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LUCHEZAR STOYANOV

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GENERALIZED HAMILTON FLOW AND POISSON RELATION FOR THE SCATTERING KERNEL

BY LUCHEZAR STOYANOV [1]

ABSTRACT. – The generalized Hamilton flow determined by a Hamilton function on a symplectic manifold with boundary is considered. A regularity property of this flow is proved, which for "Sard's theorem type applications" is as good as smoothness. It implies in particular that the generalized Hamilton flow preserves the Hausdorff dimension of Borel subsets of its phase space. As an application, it is shown that in scattering by obstacles, the so-called Poisson relation for the scattering kernel $s(t,\theta,\omega)$ becomes an equality for almost all pairs of unit vectors (θ,ω). © 2000 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – On considère le flot hamiltonien généralisé déterminé par une fonction de Hamilton sur une variété symplectique à bord. On prouve une propriété de régularité du flot qui pourrait remplacer la régularité usuelle dans les applications associées au théorème de Sard. En particulier, cette propriété implique que le flot généralisé hamiltonien préserve la dimension de Hausdorff des ensembles de Borel sur l'espace de phase. Comme application on montre que dans le cas de diffusion par des obstacles la relation de Poisson pour le noyau de diffusion $s(t,\theta,\omega)$ devient une égalité pour presque tout $(\theta,\omega) \in S^{n-1}$. © 2000 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The *generalized Hamilton (bicharacteristic) flow* (GHF) $\mathcal{F}_t$ generated by a Hamilton function p on a symplectic manifold with boundary S was introduced by Melrose and Sjöstrand (see [8, 9]) motivated by investigations on propagation of singularities of differential operators. This flow appears naturally in a variety of problems in spectral and scattering theory. In fact, special cases of it had been studied and used in one way or another before the works of Melrose and Sjöstrand (see e.g. [6,10,16]). A detailed consideration of this flow was later given by Hörmander in [4] (see Section 24.3 there).

In general the behavior of the GHF is rather complicated. In fact, as an example of M. Taylor [16] shows (see also Section 24 in [4]), in some cases this is not a flow in the usual sense of dynamical systems, since there may exist two different integral curves issued from the same point in the phase space. Even when this flow is well-defined, there are still essential difficulties that encounter — presence of diffractive points, integral curves contaning non-trivial gliding segments on the boundary, etc.

An additional condition on S and p (cf. **A3** in Section 2) guarantees [9] that the GHF is well-defined. It turns out that, though $\mathcal{F}_t$ is not smooth in general, it has some of the important features of a smooth flow. Clearly, at points of transversal reflection at ∂S the flow $\mathcal{F}_t$ is discontinuous.

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To make it continuous, consider the quotient space $\widetilde{S} = S/\sim$ of S with respect to the following equivalence relation: $\rho \sim \sigma$ if and only if $\rho = \sigma$ or $\rho, \sigma \in \partial S$ and either $\lim_{t \nearrow 0} \mathcal{F}_t(\rho) = \sigma$ or $\lim_{t \searrow 0} \mathcal{F}_t(\rho) = \sigma$. Melrose and Sjöstrand [9] proved that the natural projection of $\mathcal{F}_t$ on $\widetilde{S}$ is continuous.

We may assume that S is contained in a symplectic space $\mathcal{V}$ of the same dimension and without boundary and p is a smooth function defined on $\mathcal{V}$. Given a *metric* d_0 on $\mathcal{V}$ that generates its topology and is equivalent to the natural distance on symplectic charts near ∂S, one can define a related to it *pseudometric* d' on $\mathcal{V}$ such that $d'(\rho, \sigma) = 0$ in $\mathcal{V}$ if and only if $\rho \sim \sigma$ (cf. Section 2 for details). Then the projection of d to $\widetilde{S}$ generates the quotient topology, so $\mathcal{F}_t$ is continuous on S with respect to d. In what follows we mainly work with $\mathcal{F}_t$ as a flow on S using the metric d_0 and the pseudometric d (whichever is more appropriate) to express Lipschitz properties and Hausdorff dimension.

The natural phase space of the GHF is the *zero bicharacteristic set* $\Sigma = p^{-1}(0)$. In Section 3 below we prove that for each $T > 0$, Σ can be represented as a countable union of Borel subsets S_i such that on each S_i , $\{\mathcal{F}_t\}_{0 \leqslant t \leqslant T}$ coincides with the restriction of an one-parameter family $\mathcal{G}_t^{(i)}$ of Lipschitz maps (with respect to the appropriate metrics) defined in a neighbourhood of S_i, taking values in $\mathcal{V}$ and such that for all but finitely many t, $\mathcal{G}_t^{(i)}$ is smooth and its restriction to smooth local cross-sections is a contact transformation (cf. Theorem 2.1 for the precise statement). For "Sard's theorem type" applications this regularity property is as good as smoothness. A consequence of it is the following.

THEOREM 1.1. – *The generalized Hamilton flow $\mathcal{F}_t$ preserves the Hausdorff dimension of Borel subsets of the phase space Σ.*

Let K be a compact subset of $\mathbb{R}^n, n \geqslant 3, n$ odd, with C^∞ boundary ∂K such that $\Omega_K = \overline{\mathbb{R}^n \setminus K}$ is connected. Such a set K is called an *obstacle* in $\mathbb{R}^n$. The *scattering operator* related to the wave equation in $\mathbb{R} \times \Omega$ with Dirichlet boundary condition on $\mathbb{R} \times \partial\Omega$ can be represented as a unitary operator $S : L^2(\mathbb{R} \times \mathbf{S}^{n-1}) \to L^2(\mathbb{R} \times \mathbf{S}^{n-1})$ (see [5]). The Schwartz kernel of $S - \mathrm{Id}$ is a distribution $s_K(t, \theta, \omega) \in \mathcal{D}'(\mathbf{R} \times \mathbf{S}^{n-1} \times \mathbf{S}^{n-1})$ called the *scattering kernel*. For fixed $\omega, \theta \in \mathbf{S}^{n-1}$ the singularities of $s_K(t, \theta, \omega)$ are related to the so-called (ω, θ)-rays in Ω_K which are generalized geodesics on the manifold with boundary Ω_K incoming from infinity with direction ω and outgoing to infinity with direction θ. The *generalized geodesics* on Ω_K are the natural projections of the integral curves of the GHF $\mathcal{F}_t$ on $S = T^*(\Omega_K \times \mathbb{R})$ generated by the principal symbol p of the wave operator (see Section 2 for details).

V. Petkov [11] established that, under certain assumptions about K, we have

$$\text{sing supp}\, s_K(t, \theta, \omega) \subset \{-T_\gamma \colon \gamma \in \mathcal{L}_{\omega,\theta}(\Omega_K)\}, \tag{1}$$

where $\mathcal{L}_{\omega,\theta}(\Omega_K)$ is *the set of all (ω, θ)-rays in Ω_K* and T_γ is the so-called *sojourn time* of γ (cf. Section 2). Apart from that, Petkov gave a sufficient condition for a number $-T$ to belong to sing supp $s_K(t, \theta, \omega)$. In the general case proofs of these results can be found in [12]. In analogy with the well-known Poisson relation for the Laplacian on Riemannian manifolds, (1) is called *the Poisson relation for the scattering kernel*. Following the same analogy, the set of all T_γ, where $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega_K)$, $(\omega, \theta) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$, is called the *scattering length spectrum* of K (or Ω_K).

Roughly speaking condition **A3** in Section 2 reads: for $(x, \xi) \in T^*(\partial K)$, if the normal curvature of ∂K at x vanishes of infinite order in direction ξ, then ∂K is convex at x in direction ξ. Denote by $\mathcal{K}$ the *class of obstacles* K which have this property. It should be mentioned that $\mathcal{K}$ is of second Baire category in the space of all obstacles with smooth boundary endowed with the Whitney C^∞ topology. Using the regularity property of the GHF mentioned above and results from [11] and [13], we prove that for most obstacles and most pairs of directions (ω,θ) the Poisson relation (1) becomes an equality.

THEOREM 1.2. – *Let $K \in \mathcal{K}$. There exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that*

$$\operatorname{sing\,supp} s_K(t,\theta,\omega) = \{-T_\gamma\colon\ \gamma \in \mathcal{L}_{\omega,\theta}\}$$

holds for all $(\omega,\theta) \in \mathcal{R}$.

The proof of this theorem uses essentially the sufficient condition from [11] (see also Theorem 9.1.2 in [12]). Applying Theorem 2.2 below which is a consequence of the regularity property of the GHF discussed above, we find a subset $\mathcal{R}$ of full measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for $(\omega,\theta) \in \mathcal{R}$ the (ω,θ)-rays in Ω_K are never tangent to the boundary and, using also a result in [13], for any $T \geqslant 0$ there is at most one (ω,θ)-ray γ with $T_\gamma = T$. Then Petkov's sufficient condition implies that $\mathcal{R}$ has the property required in Theorem 1.2. It can be seen from the proof that the set $\mathcal{R}$ is also of second Baire category in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$.

In the special case when K is a finite disjoint union of convex bodies the statement of Theorem 1.2 was proved in [13].

The above theorem is related to some problems in the inverse scattering theory. It shows that for most obstacles K the singularities of the scattering kernel are completely determined by some geometrical objects — the scattering length spectrum of K. In general these objects are not enough to recover the obstacle (see Chapter 5 in [7]). One may conjecture that for obstacles K satisfying a natural accessibility condition the scattering length spectrum completely determines K. This is indeed so for some special classes of obstacles (cf. [14]).

Part of the results in this paper were announced in [15].

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2. Generalized Hamilton flow

For convenience of the reader we present here the definition of Melrose and Sjöstrand [8,9] of the generalized Hamiltonian flow in the symplectic invariant form given by Hörmander in [4] (see Section 24.3 there).

Let S be a symplectic manifold with boundary ∂S, $\dim S = 2n$, $n \geqslant 2$, and let $p\colon S \to \mathbb{R}$ be a smooth function. Throughout the paper "smooth" means C^∞. We assume that S is *σ-compact*, i.e. a countable union of compact subsets. Let $\varphi \in C^\infty(S)$ be a *defining function* of ∂S, i.e. $\varphi > 0$ in $S \setminus \partial S$, $\varphi = 0$ on ∂S and $d\varphi \neq 0$ on ∂S (φ may be only locally defined on a neighbourhood of ∂S). The first assumption that we make about S and p is the following.

A1. – $dp_{|\partial S} \neq 0$ and $\{\varphi,\{\varphi,p\}\}(\rho) \neq 0$ whenever $\rho \in \partial S$ and $\{\varphi,p\}(\rho) = 0$.

Here $\{f,g\}$ denotes the *Poisson bracket* of f and g. Denote by H_p the *Hamilton vector field* determined by the function p and consider the following subsets of S: $G = \{\sigma \in S\colon \varphi(\sigma) = H_p\varphi(\sigma) = 0\}$ (glancing set), $G_d = \{\sigma \in G\colon H_p^2\varphi(\sigma) > 0\}$ (diffractive set), $G_g = \{\sigma \in G\colon H_p^2\varphi(\sigma) < 0\}$ (gliding set), $G^k = \{\sigma \in G\colon H_p^j\varphi(\sigma) = 0,\ j = 0,1,\dots,k-1\}$ and $G^\infty = \bigcap_{k=2}^\infty G^k$. The *gliding vector field* H_p^G on G is defined by $H_p^G = H_p + \frac{H_p^2\varphi}{H_\varphi^2 p} H_\varphi$. In fact, H_p^G is a well-defined and smooth vector field in a neighbourhood of G in S.

In order to properly define the GHF, one should be able to define a "reflected trajectory" at a point $\rho \in \partial S$ where the flow of H_p hits transversally ∂S. This requires some sort of hyperbolic structure of H_p near such points.

In what follows we make the following assumption about the symplectic manifold S and the function p:

A2. – For every point $\rho_0 \in \partial S \cap p^{-1}(0)$ there exists an open neighbourhood $\mathcal{O}$ of ρ_0 in S and symplectic coordinates $(x,\xi) = (x_1,\dots,x_n;\xi_1,\dots,\xi_n)$ in $\mathcal{O}$ such that $\varphi(x,\xi) = x_1$ in $\mathcal{O}$, that is $S \cap \mathcal{O} = \{(x,\xi)\colon\ x_1 \geqslant 0\}$, $\partial S \cap \mathcal{O} = \{(x,\xi)\colon\ x_1 = 0\}$, and such that

$$p(x,\xi) = g(x,\xi)\big[\xi_1^2 - r(x,\xi')\big], \quad (x,\xi) \in \mathcal{O}, \tag{2}$$

for some smooth functions $g(x,\xi)$ and $r(x,\xi')$ with $|g(x,\xi)| \geqslant a > 0$ in $\mathcal{O}$ for some constant $a > 0$.

Here we use the notation $x' = (x_2,\dots,x_n)$, $\xi' = (\xi_2,\dots,\xi_n)$. In all cases known to the author where the generalized Hamilton flow has been involved (e.g. propagation of singularities for second-order linear differential operators), the condition **A2** has been satisfied. Notice that when $\rho_0 \in G$, then **A2** follows from **A1** and the Malgrange preparation theorem (see [8] or Section 24.3 in [4]). Moreover in this case we can choose the coordinates (x,ξ) so that they are centered at ρ_0, i.e. $\rho_0 = (0,0)$.

The following definition is due to Melrose and Sjöstrand [8,9]. Here we consider it in the form given by Hörmander (cf. Definition 24.3.6 in [4]).

DEFINITION. – *Let $I \subset \mathbf{R}$ be an interval. A curve $\Gamma : I \to S$ is called a generalized integral curve of p if there exists a discrete subset B of I such that*:

(i) *if $t \in I \setminus B$ and $\Gamma(t) \in (S \setminus \partial S) \cup G_d$, then there exists $\Gamma'(t) = H_p(\Gamma(t))$*;
(ii) *if $t \in I \setminus B$ and $\Gamma(t) \in G \setminus G_d$, then there exists $\Gamma'(t) = H_p^G(\Gamma(t))$*;
(iii) *for each $t \in B$, $\Gamma(t+s) \in S \setminus \partial S$ for all small $s \neq 0$ and there exist the limits $\Gamma(t-0) \neq \Gamma(t+0)$ which are points of one and the same integral curve of φ on ∂S.*

We will consider integral curves mainly on the *zero bicharacteristic set* $\Sigma = p^{-1}(0)$. Set

$$G_-^k = \{\sigma \in G^k\colon\ H_p^k(\sigma) < 0\}, \qquad G_+^k = \{\sigma \in G^k\colon\ H_p^k(\sigma) > 0\}, \quad k \geqslant 2.$$

The third assumption that we make about S and p is the following:

A3. – $G^\infty \cap \overline{\bigcup_{k=2}^\infty G_-^k} = \emptyset$.

In this case one can define a (local) flow $\mathcal{F}_t = \mathcal{F}_t^{(K)} : S \to S$, $t \in \mathbb{R}$, such that $\{\mathcal{F}_t(\sigma)\colon\ t \in \mathbf{R}\}$ is an integral curve of p for each $\sigma \in S$. This flow is called the *generalized Hamilton flow* (GHF) generated by p.

