpha \cup [Y])(M) = 0$ for any $\alpha \in H^1(k,S)$. By (2) (applied to X_{red} and λ'), we have $\gamma'(M) = \gamma'(N)$, hence $\gamma(M) = \gamma(N)$. $\square$

Let X be a proper, smooth, and geometrically integral k-variety and let A be the Albanese variety of X (cf. [22], II.3). Let $\bar{f} : \overline{X} \to \overline{A}$ be an Albanese map. Set:

$$q(X) := \dim A = \dim_k H^1(X, \mathcal{O}_X), \qquad d(X) := \dim X - q(X) \in \mathbf{Z},$$
$$d'(X) := \dim X - \dim \bar{f}\left(\overline{X}\right) \in \mathbf{N}.$$

In particular $d'(X) \geqslant d(X)$.

If X is a (smooth, proper, and geometrically integral) surface, we let $p_g(X) := \dim_k H^2(X, \mathcal{O}_X)$ denote its geometric genus.

PROPOSITION 3.2. – *Let X be a smooth, proper, and geometrically integral k-variety which contains a k-rational point M. Assume that $(\mathrm{Br}\,\overline{X})^{\mathcal{G}_k}$ is finite (e.g., $H^2(X,\mathcal{O}_X)=0$). Let:*

$$\mathrm{Pic}\,\overline{X} \xrightarrow{p} \mathrm{NS}\,\overline{X} \xrightarrow{q} \mathrm{NS}\,\overline{X}/\mathrm{NS}\,\overline{X}_{\mathrm{tors}}$$

be the canonical surjections and denote by $C \subset \mathrm{Br}_a X$ the kernel of the map $q_* \circ p_* \circ r : \mathrm{Br}_a X \to H^1(k, \mathrm{NS}\,\overline{X}/\mathrm{NS}\,\overline{X}_{\mathrm{tors}})$. Then:

(1) *The subgroup C is of finite index in* $\mathrm{Br}\,X$.

(2) *Assume from now on that $d'(X)>0$. Let A be the Albanese variety of X and denote by $f:X\to A$ the Albanese map sending M to 0_A; let X_0 be the fibre of f at 0_A (equipped with the embedding $s:X_0\hookrightarrow X$) and s^* (respectively s'^*) be the map $\mathrm{Br}_a X\to \mathrm{Br}_a X_0$ (respectively $H^1(k,\mathrm{Pic}\,\overline{X})\to H^1(k,\mathrm{Pic}\,\overline{X}_0)$) induced by s. Then there is a commutative diagram:*

$$\begin{array}{ccc} \mathrm{Br}_a X & \xrightarrow{r} & H^1(k,\mathrm{Pic}\,\overline{X}) \\ \downarrow{\scriptstyle s^*} & & \downarrow{\scriptstyle s'^*} \\ \mathrm{Br}_a X_0 & \xrightarrow{r_0} & H^1(k,\mathrm{Pic}\,\overline{X}_0) \end{array}$$

The specialization map $s^0:\mathrm{Pic}\,\overline{X}\to\mathrm{Pic}\,\overline{X}_0$ induces a map $\theta:\mathrm{NS}\,\overline{X}\to\mathrm{Pic}\,\overline{X}_0$ such that $\theta\circ p=s^0$.

(3) *Set $F=\mathrm{NS}\,\overline{X}_{\mathrm{tors}}$ and denote by $\lambda:F\to\mathrm{Pic}\,\overline{X}_0$ the $\mathcal{G}_k$-homomorphism obtained by composing the canonical embedding $i:F\hookrightarrow\mathrm{NS}\,\overline{X}$ and θ. Let Γ^0 be the subgroup of $\mathrm{Br}_a X_0$ consisting of those elements γ_0 such that $r_0(\gamma_0)$ comes from $H^1(k,F)$ (via λ_*). Then $s^*(C)\subset\Gamma^0$.*

(4) *Let V_0 be the connected component of X_0 containing M and $U_0:=(V_0)_{\mathrm{red}}$. Let $\lambda_0:F\to\mathrm{Pic}\,\overline{U}_0$ be the map obtained by composing λ and the canonical restriction $\mathrm{Pic}\,\overline{X}_0\to\mathrm{Pic}\,\overline{U}_0$. Then there exists a U_0-torsor Y_0 of type λ_0 such that Y_0 splits at M.*
Let $K\supset k$ be a field and let $N\in V_0(K)$. Then the condition that Y_0 splits at N implies $\gamma(M)=\gamma(N)$ for any $\gamma\in C$.

Proof. – (1) The image of $\mathrm{Br}\,X$ in $\mathrm{Br}\,\overline{X}$ obviously is contained in $(\mathrm{Br}\,\overline{X})^{\mathcal{G}_k}$, hence $\mathrm{Br}_a X$ is of finite index in $\mathrm{Br}\,X$. By definition the following sequence is exact:

$$0\to C\to\mathrm{Br}_a X\to H^1\big(k,\mathrm{NS}\,\overline{X}/\mathrm{NS}\,\overline{X}_{\mathrm{tors}}\big)$$

and $H^1(k,\mathrm{NS}\,\overline{X}/\mathrm{NS}\,\overline{X}_{\mathrm{tors}})$ is finite because $\mathrm{NS}\,\overline{X}/\mathrm{NS}\,\overline{X}_{\mathrm{tors}}$ is of finite type and torsion-free. So C is of finite index in $\mathrm{Br}\,X$.

