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THE TOPOLOGY OF LARGE H-SURFACES BOUNDED BY A CONVEX CURVE

BY BEATE SEMMLER

ABSTRACT. – We shall consider embedded compact surfaces M of constant non-zero mean curvature H (H-surfaces) in hyperbolic space $\mathbb{H}^3$. Let Ł denote a horosphere of $\mathbb{H}^3$. Assume that M is contained in the horoball bounded by Ł and that the boundary of M is a strictly convex Jordan curve Γ in Ł. We establish the following:

(i) case $H > 1$. There is an $\mathfrak{H}(\Gamma)$, depending only on the geometry of Γ, such that whenever M is a H-surface bounded by Γ, with $1 < H < \mathfrak{H}(\Gamma)$, then M is topologically a disk.

(ii) case $H \leqslant 1$. Then M is a graph over the domain $\Omega \subset$ Ł bounded by Γ with respect to the geodesics orthogonal to Ω; in particular, M is topologically a disk.

RÉSUMÉ. – On considère une surface M plongée, compacte, à courbure moyenne constante (non nulle) dans l'espace hyperbolique $\mathbb{H}^3$. Soit Ł une horosphère de $\mathbb{H}^3$. On suppose que M est contenue dans l'horoballe bordée par Ł, et que le bord de M est une courbe de Jordan strictement convexe dans Ł. On établit les résultats suivants :

(i) Cas $H > 1$. Il existe un nombre $\mathfrak{H}(\Gamma)$, qui dépende uniquement de la géométrie de Γ, tel que, quand M est une H-surface bordée par Γ, avec $1 < H < \mathfrak{H}(\Gamma)$, alors M est topologiquement un disque.

(ii) Cas $H \leqslant 1$. Alors M est un graphe géodésique orthogonal au-dessus du domaine $\Omega \subset$ Ł bordé par Γ ; en particulier, M est topologiquement un disque.

1. Introduction

Let P be a plane in Euclidean space $\mathbb{R}^3$ and let $\mathbb{R}^3_+$ be one of the two halfspaces determined by P. Consider embedded compact surfaces M of constant non-zero mean curvature H (H-surfaces) in $\mathbb{R}^3_+$ with boundary $\partial M = \Gamma$ a convex curve in P. It is known that, if H is sufficiently small in terms of the geometry of Γ, then a H-surface M has genus zero. This result is established in [3] where they use a rescaling and a version of a compactness theorem to show this. Our proof of the same result will use another technique and will also work in the hyperbolic case. Indeed, in Hyperbolic space $\mathbb{H}^3$, homotheties do not exist, hence we can not apply the compactness theorem for H-surfaces in $\mathbb{H}^3$ proved in [2] to give a similar proof as in [3].

In this paper we shall mainly investigate the hyperbolic case to obtain a result in the same spirit as in $\mathbb{R}^3$.

Let Ł be a horosphere in $\mathbb{H}^3$ and let $\mathfrak{L}$ be the horoball of $\mathbb{H}^3$ bounded by Ł; the mean curvature of Ł is one and the mean curvature vector of Ł points into $\mathfrak{L}$. We consider embedded compact H-surfaces M, H greater than one, in $\mathfrak{L}$ with boundary $\partial M = \Gamma$ a convex curve in Ł. We will show that, if H is sufficiently close to one in terms of the geometry of Γ, then M has genus zero (Theorem 2). If M is an embedded compact H-surface in $\mathfrak{L}$, bounded by Γ and $H \leqslant 1$, then M is a geodesic graph (Theorem 4). The case for H less than one and Γ in a hyperbolic plane is treated in [2].

2. The Euclidean case

THEOREM 1 ([3, Theorem 2]). – *Let $\Gamma \subset P = \{x_3 = 0\}$ be a strictly convex curve. There is an $\mathfrak{H}(\Gamma)$, depending only on the geometry of Γ, such that whenever $M \subset \mathbb{R}^3_+$ is a compact embedded H-surface bounded by Γ, with $0 \leqslant H < \mathfrak{H}(\Gamma)$, then M is topologically a disk and either M is a graph over the domain $\Omega \subset P$ bounded by Γ or $M \cap (\Omega \times [0,\infty))$ is a graph over Ω and $M \setminus (\Omega \times [0,\infty))$ is a graph over $\partial\Omega \times [0,\infty) = \Gamma \times \mathbb{R}_+$, with respect to the lines normal to $\Gamma \times \mathbb{R}_+$.*

We need the following lemma which is proved in [3]:

LEMMA 2.1 ([3]). – *Let $\Gamma \subset P$ be a strictly convex curve. There is a $r > 0$, depending only upon the extreme values of the curvature of Γ, such that whenever $M \subset \mathbb{R}^3_+$ is an H-surface with boundary Γ, there is a $p \in \Omega$ (p depends on M) such that $M \cap (D(p,r) \times [0,\infty))$ is a graph over $D(p,r)$. (Here $D(p,r)$ denotes the Euclidean disk in P centered at p, of radius r.)*

Proof of Theorem 1. – Let M be an H-surface. Let $r > 0$ and $p \in \Omega$ be given by the lemma. Let G be the unique vertical catenoid meeting P in the circle $C_0 = \partial D(p,\rho)$ where $\rho < r$ and ρ is smaller than the smallest radius of curvature of Γ (the latter condition allows us to translate C_0 horizontally in Ω so as to touch every point of Γ), and the angle between G and P along C_0 is $\pi/2$. Let $\Sigma = G \cap (P \times [0,1])$ and let C_1 be the circle of Σ at height one. Let $V = \{v \in P \mid C_0 + v \subset \Omega\}$ and let $D(R)$ be a sufficiently large disk in P such that $C_1 + v \subset D(R) \times \{1\}$ for all $v \in V$.

We know that a highest point q of M is in $\Omega \times [0,\infty)$, and the height d of q is at most $2/H$. The part of M over $P(d/2) = \{x_3 = d/2\}$ is a vertical graph. Also $M \setminus (\Omega \times [0,\infty))$ is a graph over $\Gamma \times \mathbb{R}_+$, with respect to the lines normal to $\Gamma \times \mathbb{R}_+$, of height at most $1/H$.

Let $\mathfrak{k}$ be the smallest value of the curvature of Γ and 2ω the circumscribed diameter of Ω. Note by $\mathfrak{c}$ the point in Ω such that $\Gamma \subset D(\mathfrak{c},\omega)$. As of now, we will work with H sufficiently small such that $H < \mathfrak{k}$ and $H < 1/2\omega$.

First of all, we will prove that, if $d < 1/H$, then M is a graph over Ω. Let $\beta(t), 0 \leqslant t < \infty$, be a line segment in $P(1/2H) = \{x_3 = 1/(2H)\}$ starting at $\beta(0) = \mathfrak{c} \times \{1/(2H)\}$. We consider a straight cylinder $Z(\tau)$ of radius $1/(2H)$ and axis α in the horizontal plane $P(1/(2H))$ where α meets $\beta(t)$ orthogonally at some $t = \tau$. Let $\tilde{Z}(\tau)$ be the half-cylinder of $Z(\tau)$ by cutting $Z(\tau)$ with a vertical plane intersecting $P(1/(2H))$ along α. We take $\tilde{Z}(\tau)$ so that $\beta(t) \cap \tilde{Z}(\tau) = \emptyset$ for $t < \tau$. For τ large, $\tilde{Z}(\tau)$ is disjoint from M. Now one can move $\tilde{Z}(\tau)$ towards M along β. By the maximum principle, as $\partial\tilde{Z} \cap P$ approaches Γ by horizontal translation, the first contact with M can not be at an interior point of M. Therefore no accident will occur before reaching Γ. This implies that the diameter of a smallest disk centered at $\mathfrak{c} \times \{t\}$ that contains $M \cap \{x_3 = h\}$ is smaller than $2\omega + (1/H)$ for $0 \leqslant h \leqslant 1/H$. Let S^+ be the upper hemisphere of the sphere of mean curvature H centered at $\mathfrak{c} \times \{1/H\}$. Translate S^+ downward, so the moving S^+ does not touch M before it arrives at P, i.e., M is below S^+ when ∂S^+ is on P. Then by the maximum principle and because $H < \mathfrak{k}$, one can translate S^+ horizontally to touch every point of Γ and that is why $M \subset \Omega \times [0,\infty)$. Hence M is a graph over Ω.