Recall from Section 1 that $\widetilde{S} = S/\sim$ is the quotient space with respect to the following equivalence relation on S: $\rho \sim \sigma$ if and only if either $\rho = \sigma$ or $\rho \in S \cap \partial S$, $\sigma \in S \cap \partial S$ and ρ and σ lie on one and the same integral curve of φ on ∂S. Using the natural map $\pi : S \to \widetilde{S}$, the flow $\mathcal{F}_t$ gives rise to another flow $\widetilde{\mathcal{F}}_t : \widetilde{S} \to \widetilde{S}$, called the *compressed Hamilton flow*.

Let $\Gamma : I \to S$ be a generalized integral curve of p. We say that Γ is *gliding* on ∂S if the set of those $t \in I$ such that $\Gamma(t) \in G_g$ is dense in I. In this case the trajectory $\{\Gamma(t)\colon\ t \in I\}$ is called a *gliding segment* of p on ∂S. If $\Gamma(I) \cap G_g = \emptyset$, then Γ is called a *reflecting integral curve* of p and $\{\Gamma(t)\colon\ t \in I\}$ a *reflecting trajectory*.

Remark 1. – The maps $\mathcal{F}_t$ depend on φ and in general φ is only locally defined near ∂S, however the integral curves of $\mathcal{F}_t$ are globally defined and do not depend on the choice of φ. Since the behaviour of $\mathcal{F}_t$ away from ∂S is trivial (a smooth Hamilton flow on a symplectic manifold without boundary), the emphasis here is on the study of $\mathcal{F}_t$ near ∂S.

Without loss of generality we may assume that S is part of a *symplectic manifold* $\mathcal{V}$ *of the same dimension and without boundary* and that p is a smooth function on $\mathcal{V}$. Denote by H_p the *Hamilton vector field* on $\mathcal{V}$ determined by p and by Φ_t the corresponding *smooth Hamilton flow* on $\mathcal{V}$. Clearly, if $\rho \in S \setminus \partial S$ and $\mathcal{F}_t(\rho) \in S \setminus \partial S$ for all $t \in I = (a,b)$, then $\mathcal{F}_t(\rho) = \Phi_t(\rho)$ for all $t \in I$. The apparent difference between $\mathcal{F}_t$ and Φ_t is that the latter is smooth and has no reflections at ∂S; in fact the trajectories of Φ_t can cross ∂S and enter $\mathcal{V} \setminus S$. Set[2] $\tilde{\mathcal{V}} = \mathcal{V}/\sim$, where $\sim$ is the same equivalence relation by means of which we defined $\widetilde{S}$.

For every point $\rho \in \partial S$ there is a symplectic chart $\mathcal{O}$ in $\mathcal{V}$ with the properties described in **A2**. (In fact, $\mathcal{V}$ can be constructed by gluing such charts around ∂S.) There exists a *metric* d_0 on $\mathcal{V}$ which is equivalent to the standard metric $\|x-y\| + \|\xi - \eta\|$ on each chart $\mathcal{O}$. In what follows d_0 will denote a fixed metric on $\mathcal{V}$ with this property.

There exists a *pseudometric* d on $\mathcal{V}$ such that

$$\begin{aligned} &c\min\{d_0(\rho',\sigma')\colon\ \pi(\rho') = \pi(\rho), \pi(\sigma') = \pi(\sigma)\} \\ &\quad \leqslant d(\rho,\sigma) \leqslant C\min\{d_0(\rho',\sigma')\colon\ \pi(\rho') = \pi(\rho), \pi(\sigma') = \pi(\sigma)\} \end{aligned} \tag{3}$$

for all $\rho, \sigma \in \mathcal{V}$, where $C > c > 0$ are constants. Given a coordinate open subset $\mathcal{O}$ of S defined by a Darboux chart as in **A2** with d_0 equivalent to $d_0'((x,\xi),(y,\eta)) = \|x-y\| + \|\xi-\eta\|$ on $\mathcal{O}$, we can define a *pseudometric* d' on $\mathcal{O}$, say by

$$d'\big((x,\xi),(y,\eta)\big) = \|x-y\| + \|x_1\xi - y_1\eta\| + \min\{\|\xi-\eta\|, \|\xi-\overline{\eta}\|\},$$

where $\overline{\eta} = (-\eta_1, \eta_2, \ldots, \eta_n)$. Being the sum of three pseudometrics on $\mathcal{O}$, d' is a pseudometric, too. Moreover, (3) holds with d_0 and d, replaced by d_0' and d', respectively. Gluing appropriately the above locally defined pseudometrics, one gets a globally defined pseudometric d on $\mathcal{V}$ satisfying the condition (3). Then *the projection* $\tilde{d}$ of d to $\tilde{\mathcal{V}}$ is a metric. Since the projection of $\mathcal{F}_t$ to $\widetilde{\Sigma}$ is continuous with respect to the metric $\tilde{d}$ [9], the flow $\mathcal{F}_t$ on S is continuous with respect to the pseudometric d.

Remark 2. – One can easily see that for any Borel subset B of Σ, $\dim_H(B)$ calculated with respect to the metric d_0 is the same as $\dim_H(B)$ calculated with respect to the pseudometric d. To check this it is enough to consider separately three cases: $B \subset S \setminus \partial S$ (trivial, since d_0 is equivalent to d locally in $S \setminus \partial S$), $B \subset G$ (trivial since d_0 is equivalent to d on G), and $B \subset \partial S \setminus G$. Consider the last case. Then $B = B_- \cup B_+$, where $B_- = \{\sigma \in B\colon\ \sigma = \lim_{t \nearrow 0} \mathcal{F}_t(\sigma)\}$ and B_+ is defined similarly. Then $B_\pm$ are Borel subsets of Σ and it is enough to show that $\dim_H(B_\pm)$ is the same with respect to d_0 and d. However this follows trivially, since d is a metric on each of the sets $B_\pm$ equivalent to d_0. From the last case one also obtains that for any Borel subset B of Σ we have $\dim_H(B) = \dim_H(\pi(B))$.

Given $\sigma \in S$, denote

$$\ell(\sigma) = \{\mathcal{F}_t(\sigma)\colon\ 0 \leqslant t \leqslant T\}.$$

[2] This notation is introduced just for convenience; the set $\tilde{\mathcal{V}}$ does not have some geometric importance and will not be used significantly below.

THEOREM 2.1. – *Let $T>0$. There exists a representation of Σ as a countable union $\Sigma=\bigcup_{i\in I} S_i$ of Borel subsets S_i such that for each $i\in I$ there exist an open neighbourhood V_i of S_i in $\mathcal{V}$ and a family of maps $\mathcal{G}_t^{(i)}: V_i\to\mathcal{V}$ $(0\leqslant t\leqslant T)$ with the following properties*:

(a) $\mathcal{G}_t^{(i)}(\sigma)=\mathcal{F}_t(\sigma)$ *for all* $\sigma\in S_i$ *and all* $t\in[0,T]$;

(b) *For every* $\sigma\in S_i$ *and every* $t\in(0,T]$ *there exists an open neighbourhood* $W=W(\sigma,i,t)$ *of* σ *in* V_i *such that* $\mathcal{G}_t^{(i)}$: $(W,d_1)\to(\mathcal{V},d_2)$ *is Lipschitz, where* $d_1=d_0$ *if* $\sigma\in(S\setminus\partial S)\cup G$ *and* $d_1=d$ *if* $\sigma\in\partial S\setminus G$, *and similarly* $d_2=d_0$ *if* $\mathcal{F}_t(\sigma)\in(S\setminus\partial S)\cup G$ *and* $d_2=d$ *if* $\mathcal{F}_t(\sigma)\in\partial S\setminus G$;

(c) *If* $\sigma\in S_i\cap[(S\setminus\partial S)\cup G]$ *and* $t\in(0,T]$ *is such that* $\mathcal{F}_t(\sigma)\in(S\setminus\partial S)\cup G$, *then there exists an open neighbourhood* $W=W(\sigma,i,t)$ *of* σ *in* V_i *such that the map* $\mathcal{G}_t^{(i)}: W\to\mathcal{V}$ *is smooth. If moreover both* σ *and* $\mathcal{F}_t(\sigma)$ *are not ends of gliding segments of* $\{\mathcal{F}_s(\sigma)\colon s\in[-\varepsilon,T+\varepsilon]\}$ *for any small* $\varepsilon>0$, *then* W *can be chosen in such a way that the restriction of* $\mathcal{G}_t^{(i)}$ *to any smooth cross-section in* W *at* σ *is a contact transformation.*

The latter means that if $\mathcal{M}$ is a smooth local submanifolds of W of codimension 1 containing σ and transversal to $\ell(\sigma)$, then ${\mathcal{G}_t^{(i)}}_{|\mathcal{M}}:\mathcal{M}\to\mathcal{G}_t^{(i)}(\mathcal{M})$ is a contact (canonical) transformation with respect to the standard contact structures on $\mathcal{M}$ and $\mathcal{G}_t^{(i)}(\mathcal{M})$ inherited from the symplectic structure of $\mathcal{V}$ (cf. for example Section 5.2 and Proposition 8.1.3 in [1]). In particular, $\mathcal{M}\cap\Sigma$ and $\mathcal{G}_t^{(i)}(\mathcal{M}\cap\Sigma)$ are symplectic submanifolds of $\mathcal{V}$ of codimension 2 and the restriction $\mathcal{G}_t^{(i)}:\mathcal{M}\cap\Sigma\to\mathcal{G}_t^{(i)}(\mathcal{M}\cap\Sigma)$ is a symplectic map. However, in general Σ is not invariant under $\mathcal{G}_t^{(i)}$, that is $\mathcal{G}_t^{(i)}(V_i\cap\Sigma)$ is not necessarily a subset of Σ.

It follows from [9] that for any given $\sigma\in\Sigma$, the trajectory $\ell(\sigma)$ has only finitely many transversal reflection points and finitely many gliding segments, so part (c) in the above theorem concerns all but finitely many $t\in[0,T]$. It can be seen from the proof of Theorem 2.1 that $\mathcal{G}_t^{(i)}$ is actually a "finite combination" of local Hamilton flows in $\mathcal{V}$.

Clearly Theorem 1.1 would have been trivial (and Theorem 2.1 would have been unnecessary for its proof) if the maps $\mathcal{F}_t$ were Lipschitz. However, it is well-known and easy to see that this is not the case. Locally near a point $\rho\in\widetilde{S}$, the map $\mathcal{F}_t$ is Lipschitz on a neighbourhood of ρ for small $|t|$ when $\rho\notin\partial S$ or ρ is a transversal reflection point. Whenever $\rho\in G$, the map $\mathcal{F}_t$ is not Lipschitz (cf. [8] or [4]). For example, in the simplest case of a diffractive tangent point $\rho\in G_d$, the map $\mathcal{F}_t$ has a singularity of "square root type" at ρ, so it is clearly not Lipschitz.

Theorem 2.1 is proved in Section 3. As a consequence of it one obtains Theorem 1.1.

Proof of Theorem 1.1. – It is enough to show that for each t the map $\mathcal{F}_t:\Sigma\to\Sigma$ does not increase the Hausdorff dimension of Borel subsets; then using the same property for $\mathcal{F}_{-t}$, one concludes that $\mathcal{F}_t$ actually preserves $\dim_H$. For a similar reason it is enough to consider the case $t>0$.

Let B be a Borel subset of Σ and let $t>0$ be a fixed number. We have to show that $\dim_H(\mathcal{F}_t(B))\leqslant\dim_H(B)$. From the properties of Hausdorff dimension (cf. for example [2]) we have $\dim_H(B)=\max_{1\leqslant i\leqslant 3}\dim_H(B_i)$, where $B=B_1\cup B_2\cup B_3$ with $B_1=B\setminus\partial S$, $B_2=B\cap G$, $B_3=(B\cap\partial S)\setminus G$. So, it is enough to prove that $\dim_H(\mathcal{F}_t(B_i))\leqslant\dim_H(B_i)$ for $i=1,2,3$. This essentially means that we have to consider separately three cases: $B\subset S\setminus\partial S$, $B\subset G$, and $B\subset\partial S\setminus G$.

First, assume that $B\subset S\setminus\partial S$. Take an arbitrary $T>t$, and let $\Sigma=\bigcup_{i\in I}S_i$ be a representation of Σ as a countable union of Borel subsets S_i of S with the properties listed in Theorem 2.1. To prove $\dim_H(\mathcal{F}_t(B))\leqslant\dim_H(B)$, it is enough to show that $\dim_H(\mathcal{F}_t(B\cap S_i))\leqslant\dim_H(B\cap S_i)$ for each i, for which in turn it is enough for each $\sigma\in B\cap S_i$ to find an open neighbourhood U of σ in S such that $\dim_H(\mathcal{F}_t(B\cap S_i\cap U))\leqslant\dim_H(B\cap S_i\cap U)$. Fix for a moment i and $\sigma\in B\cap S_i$. Then by Theorem 2.1(b), there exists a neighbourhood W of σ in V_i such that $\mathcal{G}_t^{(i)}:(W,d_0)\to(\mathcal{V},d_2)$ is Lipschitz, where $d_2=d$ or d_0 depending on where $\mathcal{F}_t(\sigma)$

belongs. Since $d \leqslant \text{const } d_0$, it follows that $\mathcal{G}_t^{(i)}:(W,d_0) \to (\mathcal{V},d)$ is Lipschitz. Moreover, we can take W such that $\overline{W}$ is compact and has no common points with ∂S. Then d_0 is equivalent to d on W, and so $\mathcal{G}_t^{(i)}:(W,d) \to (\mathcal{V},d)$ is Lipschitz. Using this and the fact that $\mathcal{G}_t^{(i)} = \mathcal{F}_t$ on S_i (cf. condition (a) in Theorem 2.1), we get

$$\dim_H\big(\mathcal{F}_t(B\cap S_i\cap W)\big) = \dim_H\big(\mathcal{G}_t^{(i)}(B\cap S_i\cap W)\big) \leqslant \dim_H(B\cap S_i\cap W).$$

Denote $U = W\cap \Sigma$; then U is a neighbourhood of σ in Σ having the desired property. This completes the proof in the case $B\subset S\setminus \partial S$.

The case $B\subset G$ is very similar to the first one — since d_0 is equivalent to d on G, one can use Theorem 2.1(b) again as above.

Finally, consider the case $B\subset \partial S\setminus G$. It is enough for each $\sigma\in B\cap S_i$ to find an open neighbourhood U of σ in Σ such that $\dim_H(\mathcal{F}_t(B\cap S_i\cap U)) \leqslant \dim_H(B\cap S_i\cap U)$. By Theorem 2.1(b) there exists an open neighbourhood W of σ in V_i (the domain of $\mathcal{G}_t^{(i)}$) such that $\mathcal{G}_t^{(i)}:(W,d)\to(\mathcal{V},d_2)$ is Lipschitz, where again $d_2 = d_0$ or d. As in the first case, one concludes that $\mathcal{G}_t^{(i)}:(W,d)\to(\mathcal{V},d)$ is Lipschitz and that $\dim_H(\mathcal{F}_t(B\cap S_i\cap U)) \leqslant \dim_H(B\cap S_i\cap U)$, where $U = W\cap\Sigma$. This completes the proof of $\dim_H(\mathcal{F}_t(B)) \leqslant \dim_H(B)$. □

Given $T>0$, denote by $\mathcal{T}_T$ *the set of those* $\rho\in\Sigma$ *such that* $\{\mathcal{F}_t(\rho)\colon\ 0\leqslant t\leqslant T\}\cap G_g\neq\emptyset$, that is the trajectory $\{\mathcal{F}_t(\rho)\colon\ 0\leqslant t\leqslant T\}$ contains a non-trivial gliding segment on ∂S.