(2) The commutative diagram is obtained using the exact sequence (2) of Lemma 3.1 and the functoriality of Hochschild–Serre's spectral sequence.

By definition the following sequence is exact:

$$0\to\mathrm{Pic}^0\overline{X}\to\mathrm{Pic}\,\overline{X}\xrightarrow{p}\mathrm{NS}\,\overline{X}\to 0$$

and the map f^* is an isomorphism from $\mathrm{Pic}^0\overline{A}$ to $\mathrm{Pic}^0\overline{X}$ ([22], VI.1, Theorem 1).

Therefore $s^0(\mathrm{Pic}^0\overline{X})=0$ and θ is well-defined. Note that the map $s^0_*:H^1(k,\mathrm{Pic}\,\overline{X})\to H^1(k,\mathrm{Pic}\,\overline{X}_0)$ induced by s^0 coincide with s'^*.

(3) The following diagram is commutative:

$$\begin{tikzcd}
 & & H^1(k,F) \arrow[d, "i_*"'] \\
\mathrm{Br}_a X \arrow[r, "r"] \arrow[d, "s^*"'] & H^1(k,\mathrm{Pic}\,\overline{X}) \arrow[r, "p_*"] \arrow[d, "s'^*"] & H^1(k,\mathrm{NS}\,\overline{X}) \arrow[d, "\theta_*"'] \\
\mathrm{Br}_a X_0 \arrow[r, "r_0"] & H^1(k,\mathrm{Pic}\,\overline{X}_0) \arrow[r, "\mathrm{id}"] & H^1(k,\mathrm{Pic}\,\overline{X}_0)
\end{tikzcd}$$

As $\lambda_* = \theta_* \circ i_*$, the result is now easy to check.

(4) Let us show that the proper k-variety V_0 is geometrically connected; Stein's factorization ([11], 4.3.4) of the structural morphism $V_0 \to \mathrm{Spec}\,k$ yields a morphism $h: V_0 \to \mathrm{Spec}\,L$, where L is a finite k-algebra and the fibres of h are non-empty and geometrically connected. But V_0 is connected, hence $\mathrm{Spec}\,L$ is connected and L is a field. The condition $V_0(k) \neq \emptyset$ implies $L = k$ and V_0 is geometrically connected. In particular Y_0 exists by Lemma 3.1. Let $N \in V_0(K)$ such that Y_0 splits at N. By Lemma 3.1, the equality $\gamma_0(M) = \gamma_0(N)$ holds for each $\gamma_0 \in \Gamma^0$. Now for each $\gamma \in C$, the specialization $s^*(\gamma)$ belongs to Γ^0 (see (3)), hence $\gamma(M) = \gamma(N)$ as well. □

COROLLARY 3.3. – *Let X be a proper, smooth, and geometrically integral variety over a number field k. Assume that $(\mathrm{Br}\,\overline{X})^{\mathcal{G}_k}$ is finite and that $d'(X) > 0$. Assume further that X contains a k-rational point M and denote by M_v the image of M in $X(k_v)$. Set $F = \mathrm{NS}\,\overline{X}_{\mathrm{tors}}$ and denote by T the k-group of multiplicative type which is dual to F. Then there exist a finite set S of places of k, a reduced and geometrically connected closed subscheme U_0 of X of dimension $\geqslant 1$, and a U_0-torsor Y_0 under T such that for each adelic point $(N_v) \in \prod_{v \in \Omega_k} U_0(k_v)$, the conditions*:

(1) *$N_v = M_v$ for $v \in S$.*

(2) *The torsor Y_0 splits at N_v for every $v \in \Omega_k$.*

Imply $(N_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$.

More precisely, if $f: X \to A$ is the Albanese map sending M to 0_A, one can take U_0 and Y_0 as in Proposition 3.2.

Proof. – Let C, U_0, and Y_0 be as in Proposition 3.2. Note that $\dim U_0 \geqslant 1$ by the semi-continuity theorem for the dimension of fibres of a proper morphism ([26], 5.13.E). The condition that Y_0 splits at $N_v \in U_0(k_v)$ implies $\gamma(M_v) = \gamma(N_v)$ for each $\gamma \in C$ and each $v \in \Omega_k$.

Take some liftings $\gamma_1, \dots, \gamma_r \in \mathrm{Br}\,X$ of the elements of $\mathrm{Br}\,X/C$. As X is proper over k, there exists a finite set of places S of k such that $\gamma_i(N_v) = 0$ for any v outside S and any $N_v \in X(k_v)$. Thus the conditions $N_v = M_v$ for $v \in S$ and $[Y_0](N_v) = 0$ for $v \in \Omega_k$ imply: $\gamma(M_v) = \gamma(N_v)$ for any $\gamma \in \mathrm{Br}\,X$. But M is k-rational, hence $(M_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$ and we are done. □

Remark. – Later on we will use Lemma 2.3 to find adelic points (N_v) satisfying simultaneously the assumptions of Corollary 3.3 and Proposition 2.2.

