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Amos Nevo

Robert J. Zimmer

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RIGIDITY OF FURSTENBERG ENTROPY FOR SEMISIMPLE LIE GROUP ACTIONS

BY AMOS NEVO [1] AND ROBERT J. ZIMMER [2]

ABSTRACT. – We consider the action of a semi-simple Lie group G on a compact manifold (and more generally a Borel space) X, with a measure ν stationary under a probability measure μ on G. We first establish some properties of the fundamental invariant associated with a (G,μ)-space (X,ν), namely the Furstenberg entropy [3], given by

$$h_\mu(X,\nu) = \int_G \int_X -\log \frac{dg^{-1}\nu}{d\nu}(x)\, d\nu(x)\, d\mu(g).$$

We then prove that when (X,ν) is a P-mixing (G,μ)-space [14], and $\mathbb{R}$-rank $(G) = r \geqslant 2$, the value of the Furstenberg entropy must coincide with one of the 2^r values $h_\mu(G/Q,\nu_0)$, where $Q \subset G$ is a parabolic subgroup. We also construct counterexamples to show that this conclusion fails for both non-P-mixing actions and actions of groups with $\mathbb{R}$-rank 1. We also characterize amenable actions with a stationary measure as the actions having the maximal possible value of the Furstenberg entropy. We give applications to geometric rigidity for actions with low Furstenberg entropy, to orbit equivalence and to the cohomology of actions with stationary measure. © 2000 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous considérons l'action d'un groupe de Lie semi-simple G sur une variété compacte, et plus généralement sur un espace borélien X muni d'une mesure ν qu'on suppose μ-stationnaire par rapport à une mesure de probabilité μ sur G.

Nous établissons tout d'abord certaines propriétés de l'invariant fondamental associé à un (G,μ)-espace (X,ν), l'entropie de Furstenberg [3], donnée par

$$h_\mu(X,\nu) = \int_G \int_X -\log \frac{dg^{-1}\nu}{d\nu}(x)\, d\nu(x)\, d\mu(g).$$

Nous prouvons alors que, lorsque (X,ν) est un (G,μ)-espace P mélangeant, [14] dont le $\mathbb{R}$-rang est $r \geqslant 2$, la valeur de l'entropie de Furstenberg doit coïncider avec une des 2^r valeurs $h_\mu(G/Q,\nu_0)$, où Q est un sous-groupe parabolique de G. Nous construisons aussi des contre-exemples qui montrent que cette conclusion est fausse dans le cas d'actions non P-mélangeantes et d'actions de groupes dont le $\mathbb{R}$-rang vaut 1. Nous caractérisons aussi les actions moyennables avec mesure stationnaire en prouvant que ce sont les actions ayant une entropie de Furstenberg maximale.

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Nous donnons des applications à la rigidité géométrique pour des actions à faible entropie de Furstenberg, à l'équivalence d'orbites et à la cohomologie des actions à mesure stationnaire.

Introduction

If a locally compact second countable (lcsc) group G acts on a compact metrizable space X, there is always a natural family of G-quasi-invariant measures on X, namely those equivalent to a probability measure which is stationary under an admissible probability measure on G. When G is a semisimple Lie group, the study of stationary probability measures on compact homogeneous spaces was developed by H. Furstenberg in [2–5] in connection with his study of harmonic functions and random walks on these groups. This study reveals a close connection between stationary measures and the parabolic subgroups of G. It is natural in light of these results to ask whether every action of G with a stationary measure is measurably equivalent to an action induced from a parabolic subgroup.

This question was answered in the affirmative in [14] for a natural class of actions with stationary measure (namely P-mixing actions — see Section 2 for the definition), when $\mathbb{R}$-rank$(G) \geqslant 2$. More precisely, the following was proved:

THEOREM A ([14]). – *Let G be a connected semisimple Lie group with finite center and $\mathbb{R}$-rank$(G) \geqslant 2$. Let (X,ν) be a (G,μ)-space, namely ν is a stationary measure with respect to an admissible measure μ on G. Suppose further that the action of G on (X,ν) is P-mixing. Then there is a parabolic subgroup $Q \subset G$ and an ergodic Q-space (Y,λ) with λ a Q-invariant probability measure, such that the G-action on (X,ν) is measurably isomorphic to the G-action induced from the Q-action on (Y,λ). We have $Q = G$ if and only if the stationary measure ν is G-invariant.*

As shown in [14, Theorem B], the conclusion in Theorem A fails to hold for any group of $\mathbb{R}$-rank 1. Thus the result stated in Theorem A is truly a higher rank phenomenon.

In this paper we develop these ideas further, providing applications of Theorem A and alternative ways of viewing the result. In particular, we consider the relationship to the Furstenberg entropy, a numerical invariant of an action with stationary measure introduced by Furstenberg in [3] (see also [4]).

The paper is organized as follows.

In Section 1, we develop some basic features of Furstenberg entropy and of the Radon–Nikodym factor (introduced in [9] for countable groups). We also explain the connection between stationarity of the measure ν and L^1-cohomolgy invariance of the entropy. In Section 2, we apply Theorem A and deduce the following result on rigidity of Furstenberg entropy for actions of higher rank groups.

THEOREM 2.7. – *Let G be a connected semisimple Lie group with finite center and $\mathbb{R}$-rank$(G) \geqslant 2$. Suppose μ is an admissible measure on G, (X,ν) is a (G,μ)-space, and the action is P-mixing. Then the Furstenberg entropy $h_\mu(X,\nu)$ takes on one of the finitely many values $h_\mu(G/Q,\nu_0)$ for some parabolic subgroup $Q \subset G$ (and the unique μ-stationary measure ν_0 on G/Q). Furthermore, for some Q with $h_\mu(X,\nu) = h_\mu(G/Q,\nu_0)$, X is induced from a finite measure preserving action of Q.*

We remark that for volume preserving smooth ergodic actions on compact manifolds, there is a different but somewhat analogous result for the possible values of the Kolmogorov–Sinai entropy, which follows from superridgidity for cocycles. See [20] for details and discussion.

In Section 3, we provide examples showing that rigidity of Furstenberg entropy fails to hold in general for $\mathbb{R}$-rank 1 groups. More precisely, we prove:

THEOREM 3.1. – *Let $G = PSL(2,\mathbb{R})$ and fix an admissible measure on G. Then there exists an infinite sequence of compact metric (G,μ)-spaces (X_i,ν_i) (in fact smooth manifolds), where the action is P-mixing, and the Furstenberg entropies $h_\mu(X_i,\nu_i)$ are all distinct.*

We remark that we do not determine precisely what these values are, nor the full set of possible values for Furstenberg entropy of G-actions. In particular, it is unknown whether the set of values contains an interval. We do note however (see Section 2) that the Furstenberg entropy is always bounded above by $h_\mu(G/P,\nu_0)$, where P is a minimal parabolic subgroup. Furthermore, if an lcsc group G has property T of Kazhdan, then $h_\mu(X,\nu) \geqslant \alpha(\mu) > 0$, for some positive constant $\alpha(\mu)$ independent of (X,ν), unless ν is G-invariant (see [13] for a proof).

The previous examples can be used to show that general non-P-mixing actions of higher-rank groups exhibit the same phenomenon:

THEOREM 3.4. – *Let G be a simple Lie group with $\mathbb{R}$-rank$(G) \geqslant 2$ with a parabolic subgroup $Q_0 \subset G$ that maps onto $PSL(2,\mathbb{R})$. Then*:

(a) *The Furstenberg entropy for actions of G on smooth compact manifolds with ergodic stationary measure takes on infinitely many values.*
(b) *There is a smooth compact manifold (X,ν) which is a (G,μ)-space with ergodic stationary measure of positive Furstenberg entropy, such that (X,ν) does not have relatively G-invariant measure over any G-space of the form G/Q, where Q is a proper parabolic subgroup.*
(c) *The Radon–Nikodyn factor $(\bar{X},\bar{\nu})$ of the space (X,ν) of (b) is not a Furstenberg boundary of G.*

In Section 4 we discuss amenable actions with stationary measure. The structure of amenable actions with a quasi-invariant measure in general was studied in [18], where it was shown that every amenable action of a semisimple Lie group is induced from an action of an amenable algebraic subgroup. If the measure is not just quasi-invariant but equivalent to a stationary measure, we establish the following sharper result. (We remark that this is independent of the rank of G, and does not require the P-mixing assumption.)

THEOREM 4.1. – *Let G be a connected semisimple Lie group with finite center and no compact factors. Suppose G acts ergodically on a space X with stationary measure ν. Then the following are equivalent*:

(i) *The G-action is amenable.*
(ii) *The G-action is induced from a probability measure preserving action of a minimal parabolic subgroup P.*
(iii) $h_\mu(X,\nu) \geqslant h_\mu(G/P,\nu_0)$.
(iv) $h_\mu(X,\nu) = h_\mu(G/P,\nu_0)$.

We will also show in Section 4 that when $h_\mu(X,\nu) < h_\mu(G/P,\nu_0)$ and the action is non-amenable, there need not exist a non-amenable parabolic subgroup which preserves the measure λ, in contrast to the case of P-mixing actions of higher-rank groups.

THEOREM 4.2. – *Let $G = SL_3(\mathbb{R})$ and μ an admissible measure on G, with $h_\mu(G/P,\nu_0) < \infty$. Then there exists a compact manifold (M,ν) which is a (G,μ)-space, with ergodic stationary measure ν satisfying $h_\mu(M,\nu) < h_\mu(G/P,\nu_0)$, but no non-amenable parabolic subgroup has an invariant probability measure on M.*

In Section 5 we will discuss some geometric applications of Theorem A. We note first that part of the utility of Theorem A is the assertion of existence of a Q-invariant probability measure λ on X, for a certain parabolic subgroup $Q \subset G$. Actions with finite invariant measure of higher real rank simple groups exhibit a number of rigidity properties both in terms of the topology of the underlying space and the structure of the actions themselves (see [11,20], for example). These derive in part from geometric consequences of the Borel density theorem and super-rigidity for cocycles, both of which depend upon the existence of a finite invariant measure. (Note however [7], where finite invariant measure is replaced by suitable recurrence conditions.) Theorem A allows us to generalize these results, in a modified form, to actions without an invariant measure, provided that the parabolic subgroup stabilizing the measure λ has a simple higher-real-rank subgroup contained in its Levi component. This condition, for a P-mixing action on (G,μ)-spaces (X,ν), amounts to the assumption that the Furstenberg entropy is sufficiently low (the case $h_\mu(X,\nu)=0$ is corresponds to $Q=G$, and ν being G-invariant). Some applications of this type are discussed in Section 5, where we will also deduce from Theorem A some results about orbit equivalence and the real cohomology group of the action.

1. Stationary measures and Furstenberg entropy

1.1. The Radon–Nikodym cocycle

We will briefly recall some definitions and results regarding stationary measures, Furstenberg entropy and Radon–Nikodym cocycles for locally compact groups. A further discussion of these matters can be found in [13].