THEOREM 2.2. – *Let $\mathcal{L}_0$ be an isotropic submanifold of $\Sigma\setminus\partial S$ of dimension $n-1$ such that $H_p(\rho)$ is not tangent to $\mathcal{L}_0$ at any $\rho\in\mathcal{L}_0$. Then for every $T>0$ we have* $\dim_H(\mathcal{F}_T(\mathcal{T}_T\cap\mathcal{L}_0)) \leqslant n-2$. *Moreover, if for a given T we have $\mathcal{F}_T(\mathcal{L}_0)\subset S\setminus\partial S$, then there exists a countable family $\{\mathcal{I}_m\}$ of smooth $(n-2)$-dimensional isotropic submanifolds of S such that $\mathcal{F}_T(\mathcal{T}_T\cap\mathcal{L}_0)\subset \bigcup_m \mathcal{I}_m$.*

Remark 3. – The above statement is not true if we replace $\mathcal{T}_T$ by the set $\widetilde{\mathcal{T}}_T$ of those $\rho\in\Sigma$ such that $\{\mathcal{F}_t(\rho)\colon\ 0\leqslant t\leqslant T\}\cap G\neq\emptyset$. Using simple caustics in the plane, one can easily construct examples when $\dim_H(\mathcal{F}_T(\widetilde{\mathcal{T}}_T\cap\mathcal{L}_0)) = n-1$.

We refer to reader to [1] or [4] for the definition of an isotropic submanifold.

Theorem 2.2 is proved in Section 4. We are now going to use it and prove Theorem 1.2.

Let Ω be a domain in $\mathbb{R}^n$. Consider the symplectic manifold $\check{S} = T^*(\Omega\times\mathbb{R})$ and the smooth function $\check{p}(x,t;\xi,\tau) = \sum_{i=1}^n \xi_i^2 - \tau^2$. Since both vector fields $H_{\check{p}}$ and $H_{\check{p}}^G$ do not depend on t, we have $\tau = \text{const}$ along each generalized integral curve of $\check{p}$. The change of τ can only affect the parametrization along a generalized integral curve which is not important for our aims. Thus, we may assume that $\tau = \pm 1$. There is a natural correspondence between the generalized integral curves of $\check{p}$ with this property and the generalized integral curves of the Hamilton function

(4) $$p(x,\xi) = \sum_{i=1}^n \xi_i^2 - 1$$

on the symplectic manifold $S = T^*(\Omega)$; the correspondence being given by $(x(t),\xi(t)) \mapsto (x(t),t;\xi(t),\pm 1)$. Let $\mathcal{F}_t$ be the generalized Hamilton flow on $T^*(\Omega)\setminus\{0\}$ generated by the function (4). Notice that the submanifold $\Sigma = p^{-1}(0)$ of S coincides with the *cosphere bundle* $S^*(\Omega)$. The restriction of $\mathcal{F}_t$ to $S^*(\Omega)$ defines the so-called *generalized geodesic flow*

(5) $$\mathcal{F}_t : S^*(\Omega)\to S^*(\Omega),\quad t\in\mathbb{R}.$$

Let $\mathrm{pr}_1 : T^*(\Omega) \to \Omega$ and $\mathrm{pr}_2 : T^*(\Omega) \to \mathbb{R}^n$ be the natural projections. A curve γ in Ω is called a *generalized geodesic* in Ω if there exist an interval I and $\rho \in T^*(\Omega)$ such that $\gamma = \{\mathrm{pr}_1(\mathcal{F}_t(\rho)):\ t \in I\}$. We will say that γ is a *gliding segment* on $\partial\Omega$ (respectively *reflecting ray* in Ω) if $\{\mathcal{F}_t(\rho):\ t \in I\}$ is a gliding trajectory (respectively reflecting trajectory) of p.

Next, assume that $\Omega = \Omega_K$ for some obstacle K in $\mathbb{R}^n$ and $\omega, \theta \in \mathbf{S}^{n-1}$. Let $\gamma = \{\mathrm{pr}_1(\Gamma(t)):\ t \in \mathbb{R}\}$, where $\Gamma : \mathbb{R} \to S^*(\Omega)$ is a trajectory of $\mathcal{F}_t$. The curve γ is called an (ω,θ)-*ray* in Ω if $\mathrm{pr}_2(\Gamma(t)) = \omega$ for $t \ll 0$ and $\mathrm{pr}_2(\Gamma(t)) = \theta$ for $t \gg 0$. If γ is a reflecting ray, i.e. it does not contain gliding segments on $\partial\Omega$, and has only finitely many reflection points, it is called a *reflecting* (ω,θ)-*ray* in Ω. By $\mathcal{L}_{\omega,\theta}(K)$ we denote the *set of all* (ω,θ)-*rays in* Ω_K.

Fix an open ball $\mathcal{U}$ which contains K. Given $\xi \in \mathbf{S}^{n-1}$ denote by Z_ξ *the hyperplane in* $\mathbb{R}^n$ *orthogonal to* ξ *and tangent to* $\mathcal{U}$ such that $\mathcal{U}$ is contained in the open half-space R_ξ determined by Z_ξ and having ξ as an inner normal. Given an (ω,θ)-ray γ in Ω, the *sojourn time* T_γ of γ is defined by $T_\gamma = T'_\gamma - 2a$, where T'_γ is the length of that part of γ which is contained in $R_\omega \cap R_{-\theta}$ and a is the radius of the ball $\mathcal{U}$. It is known (cf. [3]) that this definition does not depend on the choice of the ball $\mathcal{U}$.

In what follows we denote by $\mathcal{K}$ *the class of obstacles* K such that condition **A3** is satisfied for $S = T^*(\Omega_K) \setminus \{0\}$ and the function (4). Assuming $\Omega = \Omega_K$ with $K \in \mathcal{K}$, it follows from results of Melrose and Sjöstrand [9] (see also [4, Theorem 24.3.9]) that every (ω,θ)-ray γ in Ω_K which does not contain gliding segments is a reflecting (ω,θ)-ray, i.e. it consists of finitely many straight line segments in Ω (two of them being infinite rays).

Proof of Theorem 1.2. – Let K be an obstacle in $\mathbb{R}^n$ of the class $\mathcal{K}$. We are going to show that there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for each $(\omega,\theta) \in \mathcal{R}$ the only (ω,θ)-rays in Ω_K are reflecting (ω,θ)-rays.

Consider the domain $\Omega = \Omega_K$, the symplectic manifold $S = T^*(\Omega)$ and the corresponding generalized geodesic flow (5) of the function (4). As above, denote by $\mathcal{U}$ an open ball in $\mathbb{R}^n$ containing the obstacle K and by C the boundary sphere of $\mathcal{U}$. Fix $\omega \in \mathbf{S}^{n-1}$, $x_0 \in C$ and consider the generalized geodesic $(x(t),\xi(t)) = \mathcal{F}_t(x_0,\omega)$. Let $T > 0$ be such that $x(T) \in C$. Denote

$$S_0 = \big\{(x,\xi) \in S:\ x \in C,\ \xi \text{ is transversal to } C\big\}.$$

Since $\Sigma = p^{-1}(0) = S^*(\Omega)$, using the notation $S^*_C(\Omega) = \{(x,\xi) \in S^*(\Omega):\ x \in C\}$, we have $S'_0 = S_0 \cap \Sigma = \{(x,\xi) \in S^*_C(\Omega):\ \xi \text{ is transversal to } C\}$. Then S'_0 is a symplectic submanifold of S. Let $\mathcal{P} : S_0 \to S_0$ be the local map defined in a neighbourhood of (x_0,ω) using the shift along the flow $\mathcal{F}_t$; then $\mathcal{P}(S'_0) \subset S'_0$. Consider the Lagrangian submanifold

$$\mathcal{L}_0 = \big\{(x,\xi) \in S'_0:\ \xi = \omega\big\}$$

of S'_0. Setting $\mathcal{T} = \mathcal{T}_T$ and applying Theorem 2.2 to $\mathcal{L}_0$, give that $\mathcal{F}_T(\mathcal{L}_0 \cap \mathcal{T})$ is contained in a countable union of isotropic $(n-2)$-dimensional submanifolds of S. Since locally near (x_0,ω) the map $\mathcal{F}_T : S_0 \to \mathcal{F}_T(S_0)$ is smooth, $\mathcal{F}_T(S_0)$ is a $(2n-1)$-dimensional submanifold of S transversal to the flow $\mathcal{F}_t$ at $\mathcal{F}_T(x_0,\omega)$. Consequently, locally near $\mathcal{F}_T(x_0,\omega) \in \mathcal{F}_T(S_0) \cap S_0$ the shift $\mathcal{Q}$ along $\mathcal{F}_t$ from $\mathcal{F}_T(S_0)$ to S_0 (forwards or backwards) is a smooth map. Moreover $\mathcal{Q}$ maps $\mathcal{F}_T(S'_0)$ into S'_0 (since $p^{-1}(0)$ is invariant under the flow $\mathcal{F}_t$), the restriction $\mathcal{Q} : \mathcal{F}_T(S'_0) \to S'_0$ is a local symplectic map, and $\mathcal{P} = \mathcal{Q} \circ \mathcal{F}_T$. Hence the set $\mathcal{P}(\mathcal{L}_0 \cap \mathcal{T}) = \mathcal{Q}(\mathcal{F}_T(\mathcal{L}_0 \cap \mathcal{T}))$ is contained in a countable union of isotropic $(n-2)$-dimensional submanifolds of S. The projection $j : S'_0 \to \mathbf{S}^{n-1}$, $j(x,\xi) = \xi$, is smooth, so Sard's theorem gives now that the set $j(\mathcal{P}(\mathcal{L}_0 \cap \mathcal{T}))$ has Lebesgue measure zero in $\mathbf{S}^{n-1}$. Hence there exist a neighbourhood U of x_0 in C and a subset $\mathcal{R}_\omega(U) = \mathbf{S}^{n-1} \setminus j(\mathcal{P}(\mathcal{L} \cap \mathcal{T}))$ of full Lebesgue measure in $\mathbf{S}^{n-1}$ such that for

$x \in U$ every generalized (ω,θ)-ray in Ω passing through x with $\theta \in \mathcal{R}_\omega(U)$ is a reflecting (ω,θ)-ray. Covering C by a finite family of neighbourhoods U_i, we find a subset $\mathcal{R}_\omega = \bigcap_i \mathcal{R}_\omega(U_i)$ of full Lebesgue measure in $\mathbf{S}^{n-1}$ such that every (ω,θ)-ray in Ω with $\theta \in \mathcal{R}_\omega$ is a reflecting (ω,θ)-ray. It now follows from Fubini's theorem that

$$\mathcal{R}' = \{(\omega,\theta) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \colon \theta \in \mathcal{R}_\omega\}$$

is a subset of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$. Moreover it is clear that for $(\omega,\theta) \in \mathcal{R}'$, all (ω,θ)-rays in Ω are reflecting ones.

According to Theorem 3.1 and Proposition 4.1 in [13], there exists a subset $\mathcal{R}''$ of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for $(\omega,\theta) \in \mathcal{R}''$ every reflecting (ω,θ)-ray in Ω_K has no tangencies to ∂K and $T_\gamma \neq T_\delta$ whenever γ and δ are different reflecting (ω,θ)-rays in Ω_K. Then $\mathcal{R} = \mathcal{R}' \cap \mathcal{R}''$ has full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$. Given $(\omega,\theta) \in \mathcal{R}$, it follows from the results of Petkov [11] (see also Section 9.1 in [12]) that $-T_\gamma \in \operatorname{sing\,supp} s_K(t,\theta,\omega)$ for all $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega_K)$. Combining this with (1) completes the proof of the theorem. □

Using Theorem 1.2 we will now give another (and more rigourous) proof of Theorem 1.6.2 of Lax and Phillips [5]: most rays incoming from infinity are not trapped by the obstacle K. Recall that $\Omega_{\widehat{K}}$ is the closure of the complement of the *convex hull* $\widehat{K}$ of K. Below it is important that we consider points $(x,\xi) \in S^*(\Omega_{\widehat{K}})$. In general it is not true that the trapped points $(x,\xi) \in S^*(\Omega_K)$ (with $x \in \widehat{K}$), i.e. the points that generate bounded trajectories, form a set of Lebesgue measure zero in $S^*(\Omega_K)$. The example of M. Livshitz (cf. Example V.4.0 in [7]) shows that in some cases the set of trapped points may even contain a non-trivial open subset of $S^*(\Omega_K)$.

PROPOSITION 2.3. – *If $K \in \mathcal{K}$, then the set of those $(x,\xi) \in S^*(\Omega_{\widehat{K}})$ such that the trajectory $\{\mathcal{F}_t(x,\xi)\colon t \geqslant 0\}$ is bounded has Lebesgue measure zero in $S^*(\Omega_{\widehat{K}})$.*

Proof. – Let $K \in \mathcal{K}$, $\mathcal{U}$ be an open ball containing K and C be the boundary sphere of $\mathcal{U}$. Set $\Omega = \Omega_K$. For $(x,\omega) \in S^*_C(\Omega)$, let $\delta(x,\omega)$ be the generalized geodesic in Ω_K issued from x in direction ω. Assume that there exists a subset W of positive Lebesgue measure in $S^*_C(\Omega)$ such that $\delta(x,\omega) \subset \mathcal{U}$ for all $(x,\omega) \in W$. According to Theorem 2.2 above and Theorem 3.1 of [13], we may assume that for all $(x,\omega) \in W$ the generalized geodesic $\delta(x,\omega)$ does not contain gliding segments on $\partial\Omega$ and has only transversal reflections at ∂K. Given $(x,\omega) \in W$, denote by x' the first common point of $\delta(x,\omega)$ with ∂K and by ω' the reflected direction of $\delta(x,\omega)$ at x', i.e. $\omega' = \omega - 2\langle\omega,\nu(x')\rangle\nu(x')$, where $\nu(x')$ is the outer unit normal to K at x'. Then the set $W' = \{(x',\omega')\colon (x,\omega) \in W\}$ is a subset of positive Lebesgue measure in $S^*_{\partial K}(\Omega)$.

Denote by M the set of those $(y,\eta) \in S^*_{\partial K}(\Omega)$ for which the standard billiard ball map B is well-defined. The map B (as a local map) preserves the so-called Liouville's measure μ on M which is absolutely continuous with respect to the Lebesgue measure on $S^*_{\partial K}(\Omega)$.