The principle of Corollary 3.3 is that outside a finite number of places of k, the Brauer–Manin obstruction is controlled by geometric abelian coverings of X. The group $\pi_1^{\mathrm{ab}}(\overline{X})$ is an extension of $(\mathrm{NS}\,\overline{X}_{\mathrm{tors}})^\vee$ by the Tate module TA of the Albanese variety A ([27], III.4.19). The condition $f(N_v) = f(M)$ takes care of TA and the condition $[Y_0](N_v) = [Y_0](M)$ deals with $(\mathrm{NS}\,\overline{X}_{\mathrm{tors}})^\vee$.

4. Non-abelian coverings and fibres of the Albanese map

Throughout this section we will suppose that the field k is algebraically closed (and of characteristic zero). Our goal here is to show that under some assumptions, a connected non-abelian covering of X does not become trivial after restriction to a fibre X_0 of an Albanese map. Note that if $q(X) \geqslant 2$, the image of an Albanese map could for example be a curve C of genus at least two; then non-abelian geometric coverings of C provide some non-abelian geometric coverings of X whose restriction to X_0 is trivial. In general, the structure of the fundamental group seems to be closely related to the Albanese map (see for example [7,21]).

To begin with, recall the following classical result (cf. [2], V.15 in the case $k = \mathbf{C}$):

LEMMA 4.1. – *Let X be a smooth, projective, and integral k-variety. Assume that $q(X) \leqslant 1$. Then the fibres of an Albanese map $f : X \to A$ are connected.*

Proof. – The result is obvious if $A = 0$, so we may assume that f is a proper and surjective map from X to an elliptic curve A. Consider Stein's factorization:

$$X \xrightarrow{h} A' \xrightarrow{g} A$$

of f (the fibres of h are connected and g is a finite covering of normal varieties). The variety A' is a normal curve, hence it is smooth over k. The genus of A' is at most 1 because the surjective map f induces an isomorphism $H^0(A, \Omega^1) \simeq H^0(X, \Omega^1)$ and $\dim_k H^0(X, \Omega^1) = \dim A = 1$. Therefore Hurwitz's formula implies that g is unramified and A' is an elliptic curve. By the universal property of the Albanese map, g is an isomorphism; since the fibres of h are connected, the fibres of f are connected as well. □

PROPOSITION 4.2. – *Let X be a proper, smooth, and integral k-variety. Fix a k-point m of X and denote by $f : X \to A$ the Albanese map sending m to 0_A. Assume that f is flat with connected and reduced fibres.*

Let $\rho : Z \to X$ be an étale and connected covering with Galois group G. Denote by $(k(Z))^{D(G)}$ the subfield of $k(Z)$ consisting of those functions on Z which are invariant by the derived subgroup $D(G)$, and take the normalization Z^{ab} of X in $k(Z)^{D(G)}$. Let X_0 (respectively Z_0, $(Z^{\mathrm{ab}})_0$) be the fibre of X (respectively Z, Z^{ab}) at 0_A.

Then the étale covering $Z_0 \to (Z^{\mathrm{ab}})_0$ induced by ρ is geometric.

Remark. – The image $f(X)$ of the proper morphism f is a closed subset of A. If f is flat, then $f(X)$ is an open subset of A ([27], I.2.12); thus $f(X) = A$ (A is connected) and f is faithfully flat.

Note that $(Z^{\mathrm{ab}})_0$ is not necessarily connected.

Proof. – Let us consider Stein's factorization $Z \xrightarrow{h} A' \xrightarrow{g} A$ of the proper morphism $f \circ \rho$ (g is finite and the fibres of h are connected). By [12] (X.1.2), the map $A' \to A$ is an étale covering (indeed f is proper, flat, and has connected and reduced fibers), hence it is abelian because A is an abelian variety. Thus we have a tower of étale coverings:

$$Z \to Z^{\mathrm{ab}} \to X \times_A A' \to X.$$

Let d be the degree of the covering $g : A' \to A$. As the fibres of $h : Z \to A'$ are connected, the variety Z_0 has d connected components (because k is algebraically closed). The fibre $(X \times_A A')_0$ of $X \times_A A' \to A$ at 0_A also has d connected components (X_0 is connected), so the covering $Z_0 \to (X \times_A A')_0$ is geometric. A fortiori the étale covering $Z_0 \to (Z^{\mathrm{ab}})_0$ is geometric. □

Remark. – In the langage of the fundamental group, this proposition corresponds to the fact that the image of $\pi_1(X_0,m)$ in $\pi_1(X,m)$ contains the derived subgroup $D(\pi_1(X,m))$. Since $\pi_1(A,0_A)$ is abelian, this follows from the exact sequence ([12], X.1.4):

$$\pi_1(X_0,m) \to \pi_1(X,m) \to \pi_1(A,0_A). \tag{3}$$

PROPOSITION 4.3. – *The Albanese map $f: X \to A$ is flat with connected and reduced fibres in the following cases*:
(1) $q(X)=0$.
(2) *The fibres of f are reduced and* $q(X)=1$.
(3) *The tangent bundle T_X is nef.*

Proof. – The first case is obvious. The second case follows from Lemma 4.1. In the third case, the morphism $f_{\mathbf{C}}: X_{\mathbf{C}} \to A_{\mathbf{C}}$ induced by f has connected fibres by [7], Proposition 3.9. On the other hand $f_{\mathbf{C}}$ is smooth (in particular flat with reduced fibres) by [7], Proposition 3.9 and [18], III.10.4. Thus f is smooth with connected fibres by faithfully flat descent. □

5. Main result

In this section we prove that under some geometric assumptions, the condition $(Q_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$ is not sufficient to ensure that the adelic point (Q_v) can be approximated (outside a finite set of places) by a rational point.