Let G denote a (Hausdorff) locally compact second countable (lcsc) group, and let $(X,\mathcal{B})$ be a compact metric space, $\mathcal{B}$ the Borel σ-algebra. Assume G has a (jointly) continuous action on X, denoted $(g,x)\mapsto gx$. There are natural associated actions on

$$C(X)=\{f:X\to\mathbb{C}\mid f \text{ continuous}\}$$

and on

$$P(X)=\{\nu:\mathcal{B}\to[0,1]\mid \nu \text{ is a Borel probability measure}\}.$$

These are given by $(gf)(x)=f(g^{-1}x)$ and $(g\nu)(A)=\nu(g^{-1}A)$, so that

$$(g\nu)(A)=\nu\bigl(g^{-1}A\bigr)=\int_X \chi_{g^{-1}A}(x)\,d\nu=\int_X \chi_A(gx)\,d\nu=\int_X \chi_A(x)\,d(g\nu)$$

and for any $f\in C(X)$:

$$(g\nu)(f)=\int_X f(x)\,d(g\nu)=\int_X f(gx)\,d\nu=\nu\bigl(g^{-1}f\bigr).$$

Let $P(G)=\{\mu\mid\mu \text{ is a Borel probability measure on } G\}$. Fix $\mu\in P(G)$ and consider the affine map $\mu*:P(X)\to P(X)$ given by $\mu*\nu(f)=\int_G \nu(g^{-1}f)\,d\mu(g)$. Define

$$P_\mu(X)=\bigl\{\nu\in P(X)\mid \mu*\nu=\nu\bigr\},$$

and note that for every $\mu\in P(G)$, $P_\mu(X)\neq\emptyset$, by [3, Lemma 1.3].

DEFINITION 1.1. –
(i) ν *will be called* μ*-stationary if* $\mu * \nu = \nu$.
(ii) μ *will be called admissible if some convolution power* μ^{*k} *is absolutely continuous with respect to Haar measure* m_G *on* G, *and the support* C *of* μ *generates* G *as a semigroup*: $\bigcup_{n\geqslant 1} C^n = G$.
(iii) $\mu \in P(G)$ *is called of class* B_∞ (*see* [2]) *if it is admissible and some convolution power has bounded density of compact support.*
(iv) *A bounded Borel function* F *on* G *will be called* (*right*)-μ*-harmonic if for every* $g \in G$, $F(g) = \int_G F(gh)\,d\mu(h)$.

The following fact is well known (see, e.g., [14] for a proof):

LEMMA 1.2. – *Let* G *be an lcsc group,* X *a Borel* G*-space. If* $\mu \in P(G)$ *is admissible and* $\nu \in P_\mu(X)$ *is* μ*-stationary, then* ν *is* G*-quasi-invariant*; *namely for all* $g \in G$, ν *and* $g\nu$ *are mutually absolutely continuous measures on* X.

Now fix an admissible measure μ on G, and let $(X, \mathcal{B}, \nu)$ denote a Borel measurable G-space with μ-stationary $\nu \in P_\mu(X)$. The pair (X, ν) is called a (G, μ)-space. We will denote by $\mathcal{L}$ the completion of the σ-algebra $\mathcal{B}$ with respect to ν.

DEFINITION 1.3. – *The function*

$$r_\nu(g, x) = \frac{dg^{-1}\nu}{d\nu}(x)$$

is called the Radon–Nikodym cocycle of (X, ν). *For each* $g \in G$, $r_\nu(g, \cdot)$ *is defined up to equivalence modulo* ν*-null functions on* X.

We note the following standard facts (see [5]):

LEMMA 1.4. –
(i) $r_\nu(g, x)$ *satisfies the cocycle identity, i.e., for each* g, h *and for* ν*-almost all* $x \in X$: $r_\nu(gh, x) = r_\nu(g, hx)\, r_\nu(h, x)$.
(ii) *For* ν*-almost all* $x \in X$, $g \mapsto r_\nu(g^{-1}, x)$ *is a right-*μ*-harmonic function on* G.

Consider now the following quantity, which was introduced in [3] and [4], and will be refered to here as the *Furstenberg entropy* of the (G, μ) space (X, ν):

$$h_\mu(X, \nu) = -\int\limits_G \int\limits_X \log r_\nu(g, x)\, d\nu(x)\, d\mu(g).$$

Here we assume that for μ-almost all $g \in G$, the function $\log r_\nu(g, x)$ is in $L^1(X, \nu)$, and the function $g \mapsto \int_X -\log r_\nu(g, x)\, d\nu$ is in $L^1(G, \mu)$. We note that if μ is of class B_∞, then in fact $\log r_\nu(g, x) \in L^\infty(X, \nu) \subset L^1(X, \nu)$ for every $g \in G$, and also $h_\mu(X, \nu)$ is finite and bounded by a fixed constant for any action. For more on these facts see [13].

We now note the following well known facts (see [3], Corollary to Lemma 8.9):

LEMMA 1.5. –
(1) $0 \leqslant h(X, \nu) \leqslant \infty$.
(2) $h(X, \nu) = 0$ *if and only if* ν *is invariant under* G.

Proof. – By Jensen's inequality and the convexity of $-\log$:

$$-\log \int_G \int_X r_\nu(g,x)\,d\nu\,d\mu = -\log 1 = 0 \leqslant \int_G \int_X -\log r_\nu(g,x)\,d\nu\,d\mu = h(X,\nu).$$

If $h(X,\nu)=0$, then $-\log r_\nu(g,x)=0$ for $\mu\times\nu$-almost all (g,x) by strict convexity of $-\log$. Hence

$$\frac{dg^{-1}\nu}{d\nu}(x)=1$$

for ν-almost all $x\in X$, for a set of μ-measure 1. It follows that $g^{-1}\nu=\nu$ for μ-almost all g. Now recall that X can be assumed to be compact metric and the G-action continuous (see [14, §1]). Then the action of G on $P(X)$ is continuous and it follows that the stability group of a measure is closed. Consequently $g\nu=\nu$ for all $g\in C=\operatorname{supp}(\mu)$, hence for all

$$g\in\bigcup_{n\geqslant 1} C^n = G. \qquad \square$$

Furstenberg entropy is a remarkably powerful numerical invariant. Before exhibiting and utilizing its properties, we note some preliminary facts on Radon–Nikodym cocycles that will be used in the sequel.

LEMMA 1.6. – *Let $\varphi:(X,\mathcal{B},\nu)\to(X',\mathcal{B}',\nu')$ be a factor map between measurable spaces, namely $\varphi(\nu)=\nu'$. Let E denote the conditional expectation operator $E:L^1(X,\nu)\to L^1(X',\nu')$. Then*

(1) *(See* [4, Lemma 5.5].*) For any $f>0$ with $\log f\in L^1(X,\mathcal{B},\nu)$,*

$$-\log E(f)(x')\leqslant E(-\log f)(x') \quad \text{for } \nu'\text{-a.e. } x'\in X',$$

with equality (ν'-a.e.) if and only if f is measurable with respect to $\varphi^{-1}(\mathcal{L}')$. Also, the following inequality holds:

$$\int_{X'} -\log Ef(x')\,d\nu' < \int_X -\log f(x)\,d\nu$$

unless f is measurable with respect to $\mathcal{L}'$, in which case equality holds.

(2) *Assume in addition that X,X' are Borel G-spaces, and ν,ν' are G-quasi-invariant. The Radon–Nikodym cocycle $r_{\nu'}(g,x')=\frac{dg^{-1}\nu'}{d\nu'}(x')$ equals $E(r_\nu(g,\cdot))(x')$, ν'-a.e.*

Proof. –

(1) When $-\log f\in L^1(X,\nu)$, Jensen's inequality and the convexity of $-\log$ imply that $-\log(Ef)(x')\leqslant E(-\log f)(x')$ for ν'-almost all $x'\in X'$. As is well known, equality holds in Jensen's inequality above iff f is measurable with respect to $\varphi^{-1}(\mathcal{L}')$ (see [4, Lemma 5.5]), and the first claim follows. The second claim follows from strict convexity of $-\log$, by integration over X.

(2) Let $x'=\varphi(x)$, and $C=\varphi^{-1}(C_1)$, $C_1\in\mathcal{L}'$. Since E is the conditional expectation on $L^1(X',\nu')$, we have:

$$\int_X \chi_C(x)f(x)\,d\nu = \int_{X'} \chi_{C_1}(x')E(f)(x')\,d\nu'$$

and hence, using the G-equivariance of φ:

$$\int_X \chi_C(gx)\,d\nu = \int_X \chi_C \frac{dg\nu}{d\nu}\,d\nu = \int_{X'} \chi_{C_1}(x') E\left(\frac{dg\nu}{d\nu}\right)(x')\,d\nu'$$
$$= \int_{X'} \chi_{C_1}(gx')\,d\nu' = \int_{X'} \chi_{C_1}(x') \frac{dg\nu'}{d\nu'}(x')\,d\nu'.$$

Hence $r_{\nu'}(g^{-1},x') = E(r_\nu(g^{-1},\cdot))(x')$, ν'-a.e. $x' \in X'$. □

We now recall the following (see [14, §2])

DEFINITION 1.7. – *Suppose $\varphi:(X,\eta) \to (Y,\nu)$ is a factor map of G-spaces with G-quasi-invariant measure, and consider the disintegration of the measure η with respect to ν. Write $\eta = \int_Y \eta_y\,d\nu(y)$ where η_y is supported on $\varphi^{-1}(y)$. We say that (X,η) is an extension with relatively G-invariant measure over (Y,ν) if the following equivalent conditions are satisfied*:

(i) *for each g, $g_*\eta_y = \eta_{gy}$ for ν-a.e. $y \in Y$.*
(ii) *If $f \in L^1(X)$ and $I_f(y) = \int f\,d\eta_y$, then $I_{g\cdot f}(y) = I_f(g^{-1}y)$, for ν-a.e. $y \in Y$.*
(iii) (ii) *holds for a dense subset of $f \in L^1(X)$.*

We now note the following facts, whose verification is included here for completeness.