Next, we use the argument from the proof of the Poincaré Recurrence Theorem in ergodic theory. It follows from the definition of W' that $B^k(W') \subset M$ and $\mu(B^k(W')) = \mu(W') > 0$ for all $k = 0,1,2,\dots$. On the other hand, in the situation under consideration we clearly have $\mu(\bigcup_{k=0}^\infty B^k(W')) < \infty$. Therefore there exist non-negative integers $k < m$ with $B^k(W') \cap B^m(W') \neq \emptyset$. Since B is invertible, this means that there exists $(x',\omega') \in W' \cap B^{m-k}(W')$. Then $(x',\omega') = B(y,\eta)$ for some $(y,\eta) \in B^{m-k-1}(W') \subset M$. Now the choice of W and the definition of W' show that W' has no common points with $B(M)$. This is a contradiction which proves the proposition. □

3. Regularity of the generalized Hamilton flow

This section is devoted to the proof of Theorem 2.1. Throughout S will be a $2n$-dimensional symplectic manifold, p a smooth function of S satisfying the conditions **A1**, **A2** and **A3**, and $T > 0$ will be a fixed real number.

Let $\rho_0 \in \partial S \setminus G$. There exist a neighbourhood U of ρ_0 in S and $T' > 0$ such that for every $\rho \in U$ the trajectory $\{\mathcal{F}_t(\rho)\colon\ 0 \leqslant t \leqslant T'\}$ has at most one common point with ∂S which is a transversal reflection point. In fact, taking $T' > 0$ and U sufficiently small, for every $\rho \in U$ there exists a unique real number $t(\rho)$ with $|t(\rho)| < T'$ such that $\mathcal{F}_{t(\rho)}(\rho) \in \partial S$.

The following fact is probably well known. We prove it here for completeness.

LEMMA 3.1. – *Under the assumptions above, if the neighbourhood U is taken sufficiently small, then the family of maps $\mathcal{F}_t : U \to S$, $0 \leqslant t \leqslant T'$, is uniformly Lipschitz with respect to the pseudometric d on S. That is, there exists a constant $C > 0$ such that $d(\mathcal{F}_t(\rho), \mathcal{F}_t(\sigma)) \leqslant C d(\rho, \sigma)$ for all $\rho, \sigma \in U$ and all $t \in [0, T']$.*

Proof. – It is enough to show that the map $U \ni \rho \mapsto t(\rho)$ is uniformly Lipschitz with respect to the pseudometric d. The rest follows from the smoothness of the Hamilton flow of H_p, its transversality to ∂S at ρ_0, and the fact that the pseudometric d is equivalent to the metric d_0 on any subset W of S such that $\sigma = \lim_{t \searrow 0} \mathcal{F}_t(\sigma)$ for any $\sigma \in W$ (or $\sigma = \lim_{t \nearrow 0} \mathcal{F}_t(\sigma)$ for any $\sigma \in W$).

Let $\mathcal{O}$ be a coordinate neighbourhood of ρ_0 of the type described in **A2**. Then for $\rho_0 = (x^{(0)}, \xi^{(0)})$ we have $\xi_1^{(0)} \neq 0$. Take U so small that $|\xi_1| > 2|\xi_1^{(0)}|/3$ for every $\rho = (x, \xi) \in U$. Notice that since $|\xi_1|$ is uniformly bounded from below, we have $|t(\rho)| \leqslant \text{const}\, x_1$ for all $\rho = (x, \xi) \in U$, where const means a positive constant that does not depend on ρ (and σ later on).

Given $\rho = (x, \xi) \in U$ and $\sigma = (y, \eta) \in U$, we have to show that

$$|t(\rho) - t(\sigma)| \leqslant \text{const}\, d(\rho, \sigma). \tag{6}$$

If both $t(\rho)$ and $t(\sigma)$ are non-negative or non-positive, this follows again from the smoothness of the flow of H_p. Assume $t(\rho) > 0$ and $t(\sigma) < 0$ (the other remaining case is similar). Then $\xi_1 < 0$ and $\eta_1 > 0$. It follows from the main property of d that $|x_1\xi_1 - y_1\eta_1| \leqslant \text{const}\, d(\rho, \sigma)$. Hence $x_1 + y_1 \leqslant \text{const}\, d(\rho, \sigma)$, so $x_1 \leqslant \text{const}\, d(\rho, \sigma)$ and $y_1 \leqslant \text{const}\, d(\rho, \sigma)$. This implies $|t(\rho)| \leqslant \text{const}\, x_1 \leqslant \text{const}\, d(\rho, \sigma)$ and similarly $|t(\sigma)| \leqslant \text{const}\, \tilde{d}(\rho, \sigma)$. Thus, (6) holds in all possible cases for $t(\rho)$ and $t(\sigma)$. $\square$

To every $\rho \in \Sigma$ we will now associate a string

$$\alpha = (k_0, k_1, \ldots, k_m, k_{m+1}; l_0, l_1, \ldots, l_m, l_{m+1}; q_0, q_1, \ldots, q_m; q), \tag{7}$$

of integers that roughly describes the geometry of the trajectory $\ell(\rho)$. For example, m will be the number of different gliding segments contained in the interior of $\ell(\rho)$, k_i and l_i will be the orders of tangency of $\ell(\rho)$ to ∂S at the initial and terminal point of the ith gliding segment, and q_i will be the number of transversal reflections of $\ell(\rho)$ between the ith and the $(i+1)$st gliding segments. The numbers k_0, l_0, k_{m+1}, l_{m+1} will describe the combinatorial type of $\ell(\rho)$ at its initial and terminal point. For example, if $\rho \notin \partial S$, we will have $k_0 = l_0 = -1$; if $\rho \in \partial S \setminus G$, then $k_0 = l_0 = 0$, if $\ell(\rho)$ begins with a gliding segment, then k_0 and l_0 will be the orders of tangency of this segment to ∂S at its initial and terminal points, etc. The pair k_{m+1}, l_{m+1} will play a similar role at the end of the trajectory $\ell(\rho)$. Finally, $1/q$ will be (roughly speaking) a lower bound of the distance to the set G at any transversal reflection of $\ell(\rho)$ at ∂S.

For the precise definition it is better to start with a given α and define the set of points $\rho \in \Sigma$ whose type is represented by α.

Notice that a point $\rho \in S$ *belongs to a gliding segment* if there exist $a < b$ such that $0 \in [a, b]$ and $\{\mathcal{F}_t(\rho)\colon a \leqslant t \leqslant b\}$ is a gliding segment on ∂S (cf. Section 2). Then $\rho \in G$ but in general we do not necessarily have $\rho \in G_g$. However, according to condition **A3**, we do have $\rho \notin G^\infty$, hence $\rho \in G^k \setminus G^{k+1}$ for some $k \geqslant 2$.

Let (7) be a string of integers, where $m = m(\alpha) \geqslant 0$, $k_i \geqslant 3$, $l_i \geqslant 3$ $(1 \leqslant i \leqslant m)$, $k_0, l_0, k_{m+1}, l_{m+1} \geqslant -1$, $q_i \geqslant 0$ $(0 \leqslant i \leqslant m)$, and $q \geqslant 1$. We will say that α is *admissible* if whenever $k_0 \leqslant 1$ (respectively $l_{m+1} \leqslant 1$) we have $l_0 = k_0$ (respectively $k_{m+1} = l_{m+1}$) and when $k_0 \geqslant 2$ (respectively $l_{m+1} \geqslant 2$) we have $l_0 \geqslant 2$ (respectively $k_{m+1} \geqslant 2$).

DEFINITION 3.2. – *Let α be an admissible string of the form* (7). *Denote by S_α the set of those $\rho \in \Sigma$ for which there exists a sequence of real numbers*

$$(8) \qquad 0 = t_0(\rho) \leqslant s_0(\rho) < t_1(\rho) < s_1(\rho) < \cdots < t_m(\rho) < s_m(\rho) < t_{m+1}(\rho) \leqslant s_{m+1}(\rho) = T$$

with the following properties:

(i) *For every $i = 0, 1, \ldots, m$ the curve $\{\mathcal{F}_t(\rho)\colon t \in [s_i(\rho), t_{i+1}(\rho)]\}$ has exactly q_i transversal reflections at ∂S and no common points with G_g*;

(ii) *For all $i = 0, 1, \ldots, m, m+1$, $\{\mathcal{F}_t(\rho)\colon t \in [t_i(\rho), s_i(\rho)]\}$ is an integral curve of the vector field H_p^G on G and $\mathcal{F}_t(\rho) \in G_g$ for almost all $t \in [t_i(\rho), s_i(\rho)]$*;

(iii) *For every $i = 1, \ldots, m$ we have $\mathcal{F}_{t_i(\rho)}(\rho) \in G^{k_i} \setminus G^{k_i+1}$ and $\mathcal{F}_{s_i(\rho)}(\rho) \in G^{l_i} \setminus G^{l_i+1}$*;

(iv) *If $k_0 \leqslant 1$, then $t_0(\rho) = s_0(\rho) = 0$ and: $\rho \notin \partial S$ for $k_0 = -1$, $\rho \in \partial S \setminus G$ for $k_0 = 0$, $\rho \in G$ but ρ does not belong to a gliding segment for $k_0 = 1$. If $k_0 \geqslant 2$, then ρ belongs to a gliding segment, $\rho \in G^{k_0} \setminus G^{k_0+1}$ and $\mathcal{F}_{s_0(\rho)}(\rho) \in G^{l_0} \setminus G^{l_0+1}$*;

(v) *If $l_{m+1} \leqslant 1$, then $t_{m+1}(\rho) = s_{m+1}(\rho) = T$ and: $\mathcal{F}_T(\rho) \notin \partial S$ for $l_{m+1} = -1$, $\mathcal{F}_T(\rho) \in \partial S \setminus G$ for $l_{m+1} = 0$, $\mathcal{F}_T(\rho) \in G$ but $\mathcal{F}_T(\rho)$ does not belong to a gliding segment for $l_{m+1} = 1$. If $l_{m+1} \geqslant 2$, then $\mathcal{F}_T(\rho)$ belongs to a gliding segment, $\mathcal{F}_{t_{m+1}(\rho)}(\rho) \in G^{k_{m+1}} \setminus G^{k_{m+1}+1}$ and $\mathcal{F}_T(\rho) \in G^{l_{m+1}} \setminus G^{l_{m+1}+1}$.*

(vi) *For every $t \in [0, T]$ such that $\mathcal{F}_t(\rho) \in \partial S \setminus G$ we have $d(\mathcal{F}_t(\rho), G) \geqslant 1/q$.*

One can check that each S_α is a Borel subset of Σ (this can be derived using arguments from the proof of Lemma 3.4 below). Notice that some of the sets S_α may be empty and any $\rho \in \Sigma$ belongs to many (in fact, infinitely many) S_α.

Remark 4 (An important remark). – Notice that condition (i) **does not** exclude the possibility that $\{\mathcal{F}_t(\rho)\colon t \in (s_i(\rho), t_{i+1}(\rho))\}$ has some other common points with ∂S apart from the q_i transversal reflections. In general $\partial S \cap \{\mathcal{F}_t(\rho)\colon t \in (s_i(\rho), t_{i+1}(\rho))\}$ may be a very complicated set (e.g. a Cantor set). Most of the points in this set (in fact all except the q_i transversal reflections) will be points from the set $G^\infty \cup \bigcup_{k=2}^\infty G_+^k$, which according to condition **A3** is far from G_g. Because of this possibility the construction of the maps $\mathcal{G}_t^{(i)}$ is a bit more complicated than perhaps anticipated.

LEMMA 3.3. – *We have $\Sigma = \bigcup_\alpha S_\alpha$, where α runs over all admissible strings.*

Proof. – Let $\rho \in \Sigma$. It follows from [8] that $\{\mathcal{F}_t(\rho)\colon 0 \leqslant t \leqslant T\}$ has only finitely many transversal reflections at ∂S and finitely many gliding segments on ∂S. Take a small $\varepsilon > 0$ and let

$$E = \overline{\{t \in [-\varepsilon, T+\varepsilon]\colon \mathcal{F}_t(\rho) \in G_g\}} \cap [0, T].$$

Then E is a finite disjoint union of closed subintervals of the interval $[0, T]$. If $0 \notin E$, set $s_0(\rho) = 0$ and: $k_0 = l_0 = -1$ if $\rho \notin \partial S$; $k_0 = l_0 = 0$ if $\rho \in \partial S \setminus G$; $k_0 = l_0 = 1$ if $\rho \in G$.

If $0 \in E$, then $[0, s_0(\rho)]$ is part of a connected component of E for some $s_0(\rho) \geqslant 0$ (i.e. ρ belongs to a gliding segment). Consequently, there exist $k_0 \geqslant 2$ and $l_0 \geqslant 2$ such that $\rho \in G^{k_0} \setminus G^{k_0+1}$ and $\mathcal{F}_{s_0(\rho)}(\rho) \in G^{l_0} \setminus G^{l_0+1}$. Notice that $s_0(\rho) > 0$ implies $l_0 \geqslant 3$ (cf. Section 24.3 in [4]), while $0 = s_0(\rho)$ yields $k_0 = l_0 \geqslant 3$. This defines completely the pair of integers k_0, l_0. In a similar way one defines the pair k_{m+1}, l_{m+1}.

Since $\ell(\rho)$ has only finitely many transversal reflections, there exists an integer $q \geqslant 1$ such that $d(\mathcal{F}_t(\rho), G) \geqslant 1/q$ whenever $\mathcal{F}_t(\rho)$ is a transversal reflection point ($0 \leqslant t \leqslant T$). If every connected component of E contains either 0 or T, set $m = 0$ and $\alpha = (k_0, k_1; l_0, l_1; q_0; q)$, with k_0, l_0 and k_1, l_1 already defined and q_0 being the number of reflections of $\ell(\rho)$.

Assume that the union of connected components of E that do not contain 0 or T is not empty; it then has the form $\bigcup_{i=1}^{m} [t_i(\rho), s_i(\rho)]$. For each $i = 1, \ldots, m$, according to assumption **A3** again, there exist integers $l_i \geqslant 3$, $k_i \geqslant 3$ such that condition (iii) in Definition 3.2 holds. Finally, denote by q_i the number of transversal reflections of $\{\mathcal{F}_t(\rho)\colon\ s_i(\rho) < t < t_{i+1}(\rho)\}$ at ∂S and define α by (7). Then $\rho \in S_\alpha$, which proves the assertion. □

Theorem 2.1 follows immediately from the following.