Let X be a proper, smooth, and geometrically integral variety over k. We shall say that X *satisfies the condition* (*Alb*) if the following property holds:

(Alb) *An Albanese map of* $\overline{X}$ *is flat with connected and reduced fibres.*

The condition (Alb) implies the surjectivity of the Albanese map. It also implies $\dim X > q(X)$ if $\bar{f}$ is not an isomorphism, that is if $\overline{X}$ is not an abelian variety. (Alb) holds in particular if $q(X)=0$, or if the tangent bundle of $\overline{X}$ is nef (Proposition 4.3). This is a somewhat technical condition, but the assumption $\dim X > q(X)$ certainly is crucial for the whole method to work. In particular, the case of curves of genus at least two seems to be unreachable without new ideas.

THEOREM 5.1. – *Let X be a proper, smooth, and geometrically integral variety over a number field k such that $X(k) \neq \emptyset$. Assume that* $(\mathrm{Br}\,\overline{X})^{\mathcal{G}_k}$ *is finite (e.g., $H^2(X,\mathcal{O}_X)=0$) and that the geometric fundamental group $\pi_1(\overline{X})$ is non-abelian. Assume further that X satisfies the condition* (*Alb*).

Then, $X(\mathbf{A}_k)^{\mathrm{Br}} \not\subset \overline{X(k)}^{\Sigma}$ *for any finite set $\Sigma \subset \Omega_k$.*

We use the following lemma:

LEMMA 5.2. – *With the notation and assumptions of Theorem* 5.1:
(1) *Fix $M \in X(k)$. Then there exists an étale covering $\rho: Z \to X$ which splits at M, and such that $\bar{\rho}: \overline{Z} \to \overline{X}$ is a non-abelian connected Galois covering.*
(2) *Let $f: X \to A$ be the Albanese map sending M to 0_A. Take X_0 and Y_0 as in Proposition* 3.2. *Then there exists an infinite set of places $S_1 \subset \Omega_k$ such that for any $v \in S_1$, there exists $N_v \in X_0(k_v)$ with the following property*: *Y_0 splits at N_v but ρ does not split at N_v.*

Proof of Theorem 5.1 assuming Lemma 5.2 is proven. – We have $d(X)>0$ because of condition (Alb): indeed $\pi_1(\overline{X})$ is not abelian, hence $\overline{X}$ is not an abelian variety. The condition (Alb) also implies that the fibre X_0 is reduced and geometrically connected of dimension at least 1, so we can take $U_0 = X_0$ in Corollary 3.3. Fix a finite set $\Sigma \subset \Omega_k$. Let S be as in

Corollary 3.3; enlarging S if necessary, we may assume (applying Proposition 2.2) that the conditions: $Q_v = M_v$ for $v \in S$ and ρ does not split at Q_v for some $v \notin S$, imply $(Q_v) \notin \overline{X(k)}^{\Sigma}$ (recall that ρ splits at $M_v = M$). Now we can find a set S_1 as in Lemma 5.2; removing a finite set of places of S_1 if necessary, we can also suppose that $S \cap S_1 = \emptyset$. For each place $w \in S_1$, choose N_w as in Lemma 5.2. The adelic point (Q_v) defined by $Q_v = M_v$ for $v \neq w$ and $Q_w = N_w$ satisfies $(Q_v) \in X(\mathbf{A}_k)^{\mathrm{Br}}$ (Lemma 5.2 and Corollary 3.3) but $(Q_v) \notin \overline{X(k)}^{\Sigma}$ (Proposition 2.2). □

Proof of Lemma 5.2. – (1) (See also [16], 3.1.) Fix a geometric point m of X associated to M. We have the fundamental exact sequence ([12], IX.6.1):

$$1 \to \pi_1(\overline{X}, m) \to \pi_1(X, m) \to \pi_1(k, \bar{k}) \to 1.$$

As $\pi_1(\overline{X})$ is not abelian, there exists a connected étale Galois covering $\overline{Z}$ of $\overline{X}$ whose Galois group G is finite and non-abelian. One can find a finite field extension L/k and a finite étale geometric covering Z_L of $X_L := X \times_k L$ such that $\overline{Z} = Z_L \times_L \bar{k}$. Let Z' be the normalization of Z_L in the Galois closure of $L(Z_L)/k(X)$; then $\overline{Z}'$ is Galois over X. Namely, replacing $\overline{Z}$ with $\overline{Z}'$ if necessary, we may assume that the following sequence is exact:

$$1 \to \mathrm{Aut}(\overline{Z}/\overline{X}) \to \mathrm{Aut}(\overline{Z}/X) \xrightarrow{p} \mathcal{G}_k \to 1. \tag{4}$$

Now fix a geometric point m' of $\overline{Z}$ lying over m. The k-point M induces a splitting $\sigma : \mathcal{G}_k \hookrightarrow \mathrm{Aut}(\overline{Z}/X)$ of the exact sequence (4): for each $g \in \mathcal{G}_k$, one can take for $\sigma(g)$ the unique element of $\mathrm{Aut}(\overline{Z}/X)$ which is mapped to g by p and fixes m'.