PROPOSITION 1.8. – *Let G be an lcsc group. Let $\psi:(X,\nu) \to (\bar{X},\bar{\nu})$ be a factor map of Borel G-spaces with G-quasi-invariant measures. Then the following conditions are equivalent*:

(i) *(X,ν) is an extension with relatively G-invariant measure over $(\bar{X},\bar{\nu})$.*
(ii) *For every $g \in G$, $r_\nu(g,x) = r_{\bar{\nu}}(g,\psi(x))$ for ν-a.e. $x \in X$.*
(iii) *In the disintegration $\nu = \int_{\bar{X}} \nu_b\,d\bar{\nu}(b)$ of ν over $\bar{\nu}$, for every $g \in G$, $\frac{dg\nu_{g^{-1}b}}{d\nu_b}(x) = 1$, for ν_b-almost all $x \in X$, for $\bar{\nu}$-almost all $b \in \bar{X}$.*

Proof. – Consider the factor map $\psi:(X,\nu) \to (\bar{X},\bar{\nu})$ and disintegrate ν with respect to $\bar{\nu}$. Denoting the variable on $\bar{X}$ by b, we have $\nu = \int_{\bar{X}} \nu_b\,d\bar{\nu}(b)$. Using the disintegration of the measure ν over the measure $\bar{\nu}$, we can write:

$$g\nu(f) = \nu(L_{g^{-1}}f) = \int_X (L_{g^{-1}}f)(x)d\nu = \int_{\bar{X}} \nu_b(L_{g^{-1}}f)\,d\bar{\nu}(b)$$
$$= \int_{\bar{X}} \left(\int_{X_b} L_{g^{-1}}f(x)\,d\nu_b\right) d\bar{\nu}(b).$$

Here X_b is a fiber of the map ψ over $b \in \bar{X}$, and $\nu_b(X_b) = 1$, $\bar{\nu}$-a.e. Now change variables in the inner integrand via the chain rule. If $X_b \xrightarrow{g} X_{gb} \xrightarrow{f} \mathbb{R}$, then

$$\int_{X_b} L_{g^{-1}}f(x)\,d\nu_b = \int_{X_b} f(gx)\,d\nu_b = \int_{X_{gb}} f(y)\frac{dg\nu_b}{d\nu_{gb}}(y)\,d\nu_{gb}(y).$$

Denoting the last expression by $F_g(b)$, set $\beta = gb$ so that

$$\int_{\bar{X}} F_g(b)\,d\bar{\nu}(b) = \int_{\bar{X}} F_g(g^{-1}\beta)\,dg\bar{\nu}(\beta) = \int_{\bar{X}} \left(\int_{X_\beta} f(x)\frac{dg\nu_{g^{-1}\beta}}{d\nu_\beta}(x)\,d\nu_\beta(x)\right) dg\bar{\nu}(\beta)$$
$$= \int_{\bar{X}} \left(\int_{X_\beta} f(x)\frac{dg\nu_{g^{-1}\beta}}{d\nu_\beta}\,d\nu_\beta(x)\right) \frac{dg\bar{\nu}}{d\bar{\nu}}(\beta)\,d\bar{\nu}(\beta).$$

On the other hand note that:

$$g\nu(f) = \int_X f(gx)\,d\nu = \int_X f(x)\frac{dg\nu}{d\nu}(x)\,d\nu.$$

Proof of (ii) ⇒ (iii). – Suppose now that the Radon–Nikodym cocycle $r_\nu(g,x)$ depends only on $\psi(x) = \beta$, namely:

$$\frac{dg\nu}{d\nu}(x) = r_\nu\big(g^{-1},x\big) = r_{\bar\nu}\big(g^{-1},\psi(x)\big)$$

so that the last expression is (disintegrating ν over $\bar\nu$)

$$g\nu(f) = \int_X f(x)r_\nu\big(g^{-1},x\big)\,d\nu(x) = \int_{\bar X}\Big(\int_{X_\beta} f(x)\,d\nu_\beta(x)\Big)r_{\bar\nu}\big(g^{-1},\beta\big)\,d\bar\nu(\beta).$$

Therefore in this case we obtain two disintegrations of $g\nu$ with respect to $g\bar\nu$, namely:

$$g\nu = \int_{\bar X}\nu_\beta r_{\bar\nu}\big(g^{-1},\beta\big)\,d\bar\nu(\beta) = \int_{\bar X}\frac{dg\nu_{g^{-1}\beta}}{d\nu_\beta}\,\nu_\beta r_{\bar\nu}\big(g^{-1},\beta\big)\,d\bar\nu(\beta).$$

By uniqueness of the disintegration, we get

$$\frac{dg\nu_{g^{-1}\beta}}{d\nu_\beta}(x) = 1$$

for ν_β-almost all $x \in X_\beta$, for $\bar\nu$-almost all $\beta \in \bar X$. Hence the action of G is measure preserving on the fibers of ψ, namely for all $g \in G$, $g \cdot \nu_{g^{-1}b} = \nu_b$ for $\bar\nu$ almost every $b \in \bar X$.

Proof of (iii) ⇔ (i). – By definition, the condition $\frac{dg\nu_{g^{-1}b}}{d\nu_b}(x) = 1$ (for every $g \in G$, ν_b-almost all $x \in X$, and $\bar\nu$-almost all $b \in \bar X$), is equivalent to $g\nu_b = \nu_{gb}$ for $\bar\nu$-almost all $b \in \bar X$, which is our definition of an extension with relatively G-invariant measure.

Proof of (i) ⇒ (ii). – As just noted, if (i) holds, then $\frac{dg\nu_{g^{-1}\beta}}{d\nu_\beta}(x) = 1$, and we obtain two disintegrations of $g\nu$ over $g\bar\nu$ given by

$$g\nu(f) = \int_{\bar X}\Big(\int_{X_b} f(x)\,d\nu_b(x)\Big)r_{\bar\nu}\big(g^{-1},b\big)\,d\bar\nu(b) = \int_{\bar X}\Big(\int_{X_b} f(x)r_\nu\big(g^{-1},x\big)\,d\nu_b(x)\Big)d\bar\nu(b).$$

Again by uniqueness of disintegration the measure $r_{\bar\nu}(g^{-1},b)\,d\nu_b$ is equal to the measure $r_\nu(g^{-1},x)\,d\nu_b$ as measures on X_b, for $\bar\nu$-almost all $b \in \bar X$. Hence the function $r_\nu(g^{-1},x)$ must be ν_b-essentially constant when restricted to X_b, for $\bar\nu$-almost all $b \in \bar X$, and the constant is equal to $r_{\bar\nu}(g^{-1},b)$. □

We now consider the case of (G,μ)-spaces (X,ν), namely when the measure ν is μ-stationary:

PROPOSITION 1.9. – *Let G be an lcsc group, μ an admissible measure on G. Let $\psi:(X,\nu) \to (\bar X,\bar\nu)$ be a measurable factor map of Borel (G,μ)-spaces. Then*

(i) $h_\mu(\bar{X},\bar{\nu}) \leqslant h_\mu(X,\nu)$.
(ii) *When* $h_\mu(X,\nu) < \infty$, *we have* $h_\mu(\bar{X},\bar{\nu}) = h_\mu(X,\nu)$ *if and only if* (X,ν) *is an extension of* $(\bar{X},\bar{\nu})$ *with relatively* G*-invariant measure.*
(iii) *Assume that* $(\bar{X},\bar{\nu})$ *is a transitive* G*-space*: $(\bar{X},\bar{\nu}) = (G/Q,\nu_0)$, Q *a closed subgroup. If* $h_\mu(X,\nu) = h_\mu(G/Q,\nu_0) < \infty$, *then* X *is induced from a probability-measure-preserving action of* Q, *i.e.* $(X,\nu) = (G/Q \ltimes X_0, \nu_0 \times \lambda)$, *where* λ *is a* Q*-invariant measure on* X_0. *Here* ν_0 *is the unique* μ*-stationary measure on* G/Q.

Proof. – To prove (i), note first that if $h_\mu(X,\nu) = \infty$ there is nothing to prove. Otherwise, for μ-almost all $g \in G$ we have by Lemma 1.6:

$$-\int_{\bar{X}} \log r_{\bar{\nu}}(g,b)\, d\bar{\nu}(b) \leqslant -\int_X \log r_\nu(g,x)\, d\nu(x)$$

and the result follows by integrating with respect to μ.

To prove (ii), it is enough, by Proposition 1.8 to show that (for ν-almost all x) $r_\nu(g,x) = r_{\bar{\nu}}(g,\psi(x))$ if and only if $h(X,\nu) = h(\bar{X},\bar{\nu})$.

Disintegrating ν over $\bar{\nu}$ we have:

$$\begin{aligned} h(X,\nu) &= -\int_G\int_X \log r_\nu(g,x)\, d\nu(x)\, d\mu(g) \\ &= -\int_G\int_{\bar{X}} \left(\int_{X_b} \log r_\nu(g,x)\, d\nu_b(x)\right) d\bar{\nu}(b)\, d\mu(g) \\ &= -\int_G\int_{\bar{X}} E\big(\log(r_\nu(g,\cdot)\big)(b)\, d\bar{\nu}(b)\, d\mu(g). \end{aligned}$$

On the other hand, by Lemma 1.6 $E(r_\nu(g,\cdot) = r_{\bar{\nu}}(g,\psi(x))$ where E is the conditional expectation operator, and therefore

$$h(\bar{X},\bar{\nu}) = -\int_G\int_{\bar{X}} \log r_{\bar{\nu}}(g,\beta)\, d\bar{\nu}(\beta)\, d\mu(g) = -\int_G\int_{\bar{X}} \log E\big(r_\nu(g,\cdot)\big)(b)\, d\bar{\nu}(b)\, d\mu(g).$$

Applying Lemma 1.6(1) to the measure space X_b, the measure ν_b and the conditional expectation corresponding to the trivial factor (namely the operator of integration with respect to ν_b), we obtain, for μ-almost all $g \in G$

$$E\big(-\log r_\nu(g,\cdot)\big)(b) > -\log r_{\bar{\nu}}(g,b)$$

unless the function $r_\nu(g,x)$ is constant ν_b-a.e. when restricted to X_b. Hence equality of entropies $h_\mu(X,\nu) = h_\mu(\bar{X},\bar{\nu})$ is equivalent to the condition $r_\nu(g,x) = r_{\bar{\nu}}(g,\psi(x))$ $\mu \times \nu$-a.e. (g,x). To see that this holds for all $g \in G$, note that ν is stationary under μ^{*n}, and furthermore, by a straightforward computation, for any (G,μ)-space space (X,ν):

$$\int_G\int_X -\log r_\nu(g,x)\, d\nu(x)\, d\big(\mu^{*n}\big)(g) = nh(X,\nu).$$

Therefore, the same conclusion applies to μ^{*n}-almost every $g \in G$. Since μ is admissible and $\bigcup_{n\geqslant 0} C^n = G$, ($C = \operatorname{supp}\mu$), it holds for Haar-almost all $g \in G$, and in particular, for a dense set in G. By Proposition 1.14 proved below, it follows that $r_\nu(g,x)$ is measurable with respect to $\varphi^{-1}(\bar{\mathcal{L}})$ for every $g \in G$ and (ii) is proved.

To prove (iii), note that if $\psi:(X,\nu) \to (G/Q,\nu_0)$ is an extension with relatively G-invariant measure, then for ν_0-almost all $b = gQ \in G/Q$, the subgroup gQg^{-1} leaves invariant the probability measure ν_b on the fiber $X_b = \psi^{-1}(gQ)$. If μ is admissible, then ν_0 is G-quasi-invariant ([14, Lemma 1.1]), and then (X,ν) is G-isomorphic to the action of G induced by the action of Q on the measure space $(\psi^{-1}([Q]),\nu_{[Q]})$. This fact is well known, and can be found in [18]. □

1.2. The Radon–Nikodym factor

Given a (G,μ)-space (X,ν), recall that $\mathcal{L}$ denotes the completion of the σ-algebra $\mathcal{B}$ with respect to the measure ν. Each function $r_\nu(g,\cdot)$ is defined up to equivalence modulo ν-null functions, and is measurable with respect to $\mathcal{L}$.