Henceforth we assume that $d \geqslant 1/H$. The part of M over $P(d/2)$ is a vertical graph.

(i) If an H-graph M' over a domain $\mathfrak{D}$ in the plane $P(t)$ where $\partial M' \subset P(t)$ has height h, then the radius of the smallest disk in $P(t)$ containing strictly $\mathfrak{D}$ is at least $\lambda(h;H) = \sqrt{(2h/H) - h^2}$. To see this, suppose, on the contrary, that the domain $\mathfrak{D}$ is contained in a disk $D(c,\tilde{r}) \subset P(t)$ where $\tilde{r} < \lambda(h;H)$. Let S be the H-sphere centered at $\mathfrak{c} \times \{t\}$ and denoted by $S(h)$ the part of S over the plane $P(t + (1/H) - h)$. M' is contained in the vertical cylinder over $\mathfrak{D}$ and the radius of $\partial S(h)$ is strictly greater than $\tilde{r}$, so by moving $S(h)$ towards M' the first contact with M' must occur at an interior point of $S(h)$ with a boundary point of M'. This means that the height of M' is less than h which gives a contradiction.

(ii) Let $\Omega(t)$ be the domain in $P(t)$ bounded by $M \cap P(t)$ for $t \in [d/2, d]$, and let $D_t(r)$ be the disk in $P(t)$ centered at $c_t = \mathfrak{c} \times \{t\}$, of radius r.

Let $r_{\max} = \inf_r\{\Omega(t) \subseteq D_t(r)\}$ and $r_{\min} = \sup_r\{\Omega(t) \supset D_t(r)\}$.

We want to prove: If $r_{\max} > 2\omega$ then $r_{\min} > r_{\max} - 2\omega$.

We know that $M \setminus (D(\mathfrak{c},\omega) \times [0,\infty))$ is a graph over $\partial(D(\mathfrak{c},\omega) \times [0,\infty))$, with respect to the lines normal to $\partial D(\mathfrak{c},\omega) \times \mathbb{R}_+$. So, for some point μ in $\partial D_t(r_{\max}) \cap M$, we consider all reflection by vertical planes and looking at the set of images of μ in $P(t)$. This set is contained in the interior of the domain in $\mathbb{R}^3_+$ bounded by $M \cup \Omega$; in particular in $\Omega(t)$ since each vertical plane is orthogonal to $P(t)$ and so the symmetry with respect to vertical planes leaves $P(t)$ invariant.

Doing elementary calculations, we see that the set of images of μ contains the disk $D_t(r_{\max} - 2\omega)$. In more detail, denoted by β_0 the half geodesic in $P(t)$ starting at c_t and passing through μ, and by β_ϕ the half geodesic in $P(t)$ where the angle between β_0 and β_ϕ at μ is ϕ, $|\phi| \in [0, \arccos(\omega/r_{\max})]$. Consider the family of vertical planes $V_\phi(s)$ orthogonal to β_ϕ at $\beta_\phi(s+\omega)$ (Fig. 1). Apply the Alexandrov reflection technique to M with the planes $V_\phi(s)$; by decreasing s from ∞, no accident will occur up till $\partial D_t(\omega)$; i.e. $s = 0$. When μ' denotes the reflected image of μ with respect to the plane $V_\phi(0)$, the line segment l joining μ to μ' is contained in $\Omega(t)$.

Now we have $\cos\phi = (\omega + \mathrm{dist}(\mu, V_\phi(0))/r_{\max}$ and $\sin\phi = \mathrm{dist}(c_t, l)/r_{\max}$.

The distance from μ' to c_t is equal to

$$x(\phi) = \mathrm{dist}(\mu', c_t) = \sqrt{\left(\mathrm{dist}\left(\mu, V_\phi(0)\right) - \omega\right)^2 + \mathrm{dist}^2(c_t, l)},$$

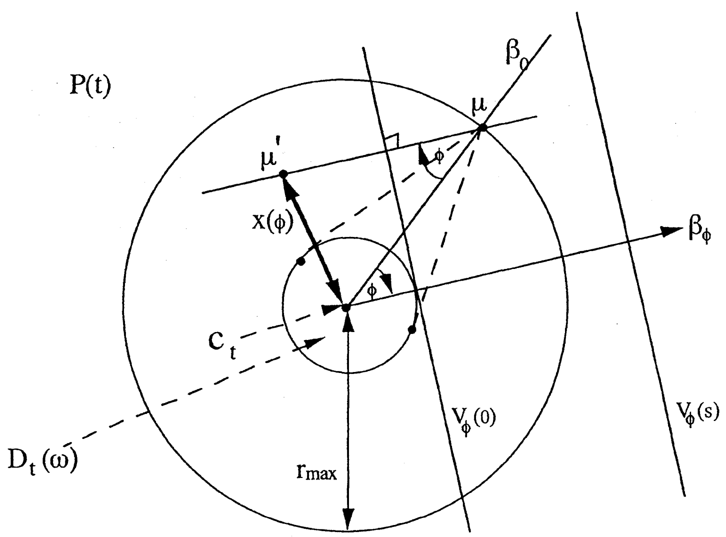

Fig. 1.

therefore $x(\phi) = \sqrt{r_{\max}^2 + 4\omega^2 - 4\omega r_{\max}\cos\phi}$, thus $x(\phi) \geqslant r_{\max} - 2\omega$ and this implies $r_{\min} > r_{\max} - 2\omega$.

(iii) Now we are able to show that, if H is sufficiently small, then $M \cap \{D(R) \times [1, d-1]\} = \emptyset$ and $M \cap \{P \times [d-1, d]\}$ is a graph over $P(d-1)$. For $h = 1$, we get from (i) that $r_{\max}$ is at least $\sqrt{(2/H) - 1}$ on $P(d-1)$; to apply (i) we need $d/2 > 1$ so we work with H such that $1/(2H) > 1$. From (ii), by assuming $1/H > (1/2) + 2\omega^2$, it follows that $r_{\min} + 2\omega \geqslant \sqrt{(2/H) - 1}$. Therefore , if $1/H > (1/2)\{(R + 4\omega)^2 + 1\}$ then $r_{\min} > R + 2\omega$.

Set

$$\mathfrak{h} = \min\left(\mathfrak{k}; \frac{1}{2}; \left(\frac{1}{2}\{(R+4\omega)^2 + 1\}\right)^{-1}\right).$$

The end of the proof is the same as in [3]. □

3. The hyperbolic case

We work in the upper half-space model of hyperbolic space, that is,

$$\mathbb{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$$

with the hyperbolic metric, i.e. the Euclidean metric divided by x_3. In the following, we will represent by dist the hyperbolic geodesic distance in $\mathbb{H}^3$; τ will be the hyperbolic parameter of arc length (in general used for geodesics and planes) and t the euclidean parameter associated with the model (used for horospheres).

Let $Ł(t)$ denote the horizontal horosphere $\{x_3 = t\}$ and let $\mathfrak{L}$ be the non compact component of $\mathbb{H}^3$ bounded by $Ł(1)$ such that the mean curvature vector of $Ł(1)$ points towards $\mathfrak{L}$.

THEOREM 2. – *Let $\Gamma \subset Ł(1)$ be a strictly convex curve. There is an $\mathfrak{H}(\Gamma)$, depending only on the geometry of Γ, such that whenever $M \subset \mathfrak{L}$ is a compact embedded H-surface bounded by Γ, with $1 < H < \mathfrak{H}(\Gamma)$, then M is topologically a disk and either M is a graph over the domain $\Omega \subset Ł(1)$ bounded by Γ with respect to the geodesics orthogonal to Ω or $M \cap (\Omega \times [1, \infty))$ is a geodesic graph over Ω and $M \setminus (\Omega \times [1, \infty))$ is a graph over $\partial\Omega \times [1, \infty) = \Gamma \times [1, \infty)$, with respect to the geodesics orthogonal to $\Gamma \times [1, \infty)$.*

3.1. Properties of compact surfaces in a horoball

Before proving Theorem 2, we give a representative example of hyperbolic calculations, we establish some basic properties of an H-surface as in Theorem 2 and we state a lemma whose proof we will give later.