This fixes a k-form Z of $\overline{Z}$ (that is: a finite and étale covering Z/X which becomes isomorphic to $\overline{Z}/\overline{X}$ over $\bar{k}$): the k-variety Z is the normalization of X in $\bar{k}(\overline{Z})^H$, where $H \simeq \mathcal{G}_k$ is the image of $\mathcal{G}_k$ by σ. Moreover Z contains a k-point lying over M because the image of m' by the canonical morphism $\overline{Z} \to Z$ belongs to $(Z(\bar{k}))^{\mathcal{G}_k}$, hence to $Z(k)$.

Note that Z/X is not necessarily Galois.

(2) Set $F = \mathrm{NS}\,\overline{X}_{\mathrm{tors}}$ and let T be the k-group of multiplicative type which is dual to F. The exact sequence:

$$0 \to \mathrm{Pic}^0 \overline{X} \to \mathrm{Pic}\, \overline{X} \to \mathrm{NS}\, \overline{X} \to 0$$

is split as a sequence of abelian groups because $\mathrm{Pic}^0 \overline{X}$ is divisible. Since $\mathrm{NS}\,\overline{X}$ is of finite type, we can therefore find a Galois extension K of k such that:

- The action of $\mathcal{G}_K$ on $T(\bar{k})$ and $\mathrm{NS}\,\overline{X}$ is trivial and the map $\mathrm{Pic}\,\overline{X} \to \mathrm{NS}\,\overline{X}$ admits a $\mathcal{G}_K$-section σ_K.
- The covering $\rho_K : Z_K \to X_K$ is Galois with group G.

Set $X_{0,K} = X_0 \times_k K$ and denote by s_K^0 the specialization map: $\mathrm{Pic}\,\overline{X} \to \mathrm{Pic}\, X_{0,K}$ (cf. Proposition 3.2). The type λ_K of the $X_{0,K}$-torsor $Y_{0,K} := Y_0 \times_k K$ is (by construction) $\lambda_K = s_K^0 \circ \sigma_K$.

Let us show that $Y_{0,K}$ can be obtained by specialization of an X_K-torsor: let Y_K be an X_K-torsor of type $\sigma_K : (\mathrm{NS}\,\overline{X})_{\mathrm{tors}} \hookrightarrow \mathrm{Pic}\,\overline{X}$, with Y_K split at M; then the specialization $Y_K \times_{X_K} X_{0,K}$ is of type λ_K and splits at M, so it is isomorphic to $Y_{0,K}$ (apply the exact sequence (1) of Section 1.2 to X_0, which is proper, reduced and geometrically connected). Thus we may assume that $Y_K \times_{X_K} X_{0,K} = Y_{0,K}$.

As the $\mathcal{G}_K$-action on $T(\bar{k})$ is trivial, the X_K-torsor Y_K (which is geometrically connected because σ_K is injective) is just a geometric étale Galois covering of X_K with group $T(\bar{k})$. Now it is sufficient to show that there exist infinitely many places w of K with $K_w \simeq k_v$, and such that there exists $N_w \in X_0(K_w)$ with the property: Y_K splits at N_w and ρ_K does not split at N_w. To do that, we may assume (replacing Z_K with a connected component of $Z_K \times_K Y_K$ if necessary) that the covering ρ_K factorizes through an étale covering $Z_K \to Y_K$. Then, as T is abelian, we have (over K) the following tower of geometric étale coverings:

$$Z_K \to Z_K^{\mathrm{ab}} \to Y_K \to X_K.$$

Taking the fibred product (over X_K) by $X_{0,K}$, we obtain étale coverings:

$$Z_{0,K} \to \left(Z_K^{\mathrm{ab}}\right)_0 \to Y_{0,K} \to X_{0,K}$$

and thanks to the condition (Alb), the covering $Z_{0,K} \to (Z_K^{\mathrm{ab}})_0$ is geometric (Proposition 4.2) of degree $d \geqslant 2$ (because G is non-abelian, hence the geometric covering $Z_K \to Z_K^{\mathrm{ab}}$ is not trivial). Now we apply Lemma 2.3 to the coverings of K-varieties:

$$Z_{0,K} \to \left(Z_K^{\mathrm{ab}}\right)_0 \xrightarrow{h_K} X_{0,K}.$$

We get a finite field extension K' of K (which may be taken Galois over k) such that for almost all places w of K which are completely split for K'/K, there exists a K_w-point N_w of $X_{0,K}$ satisfying: h_K splits at N_w and ρ_K does not split at N_w; a fortiori $Y_{0,K}$ splits at N_w. Let S_1 be the set of places of k which are totally split for K'/k. The set S_1 is infinite because of Tchebotarev's density theorem. Thus (removing a finite set of places of S_1 if necessary) we get for any place v of S_1 a k_v-point $N_v \in X_0(k_v)$ satisfying: ρ does not split at N_v and Y_0 splits at N_v. □