We now define, following [9], a G-invariant sub-σ-algebra of $\mathcal{L}$.

DEFINITION 1.10. – *The Radon–Nikodym σ-algebra associated with the (G,μ)-space (X,ν) is defined by: $\mathcal{RN} = \bigcap\{\mathcal{L}' \subset \mathcal{L} \mid \mathcal{L}'$ is a ν-complete sub-σ-algebra, and for every $g \in G$, $r_\nu(g,\cdot)$ is measurable with respect to $\mathcal{L}'\}$.*

We then have the following:

LEMMA 1.11. –
(1) *$\mathcal{RN} \subset \mathcal{L}$ is a ν-complete G-invariant σ-algebra, and each function $r_\nu(g,\cdot)$ is measurable with respect to $\mathcal{RN}$.*
(2) *There exist a standard Borel G-space $(\bar{X},\bar{\mathcal{B}},\bar{\nu})$ and a Borel measurable G-factor map $\varphi:(X_1,\mathcal{B},\nu) \to (\bar{X},\bar{\mathcal{B}},\bar{\nu})$, satisfying $\varphi^{-1}(\bar{\mathcal{L}}) = \mathcal{RN}$, and $\varphi(\nu) = \bar{\nu}$. Here X_1 is a G-invariant co-null set, and $\bar{\mathcal{L}}$ is the completion of $\bar{\mathcal{B}}$ with respect to $\bar{\nu}$.*
(3) *The Radon–Nikodym cocycles satisfy: $r_\nu(g,x) = r_{\bar{\nu}}(g,\varphi(x))$ for ν-almost all $x \in X$. Furthermore, if $\psi : (X,\mathcal{B},\nu) \to (X',\mathcal{B}',\nu')$ is another G-factor map, with $r_\nu(g,x) = r_{\nu'}(g,\psi(x))$ ν-a.e., then $(\bar{X},\bar{\mathcal{B}},\bar{\nu})$ is a measurable G-factor of $(X',\mathcal{B}',\nu')$.*

Proof. –
(1) If f_1 and f_2 are measurable with respect to $\mathcal{L}'$, then so is f_1/f_2. Therefore if $r(g,\cdot)$ is measurable with respect to $\mathcal{L}'$ for all $g \in G$, so is $r(gh,\cdot)/r(h,\cdot) = r(g,h\cdot)$ for every (g,h). Hence, if $A = r(g,\cdot)^{-1}(U) \in \mathcal{L}'$ for some open $U \subset \mathbb{R}$, it follows that $h^{-1}A \in \mathcal{L}'$ also, for all $h \in G$. Since these sets generate $\mathcal{RN}$ (modulo null sets), it is G-invariant. Finally, since each $\mathcal{L}'$ measures each of the functions $r_\nu(g,\cdot)$, so does their intersection.
(2) This part follows from the point realization theorem due to G.W. Mackey [20, Theorem B.10] using the fact that $\mathcal{RN}$ is G-invariant σ-algebra. (Note however that the choice of the Borel structure, namely a countably generated and countably separated σ-algebra $\bar{\mathcal{B}}$ with completion $\bar{\mathcal{L}} \cong \mathcal{RN}$ is not unique.)
(3) The equality $r_\nu(g,x) = r_{\bar{\nu}}(g,\varphi(x))$ ν-a.e. for each $g \in G$ holds by definition. If also $r_\nu(g,x) = r_{\nu'}(g,\psi(x))$ ν-a.e. then the σ-algebra $\psi^{-1}(\mathcal{L}')$ measures all the Radon–Nikodym derivatives $r_\nu(g,\cdot)$, and hence contains $\mathcal{RN}$. The claim now follows as in (ii). □

DEFINITION 1.12. – *The G-space $(\bar{X},\mathcal{RN},\bar{\nu})$ is the Radon–Nikodym factor of $(X,\mathcal{B},\nu)$. It was introduced for countable groups in* [9].

We note the following properties:

LEMMA 1.13. – *Assume (X,ν) is a (G,μ)-space, and $h_\mu(X,\nu) < \infty$. Then*

(1) *(X,ν) and the Radon–Nikodym factor $(\bar{X},\bar{\nu})$ have the same Furstenberg entropy, $h_\mu(X,\nu) = h_\mu(\bar{X},\bar{\nu})$. Therefore (X,ν) is an extension with relatively G-invariant measure of its Radon–Nikodym factor $(\bar{X},\bar{\nu})$.*

(2) *Every non-trivial G-factor of the Radon–Nikodym factor $(\bar{X},\bar{\nu})$ has strictly smaller Furstenberg entropy.*

Proof. –

(1) By Proposition 1.9 $h_\mu(\bar{X},\bar{\nu}) \leqslant h_\mu(X,\nu)$, since it is a factor. By Lemma 1.6 equality holds, since each function $r_\nu(g,\cdot)$ is measurable with respect to the factor, by definition.

(2) Let $\varphi:(\bar{X},\bar{\nu}) \to (Y,\nu')$ be a G-factor map. By Lemma 1.6 we have $E(r_{\bar{\nu}}(g,\bar{x})) = r_{\nu'}(g,y)$ $(y=\varphi(\bar{x}))$, and

$$0 \leqslant -\int\limits_Y \log r_{\nu'}(g,y)\,d\nu' < -\int\limits_{\bar{X}} \log r_{\bar{\nu}}(g,\bar{x})\,d\bar{\nu},$$

unless $r_{\bar{\nu}}(g,\bar{x})$ is measurable with respect to $\varphi^{-1}(\mathcal{L}(Y))$. Hence if $h_\mu(\bar{X},\bar{\nu}) = h_\mu(Y,\nu')$ it follows that $r_{\bar{\nu}}(g,\bar{x})$ is measurable with respect to $\varphi^{-1}(\mathcal{L}(Y))$ for μ-almost all $g\in G$. To see that this holds for all $g \in G$, again since ν is stationary under μ^{*n}, and

$$\int\limits_G \int\limits_X -\log r_\nu(g,x)\,d\nu(x)\,d\big(\mu^{*n}\big)(g) = nh(X,\nu)$$

the same conclusion applies to μ^{*n}-almost every $g\in G$. Since μ is admissible and $\bigcup_{n\geqslant 0} C^n = G$, $(C=\operatorname{supp}\mu)$, it holds for Haar-almost all $g\in G$, and in particular, for a dense set in G. By Proposition 1.14 proved below, it follows that $r_{\bar{\nu}}(g,\bar{x})$ is measurable with respect to $\varphi^{-1}(\mathcal{L}(Y)$ for every $g\in G$. Hence $\varphi^{-1}\mathcal{L}(Y) = \mathcal{L}(\bar{X})$ and so $(\bar{X},\bar{\nu})$ and (Y,ν') are measurably G-isomorphic. □

We now prove the following result, used in Propositions 1.9 and 1.13:

PROPOSITION 1.14. – *Let $(X,\mathcal{L},\nu)$ be a G-space with a quasi-invariant measure ν, $r_\nu(g,x)$ the Radon–Nikodym cocycle. If $\mathcal{D}\subset\mathcal{L}$ is a ν-complete sub-σ-algebra, and the functions $r_\nu(g,\cdot)$ for every g in a dense subset of G are measurable with respect to $\mathcal{D}$, then the functions $r_\nu(g,\cdot)$ are measurable with respect to $\mathcal{D}$ for every $g\in G$.*

Proof. – Consider the unitary representation $\pi_{\frac12}$ of G in the space $L^2(X,\mathcal{B},\nu)$, defined by the formula:

$$\pi_{\frac12}(g)f(x) = \sqrt{r_\nu\big(g^{-1},x\big)}\,f\big(g^{-1}x\big).$$

The representation is strongly continuous, so that if $h_n\to h$ in G, then $\pi_{\frac12}(h_n)1 \to \pi_{\frac12}(h)1$ in L^2-norm. Assume that the functions $r_\nu(g_k,\cdot)$ are measurable with respect to $\mathcal{D}$ for some sequence g_k dense in G. Then if $h_n = g_{k_n}$ converge to $h\in G$, it follows that for some subsequence h_{n_j}, the sequence $\sqrt{r_\nu(h_{n_j},x)}$ converges pointwise almost everywhere to $\sqrt{r_\nu(h,x)}$. It follows that $r_\nu(h,x)$ is measurable with respect to $\mathcal{D}$ for every $h\in G$. □

1.3. Cohomology invariants and stationary measures

In this section we show the relevance of the assumption that the measure ν on a G-space X is stationary, rather than just quasi-invariant. We assume throughout this section that the measure μ is of class B_∞, namely it is admissible and has a convolution power with bounded density of compact support. Then $r_\nu(g,x)$ is a bounded function of $x\in X$, for every $g\in G$, and the same holds for $r_\nu(g,x)^{-1}$ (see [13]). It follows that if $F \in L^1(X,\nu)$, then also $F \circ g \in L^1(X,\nu)$ for every $g \in G$, so that $L^1(X,\nu)$ is a G-module with G acting by translations.

Let $Z^1(G,X,\nu)$ denote the space of real-valued measurable additive cocycles for the action. Define now:

$$L^1Z^1(G,X,\nu) = \left\{ s(g,x) \in Z^1(G,X,\nu) \colon \int_X |s(g,x)|\, d\nu < \infty \text{ for all } g \in G \right\}.$$

Two cocycles in $L^1Z^1(G,X,\nu)$ are called L^1-equivalent if they satisfy $s_1(g,x) = s_2(g,x) + F(gx) - F(x)$, where $F \in L^1(X,\nu)$. The set of equivalence classes will be called the L^1-cohomology space (of measurable real valued additive integrable cocycles), denoted $L^1H^1(G,X,\nu)$.

Given any cocycle $s(g,x) \in L^1Z^1(G,X,\nu)$, we can consider the expression $h_\mu(X,\nu,s) = \int_G \int_X s(g,x)\, d\nu(x)\, d\mu(g)$, which may be finite or $\pm\infty$.

The Furstenberg entropy of a stationary measure of class B_∞ constitutes an invariant of the L^1-cohomology class of the cocycle $\log r_\nu(g,x)$. More precisely:

PROPOSITION 1.15. – *Let G be an lcsc group, μ a probability measure of class B_∞. Let (X,ν) be a (G,μ) space. Then*:

(1) *The additive cocycle $-\log r_\nu(g,x)$ belongs to the space $L^1Z^1(G,X,\nu)$, and the Furstenberg entropy $h_\mu(X,\nu)$ is finite.*

(2) *Let ν' be a probability measure equivalent to ν, satisfying $\frac{d\nu'}{d\nu} = f(x)$ where $f > 0, \log f \in L^1(X,\nu)$. Then $r_{\nu'}(g,x) = \frac{dg^{-1}\nu'}{d\nu'}(x)$ satisfies $h_\mu(X,\nu) = h_\mu(X,\nu')$.*

(3) *Conversely, if for some G-quasi-invariant probability measure ν on a G-space X, $h_\mu(X,\nu)$ is finite and constitutes an invariant of the L^1-cohomology class of the cocycle $\log r_\nu(g,x)$, then the measure ν is necessarily μ-stationary.*

Proof. –

(1) The function $g \mapsto \int_X \log r_\nu(g,x)\, d\nu$ is a continuous function on G, provided ν is stationary with respect to a measure of class B_∞ (see [13]). Hence it is integrable over G with respect to the measure μ, and $h_\mu(X,\nu)$ is finite.