Notation and Example. Let $q \in Ł(1)$ be the point $(0, 0, 1)$ and let $\gamma(\tau) \subset \mathfrak{L}$ denote the vertical geodesic through q orthogonal to $Ł(1)$ parametrized such that $\tau = \text{dist}(\gamma(\tau), Ł(1))$. Consider the family $P_\gamma(\tau) \subset \mathbb{H}^3$ of planes orthogonal to γ at $\gamma(\tau)$. Let $p \notin \gamma(\tau)$ be a point in some $Ł(t), t > 1$, denoted by R the geodesic distance from p to $\gamma(\tau)$ and by α the angle between the x_3-axis and the euclidean line joining $(0, 0, 0)$ to p; (Fig. 2).

$Ł(t)$ intersects $\gamma(\tau)$ at $\tau = \ln t$. R is related to α by $\tan\alpha = \sinh R$; since the hyperbolic metric on $Ł(t)$ is the euclidean metric divided by t, the hyperbolic length from p to $\gamma(\ln t)$ in $Ł(t)$ is equal to $\sinh R$ (notice that the geodesic distance from p to $\gamma(\ln t)$ which is $2\text{arcsinh}(\sinh R/2)$, is naturally smaller than the former). The geodesic passing through p and realizing the distance R from p to $\gamma(\tau)$ is lying on $P_\gamma(\ln(t\cosh R)$ and this implies that the length of the segment of γ joining $Ł(t)$ to this plane is equal to $\ln\cosh R$. The intersection between $Ł(t)$ and $P_\gamma(\ln(t\cosh R)$

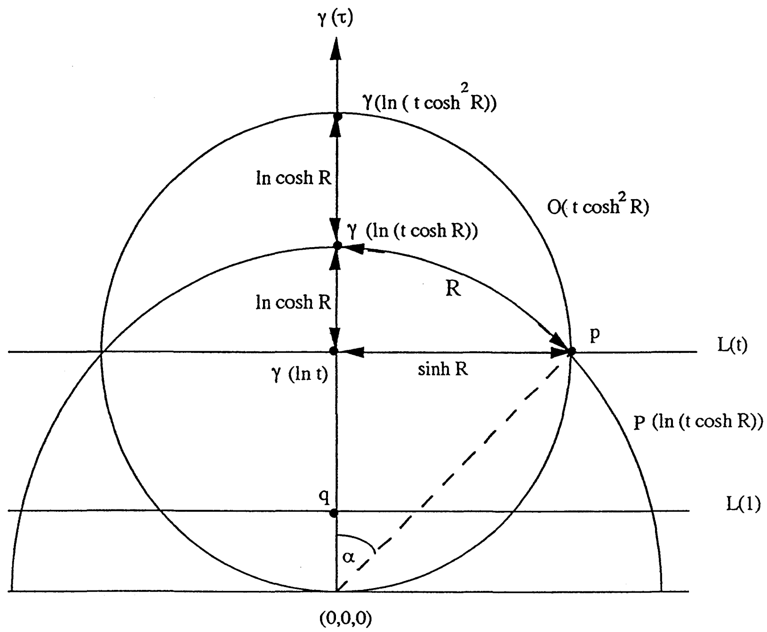

Fig. 2.

is a hyperbolic circle C with hyperbolic center at $\gamma(\ln(t\cosh R)$, of hyperbolic radius R. By hyperbolic reflection with respect to $P_\gamma(\ln(t\cosh R)$, the image of $Ł(t)$ is a horosphere, denoted by $O(t\cosh^2 R)$, containing also C and intersecting $\gamma(\tau)$ at $\tau = \ln(t\cosh^2 R)$; so the distance between both horospheres on $\gamma(\tau)$ is $2\ln\cosh R$.

Basic properties. Let M be defined as in Theorem 2. Let $\mathfrak{B}$ be the compact component of $\mathfrak{L}$ bounded by M and the domain $\Omega \subset Ł(1)$ such that $\partial\Omega = \partial M$. Let $\mathbf{H}$ be the mean curvature vector of M; we orient M by $\mathbf{H}$. Then:

(i) $\mathbf{H}$ points towards $\mathfrak{U} = \mathfrak{B} \cup (\Omega \times (0,1])$.

(ii) Each point $q \in M$ at maximal distance from $Ł(1)$ is contained in the solid vertical geodesic cylinder over Ω denoted by $\mathfrak{C}$.

(iii) Let γ be any geodesic orthogonal to $Ł(1)$ passing through a point of Ω; if M is contained in the solid Killing cylinder over Ω with respect to γ (i.e. the integral curves of the Killing vector field associated to the hyperbolic translation along γ) then M is a Killing graph over Ω with respect to γ.

(iv) $M \setminus (\Omega \times [1,\infty))$ is a graph over $\Gamma \times [1,\infty)$, with respect to the geodesics orthogonal to $\Gamma \times [1,\infty)$; this part of M outside $\mathfrak{C}$ is also a graph over $\Gamma \times [1,\infty)$ with respect to the horocycles in $Ł(t), t \in [1,\infty)$, normal to $\Gamma \times [1,\infty)$.

(v) Let $q \in M$ be a point at maximal distance d from $Ł(1)$ and let $\gamma(\tau) \subset \mathfrak{L}$ be the geodesic through q orthogonal to $Ł(1)$ parametrized such that $\tau = \text{dist}(\gamma(\tau), Ł(1))$. Consider the family $P_\gamma(\tau) \subset \mathbb{H}^3$ of planes orthogonal to γ at $\gamma(\tau)$. Let $R = \max_{p\in\Gamma} \text{dist}(p,\gamma)$. Then the part of M lying above $P_\gamma((d/2) + \ln\cosh R)$ is a Killing graph with respect to γ.

Proof. – (i) Consider the family of horospheres $Ł(t) = \{x_3 = t\}$; if t is large enough, then $Ł(t) \cap M = \emptyset$; decrease t and consider the first horosphere that touches M. At this point of contact, the mean curvature vector of $Ł(t)$ points upward and since the mean curvature of $Ł(t)$

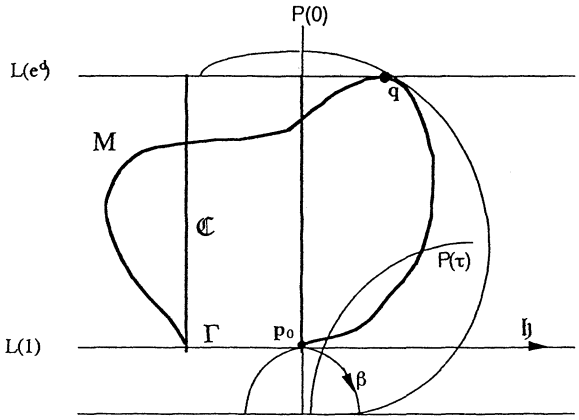

Fig. 3.

(which is equal to one) is smaller than the mean curvature of M, the maximum principle implies that $\mathbf{H}$ points towards $\mathfrak{U}$, hence the same is true at each point of M.

(ii) Let $d = \text{dist}(q, Ł(1))$ and let γ be the geodesic through q orthogonal to Ł(1). Suppose, on the contrary, that $q \notin \mathfrak{C}$ so $q_0 = \gamma \cap Ł(1)$ is not in Ω. Let $\mathfrak{h}$ be a half horocycle in Ł(1) starting at a point p_0 of Γ passing through q_0 and such that $\text{dist}(p_0, q_0) = \inf_{p \in \Gamma} \text{dist}(p, q_0)$. Now consider the unique geodesic plane $E \subset \mathbb{H}^3 \setminus \mathfrak{L}$ tangent to Ł(1) at p_0; note by β the half geodesic in E, starting at p_0, which is contained in the vertical half plane determined by $\mathfrak{h}$ and q. Let $P(\tau)$ be the family of planes in $\mathbb{H}^3$, $0 \leqslant \tau < \infty$, such that for each point b of β, there exists one $P(\tau)$ intersecting β orthogonally at b. Parametrize so that $P(0)$ contains the initial point p_0 of β (Fig. 3).