LEMMA 3.4. – *Let α be an admissible string of the form* (7). *For every $\rho \in S_\alpha$ there exist an open neighbourhood $V(\alpha, \rho)$ of ρ in $\mathcal{V}$ and a family of maps $\mathcal{G}_t^{(\alpha,\rho)}\colon V(\alpha, \rho) \to \mathcal{V}$, $0 \leqslant t \leqslant T$, such that*:

(a) $\mathcal{G}_t^{(\alpha,\rho)}(\sigma) = \mathcal{F}_t(\sigma)$ *for all $\sigma \in S_\alpha \cap V(\alpha, \rho)$ and all $t \in [0, T]$*;

(b) *For every $\sigma \in S_\alpha \cap V(\alpha, \rho)$ and every $t \in (0, T]$ there exists an open neighbourhood $W = W(\sigma, \alpha, t)$ of σ in $V(\alpha, \rho)$ such that $\mathcal{G}_t^{(\alpha,\rho)}\colon\ (W, d_1) \to (\mathcal{V}, d_2)$ is Lipschitz, where the meaning of d_1 and d_2 is the same as in Theorem* 2.1(b);

(c) *If $\sigma \in S_\alpha \cap V(\alpha, \rho) \cap [(S \setminus \partial S) \cup G]$ and $t \in (0, T]$ is such that $\mathcal{F}_t(\sigma) \in (S \setminus \partial S) \cup G$, then there exists an open neighbourhood $W = W(\sigma, \alpha, t)$ of σ in $V(\alpha, \rho)$ such that the map $\mathcal{G}_t^{(\alpha,\rho)}\colon W \to \mathcal{V}$ is smooth. If moreover both σ and $\mathcal{F}_t(\sigma)$ are not ends of gliding segments of $\{\mathcal{F}_s(\sigma)\colon\ s \in [-\varepsilon, T + \varepsilon]\}$ for any $\varepsilon > 0$, then W can be chosen in such a way that the restriction of $\mathcal{G}_t^{(\alpha,\rho)}$ to any smooth local cross-section at σ in W is a contact transformation.*

Proof of Theorem 2.1. – Fix for each $\rho \in S_\alpha$ a neighbourhood $V(\alpha, \rho)$ and a family of maps $\mathcal{G}^{(\alpha,\rho)}$ as in the above lemma. For each α there exists a countable open cover of S_α consisting of sets of the form $V(\alpha, \rho_j(\alpha))$ ($j = 1, 2, \ldots$). Then the set I of pairs $i = (\alpha, j)$ with α an admissible string of the form (7) and j a positive integer is countable. For each $i = (\alpha, j)$ set $V_i = V(\alpha, \rho_j(\alpha))$, $S_i = S_\alpha \cap V_i$ and $\mathcal{G}_t^{(i)} = \mathcal{G}_t^{(\alpha,\rho_j(\alpha))}$ ($0 \leqslant t \leqslant T$). According to Lemma 3.4, these objects have all the properties required in Theorem 2.1. □

The rest of this section is devoted to the proof of Lemma 3.4.

Before we go on let us briefly describe the idea of the construction of the maps $\mathcal{G}_t^{(\alpha,\rho)}$.

Recall the gliding vector field H_p^G from Section 2. We will slightly change it to make a Hamilton vector field in $\mathcal{V}$. The function $\frac{H_p^2 \varphi}{H_\varphi^2 p}$ is well-defined and smooth near ∂S. Fix an arbitrary smooth extension f of $\frac{H_p^2 \varphi}{H_\varphi^2 p}$ to $\mathcal{V}$ and denote

$$\widehat{p} = p + f\varphi,$$

thus obtaining another smooth function on $\mathcal{V}$. Notice that $\widehat{p} = p$ on ∂S, so $p^{-1}(0) \cap \partial S = \widehat{p}^{-1}(0) \cap \partial S$. Moreover, $H_{\widehat{p}} = H_p + f H_\varphi = H_p^G$ on ∂S.

Denote by Ψ_t the *flow of the Hamilton vector field* $H_{\widehat{p}}$ on $\mathcal{V}$. Since the flows Φ_t and Ψ_t are smooth on $\mathcal{V}$, the families of maps $\{\Phi_t\}_{0\leqslant t\leqslant T}$ and $\{\Psi_t\}_{0\leqslant t\leqslant T}$ are uniformly Lipschitz on any subset U' of $\mathcal{V}$ with $\overline{U'}$ compact.

Idea of the construction of the maps $\mathcal{G}_t^{(\alpha,\rho)}$. For simplicity consider the case $\alpha = (0,k_1,0;0,l_1,0;1,0;q)$. Given $\rho \in S_\alpha$, we have $0 = s_0(\rho) < t_1(\rho) < s_1(\rho) < t_2(\rho) = T$, and $\ell(\rho)$ has exactly one transversal reflection whose time $a \in (0,t_1(\rho))$. Take b and c very close to $t_1(\rho)$ such that $0 < b < a < c < t_1(\rho)$ and $\mathcal{F}_t(\rho) \in S \setminus \partial S$ for all $t \in [b,a)$ and $t \in (a,c]$. Consider arbitrary smooth local cross-sections $\mathcal{B}$ and $\mathcal{C}$ in S to $\mathcal{F}_t$ containing the points $\mathcal{F}_b(\rho)$ and $\mathcal{F}_c(\rho)$, respectively. Choosing appropriately small neighbourhoods U_1 and W_1 of $\mathcal{F}_{t_1(\rho)}(\rho)$ and $\mathcal{F}_{s_1(\rho)}(\rho)$ in $\mathcal{V}$, set $\mathcal{M}_1 = \{\sigma \in U_1\colon\ H_p^{k_1-1}\varphi(\sigma) = 0\}$ and $\mathcal{N}_1 = \{\sigma \in W_1\colon\ H_p^{l_1-1}\varphi(\sigma) = 0\}$; these are then smooth local cross-sections to $\mathcal{F}_t$ at $\mathcal{F}_{t_1(\rho)}(\rho)$ and $\mathcal{F}_{s_1(\rho)}(\rho)$, respectively. On a small neighbourhood $V = V(\alpha,\rho)$ of ρ in $\mathcal{V}$ "the flow" $\mathcal{G}_t = \mathcal{G}_t^{(\alpha,\rho)}$ is defined as follows: it carries $\sigma \in V$ along the trajectory $\Phi_t(\sigma)$ until it hits the hypersurface $\mathcal{B}$; between the hypersurfaces $\mathcal{B}$ and $\mathcal{C}$ "the flow" $\mathcal{G}_t$ coincides with $\mathcal{F}_t$; between $\mathcal{C}$ and $\mathcal{M}_1$, $\mathcal{G}_t$ acts as Φ_t again, between $\mathcal{M}_1$ and $\mathcal{N}_1$, $\mathcal{G}_t$ coincides with the flow Ψ_t of $H_{\widehat{p}}$; and finally from $\mathcal{N}_1$ "onwards" $\mathcal{G}_t$ coincides with Φ_t. As one can see, the idea is quite simple — the action of $\mathcal{G}_t$ between any two consecutive distinguished cross-sections ($\mathcal{B}$, $\mathcal{C}$, $\mathcal{M}_1$, $\mathcal{N}_1$) coincides with the action of one of the flows Φ_t, Ψ_t and $\mathcal{F}_t$ (the first two being smooth Hamilton flows in $\mathcal{V}$). Notice that $\Phi_t = \mathcal{F}_t$ near $\mathcal{B}$ and $\mathcal{C}$, so there is no loss of smoothness there. The places where we can (and actually do) loose smoothness are the transversal reflections and the cross-sections at ends of gliding segments ($\mathcal{M}_1$ and $\mathcal{N}_1$ in our example). One can easily observe that, if V is chosen sufficiently small, then $\mathcal{G}_t(\sigma) = \mathcal{F}_t(\sigma)$ whenever $\sigma \in S_\alpha \cap V$.

Fix $\rho \in \Sigma$ and an admissible string α of the form (7) such that $\rho \in S_\alpha$. We are going to define the neighbourhood $V = V(\alpha,\rho)$ and the family of maps $\mathcal{G}_t = \mathcal{G}_t^{(\alpha,\rho)}$ required in Lemma 3.4.

There are several possible cases for the pairs k_0, l_0 and k_{m+1}, l_{m+1} described in (iv) and (v) in Definition 3.2. We will consider in details one of these; the others can be dealt with in the same way with minor modifications at the ends of the trajectory $\ell(\rho)$; see the end of this section for some details.

We will assume that

$$k_0 = -1, \qquad l_{m+1} \geqslant 2. \tag{9}$$

Let $t_i = t_i(\rho)$, $s_i = s_i(\rho)$ **be the corresponding numbers** from (8). The assumption (9) implies that (cf. (iv) and (v) in Definition 3.2) $0 = t_0(\rho) = s_0(\rho)$, $\rho \notin \partial S$ and $\mathcal{F}_T(\rho)$ belongs to a gliding segment. Thus, $\{\mathcal{F}_t(\rho)\colon\ t_{m+1} \leqslant t \leqslant T\}$ is a gliding segment on ∂S if $t_{m+1} < s_{m+1} = T$, and $\{\mathcal{F}_t(\rho)\colon\ T \leqslant t \leqslant T+\varepsilon\}$ is a gliding segment on ∂S for some $\varepsilon > 0$ if $t_{m+1} = s_{m+1} = T$.

For every $i = 1,2,\ldots,m+1$, $\rho \in S_\alpha$ gives that $\mathcal{F}_{t_i}(\rho) \in G^{k_i} \setminus G^{k_i+1}$, thus $H_p^{k_i}\varphi(\mathcal{F}_{t_i}(\rho)) \neq 0$. From Definition 3.2 it also follows that $\mathcal{F}_t(\rho) \in S \setminus \partial S$ for $t < t_i$ sufficiently close to t_i. This is only possible if $H_p^{k_i}\varphi(\mathcal{F}_{t_i}(\rho)) < 0$ (cf. Section 24.3 in [4]). In the same way one gets $H_p^{l_i}\varphi(\mathcal{F}_{s_i}(\rho)) > 0$.

Fix small open neighbourhoods U_i of $\mathcal{F}_{t_i}(\rho)$ $(1 \leqslant i \leqslant m+1)$ and W_i of $\mathcal{F}_{s_i}(\rho)$ $(1 \leqslant i \leqslant m)$ in $\mathcal{V}$ such that $H_p^{k_i}\varphi(\sigma) < 0$ for $\sigma \in U_i$ $(1 \leqslant i \leqslant m+1)$ and $H_p^{l_i}\varphi(\sigma) > 0$ for $\sigma \in W_i$ $(1 \leqslant i \leqslant m)$ and $H_p^{l_{m+1}}\varphi(\sigma) < 0$ for $\sigma \in W_{m+1}$. Define

$$\mathcal{M}_i = \{\rho \in U_i\colon\ H_p^{k_i-1}\varphi(\rho) = 0\}, \qquad \mathcal{N}_i = \{\rho \in W_i\colon\ H_p^{l_i-1}\varphi(\rho) = 0\}$$

for $1 \leqslant i \leqslant m+1$ and $1 \leqslant i \leqslant m$, respectively. Since $\{p, H_p^{k_i-1}\varphi\}(\mathcal{F}_{t_i}(\rho)) = H_p^{k_i}(\mathcal{F}_{t_i}(\rho)) \neq 0$, shrinking the neighbourhood U_i if necessary, we have that $\mathcal{M}_i$ is a smooth $(2n-1)$-dimensional submanifold of $\mathcal{V}$ containing $\mathcal{F}_{t_i}(\rho_0)$ and transversal to the flow $\mathcal{F}_t$ at this point. Similarly, $\mathcal{N}_i$ is a smooth $(2n-1)$-dimensional submanifold of $\mathcal{V}$ containing $\mathcal{F}_{s_i}(\rho_0)$ and transversal to $\mathcal{F}_t$ at $\mathcal{F}_{s_i}(\rho_0)$.

It follows from the definition of the numbers t_i, s_i that the part $\{\mathcal{F}_t(\rho)\colon\ s_i \leqslant t \leqslant t_{i+1}\}$ of the trajectory of ρ does not contain gliding segments to ∂S and has exactly q_i transversal reflections at ∂S. However it may have some other common points with ∂S (cf. Remark 4 earlier in this section). Our plan is to isolate the times of transversal reflections in small open intervals; then on the rest of $[s_i, t_{i+1}]$, which we denote by $I_i(\rho)$, the trajectory of ρ will be an integral curve of H_p in S and therefore of Φ_t in $\mathcal{V}$. The latter is smooth and we can use it to define the orbit of $\mathcal{G}_t$ over $I_i(\rho)$ for any point $\sigma \in \mathcal{V}$ sufficiently close to ρ.

Let $i = 0, 1, \ldots, m$ be such that $q_i > 0$ and let $a_i^{(1)} < \cdots < a_i^{(q_i)}$ be the *times of the transversal reflections* of $\{\mathcal{F}_t(\rho)\colon\ s_i \leqslant t \leqslant t_{i+1}\}$. For each $j = 1, \ldots, q_i$, fix arbitrary numbers $b_i^{(j)}$ and $c_i^{(j)}$ close to $a_i^{(j)}$ such that

$$t_i < b_i^{(1)} < a_i^{(1)} < c_i^{(1)} < b_i^{(2)} < a_i^{(2)} < c_i^{(2)} < \cdots < b_i^{(q_i)} < a_i^{(q_i)} < c_i^{(q_i)} < s_{i+1},$$

and $\mathcal{F}_t(\rho) \in S \setminus \partial S$ for $t \in [b_i^{(j)}, a_i^{(j)}) \cup (a_i^{(j)}, c_i^{(j)}]$, $j = 1, 2, \ldots, q_i$.

Next, choose arbitrary smooth local $(2n-1)$-dimensional submanifolds $\mathcal{B}_i^{(j)}$ and $\mathcal{C}_i^{(j)}$ of S so that $\mathcal{B}_i^{(j)}$ (respectively $\mathcal{C}_i^{(j)}$) contains $\mathcal{F}_{b_i^{(j)}}(\rho)$ (respectively $\mathcal{F}_{c_i^{(j)}}(\rho)$) and is transversal to H_p at $\mathcal{F}_{b_i^{(j)}}(\rho)$ (respectively at $\mathcal{F}_{c_i^{(j)}}(\rho)$). We take these submanifolds in such a way that $\overline{\mathcal{B}_i^{(j)}} \cap \partial S = \overline{\mathcal{C}_i^{(j)}} \cap \partial S = \emptyset$. Using the continuity of the flows $\mathcal{F}_t$, Φ_t and Ψ_t, and a simple (backward) induction, we may assume that these local cross-sections are such that:

(C) the shift along the flow Φ_t maps $\mathcal{C}_i^{(q_{i-1})}$ to $\mathcal{M}_i$ ($1 \leqslant i \leqslant m$ with $q_i > 0$);

(M) the shift along the flow Ψ_t maps $\mathcal{M}_i$ to $\mathcal{N}_i$ ($1 \leqslant i \leqslant m$);

(N) the shift along the flow Φ_t maps $\mathcal{N}_i$ to $\mathcal{B}_i^{(1)}$ if $q_i > 0$ and to $\mathcal{M}_{i+1}$ if $q_i = 0$ ($1 \leqslant i \leqslant m$);

(B) the shift along the flow $\mathcal{F}_t$ maps $\mathcal{B}_i^{(j)}$ to $\mathcal{C}_i^{(j)}$ ($0 \leqslant i \leqslant m$ with $q_i > 0$, $1 \leqslant j \leqslant q_i$).

Finally, using again the continuity of $\mathcal{F}_t$, choose an *open neighbourhood* $V = V(\alpha, \rho)$ of ρ in S (hence in $\mathcal{V}$) such that the shift along the flow Φ_t maps V into $\mathcal{B}_0^{(1)}$ if $q_0 > 0$. If $q_0 = 0$, we choose V so small that $\Phi_{t_1(\sigma)}(\sigma) \in \mathcal{M}_1$ for all $\sigma \in V$.