((2) and (3)). To show that the Furstenberg entropy is an L^1-cohomology invariant of r_ν (when finite) if and only if ν is stationary, we compute, putting $r_{\nu'}(g,x) = f(gx) r_\nu(g,x) f(x)^{-1}$:

$$\begin{aligned} h_\mu(X,\nu') &= -\int_G \int_X \log r_{\nu'}(g,x)\, d\nu'(x)\, d\mu(g) \\ &= -\int_G \int_X \log r_\nu(g,x)\, d\nu(x)\, d\mu(g) - \int_G \int_X \log f(gx)\, d\nu(x)\, d\mu(g) \\ &\quad + \int_G \int_X \log f(x)\, d\nu(x)\, d\mu(g) \\ &= h_\mu(X,\nu) - \int_X \log f(x)\, d(\mu * \nu)(x) + \int_X \log f(x)\, d\nu(x) \end{aligned}$$

by definition of the convolution $\mu * \nu$. It follows that if $\mu * \nu = \nu$ and $\log f \in L^1(X,\nu)$, then $h_\mu(X,\nu) = h_\mu(X,\nu')$. Conversely, when the entropy is finite and is an L^1-cohomology invariant of the cocycle $\log r_\nu(g,x)$, it follows that for any $F = \log f \in L^1(X,\nu)$

$$\int_G \int_X F(gx)\, d\nu(x)\, d\mu(g) = \int_X F(x)\, d(\mu * \nu)(x) = \int_X F(x)\, d\nu$$

so that $\mu * \nu = \nu$. $\square$

Remark. –

(1) Let q be any cocycle $q \in L^1 Z^1(G, X, \nu)$, such that the function $g \mapsto \int_X |q(g,x)|\, d\nu$ is bounded on compact sets in G. If μ is of class B_∞, the (finite) quantity

$$h_\mu(X,\nu,q) = \int_G \int_X q(g,x)\, d\nu(x)\, d\mu(g)$$

is an invariant of the L^1-cohomology class of q if and only if ν is μ-stationary, as follows from the previous argument.

(2) For an arbitrary admissible μ, it is natural to consider the space of $(\mu \times \nu)$-integrable cocycles:

$$\left\{ s \in Z^1(G,X,\nu) \colon \int_G \int_X |s(g,x)| d\nu\, d\mu < \infty \right\}.$$

This space was introduced by H. Furstenberg in [6], and it was shown there that $(\mu \times \nu)$-integrable cocycles satisfy a strong law of large numbers on (X,ν).

2. Rigidity of Furstenberg entropy

We now specialize to the case in which G is a semisimple Lie group. We begin by recalling some important facts regarding the structure of a stationary measure ν on a general (G,μ)-space (X,ν), which are due to H. Furstenberg (see [2,3]).

THEOREM 2.1 ([2,3] see also [14]). – *Let G be a connected non-compact semi-simple Lie group with finite center, and μ an admissible measure on G. Let X be an lcsc space with a continuous G-action, and consider the w^*-toplogy on $P(X) = C_0(X)^*$. Then*

(1) *There exists a w^*-continuous affine isomorphism between the set $P_\mu(X)$ of probability measures on X stationary under μ, and the set $P_P(X)$ of probability measures on X invariant under a given minimal parabolic subgroup P* [3, Theorem 2.1].

(2) *Let $\tilde\nu_0$ be any probability measure on G, which under $p: G \to G/P$ satisfies $p(\tilde\nu_0) = \nu_0$, where ν_0 is the* (*unique, see* [3, Theorem 2.2]) *μ-stationary measure on G/P. Then the map $\lambda \mapsto \tilde\nu_0 * \lambda$ implements the isomorphism above $P_P(X) \mapsto P_\mu(X)$. In particular, if λ is P-invariant, then $\tilde\nu_0 * \lambda$ is μ-stationary, and conversely, every μ-stationary measure ν has a representation in the form $\nu = \tilde\nu_0 * \lambda$, where λ is a uniquely determined P-invariant measure* [3, Lemma 2.1].

(3) *Every $\lambda \in P_P(X)$ determines a continuous map $G/P \mapsto P(X)$, given by $gP \mapsto \lambda_{gP} = g\lambda$. Every $\nu \in P_\mu(X)$ is a convex combination of the form $\nu = \int_{G/P} \lambda_{gP}\, d\nu_0(gP)$, and λ and ν determine each other uniquely.*

Given a P-invariant probability measure λ on an lcsc G-space X, and any measure ν_0 on G/P, we denote by $\nu_0 * \lambda$ the measure $\tilde\nu_0 * \lambda$. This definition is unambiguous by Theorem 2.1.

It is useful to extend Theorem 2.1 to measurable actions on measurable spaces. The following result is proved in detail in [15].

PROPOSITION 2.2. – *Let G be a connected non-compact semi-simple Lie group with finite center, μ an admissible measure on G. Let $(X,\mathcal{B})$ be a Borel G-space. Then*

(1) *The set of P-invariant Borel measures λ on $\mathcal{B}$ is in one-to-one affine correspondence with the set of μ-stationary measures ν on $\mathcal{B}$. The correspondence satisfies: For every bounded Borel function f, $\nu(f)=\int_{G/P} g\lambda(f)\,d\nu_0(gP)$, or equivalently $\nu=\nu_0*\lambda$.*
(2) *If $\phi:(X,\mathcal{B},\nu)\to(\tilde{X},\tilde{\mathcal{B}},\tilde{\nu})$ is a G-equivariant Borel factor map, and $\nu=\nu_0*\lambda$, $\tilde{\nu}=\nu_0*\tilde{\lambda}$, where λ and $\tilde{\lambda}$ are P-invariant probability measures, then $\phi(\lambda)=\tilde{\lambda}$.*
(3) *In particular, $\phi:(X,\mathcal{B},\lambda)\to(\tilde{X},\tilde{\mathcal{B}},\tilde{\lambda})$ is a P-equivariant measurable factor map.*

DEFINITION 2.3. – *The G action on a (G,μ)-space (X,ν) will be called a* P-mixing action *if the P-action on (X,λ) is mixing in the usual sense for probability-preserving actions, namely for every $f\in L_0^2(X,\lambda)$, the matrix coefficients $\langle\pi(p)f,f\rangle\to 0$ as p leaves compact sets in P. Note that when the action is P-mixing, λ is P-ergodic, so that ν is ergodic under G by Theorem 2.1(1).*

Now let (X,ν) be a (G,μ)-space, and λ the P-invariant measure on X corresponding to ν under Theorem 2.1. Let X_0 be the support of λ, and $G\times_P X_0\cong G/P\ltimes X_0$ the G-space induced by the P-action on X_0. (See [14] for details, discussion and references.) We have:

PROPOSITION 2.4 ([14, Proposition 2.5]). –

(1) *The G-space (X,ν) is a measurable G-factor of the G-space $(G/P\ltimes X_0,\nu_0\times\lambda)$.*
(2) *If the measure λ corresponding to ν in Theorem 2.1 is invariant under a parabolic subgroup Q, then (X,ν) is a G-factor of the G-space $(G/Q\ltimes X_0,\nu_0\times\lambda)$. (We also denote by ν_0 the canonical projection of $\nu_0\in P(G/P)$ to a measure on G/Q).*

We therefore have

COROLLARY 2.5. –

(1) *For any (G,μ)-space (X,ν)*

$$0\leqslant h_\mu(X,\nu)\leqslant h_\mu(G/P,\nu_0).$$

(2) *If the measure λ corresponding to ν in Theorem 2.1 is invariant under a parabolic subgroup Q, then $h_\mu(X,\nu)\leqslant h_\mu(G/Q,\nu_0)$.*

Proof. – $(G/P\ltimes X_0,\nu_0\times\lambda)\to(G/P,\nu_0)$ is a G-factor map with a relatively invariant measure. Now apply Proposition 1.9 twice. The same argument applies when λ is Q-invariant, using also Theorem 2.1. □

We now note the following useful fact, which is an immediate corollary of Proposition 2.2.

LEMMA 2.6. – *Let (G,μ) be as in Proposition 2.2. If (X,ν) is (G,μ)-space and $\nu=\nu_0*\lambda$, then for any (G,μ)-factor space $\psi:(X,\nu)\to(X',\nu')$,*

(1) *(X_0',λ') is a P-factor of (X_0,λ), where $\nu'=\nu_0*\lambda'$.*
(2) *In particular, if (X,ν) is a P-mixing action, so is any factor (X',ν').*

For G of $\mathbb{R}$-rank $\geqslant 2$, we now deduce from Theorem A and Proposition 1.9:

THEOREM 2.7. – *Let G be a connected semisimple Lie group with finite center, and $\mathbb{R}$-rank$(G)\geqslant 2$. Suppose $\mu\in P(G)$ is admissible, $h_\mu(G/P,\nu_0)<\infty$, (X,ν) is a (G,μ)-space and the action is P-mixing. Then the Furstenberg entropy $h_\mu(X,\nu)$ takes on one of the finitely many* *values $h_\mu(G/Q,\nu_0)$ for some parabolic subgroup $Q \subset G$ (and the unique μ-stationary measure ν_0 on G/Q.) Furthermore, for some Q with $h_\mu(X,\nu) = h_\mu(G/Q,\nu_0)$, (X,ν) is induced from a finite measure preserving action of Q.*

We now consider the following corollaries:

COROLLARY 2.8. – *Let (G,μ) be as in Theorem 2.7, and let $r = \mathbb{R}$-rank (G). For each i, let (X_i,ν_i) be (G,μ)-space, and assume there exists a G-factor map $p_i:(X_i,\nu_i) \to (X_{i-1},\nu_{i-1})$ for each i. If the G-actions on (X_i,ν_i) are P-mixing, then for all but at most 2^r-values of i, (X_i,ν_i) is an extension with relatively G-invariant measure over (X_{i-1},ν_{i-1}).*

Proof. – By Theorem 2.7, the entropies $h_\mu(X_i,\nu_0)$ can take at most 2^r different values. Therefore the result follows from Proposition 1.9. □

COROLLARY 2.9. – *Let (G,μ) be as in Theorem 2.7. Then for any P-mixing action with stationary measure (X,ν), the Radon–Nikodym factor $(\bar{X},\mathcal{RN},\bar{\nu})$ of (X,ν) is a transitive G-space, of the form $(G/Q,\nu_0)$, Q a parabolic subgroup, where $h_\mu(X,\nu) = h_\mu(\bar{X},\bar{\nu}) = h_\mu(G/Q,\nu_0)$.*

Proof. – By Proposition 2.6 the Radon–Nikodym factor is a P-mixing action and by Proposition 1.9 it has the same Furstenberg entropy as (X,ν). By Theorem 2.7 and Proposition 1.9 it is an extension of a space $(G/Q,\nu_0)$ with relatively G-invariant measure, and in particular with the same Furstenberg entropy. But by Lemma 1.13, any proper factor of the Radon–Nikodym factor has strictly smaller Furstenberg entropy, and the corollary follows. □

Another corollary of Theorem 2.7 is the following:

COROLLARY 2.10. – *Let (G,μ) be as in Theorem 2.7, and define*

$$h_m(\mu) = \min\bigl\{h_\mu(G/Q,\nu_0)\colon\ Q \text{ is a proper parabolic subgroup of } G\bigr\}.$$

Let (X,ν) be a P-mixing (G,μ)-space, with $h_\mu(X,\nu) < h_m(\mu)$. Then the measure ν is G-invariant, namely $h_\mu(X,\nu) = 0$.