Apply the Alexandrov reflection technique to M with the planes $P(\tau)$ (cf. [5]). For τ large, $P(\tau)$ is disjoint from M. Now, if we approach M by $P(\tau)$, there will be a first contact point of some $P(\tau)$ with M. One continues to decrease τ and considers the symmetry of the part of M swept out by $P(\tau)$ with respect to $P(\tau)$. These symmetries of M are in $\mathfrak{B}$. Notice that the symmetry through $P(\tau)$ of the relevant part of Ł(1) is contained in $\mathfrak{L}$. (Here relevant part means the part of Ł(1) lying on the same side of $P(\tau)$ as the part of M swept out by $P(\tau)$.) So, by the maximum principle, no accident can occur until $P(0)$ and the part of M in question is a Killing graph over $P(0)$ with respect to the geodesic β (the integral curves of the Killing vector field associated to the hyperbolic translation along β are invariant by reflection with respect to $P(\tau)$). But the Killing segment joining q to $P(0)$ and its symmetry through $P(0)$ are lying above Ł(e^d) whereas $\mathfrak{B}$ is below this horosphere which gives a contradiction and therefore q must be in $\mathfrak{C}$.

(iii) In this case we can do Alexandrov reflection with the family of planes orthogonal to γ until a plane below Ł(1) without any accident, so M is a Killing graph over Ł(1).

(iv) Let γ_p be a geodesic through a point p of Γ orthogonal to Ł(1) at p and let T_p be the vertical plane tangent to Γ at p. Consider any half geodesic β_g orthogonal to T_p at some point g of γ_p where Γ and β_g are on opposite sides of T_p. As in (ii), we apply the Alexandrov reflection technique to M with the family $P(\tau)$ of planes orthogonal to β_g. Therefore the relevant part of M is a Killing graph over $P(0) = T_p$ with respect to the geodesic β_g.

We can do this for each point g of γ_p and each such half geodesic β_g; and also for γ_p associated to each point p of Γ; this means that on each geodesic orthogonal to $\Gamma \times [1, \infty)$ there is only one point of M, hence $M \setminus (\Omega \times [1, \infty))$ is a geodesic graph and the first assertion of (iv) follows.

Let p be a point of Γ and let T_p be as above. Note by $\mathfrak{h}(s)$ the half horocycle in Ł(1) starting at $p = \mathfrak{h}(0)$ and orthogonal to $T_p \cup Ł(1)$ at p where Γ and $\mathfrak{h}(s)$ are on opposite sides of T_p. Let $T(s)$ be a family of vertical planes such that $T(s)$ intersects $\mathfrak{h}$ orthogonally at $\mathfrak{h}(s)$ and

$T(0) = T_p$. Apply the Alexandrov reflection process to M and the planes $T(s)$. Notice that hyperbolic symmetry through each $T(s)$ leaves $Ł(1)$ and all horocycles orthogonal to $T(s)$ in all $Ł(t)$ invariant. One can translate $T(s)$ along $\mathfrak{h}$ until $\partial\Omega = \Gamma$ and the part of M swept out by $T(0) = T_p$ is a graph over T_p with respect to the horocycle orthogonal to T_p.

(v) From (ii) we know that q lies in the solid geodesic cylinder $\mathfrak{C}$ over Ω. We apply the Alexandrov reflection technique to M with the planes $P_\gamma(\tau)$; the first accident occurs when the image of an interior point p_i of M touches Γ. This point p_i is situated on an integral curve of the Killing vector field with respect to $\gamma(\tau)$ over Γ; the Killing coordinate of such point p_i is at most $d + \ln\cosh R$ where $R = \max_{p\in\Gamma} \operatorname{dist}(p, \gamma)$ (see *Notation and Example*). Therefore the result follows. □

We will prove the following Lemma 3 after the proof of Theorem 4.

LEMMA 3. – *Let $\Gamma \subset Ł(1)$ be a strictly convex curve. There is a $r > 0$, depending only upon the extreme values of the curvature of Γ, such that whenever $M \subset \mathfrak{L}$ is an H-surface, $H > 1$, with boundary Γ, there is a $p \in \Omega$ (p depends on M) such that the part of M in the solid Killing cylinder over $D(p, \sinh r) \subset Ł(1)$ with respect to the vertical geodesic γ_p passing through p is a Killing graph over $D(p, \sinh r)$ with respect to γ_p.*

(Here $D(p, \sinh r)$ denotes the disk in $L(1)$ centered at p such that $\partial D(p, \sinh r)$ is the hyperbolic circle centered at $\gamma_p(\tau = \ln\cosh r)$ of hyperbolic radius r.)

3.2. Proof of the main result

Proof of Theorem 2. – Let M be an H-surface as in Theorem 2. Let ω be the hyperbolic radius of a smallest hyperbolic circle such that the domain in $Ł(1)$ bounded by this circle contains Ω. Note by $\mathfrak{c}$ the point in Ω such that $\Gamma \subset D(\mathfrak{c}, \sinh\omega)$, and by $\gamma_c(\tau)$ the vertical geodesic passing through $\mathfrak{c}$; parametrized such that $\tau = \operatorname{dist}(\gamma_c(\tau), Ł(1))$.

By property (ii) we know that the points at maximal distance d from $Ł(1)$ are contained in the solid vertical geodesic cylinder $\mathfrak{C}$ over Ω. In the proof of property (iv), we saw that, if T_p is a vertical plane tangent to Γ at a point $p \in \Gamma$, then the part of M in the half space determined by T_p which does not contain Γ is a Killing graph over T_p with respect to any geodesic orthogonal to T_p at some point of $T_p \cap \mathfrak{C}$.

The same is still true if we choose some point p' in $\partial D(\mathfrak{c}, \sinh\omega) \subset Ł(1)$, $T_{p'}$ the vertical plane tangent to $\partial D(\mathfrak{c}, \sinh\omega)$ at p' and $\beta_{p'}$ the geodesic orthogonal to $T_{p'}$ at p'. Now consider the point g in $Ł(1)$ where the Killing segment k (with respect to $\beta_{p'}$) that joins the point $T_{p'} \cap \{D(\mathfrak{c}, \sinh\omega) \times [1, \infty)\} \cap Ł(e^d) = p' \times \{e^d\}$, intersects $Ł(1)$ (Fig. 4).

We want to evaluate the hyperbolic distance κ between g and the geodesic $\gamma_{p'} = T_{p'} \cap \{D(\mathfrak{c}, \sinh\omega) \times [1, \infty)\}$. Recall that sinh of κ is the hyperbolic length in $Ł(1)$ from g to $\gamma_{p'} \cap Ł(1) = p'$ and this value is also equal to the euclidean distance between g and p' in $Ł(1)$. Let a be the euclidean center of the Killing segment k (this makes sense since k looks like a part of a circle) and let b be the euclidean radius of k. We have $b^2 = 1 + (e^d - b)^2$ and $\sinh^2\kappa = b^2 - (e^d - b - 1)^2$. Therefore $\sinh\kappa$ is equal to $\sqrt{2\sinh d}$.

Since the part of M outside the vertical geodesic cylinder over $D(\mathfrak{c}, \sinh\omega) \subset Ł(1)$ is a Killing graph with respect to $\beta_{p'}$ and $T_{p'}$ for each point $p' \in \partial D(\mathfrak{c}, \sinh\omega)$, no point of M in the vertical half plane containing $\beta_{p'}$ with boundary $\gamma_{p'}$ can be a distance greater than κ from $\gamma_{p'}$, for each p'. This implies that M is contained in the solid Killing cylinder over $D(\mathfrak{c}, \sqrt{2\sinh d} + \sinh\omega) \subset Ł(1)$ with respect to γ_c.

Now we will distinguish two cases.

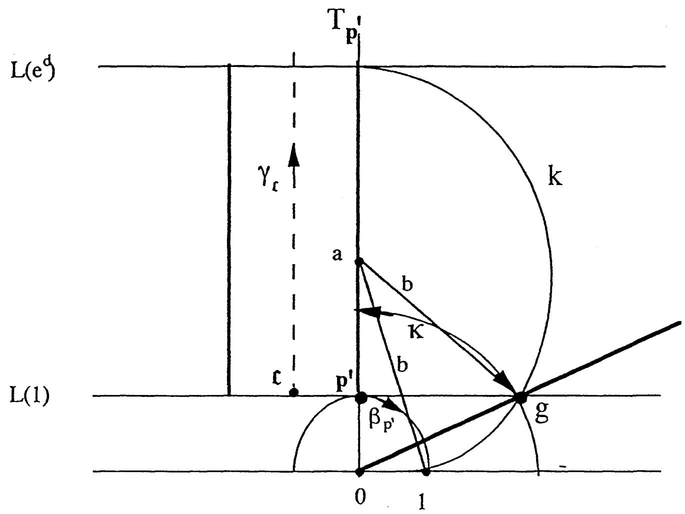

Fig. 4.