DEFINITION 3.5. – *Given $\sigma \in V$, consider the curve $\{\mathcal{G}_t(\sigma)\colon\ 0 \leqslant t \leqslant T\}$ in $\mathcal{V}$ with the following properties*:

(i) *There exist numbers $t_i(\sigma)$ ($1 \leqslant i \leqslant m+1$) and $s_i(\sigma)$ ($1 \leqslant i \leqslant m$) with*

$$0 < t_1(\sigma) < s_1(\sigma) < \cdots < t_m(\sigma) < s_m(\sigma) < \cdots < t_{m+1}(\sigma)$$

such that $\mathcal{G}_{t_i(\sigma)}(\sigma) \in \mathcal{M}_i$ for $i = 1, \ldots, m, m+1$, and $\mathcal{G}_{s_i(\sigma)}(\sigma) \in \mathcal{N}_i$ for $i = 1, \ldots, m$.

(ii) *For every $i = 0, 1, \ldots, m$ with $q_i > 0$ and every $j = 1, \ldots, q_i$ there exist numbers $a_i^{(j)}(\sigma)$, $b_i^{(j)}(\sigma)$ and $c_i^{(j)}(\sigma)$ such that*

$$s_i(\sigma) < b_i^{(1)}(\sigma) < a_i^{(1)}(\sigma) < c_i^{(1)}(\sigma) < \cdots < b_i^{(q_i)}(\sigma) < a_i^{(q_i)}(\sigma) < c_i^{(q_i)}(\sigma) < t_{i+1}(\sigma),$$

$$\mathcal{G}_{b_i^{(j)}(\sigma)}(\sigma) \in \mathcal{B}_i^{(j)},\ \mathcal{G}_{c_i^{(j)}(\sigma)}(\sigma) \in \mathcal{C}_i^{(j)} \text{ and } \mathcal{G}_{a_i^{(j)}(\sigma)}(\sigma) \in \partial S;$$

(iii) $\{\mathcal{G}_t(\sigma)\colon\ t \in I\}$ *is a trajectory of:*

- Φ_t *for any interval I contained in*

$$I_i(\sigma) = \bigl[s_i(\sigma), t_{i+1}(\sigma)\bigr] \setminus \bigcup_{j=1}^{q_i} \bigl(b_i^{(j)}(\sigma), c_i^{(j)}(\sigma)\bigr)$$

for some $i = 0, 1, \dots, m$;

- Ψ_t *for* $I = [t_i(\sigma), s_i(\sigma)]$, $i = 1, \dots, m, m+1$;
- $\mathcal{F}_t$ *for* $I = [b_i^{(j)}(\sigma), a_i^{(j)}(\sigma))$ *or* $I = (a_i^{(j)}(\sigma), c_i^{(j)}(\sigma)]$ *for any* $i = 0, 1, \dots, m$ *with* $q_i > 0$ *and* $j = 1, \dots, q_i$.

Clearly the definition of $\mathcal{G}_t$ can be carried out step by step — first on the interval $[0, b_0^{(1)}(\sigma)]$ (assuming $q_0 > 0$), then on $[b_0^{(1)}(\sigma), c_0^{(1)}(\sigma)]$, $[c_0^{(1)}(\sigma), b_0^{(2)}(\sigma)]$, etc. The numbers $b_0^{(j)}(\sigma)$, $a_0^{(j)}(\sigma)$, $c_0^{(j)}(\sigma)$, $t_1(\sigma)$, $s_1(\sigma)$, etc., are also defined step by step following the inductive construction. From this procedure, which is effectively described by Definition 3.5, one can see that if the neighbourhood $V = V(\alpha, \rho)$ of ρ in $\mathcal{V}$ is taken sufficiently small, then the curve $\{\mathcal{G}_t(\sigma)\colon\ 0 \leqslant t \leqslant T\}$ is well-defined for all $\sigma \in V$. Moreover, we can choose V in such a way, that

$$d(\mathcal{G}_{a_i^{(j)}(\sigma)}(\sigma), G) \geqslant \frac{1}{2q}, \quad \sigma \in V.$$

for all $i = 0, 1, \dots, m$ with $q_i > 0$ and $j = 1, \dots, q_i$. Notice that in (i) it may happen that $t_{m+1}(\sigma) > T$. For such σ in the corresponding parts in (iii) the last interval involved will be $[s_m(\sigma), T]$ if $q_m = 0$ and $[c_m^{(q_m)}(\sigma), T]$ if $q_m > 0$.

In what follows we will use the notation $I_i^{(j)}(\sigma) = (b_i^{(j)}(\sigma), c_i^{(j)}(\sigma))$. Clearly, it makes sense only when $q_i > 0$.

Proof of Lemma 3.4. – We will show that $V = V(\alpha, \rho)$ and the maps $\mathcal{G}_t = \mathcal{G}^{(\alpha,\rho)}$ have the properties listed in Lemma 3.4. We are still considering the case (9). As promised earlier, at the end of the proof we will say how to deal with the other possible cases.

Step 1. We will show that the real-valued functions $t_i(\sigma)$, $s_i(\sigma)$, $a_i^{(j)}(\sigma)$, $b_i^{(j)}(\sigma)$, $c_i^{(j)}(\sigma)$ ($i \leqslant m$) and the corresponding points $\mathcal{G}_{t_i(\sigma)}(\sigma)$, $\mathcal{G}_{s_i(\sigma)}(\sigma)$, $\mathcal{G}_{a_i^{(j)}(\sigma)}(\sigma)$, $\mathcal{G}_{b_i^{(j)}(\sigma)}(\sigma)$, $\mathcal{G}_{c_i^{(j)}(\sigma)}(\sigma)$ depend smoothly on $\sigma \in V$. If $q_0 = 0$, then $t_1(\sigma)$ is just the (first) time when the trajectory $\{\Phi_t(\sigma)\colon\ 0 \leqslant t \leqslant T\}$ hits the cross-section $\mathcal{M}_1$. Since Φ_t is a smooth (Hamilton) flow in $\mathcal{V}$, it follows that both $t_1(\sigma)$ and $\mathcal{G}_{t_1(\sigma)}(\sigma)$ depend smoothly on $\sigma \in V$. If $q_0 > 0$, the first number we have to define is $b_0^{(1)}(\sigma)$. This is the time when $\{\Phi_t(\sigma)\colon\ 0 \leqslant t \leqslant T\}$ hits the cross-section $\mathcal{B}_0^{(1)}$, so for the same reason as above, $b_0^{(1)}(\sigma)$ and $\mathcal{G}_{b_0^{(1)}(\sigma)}(\sigma)$ depend smoothly on σ. From $\mathcal{B}_0^{(1)}$ to ∂S our trajectory follows $\mathcal{F}_t$ which in $S \setminus \partial S$ is a smooth Hamilton flow ($\equiv \Phi_t$ in $S \setminus \partial S$) transversal to ∂S at $\mathcal{G}_{a_0^{(1)}(\sigma)}(\sigma)$. Hence $a_0^{(1)}(\sigma) - b_0^{(1)}(\sigma)$ and therefore $a_0^{(1)}(\sigma)$ depend smoothly on σ. This also implies that $\mathcal{G}_{a_0^{(1)}(\sigma)}(\sigma)$ is smooth. The corresponding statement for $c_0^{(1)}(\sigma)$ follows similarly.

Next, suppose we have shown that $t_1(\sigma)$ and $\mathcal{G}_{t_1(\sigma)}(\sigma) \in \mathcal{M}_1$ depend smoothly on $\sigma \in V$. From the cross-section $\mathcal{M}_1$ to the cross-section $\mathcal{N}_1$, $\mathcal{G}_t$ acts as the smooth Hamilton flow Ψ_t in $\mathcal{V}$. Thus, $s_1(\sigma) - t_1(\sigma)$ (and therefore $s_1(\sigma)$) and $\mathcal{G}_{s_1(\sigma)}(\sigma)$ depend smoothly on σ. Proceeding in this way inductively, one completes Step 1. By the same procedure it follows that $t_{m+1}(\sigma)$ is a smooth function of $\sigma \in V(\alpha, \rho)$. However, as mentioned earlier if $t_{m+1}(\rho) = T$, then we may have $t_{m+1}(\sigma) > T$ for some $\sigma \in V(\alpha, \rho)$ arbitrarily close to ρ.

Step 2. We are going to show that $\mathcal{G}_t = \mathcal{F}_t$ on $S_\alpha \cap V$; this will prove (a) of Lemma 3.4. Let $\sigma \in S_\alpha \cap V$. It follows from the choice of the neighbourhood V and the definition of the numbers

$t_i(\sigma)$ and $s_i(\sigma)$, that $\mathcal{F}_{t_i(\sigma)}(\sigma) \in \mathcal{M}_i$, $\mathcal{F}_{s_i(\sigma)}(\sigma) \in \mathcal{N}_i$. Moreover, the definition of $\mathcal{G}_t$ gives that on each interval contained in $I_0(\sigma)$, $\mathcal{G}_t$ acts as the flow Φ_t. However, $\sigma \in S_\alpha$ implies that $\mathcal{F}_t(\sigma)$ has no transversal reflections or gliding segments on $I_0(\sigma)$; so on any time interval contained in $I_0(\sigma)$ the action of the flow $\mathcal{F}$ is the same as that of Φ. On the intervals $I_0^{(j)}(\sigma)$ containing the times of transversal reflections, $\mathcal{G}_t$ acts as $\mathcal{F}_t$ by definition. Therefore $\mathcal{G}_t(\sigma) = \mathcal{F}_t(\sigma)$ for all $t \in [0, t_1(\sigma)]$.

Next, $\sigma \in S_\alpha$ implies that $\{\mathcal{F}_t(\sigma)\colon\ t \in [t_1(\sigma), s_1(\sigma)]\}$ is an integral curve of the vector field H_p^G contained in $G \subset \partial S$. Since $H_{\widehat{p}} = H_p^G$ on ∂S, it follows that $\{\mathcal{F}_t(\sigma)\colon\ t \in [t_1(\sigma), s_1(\sigma)]\}$ is an integral curve of the vector field $H_{\widehat{p}}$, too. This agrees with the definition of $\mathcal{G}_t$, so $\mathcal{F}_t(\sigma) = \mathcal{G}_t(\sigma)$ for all $t \in [0, s_1(\sigma)]$. This and the definition of $\mathcal{G}_t$ yield $\mathcal{F}_t(\sigma) = \mathcal{G}_t(\sigma)$ for $t \in [0, s_2(\sigma)]$, etc. Applying the above procedure inductively, we get $\mathcal{F}_t(\sigma) = \mathcal{G}_t(\sigma)$ for all $t \in [0, T]$.

Step 3. Next, we check condition (c) of Lemma 3.4. Let $\sigma \in S_\alpha \cap V \cap [(S \setminus \partial S) \cup G]$ and $t \in (0, T]$ be such that $\mathcal{F}_t(\sigma) \in (S \setminus \partial S) \cup G$, and let $\mathcal{M}$ be an arbitrary smooth cross-section to the flow $\mathcal{F}_t$ with $\sigma \in \mathcal{M} \subset V$. First, consider the case $0 < t < t_1(\sigma)$. If $q_0 = 0$, then $I_0(\sigma) = [0, t_1(\sigma)]$, and so $\mathcal{G}_s(\sigma) = \Phi_s(\sigma)$ for all $s \in [0, t_1(\sigma)]$. Moreover for σ' in a small open neighbourhood W of σ in V we have $\mathcal{G}_s(\sigma') = \Phi_s(\sigma')$ for all $s \in [0, t]$. Since Φ_s is a smooth Hamilton flow in $\mathcal{V}$, it follows that $\mathcal{G}_t : W \to \mathcal{V}$ is smooth and $\mathcal{G}_t : \mathcal{M} \cap W \to \mathcal{G}_t(\mathcal{M} \cap W)$ is a contact transformation. Let $q_0 \geqslant 1$. Then $\{\mathcal{F}_s(\sigma)\colon\ 0 \leqslant s \leqslant t_1(\sigma)\}$ has transversal reflections for $s = a_0^{(j)}$ $(j = 1, \ldots, q_0)$ and possibly some other common points with ∂S. For $s \in [0, b_0^{(1)}(\sigma)]$ we have $\mathcal{G}_s(\sigma) = \Phi_s(\sigma)$, so for $t \leqslant b_0^{(1)}(\sigma)$, the map $\mathcal{G}_t : V \to \mathcal{V}$ is smooth and $\mathcal{G}_t : \mathcal{M} \to \mathcal{G}_t(\mathcal{M})$ is a contact transformation. Also notice that

$$\mathcal{G}_{b_0^{(1)}(\sigma')}(\sigma') = \Phi_{b_0^{(1)}(\sigma')}(\sigma') \in \mathcal{B}_0^{(1)}$$

depends smoothly on $\sigma' \in V$ and defines a contact transformation from $\mathcal{M}$ to $\mathcal{B}_0^{(1)}$.

Let $b_0^{(1)}(\sigma) < t \leqslant c_0^{(1)}(\sigma)$; then the definition of $\mathcal{G}$ implies

$$\mathcal{G}_t(\sigma') = \mathcal{F}_{t - b_0^{(1)}(\sigma')} \circ \Phi_{b_0^{(1)}(\sigma')}(\sigma')$$

on a small neighbourhood W of σ in V. The only $s \in [b_0^{(1)}(\sigma'), c_0^{(1)}(\sigma')]$ with $\mathcal{F}_s(\sigma') = \mathcal{G}_s(\sigma') \in \partial S$ is $s = a_0^{(1)}(\sigma')$, the time of the corresponding transversal reflection. Assuming $t \neq a_0^{(1)}(\sigma)$, we can take the neighbourhood W so small that $\mathcal{G}_t(W) \cap \partial S = \emptyset$; then $\mathcal{G}_t$ is smooth on W and $\mathcal{G}_t : \mathcal{M} \cap W \to \mathcal{G}_t(\mathcal{M} \cap W)$ is a contact transformation. Moreover,

$$\mathcal{G}_{c_0^{(1)}(\sigma')}(\sigma') = \mathcal{F}_{c_0^{(1)}(\sigma') - b_0^{(1)}(\sigma')} \circ \Phi_{b_0^{(1)}(\sigma')}(\sigma')$$

is smooth on the whole V and defines a contact transformation between $\mathcal{M}$ and $\mathcal{C}_0^{(1)}$. Continuing in this way by induction, one checks that condition (ii) in Lemma 3.4 holds for $t < t_1(\sigma)$. Apart from that, we get that the map $V \ni \sigma' \mapsto \mathcal{G}_{t_1(\sigma')}(\sigma') \in \mathcal{M}_1$ (which is smooth by Step 1) defines a contact transformation between $\mathcal{M}$ and $\mathcal{M}_1$.

Next, consider the case $t_1(\sigma) < t < s_1(\sigma)$. On this time interval $\mathcal{G}_t$ acts as the smooth Hamilton flow Ψ_t, so condition (ii) is again trivially satisfied. More precisely, we have $\mathcal{G}_t(\sigma') = \Psi_{t - t_1(\sigma')} \circ \mathcal{G}_{t_1(\sigma')}(\sigma')$ on a small neighbourhood W of σ in V. Moreover, $V \ni \sigma' \mapsto \mathcal{G}_{s_1(\sigma')}(\sigma') \in \mathcal{N}_1$ is smooth and its restriction to $\mathcal{M}$ defines a contact transformation.