Remark. – It is possible to replace the condition (Alb) of Theorem 5.1 by the condition (Alb′): $\Pi \cap D(\pi_1(\overline{X}, m)) \neq \{e\}$, where m is a geometric point of X associated to a k-rational point M of X, and Π is the image of $\pi_1(\overline{V_0}, m)$ in $\pi_1(\overline{X}, m)$ (recall that V_0 is the connected component containing M of the fibre X_0 of the Albanese map f at 0_A); indeed Proposition 4.2 does not necessarily hold anymore, but it is easy to see that (Alb′) implies the existence of an étale covering $Z \to X$ such that $Z \times_X V_0 \to (Z^{\mathrm{ab}}) \times_X V_0$ is geometrically non-trivial (with the notation of Proposition 4.2), and this hypothesis is sufficient to apply Lemma 2.3 to $U_0 = (V_0)_{\mathrm{red}}$ in the proof of Theorem 5.1.

The condition (Alb′) holds in particular if Π is infinite (e.g. if $d'(X) > 0$ and $\Pi := \pi_1(\overline{X})$ is large in the sense of [21], I.4.1.2); indeed let D' be the subgroup of Π corresponding to the coverings coming from the Albanese variety $\overline{A}$ of $\overline{X}$. We have $\Pi \subset D'$ because such coverings induce trivial coverings of the fibre $\overline{X_0}$; moreover $D'/D(\Pi)$ is finite (because $\mathrm{NS}\,\overline{X}_{\mathrm{tors}}$ is finite; see the remark at the end of Corollary 3.3), hence Π infinite implies $\Pi \cap D(\Pi) \neq \{e\}$.

For varieties with non-abelian geometric fundamental group, the condition (Alb′) is weaker than (Alb) (the latter implies $\Pi \supset \pi_1(\overline{X}, m)$ by Proposition 4.2), but it seems difficult in practice to check that (Alb′) holds when (Alb) does not hold.

6. Examples

First of all, we can use the following result [31] to construct examples in dimension $\geqslant 3$:

THEOREM (Serre). – *Let G be a finite group and k be a field. For any $r \geqslant 2$, there exist $s \in \mathbf{N}$ and a smooth complete intersection $V \subset \mathbf{P}_k^s$ of dimension r on which G acts without fixed points (i.e., all stabilizers are trivial).*

COROLLARY 6.1. – *Let $r \geqslant 3$. There exist a number field k and a smooth, projective and geometrically integral k-variety X of dimension r, such that $X(\mathbf{A}_k)^{\mathrm{Br}} \not\subset \overline{X(k)}^{\Sigma}$ for any finite set $\Sigma \subset \Omega_k$. Moreover, any non-abelian finite group may occur as $\pi_1(\overline{X})$.*

Proof. – Let G be a non-abelian finite group. Take a smooth complete intersection $V \subset \mathbf{P}^s_{\mathbf{Q}}$ of dimension r on which G acts without fixed points and set $X = V/G$. Now the assumption $r \geqslant 3$ implies that $\pi_1(\overline{V})$, $H^1(V, \mathcal{O}_V)$ and $H^2(V, \mathcal{O}_V)$ are trivial because of the Lefschetz theorems ([13], X.3 and XII.3). So $q(X) = 0$, $H^2(X, \mathcal{O}_X) = 0$, and the étale fundamental group of $\overline{X}$ is G. It remains to take a finite extension $k/\mathbf{Q}$ such that $X(k) \neq \emptyset$ to apply Theorem 5.1. □

Remark. – If we assume further the group G perfect (that is $D(G) = G$, e.g., $G = \mathcal{A}_n$ with $n \geqslant 5$), we have $\mathrm{Br}\,\overline{X} = 0$ because $\mathrm{Pic}\,\overline{X}$ is torsion-free and $H^2(X, \mathcal{O}_X) = 0$ (use [27], III.4.19, [14], Corollary 3.4 and [15], 8.12). Thus $\mathrm{Br}\,X/\mathrm{Br}\,\mathbf{Q}$ is finite by Lemma 3.1(1). Taking k sufficiently large, we obtain examples with $\mathrm{Br}\,X/\mathrm{Br}\,k = 0$; in this case the Brauer–Manin obstruction to weak approximation is empty.

It is not clear whether the previous construction provides varieties with $X(k)$ Zariski-dense, so the next example is perhaps more significant:

PROPOSITION 6.2. – *Let k be a number field and G be a finite group acting freely on an abelian k-variety A such that $\dim A \geqslant 2$. Assume that $A(k)$ is Zariski-dense in A. Let $X = A/G$ be the étale quotient of A by G, assume $H^2(X, \mathcal{O}_X) = 0$. Then $X(k)$ is Zariski-dense in X but $X(\mathbf{A}_k)^{\mathrm{Br}} \not\subset \overline{X(k)}^{\Sigma}$ for any finite set $\Sigma \subset \Omega_k$.*

Remark. – Let k be a number field. For any abelian k-variety A, there exists a finite field extension K/k such that $A(K)$ is Zariski-dense in A ([10], Theorem 10.1).