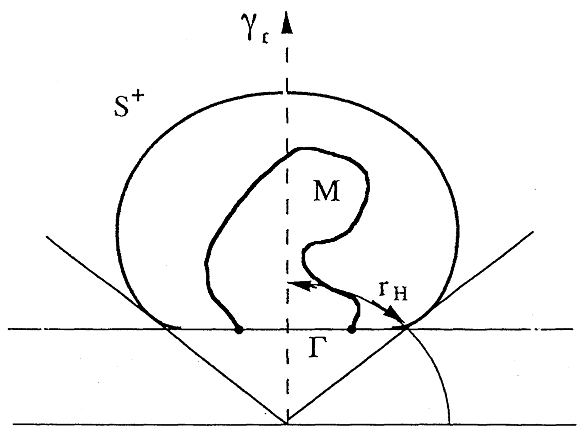

Fig. 5.

(1) Small case

Suppose that $\sqrt{2}\sinh d + \sinh\omega$ is strictly smaller than $\sinh r_H$ where r_H is the radius of the sphere of mean curvature H.

We will first see that M stays inside in the Killing cylinder with respect to $\gamma_c(\tau)$ over the domain in Ⱡ(1) bounded by $D(\mathfrak{c}, \sinh\omega)$. Let S^+ be the upper hemisphere of the H-sphere centered at $\gamma_c(d + \ln\cosh(r_H))$. Translate S^+ downward, so the moving S^+ does not touch M before it arrives at Ⱡ(1), i.e., M is below S^+ when ∂S^+ is on Ⱡ(1) (Fig. 5).

Next consider the family $S^+(\tau)$ of upper half spheres with center at $\gamma_c(\tau)$ for $\tau \in [\tau_0, \tau_1] = [\ln\cosh\omega, \ln\cosh r_H]$ and $\partial S^+(\tau)$ on Ⱡ(1). This continuous family consists of surfaces in $\mathfrak{L}$ where the mean curvature starts from $\coth\omega$, decreases to $\coth r_H = H$ and where $\partial S^+(\tau)$ is a foliation of the compact region in Ⱡ(1) bounded by $\partial D(\mathfrak{c}, \sinh r_H) \cup \partial D(\mathfrak{c}, \sinh\omega)$. So, for each τ, Γ is contained in the domain of Ⱡ(1) bounded by $\partial S^+(\tau)$. Since M is below $S^+ = S^+(\tau_1)$ and when we decrease τ from τ_1 to τ_0, the maximum principle implies that M is still below

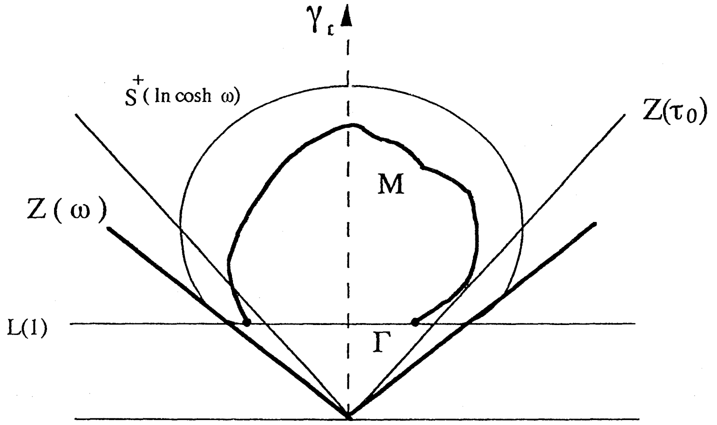

Fig. 6.

$S^+(\tau_0)$, the upper half sphere of radius ω. Therefore M is contained in the Killing cylinder with respect to γ_c over $\partial D(\mathfrak{c}, \sinh\omega) \subset Ł(1)$.

Our aim is now to show that, if H is sufficiently small in terms of ω then M is even contained in the Killing cylinder over Ω with respect to γ_c and so we can conclude by property (iii) that M is a Killing graph over Ω.

Let $\delta = \sup_r\{\Omega \supset D(\mathfrak{c}, \sinh r)\}$. To establish the result we consider the family $Z(\tau)$ of Killing cylinders over $\partial D(\mathfrak{c}, \sinh\tau) \subset Ł(1)$ with respect to γ_c for $\tau \in [\delta, \omega)]$. The mean curvature vector of $Z(\tau)$ points into the component of $\mathbb{H}^3$ bounded by $Z(\tau)$ which contains γ_c and the mean curvature varies continuously in τ from coth(2δ) decreasing to coth(2ω) (Fig. 6).

Now, suppose on the contrary, that M is not in the solid Killing cylinder over Ω. Since M is lying in $Z(\omega)$ and by decreasing τ from ω to δ there will be some τ_0 where $Z(\tau_0)$ touches M for the first time at an interior point of M such that $Z(\tau_0)$ is tangent to M and the mean curvature vector of both surfaces points in the same direction. However, if $H < \coth 2\tau_0$, this is impossible by the maximum principle. Therefore M is contained in the Killing cylinder over Ω with respect to γ_c for H smaller than coth(2ω).

Thus M is a Killing graph over Ω with respect to γ_c.

To finish our investigation for small H-surfaces, we will show that M is even a geodesic graph over Ω with respect to the geodesics orthogonal to Ω. Let p be a point in Ω and let γ_p be the vertical geodesic passing through p. Since M is below the upper half sphere of radius ω centered on γ_c with boundary $D(\mathfrak{c}, \sinh\omega) \subset Ł(1)$, M is also below the upper half sphere of radius 2ω centered on γ_p with boundary in $Ł(1)$. We consider Killing cylinders with axes γ_p and conclude by the same argument as before that M is a Killing graph over Ω with respect to γ_p. We can repeat this for each point p in Ω; this means that on each vertical geodesic there is only one point of M, hence M is a geodesic graph over Ω; in particular M is topologically a disk.

(2) Large case

Henceforth we assume that d is bounded from below in terms of H, i.e.

$$\sqrt{2}\sinh d + \sinh\omega \geqslant \sinh r_H = \frac{1}{\sqrt{H^2-1}}.$$

Let $r > 0$ and $p \in \Omega$ be given by Lemma 3. Let G be the unique vertical catenoid cousin meeting $Ł(1)$ in the circle $C_0 = \partial D(p, \sinh\rho)$ where $\rho < r$ and $\sinh\rho$ is smaller than the smallest radius of curvature of Γ in $Ł(1)$ (the latter condition allows us to translate C_0 horizontally in Ω so as to touch every point of Γ), and G has its waist at $Ł(1)$ (see [4] for catenoid cousins).

Let $\Sigma = G \cap (Ł(1) \times [1, x_3 = e \cdot \cosh^3 \omega])$ and let C_1 be the circle of Σ at euclidean height $\{x_3 = e \cdot \cosh^3 \omega\}$. (The hyperbolic height of Σ is equal to $1 + 3 \ln \cosh \omega$.) Σ is a Killing graph with respect to γ_p over the non compact component of $Ł(1) \cap C_0$. Let $V = \{v \in Ł(1) \mid C_0 + v \subset \Omega\}$ and let $D(\mathfrak{c}, \sinh R)$ be a sufficiently large disk in $Ł(1)$ centered at $\mathfrak{c}$ such that $C_1 + \tilde{v} \subset \tilde{D}(\mathfrak{c}, \sinh R) \times \{x_3 = e \cdot \cosh^3 \omega\}$ for all $v \in V$ (here, we translate the hyperbolic objects v, respectively $D(\mathfrak{c}, \sinh R)$, from $Ł(1)$ to $Ł(e \cdot \cosh^3 \omega)$ with respect to the vertical geodesic γ_p, respectively γ_c, and note them by $\tilde{v}$, respectively $\tilde{D}(\mathfrak{c}, \sinh R)$).

As of now, we choose H such that $d/2 > 1 + 3 \ln \cosh \omega$.

Let $O(t)$ be the family of horospheres in $\mathbb{H}^3$ such that $O(t)$ is tangent to the horosphere $Ł(t)$ at $Ł(t) \cap \gamma_c$; $O(t) \neq Ł(t)$.