Proceeding in this way we show that for every t which (in the case under consideration) is different from $t_i(\sigma)$ $(i = 1, \ldots, m+1)$, $s_i(\sigma)$ $(i = 1, \ldots, m)$ and $a_i^{(j)}(\sigma)$ $(i = 0, 1, \ldots, m$ with $q_i > 0$ and $j = 1, \ldots, q_i)$ there exists an open neighbourhood W of σ in V such that $\mathcal{G}_t : W \to \mathcal{V}$ is smooth and $\mathcal{G}_t : \mathcal{M} \cap W \to \mathcal{G}_t(\mathcal{M} \cap W)$ is a contact transformation.

Step 4. Let us now prove (b) of Lemma 3.4. Let $\sigma \in S_\alpha \cap V$ and $t \in (0,T]$. If t is different from all $t_i(\sigma)$, $s_i(\sigma)$ and $a_i^{(j)}(\sigma)$, then it follows from the previous step that $\mathcal{G}_t : W \to \mathcal{V}$ is smooth for some neighbourhood W of σ in V, thus (possibly shrinking W so that $\overline{W}$ is contained in the domain of smoothness of $\mathcal{G}_t$), $\mathcal{G}_t : (W, d_0) \to (\mathcal{V}, d_0)$ is Lipschitz.

Next, assume that $t = a_i^{(j)}(\sigma)$ for some $i = 0, 1, \ldots, m$ with $q_i > 0$ and some $j = 1, \ldots, q_i$. Then by Definition 3.5, $\mathcal{G}_t(\sigma') = \mathcal{F}_{t - b_i^{(j)}(\sigma')} \circ \mathcal{G}_{b_i^{(j)}(\sigma')}(\sigma')$ for all $\sigma' \in V$. Since

$$V \ni \sigma' \mapsto \mathcal{G}_{b_i^{(j)}(\sigma')}(\sigma') \in \mathcal{B}_i^{(j)}$$

is smooth, it is Lipschitz with respect to the metric d_0 on every neighbourhood W of σ in V with $\overline{W}$ compact and contained in V. On the other hand, $\overline{\mathcal{B}_i^{(j)}} \cap \partial S = \emptyset$ shows that d and d_0 are equivalent on $\mathcal{B}_i^{(j)}$. Thus, taking W sufficiently small, Lemma 3.1 gives that $\mathcal{G}_t : (W, d_0) \to (\mathcal{V}, d)$ is Lipschitz.

Finally, assume that $t = t_i(\sigma)$ for some $i = 1, \ldots, m, m+1$ (the case $t = s_i(\sigma)$ is almost identical). Take some $\tau < t$ close to t so that $\mathcal{G}_\tau(\sigma) \in S \setminus \partial S$ (such τ exists according to Proposition 24.3.8 in [4]). Then by Step 3, $\mathcal{G}_\tau : W \to \mathcal{V}$ is smooth for some small neighbourhood W of σ in V. For $\sigma' \in W$ we have $\mathcal{G}_t(\sigma') = \Phi_{t-\tau} \circ \mathcal{G}_\tau(\sigma')$ if $t_i(\sigma') \geqslant t$ and $\mathcal{G}_t(\sigma') = \Psi_{t - t_i(\sigma')} \circ \Phi_{t_i(\sigma') - \tau} \circ G_\tau(\sigma')$ if $t_i(\sigma') < t$. From this it follows easily that $\mathcal{G}_t : (W, d_0) \to (\mathcal{V}, d_0)$ is Lipschitz.

With this the proof of Lemma 3.4 in the case $k_0 = -1$ and $l_{m+1} \geqslant 2$ is complete.

Step 5. Let us now explain how to deal with the other possible cases for k_0 and l_{m+1}.

Case 2. $k_0 = 0$, $l_{m+1} \geqslant 2$. Given $\rho \in S_\alpha$, we have that $\rho \in \partial S \setminus G$. Take an arbitrary $c_0 > 0$ close to 0 such that $\{\mathcal{F}_t(\rho)\colon 0 < t \leqslant c_0\} \subset S \setminus \partial S$ and a smooth local cross-section $\mathcal{C}_0$ at $\mathcal{F}_{c_0}(\rho)$. We take $\mathcal{C}_0$ such that $\overline{\mathcal{C}_0} \cap \partial S = \emptyset$. Then d_0 is equivalent to d on $\mathcal{C}_0$. Taking a sufficiently small neighbourhood V of ρ in $\mathcal{V}$ we now define $\mathcal{G}_t$ slightly changing Definition 3.5 in the following way: $\mathcal{G}_t(\sigma) = \mathcal{F}_t(\sigma)$ for $t \leqslant c_0(\sigma)$, where $\mathcal{F}_{c_0(\sigma)}(\sigma) \in \mathcal{C}_0$. From the cross-section $\mathcal{C}_0$ "onwards" we define the action of $\mathcal{G}_t$ as in Definition 3.5. One proves (a) of Lemma 3.4 as in Step 1. To prove (b), consider arbitrary $\sigma \in V$ and $t \in (0,T]$. Using Lemma 3.1 as in Step 4, one shows that if $t \leqslant c_0(\sigma)$, then $\mathcal{G}_t : (W, d) \to (\mathcal{V}, d)$ is Lipschitz. Let $t > c_0(\sigma)$. Then for $t'(\sigma') = t - c_0(\sigma')$ we have $\mathcal{G}_t(\sigma') = \mathcal{G}_{t'(\sigma')} \circ \mathcal{G}_{c_0(\sigma')}(\sigma')$. For the map $\sigma' \mapsto \mathcal{G}_{t'(\sigma')}$ we can apply the arguments in the previous steps. Since d is equivalent to d_0 on $\mathcal{C}_0$, condition (b) of Lemma 3.4. follows. Condition (c) does not apply to the case under consideration.

Case 3. $k_0 \geqslant 2$, $l_{m+1} \geqslant 2$. Then $\rho \in S_\alpha$ implies that ρ belongs to a gliding segment. Taking a small open neighbourhood V of ρ in $\mathcal{V}$, set $\mathcal{M}_0 = \{\rho' \in V\colon H_p^{l_0 - 1}\varphi(\rho') = 0\}$. Change Definition 3.5. in the following way: for $\sigma \in V$ there exists $s_0(\sigma)$ (which may be negative if $s_0(\rho) = 0$) such that $\mathcal{F}_{s_0(\sigma)}(\sigma) \in \mathcal{M}_0$; if $s_0(\sigma) > 0$, then $\mathcal{G}_t(\sigma) = \Psi_t(\sigma)$ for $0 \leqslant t \leqslant s_0(\sigma)\}$, while for $t > s_0(\sigma)$ the orbit $\mathcal{G}_t(\sigma)$ is defined as in Definition 3.5; if $s_0(\sigma) \leqslant 0$, then the orbit $\mathcal{G}_t(\sigma)$ is defined as in Definition 3.5 One proves (a) and (b) as in the first case with minor modifications. The only difference comes when one deals with condition (b) of Lemma 3.4. Now W has to be considered with the metric d_0. The rest is the same.

Case 4. $k_0 = -1$, $l_{m+1} = -1$. This is in fact the easiest case to deal with. Now $\rho \in S_\alpha$ implies that both ρ and $\mathcal{F}_T(\rho)$ are in $S \setminus \partial S$, and we can take $V = V(\alpha, \rho)$ in such a way that $\overline{V} \cap \partial S = \emptyset$ and $\mathcal{F}_T(\overline{V}) \cap \partial S = \emptyset$. The rest is the same.

Case 5. $k_0 = -1$, $l_{m+1} = 0$. Similarly to Case 2, take a smooth local cross-section $\mathcal{B}_{m+1}$ at some point $\mathcal{F}_{b_{m+1}(\rho)}(\rho)$, where $b_{m+1}(\rho)$ is less than but very close to T. We take $\mathcal{B}_{m+1}$ such that $\overline{\mathcal{B}_{m+1}} \cap \partial S = \emptyset$; then d_0 is equivalent to d on $\mathcal{B}_{m+1}$. We change Definition 3.5 so that for any $\sigma \in V$, $\mathcal{G}_t$ acts as $\mathcal{F}_t$ on the interval $[b_{m+1}(\sigma), T]$, where $\mathcal{G}_{b_{m+1}(\sigma)}(\sigma) \in \mathcal{B}_{m+1}$. Given $\sigma \in V$ and

$t \in (0, b_{m+1}(\sigma)]$, the corresponding statements in Lemma 3.4 follow immediately from the first case considered. For $t > b_{m+1}(\sigma)$ we have $\mathcal{G}_t(\sigma') = \mathcal{F}_{t-b_{m+1}(\sigma')} \circ \mathcal{G}_{b_{m+1}(\sigma')}(\sigma')$ on a sufficiently small neighbourhood W of σ in V. Moreover, as in the first case one shows that $\mathcal{G}_{b_{m+1}(\sigma')}(\sigma')$ is smooth on V (provided the latter is small enough). Combining this with Lemma 3.1, one derives that $\mathcal{G}_t : (W, d_0) \to (S, d)$ is Lipschitz for a sufficiently small neighbourhood W of σ. If $\sigma \in S_\alpha \cap V$ and $t > b_{m+1}(\sigma)$ is such that $\mathcal{F}_t(\sigma) = \mathcal{G}_t(\sigma) \in (S \setminus \partial S) \cup G$, then as in the first case we derive that $\mathcal{G}_t$ is smooth on a small neighbourhood W.

Cases 6, 7, 8, 9. The remaining cases can be easily dealt with combining arguments from the previous cases considered. We leave the details to the reader. □

4. Proof of Theorem 2.2

Let $\mathcal{L}_0$ be an isotropic submanifold of $\Sigma \setminus \partial S = p^{-1}(0) \setminus \partial S$ of dimension $n-1$ such that $H_p(\rho)$ is not tangent to $\mathcal{L}_0$ at each $\rho \in \mathcal{L}_0$ and let $T > 0$. It is sufficient to consider the case when $\mathcal{L}_0$ is contained in a small open neighbourhood of some of its points. That is why, we may assume that there exists a $(2n-1)$-dimensional submanifold S_0 of S which is transversal to H_p and such that $S_0' = S_0 \cap p^{-1}(0)$ is a $(2n-2)$-dimensional symplectic submanifold of S containing $\mathcal{L}_0$. The main point is to prove the following local version of Theorem 2.2.

LEMMA 4.1. – *For every admissible string α of the form* (7) *and every $\rho \in \mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$ there exists an open neighbourhood $W = W(\alpha, \rho)$ of ρ in S such that $\dim_H(\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \cap W)) \leqslant n-2$. Moreover, if $\mathcal{F}_T(\rho) \notin \partial S$, then W can be chosen in such a way that $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \cap W)$ is contained in an $(n-2)$-dimensional isotropic submanifold of S.*

Let us first show that Theorem 2.2 is a consequence of Lemma 4.1.

Proof of Theorem 2.2. – Assume that for each string α of the form (7) and each $\rho \in \mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$ there exists a neighbourhood $W(\alpha, \rho)$ as stated in Lemma 4.1. Since $\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$ is a separable metric space, there exists a sequence $\rho_1(\alpha), \ldots, \rho_m(\alpha), \ldots$ of elements of $\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$ such that $\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \subset \bigcup_{m=1}^\infty W(\alpha, \rho_m(\alpha))$. Thus, we have

$$\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha) \subset \bigcup_{m=1}^{\infty} \mathcal{F}_T\big(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \cap W\big(\alpha, \rho_m(\alpha)\big)\big),$$

which implies $\dim_H \mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha) \leqslant n-2$. Since $\mathcal{T}_T \cap \mathcal{L}_0 \subset \bigcup_\alpha (\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha)$, where α runs over the countable set of all configuartions of the form (7), it now follows that $\dim_H(\mathcal{T}_T \cap \mathcal{L}_0) \leqslant n-2$.

In the case $\mathcal{F}_T(\mathcal{L}_0) \subset S \setminus \partial S$, we may assume (according to Lemma 4.1) that each $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha \cap W(\alpha, \rho_m(\alpha)))$ is contained in an $(n-2)$-dimensional isotropic submanifold of S. Then $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha)$ is contained in a countable union of $(n-2)$-dimensional isotropic submanifolds of S, and so $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0)$ has the same property. □

For the proof of Lemma 4.1 we need the following fact.

PROPOSITION 4.2. – *Let $\mathcal{N}$ be a symplectic manifold without boundary with $\dim \mathcal{N} = 2k$, $k \geqslant 2$, and let $\mathcal{E}$ be a symplectic submanifold of $\mathcal{N}$ with $\dim \mathcal{E} = 2k-2$. For every Lagrangian submanifold $\mathcal{L}$ of $\mathcal{N}$ and every $\rho_0 \in \mathcal{L} \cap \mathcal{E}$ there exist an open neighbourhood $\mathcal{U}$ of ρ_0 in $\mathcal{N}$ and a Lagrangian submanifold $\mathcal{L}'$ of $\mathcal{E}$ such that $\rho_0 \in \mathcal{L} \cap \mathcal{E} \cap \mathcal{U} \subset \mathcal{L}'$.*

Proof. – Since the statement is of a local nature, we may assume $\mathcal{N} = \mathbb{R}^k \times \mathbb{R}^k$ with the standard symplectic form ω and $\rho_0 = 0$. Given $\mathcal{L}$ with $0 \in \mathcal{L} \cap \mathcal{E}$, denote $E = T_0\mathcal{E}$ and $L = T_0\mathcal{L}$. Then E is a symplectic linear subspace of $N = T_0\mathcal{N} = \mathbb{R}^k \times \mathbb{R}^k$ with $\dim E = 2k-2$, while L is a Lagrangian subspace of N. For $A \subset N$ set $A^\perp = \{v \in N\colon\ \omega(v,u) = 0\ \forall u \in A\}$. Now the assumptions on L and E imply $L = L^\perp$ and $E \cap E^\perp = \{0\}$. It then follows that $E^\perp$ is not contained in L. Indeed, if $E^\perp \subset L$, then $L = L^\perp \subset E$ and therefore $E^\perp \subset L \subset E$ which is a contradiction since $\dim E^\perp = 2$ and $E \cap E^\perp = \{0\}$. Hence the linear subspace $E^\perp \cap L$ is either zero- or one-dimensional.

Case 1. $\dim(E^\perp \cap L) = 0$. Then $N = E + L$, so $\mathcal{E}$ and $\mathcal{L}$ are transversal at 0. Hence there exists a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{N}$ such that $\mathcal{L}'' = \mathcal{E} \cap \mathcal{L} \cap \mathcal{U}$ is smooth submanifold of $\mathcal{L}$ with codimension 2 in $\mathcal{L}$, i.e. $\dim \mathcal{L}'' = k-2$. Being a submanifold of $\mathcal{L}$, $\mathcal{L}''$ is isotropic, so (possibly shrinking $\mathcal{U}$) it is contained in a Lagrangian submanifold $\mathcal{L}'$ of $\mathcal{E}$.