Proof. – The variety $X_{\mathbf{C}}$ has a nef tangent bundle by [7] (3.2.ii and 3.4), so this is also the case for $\overline{X}$. Thus it just remains to check that $\pi_1(\overline{X})$ is not abelian to apply Theorem 5.1. The following sequence of groups is exact:

$$1 \to \pi_1(\overline{A}) \to \pi_1(\overline{X}) \to G \to 1. \tag{5}$$

The group $\pi_1(\overline{A})$ is abelian; the homomorphism $\Phi : G \to \mathrm{Aut}(\pi_1(\overline{A}))$ given by the sequence (5) is induced by the action of G on $\overline{A}$. Let g be an automorphism of the algebraic variety A; by [29] (II.4, Corollary 1), we have $g = \tau \circ g_0$, where τ acts by translation and g_0 is an automorphism of the abelian variety A (that is: g_0 is compatible with the group structure on A). The action induced by τ on $H^1(\overline{A}, \mathbf{Q}/\mathbf{Z})^{\vee}$ is trivial and the action induced by g_0 on $(H^1(\overline{A}, \mathbf{Z}/n)^{\vee} =_n \overline{A}$ is simply the natural action of $g_0 \in \mathrm{Aut}(A)$ on $\overline{A}$; thus the homomorphism Φ is non-trivial if and only if G contains an automorphism of the algebraic variety A which does not act by translation. But this condition is ensured by the fact that $\overline{X}$ is not an abelian variety ([29], III.12, Corollary 1), which follows from the assumptions $H^2(X, \mathcal{O}_X) = 0$ and $\dim X \geqslant 2$: indeed, for any abelian $\bar{k}$-variety B, the $\bar{k}$-vector space $H^2(B, \mathcal{O}_B)$ has dimension $C^2_{\dim B}$ by [29], III.13, Corollary 2. As Φ is not trivial, the subgroup $\pi_1(\overline{A})$ is not central in $\pi_1(\overline{X})$ and in particular $\pi_1(\overline{X})$ is not abelian. □

Recall ([2], VI.20) that étale quotients of abelian surfaces (except abelian surfaces themselves) over $\bar{k}$ are of the form $(E \times F)/G$, where E and F are elliptic curves (and there are seven possible cases for the group G). Such surfaces (the so-called bielliptic surfaces) have geometric genus 0, hence Proposition 6.2 applies and we obtain:

COROLLARY 6.3. – *Let $X = (C_1 \times C_2)/G$ be a bielliptic surface over a number field k such that the elliptic curves C_1 and C_2 are of positive rank. Then $X(k)$ is Zariski dense in X but $X(\mathbf{A}_k)^{\mathrm{Br}} \not\subset \overline{X(k)}^{\Sigma}$ for any finite set $\Sigma \subset \Omega_k$.*

Remarks. – (1) Skorobogatov's counterexample to the Hasse principle [32] is (geometrically) a bielliptic surface $X=(C\times D)/G$, where C and D are curves of genus one and $G=\mathbf{Z}/2$; the absence of k-rational points on X is not accounted for by the Brauer–Manin obstruction, but can be explained using (as in Proposition 2.2) an obstruction related to a geometric non-abelian étale covering of X. See [16] (5.1) for more details.

(2) Bielliptic surfaces were used as counterexamples to arithmetic properties for the first time by Colliot-Thélène, Skorobogatov and Swinnerton-Dyer [6].

(3) There are varieties of dimension $\geqslant 3$ to which Proposition 6.2 applies: see [7] (3.3 and 3.11) for such an example in dimension 3, with $H^1(X,\mathcal{O}_X)=0$ and $H^2(X,\mathcal{O}_X)=0$.

Of similar kind is the following example (cf. [21], I.4.4.7):

PROPOSITION 6.4. – *Let E be an elliptic curve and S be a $K3$-surface with a fixed-point free involution τ such that $E(k)$ (respectively $S(k)$) is Zariski-dense in E (respectively in S). Let $X=E\times S/(-1,\tau)$. Then $X(k)$ is Zariski-dense in X but $X(\mathbf{A}_k)^{\mathrm{Br}}\not\subset \overline{X(k)}^{\Sigma}$ for any finite set $\Sigma\subset\Omega_k$.*