First we will show that, if H is sufficiently small, then $M \cap \{$the region in the solid Killing cylinder over $D(\mathfrak{c}, \sinh R)$ with respect to γ_c bounded below by $Ł(e \cdot \cosh^3 \omega)$ and from above by $O(e^{d-1}/\cosh \omega)\}$ is empty. To establish this result we adapt our strategy from the proof in the euclidean case; we work this out in three steps in the same spirit as in (i)–(iii) Theorem 1.

By property (v), the part of M lying above $P_{\gamma_c}((d/2) + \ln \cosh \omega)$ is a Killing graph with respect to γ_c. M is below $Ł(e^d)$. Note by E the domain in $\mathfrak{L}$ bounded by $P_{\gamma_c}((d/2) + \ln \cosh \omega)$ and $Ł(e^d)$. The hyperbolic distance between this plane and this horosphere is realized on γ_c and equal to $(d/2) - \ln \cosh \omega$.

(i) Let $\Omega(t)$ be the domain in $O(t)$ bounded by $M \cap O(t)$ for $t \in [e^{d/2} \cosh \omega, e^d]$. The part of $M \cap E$ above $O(t)$ is also a Killing graph with respect to γ_c. Our aim is now to show that the radius of the smallest disk in $O(t)$ containing $\Omega(t)$ and centered on γ_c can not be *too small* in terms of $h = d - \ln t$ and H. Let M' be a H-Killing graph over $O(e^{d-h})$ with respect to γ_c, $\partial M' \subset O(e^{d-h})$ and with a highest point on $Ł(e^d)$ (here highest means the x_3 coordinate). Then the hyperbolic radius of the smallest disk in $O(e^{d-h})$ centered on γ_c containing strictly $\Omega(e^{d-h})$ is at least

$$\lambda(h; H) = \operatorname{arcosh}\left(\sqrt{\frac{H+1}{H-1}\left(e^{-h} - e^{-2h}\right) + e^{-h}}\right).$$

To see this, suppose, on the contrary, that $\partial M'$ is contained in a disk of $O(e^{d-h})$ of radius smaller than $\lambda(h; H)$. Therefore M' must lie in the Killing cylinder over the disk of radius $\lambda(h; H)$ with respect to γ_c. Next consider the H-sphere S with center at $\gamma_c(d - \operatorname{arcoth} H)$, tangent to $Ł(e^d)$ at $\gamma_c(d) = \gamma_c \cap Ł(e^d)$ and denoted by $S(h)$ the part of S over $O(e^{d-h})$ (Fig. 7).

We will show that the hyperbolic radius of the hyperbolic circle $\partial S(h) = S(h) \cap O(e^{d-h})$ is exactly $\lambda(h; H)$. Let $a \in \gamma_c$ be the hyperbolic center of S and $b \in \gamma_c$ the hyperbolic center of $\partial S(h)$. When q is some point in $\partial S(h)$, consider the geodesic triangle $\triangle a, b, q$. The angle at b is $\pi/2$; by using hyperbolic trigonometry formulas [1] we obtain that

$$\cosh r_H = \cosh \operatorname{dist}(a, b) \cdot \cosh \operatorname{dist}(b, q)$$

(here r_H is the radius of the H-sphere). On the other hand, the distance from b to $O(e^{d-h})$ is $\ln \cosh \operatorname{dist}(b, q)$ (see in *Notation and Example* above) and so $\operatorname{dist}(a, b) = d - h - \ln \cosh \operatorname{dist}(b, q)$. It is straightforward to check that

$$\begin{aligned}
&\cosh^2 \operatorname{dist}(b, q) \\
&\quad = \left(2 \cosh r_H - \cosh(r_H - h) - \sinh(r_H - h)\right)\left(\cosh(r_H - h) - \sinh(r_H - h)\right) \\
&\quad = e^{2r_H}\left(e^{-h} - e^{-2h}\right) + e^{-h};
\end{aligned}$$

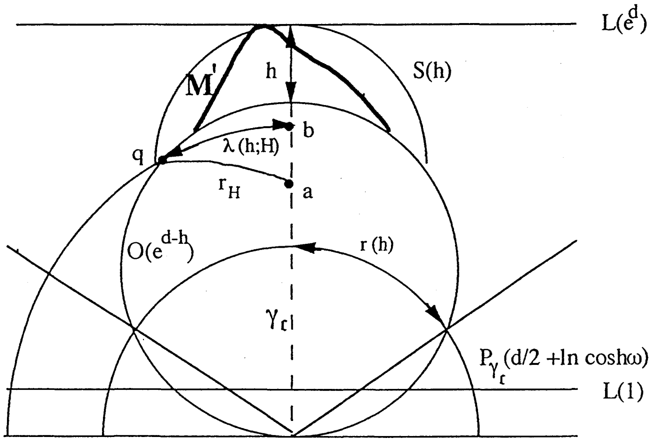

Fig. 7.

so, by taking $H = \coth r_H$ into account, we find out that

$$\operatorname{dist}(b, q) = \lambda(h; H).$$

(Notice that by construction $\partial S(h)$ is also the intersection $O(e^{d-h}) \cap P(d - h - \ln\cosh\lambda)$ or $O(e^{d-h}) \cap Ł(e^{d-h}/\cosh^2\lambda)$.)

We continue the proof of (i). Translate $S(h)$ along γ_c upward to be disjoint from M'. Now come back down; the moving $S(h)$ does not touch M' before it arrives at $O(e^{d-h})$ again; one continues displacement of $S(h)$ along γ_c and the first contact with M' must occur at an interior point of $S(h)$ with a boundary point of M'. This means that no point of M' is on $Ł(e^d)$ which gives a contradiction.

In the following, when we desire to use this result, there is some obstacle (quite different from the euclidean case): how one can ensure that, for fixed h, the part of M over $O(e^{d-h})$ in E has its boundary on $O(e^{d-h})$? However what we need in (iii) below, is only to find a disk in $O(e^{d-h})$ of hyperbolic radius at least $\operatorname{arcsinh}(\sinh R + 2\sinh\omega)$. Since, for h fixed, the largest radius on $O(e^{d-h})$ in E is equal to $r(h) = \operatorname{arcosh}(e^{(d/2)-h}/\cosh\omega)$; we assume up to now that d is big enough (or H is small enough) such that $\sqrt{\sinh r(h)} > (\sinh R + 2\sinh\omega)$. (To evaluate $r(h)$ we apply again *Notation and Example*: the distance on γ_c between $O(e^{d-h})$ and $P_{\gamma_c}((d/2) + \ln\cosh\omega)$ which is $(d/2) - h - \ln\cosh\omega$, must be equal to $\ln\cosh r(h)$.)

The assumption above implies that if each point of M in $O(e^{d-h})$ is at most a distance

$$\operatorname{arcsinh}\big((\sinh R + 2\sinh\omega)^2\big)$$

from γ_c then M is a Killing graph over $O(e^{d-h})$ with boundary on $O(e^{d-h})$.

(ii) Let $\tilde{\Omega}(t)$ be the domain in $Ł(t)$ bounded by $M \cap Ł(t)$ for $t \in [e^{d/2}\cosh\omega, e^d]$ and let $D_t(\sinh r)$ be the disk in $Ł(t)$ centered on γ_c of hyperbolic radius r. Let $r_{\max} = \inf_r\{\tilde{\Omega}(t) \subseteq D_t(\sinh r)\}$ and $r_{\min} = \sup_r\{\tilde{\Omega}(t) \supset D_t(\sinh r)\}$. We want to prove: If $\sinh r_{\max} > (2/t)\sinh\omega$ then $\sinh r_{\min} > \sinh r_{\max} - (2/t)\sinh\omega$.

By property (iv) we know that $M \setminus \{D(\mathfrak{c}, \sinh\omega) \times [1, \infty)\}$ is a graph over $\partial\{D(\mathfrak{c}, \sinh\omega) \times [1, \infty)\}$ with respect to the horocycles in $Ł(t), t \in [1, \infty)$, normal to $\partial\{D(\mathfrak{c}, \sinh\omega) \times [1, \infty)\}$.