We remark that a similar conclusion holds for any (G,μ)-space (X,ν) provided G is a lcsc group satisfying property T of Kazhdan (see Section 1.1. and [13]).

3. Furstenberg entropy for actions in $\mathbb{R}$-rank 1 and non-mixing actions in higher rank

The aim of this section is to prove that rigidity of Furstenberg entropy stated in Theorem 2.7 fails for P-mixing actions of $\mathbb{R}$-rank one groups. Furthermore, we will show that it fails also for actions of higher rank groups, which are P-ergodic but not P-mixing.

THEOREM 3.1. – *Let $G = PSL(2,\mathbb{R})$ and fix an admissible measure on G. Then there exists an infinite sequence of compact metric (G,μ)-spaces (X_i,ν_i) (in fact smooth manifolds), where the action is P-mixing, and the Furstenberg entropies $h_\mu(X_i,\nu_i)$ are all distinct.*

We first note the following fact:

LEMMA 3.2. – *Let $\varphi:(X,\mathcal{B},\eta) \to (X',\mathcal{B}',\eta')$ be a Borel measurable G-factor map between two Borel measurable G-spaces with quasi-invariant probability measures. Then*

(1) *If (X,η) is an extension of (X',η') with relatively G-invariant measure then there exists a measurable G-equivariant map $\psi: X' \to P(X)$, such that the $\psi(x')(\varphi^{-1}(x')) = 1$ for η'-almost all $x' \in X'$, namely the measure $\psi(x')$ is almost always supported in $\varphi^{-1}(x')$.*

(2) *Conversely, if such a measurable G-equivariant map $\psi: X' \to P(X)$ exists, then the measure $\tilde{\eta} = \int_{X'} \psi(x')\, d\eta'$ is G-quasi-invariant, and $(X, \tilde{\eta})$ is an extension of (X', η') with relatively G-invariant measure.*

Proof. – (1) The map $\psi: X' \to P(X)$ given by $\psi: x' \mapsto \eta_{x'}$, where $\eta = \int_{X'} \eta_{x'}\, d\eta'(x')$ is the disintegration of η with respect to η', is measurable. Furthermore, ψ is G-equivariant if and only if the extension has relatively G-invariant measure, and the claim follows.

(2) Here the condition of disjoint supports, $\varphi^{-1}(x_1) \cap \varphi^{-1}(x_2) = \emptyset$, as well as $\psi(x_1')(\varphi^{-1}(x_2')) = 0$ if $x_1' \neq x_2'$ are satisfied (η'-a.e.). Together with uniqueness of disintegration of measures, they imply that the disintegration of $\tilde{\eta}$ with respect to η' is given by the integral defining $\tilde{\eta}$. Since ψ is G-equivariant, the extension has relatively G-invariant measure by definition, and the claim follows. □

Proof. – To prove Theorem 3.1, it suffices by Propositions 1.8 and 1.9 to show that for each N, there is a length N sequence of P-mixing (G, μ)-spaces (X_i, ν_i), $1 \leqslant i \leqslant N$, and G-factor maps $\varphi_i: (X_i, \nu_i) \to (X_{i-1}, \nu_{i-1})$, such that (X_i, ν_i) does not have relatively G-invariant measure over (X_{i-1}, ν_{i-1}).

Fix N simple non-compact Lie groups, say L_i, $1 \leqslant i \leqslant N$, which may be the same or different. Let $P_i \subset L_i$ be a minimal parabolic subgroup. We can then find a finitely generated free group F and a homomorphism

$$\pi: F \to \prod_1^N L_i$$

(with projections denoted by $\pi_i: F \to L_i$) such that:

(i) $\pi(F)$ is dense in $\prod_1^N L_i$; and

(ii) for each j, there is some $g_j \in F$ such that $\pi_i(g_j) = e$ for $i < j$, and $\pi_j(g_j)$ is not of finite order.

Now choose a cocompact lattice $\Gamma \subset G = PSL_2(\mathbb{R})$ and a surjective homomorphism $\theta: \Gamma \to F$.

Let

$$Y_j = \prod_1^j L_i/P_i,$$

which is a compact Γ-space via the homomorphism $\pi \circ \theta$. By projection, $Y_j \to Y_{j-1}$ is an extension of Γ-spaces, with fiber L_j/P_j. Let X_j be the G-space induced from the Γ-action on Y_j, namely $X_j = G/\Gamma \ltimes Y_j$. Using the previous projection, we construct the natural G-factor map $\varphi_j: X_j \to X_{j-1}$, again with the same fiber L_j/P_j. Fixing an admissible measure μ on G, and an ergodic μ-stationary measure ν_N on X_N, we let $\nu_{j-1} = \varphi_j(\nu_j)$, so that φ_j becomes a factor map of G-spaces with stationary measures.

We note that a (G, μ)-stationary measure ν_j on X_j projects to the invariant probability measure m on G/Γ, under the natural map $X_j = G/\Gamma \ltimes Y_j \to G/\Gamma$ since the latter is the unique (G, μ)-stationary measure (see [14, Lemma 6.3]). However the fiber measures $\alpha^j_{h\Gamma}$ on Y_j occuring in the disintegration $(X_j, \nu_j) \to (G/\Gamma, m)$ may be singular with respect to the Lebesgue measure class on Y_j.

We now claim that the extension $(X_j, \nu_j) \to (X_{j-1}, \nu_{j-1})$ does not have relatively G-invariant measure.

If it did, then consider the disintegration $\nu_j = \int_{G/\Gamma \times Y_{j-1}} \nu_{(h\Gamma, y_{j-1})}\, d\nu_{j-1}$. The measure $\nu_{(h\Gamma, y_{j-1})}$ is supported on the set

$$\varphi_j^{-1}(h\Gamma, y_{j-1}) = \{(h\Gamma, y_{j-1})\} \times L_j/P_j \subset X_j.$$

The map $\psi_j:(h\Gamma, y_{j-1}) \mapsto \nu_{(h\Gamma, y_{j-1})}$ is a G-equivariant measurable map $\psi_j : X_{j-1} \to P(X_j)$.

Since $X_{j-1} = G/\Gamma \ltimes Y_{j-1}$, for m-almost all $h\Gamma \in G/\Gamma$ the restriction of ψ_j to the set $\{h\Gamma\} \times Y_{j-1}$ satisfies $h\gamma h^{-1}\psi_j(h\Gamma, y_{j-1}) = \psi_j(h\Gamma, \tau(h)\gamma\tau(h)^{-1}y_{j-1})$, ($\alpha^{j-1}_{h\Gamma}$-a.e.), where $\tau(h)$ is a fixed element of Γ (see [14, §6]).

Fixing such a point $h\Gamma$, the restriction of ψ_j implements a Γ-equivariant map $\tilde{\psi}_j : Y_{j-1} \to P(X_j)$. Here Γ acts ($\alpha^{j-1}_{h\Gamma}$-ergodically) on the first space via conjugation by $\tau(h)$, and on the second via conjugation by h, and $\tilde{\psi}_j$ is defined $\alpha^{j-1}_{h\Gamma} = \alpha^{j-1}$-almost everwhere.

Now project the measure $\nu_{(h\Gamma, y_{j-1})}$ supported on $\{(h\Gamma, y_{j-1})\} \times L_j/P_j$, to a measure on L_j/P_j. The projection is Γ-equivariant, where Γ acts on L_j/P_j via conjugation by $\tau(h) \in \Gamma$. Hence by composition we obtain a measurable Γ-map, defined almost everywhere with respect to α^{j-1}:

$$\zeta_j : Y_{j-1} \to P(L_j/P_j),$$

where as above $P(\cdot)$ denotes the space of probability measures. The action of L_j on $P(L_j/P_j)$ is tame with real algebraic amenable stabilizers (see [20, Chapter 3]). The Γ action on (Y_{j-1}, α^{j-1}) is ergodic, and it follows that for α^{j-1}-almost all y_{j-1}, $\zeta_j(y_{j-1})$ is belongs to a single L_j-orbit, and so we can view ζ_j as a Γ-map

$$\zeta_j : Y_{j-1} \to L_j/H_j$$

for some amenable algebraic subgroup H_j. The measure $(\zeta_j)_*(\alpha^{j-1})$ is a Γ-quasi-invariant measure on L_j/H_j. Since the image of Γ is dense in L_j, the support of this measure must coincide with L_j/H_j. However, since g_j acts trivially on Y_{j-1}, and ζ_j is a Γ-map, g_j must act trivially on almost all points of L_j/H_j with respect to the pushed forward measure. Since this measure has full support, g_j acts trivially on L_j/H_j. Hence $\pi_j(g_j)$ is contained in the intersection of all conjugates of H_j, a closed normal subgroup of L_j. But since H_j is amenable, it is a proper subgroup, hence $\pi_j(g_j)$ is contained in the finite center of L_j. Since $\pi_j(g_j)$ has infinite order, this is impossible, and so $(X_j, \nu_j) \to (X_{j-1}, \nu_{j-1})$ does not have relatively G-invariant measure.