Case 2. $\dim(E^\perp \cap L) = 1$. Locally near 0, we may assume that $\mathcal{E} = f^{-1}(0) \cap g^{-1}(0)$, where f and g are smooth functions such that $df(\rho)$ and $dg(\rho)$ are linearly independent and $\{f,g\}(\rho) = 1$ for each ρ in an open ball $\mathcal{U}$ with centre 0 in $\mathcal{N}$. Then $E^\perp = \operatorname{span}\{X_f(0), X_g(0)\}$ and therefore there exist $a, b \in \mathbb{R}$, $(a,b) \neq (0,0)$, with $aX_f(0) + bX_g(0) \in L$. We may assume that $X_g(0) \in L$; otherwise one can replace g by an appropriate linear combination of f and g. With this assumption we have $X_g(0) \in E^\perp \cap L$.

Since $(E+L)^\perp = E^\perp \cap L$ is one-dimensional, $\dim(E+L) = 2k-1$ and therefore $\dim(E \cap L) = k-1$. Fix an arbitrary basis $v_2, \ldots, v_k$ in $E \cap L$ and set $v_1 = X_g(0)$. Then $v_1 \in E^\perp \cap L$, and $E \cap E^\perp = \{0\}$ implies that $v_1, v_2, \ldots, v_k$ is a basis in L.

Set $u_1 = X_f(0)$. Then $u_1 \in E^\perp$ gives $\omega(u_1, v_i) = 0$ for all $i = 2, \ldots, k$. Moreover $\omega(u_1, v_1) = \omega(X_f(0), X_g(0)) = \{f,g\}(0) = 1$. There exists $u_2, \ldots, u_k \in E$ such that $u_2, \ldots, u_k, v_2, \ldots, v_k$ form a symplectic basis in E. From $u_1, v_1 \in E^\perp$, we get $\omega(u_1, u_i) = \omega(v_1, u_i) = 0$ for all $i = 2, \ldots, k$ which shows that $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ is a symplectic basis in N. Then shrinking $\mathcal{U}$ again if necessary, there exist symplectic coordinates $x_1 = g, x_2, \ldots, x_k, \xi_1 = f, \xi_2, \ldots, \xi_k$ such that $u_i = X_{x_i}(0) = -\frac{\partial}{\partial \xi_i}$ $v_i = X_{\xi_i}(0) = \frac{\partial}{\partial x_i}$ for all $i = 1, \ldots, k$. In these coordinates we have $\mathcal{E} \cap \mathcal{U} = \{\rho = (x,\xi) \in \mathcal{U}\colon\ x_1 = \xi_1 = 0\}$. Moreover,

$$L = \operatorname{span}\{v_1, \ldots, v_k\} = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}\right\} = \mathbb{R}^k \times \{0\}.$$

Therefore, taking $\mathcal{U}$ small enough, the Lagrangian submanifold $\mathcal{L} \cap \mathcal{U}$ can be written as a graph of a smooth map: $\mathcal{L} \cap \mathcal{U} = \{(x, h(x))\colon\ x \in W\}$, where W is a neighbourhood of 0 in $\mathbb{R}^k$ and $h(x) = (h_1(x), \ldots, h_k(x))$ is smooth in W. Then

$$\frac{\partial h_i}{\partial x_j}(x) = \frac{\partial h_j}{\partial x_i}(x), \quad i,j = 1, \ldots, k,\ x \in W. \tag{10}$$

This follows, for example, from the fact that $\mathcal{L} \cap \mathcal{U}$ is the graph of the 1-form $\beta(x) = \sum_{i=1}^k h_i(x)\,dx_i$. It is known (cf. for example [1]) that in such a case $\mathcal{L} \cap \mathcal{U}$ is Lagrangian if and only if β is closed, i.e. $d\beta = 0$ on $\mathcal{U}$. Since

$$d\beta = \sum_{i=1}^k dh_i \wedge dx_i = \sum_{i=1}^k \sum_{j=1}^n \frac{\partial h_i}{\partial x_j} dx_j \wedge dx_i = \sum_{j<i} \left(\frac{\partial h_i}{\partial x_j} - \frac{\partial h_j}{\partial x_i}\right) dx_j \wedge dx_i,$$

it is clear that $d\beta = 0$ on $\mathcal{U}$ is equivalent to (10).

Locally near 0 we have

$$\mathcal{L} \cap \mathcal{E} = \bigl\{(0, x'; 0, h_2(0,x'), \ldots, h_k(0,x'))\colon\ h_1(0,x') = 0\bigr\},$$

where $x' = (x_2, \dots, x_k) \in \mathbb{R}^{k-1}$. Consider the local submanifold

$$\mathcal{L}' = \{(0, x', 0, h_2(0, x'), \dots, h_k(0, x')): \ x' \in \mathbb{R}^{k-1}\}.$$

It follows from the above argument and (10) that $\mathcal{L}'$ is a Lagrangian submanifold of $\mathbb{R}^{k-1} \times \mathbb{R}^{k-1} = (\{0\} \times \mathbb{R}^{k-1}) \times (\{0\} \times \mathbb{R}^{k-1}) \subset \mathcal{N}$. Since $\mathcal{L} \cap \mathcal{E} \subset \mathcal{L}'$ locally near 0, this proves the assertion. $\square$

Proof of Lemma 4.1. – Let α be a string of the form (7) and let $\rho \in \mathcal{T}_T \cap \mathcal{L}_0 \cap S_\alpha$. We have to find a neighbourhood $W = W(\alpha, \rho)$ of ρ in S with $\dim_H(\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap W \cap S_\alpha)) \leqslant n - 2$, and in the case $\mathcal{F}_T(\rho) \notin \partial S$ such that $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0 \cap W \cap S_\alpha)$ is contained in a smooth $(n-2)$-dimensional isotropic submanifold of S. Using Lemma 3.4, there exists an open neighbourhood $V = V(\alpha, \rho)$ of ρ in $\mathcal{V}$ and a family of maps $\mathcal{G}_t^{(\alpha,\rho)} : V(\alpha, \rho) \to \mathcal{V}$, $0 \leqslant t \leqslant T$, with the properties listed in Lemma 3.4. Since $\rho \in S \setminus \partial S$, we can take $V \subset \overline{V} \subset S \setminus \partial S$. Moreover, we have $k_0 = -1$.

From $\rho \in \mathcal{T}_T$ we get $\mathcal{F}_t(\rho) \in G_g$ for some $t \in [0, T]$. Therefore either the number $m = m(\alpha)$ in (7) is positive or $m = 0$ and $\mathcal{F}_T(\rho)$ belongs to a gliding segment, so we must have $t_1(\rho) < s_1(\rho) = T$. (It is impossible to have $t_1(\rho) = T$, since $\mathcal{F}_t(\rho) \in G_g$ for some $t \leqslant T$.) Here $t_i = t_i(\rho)$ and $s_i = s_i(\rho)$ are the numbers given by (8). In both cases there exists c with $t_1 < c < s_1$ and $\mathcal{F}_c(\rho) \in G_g$. Then we can take an open neighbourhood W of ρ in V so small that for all $\sigma \in W \cap S_\alpha$ we have $t_1(\sigma) < c < s_1(\sigma)$ and $\mathcal{F}_c(\sigma) \in G_g$ (the latter is possible, since G_g is an open subset of G). Using Lemma 3.4 (c) with $\sigma = \rho$ and $t = c$, we can take the neighbourhood W of ρ in such a way that $\mathcal{G}_c : W \to \mathcal{V}$ is smooth and $\mathcal{G}_c : S_0 \cap W \to \mathcal{G}_c(S_0 \cap W)$ is a contact transformation.

For $\rho' = \mathcal{G}_c(\rho) = \mathcal{F}_c(\rho)$ there exists a symplectic submanifold $\mathcal{N}'$ of S of dimension $2n - 2$ such that $\mathcal{N}' \subset p^{-1}(0)$ and $\mathcal{E} = \mathcal{N}' \cap G$ is a symplectic submanifold of S of dimension $2n - 4$. Indeed, take local symplectic coordinates x, ξ in a neighbourhood $\mathcal{O}$ of ρ' in $\mathcal{V}$ as in condition **A2** in Section 2. Then $G = \{(x, \xi): \ x_1 = \xi_1 = 0\}$. Since $\{x_1, p\} = \{\xi_1, p\} = 0$ on G, the Darboux lemma implies that there exists a smooth function g in $\mathcal{O}$ (possibly shrinking $\mathcal{O}$) such that $\{x_1, g\} = \{\xi_1, g\} = 0$, $dg \neq 0$ and $\{p, g\} = 1$. Consequently

$$\mathcal{N}' = \{(x, \xi) \in \mathcal{O}: \ p(x, \xi) = g(x, \xi) = 0\}$$

and $\mathcal{E} = \mathcal{N}' \cap G$ are symplectic submanifolds of $\mathcal{V}$ of dimension $2n - 2$ and $2n - 4$, respectively. We will also need the submanifold $\mathcal{N} = g^{-1}(0)$. Clearly, this is a $(2n - 1)$-dimensional submanifold of $\mathcal{V}$ containing the point $\mathcal{F}_c(\rho)$ and transversal to $H_p(\rho)$. Notice that

$$\mathcal{N}' = \mathcal{N} \cap p^{-1}(0) = \mathcal{N} \cap \Sigma.$$

Assuming W is small enough, for each $\sigma \in W$ the curve $\{\mathcal{G}_t(\sigma): \ t \in [0, T]\}$ intersects transversally $\mathcal{N}$ at some point $\mathcal{P}'_\alpha(\sigma)$. As in the proof of Lemma 3.4 one shows that the resulting map $\mathcal{P}'_\alpha : W \to \mathcal{N}$ is smooth and its restriction $\mathcal{P}'_\alpha : W \cap S_0 \to \mathcal{P}'_\alpha(W \cap S_0)$ is a contact transformation. Next, define the map $\mathcal{P}''_\alpha : \mathcal{N} \to \mathcal{V}$ by $\mathcal{P}''_\alpha(\mathcal{P}'_\alpha(\sigma)) = \mathcal{G}_T(\sigma)$. Using again an argument from the proof of Lemma 3.4, it follows that $\mathcal{P}''_\alpha : (\mathcal{N}, d_0) \to (\mathcal{V}, d)$ is a Lipschitz map (cf. also the argument in the proof of Theorem 1.1 in Section 2). Finally, define $\mathcal{P}' : W \cap S_0 \to \mathcal{N}$ like $\mathcal{P}'_\alpha$ using the flow $\mathcal{F}_t$ instead of $\mathcal{G}_t$, and set $\mathcal{P}''(\mathcal{P}'(\sigma)) = \mathcal{F}_T(\rho)$ for $\sigma \in W$.

It follows from Lemma 3.4(a) that $\mathcal{G}_t = \mathcal{F}_t$ on $V \cap S_\alpha$ for all $t \in [0, T]$. Consequently $\mathcal{P}'_\alpha = \mathcal{P}'$ on $W \cap S_\alpha$ and $\mathcal{P}''_\alpha = \mathcal{P}''$ on $\mathcal{P}'(W \cap S_\alpha)$. On the other hand, it follows from the choice of W and the definition of $\mathcal{P}'$ that $\mathcal{P}'(W \cap S_\alpha) \subset G$. Since $p^{-1}(0)$ is invariant under the flow $\mathcal{F}_t$ and $\mathcal{N}' = \mathcal{N} \cap p^{-1}(0)$, we have $\mathcal{P}'(S'_0 \cap W \cap S_\alpha) \subset \mathcal{N}' \cap G = \mathcal{E}$.

Recall from above that $\mathcal{P}'_\alpha : W \cap S_0 \to \mathcal{P}'_\alpha(W \cap S_0)$ is a contact transformation. Hence $\mathcal{L} = \mathcal{P}'_\alpha(W \cap \mathcal{L}_0)$ is a Lagrangian submanifold of $\mathcal{P}'_\alpha(W \cap S'_0)$, and

$$\mathcal{P}'(\mathcal{L}_0 \cap W \cap S_\alpha) = \mathcal{P}'_\alpha(\mathcal{L}_0 \cap W \cap S_\alpha) \subset \mathcal{L} \cap \mathcal{E}.$$

Now Proposition 4.2 implies that if the neighbourhood $\mathcal{O}$ of $\rho' = \mathcal{F}_c(\rho)$ is sufficiently small, then $\mathcal{O} \cap \mathcal{L} \cap \mathcal{E}$ is contained in a Lagrangian submanifold $\mathcal{L}'$ of $\mathcal{E}$. Without loss of generality we may assume that $\mathcal{P}'_\alpha(W) \subset \mathcal{O}$; then $\mathcal{P}'_\alpha(\mathcal{L}_0 \cap W \cap S_\alpha) \subset \mathcal{L}'$. Since $\dim \mathcal{L}' = n-2$, we have $\dim_H(\mathcal{L}') = n-2$. As we observed above $\mathcal{P}''_\alpha : (\mathcal{N}, d_0) \to (\mathcal{V}, d)$ is a Lipschitz map. Moreover (as we mentioned in Section 2), for Borel subsets of $\Sigma = p^{-1}(0)$, $\dim_H$ calculated with respect to d_0 or d is the same. Hence $\dim_H(\mathcal{P}''_\alpha(\mathcal{L}')) \leqslant n-2$. This and

$$\mathcal{F}_T(\mathcal{L}_0 \cap W \cap S_\alpha) = \mathcal{P}''_\alpha \circ \mathcal{P}'_\alpha(\mathcal{L}_0 \cap W \cap S_\alpha) \subset \mathcal{P}''_\alpha(\mathcal{L}')$$

yield $\dim_H(\mathcal{F}_T(\mathcal{L}_0 \cap W \cap S_\alpha)) \leqslant n-2$.

Next, assume that $\mathcal{F}_T(\rho) \in S \setminus \partial S$. Shrinking W if necessary, we may assume that

$$\mathcal{F}_T(\overline{W}) \subset S \setminus \partial S.$$

In this case, as for $\mathcal{P}'_\alpha$, one shows that $\mathcal{P}''_\alpha$ is smooth and its restriction $\mathcal{P}''_\alpha : \mathcal{N} \to \mathcal{P}''_\alpha(\mathcal{N})$ is contact. Consequently, $S_T = \mathcal{P}''_\alpha(\mathcal{N}')$ is a symplectic submanifold of S of dimension $2n-2$ and $\mathcal{P}''_\alpha : \mathcal{N}' \to S_T$ is a local symplectic map. Clearly $\mathcal{L}'$ (being a Lagrangian submanifold of $\mathcal{E}$) is an $(n-2)$-dimensional isotropic submanifold of $\mathcal{N}$, so $\mathcal{I} = \mathcal{P}''_\alpha(\mathcal{L}')$ is an $(n-2)$-dimensional isotropic submanifold of S_T. Moreover, it follows from above that $\mathcal{F}_T(\mathcal{L}_0 \cap W \cap S_\alpha) \subset \mathcal{I}$. This proves the lemma. □

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Luchezar STOYANOV
Department of Mathematics,
University of Western Australia, Nedlands,
Perth 6907, Western Australia, Australia
E-mail: stoyanov@maths.uwa.edu.au