As the hyperbolic metric on $Ł(t)$ is the Euclidean metric divided by t, and the hyperbolic symmetries in vertical planes induce euclidean symmetries in $Ł(t)$, the euclidean calculation in (ii), Proof of Theorem 1, yields (ii) here.

(iii) Now we apply (i) for $h = \ln\cosh\omega$; so we need that

$$\sinh r(h) = \sqrt{\frac{e^d}{\cosh^4\omega} - 1} > (\sinh R + 2\sinh\omega)^2$$

and because $\sqrt{2\sinh d} + \sinh\omega \geqslant 1/\sqrt{H^2-1}$ we work with H such that

$$\frac{1}{\sqrt{H^2-1}} > \sinh\omega + \cosh^2\omega\sqrt{1 + (\sinh R + 2\sinh\omega)^4}.$$

Let q be a point in $O(e^{d-h}) \cap M$ at maximal distance from γ_c; (i) implies that this hyperbolic distance from q to γ_c is greater than

$$r_0 = \min\big(\operatorname{arcsinh}\big((\sinh R + 2\sinh\omega)^2\big), \lambda(\ln\cosh\omega; H)\big).$$

The point q is also lying on some horizontal horosphere $Ł(t_0)$ for t_0 smaller than $e^{d-h}/\cosh^2 r_0$ and the hyperbolic length, denoted by $\sinh r_1$, from q to γ_c in $Ł(t_0)$ is greater than $\sinh r_0$. Now (ii) implies that there is a disk in $Ł(t_0)$ centered at $\gamma_c \cap Ł(t_0)$ of radius (the hyperbolic length in $Ł(t_0)$) greater than $\sinh r_1 - (2/t_0)\sinh\omega$ and this disk is contained in the interior of the domain in $\mathbb{H}^3$ bounded by $M \cup \Omega$ (Fig. 8).

Next consider the horosphere $O(t')$ which intersects $Ł(t_0)$ in the hyperbolic circle centered on γ_c of radius $r_2 = \operatorname{arcsinh}\{\sinh r_1 - (2/t_0)\sinh\omega\}$ and denoted by O^+ the part of $O(t')$ above $Ł(t_0)$. When $\mathfrak{B}'$ is the domain in $\mathbb{H}^3$ bounded by $Ł(t_0)$ and the part of M above $Ł(t_0)$; we observe that O^+ is contained in $\mathfrak{B}'$. To see this we move O^+ downward to be disjoint from $\mathfrak{B}'$; then come back upward; by the maximum principle the moving O^+ can not touch M before it arrives again at its starting position.

Remark that the distance between $O(t')$ and $O(e^{d-h})$ on γ_c is equal to $2\ln(\cosh r_1/\cosh r_2)$. Since $\sinh r_2 = \sinh r_1 - (2/t_0)\sinh\omega$; $t_0 = e^d/(\cosh\omega\cosh^2 r_1)$ and if we assume furthermore that r_1 is sufficiently large in terms of H and ω then $t' = e^d\cosh^2 r_2/(\cosh\omega\cosh^2 r_1) > e^{d-h-1}$.

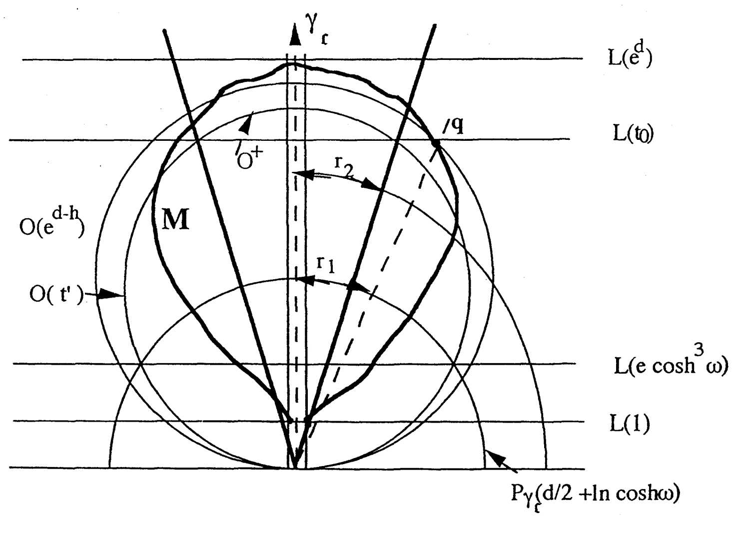

Fig. 8.

Therefore the part of M lying in the Killing cylinder with respect to γ_c over the hyperbolic disk in $P_{\gamma_c}((d/2)+\ln\cosh\omega)$ of radius r_2 is contained in the slice between $Ł(e^d)$ and $O(e^{d-h-1})$. (Since the highest points of M are in the vertical geodesic cylinder over Ω, M has points in this domain for r_2 large enough.)

Let $P_{\gamma_c}(\tau) \subset \mathbb{H}^3$ be the family of planes orthogonal to γ_c at $\gamma_c(\tau)$; we can apply the Alexandrov reflection technique to M with $P_{\gamma_c}(\tau)$; by decreasing τ from ∞ until $\tau_0 = (d/2) + \ln\cosh\omega$ no accident will occur. The symmetry of $O(e^{d-h-1})$ through $P_{\gamma_c}(\tau_0)$ is exactly $Ł(e \cdot \cosh^3\omega)$ and this implies that the intersection between M and the part of the solid Killing cylinder over $D(\mathfrak{c}, \sinh r_2) \subset Ł(1)$ bounded below by $Ł(e \cdot \cosh^3\omega)$ and from above by $O(e^{d-1}/\cosh\omega)$ is empty.

To finish our investigation, we will choose H such that $\sinh r_2 > \sinh R$. We know that

$$\sinh r_2 = \sinh r_1 - \frac{2}{t_0}\sinh\omega > \sinh r_1 - 2\sinh\omega > \sinh r_0 - 2\sinh\omega$$

hence we take H such that $\lambda(\ln\cosh\omega; H) > \operatorname{arcsinh}(\sinh R + 2\sinh\omega)$, i.e.,

$$\frac{H+1}{H-1} > \cosh\omega + \frac{\cosh^2\omega}{\cosh\omega - 1}(\sinh R + 2\sinh\omega)^2.$$

Now, in the second part of the proof, we will show that $\Omega \times [1, (e \cdot \cosh^3\omega] \subset \mathfrak{B}$.

Recall that by Lemma 3 the family $C_0(t)$ of disks obtained by translating C_0 with respect to the vertical geodesic γ_p, (i.e. $C_0(t) = \partial D(p(t), \sinh\rho) \subset Ł(t)$, for $t \in [t_1, t_2] = [1, e \cdot \cosh^3\omega]$), is contained in $\mathfrak{B}$. Let $\Sigma(t)$ denote the family of the translated Σ where $\partial\Sigma(t) \cap Ł(t) = C_0(t)$. Our result above implies that the upper boundary of $\Sigma(t)$ for all $t \in [t_1, t_2]$ and $\Sigma(t_2)$ are contained in $\mathfrak{B}$ and therefore $\Sigma(t)$ must also lie in $\mathfrak{B}$. Otherwise when one translates $\Sigma(t_2)$ down to $\Sigma(t_1)$, there would be a first point of contact of some $\Sigma(t)$ with M. This contact point occurs on the inner side of M; the mean curvature vector of both surfaces points in the same direction. This is impossible since the point of contact is an interior point of both M and $\Sigma(t)$ and the mean curvature of $\Sigma(t)$ (which is equal to one) is smaller than H.

We know that the upper boundary component of $\Sigma + v$, for $v \in V$, at height t_2 is contained in $\mathfrak{B}$. Hence $\Sigma + v \subset \mathfrak{B}$ for each $v \in V$ by similar reasoning as above: the family $\Sigma + sv, s \in [0,1]$ can have no first point of interior contact with M as s goes from 0 to 1.

Our choice of C_0 guarantees that for each $q \in \Gamma$, there is a $v \in V$ such that $C_0 + v$ is tangent to Γ at q. The angle θ between Σ and non compact component of $Ł(1) \cap C_0$ along C_0 is equal to $\arcsin(\cosh^{-1}\rho)$. Therefore the outer angle that $\mathfrak{B}$ makes with $Ł(1)$ at q is smaller than θ; in particular M stays outside the solid vertical geodesic cylinder over Γ between $Ł(1)$ and $Ł(t_2)$.