Now suppose (as we may) that each L_i is of $\mathbb{R}$-rank 1. To complete the proof of the theorem, it suffices to see that the action of G on (X_j, ν_j) is P-mixing. Fix j, and if λ is a P-invariant measure on $X_j = G/\Gamma \ltimes Y_j$, then write

$$\lambda = \int_{G/\Gamma} \lambda_z \, dm(z),$$

where λ_z is supported on the fiber of $X_j \to G/\Gamma$ over z, which we can indentify with a measure on Y_j. For each $i \leqslant j$, let λ^i_z denote the projection of λ_z to a measure on L_i/P_i. The argument in the proof of [14, §6, Theorem B] shows that for a.e. $z \in G/\Gamma$, λ^i_z is supported on a single point of L_i/P_i. This implies that λ_z itself is supported on a point, and hence as in [14, §6], the P-action on (X_j, λ) is P-isomorphic to the P-action on $(G/\Gamma, m)$. It follows that $L^2(X_j, \lambda)$ is a mixing representation of P (see, e.g., [20]). □

COROLLARY 3.3. – *There is a $PSL(2,\mathbb{R})$ action on a compact manifold M with measure ν, stationary under an admissible measure μ, which is P-mixing, but whose Radon–Nikodym factor $(\bar{M}, \mathcal{RN}, \bar{\nu})$ is not a transitive action of $PSL(2,\mathbb{R})$, and in particular, it is not a Furstenberg boundary of G.*

Remark. – Theorem 3.1 holds for the groups $SO(n,1)$, $N \geqslant 2$ as well, since they also contain a uniform lattice mapping onto a free non-Abelian group, by [10].

For actions of higher rank groups, both Theorem A and Theorem 2.7 fail if one assumes only P-ergodicity and not P-mixing. We will show:

THEOREM 3.4. – *Let G be a simple Lie group with $\mathbb{R}$-rank$(G) \geqslant 2$, containing a parabolic subgroup $Q_0 \subset G$ that maps onto $PSL(2,\mathbb{R})$. Then*:

(a) *The Furstenberg entropy for actions of G on smooth compact manifolds with ergodic stationary measure takes on infinitely many values.*
(b) *There is a smooth compact manifold (X,ν) which is a (G,μ)-space with ergodic stationary measure of positive Furstenberg entropy, such that (X,ν) does not have relatively G-invariant measure over any G-space of the form G/Q, where Q is a parabolic subgroup.*
(c) *The Radon–Nikodyn factor $(\bar{X},\bar{\nu})$ of the space (X,ν) of* (b) *is not a Furstenberg boundary of G.*

Proof. – To prove (a), let X_i be as in the proof of Theorem 3.1, so that X_i is a compact $PSL(2,\mathbb{R})$-space. We can then view X_i as a Q_0-space via the (surjective) homomorphism $Q_0 \to PSL(2,\mathbb{R})$. Let W_i be the G-space induced from the Q_0-action on X_i. Then we have a continuous G-map of compact G-spaces, $W_i \to W_{i-1}$, with fiber L_i/P_i, as before. It suffices to show that no G-quasi-invariant measure (in fact stationary measure) on W_i is a relatively G-invariant measure over W_{i-1}.

However, the existence of such measure on W_i implies the existence of a Q_0-relatively invariant measure for $X_i \to X_{i-1}$. This follows for the pair (G,Q_0) employing the same argument used the proof of Theorem 3.1 to establish the same property for the pair $(PSL_2(\mathbb{R}),\Gamma)$, namely by restricting to an appropriately generic fiber. Clearly, it then follows that $X_i \to X_{i-1}$ is an extension with relatively $PSL_2(\mathbb{R})$-invariant measure (since Q_0 acts via the homomorphism $Q_0 \to PSL_2(\mathbb{R})$), which the proof of Theorem 3.1 shows is impossible.

As to (b), if (X,ν) is a measure-preserving extension of $(G/Q,\nu_0)$ then their Furstenberg entropies are equal by Proposition 1.9. Since there are finitely many parabolic subgroups (containing a given minimal parabolic P), and infinitely many distinct values for the entropy by (a), the conclusion follows.

Finally (c) follows from (b), since by Proposition 1.13 and 1.9 (X,ν) is an extension with relatively G-invariant measure of its Radon–Nikodym factor. $\square$

4. Amenable actions with stationary measure

The aim of this section is to describe the amenable actions with stationary measure. We recall that any amenable ergodic action with quasi-invariant measure of a semisimple Lie group is induced from an action of an amenable algebraic subgroup [18]. When the measure is stationary, we have:

THEOREM 4.1. – *Let G be a connected semisimple Lie group with finite center and no compact factors. Suppose $\mu \in P(G)$ is admissible, $h_\mu(G/P,\nu_0) < \infty$, and G acts ergodically on a space X with μ-stationary measure ν. Then the following are equivalent*:

(i) *The G action on (X,ν) is amenable.*
(ii) *The G-action on (X,ν) is induced from a probability measure preserving action of a minimal parabolic subgroup P.*
(iii) $h_\mu(X,\nu) \geqslant h_\mu(G/P,\nu_0)$.
(iv) $h_\mu(X,\nu) = h_\mu(G/P,\nu_0)$.

Proof. – If G acts amenably on (X,ν), then by [18] there is a G-map $X \to G/S$ where S is amenable and algebraic. If ν is stationary, this implies there is a stationary measure on

G/S, which by a result of Furstenberg [3, Lemma 2.4] implies $S \subset P$. Thus, there is a G-map $X \to G/P$. By Corollary 2.5 it follows that $h_\mu(X,\nu) = h_\mu(G/P,\nu_0)$ and by Proposition 1.9 $X \to G/P$ is an extension with relatively G-invariant measure. This implies (cf. [14, Proposition 2.3]) that X is induced from a probability measure preserving action of P.

In light of Propositions 1.9 and 2.5, to prove the theorem it suffices to show that $h_\mu(X,\nu) = h_\mu(G/P,\nu_0)$ implies that (X,ν) is an amenable action. Let (X_0,λ) be the P-space as in Proposition 2.4 (where λ is a P-invariant measure), so that $G/P \ltimes X_0 \cong G \times_P X_0 \to X$ is a factor map. By Corollary 2.5, $h_\mu((G/P \ltimes X_0), \nu_0 \times \lambda) = h_\mu(X,\nu)$ and hence (by Proposition 1.9) $(G/P \ltimes X_0, \nu_0 \times \lambda)$ has relatively invariant measure over (X,ν). This implies there is a measurable G-map $X \to P(G/P \ltimes X_0)$, where $P(\cdot)$ denotes the space of probability measures. By projecting $G/P \ltimes X_0$ to G/P, we obtain a measurable G-map $\psi : X \to P(G/P)$. The action of G on $P(G/P)$ is tame with amenable algebraic stablizers. Since the G-action on (X,ν) is ergodic, $\psi(x)$ will lie ν-a.e. in a single G-orbit and so we can view ψ as a map $\psi : X \to G/S$ where S is amenable. Thus X is induced from an action of an amenable subgroup of G, and hence the G-action on X is amenable [18]. □

It is interesting to determine whether, when the Furstenberg entropy is strictly smaller than the maximum value (i.e. $h_\mu(X,\nu) < h_\mu(G/P,\nu_0)$) and the action of G is therefore not amenable, the probability measure λ satisfying $\nu = \nu_0 * \lambda$ is invariant under a non-amenable parabolic subgroup Q. Even for higher rank simple groups, however, this is not always the case:

THEOREM 4.2. – *Let $G = SL_3(\mathbb{R})$ and μ an admissible measure on G, with $h_\mu(G/P,\nu_0) < \infty$. Then there exists a compact manifold (M,ν) which is a (G,μ)-space, with ergodic stationary measure ν satisfying $h_\mu(X,\nu) < h_\mu(G/P,\nu_0)$, but no non-amenable parabolic subgroup has an invariant probability measure on M.*

Proof. – Let Q_0 and Q_1 denote the two maximal parabolic subgroups of G containing a given minimal parabolic subgroup P. Fix a surjective homomorphism $Q_0 \to PSL_2(\mathbb{R})$. Let X_j be the compact $PSL_2(\mathbb{R})$-spaces constructed in the proof of Theorem 3.1, where we assume that all the groups L_j have real rank one. Consider X_j as a Q_0-space via the surjective homomorphism $Q_0 \to PSL_2(\mathbb{R})$. We note that by construction, $PSL_2(\mathbb{R})$ does not have an invariant probability measure on X_j, as follows from [14, Lemma 6.1]. As in the proof of Theorem 3.4 let W_j be the action induced to G. By Proposition 2.5 the values of $h_\mu(W_j,\nu_j)$ are all bounded by $h_\mu(G/P,\nu_0)$, and since there are infinitely many distinct values, we can choose $(W,\nu) = (W_{j_0},\nu_{j_0})$ satisfying $0 < h_\mu(W,\mu) < h_\mu(G/P,\nu_0)$. W is a compact G-space, with a continuous G-factor map $\varphi : W \to G/Q_0$. Hence any probability measure λ on W, which is invariant under a parabolic subgroup Q, gives rise to a probability measure $\varphi_*(\lambda)$ on G/Q_0 which is invariant under Q. As is well known, such a measure is supported of the fixed points of Q in the variety G/Q_0. Since Q_1 is not conjugate to a subgroup of Q_0, Q_1 has no fixed points on G/Q_0, and hence there exists no Q_1-invariant probability measure λ on W. If λ is invariant under Q_0, the the support of $\varphi_*(\lambda)$ on G/Q_0 is the unique fixed point of Q_0, namely the coset $[Q_0]$. Since $W = G/Q_0 \ltimes X$, it follows that Q_0 and hence $PSL_2(\mathbb{R})$ has an invariant probability measure on X, which is a contradiction. Clearly, G itself does not have an invariant probability measure on W, since there is no G-invariant probability measure on G/Q_0. Hence no non-amenable parabolic subgroup leaves invariant a probability measure on W. □

Remark 4.3. – In [14] and the present paper we have restricted attention to proving the existence of quotients of the form G/Q, Q parabolic, for actions of G with stationary measures. For an action of G with a general G-quasi-invariant measure on a compact manifold, it is easy to see that such quotients usually do not exist. Namely, given any algebraic subgroup $H \subset G$, by Chevalley's theorem there will be an action on a compact projective variety which has an open orbit of the form G/H. Thus, if H is not contained in a parabolic subgroup, the action on this variety (with the Lebesgue measure class on G/H) will not admit a measurable quotient of the form G/Q.

5. Geometric applications

5.1. P-mixing actions with low Furstenberg entropy

As we briefly discussed in the introduction, actions of semisimple Lie groups with finite invariant measure exhibit strong topological and geometric rigidity properties. When there is no finite measure preserved, much less has been known. Theorem A implies that in a P-mixing action on a (G,μ)-space (X,ν) with $h_\mu(X,\nu) < h_\mu(G/P,\nu_0)$ the P-invariant measure λ is in fact invariant under a non-amenable parabolic subgroup Q, whose Levi component contains a non-trivial semisimple group L. Thus, one can apply rigidity results to the ergodic measure-preserving action of L on (X_0,λ), if the $\mathbb{R}$-rank of L is $\geqslant 2$. Furthermore, the lower the Furstenberg entropy is, the larger is the stability group for λ, and the higher is the $\mathbb{R}$-rank of its Levi component.

We will develop this theme in greater detail elsewhere, and present here some applications to illustrate the point.