Proof. – We have $\pi_1^{\mathrm{top}}(S_{\mathbf{C}})=0$ ([1], Corollary VIII.8.6), so by the comparison theorem $\pi_1(\overline{S})=0$. Thus the following sequence of groups is exact:

$$1\to\pi_1(\overline{E})\to\pi_1(\overline{X})\to\mathbf{Z}/2\to 1$$

and $\mathbf{Z}/2$ acts by conjugation on $\pi_1(\overline{E})=\widehat{\mathbf{Z}}^2$ via $x\mapsto -x$. Therefore $\pi_1(\overline{X})$ is infinite but $\pi_1(\overline{X})^{\mathrm{ab}}=\mathbf{Z}/2$. In particular $q(X)=0$ and $\pi_1(\overline{X})$ is not abelian. It remains to show $H^2(X,\mathcal{O}_X)=0$. Set $Y=E\times S$. We have $H^1(S,\mathcal{O}_S)=H^2(E,\mathcal{O}_E)=0$, hence (by Künneth formula) the natural map: $H^2(S,\mathcal{O}_S)\to H^2(Y,\mathcal{O}_Y)$ is an isomorphism. Now $H^2(X,\mathcal{O}_X)$ is isomorphic to the group of those 2-forms on Y which are invariant by the action of $\mathbf{Z}/2$ induced by $(-1,\tau)$. In particular it is isomorphic to $H^2(S',\mathcal{O}_{S'})$, where $S':=S/\tau$. But S' is an Enriques surface, hence $H^2(S'_{\mathbf{C}},\mathcal{O}_{S'_{\mathbf{C}}})=0$ ([2], Theorem VIII.2 and Proposition VIII.17). Thus $H^2(S',\mathcal{O}_{S'})=0$. □

Remark. – Let S be a *Kummer surface* over k, that is a minimal projective and smooth model of the (singular) quotient of an abelian surface by the multiplication by -1 (cf. [2], VIII.10 and VIII.12 for the case $k=\mathbf{C}$). Keum [19] has proved that $S_{\mathbf{C}}$ admits a fixed-point free involution. Thus there exists a finite field extension K/k such that $S(K)$ is Zariski-dense in S and $S\times_k K$ admits a fixed-point free involution.

More generally, it has been proved very recently by Bogomolov and Tschinkel [4] that for every Enriques surface S', the set $S'(K)$ is Zariski-dense in S' for some finite field extension K/k. In particular, the same property holds for any $K3$-surface S which admits a fixed-point free involution.

It is an open question to know if this is still true for every variety X such that $-K_X=\bigwedge^{\dim X}T_X$ is nef. Some special cases (certain $K3$ surfaces, Fano threefolds which are not double covers of $\mathbf{P}^3$ ramified in smooth surfaces of degree 6) were solved by Bogomolov, Harris, and Tschinkel [17,3]. It is also expected that for a smooth, proper and geometrically integral variety X such that $-K_X$ is nef, the Albanese map is a smooth morphism; thus it seems reasonable to hope that Theorem 5.1 still holds in some of these situations.

An *elliptic surface* over a curve C is a smooth and proper surface X equipped with a surjective morphism $\pi:X\to C$, and such that the generic fibre of π is a smooth curve of genus one (we do not require that π admits a section). A fiber of π is said to be *multiple* if it is divisible by n, for some $n\geqslant 2$, in the group of divisors of X.

Bielliptic surfaces are special cases of elliptic surfaces. Recall the following result about the fundamental group of an elliptic surface over $\mathbf{P}^1$:

PROPOSITION 6.5. – *Let $\pi : X \to \mathbf{P}^1_k$ be an elliptic surface. Assume that $\bar{\pi} : \overline{X} \to \mathbf{P}^1_{\bar{k}}$ has at least three multiple fibres. Then $\pi_1(\overline{X})$ is not abelian.*

Proof. – Use [8], p. 146 and the comparison theorem for the fundamental group. □

COROLLARY 6.6. – *Let $\pi : X \to \mathbf{P}^1_k$ be an elliptic surface such that $p_g(X) = q(X) = 0$ and $X(k) \neq \emptyset$. If π has at least three multiple fibres, then $X(\mathbf{A}_k)^{\mathrm{Br}} \not\subset \overline{X(k)}^{\Sigma}$ for any finite set $\Sigma \subset \Omega_k$.*

Remarks. – (1) Such surfaces exist ([8], Corollary 2, p. 139). It would be interesting to construct examples of such elliptic fibrations, with three or four multiple fibres and $X(k)$ Zariski-dense. The condition $X(k)$ Zariski-dense probably never holds if there are at least five multiple fibres; for example $X(k)$ is never Zariski-dense if there are at least five double fibres ([6], Corollary 2.4).

(2) There are also examples of surfaces of general type to which Theorem 5.1 applies: actually there exist (proper and smooth) surfaces of general type X such that $p_g(X) = q(X) = 0$ and $\pi_1(\overline{X})$ is non-abelian (such examples can be found in [1] or [8]). But if one believes Lang's conjecture (that is: for any k-variety of general type, the set of rational points is not Zariski-dense), the conclusion of Theorem 5.1 is somewhat weak for such surfaces.

It would be nice (using a "non-abelian" obstruction as in Proposition 2.2) to construct a surface of general type X such that $X(\mathbf{A}_k)^{\mathrm{Br}} \neq \emptyset$ but $X(k) = \emptyset$, that is to construct a counterexample to the Hasse principle not accounted for by the Brauer–Manin obstruction among the surfaces of general type. The similar problem for curves of genus at least 2 is probably more difficult because constructing points in $X(\mathbf{A}_k)^{\mathrm{Br}}$ seems to be a tricky task.

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D. HARARI
IRMA, Université Louis-Pasteur et CNRS,
7, rue René-Descartes,
67084 Strasbourg cedex, France
E-mail: harari@math.u-strasbg.fr