Since the horizontal translations $\Sigma + sv, v \in V, 0 \leqslant s < 1$ are all in $\mathfrak{B}$ and $D(p, \sinh r) \times [t_1, t_2] \subset \mathfrak{B}$ by Lemma 3, we conclude that $\Omega \times [t_1, t_2] \subset \mathfrak{B}$. Also M meets the solid Killing cylinder over $D(\gamma_c(\tau = \ln t_2), \sinh R) \subset Ł(t_2)$ with respect to γ_c in a Killing graph above $O(e^{d-1}/\cosh\omega)$.

The part of M in $(\Omega \times [1, \infty))$ is even a geodesic graph over Ω; we find this out by coming down with planes P_q orthogonal to the vertical geodesic γ_q passing through q, q any point in Ω; and we consider the symmetries of M with respect to P_q until P_q is below $O(e^{d-1}/\cosh\omega) \cap (\Omega \times [1, \infty))$. For H small, we are far from Γ, so no accident will occur and on any geodesic γ_q is exactly one point of M.

The part of M outside $\Omega \times [1, \infty)$ is of genus zero, so M is topologically a disk and Theorem 2 is established. □

THEOREM 4. – *Let $\Gamma \subset Ł(1)$ be a strictly convex curve. If $M \subset \mathfrak{L}$ is a compact embedded H-surface bounded by Γ, with $H \leqslant 1$ then M is a graph over the domain $\Omega \subset Ł(1)$ bounded by Γ with respect to the geodesics orthogonal to Ω; in particular, M is topologically a disk.*

Proof. – M is compact hence there exists a compact half sphere S^+ with ∂S^+ on $Ł(1)$ and M below S^+. The mean curvature of S^+ is greater than H. We conclude by the same argument as in the proof of Theorem 2: we are in the situation of the *Small case.* □

The following proof is quite similar to the proof in the euclidean case of Lemma 2.1 in [3].

Proof of Lemma 3. – Let ω be the hyperbolic radius of a smallest hyperbolic circle such that the domain in $Ł(1)$ bounded by this circle contains Ω. Note by $\mathfrak{c}$ the point in Ω such that $\Gamma \subset D(\mathfrak{c}, \sinh\omega)$, and by $\gamma(\tau)$ the vertical geodesic passing through $\mathfrak{c}$; parametrized such that $\tau = \mathrm{dist}(\gamma(\tau), Ł(1))$. Consider the family $P_\gamma(\tau) \subset \mathbb{H}^3$ of planes orthogonal to γ at $\gamma(\tau)$. For $p \in \Omega$, let $\eta_p(\tau)$ be the orbit through p of the hyperbolic translation along γ, i.e. the integral curve of the Killing vector field associated to the hyperbolic translation.

Apply the Alexandrov reflection technique to M with the planes $P_\gamma(\tau)$ by decreasing τ from ∞. If we can come down to $P_\gamma(0)$, then M is a Killing graph above Ω and the lemma is clear. Otherwise there is a τ_0 where the reflected surfaces with respect to $P_\gamma(\tau_0)$ touches Γ for the first time at a point $q \in \Gamma$. So η_q intersects M exactly once; and the segment of $\eta_q(\tau)$ for $\tau \in [\ln\cosh\rho_q, 2\cdot\tau_0 - \ln\cosh\rho_q]$ is contained in $\mathrm{int}\,\mathfrak{B}$ where ρ_q denote the hyperbolic distance from q to γ (to find the values of τ see in *Notation and Example*). Also the part of M above $P_\gamma(\tau_0)$ is a Killing graph with respect to γ.

Next consider Alexandrov reflection with vertical planes Q; let v be the normal to Q in $Ł(1)$, $|v| = 1$. Suppose one can do Alexandrov reflection of M, moving the plane Q slightly beyond q, and denote by $J(v)$ the segment in Ω joining q to its reflected image by this plane Q'. Since the part of M swept out by Q is a geodesic graph over Q with respect to the geodesics orthogonal to Q (property (iv)), the vertical domain $G(v)$ bounded by $J(v)$, the segment $\eta_q(\tau)$ and its reflected image through Q', $\tau \in [\ln\cosh\rho_q, 2\cdot\tau_0 - \ln\cosh\rho_q]$, and by the segment of the geodesic orthogonal to Q' joining the point $\eta_q(2\cdot\tau_0 - \ln\cosh\rho_q)$ to its reflected image is contained in $\mathrm{int}\,\mathfrak{B}$.

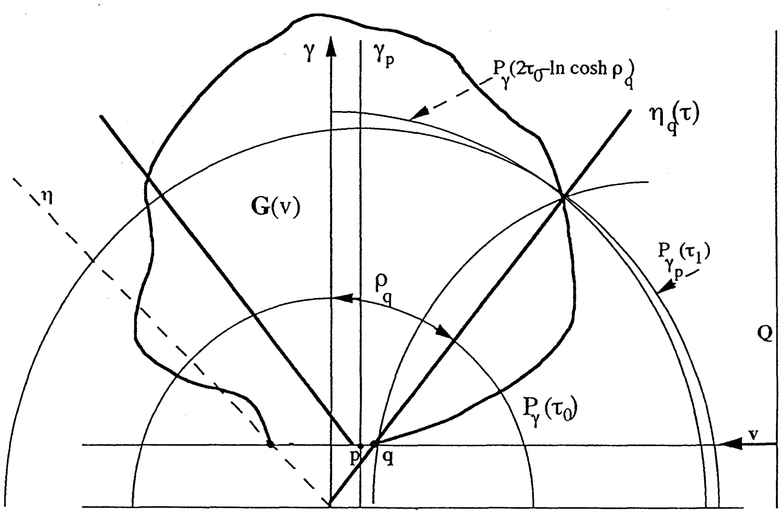

Fig. 9.

Suppose we could repeat this reasoning for a family of directions $v \in Ł(1)$, $|v| = 1$, such that, for some $p \in \Omega$ and $r > 0$, we have $D(p, \sinh r) \subset \bigcup_v J(v)$. Note by γ_p the vertical geodesic passing through p and by $P_{\gamma_p}(\tau)$ the family of planes orthogonal to γ_p; then we would have that the domain in the solid Killing cylinder over $D(p, \sinh r)$ with respect to γ_p between $Ł(1)$ and $P_{\gamma_p}(\tau_1)$ is also contained in $\operatorname{int} \mathfrak{B}$ where $P_{\gamma_p}(\tau_1)$ is the plane orthogonal to γ_p that intersects $\eta_q(\tau)$ at $\tau = 2\tau_0 - \ln\cosh \rho_q$ (i.e., the point of M that touches Γ for the first time by applying Alexandrov technique with respect to the planes $P_\gamma(\tau)$ above). Hence the points of M in this Killing cylinder are only above $P_{\gamma_p}(\tau_1)$. Now we can apply the Alexandrov reflection technique to M with the planes $P_{\gamma_p}(\tau)$; by decreasing τ from ∞ until τ_1 no accident will occur (the plane $P_\gamma(\tau_0)$ is always below $P_{\gamma_p}(\tau_1)$), so the part of M in the solid Killing cylinder over $D(p, \sinh r) \subset Ł(1)$ is a Killing graph with respect to γ_p as desired (see Fig. 9). So we have to understand the horizontal directions v for which Alexandrov reflection goes beyond a point $q \in \Gamma$.

First recall, that for horizontal directions v, one can always do Alexandrov reflection up till Γ. Let k be the minimum curvature of Γ and let $C \subset Ł(1)$ be a circle of curvature k. So if C is tangent to Γ at q, then Γ is inside C. Let ρ be chosen so that the tubular neighborhood of Γ of radius ρ is an embedded annulus.

Then for each horizontal v, we can do Alexandrov reflection with vertical planes at least a distance $\rho/2$ beyond each point of Γ and so at least a distance $\rho/2$ beyond the first time the horizontal plane meets the circle C. Now consider those planes which left behind q. This will hold for those directions in some neighborhood $V = \{v \in Ł(1);\ |v| = 1\}$ of the inward pointing normal to C at q. It is clear from the geometry of the circle, that $\bigcup_{v \in V} J(v)$ contains a disk $D(p, \sinh r), r > 0$ which depends on ρ and C but not on q. This completes the proof of Lemma 3. □

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