DEFINITION 5.1. – *Let G be a connected semisimple Lie group with no compact factors, $\widetilde{G}$ the universal covering group, and μ an admissible measure on $\widetilde{G}$. Let X be a smooth compact manifold (or more generally, a compact space for which covering space theory holds) which is a G-space. Let ν be a μ-stationary P-mixing measure on X. We say the action of G on (X,ν) is engaging if the action of G' on (X',ν') is P'-mixing for all finite covers $X' \to X$, where ν' is the lift of ν to X', and G' is a finite covering group of G that acts on X'.*

Remark 5.2. – For actions G with finite invariant measure, this definition coincides with the notion of engaging used in [11] for example (and is slightly more restrictive than the one introduced in [21]).

DEFINITION 5.3. – *Let G be a connected semisimple Lie group with finite center and no compact factors, and fix an admissible measure μ of finite entropy. Define $\xi(G) = \sup\{c \mid$ for any parabolic Q with $h_\mu(G/Q,\nu_0) \leqslant c$, we have $Q \supset L$ where L is a simple group of $\mathbb{R}$-rank at least $2\}$.*

Remark 5.4. –
- (i) $\xi(G)$ is explicitly computable since $h_\mu(G/Q,\nu_0)$ takes on only finitely many values. Of course, $\xi(G) = 0$ unless the real-rank of $G \geqslant 3$.
- (ii) For $n \geqslant 5$, $\xi(SL(n,\mathbb{R})) > 0$.

THEOREM 5.5. – *Suppose G is a connected semisimple Lie group with finite center and no compact factors and that (X,ν) is a connected compact manifold which is a (G,μ)-space, μ an admissible measure. Suppose the action is engaging. Let σ be any linear representation of $\pi_1(X)$ over $\mathbb{C}$ and suppose $\sigma(\pi_1(X)) = \Gamma$ is infinite. Finally, suppose $h_\mu(X,\nu) < \xi(G)$. Then Γ contains an arithmetic group Λ which is commensurable to $H_{\mathbb{Z}}$ where H is a real-algebraic $\mathbb{Q}$-group with $\mathfrak{h} \supset \mathfrak{l}$. Here $\mathfrak{l}$ is the (non-empty) product of the simple factors of $\mathbb{R}$-rank $\geqslant 2$ in the Levi component of a parabolic subalgebra of $\mathfrak{g}$.*

Proof. – By Theorem A, since the action on (X,ν) is P-mixing, it is induced from a measure-preserving action of a parabolic subgroup Q. Since $h_\mu(X,\nu) < \xi(G)$, the parabolic Q contains a simple subgroup of real rank at least two. Clearly, Q has an invariant probability measure λ on X (satisfying $\nu = \nu_0 * \lambda$). Let $\varphi : X' \to X$ be a continuous finite cover, and λ' the canonical lift of λ with $\varphi(\lambda')=\lambda$. Let G' denote a finite cover of G that acts on X', and μ' the canonical lift of μ. Then $\nu'=\nu_0*\lambda'$ is a (G',μ')-stationary measure, satisfying $\varphi(\nu')=\nu$, by Theorem 2.1. (Note that G' also has finite center, so that Theorem 2.1 applies.) Since the G-action on (X,ν) is assumed engaging, the action of P' on (X',λ') is mixing, hence ergodic. It follows that Q' acts ergodically on (X',λ'), and so the Q action on (X,λ) is engaging in the sense of [11]. Therefore the conclusion follows from [11, Theorem B]. □

Another immediate consequence of Theorem A is:

COROLLARY 5.6. – *Let G be as in Theorem A. Then any G action with a P-mixing stationary measure is orbit-equivalent to a finite measure preserving action of a proper parabolic subgroup.*

We conclude this section with an application to cohomology. Let the group G act on a space (X,ν) with ν quasi-invariant, and let R be an lcsc group. Consider the cohomology space of R-valued measurable cocycles, denoted by $H^1(G,X,\nu,R)$. When $R=\mathbb{R}$, (or more generally when R is Abelian), it can alternatively be described as $H^1_{meas}(G;F(X;\mathbb{R}))$, where H^1_{meas} denotes group cohomology with measurable cocycles, and $F(X,\mathbb{R})$ is the Abelian G-module of measurable $\mathbb{R}$-valued functions with the topology of convergence in measure. If $(X,\nu)=\mathrm{ind}_H^G(X_0,\lambda)$ for some closed subgroup $Q\subset G$, then by [19, Proposition 2.2], for any lcsc group R

$$H^1(G,X,\nu,R)\cong H^1(Q,X_0,\lambda,R).$$

(This can be viewed as a version of Shapiro's Lemma in a measurable setting.)

THEOREM 5.7. – *Let (G,μ), (X,ν), Q and λ be as in Theorem* 2.7. *Suppose $h_\mu(X,\nu)<\xi(G)$* (*Definition* 5.3).

(1) *If R is any lcsc amenable group without non-trivial compact subgroups, then there exists a canonical surjective map*

$$\mathrm{Hom}(Q;R)\to H^1(G,X,\nu,R).$$

(2) *If R is in addition Abelian, then the canonical map above is an isomorphism* (*of Abelian groups*).

Proof. – We define the map between the cohomology spaces as follows. Let λ be the Q-invariant measure on X_0, where we write $(X,\nu)=\mathrm{ind}_Q^G(X_0,\lambda)$. Then (X_0,λ) is a mixing action of Q with finite invariant measure. We have a natural map $\phi:\mathrm{Hom}(Q;R)\to H^1(Q,X_0,\lambda,R)$ given by $\pi\to\alpha_\pi$, where $\alpha_\pi(q,x)=\pi(q)$. As indicated above, we also have a bijective correspondence $H^1(Q,X_0,\lambda,R)\cong H^1(G,X,\nu,R)$. To establish (1) it suffices to show that ϕ is a surjection. We begin by establishing the following:

LEMMA 5.8. – *Let Q be an lcsc group, acting on a measure space (Y,η), where η is Q-quasi-invariant. Let $\alpha:Q\times Y\to R$ be a measurable cocycle into an Abelian lcsc group R. Let $L\subset Q$ be a subgroup acting ergodically and suppose $\alpha:L\times Y\to R$ is a constant cocycle, namely $\alpha(l,y)$ is (a.e.) independent of $y\in Y$ for each $l\in L$. If L centralizes a subgroup $H\subset Q$, then $\alpha:S\times Y\to R$ is also a constant cocycle for all $s\in S=HL=LH$.*

Furthermore, under the stronger condition that $\alpha(l,y)=e$ for all $l\in L$ (and a.e. $y\in Y$), the same conclusion holds for any lcsc target group R.

Proof. – Consider first the case of an Abelian target group R, and write the group operation in R additively. Given $h\in H$ consider the function $\psi(y)=\alpha(h,y)$. Then by the cocycle identity, the fact that $\alpha(l,hy)=\alpha(l,y)$ and $hl=lh$ for all $l\in L$, $h\in H$, we obtain:

$$\begin{aligned}\psi(ly) &= \alpha(h, ly) = \alpha(hl, y) - \alpha(l, y) = \alpha(lh, y) - \alpha(l, y)\\ &= \alpha(l, hy) + \alpha(h, y) - \alpha(l, y) = \psi(y).\end{aligned}$$

Hence $\psi(y)$ is invariant under the action of L, and by ergodicity of L it is constant. Therefore $\alpha(lh, y) = \alpha(l, y) + \alpha(h, y)$ is a sum of two constant functions of y, so the cocycle α is constant on the subgroup S, as asserted.

For a general target group R, the previous argument gives

$$\psi(ly) = \alpha(l, y)\psi(y)\alpha(l, y)^{-1}.$$

Hence if we assume that if fact $\alpha(l, y) = e$ for all $l \in L$ and almost all $y \in Y$, the same argument applies. □

Returning to the proof of Theorem 5.7, we apply Lemma 5.8 to the action of Q on (X_0, λ), which is a P-mixing measure preserving action. Let $\alpha : Q \times X_0 \to R$ be a cocycle. By the hypothesis on entropy, $Q \supset L$ where L is a non-compact simple Kazhdan group. By the P-mixing hypothesis, L will be ergodic on (X_0, λ). Therefore, since R is assumed amenable without non-trivial compact subgroups, and L is a Kazhdan group preserving a finite measure and acting ergodically, then $H^1(L, X_0, \lambda, R) = 0$ [20, Theorem 9.1.1]. We therefore may and will assume by passing to a cohomologous cocycle that $\alpha | L \times X_0 = e$.

Now let $Q = MAN$ be the Langlands decomposition of the parabolic Q. As noted already, by the P-mixing assumption, L is ergodic on (X_0, λ), and clearly Lemma 5.8 implies that $\alpha | MA \times X_0$ is a constant cocycle, namely independent of $x_0 \in X_0$.

Again by the P-mixing assumption, every non-trivial closed subgroup of A acts ergodically on (X_0, λ). Let $N_1 \subset N$ be a subgroup which has a non-trivial centralizer A_1 in $A \cap L$. Since $\alpha | A_1 \times X_0 = e$, we can apply Lemma 5.8 again, and conclude that the cocycle $\alpha : N_1 \times X_0 \to R$ is constant. We claim that N is generated by finitely many 1-parameter subgroups N_i each of which has a non-trivial centralizer A_i in $A \cap L$. Indeed L normalizes N and acts by the adjoint representation on the Lie algebra of N. The representation space is the direct sum of eigenspaces Y_β of $Ad(A \cap L)$, where β is a character of $A \cap L$. Since we assume $\mathbb{R}$-rank $L \geqslant 2$, it follows that each β has a positive-dimension kernel. Hence each Y_β generates a subgroup N_β of N whose centralizer in $A \cap L$ is non-trivial. Since the subgroups N_β generate N, the claim follows.

To complete the discussion of $\alpha | N \times X_0$, note that if $\alpha : N_i \times X_0 \to R$ is constant for each i, it is also constant on the group generated by the N_i. Applying the same argument to the group $Q = MAN$, we finally conclude that $\alpha : Q \times X_0 \to R$ is a constant cocycle, and therefore corresponds to a homomorphism $Q \to R$. This completes the proof that ϕ is surjective.

To prove (2) it suffices to show that when R is Abelian ϕ is injective. Assume then that $\pi_1, \pi_2 : Q \to R$ are homomorphisms, and

$$f(qx_0) + \pi_1(q) - f(x_0) = \pi_2(q) \quad \text{for } f : X_0 \to R \text{ measurable.}$$

Now choose a set $B \subset X$ of positive measure on which f is bounded. If $\pi_1(q) - \pi_2(q)$ is not identically e, choose $q_0 \in Q$ such that $(\pi_1 - \pi_2)(q_0^n) \to \infty$. Since Q leaves the finite measure λ invariant on X_0, for almost any $x_0 \in B$, we can find infinitely many positive n such that $q_0^n \cdot x_0 \in B$. This clearly contradicts $f | B$ is bounded, establishing injectivity of ϕ. □

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Amos NEVO
Technion Israël Institute of Technology,
Department of Mathematics,
32000, Haifa, Israel
E-mail: anevo@tx.technion.ac.il

Robert J. ZIMMER
University of Chicago,
Department of Mathematics,
Chicago, IL 60637, USA
E-mail: zimmer@math.uchicago.edu